MATHEMATICAL FINANCE
An International Journal of Mathematics, Statistics and Financial Economics

Mathematical Finance, Vol. 27, No. 1 (January 2017), 115–150

SENSITIVITY ANALYSIS OF NONLINEAR BEHAVIOR WITH DISTORTED PROBABILITY

XI-REN CAO

Antai College of Economics and Management and Department of Automation, Shanghai Jiao Tong University, and Institute of Advanced Study, Hong Kong University of Science and Technology

XIANGWEI WAN

Antai College of Economics and Management, Shanghai Jiao Tong University

In this paper, we propose a sensitivity-based analysis to study the nonlinear behavior under nonexpected utility with probability distortions (or "distorted utility" for short). We first discover the "monolinearity" of distorted utility, which means that after properly changing the underlying probability measure, distorted utility becomes locally linear in probabilities, and the derivative of distorted utility is simply an expectation of the sample path derivative under the new measure. From the monolinearity property, simulation algorithms for estimating the derivative of distorted utility can be developed, leading to gradient-based search algorithms for the optimum of distorted utility. We then apply the sensitivity-based approach to the portfolio selection problem under distorted utility with complete and incomplete markets. For the complete markets case, the first-order condition is derived and optimal wealth deduced. For the incomplete markets case, a dual characterization of optimal policies is provided; a solvable incomplete market example with unhedgeable interest rate risk is also presented. We expect this sensitivity-based approach to be generally applicable to optimization problems involving probability distortions.

KEY WORDS: probability distortion, monolinearity, portfolio selection, perturbation analysis, sensitivity-based optimization, incomplete market.

1. INTRODUCTION

The portfolio selection problem using expected utility (EU) has been dominant since the seminal papers of Samuelson (1969) and Merton (1969, 1971). The rationality of its economic postulates, in conjunction with its mathematical tractability, makes EU an attractive model to depict decision making under risk (Schoemaker 1982). Despite great successes since creation by von Neumann and Morgenstern (1944), researchers as early

The authors are grateful to two anonymous referees, the associate editor, and Prof. Jerome Detemple, the editor, for their valuable comments. We thank Prof. Xun-Yu Zhou from Oxford University and the Chinese University of Hong Kong and Dr. Hanqing Jin from Oxford University for introducing this subject to us and for insightful discussions. The first author was supported in part by National Natural Science Foundation of China (NSFC) under Grant 61221003 and the Collaborative Research Fund of the Research Grants Council, Hong Kong Special Administrative Region, China, under Grant No. HKUST11/CRF/10. The second author was sponsored by Shanghai Pujiang Program (No. 12PJC063), the National Natural Science Foundation of China (Grant 71401103), and start-up funds from Shanghai Jiao Tong University.

Manuscript received October 2012; final revision received April 2014.

Address correspondence to Xiangwei Wan, Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai, China; e-mail: xwwan@sjtu.edu.cn.

DOI: 10.1111/mafi.12076

as Friedman and Savage (1948), Allais (1953), Ellsberg (1961), and Mehra and Prescott (1985) empirically or experimentally find that investor behaviors may systematically violate the hypotheses of EU. To explain these paradoxes or "irrational" behaviors, several alternative models, labeled nonexpected utility theories, have been proposed (Starmer 2000). Among them, the dual theory of Yaari (1987) and the rank-dependent expected utility (RDEU) of Quiggin (1982, 1993) are of great interest.

In contrast with EU, Yaari's dual theory maintains the invariance of payments while distorting the probabilities of payments. As an alternative model, the dual theory can explain a number of puzzles and paradoxes associated with EU. However, as pointed out by Yaari (1987), new paradoxes arise within the dual theory when the roles of payments and probabilities are reversed, possibly owing to the linearity in payments. RDEU proposed by Quiggin (1982) is a generalization of EU and dual theory, which distorts both payments and probabilities. Distortions in probabilities are consistent with the intuition that investors tend to inflate the effects of rare events and diminish those of common events. The existence of probability distortions is also empirically or experimentally supported by studies of Quiggin (1987), Tversky and Kahneman (1992), Wu and Gonzalez (1996), Prelec (1998), and Polkovnichenko and Zhao (2013). However, a rigorous theory for utility optimization with probability distortions is difficult to establish. Distorted probabilities are nonlinear expectations and thus invalidate the law of iterated expectations. Particularly, the dynamic programming principle, which is a fundamental tool for addressing the dynamic control problem under EU, does not apply in this case.

Significant progress has been made in portfolio optimization based on utility with distorted probabilities, or "distorted utility" for short. In a single-period setting, Bernard and Ghossoub (2010) and He and Zhou (2011a) provide closed-form solutions for several special models with a single risky asset under Tversky and Kahneman (1992)'s cumulative prospect theory (CPT). For the continuous-time model, Carlier and Dana (2006, 2011) use the quantile approach to study the portfolio selection problem under RDEU with a convex distortion function. A breakthrough comes from Jin and Zhou (2008), who propose a quantile approach to manage the behavioral portfolio selection problem under CPT, and obtain analytical solutions under a general complete market setting. He and Zhou (2011b) use the quantile approach to provide a unified formulation for a wide class of portfolio selection problems with law-invariant utility. Similar to Jin and Zhou (2008), He and Zhou (2011b) first assume that the market is complete, enabling the dynamic control problem to be transformed into a static stochastic optimization problem with a constraint formulated by the unique pricing kernel. In an incomplete market, the quantile formulation is available for the case of a deterministic investment opportunity set (He and Zhou 2011b).

Dynamic programming is not applicable in the optimization of distorted utility primarily because such an optimization problem loses the time-consistency property; specifically, if a policy is optimal for distorted utility when the process starts from time t, this policy is no longer optimal from time t onward if the process begins from time $t' < t$. We must seek other approaches that do not rely on this property.

Meanwhile, a sensitivity-based approach to performance optimization has been developed (see Cao 2007, or Cao 2009 for a short survey). Unlike dynamic programming, which essentially works backward in time (finding an optimal rule at t based on that at $t + \Delta t$) and therefore requires time consistency, the sensitivity-based approach is based on a direct comparison of the performances of alternative policies and thus is not subject to the issue of time consistency. This approach includes perturbation analysis (see Cao 1985; Ho and Cao 1983, 1991; and Glasserman 1991), which compares two policies that are infinitesimally close to each other. As the performance criterion and its derivatives are measured on a sample path, the method is a sample-path-based approach.

Models with probability distortions are becoming increasingly attractive in finance and few analytical studies focus on portfolio optimization in that context. Accordingly, we aim to investigate whether the sensitivity-based approach, as an alternative to the quantile method, may help in studying the problem with distorted utility. The results presented in this paper suggest an affirmative answer.

In this paper, we first formulate the optimization problem under distorted utility using the sensitivity-based approach. We then demonstrate that after changing the underlying probability measure to another measure determined by the distortion function, distorted utility shows some sort of local linearity in probabilities. We call this property *monolinearity*. Monolinearity facilitates the analysis and forms the basis of the approach developed in this paper. In particular, it implies that the derivative of distorted utility is an expectation of the sample path derivative under the new measure; monolinearity thus allows us to use perturbation analysis, leading to efficient simulation algorithms for the derivative of distorted utility. The first- and second-order derivatives of distorted utility are also obtained, which characterize the geometric properties of distorted utility as a function of the control parameters.

We then apply the above analysis to the initial asset allocation problem as well as the continuous asset allocation problem under distorted utility. For the initial asset allocation problem, we obtain the first-order condition, from which the optimal allocation policy can be derived for the model with a single risky asset. Efficient simulation algorithms for estimating the derivatives of distorted utility are presented, and simulation-based optimization algorithms can be derived from these derivatives when a closed-form solution is not available. Just as one could expect, monolinearity, in conjunction with the linearity in payments, drives the optimal solutions to the boundary under dual theory. Within the framework of RDEU, we provide an example, in which it is optimal to invest in both the stock and the bond, i.e., an example where diversification is optimal. For the continuous asset allocation problem under RDEU, monolinearity allows us to use the calculus of variations to implement a perturbation analysis and thus discover a structural property of the optimal policies; that is, the distorted marginal utility over optimal terminal wealth plays the role of the pricing kernel in both the complete and the incomplete markets cases. Based on this structural property, we obtain a closed-form solution for the optimal terminal wealth in the complete markets case; an equation for the optimal allocation strategy is provided in Appendix B (alternative characterizations, as in Ocone and Karatzas 1991 and Detemple, Garcia, and Rindisbache 2003, can also be provided). For the incomplete markets case, where the number of stocks is strictly less than the number of Brownian motions, the static optimization problem involves an infinite number of budget constraints because of the infinite number of pricing kernels. Under EU, He and Pearson (1991), Karatzas et al. (1991), Xu and Shreve (1992a,b), and Cvitanic and Karatzas (1992) develop a dual method to transform the portfolio problem into a dual minimization problem over the space of pricing kernels. Based on RDEU, with the help of monolinearity, we develop a dual minimization problem to seek the "right" pricing kernel. Examples illustrate the dual problem. In particular, in Example 5.9, when the coefficients of asset returns are constant and a special distortion function is chosen, closed-form solutions for the optimal portfolio are derived, which show how distorted probabilities affect investment strategies; an increasing effect on the chance of small payments reduces risk-taking. In Example 5.12, we solve explicitly an incomplete market model with a stochastic investment opportunity set.

This paper is organized as follows. In Section 2, we formulate distorted utility as a weighted expectation, leading to a new measure under which the distorted utility becomes a regular expectation. In Section 3, we derive the first- and second-order derivatives of distorted utility and prove one of our main results: the derivatives of distorted utility are expectations of the sample path derivatives under the new measure. Asset allocation problems with distorted probabilities are presented in Sections 4 and 5. Section 6 concludes with some remarks and a discussion. Appendix A presents monolinearity and discusses its properties. Appendix B characterizes the optimal policy in the complete markets case. Appendix C proves Theorem 5.4.

2. REFORMULATION OF DISTORTED UTILITY

Consider nonnegative, atom-less, a.s. finite random variables X_θ, $\theta \in \Theta$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where θ denotes a real parameter that we are concerned with. In EU, the objective is to maximize

$$E_{\mathcal{P}}[U(X_\theta)] = \int_0^{+\infty} \mathcal{P}[U(X_\theta) > x]dx,$$

where $E_{\mathcal{P}}$ denotes the expectation under probability measure $\mathcal{P}$, and $U(x)$ is called a *utility function*, which we assume to take a positive value in the above expectation.

EU uses an event's real probability while distorting payments by the utility function; that is, it models the "satisfactions" for a payment X_θ as $U(X_\theta)$. A concave utility is used by a risk-averse investor. The dual theory proposed by Yaari (1987) maintains the invariance of payments while distorting the probabilities of events that payments appear. This approach models the fact that investors usually subconsciously overestimate the chance of rare events while underestimating the change of common events. Therefore, attaching a weight to the real probability may more closely model investors' normal behaviors. Therefore, in the dual theory, we wish to maximize the following *"distorted utility"*:

$$\eta_{X_\theta} := \widetilde{E}_{\mathcal{P}}[X_\theta] = \int_0^{+\infty} w\{\mathcal{P}[X_\theta > x]\}dx, \tag{2.1}$$

where w is a nonlinear *distortion function*. In RDEU, we distort both payments and probabilities, and thus must optimize the distorted utility over $U(X_\theta)$, that is,

$$\eta_{U(X_\theta)} := \widetilde{E}_{\mathcal{P}}[U(X_\theta)] = \int_0^{+\infty} w\{\mathcal{P}[U(X_\theta) > x]\}dx. \tag{2.2}$$

In this paper, we use sensitivity analysis to study the optimization problem under both dual theory and RDEU, with distorted utility of (2.1) and (2.2). To derive the sample path derivative, we treat (2.1) and (2.2) in almost the same way by replacing X_θ with $U(X_\theta)$. Therefore, we will first consider the dual utility (2.1) for simplicity.

Throughout the paper, we assume that both $w(z) : z \in [0, 1] \to w \in [0, 1]$ and $U(x)$: $x \in [0, +\infty) \to U \in [0, +\infty)$ are strictly increasing and continuous and that they are twice continuously differentiable in the interior of their domain, with $w(0) = 0$, $w(1) = 1$ representing no distortion on the certain event and $U(0) = 0$. To make the model well-posed, we assume that distorted utility (2.1) and (2.2) are finite for any $\theta \in \Theta$. Let $H_\theta(x) = \mathcal{P}[X_\theta > x]$ be the decumulative distribution function of X_θ. A sufficient condition to guarantee the finiteness of (2.1) or (2.2) is that there exists $\epsilon_0 > 0$, s.t.,

$w(H_\theta(x)) = O(x^{-1-\epsilon_0})$ or $w(H_\theta(U^{-1}(x))) = O(x^{-1-\epsilon_0})$ for x large enough and all $\theta \in \Theta$ (see Assumption 3 of He and Zhou 2011a for similar technique conditions).

2.1. Weighted Expectation Form

The distorted utility (2.1) can be reformulated as

$$\widetilde{E}_{\mathcal{P}}[X_\theta] = \int_0^{+\infty} x d(-w(H_\theta(x))) \tag{2.3}$$

$$= E_{\mathcal{P}}\Big[X_\theta \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big]. \tag{2.4}$$

REMARK 2.1. A straightforward computation implies that (2.4) applies to random variables that can take negative values, in which case (2.1) becomes a Choquet integral over the whole real line. Therefore, our results apply to utility functions that can take negative values.

REMARK 2.2. It is easy to obtain some preliminary properties for the distorted utility (2.1). For example, a) linearity for single random variable $X \geq 0$: $\eta_{\kappa X+c} = \kappa\eta_X + c$, $c > 0$, $\kappa > 0$; b) monotonicity: $\eta_{X_1} \geq \eta_{X_2}$ if $X_1 \geq X_2$ a.s.

2.2. Change of Measure

We can use the distorted term $\frac{dw(z)}{dz}\big|_{z=H_\theta(X_\theta)}$ in (2.4) as a Radon–Nikodym derivative to define another measure $\mathcal{Q}$ on Ω:

$$\frac{d\mathcal{Q}}{d\mathcal{P}} = \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}. \tag{2.5}$$

Indeed, w is strictly increasing, and we have

$$E_{\mathcal{P}}\Big[\frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big] = \int_0^1 \frac{dw(z)}{dz} dz = w(1) - w(0) = 1.$$

Thus, the distorted utility (2.4) becomes

$$\widetilde{E}_{\mathcal{P}}[X_\theta] = E_{\mathcal{P}}\Big[X_\theta \frac{d\mathcal{Q}}{d\mathcal{P}}\Big] = E_{\mathcal{Q}}[X_\theta]. \tag{2.6}$$

That is, the nonlinear expectation under $\mathcal{P}$, $\widetilde{E}_{\mathcal{P}}$, is the standard expectation under the changed measure $\mathcal{Q}$, $E_{\mathcal{Q}}$. In fact, the nonlinear expectation $\widetilde{E}_{\mathcal{P}}$ is decomposed into two parts, the linear part X_θ and the nonlinear part $\frac{d\mathcal{Q}}{d\mathcal{P}}$, where the latter is "buried" in the new measure $\mathcal{Q}$.

For any two nonnegative random variables X_1 and X_2, we have the following "pseudolinearity":

$$E_{\mathcal{Q}_{X_1+X_2}}[X_1 + X_2] = E_{\mathcal{Q}_{X_1+X_2}}[X_1] + E_{\mathcal{Q}_{X_1+X_2}}[X_2].$$

However, as indicated, $\mathcal{Q}_{X_1+X_2}$ corresponds to $X_1 + X_2$, and not to X_1 or X_2 individually.

3. DERIVATION OF DERIVATIVES FOR DISTORTED UTILITY

In this section, we derive the derivatives of distorted utility with respect to (w.r.t.) some real value parameter θ. In (2.4), the distorted utility is reformulated as a weighted expectation. We use the pathwise method to express the derivatives of distorted utility as expectations of the sample path derivatives. This method was proposed by Ho and Cao (1983) to study discrete-event systems (see also Ho and Cao 1991 and Glasserman 1991), and it was used by Broadie and Glasserman (1996) to estimate the derivatives of security prices in financial engineering.

3.1. First-Order Derivatives

To follow the pathwise method, we need certain technical conditions that guarantee the exchangeability of differentiation and expectation. Cao (1985) and Heidelberger et al. (1988) did rigorous studies to handle this issue in perturbation analysis for dynamic systems. In general, the exchangeability is guaranteed by the dominated convergence theorem (see, e.g., theorem 1.21 in Kallenberg 2002), provided that a certain type of Lipschitz condition holds.

In our case, because the distorted utility (2.4) involves the distortion function $w(\cdot)$ and the distribution function $H_\theta(\cdot)$, we require additional assumptions. Let $\vec{W} = (W_1, W_2, \cdots)$ be a sequence of random variables, or $\vec{W} = \{W_t, t \in \mathcal{T}\}$ be a family of random variables, independent with parameter θ, and $X_\theta = X_\theta(\vec{W}) = h(\theta, \vec{W})$. Set $F(\theta, x) = \mathcal{P}[X_\theta \leq x]$ to be the distribution function and hence $H_\theta(X_\theta) = 1 - F(\theta, X_\theta)$. To calculate the derivatives of the distribution function, we must have the following technical assumptions (see Hong 2009).

ASSUMPTION 3.1. *$F(\theta, x)$ is continuously differentiable w.r.t. θ and x, and $E_{\mathcal{P}}[\frac{d}{d\theta}\{X_\theta\} | X_\theta = x]$ is continuous in x.*

Later, we will demonstrate that these assumptions are weak for continuous random variables (cf. footnote 1). Last, we must apply a restriction on the distortion function(cf. footnote 2).

ASSUMPTION 3.2. *For any θ, $\frac{d}{d\theta}\{X_\theta\} = \lim_{\Delta\theta \to 0} \frac{X_{\theta+\Delta\theta} - X_\theta}{\Delta\theta}$ exists a.s. and there exists a random variable $\bar{K}_\theta$ with finite expectation, s.t.,*

$$|X_{\theta+\Delta\theta} \frac{dw(z)}{dz}\Big|_{z=H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta})} - X_\theta \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}| \leq \bar{K}_\theta |\Delta\theta|, a.s.$$

for $|\Delta\theta|$ small enough.

THEOREM 3.3. *Under Assumption 3.1 and 3.2, we have*

$$\frac{d}{d\theta}\widetilde{E}[X_\theta] = E_{\mathcal{P}}\Big[\frac{d}{d\theta}\{X_\theta\} \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big] = E_{\mathcal{Q}}\Big[\frac{d}{d\theta}\{X_\theta\}\Big]. \tag{3.1}$$

Proof. From (2.4) and Assumption 3.2, applying the dominated convergence theorem, we have

$$\frac{d}{d\theta}\widetilde{E}[X_\theta] = E_{\mathcal{P}}\Big[\frac{d}{d\theta}\Big\{X_\theta \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big\}\Big]$$

$$= E_{\mathcal{P}}\Big[\frac{d}{d\theta}\Big\{X_\theta\Big\}\frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big] + E_{\mathcal{P}}\Big[X_\theta\frac{d^2w(z)}{dz^2}\Big|_{z=H_\theta(X_\theta)}\frac{d}{d\theta}\Big\{H_\theta(X_\theta)\Big\}\Big], \tag{3.2}$$

and

$$-\frac{d}{d\theta}\Big\{H_\theta(X_\theta)\Big\} = \frac{d}{d\theta}\Big\{F(\theta,X_\theta)\Big\} = \frac{\partial}{\partial\theta}F(\theta,X_\theta) + f_\theta(X_\theta)\frac{\partial}{\partial\theta}h(\theta,\vec{W}),$$

where $f_\theta(X_\theta) = \frac{\partial}{\partial x}F(\theta,X_\theta)$, with f_θ being the density of $X_\theta = h(\theta,\vec{W})$. For any realization $\vec{W}_*$ of $\vec{W}$, let $x_* = X_\theta^* = h(\theta,\vec{W}_*)$. It follows from theorem 1 in Hong (2009) that

$$\frac{\partial}{\partial\theta}F(\theta,x_*) = -f_\theta(x_*)E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta}h(\theta,\vec{W})\Big|h(\theta,\vec{W}) = x_*\Big].$$

Therefore, the sample path derivative is given by

$$\frac{d}{d\theta}\Big\{H_\theta(X_\theta^*)\Big\} = f_\theta(x_*)\Big\{E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta}h(\theta,\vec{W})\Big|h(\theta,\vec{W}) = x_*\Big] - \frac{\partial}{\partial\theta}h(\theta,\vec{W}_*)\Big\}. \tag{3.3}$$

It follows that $E_{\mathcal{P}}[\frac{d}{d\theta}\{H_\theta(X_\theta)\}|X_\theta] = 0$. Therefore,

$$E_{\mathcal{P}}\Big[X_\theta\frac{d^2w(z)}{dz^2}\Big|_{z=H_\theta(X_\theta)}\frac{d}{d\theta}\Big\{H_\theta(X_\theta)\Big\}\Big] = E_{\mathcal{P}}\Big[X_\theta\frac{d^2w(z)}{dz^2}\Big|_{z=H_\theta(X_\theta)}E_{\mathcal{P}}\Big[\frac{d}{d\theta}\Big\{H_\theta(X_\theta)\Big\}\Big|X_\theta\Big]\Big] = 0.$$

Thus, from (3.2), the theorem holds. □

REMARK 3.4. Equation (3.1) indicates that the derivative of distorted utility equals the expectation of the sample path derivative $\frac{d}{d\theta}X_\theta$ under the new measure. We call this property "monolinearity" and its properties are further studied in Appendix A.

3.2. Second-Order Derivatives

Now, we derive the second-order derivatives. Set $\vec{\theta} = (\theta_1,\cdots,\theta_K)^\top$. We consider multivariable sample functions $X_{\vec{\theta}} := X_{\theta_1,\cdots,\theta_K} = h(\vec{\theta},\vec{W}) = h(\theta_1,\cdots,\theta_K,\vec{W})$.

ASSUMPTION 3.5. *$F(\vec{\theta},x)$ is continuously differentiable w.r.t. $\vec{\theta}$ and x; $\frac{\partial}{\partial\theta_i}\{X_{\vec{\theta}}\}$ and $\frac{\partial^2}{\partial\theta_i\partial\theta_j}\{X_{\vec{\theta}}\}$ exist a.s.; $E_{\mathcal{P}}[\frac{\partial}{\partial\theta_i}\{X_{\vec{\theta}}\}|X_{\vec{\theta}} = x]$ is continuous in x; letting $\Delta\vec{\theta}_i$ be the vector with $\Delta\theta$ being its ith component and 0 for others, there exists random variables $\bar{K}_{\theta_i}$ and $\hat{K}_{\theta_{i,j}}$ with finite expectation, s.t.,*

$$\Big|X_{\vec{\theta}+\Delta\vec{\theta}_i}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}+\Delta\vec{\theta}_i}(X_{\vec{\theta}+\Delta\vec{\theta}_i})} - X_{\vec{\theta}}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\Big| \le \bar{K}_{\theta_i}|\Delta\theta|, a.s.$$

and

$$\Big|\frac{\partial}{\partial\theta_j}X_{\vec{\theta}+\Delta\vec{\theta}_i}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}+\Delta\vec{\theta}_i}(X_{\vec{\theta}+\Delta\vec{\theta}_i})} - \frac{\partial}{\partial\theta_j}X_{\vec{\theta}}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\Big| \le \hat{K}_{\theta_{i,j}}|\Delta\theta|, a.s.$$

for $|\Delta\theta|$ small enough, $i, j = 1,\ldots,K$.

REMARK 3.6. If $X_{\vec{\theta}} = X + (\vec{\theta})^\top\vec{Y}$, then $\frac{\partial}{\partial\theta_i}\{X_{\vec{\theta}}\} = Y_i$ and $\frac{\partial^2}{\partial\theta_i\partial\theta_j}\{X_{\vec{\theta}}\} = 0$. Assumption 3.5 is not much more restrictive than Assumptions 3.1 and 3.2. □

Under Assumption 3.5, by Theorem 3.3, we have

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}\widetilde{E}_{\mathcal{P}}[X_\theta] = \frac{\partial}{\partial\theta_i}\Big\{E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))}\Big]\Big\}$$

$$= E_{\mathcal{P}}\Big[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))}\Big]$$

$$+E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{d^2w(z)}{dz^2}\Big|_{z=H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))}\frac{\partial}{\partial\theta_i}\Big\{H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))\Big\}\Big]. \tag{3.4}$$

By (3.3), in the last term we have

$$\frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{\partial}{\partial\theta_i}\Big\{H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))\Big\}$$

$$= f_{\vec{\theta}}(X_{\vec{\theta}})\Big\{\frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta_i}\{h(\vec{\theta},\vec{W})\}\Big|X_{\vec{\theta}}\Big] - \frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{\partial}{\partial\theta_i}\{h(\vec{\theta},\vec{W})\}\Big\}.$$

Thus,

$$E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{\partial}{\partial\theta_i}\Big\{H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))\Big\}\Big|X_{\vec{\theta}}\Big] = -f_{\vec{\theta}}(X_{\vec{\theta}})\mathrm{Cov}\Big[\frac{\partial}{\partial\theta_i}X_{\vec{\theta}},\frac{\partial}{\partial\theta_j}X_{\vec{\theta}}\Big|X_{\vec{\theta}}\Big],$$

where $\mathrm{Cov}[\cdot,\cdot|X_{\vec{\theta}}]$ is the covariance conditioning on $X_{\vec{\theta}}$. The last term in (3.4) is

$$E_{\mathcal{P}}\Big\{f_{\vec{\theta}}(X_{\vec{\theta}})\frac{d^2w(z)}{dz^2}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta_j}\{h(\vec{\theta},\vec{W})\}\frac{\partial}{\partial\theta_i}\{H_{\vec{\theta}}(h(\vec{\theta},\vec{W}))\}\Big|X_{\vec{\theta}}\Big]\Big\}$$

$$= -E_{\mathcal{P}}\Big[f_{\vec{\theta}}(X_{\vec{\theta}})\frac{d^2w(z)}{dz^2}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\mathrm{Cov}\Big[\frac{\partial}{\partial\theta_i}X_{\vec{\theta}},\frac{\partial}{\partial\theta_j}X_{\vec{\theta}}\Big|X_{\vec{\theta}}\Big]\Big].$$

Then, the Hessian matrix is

$$D^2_{\vec{\theta}}\widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}}] = E_{\mathcal{Q}}[D^2_{\vec{\theta}}X_{\vec{\theta}}] + 2\Upsilon_{X_{\vec{\theta}}}, \tag{3.5}$$

where $D_{\vec{\theta}} = (\frac{\partial}{\partial\theta_1},\dots,\frac{\partial}{\partial\theta_K})^\top$ is the gradient operator, $D^2_{\vec{\theta}} = \{\frac{\partial^2}{\partial\theta_i\partial\theta_j}\}_{i,j=1,\dots,K}$ is the second-order derivative operator matrix, and

$$\Upsilon_{X_{\vec{\theta}}} = \frac{1}{2}E_{\mathcal{Q}}\Big[f_{\vec{\theta}}(X_{\vec{\theta}})\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}}))Cov\Big[D_{\vec{\theta}}X_{\vec{\theta}},D_{\vec{\theta}}X_{\vec{\theta}}\Big|X_{\vec{\theta}}\Big]\Big] \tag{3.6}$$

with $\mathcal{I}_w(z) := -\frac{d^2w(z)}{dz^2}/\frac{dw(z)}{dz}$ analogous to the Arrow–Pratt measure of absolute risk aversion.

That is, distortions in probabilities add an additional $2\Upsilon_{X_{\vec{\theta}}}$ to the curvature of the utility surface; it is positive definite everywhere if $w(z)$ is concave, and it is negative definite everywhere if $w(z)$ is convex.

Now, we have the following expansion at $\vec{\theta}$:

$$\begin{aligned}\widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}+\Delta\vec{\theta}}] - \widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}}] &= (D_{\vec{\theta}}\widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}}])^\top\Delta\vec{\theta} + \tfrac{1}{2}(\Delta\vec{\theta})^\top D^2_{\vec{\theta}}\widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}}]\Delta\vec{\theta} + o(|\Delta\vec{\theta}|^2)\\ &= E_{\mathcal{Q}}[D_{\vec{\theta}}X_{\vec{\theta}}]^\top\Delta\vec{\theta} + \tfrac{1}{2}(\Delta\vec{\theta})^\top E_{\mathcal{Q}}[D^2_{\vec{\theta}}X_{\vec{\theta}}]\Delta\vec{\theta} + (\Delta\vec{\theta})^\top\Upsilon_{X_{\vec{\theta}}}\Delta\vec{\theta}\\ &\quad + o(|\Delta\vec{\theta}|^2)\\ &= E_{\mathcal{Q}}[X_{\vec{\theta}+\Delta\vec{\theta}} - X_{\vec{\theta}}] + (\Delta\vec{\theta})^\top\Upsilon_{X_{\vec{\theta}}}\Delta\vec{\theta} + o(|\Delta\vec{\theta}|^2);\end{aligned}$$

that is, the linearity holds at the second order with an error $(\Delta\vec{\theta})^\top \Upsilon_{X_{\vec{\theta}}} \Delta\vec{\theta}$, or we may use the sample function expansion to approximate the distorted mean expansion in the second order with an error $(\Delta\vec{\theta})^\top \Upsilon_{X_{\vec{\theta}}} \Delta\vec{\theta}$.

We summarize the above results in the following theorem.

THEOREM 3.7. *For the distorted utility* (2.1) *, we have*

$$\begin{aligned}\widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}+\Delta\vec{\theta}}] - \widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}}] &= E_{\mathcal{Q}}[X_{\vec{\theta}+\Delta\vec{\theta}} - X_{\vec{\theta}}] + o(|\Delta\vec{\theta}|)\\ &= E_{\mathcal{Q}}[X_{\vec{\theta}+\Delta\vec{\theta}} - X_{\vec{\theta}}] + (\Delta\vec{\theta})^\top \Upsilon_{X_{\vec{\theta}}} \Delta\vec{\theta} + o(|\Delta\vec{\theta}|^2);\end{aligned}$$

with an error $\Upsilon_{X_{\vec{\theta}}}$ *in* (3.6).

LEMMA 3.8. *The distorted utility* (2.1) *is concave (convex) in* $\vec{\theta}$ *if the sample* $h(\vec{\theta}, \vec{W})$ *is concave (convex) in* $\vec{\theta}$ *and* $w(z)$ *is convex (concave).*

This follows from (3.5) and the fact that $(\Delta\vec{\theta})^\top \mathrm{Cov}[D_{\vec{\theta}} X_{\vec{\theta}}, D_{\vec{\theta}} X_{\vec{\theta}} | X_{\vec{\theta}}]\Delta\vec{\theta} \geq 0$. Furthermore,

LEMMA 3.9. *When the sample* $h(\vec{\theta}, \vec{W})$ *is linear w.r.t.* $\vec{\theta}$ *, the distorted utility* (2.1) *is concave (convex) in* $\vec{\theta}$ *if* $w(z)$ *is convex (concave).*

3.3. Derivatives under RDEU

In this subsection, we impose a utility function $U(\cdot)$ to distort the payments and consider the distorted utility (2.2). Define $H^U_{\vec{\theta}}(x) = \mathcal{P}[U(X_{\vec{\theta}}) > x]$. Since U is strictly increasing, we have

$$H^U_{\vec{\theta}}(U(X_{\vec{\theta}})) = H_{\vec{\theta}}(X_{\vec{\theta}}). \tag{3.7}$$

Thus, from (2.4), the weighted expected form for the distorted utility (2.2) is given by

$$\widetilde{E}[U(X_{\vec{\theta}})] = E_{\mathcal{P}}\Big[U(X_{\vec{\theta}})\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\Big].$$

To compute the first- and second-order derivatives, we assume that the following assumptions are valid throughout this subsection.

ASSUMPTION 3.10. *$F(\theta, x)$ is continuously differentiable w.r.t. θ and x; $\frac{\partial}{\partial\theta_i} X_\theta$ and $\frac{\partial^2}{\partial\theta_i\partial\theta_j} X_\theta$ exist a.s.; $E_{\mathcal{P}}[\frac{\partial}{\partial\theta_i} X_\theta | X_\theta = x]$ is continuous in x; letting $\Delta\vec{\theta}_i$ be the vector with $\Delta\theta$ being its ith component and 0 for others, there exists random variables $\bar{K}_{\theta_i}$ and $\hat{K}_{\theta_{i,j}}$ with finite expectation, s.t.,*

$$\Big|U(X_{\vec{\theta}+\Delta\vec{\theta}_i})\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}+\Delta\vec{\theta}_i}(X_{\vec{\theta}+\Delta\vec{\theta}_i})} - U(X_{\vec{\theta}+\Delta\vec{\theta}_i})\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\Big| \leq \bar{K}_{\theta_i}|\Delta\theta|, a.s.$$

and

$$\Big|\frac{\partial}{\partial\theta_j}\{U(X_{\vec{\theta}+\Delta\vec{\theta}_i})\}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}+\Delta\vec{\theta}_i}(X_{\vec{\theta}+\Delta\vec{\theta}_i})} - \frac{\partial}{\partial\theta_j}\{U(X_{\vec{\theta}})\}\frac{dw(z)}{dz}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\Big| \leq \hat{K}_{\theta_{i,j}}|\Delta\theta|, a.s.$$

for $|\Delta\theta|$ small enough, $i, j = 1, \ldots, K$.

REMARK 3.11. If U is concave, then $|U(X_{\vec{\theta}+\Delta\vec{\theta}_i}) - U(X_{\vec{\theta}})| \leq \frac{dU(y)}{dy}|_{y=\min\{X_{\vec{\theta}+\Delta\vec{\theta}_i}, X_{\vec{\theta}}\}} |X_{\vec{\theta}+\Delta\vec{\theta}_i} - X_{\vec{\theta}}|$. Based on similar arguments as those in footnote 2, we can see that Assumption 3.10 is not much more restrictive than assumptions 3.1 and 3.2, except that we require $U(X_{\vec{\theta}})\frac{d^2w(z)}{dz^2}|_{z=\tilde{Z}}$ and $\frac{\partial}{\partial\theta_j}\{U(X_{\vec{\theta}})\}\frac{d^2w(z)}{dz^2}|_{z=\tilde{Z}}$ have finite expectations, where $\tilde{Z} := \min\{H_{\vec{\theta}+\Delta\vec{\theta}_i}(X_{\vec{\theta}+\Delta\vec{\theta}_i}), H_{\vec{\theta}}(X_{\vec{\theta}})\}$.

Under Assumption 3.10, Theorem 3.3 takes the following form:

$$\frac{\partial}{\partial\theta_i}\widetilde{E}[U(h(\vec{\theta}, \vec{W}))] = E_{\mathcal{Q}}\Big[\frac{dU(y)}{dy}\Big|_{y=h(\vec{\theta},\vec{W})}\frac{\partial}{\partial\theta_i}\{h(\vec{\theta}, \vec{W})\}\Big]. \tag{3.8}$$

Repeating the process from (3.4) to (3.6), we get the Hessian matrix:

$$D^2_{\vec{\theta}}\widetilde{E}_{\mathcal{P}}[U(X_{\vec{\theta}})] = E_{\mathcal{Q}}[D^2_{\vec{\theta}}U(X_{\vec{\theta}})] + 2\Upsilon^U_{X_{\vec{\theta}}},$$

where

$$\Upsilon^U_{X_{\vec{\theta}}} = \frac{1}{2}E_{\mathcal{Q}}\Big[f_{\vec{\theta}}(X_{\vec{\theta}})\frac{dU(y)}{dy}\Big|_{y=X_{\vec{\theta}}}\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}}))Cov\Big[D_{\vec{\theta}}X_{\vec{\theta}}, D_{\vec{\theta}}X_{\vec{\theta}}\Big|X_{\vec{\theta}}\Big]\Big]. \tag{3.9}$$

Expansion at $\vec{\theta}$ now becomes

$$\begin{aligned}\widetilde{E}_{\mathcal{P}}[U(X_{\vec{\theta}+\Delta\vec{\theta}})] - \widetilde{E}_{\mathcal{P}}[U(X_{\vec{\theta}})] &= E_{\mathcal{Q}}[U(X_{\vec{\theta}+\Delta\vec{\theta}}) - U(X_{\vec{\theta}})] + (\Delta\vec{\theta})^\top\Upsilon^U_{X_{\vec{\theta}}}\Delta\vec{\theta} + o(|\Delta\vec{\theta}|^2)\\ &= E_{\mathcal{Q}}[D_{\vec{\theta}}U(X_{\vec{\theta}})]^\top\Delta\vec{\theta} + \frac{1}{2}E_{\mathcal{Q}}\Big[\frac{d^2U(y)}{dy^2}\Big|_{y=X_{\vec{\theta}}}\Big((D_{\vec{\theta}}X_{\vec{\theta}})^\top\Delta\vec{\theta}\Big)^2\Big]\\ &\quad+\frac{1}{2}E_{\mathcal{Q}}\Big[\frac{dU(y)}{dy}\Big|_{y=X_{\vec{\theta}}}(\Delta\vec{\theta})^\top D^2_{\vec{\theta}}X_{\vec{\theta}}\Delta\vec{\theta}\Big]\\ &\quad+(\Delta\vec{\theta})^\top\Upsilon^U_{X_{\vec{\theta}}}\Delta\vec{\theta} + o(|\Delta\vec{\theta}|^2).\end{aligned}$$

If $h(\vec{\theta}, \vec{W})$ is a linear function of $\vec{\theta}$, the difference becomes

$$\begin{aligned}\widetilde{E}_{\mathcal{P}}[U(X_{\vec{\theta}+\Delta\vec{\theta}})] - \widetilde{E}_{\mathcal{P}}[U(X_{\vec{\theta}})] &= E_{\mathcal{Q}}[D_{\vec{\theta}}U(X_{\vec{\theta}})]^\top\Delta\vec{\theta} + \frac{1}{2}E_{\mathcal{Q}}\Big[\frac{d^2U(y)}{dy^2}\Big|_{y=X_{\vec{\theta}}}\Big((D_{\vec{\theta}}X_{\vec{\theta}})^\top\Delta\vec{\theta}\Big)^2\Big]\\ &\quad+(\Delta\vec{\theta})^\top\Upsilon^U_{X_{\vec{\theta}}}\Delta\vec{\theta} + o(|\Delta\vec{\theta}|^2).\end{aligned} \tag{3.10}$$

LEMMA 3.12. *If the sample $h(\vec{\theta}, \vec{W})$ is linear in $\vec{\theta}$, the distorted utility* (2.2) *is concave (convex) in $\vec{\theta}$ if $U(y)$ is concave (convex) and $w(z)$ is convex (concave).*

3.4. Simulation Algorithms for Estimating Derivatives of Distorted Utility

Based on (3.1), we develop a simulation algorithm for estimating the (first-order) derivative. Assume that θ is given and fixed.

(1) Simulate N independent samples $\vec{W}_n$, $n = 1, 2, \cdots, N$.
(2) Obtain both the sample value $X_{\theta,n} = h(\theta, \vec{W}_n)$ and the sample path derivative $\frac{d}{d\theta}X_{\theta,n} = \frac{\partial}{\partial\theta}h(\theta, \vec{W}_n)$ $(n = 1, 2, \cdots, N)$ via perturbation analysis (Ho and Cao 1983).

(3) Reorder the samples according to their values, and denote $X_{\theta,(n)}$ as the nth largest number among $\{X_{\theta,n}, n = 1, 2, \cdots, N\}$ and the corresponding sample path derivatives as $\frac{d}{d\theta} X_{\theta,(n)}$.
(4) Estimate the derivative of distorted utility by (cf. (3.1))

$$\frac{d}{d\theta}\widetilde{E}_{\mathcal{P}}[X_\theta] = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{d}{d\theta}\{X_{\theta,(n)}\}\frac{dw(z)}{dz}\Big|_{z=\frac{n-1}{N}}. \tag{3.11}$$

REMARK 3.13. In step 3, we reorder the sample value of X_θ to approximate its quantile function H_θ^{-1}. According to Serfling (1980), we have

$$X_{\theta,(\lfloor \alpha N\rfloor)} \to H_\theta^{-1}(\alpha), \text{ as } N \to +\infty, \quad a.s., \quad \forall\alpha \in [0, 1],$$

where $\lfloor x\rfloor$ is the largest integer not greater than x.

REMARK 3.14. The algorithm follows the spirit of perturbation analysis. In many cases, the sample path derivative is easily estimated based on a single sample path with only parameter θ (i.e., no sample path with $\theta + \Delta\theta$ is needed); a classical example is the estimation for the sample path derivative of the performance criterion w.r.t. a mean service time in a closed queueing network (see Ho and Cao 1983, Glasserman 1991, and Cao 2007). We will see in Section 4 that the sample path derivatives of the terminal value of a portfolio w.r.t. the money allocated to each stock can be easily estimated or calculated.

REMARK 3.15. The algorithm belongs to the class of those referred to in the literature as perturbation analysis algorithms (see Ho and Cao 1991, Glasserman 1991, and Cao 2007). These algorithms are efficient because no sample path with $\theta + \Delta\theta$ must be simulated. The optimization algorithms based on perturbation analysis algorithms are usually more efficient than Monte-Carlo-based optimization algorithms, as the former are of the Robbins–Monro type and the latter, the Kiefer–Wolfowitz type (see Robbins and Monro 1951 and Kiefer and Wolfowitz 1952). Monolinearity makes it possible to apply perturbation analysis to distorted utility under the new measure.

4. PORTFOLIO OPTIMIZATION: INITIAL ALLOCATION

In this section, we apply the results developed in Section 3, especially the monolinearity of the distorted utility $\eta = \widetilde{E}_{\mathcal{P}}[X]$, to the portfolio selection problem. We will see that, because of this property, many results under dual theory or RDEU are similar to those under EU. The essential point is to find the sample path derivatives that are easily obtained analytically.

4.1. The Continuous Financial Market

We use the financial market in Karatzas and Shreve (1998). Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a probability space and $T > 0$ be the terminal time. An L-dimensional Brownian motion $\{W(t) = \{W_1(t), \ldots, W_L(t)\}, \mathcal{F}_t; 0 \le t \le T\}$ is defined on $\{\Omega, \mathcal{F}, \mathcal{P}\}$, where $\{\mathcal{F}_t\}_{0\le t\le T}$ is the augmentation (by the null sets in $\mathcal{F}_T^W$) of the filtration $\{\mathcal{F}_t^W\}_{0\le t\le T}$ generated by $W(\cdot)$.

In the market, one bond and K stocks exist whose price dynamics are given exogenously. The bond price is given by

$$dS_0(t) = S_0(t)r(t)dt, \quad t \in [0, T]; \quad S_0(0) = s_0 = 1,$$

where the risk-free interest rate $r(\cdot)$ is $\mathcal{F}_t$ progressively measurable, satisfying $\int_0^T |r(t)|dt < \infty$ a.s., and the prices of stock k, $k = 1, 2 \cdots, K$, follow

$$dS_k(t) = S_k(t)\{\mu_k(t)dt + \sum_{l=1}^{L} \sigma_{k,l}(t)dW_l(t)\}, \quad t \in [0, T]; \quad S_k(0) = s_i > 0, \tag{4.1}$$

where the drift $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_K(\cdot))^\top$ and volatility $\sigma(\cdot) = \{\sigma_{k,l}(\cdot)\}_{1\le k\le K, 1\le l\le L}$ are $\mathcal{F}_t$ progressively measurable, satisfying

$$\int_0^T \left\{ \sum_{k=1}^{K} |\mu_k(t)| + \sum_{k=1}^{K}\sum_{l=1}^{L} |\sigma_{k,l}|^2 \right\} dt < \infty, a.s.,$$

and the volatility matrix $\sigma(t)$ has a full rank for every $0 \le t \le T$.

4.2. Initial Allocation Problem

In this section, we consider the allocation of the initial wealth. Let α_k be the amount of money allocated in stock k at time $t = 0$, $k = 0, 1, \cdots, K$, with $\sum_{k=0}^{K} \alpha_k = v$, where v is the initial budget and is fixed. We assume that in $(0, T]$, the investor cannot reallocate (buy or sell) her/his wealth among the bond and stocks. The case allowing continuously reallocation in $[0, T]$ will be considered in the next section.

At $t = 0$, the investor owns α_k/s_k shares of stock k, $k = 1, 2, \cdots, K$. At time T, the investor's total wealth is

$$V(T; \alpha) = \alpha_0 e^{\int_0^T r(t)dt} + \sum_{k=1}^{K} \alpha_k \frac{S_k(T)}{s_k}. \tag{4.2}$$

Let $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_K)$ be the parameter vector. Assume that short selling is prohibited, i.e., $\alpha_k \ge 0$, $k = 0, 1, \ldots, K$. Then, the final wealth $V(T; \alpha)$ is nonnegative a.s. Our problem consists of finding the right allocation α^* by solving the following nonlinear portfolio optimization problem

$$\begin{array}{ll} \underset{\alpha}{\text{Maximize}} & \eta := \widetilde{E}_{\mathcal{P}}[V(T; \alpha)] \\ \text{subject to} & \sum_{k=0}^{K} \alpha_k = v, \\ & \alpha_k \ge 0, k = 0, 1, \ldots, K. \end{array} \tag{4.3}$$

4.3. First-Order Condition

Applying the Lagrange method, if the optimal portfolio α^* is in the interior of the domain, i.e., $\alpha_k^* > 0$, $k = 0, 1, \ldots, K$, there exists a λ, s.t.,

$$\frac{\partial}{\partial \alpha_k} \widetilde{E}_{\mathcal{P}}[V(T; \alpha)]\Big|_{\alpha=\alpha^*} = \lambda, \quad k = 0, 1, \ldots, K. \tag{4.4}$$

We apply (3.1) to obtain the derivative in (4.4). Apparently, in this problem, we have $X_\theta = V(T;\alpha)$, and from (4.2), the sample path derivatives are

$$\frac{\partial}{\partial \alpha_0}\{V(T;\alpha)\} = e^{\int_0^T r(t)dt}, \text{ and } \frac{\partial}{\partial \alpha_k}\{V(T;\alpha)\} = \frac{S_k(T)}{s_k}, \quad k = 1, 2, \cdots, K.$$

Since the stock returns at T are continuous random variables and $V(T;\alpha)$ is a linear combination of the stock returns, Assumption 3.1 holds.[1] If we further choose the distortion function to satisfy Assumption 3.2,[2] we may use (3.1):

$$\frac{\partial}{\partial \alpha_0}\widetilde{E}_{\mathcal{P}}[V(T,\alpha)] = E_{\mathcal{P}}\left[e^{\int_0^T r(t)dt}\frac{dw(z)}{dz}\Big|_{z=H_\alpha(V(T;\alpha))}\right],$$

and

$$\frac{\partial}{\partial \alpha_k}\widetilde{E}_{\mathcal{P}}[V(T,\alpha)] = E_{\mathcal{P}}\Big[\frac{S_k(T)}{s_k}\frac{dw(z)}{dz}\Big|_{z=H_\alpha(V(T;\alpha))}\Big], \quad k = 1, 2, \cdots, K.$$

Plugging them into the first-order condition (4.4), we have, for $k = 1, 2, \cdots, K$,

$$E_{\mathcal{P}}\Big[\Big(\frac{S_k(T)}{s_k} - e^{\int_0^T r(t)dt}\Big)\frac{dw(z)}{dz}\Big|_{z=H_{\alpha^*}(V(T;\alpha^*))}\Big] = 0. \tag{4.5}$$

If the interest rate is nonstochastic, we have, for $k = 1, 2, \cdots, K$,

$$E_{\mathcal{P}}\Big[\frac{S_k(T)}{s_k}\frac{dw(z)}{dz}\Big|_{z=H_{\alpha^*}(V(T;\alpha^*))}\Big] = e^{\int_0^T r(t)dt}. \tag{4.6}$$

These two equations provide the necessary conditions for an interior point to be optimal. They may not have a solution, indicating that the optimal point lies on the boundary, i.e., with some α_k, $k = 0, 1, \cdots, K$, being zero. Although it may be difficult to solve the first-order optimality conditions (4.5) or (4.6), by the linearity in payments, we can easily obtain the following linearity for optimal policies regarding the initial wealth.

[1]In this case, X_θ is of form $X + \theta Y$ for two continuous random variables X and Y, which have continuous joint density $f(x, y)$ and finite expectation. The distribution of X_θ is $F(\theta, z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{z-\theta y} f(x, y)dxdy$. Then, $\frac{\partial}{\partial z}F(\theta, z) = \int_{-\infty}^{+\infty} f(z - \theta y, y)dy$ and $\frac{\partial}{\partial \theta}F(\theta, z) = -\int_{-\infty}^{+\infty} yf(z - \theta y, y)dy$, which are both continuous. And $E_{\mathcal{P}}[\frac{d}{d\theta}\{X_\theta\}|X_\theta = x] = E_{\mathcal{P}}[Y|X_\theta = z] = -\frac{\partial}{\partial \theta}F(\theta, z)/\frac{\partial}{\partial z}F(\theta, z)$ is continuous.

[2]Note that in Assumption 3.2,

$$\begin{aligned}&|X_{\theta+\Delta\theta}\frac{dw(z)}{dz}\Big|_{z=H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta})} - X_\theta\frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}|\\ \le\ &|X_{\theta+\Delta\theta} - X_\theta|\,|\frac{dw(z)}{dz}\Big|_{z=H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta})} + X_\theta\Big|\frac{dw(z)}{dz}\Big|_{z=H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta})} - \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big|.\end{aligned}$$

For $X_\theta = X + \theta Y$, we have $|H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta}) - H_\theta(X_\theta)| = |\Delta\theta|\big|\int_{-\infty}^0\int_y^0 f(x\Delta\theta + X + \theta y, Y - y)dxdy) - \int_0^{+\infty}\int_0^y f(x\Delta\theta + X + \theta y, Y - y)dxdy\big| \le |\Delta\theta|$. Thus, Assumption 3.2 holds if we choose a distortion function with bounded second-order derivatives (in this case, $dw(z)/dz$ satisfies a global Lipschitz condition). If we choose a power-type distortion function, i.e., $w(z) = z^\gamma$, $0 < \gamma < 1$, its second-order derivative is unbounded toward $z = 0$. However, $|d^2w(z)/dz^2|$ is decreasing on (0, 1). By Lagrange mean value theorem,

$$\left|\frac{dw(z)}{dz}\Big|_{z=H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta})} - \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\right| \le \left|\frac{d^2w(z)}{dz^2}\Big|_{z=\min\{H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta}),H_\theta(X_\theta)\}}\right||H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta}) - H_\theta(X_\theta)|.$$

Then, Assumption 3.2 will hold provided $X_\theta\big|\frac{d^2w(z)}{dz^2}\big|_{z=\min\{H_{\theta+\Delta\theta}(X_{\theta+\Delta\theta}),H_\theta(X_\theta)\}}$ has finite expectation. This regularity holds if X and Y are joint normal distributed.

PROPOSITION 4.1. *If $\alpha^*(t) = (\alpha_1^*(t), \alpha_2^*(t), \cdots, \alpha_K^*(t))$ is an optimal allocation for the optimization problem* (4.3) *with initial wealth v, then $\kappa\alpha^*(t) = (\kappa\alpha_1^*(t), \kappa\alpha_2^*(t), \cdots, \kappa\alpha_K^*(t))$ is an optimal allocation for the optimization problem* (4.3) *with initial wealth κv.*

Note that the optimal wealth $V(T;\alpha^*)$ is linear w.r.t. $S_k(T)/s_k$. Multiplying both sides of (4.6) by α_k^* and taking summation along $k = 1, \dots, K$ leads to

$$\widetilde{E}_{\mathcal{P}}[V(T;\alpha^*)] = v e^{\int_0^T r(t)dt}. \tag{4.7}$$

This implies that if a policy of investing in the stocks and the bond is optimal, the "satisfaction" obtainable with this optimal policy is equal to that gained when investing entirely in the bond. That is, the optimal solution lies on the boundary, as in the case of EU.

In the following, we use a simple example to illustrate this general result under dual theory.

EXAMPLE 4.1. We consider the case that only one stock and one bond exist in the market. The interest rate is constant. In this case, (4.6) for the optimal portfolio becomes

$$\widetilde{E}_{\mathcal{P}}[S_1(T)/s_1] = e^{rT}.$$

The left- (right)-hand side of the above equation is the distorted expected return of the stock (bond). The equation does not involve the portfolio $\{\alpha_0, \alpha_1\}$. All the solutions are extreme: if the distorted expected return of the stock is equal to that of the bond, the investor is indifferent between the bond and the stock; if the distorted expected return of the stock is strictly bigger (smaller) than that of the bond, the investor puts all the money in the stock (the bond). These results are similar to those for the portfolio selection problem under EU.

4.4. Initial Allocation Problem under RDEU

As shown in Example 4.1, because of the linearity in payments and the monolinearity in distorted utility, the optimal portfolio usually takes the boundary values. It follows that the fundamental principle of diversification fails. Therefore, it may be more proper to consider a strictly concave utility to reflect diminishing marginal utility of wealth, as in the case of RDEU. We will provide an example below to demonstrate that a diversified strategy (an interior solution) could be optimal under RDEU. In this subsection, we assume that $U(\cdot)$ is strictly concave and consider the portfolio selection problem with the following objective function:

$$\eta := \widetilde{E}_{\mathcal{P}}[U(V(T;\alpha))].$$

Under Assumption 3.10, according to (3.8), the derivative becomes (cf. (3.7))

$$\frac{d}{d\alpha}\widetilde{E}_{\mathcal{P}}[U(V(T;\alpha))] = E_{\mathcal{P}}\Big[\frac{dU(x)}{dx}\Big|_{x=V(T;\alpha)}\frac{d}{d\alpha}\{V(T;\alpha)\}\frac{dw(z)}{dz}\Big|_{z=H_\alpha(V(T;\alpha))}\Big]. \tag{4.8}$$

The optimal criterion (4.5) becomes, for $k = 1, 2, \cdots, K$,

$$E_{\mathcal{P}}\Big[\Big(\frac{S_k(T)}{s_k} - e^{\int_0^T r(t)dt}\Big)\frac{dw(z)}{dz}\Big|_{z=H_{\alpha^*}(V(T;\alpha^*))}\frac{dU(x)}{dx}\Big|_{x=V(T;\alpha^*)}\Big] = 0.$$

Multiplying both sides by α_k^* and taking the summation yields

$$E_{\mathcal{P}}\Big[\Big(V(T;\alpha^*)) - ve^{\int_0^T r(t)dt}\Big)\frac{dw(z)}{dz}\Big|_{z=H_{\alpha^*}(V(T;\alpha^*))}\frac{dU(x)}{dx}\Big|_{x=V(T;\alpha^*)}\Big] = 0.$$

EXAMPLE 4.2. We consider the two-asset financial market in Example 4.1. Now, we have a strictly concave utility $U(\cdot)$. Using $\theta = \alpha_1 = v - \alpha_0$, and setting $R = e^{-rT}S_1(T)/s_1$, we have $V(T;\theta) = \alpha_0 e^{rT} + \alpha_1 S_1(T)/s_1 = e^{rT}(v + \theta(R-1))$. Define $\varphi(\theta) := e^{-rT}\widetilde{E}_{\mathcal{P}}[U(V(T;\theta))]$. Noting that $H_\alpha(V(T;\alpha)) = H_R(R)$, where $H_R(x) = \mathcal{P}[R > x]$, we have

$$\frac{d\varphi(\theta)}{d\theta} = E_{\mathcal{P}}\Big[(R-1)\frac{dw(z)}{dz}\Big|_{z=H_R(R)}\frac{dU(x)}{dx}\Big|_{x=e^{rT}(v+\theta(R-1))}\Big],$$

and

$$\frac{d^2\varphi(\theta)}{d\theta^2} = e^{rT}E_{\mathcal{P}}\Big[(R-1)^2\frac{dw(z)}{dz}\Big|_{z=H_R(R)}\frac{d^2U(x)}{dx^2}\Big|_{x=e^{rT}(v+\theta(R-1))}\Big] < 0, \text{ for } 0 \le \vartheta \le v.$$

Hence, the optimal allocation θ^* is the unique solution of the first-order condition $d\varphi(\theta)/d\theta = 0$, provided that $d\varphi(\theta)/d\theta\big|_{\theta=0} > 0$ and $d\varphi(\theta)/d\theta\big|_{\theta=1} < 0$.

To summarize, if

$$E_{\mathcal{P}}\Big[(R-1)\frac{dw(z)}{dz}\Big|_{z=H_R(R)}\Big] > 0 \text{ and } E_{\mathcal{P}}\Big[(R-1)\frac{dw(z)}{dz}\Big|_{z=H_R(R)}\frac{dU(x)}{dx}\Big|_{x=e^{rT}vR}\Big] < 0,$$

it is optimal to invest in both the bond and the stock. More precisely, there exists a unique $\theta^*(0 < \theta^* < v)$, satisfying $d\varphi(\theta)/d\theta\big|_{\theta=\theta^*} = 0$, s.t., it is optimal to invest θ^* in the stock, and $v - \theta^*$ in the bond.

5. PORTFOLIO OPTIMIZATION: CONTINUOUS ALLOCATION UNDER RDEU

In this section, we use the setting of the financial market in Subsection 4.1 and assume that the market price of risk process

$$\theta_0(t) := \sigma(t)^\top \left(\sigma(t)\sigma(t)^\top\right)^{-1}(\mu(t) - r(t))$$

is $\mathcal{F}_t$ progressively measurable, and the Novikov condition $E_{\mathcal{P}}[e^{\frac{1}{2}\int_0^T \theta_0(t)^\top\theta_0(t)dt}] < \infty$ holds, s.t.,

$$Z_0(t) := \exp\left\{-\int_0^t \frac{1}{2}\theta_0(s)^\top\theta_0(s)ds - \int_0^t \theta_0(s)^\top dW(s)\right\}, 0 \le t \le T,$$

is a martingale.

In the continuous allocation problem, the investor may dynamically update her/his wealth allocation based on up-to-date information. Let $\alpha_k(t)$ be the amount of money that is allocated in stock k at time t, $k = 1, 2, \cdots, K$, and assume that there exists a feedback policy $\alpha_t = (\alpha_1(t), \cdots, \alpha_K(t))^\top \in \mathcal{F}_t$, satisfying

$$\int_0^T \left\{|\alpha^\top(t)(\mu(t) - r(t))| + |\sigma^\top(t)\alpha(t)|^2\right\}dt < \infty, a.s.$$

Denote $V^{v,\alpha}(t)$ as the wealth process corresponding to the policy α with initial wealth $V^{v,\alpha}(0) = v > 0$. We further restrict our policy α to be the portfolio for which the wealth process $V^{v,\alpha}(t)$ is positive a.s. for every $0 \le t \le T$. Assume that there is no money infusion and extraction and that there are no transaction costs. According to section 1.3 of Karatzas and Shreve (1998), denoting $W^{(0)}(t) = \int_0^t \theta_0(s)ds + W(t)$, $0 \le t \le T$, the wealth process follows

$$dV^{v,\alpha}(t) = V^{v,\alpha}(t)r(t)dt + \alpha^\top(t)\sigma(t)dW^{(0)}(t),\ V^{v,\alpha}(0) = v. \tag{5.1}$$

We wish to optimize (maximize)

$$\eta(v,\alpha) := \widetilde{E}_{\mathcal{P}}[U(V^{v,\alpha}(T))] = \int_0^{+\infty} w\{\mathcal{P}[U(V^{v,\alpha}(T)) > x]\}dx, \tag{5.2}$$

by choosing the portfolio α, where the utility function U is strictly concave and satisfies the Inada condition: $\frac{dU(x)}{dx}|_{x\to 0+} = +\infty$, $\frac{dU(x)}{dx}|_{x\to +\infty} = 0$.

5.1. Sample Path Derivatives

To apply the sensitivity-based approach, we must find the sample path derivative of terminal wealth $V^{v,\alpha}(T)$ w.r.t. any possible parameter of the policy α. We apply the calculus of variations to a sample path of the system. In the spirit of perturbation analysis, we perturb the optimal policy α^* to $\alpha(\epsilon)$: $\alpha_k^*(t)$ is perturbed to $\alpha_k^*(t) + \epsilon_k\beta_k(t)$, for $\mathcal{F}_t$-progressively measurable $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_K(\cdot))^\top$ and small real numbers $\epsilon = (\epsilon_1, \cdots, \epsilon_K)$, s.t., the terminal wealth with the new policy $\alpha(\epsilon)$ is nonnegative and atomless. Then, the dynamics of the wealth changes from (5.1) to

$$dV^{v,\alpha(\epsilon)}(t) = V^{v,\alpha(\epsilon)}(t)r(t)dt + \sum_{k=1}^{K}\{\alpha_k(t) + \epsilon_k\beta_k(t)\}\sigma_{k,\cdot}(t)dW^{(0)}(t),$$

where $\sigma_{k,\cdot}(t)$ is the kth row of matrix $\sigma(t)$. Using Ito's formula to $S_0^{-1}(t)V^{v,\alpha(\epsilon)}(t)$, it follows

$$\begin{aligned} &S_0^{-1}(t)V^{v,\alpha(\epsilon)}(t) = v \\ &+\sum_{k=1}^{K}\int_0^t S_0^{-1}(s)\alpha_k(s)\sigma_{k,\cdot}(s)dW^{(0)}(s) + \sum_{k=1}^{K}\epsilon_k\int_0^t S_0^{-1}(s)\beta_k(s)\sigma_{k,\cdot}(s)dW^{(0)}(s). \end{aligned} \tag{5.3}$$

Taking derivatives w.r.t. ϵ_k on both sides and denoting $\frac{\partial}{\partial\epsilon_k}V^{v,\alpha(\epsilon)}(t) = \dot{V}_{\epsilon_k}(t)$, we get

$$S_0^{-1}(t)\dot{V}_{\epsilon_k}(t) = \int_0^t S_0^{-1}(s)\beta_k(s)\sigma_{k,\cdot}(s)dW^{(0)}(s). \tag{5.4}$$

5.2. First-Order Condition

Recall that our problem consists of maximizing $\eta(v,\alpha)$ given by (5.2), where the nonnegative wealth process $V^{v,\alpha}(t)$, $0 \le t \le T$, is governed by (5.1). The nonnegativity constraint usually holds automatically if we obtain a nonnegative terminal wealth.

Therefore, we first disregard this constraint and plug the wealth process (5.3) into (5.2). The first-order condition yields

$$\frac{d\eta(v,\alpha(\epsilon))}{d\epsilon_k}\Big|_{\epsilon=0} = 0.$$

To apply the monolinearity (3.8), we may assume that Assumption 3.10 holds.[3] Then, plugging the sample path derivative (5.4) into (3.8), we get the (necessary) optimality condition

$$E_{\mathcal{P}}\Big[\frac{dU(x)}{dx}\Big|_{x=V^{v,\alpha^*}(T)}\frac{dw(z)}{dz}\Big|_{z=H^{v,\alpha^*}(V^{v,\alpha^*})}S_0(T)\int_0^T S_0^{-1}(t)\beta_k(t)\sigma_{k,\cdot}(t)dW^{(0)}(t)\Big] = 0, \tag{5.5}$$

where $H^{v,\alpha^*}(\cdot)$ is the decumulative distribution function of $V^{v,\alpha^*}(T)$.

In EU, it is usually true that $\frac{dU(x)}{dx}|_{x=V^{v,\alpha^*}(T)}$ will work as a pricing kernel. In the following theorem, we will demonstrate that when extending to RDEU, $\frac{dU(x)}{dx}|_{x=V^{v,\alpha^*}(T)}\frac{dw(z)}{dz}|_{z=H^{v,\alpha^*}(V^{v,\alpha^*})}$ will take the position. Define $\lambda(v) := E_{\mathcal{P}}[S_0(T)\frac{dU(x)}{dx}|_{x=V^{v,\alpha^*}(T)}\frac{dw(z)}{dz}|_{z=H^{v,\alpha^*}(V^{v,\alpha^*}(T))}]$,

$$\xi^{\alpha^*}(T) := \frac{S_0(T)}{\lambda(v)}\frac{dU(x)}{dx}\Big|_{x=V^{v,\alpha^*}(T)}\frac{dw(z)}{dz}\Big|_{z=H^{v,\alpha^*}(V^{v,\alpha^*}(T))}, \tag{5.6}$$

and

$$\xi^{\alpha^*}(t) = E_{\mathcal{P}}[\xi^{\alpha^*}(T)|\mathcal{F}_t], \quad 0 \le t \le T, \tag{5.7}$$

with $\xi^{\alpha^*}(0) = E_{\mathcal{P}}[\xi^{\alpha^*}(T)|\mathcal{F}_0] = 1$.

THEOREM 5.1. *If $\alpha^*(t)$ is an optimal $\mathcal{F}_t$ policy, and $\xi^{\alpha^*}(t)$ is defined by* (5.6)–(5.7)*, then*

$$\xi^{\alpha^*}(t) = \exp\left\{-\frac{1}{2}\int_0^t(\theta_0(s)^\top\theta_0(s) + \nu(s)^\top\nu(s))ds - \int_0^t(\theta_0(s)+\nu(s))^\top dW(s)\right\}, \tag{5.8}$$

for some $\nu(t)$ in $Kernel(\sigma(t)) := \{x : \sigma(t)x = 0\}$. Furthermore, $\xi^{\alpha^*}(t)S_0^{-1}(t)V^{v,\alpha}(t)$ is a local martingale and

$$E_{\mathcal{P}}[\xi^{\alpha^*}(T)S_0^{-1}(T)V^{v,\alpha}(T)] \le v, \forall\alpha. \tag{5.9}$$

[3]From (5.3), X_θ is of form $X+\theta Y$, where X is the optimal terminal wealth and ultimately is found to be atomless. Following a similar argument used in footnote 1, the first two conditions in Assumption 3.10 are not restrictive. For the other conditions, we may follow the arguments in footnote 2 in choosing the distortion function $w(\cdot)$. In addition, a localization technique may be used to weaken the constraint on $w(\cdot)$. To obtain this condition, for any β in Subsection 5.1, define

$$\tau_n := T\wedge\inf\Big\{t\in[0,T] : V^{v,\alpha^*}(t)\in(0,\tfrac{1}{n}]\cup[n,+\infty),\ \text{or}\ \ V^{v,\alpha(1)}(t)\in(0,\tfrac{1}{n}]\cup[n,+\infty),\ \text{or}$$
$$\sum_{k=1}^K\Big|\int_0^t\beta_k(s)S_0^{-1}(s)\xi^{\alpha^*}(s)\sigma_{k,\cdot}(s)dW(s) - \xi^{\alpha^*}(s)S_0^{-1}(s)\dot{V}_{\epsilon_k}(s)\varsigma^\top(s)dW(s)\Big| \le n\Big\},$$

where $\xi^{\alpha^*}(\cdot)$ and $\varsigma(\cdot)$ are defined by (5.7) and (5.10), respectively. And consider the perturbation $\alpha_k^*(t) + \epsilon_k\beta_k^{(n)}(t)$, for $\beta_k^{(n)}(t); = \beta_k(t)1_{t\le\tau_n}$. Then, to exchange the order of differentiation and expectation, as the similar argument on page 19 of Karatzas et al. (1991), we only need (cf. Remark 3.11) finite expectation requirements for

$$U(X_{\bar\theta})\frac{d^2w(z)}{dz^2}\Big|_{z=H_{\bar\theta}(X_{\bar\theta})}\ \text{and}\ \frac{\partial}{\partial\theta_j}\{U(X_{\bar\theta})\}\frac{d^2w(z)}{dz^2}\Big|_{z=H_{\bar\theta}(X_{\bar\theta})}.$$

Proof. By definition (5.6)–(5.7), $\{\xi^{\alpha^*}(t), 0 \leq t \leq T\}$ is a positive martingale defined on $(\Omega, \mathcal{F}, \mathcal{P})$. By the martingale representation theorem (see problem 3.4.16, Karatzas and Shreve 1991), there exists an $\mathcal{F}_t$-adapted and $\mathcal{R}^L$-valued process $\varsigma(t) = (\varsigma_1(t), \ldots, \varsigma_L(t))^\top$ with

$$\sum_{l=1}^{L} \int_0^T \xi^{\alpha^*}(t)^2 \varsigma_l^2(t) dt < \infty,$$

s.t.,

$$\xi^{\alpha^*}(t) = \exp\left\{-\int_0^t \frac{1}{2}\varsigma(s)^\top \varsigma(s) ds - \int_0^t \varsigma(s)^\top dW(s)\right\} \tag{5.10}$$

holds a.s. Based on (5.10) and (5.4), we have

$$\begin{aligned} d\{\xi^{\alpha^*}(t) S_0^{-1}(t) \dot{V}_{\epsilon_k}(t)\} = {} & \beta_k(t) S_0^{-1}(t) \xi^{\alpha^*}(t) \sigma_{k,\cdot}(t)(\theta_0(t) - \varsigma(t)) dt \\ & + \beta_k(t) S_0^{-1}(t) \xi^{\alpha^*}(t) \sigma_{k,\cdot}(t) dW(t) - \xi^{\alpha^*}(t) S_0^{-1}(t) \dot{V}_{\epsilon_k}(t) \varsigma^\top(t) dW(t). \end{aligned}$$

According to our optimal condition (5.5), for $k = 1, \ldots, K$, we have

$$0 = E_{\mathcal{P}}[\xi^{\alpha^*}(T) S_0^{-1}(T) \dot{V}_{\epsilon_k}(T)] = E_{\mathcal{P}}\left[\int_0^T \beta_k(t) S_0^{-1}(t) \xi^{\alpha^*}(t) \sigma_{k,\cdot}(t)(\theta_0(t) - \varsigma(t)) dt\right];$$

the last equation follows from the fact that the expectation of an Ito integral equals zero (the localization technique in footnote 3 may be used if the Ito processes are local martingales). Because $\beta_k(\cdot)$ is arbitrary, $k = 1, \ldots, K$, we have

$$\sigma(t)(\theta_0(t) - \varsigma(t)) = 0, \quad \text{a.s., a.e.} \quad t \in [0, T].$$

Then, $\varsigma(t) = \theta_0(t) + \nu(t)$, for some $\nu(t)$ satisfying $\sigma(t)\nu(t) = 0$. Therefore, (5.8) follows from (5.10) and the fact that $\theta_0(s)^\top \nu(s) = 0$. Using Ito's formula, following (5.1) and (5.8), we have

$$\xi^{\alpha^*}(t) S_0^{-1}(t) V^{v,\alpha}(t) = v + \int_0^t \xi^{\alpha^*}(s) S_0^{-1}(s)(\sigma^\top(t)\alpha(t) - V^{v,\alpha}(t)(\theta_0(s) + \nu(s)))^\top dW(t),$$

which demonstrates that $\xi^{\alpha^*}(t) S_0^{-1}(t) V^{v,\alpha}(t)$ is a positive local martingale and hence a supermartingale. Thus, (5.9) holds. □

5.2.1. Solutions in the complete markets case. To obtain the optimal allocation, we first consider the case with a complete market. Following theorem 6.6 of Karatzas and Shreve (1998), we must make the following assumption:

ASSUMPTION 5.2. *The volatility matrix $\sigma(t)$ is invertible a.s. for a.e. $t \in [0, T]$.*

When Assumption 5.2 holds, $Kernel(\sigma(t)) = \{0\}$ and $\xi^{\alpha^*}(t)$ in Theorem 5.1 is uniquely determined, that is,

$$\xi^{\alpha^*}(t) = Z_0(t), a.e. \quad t \in [0, T], a.s., \tag{5.11}$$

where $Z_0(\cdot)$ is defined at the beginning of Section 5. Define $p_0(t) := S_0^{-1}(t)Z_0(t)$ as the unique pricing kernel. By (5.6) and (5.11), the optimal final wealth $V^{v,\alpha^*}(T)$ satisfies

$$\lambda(v)p_0(T) = \frac{dU(x)}{dx}\bigg|_{x=V^{v,\alpha^*}(T)} \frac{dw(z)}{dz}\bigg|_{z=H^{v,\alpha^*}(V^{v,\alpha^*}(T))}. \tag{5.12}$$

Let $\dot{U}^{-1}(\cdot)$ be the inverse of the derivative function of $U(\cdot)$, then $\dot{U}^{-1}(\cdot)$ is strictly decreasing owing to the strict convexity of $U(\cdot)$. We may rewrite (5.12) as

$$V^{v,\alpha^*}(T) = \dot{U}^{-1}\Big(\lambda(v)p_0(T)/\frac{dw(z)}{dz}\big|_{z=H^{v,\alpha^*}(V^{v,\alpha^*}(T))}\Big). \tag{5.13}$$

This is the optimal final wealth if we know some structure of $H^{v,\alpha^*}(V^{v,\alpha^*}(T))$ on the right-hand side. To obtain an explicit solution, we further assume

ASSUMPTION 5.3. *$p_0(T)$ has no atom; $F^{-1}_{p_0(T)}(z)/\frac{dw(z)}{dz}$, $0 \le z \le 1$ is nondecreasing.*

According to Jin and Zhou (2010), the optimal terminal wealth must be anti-comonotonic w.r.t. the pricing kernel, that is,

$$V^{v,\alpha^*}(T) = (H^{v,\alpha^*})^{-1}(F_{p_0(T)}(p_0(T))),$$

where $F_{p_0(T)}(\cdot)$ is the distribution function of $p_0(T)$. Then, (5.13) becomes $V^{v,\alpha^*}(T) = \chi_0^v$, with

$$\chi_0^v := \dot{U}^{-1}\left(\lambda_0(v)p_0(T)/\frac{dw(z)}{dz}\Big|_{z=F_{p_0(T)}(p_0(T))}\right), \tag{5.14}$$

where $\lambda(v) = \lambda_0(v)$ is defined by the initial budget constraint: $v = E_{\mathcal{P}}[p_0(T)\chi_0^v]$.

Since $\dot{U}^{-1}(\cdot)$ is strictly decreasing, the anti-comonotonicity of $V^{v,\alpha^*}(T)$ and $p_0(T)$ requires that $x/\frac{dw(z)}{dz}|_{z=F_{p_0(T)}(x)}$ or $F^{-1}_{p_0(T)}(z)/\frac{dw(z)}{dz}$, $0 \le z \le 1$ is nondecreasing (this is the necessary condition to guarantee that our first-order condition has a solution).

χ_0^v in (5.14) is the unique solution to our first-order condition, and the uniqueness will help us to prove its global optimality. The assumption of market completeness will guarantee that there exists a portfolio α^* (an equation for the optimal policy is presented in Appendix B) s.t., its wealth process matches χ_0^v at terminal time T and

$$(H^{v,\alpha^*})^{-1}(z) = \dot{U}^{-1}\left(\lambda_0(v)F^{-1}_{p_0(T)}(z)/\frac{dw(z)}{dz}\right), 0 \le z \le 1. \tag{5.15}$$

To summarize, we have

THEOREM 5.4. *In the continuous allocation problem with objective* (5.2)*, under Assumption 3.10, Assumption 5.2, and Assumption 5.3,*[4] *the optimal final wealth is χ_0^v defined by* (5.14).

Proof. See Appendix C. □

REMARK 5.5. In a complete market, our result in Theorem 5.4 is consistent with that in Jin and Zhou (2008). We use a different method, the sensitivity-based approach, to manage the problem, which may be further extended to incomplete markets, as shown in the next subsection.

[4]By the quantile method, Carlier and Dana (2011) and Xia and Zhou (2016) provided the solution for the case in which the monotonicity of $F^{-1}_{p_0(T)}(z)/\frac{dw(z)}{dz}$ does not hold. Xia and Zhou (2016) characterized this solution via the concave envelope of an integral of $F^{-1}_{p_0(T)}(z)/\frac{dw(z)}{dz}$.

5.2.2. Solutions in the incomplete markets case. In this subsection, we assume that $K < L$, i.e., the number of stocks is strictly less than the number of Brownian motions. In this case, the portfolios of stocks may not completely hedge the risk arising from the Brownian motions, so the market is incomplete.

Through sensitivity analysis, we have discovered the structural property of the terminal wealth (demonstrated in Theorem 5.1, i.e., $\frac{dU(x)}{dx}\big|_{x=V^{\nu,\alpha^*}(T)} \frac{dw(z)}{dz}\big|_{z=H^{\nu,\alpha^*}(V^{\nu,\alpha^*})}$ is proportional to a pricing kernel). In this subsection, we will demonstrate that this property also enables us to find the optimal final wealth for incomplete markets. We prove that this objective is achieved by following the method of "fictitious completion," which was used in Karatzas et al. (1991) to address the portfolio selection problem in incomplete markets under EU. We demonstrate that a result similar to that proposed by Theorem 5.4 holds for incomplete markets under RDEU, provided that an appropriate pricing kernel is selected.

Because the $K \times L$ volatility matrix $\sigma(t)$ has rank K for all $t \in [0, T]$ a.s., i.e., there are no redundant stocks in the market, to obtain a complete market, we must have $L - K$ additional stocks. Therefore, we create $L - K$ fictitious stocks, whose prices $P_i(t)$ are governed by

$$dP_i(t) = P_i(t)\{b_i(t)dt + \sum_{l=1}^{L} \rho_{i,l}(t)dW_l(t)\}, \quad t \in [0, T], i = 1, \dots, L - K,$$

where the $(L - K) \times L$ matrix $\rho(t)$ has full rank and its row vectors are chosen from the kernel of $\sigma(t)$, i.e., $\sigma(t)\rho(t)^\top = 0$.

With the fictitious stocks, we obtain a fictitious complete market, in which the optimal solution is given as that in Subsection 5.2.1. In general, the optimal "satisfaction" in the fictitious complete market will be better than that in the original incomplete market, owing to an expansion of choice. If we choose appropriate parameters for the fictitious stocks $\{b(t), \rho(t)\}$, s.t., the optimal allocation in the fictitious complete market actually does not invest any wealth in fictitious stocks, then the solution to the problem in the fictitious complete market is also an optimal solution for the original problem in an incomplete market.

In the fictitious complete market, the market price of risk is

$$\theta_\nu(t) = \theta_0(t) + \nu(t), \tag{5.16}$$

where $\theta_0(t) = \sigma(t)^\top \left(\sigma(t)\sigma(t)^\top\right)^{-1} (\mu(t) - r(t))$ and $\nu(t) = \rho(t)^\top \left(\rho(t)\rho(t)^\top\right)^{-1} (b(t) - r(t))$. Note that $\theta_0(t) \in Range(\sigma(t)^\top) = Kernel^\perp(\sigma(t))$, $\nu(t) \in Kernel(\sigma(t))$ and $\theta_\nu(t)$ satisfies

$$\sigma(t)\theta_\nu(t) = \mu(t) - r(t).$$

For $0 \le t \le T$, define

$$\begin{aligned} Z_\nu(t) &= \exp\left\{-\int_0^t \frac{1}{2}\theta_\nu(s)^\top\theta_\nu(s)ds - \int_0^t \theta_\nu(s)^\top dW(s)\right\} \\ &= \exp\left\{-\frac{1}{2}\int_0^t (\theta_0(s)^\top\theta_0(s) + \nu(s)^\top\nu(s))ds - \int_0^t (\theta_0(s)^\top + \nu(s)^\top)dW(s)\right\}, \end{aligned} \tag{5.17}$$

$p_\nu(t) = S_0^{-1}(t)Z_\nu(t)$, and

$$\kappa_\nu(y) := E_{\mathcal{P}}\left[p_\nu(T)\dot{U}^{-1}\left(yp_\nu(T)/\frac{dw(z)}{dz}\Big|_{z=F_{p_\nu(T)}(p_\nu(T))}\right)\right]. \tag{5.18}$$

ASSUMPTION 5.6. *$Z_\nu(t)$ is a martingale; $p_\nu(T)$ has no atom; $F^{-1}_{p_\nu(T)}(z)/\frac{dw(z)}{dz}$, $0 \le z \le 1$ is nondecreasing;*[5] *the utility U is strictly concave; $\kappa_\nu(y) < +\infty$.*

Under Assumption 5.6, $\kappa_\nu(\cdot)$ inherits the decreasing property of $\dot{U}^{-1}(\cdot)$ and we define $\lambda_\nu(\cdot)$ to be its inverse function. Define

$$\chi_\nu^v := \dot{U}^{-1}\left(\lambda_\nu(v)p_\nu(T)\Big/\frac{dw(z)}{dz}\Big|_{z=F_{p_\nu(T)}(p_\nu(T))}\right). \tag{5.19}$$

From Subsection 5.2.1, in the fictitious complete market involving $\nu(t) \in Kernel(\sigma(t))$, $p_\nu(t)$ is the unique pricing kernel, and χ_ν^v will be the optimal final wealth of our continuous allocation problem provided that Assumptions 3.10 and 5.6 hold.

To solve our original problem in the incomplete market, we must find the appropriate $\nu_*(t) \in Kernel(\sigma(t))$, s.t., $\chi_{\nu_*}^v$ is attainable from the original incomplete market. It turns out that the following conditions are equivalent and each of them can characterize ν_*.

(A) FINANCIBILITY OF $\chi_{\nu_*}^v$: There exists a portfolio α^* in the original incomplete market, s.t., $V^{x,\alpha^*}(T) = \chi_{\nu_*}^v$ a.s.
(B) LEAST-FAVORABILITY OF ν_*: $\widetilde{E}_{\mathcal{P}}[U(\chi_{\nu_*}^v)] \le \widetilde{E}_{\mathcal{P}}[U(\chi_\nu^v)]$, for all $\nu \in Kernel(\sigma)$.
(C) DUAL OPTIMALITY OF ν_*: Let $\tilde{U}$ be the conjugate of U:$\tilde{U}(y) = \max_{x>0}(U(x) - xy)$ and define $\dot{w}_\nu(T) := \frac{dw(z)}{dz}\big|_{z=F_{p_\nu(T)}(p_\nu(T))}$. For all $\nu \in Kernel(\sigma)$

$$E_{\mathcal{P}}[\tilde{U}(\lambda_{\nu_*}(v)p_{\nu_*}(T)/\dot{w}_{\nu_*}(T))\dot{w}_{\nu_*}(T)] \le E_{\mathcal{P}}[\tilde{U}(\lambda_{\nu_*}(v)p_\nu(T)/\dot{w}_\nu(T))\dot{w}_\nu(T)]. \tag{5.20}$$

(D) PARSIMONY OF ν_*: $E_{\mathcal{P}}[p_\nu(T)\chi_{\nu_*}^v] \le v = E_{\mathcal{P}}[p_{\nu_*}(T)\chi_{\nu_*}^v]$, for all $\nu \in Kernel(\sigma)$.

THEOREM 5.7. *Assume that Assumptions 3.10 and 5.6 hold. Conditions (A)–(D) are equivalent, and if (A) holds, the portfolio α^* provided by* (5.25) *is the optimal portfolio of the continuous allocation problem in the incomplete market.*

Proof. "(A) ⇒ (B)": This part follows the facts that the optimal "satisfaction" in the fictitious complete market is no worse than that in the original incomplete market, χ_ν^v is optimal in the fictitious complete market involving ν for $\nu \in Kernel(\sigma)$, and $\chi_{\nu_*}^v$ is attainable in the original incomplete market, according to Condition (A). "(B) ⇒

[5] To apply Theorem 5.4 for the fictitious complete market with pricing kernel $p_\nu(T)$, we assume here that $F^{-1}_{p_\nu(T)}(z)/\frac{dw(z)}{dz}$, $0 \le z \le 1$ is nondecreasing. However, as noted in footnote 4, this monotonicity assumption (and, moreover, the no-atom assumption of terminal wealth) may be removed. Consequently, we may present our results in the incomplete market by applying Xia and Zhou's (2016) result for the fictitious complete market without the monotonicity assumption of $F^{-1}_{p_\nu(T)}(z)/\frac{dw(z)}{dz}$. In this case, χ_ν^v in (5.19) will be replaced by $\dot{U}^{-1}(\lambda_\nu(v)\hat{N}(1 - F_{p_\nu(T)}(p_\nu(T))))$, where $\hat{N}(q)$ is the concave envelope of $N(q)(:= -\int_q^1 F^{-1}_{p_\nu(T)}(w^{-1}(1-p))/\frac{dw(z)}{dz}|_{z=w^{-1}(1-p)}dp)$. See theorem 3.5 and remark 3.15 (Xia and Zhou 2016).

(C)": Noting that $\tilde{U}(y) = U(\dot{U}^{-1}(y)) - y\dot{U}^{-1}(y)$ and $d\tilde{U}(y)/dy = -\dot{U}^{-1}(y)$. Hence, $\tilde{U}(\cdot)$ is decreasing, convex, and we have

$$\frac{d}{dy} E_{\mathcal{P}}[\tilde{U}(yp_\nu(T)/\dot{w}_\nu)\dot{w}_\nu] = -E_{\mathcal{P}}[yp_\nu(T)\dot{U}^{-1}(yp_\nu(T)/\dot{w}_\nu)] = -\kappa_\nu(y),$$

where the interchangeability of the derivative and the expectation follows from the dominated convergence theorem and

$$|\tilde{U}((y+\Delta y)p_\nu(T)/\dot{w}_\nu) - \tilde{U}((y)p_\nu(T)/\dot{w}_\nu)|\dot{w}_\nu \leq p_\nu(T)\dot{U}^{-1}((y/2)p_\nu(T)/\dot{w}_\nu)|\Delta y|,$$
$$\forall \Delta y > -y/2.$$

Therefore, for any $v > 0$, the convex function

$$f_\nu(y) := E_{\mathcal{P}}[\tilde{U}(yp_\nu(T)/\dot{w}_\nu)\dot{w}_\nu] + vy$$

attains its minimum at $\lambda_\nu(v)$. From Condition (B) and (5.18), for any $y > 0$, $\nu \in Kernel(\sigma)$,

$$\begin{aligned} f_\nu(y) \geq f_\nu(\lambda_\nu(v)) &= E_{\mathcal{P}}[\tilde{U}(\lambda_\nu(v)p_\nu(T)/\dot{w}_\nu)\dot{w}_\nu] + \lambda_\nu(v)v \\ &= E_{\mathcal{P}}[\dot{w}_\nu U(\chi_\nu^v)] = \widetilde{E}_{\mathcal{P}}[U(\chi_\nu^v)] \geq \widetilde{E}_{\mathcal{P}}[U(\chi_{\nu_*}^v)] = f_{\nu_*}(\lambda_{\nu_*}(v)). \end{aligned} \tag{5.21}$$

Then,

$$\begin{aligned} E_{\mathcal{P}}[\tilde{U}(\lambda_{\nu_*}(v)p_{\nu_*}(T)/\dot{w}_{\nu_*})\dot{w}_{\nu_*}] &= E_{\mathcal{P}}[U(\chi_{\nu_*}^v)\dot{w}_{\nu_*} - \lambda_{\nu_*}(v)p_{\nu_*}(T)\chi_{\nu_*}^v] \\ &= \widetilde{E}_{\mathcal{P}}[U(\chi_{\nu_*}^v)] - \lambda_{\nu_*}(v)v = f_{\nu_*}(\lambda_{\nu_*}(v)) - \lambda_{\nu_*}(v)v \\ &\leq f_\nu(\lambda_{\nu_*}(v)) - \lambda_{\nu_*}(v)v = E_{\mathcal{P}}[\tilde{U}(\lambda_{\nu_*}(v)p_\nu(T)/\dot{w}_\nu)\dot{w}_\nu]. \end{aligned}$$

"(C) $\Rightarrow$ (A)": For $\nu \in Kernel(\sigma)$, $n > 0$, define

$$\begin{aligned} \tau_n := \quad & \inf\Big\{t \in [0, T] : \int_0^t (r(s) + \theta_0(s)^\top\theta_0(s) + \nu_*(s)^\top\nu_*(s) + \nu(s)^\top\nu(s) \\ & \qquad + \nu_*(s)^\top\nu(s))ds \geq n, \\ & \text{or } \Big|\int_0^t (\theta_0(s)^\top + \nu_*(s)^\top)dW(s)| \geq n, \text{ or } \Big|\int_0^t \nu(s)^\top dW(s)| \geq n\Big\} \wedge T, \end{aligned}$$

and $\nu_n(t) := \nu(t)1_{[0,\tau_n]}(t)$. Consider the perturbation $\nu_* + \epsilon\nu_n$. Then, $\nu_* + \epsilon\nu_n \in Kernel(\sigma)$ and

$$p_{\nu_*+\epsilon\nu_n}(t) = p_{\nu_*}(t)\exp\left\{-\frac{\epsilon^2}{2}\int_0^t \nu_n(s)^\top\nu_n(s)ds - \epsilon\int_0^t \nu_n(s)^\top(dW(s) + \nu_*(s)ds)\right\}.$$

Hence, $\frac{d}{d\epsilon}p_{\nu_*+\epsilon\nu_n}(T)|_{\epsilon=0} = -p_{\nu_*}(T)\int_0^T \nu_n(s)^\top(dW(s) + \nu_*(s)ds$. Let $f(x,z) = \tilde{U}(\frac{\lambda_{\nu_*}(v)x}{dw(z)/dz})\frac{dw(z)}{dz}$.

$$\begin{aligned} & E_{\mathcal{P}}[\frac{d}{d\epsilon}f(p_{\nu_*+\epsilon\nu_n}(T), F_{p_{\nu_*+\epsilon\nu_n}(T)}(p_{\nu_*+\epsilon\nu_n}(T)))] \\ & = E_{\mathcal{P}}[\frac{\partial f}{\partial x}f(p_{\nu_*+\epsilon\nu_n}(T), F_{p_{\nu_*+\epsilon\nu_n}(T)}(p_{\nu_*+\epsilon\nu_n}(T)))\frac{d}{d\epsilon}p_{\nu_*+\epsilon\nu_n}(T)] \end{aligned}$$

$$+E_{\mathcal{P}}[\frac{\partial f}{\partial z} f(p_{\nu_*+\epsilon\nu_n}(T), F_{p_{\nu_*+\epsilon\nu_n}(T)}(p_{\nu_*+\epsilon\nu_n}(T)))\frac{d}{d\epsilon}F_{p_{\nu_*+\epsilon\nu_n}(T)}(p_{\nu_*+\epsilon\nu_n}(T))],$$

where the second term is zero, owing to (3.3).[6] Then, from Condition (C), we have

$$\begin{aligned}
0 = & \frac{d}{d\epsilon}E_{\mathcal{P}}[\tilde{U}(\lambda_{\nu_*}(v)p_{\nu_*+\epsilon\nu_n}(T)/\dot{w}_{\nu_*+\epsilon\nu_n}(T))\dot{w}_{\nu_*+\epsilon\nu_n}(T)]\Big|_{\epsilon=0}\\
= & E_{\mathcal{P}}\left[\frac{d\tilde{U}(y)}{dy}\Big|_{y=\lambda_{\nu_*}(v)p_{\nu_*}(T)/\dot{w}_{\nu_*}(T)}\lambda_{\nu_*}(v)\frac{d}{d\epsilon}p_{\nu_*+\epsilon\nu_n}(T)\Big|_{\epsilon=0}\right]\\
= & \lambda_{\nu_*}(v)E_{\mathcal{P}}\left[p_{\nu_*}(T)\chi^v_{\nu_*}\int_0^T \nu_n(s)^\top(dW(s)+\nu_*(s)ds\right],
\end{aligned}$$

where the exchangeability is guaranteed by $p_{\nu_*+\epsilon\nu_n}(T)\in[e^{-3n}, e^{3n}]$ for small ϵ.

Define $M(T) := p_{\nu_*}(T)\chi^v_{\nu_*}$ and

$$M(t) := E_{\mathcal{P}}[M(T)|\mathcal{F}_t] = v - \int_0^t M(s)\zeta_*^\top(s)dW(s),\ t\in[0, T], \tag{5.22}$$

for some $\mathcal{F}_t$ adapted process ζ_* satisfying $\int_0^T \zeta_*^\top(s)\zeta_*(s)ds < \infty$ a.s. Similar to the proof of Theorem 5.1, we have

$$0 = E_{\mathcal{P}}\left[M(T)\int_0^T \nu_n(s)^\top(dW(s)+\nu_*(s)ds\right] = E_{\mathcal{P}}\left[\int_0^{\tau_n} M(t)\nu^\top(t)(\nu_*(t)-\zeta_*(t))dt\right].$$

Because $\nu(\cdot)\in Kernel(\sigma)$ is arbitrary and $\lim_{n\to+\infty}\tau_n = T$ a.s., we have

$$\nu_*(t)-\zeta_*(t)\in Kernel^\perp(\sigma(t)) = Range(\sigma^\top(t)). \tag{5.23}$$

In conjunction with the fact that $\theta_0 = Range(\sigma^\top)$, there exists α^* solving the linear equation

$$\sigma^\top(t)\tilde{\alpha}^*(t) = \theta_0(t)+\nu_*(t)-\zeta_*(t). \tag{5.24}$$

Let $\tilde{\alpha}_*$ be the portfolio proportion in stocks, i.e., $\tilde{\alpha}_*(t) = \alpha^*(t)/V^{v,\alpha^*}(t)$. Using Ito's formula to $p_{\nu_*}(t)V^{v,\alpha^*}(t)$ and comparing it with (5.22), we have that $V^{v,\alpha^*}(t) = p^{-1}_{\nu_*}(t)M(t), 0\le t\le T$. Specially, $V^{v,\alpha^*}(T) = \chi^v_{\nu_*}$, that is, Condition (A) holds, and the corresponding wealth $V^{v,\alpha^*}(t) = p^{-1}_{\nu_*}(t)M(t)$, with portfolio given by (cf. (5.24))

$$\alpha^*(t) = V^{v,\alpha^*}(t)\left(\sigma(t)\sigma(t)^\top\right)^{-1}\sigma(t)(\theta_0(t)+\nu_*(t)-\zeta_*(t)). \tag{5.25}$$

"(A) $\Rightarrow$ (D)": Applying Ito's formula to $p_\nu(t)V^{x,\alpha^*}(t)$, we have

$$p_\nu(t)V^{v,\alpha^*}(t) = v + \int_0^t p_\nu(s)(\sigma^\top(t)\alpha^*(t) - V^{v,\alpha^*}(t)(\theta_0(s)+\nu(s)))^\top dW(t),$$

which demonstrates that $p_\nu(t)V^{x,\alpha^*}(t)$ is a positive local martingale and hence a supermartingale. Thus, Condition (D) holds.

"(D) $\Rightarrow$ (A)": This is the classical result to verify the attainability of wealth in an incomplete market, and it follows by repeating the steps in the proof of "(C) $\Rightarrow$ (A)". □

[6]This requires the condition that Assumption 3.1 holds for $p_{\nu_*+\epsilon\nu_n}(T)$, while, as stated in footnote 1, this is not restrictive.

REMARK 5.8. Based on (5.21), the minimization dual problem (5.20) in Condition (C) is equivalent to the following minimization problem
$\min_{\lambda>0}\left\{\min_{\nu\in Kernel(\sigma)} E_{\mathcal{P}}[\tilde{U}(\lambda p_\nu(T)/\dot{w}_\nu)\dot{w}_\nu]+\lambda v\right\}$.

In the remainder of this subsection, we provide some examples to illustrate the dual problem. The first example is the case in which asset returns have constant coefficients. From this example, we will demonstrate how the distorted probability affects investors' investment strategy.

EXAMPLE 5.9. (A deterministic investment opportunity set): Suppose that the coefficient processes $r(\cdot)$, $\mu(\cdot)$, and $\sigma(\cdot)$ are deterministic. We will use the above conditions to demonstrate that $\nu_*\equiv 0$ is optimal and that the optimal terminal wealth is χ_0^v. In fact, because $p_0(t)$ is Markovian and χ_0^v is a function of $p_0(T)$, there exists a function g, s.t.,

$$M(t)=E_{\mathcal{P}}[p_0(T)\chi_0^v|\mathcal{F}_t]=E_{\mathcal{P}}[p_0(T)\chi_0^v|p_0(t)]=g(t,p_0(t)).$$

If $g\in C^{1,2}[[0,T)\times(0,\infty)]$, using Ito's formula to $g(t,p_0(t))$ and comparing with (5.22), we have

$$\begin{cases}\frac{\partial}{\partial t}g(t,p)+rp\frac{\partial}{\partial p}g(t,p)+\frac{1}{2}\theta_0{}^\top\theta_0\frac{\partial^2}{\partial p^2}g(t,p)=0, \text{ for any } (t,p)\in[0,T)\times(0,\infty)\\ p_0(t)\frac{\partial}{\partial p}g(t,p_0(t))\theta_0=f(t,p_0(t)\zeta_*,\end{cases}$$

(5.26)

with the boundary condition $g(T,p)=p\dot{U}^{-1}(\lambda_0(v)p/\frac{\partial w(z)}{\partial z}|_{z=F_{p_0(T)}(p)})$, $\forall p>0$.
From the second equation of (5.26) and $\nu_*\equiv 0$, we have

$$\nu_*-\zeta_*=-p_0(t)\frac{\partial}{\partial p}g(t,p_0(t))\theta_0/f(t,p_0(t)\in Range(\sigma(t)).$$

Thus, by setting the portfolio α^* as (5.25), we have $V^{x,\alpha^*}(T)=\chi_{\nu_*}^v$, i.e., Condition (A) holds. This outcome confirms our conclusion that $\nu_*\equiv 0$ is optimal and that the optimal terminal wealth is χ_0^v. This finding is consistent with the result of He and Zhou (2011b), where the model with a deterministic investment opportunity set is the only case in incomplete markets that can be managed by the quantile method.

To see how probability distortions will change the pattern of portfolio selection, we further assume that the utility function $U(x)=x^\gamma/\gamma$, $0<\gamma<1$ and that r, μ, and σ are constant. Then, $p_0(T)$ is lognormal distributed with $F_{p_0(T)}(x)=N((\ln(x)-\bar{\mu})/\bar{\sigma})$, where $\bar{\mu}=-(r+|\theta_0|^2/2)T$, $\bar{\sigma}=|\theta_0|\sqrt{T}$ and $N(\cdot)$ is the cumulative distribution function of the standard normal random variable. Following example 6.1 of Jin and Zhou (2008), we set the distortion function $w(z):=N(N^{-1}(z)-a\bar{\sigma})$ for $a<1$, which is strictly convex (concave) when $0<a<1$ ($a<0$). It follows that (note that $\frac{dw(z)}{dz}|_{z=H_\theta(x)}$ is of power type and X_θ is lognormal distributed),

$$\begin{aligned}\chi_0^v&=\frac{v}{E_{\mathcal{P}}\left[(p_0(T))^{1-\frac{1-a}{1-\gamma}}\right]}(p_0(T))^{-\frac{1-a}{1-\gamma}}, \text{ and } \alpha^*(t)\\ &=V^{v,\alpha^*}(t)\frac{1-a}{1-\gamma}\left(\sigma\sigma^\top\right)^{-1}(\mu-r), \forall t\in[0,T].\end{aligned}$$

This finding reveals that, at any time, the optimal strategy is to invest a proportion $\frac{1-a}{1-\gamma}\left(\sigma\sigma^\top\right)^{-1}(\mu-r)$ of the total wealth in stock. If $a=0$, there is no distortion in probabilities, and we obtain Merton's (1971) classic result under EU. If $0 < a < 1 (a < 0)$, the distortion function is convex (concave), and the investor will put less (more) weight on stocks, i.e., investors with a convex (concave) probability distortion will be more risk-averse (risk-seeking) than those with EU. □

The above portfolio strategy matches the intuition that an amplification effect on the chance of small payments will make investors reduce risk-taking. From the weighted expectation form (2.4), distorted utility differs from EU by placing a weight $\frac{dw(z)}{dz}\big|_{z=H_\theta(X_\theta)}$ on payment X_θ. If $w(\cdot)$ is convex, then $dw(z)/dz > 1$ for z near 1. Therefore, the weight of a small X_θ (bad event) is greater than 1 ($H_\theta(X_\theta)$ is near 1 for small values of X_θ), i.e., convex probability distortion functions increase the effect of bad events, rendering investors with convex distorted utility more pessimistic.

The second example is the case in which the stock prices and some auxiliary state variables form a joint Markov process. In this case, we provide a verification algorithm for the solution of the dual minimization problem.

EXAMPLE 5.10. (A joint Markov process): Suppose that the coefficient processes $r(\cdot)$, $\mu(\cdot)$, and $\sigma(\cdot)$ are functions of time t, K stock prices $S(t)$, and the other $L-K$ state variables $Y(t)$; and the $L-K$-dimensional state variables $Y(t)$ are nontradable with drift μ^Y and σ^Y dependent only upon the current values of state variables, current stock prices, and time. Thus, $\{S(t), Y(t)\}$ form a Markov process. We further assume that the joint volatility matrix of $\{S(t), Y(t)\}$ is nonsingular.

According to Condition (C), ν_* is the solution to the following minimization dual problem:

$$\min_{\nu\in Kernel(\sigma)} \varpi(\nu), \text{ where} \varpi(\nu) := E_{\mathcal{P}}\left[\tilde{U}(\lambda p_\nu(T)/\dot{w}_\nu(T))\,\dot{w}_\nu(T)\right], \tag{5.27}$$

for some λ. Let $\dot{w}_\nu(t) := E_{\mathcal{P}}[\dot{w}_\nu(T)|\mathcal{F}_t]$, then $\dot{w}_\nu(t)$ is a martingale and $\dot{w}_\nu(0) = 1$. By the martingale representation theorem, we have

$$\dot{w}_\nu(t) = 1 - \int_0^t \dot{w}_\nu(t)\varrho_\nu^\top(s)dW(s) = \exp\left\{-\frac{1}{2}\int_0^t \varrho_\nu(s)^\top\varrho_\nu(s)ds - \int_0^t \varrho_\nu(s)^\top dW(s)\right\}, \tag{5.28}$$

for some $\mathcal{F}_t$ adapted process ϱ_ν satisfying $\int_0^T \varrho_\nu^\top(s)\varrho_\nu(s)ds < \infty$ a.s. Define measure $\mathcal{Q}$ on Ω by $d\mathcal{Q}/d\mathcal{P} = \dot{w}_\nu(T)$. By Girsanov theorem, the Brownian motion $W_\nu(t)$ under $\mathcal{Q}$ is given by

$$dW_\nu(t) = dW(t) + \varrho_\nu(t)dt.$$

Let θ be a parameter of ν. From the proof "(C) ⇒ (A)" of Theorem 5.7, at the optimal, the first-order condition gives

$$\begin{aligned} 0 = \frac{d\varpi}{d\theta}\Big|_{\theta=\theta_*} &= E_{\mathcal{P}}\left[\frac{d\tilde{U}(y)}{dy}\Big|_{y=\lambda p_\nu(T)/\dot{w}_{\nu_*}(T)}\lambda\frac{d}{d\theta}p_\nu(T)\Big|_{\theta=\theta_*}\right] \\ &= E_{\mathcal{P}}\left[\frac{d}{d\theta}\tilde{U}(\lambda p_\nu(T)/\dot{w}_{\nu_*}(T))\dot{w}_{\nu_*}(T)\Big|_{\theta=\theta_*}\right] \\ &= \frac{d}{d\theta}E_{\mathcal{Q}^*}\left[\tilde{U}(\lambda p_\nu(T)/\dot{w}_{\nu_*}(T))\right]\Big|_{\theta=\theta_*}. \end{aligned} \tag{5.29}$$

Note that (5.29) has the same form as the first-order condition for the minimization dual problem with no distortion under the changed probability measure $\mathcal{Q}^*$.

To characterize the measure $\mathcal{Q}^*$, we use the Markov property of $\{p_\nu(t), S(t), Y(t)\}$. It follows that we may define

$$f_\nu(p_\nu(t), S(t), Y(t), t) := \dot{w}_\nu(t) = E_{\mathcal{P}}[\dot{w}_\nu(T)|\mathcal{F}_t] = E_{\mathcal{P}}[\dot{w}_\nu(T)|p_\nu(t), S(t), Y(t)].$$

If f_ν is continuously differentiable in t and second-order continuously differentiable in p, s, and y (see the footnote in Appendix B for the similar sufficient conditions that guarantee the differentiability of f_ν), then applying Ito's formula to $f_\nu(p_\nu(t), S(t), Y(t), t)$ and comparing the result with (5.28), we have

$$\mathbb{A}_\nu f_\nu(p, s, y, t) = 0, \text{ with the boundary condition } f_\nu(p, s, y, T) = \frac{dw(z)}{dz}\Big|_{z=F_{p_\nu(T)}(p)}, \tag{5.30}$$

$$\varrho_\nu(t) = \left\{-p_\nu(t)(\theta_0(t)+\nu(t))\frac{\partial f_\nu}{\partial p} + \left(\frac{\partial f_\nu}{\partial s}\right)^\top \tilde{\sigma}(t) + \left(\frac{\partial f_\nu}{\partial y}\right)^\top \tilde{\sigma}^Y(t)\right\} / f_\nu, \tag{5.31}$$

where $\tilde{\sigma}_{k,\cdot}(t) = S_k(t)\sigma_{k,\cdot}(t)$, $k = 1, \dots, K$, $\tilde{\sigma}^Y_{i,\cdot}(t) = Y_i(t)\sigma_{i,\cdot}(t)$, $i = 1, \dots, L-K$ and $\mathbb{A}_\nu$ is the infinitesimal generator of joint processes $\{p_\nu(t), S(t), Y(t)\}$.

We consider a feedback control to be a candidate optimal solution, i.e., let $\nu_*(t)$ be the solution of (5.27) and be determined by $\{p_{\nu_*}(t), S(t), Y(t)\}$. Then, from (5.31), $\varrho_{\nu_*}(t)$ is determined by $\{p_{\nu_*}(t), S(t), Y(t)\}$. With a fixed ν_*, consider the optimization problem under measure $\mathcal{Q}^*$

$$\min_{\nu\in Kernel(\sigma)} \breve{\varpi}(\nu), \text{ where } \breve{\varpi}(\nu) := E_{\mathcal{Q}^*}\left[\tilde{U}(\lambda p_\nu(T)/\dot{w}_{\nu_*}(T))\right], \tag{5.32}$$

There is no probability distortion in (5.32). Following the usual argument of dynamic programming, the solution of (5.32) is a feedback control. In conjunction with the argument of (5.29), we may conclude that the solution of (5.32) is ν_*, coinciding with the solution of (5.27).

Define $q_\nu(t) = p_\nu(t)/\dot{w}_{\nu_*}(t)$, $0 \le t \le T$, then

$$dq_\nu(t) = -q_\nu(t)(r(t)dt + (\theta_0(t)+\nu(t)-\varrho_{\nu_*}(t))dW_{\nu_*}(t)).$$

By the Markov property, define

$$g_\nu(q_\nu(t), S(t), Y(t), t) := E_{\mathcal{Q}^*}\left[\tilde{U}(\lambda p_\nu(T)/\dot{w}_{\nu_*}(T))\,\Big|\,q_\nu(t), S(t), Y(t)\right].$$

If g_ν is continuously differentiable in t and second-order continuously differentiable in q, s, and y, then g_ν satisfies the Poisson equation:

$$\breve{\mathbb{A}}_\nu g_\nu(p, s, y, t) = 0, \text{ with the boundary condition } g_\nu(q, s, y, T) = \tilde{U}(\lambda q), \tag{5.33}$$

where $\breve{\mathbb{A}}_\nu$ is the infinitesimal generator of joint processes $\{q_\nu(t), S(t), Y(t)\}$ under $\mathcal{Q}^*$.

Similar to equation (6.6.15) in Karatzas and Shreve (1998), g_{ν_*} solves the HJB equation:

$$\min_{\nu\in Kernel(\sigma)} \breve{\mathbb{A}}_\nu g_{\nu_*}(p, s, y, t) = 0, \text{ with the boundary condition } g_{\nu_*}(q, s, y, T) = \tilde{U}(\lambda q).$$

(5.34)

Then, by the perturbation $\nu_* + \epsilon\nu$, where $\nu \in Kernel(\sigma)$, we have

$$0 = \frac{\partial \breve{\mathbb{A}}_{\nu_*+\epsilon\nu}}{\partial \epsilon}\Big|_{\epsilon=0} g_{\nu_*} = \nu^\top \left(q^2 \frac{\partial^2 g_{\nu_*}}{\partial q^2}(\nu_* - \varrho_*) - q\frac{\partial}{\partial q}\left(\sigma^\top \frac{\partial g_{\nu_*}}{\partial s} + (\sigma^Y)^\top \frac{\partial g_{\nu_*}}{\partial y}\right)\right).$$

It follows that $\vec{v}_*(t) := q^2 \frac{\partial^2 g_{\nu_*}(p,s,y,t)}{\partial q^2}(\nu_*(t) - \varrho_*(t)) - q\frac{\partial}{\partial q}\left((\sigma^Y(t))^\top \frac{\partial g_{\nu_*}(p,s,y,t)}{\partial y}\right) \in Range(\sigma^\top(t))$, which is equivalent to

$$\sigma^\top(t)(\sigma(t)\sigma^\top(t))^{-1}\sigma(t)\vec{v}_*(t) = \vec{v}_*(t),\ t \in [0, 1] a.e. \tag{5.35}$$

Verification would proceed as follows: Given $\nu \in Kernel(\sigma)$,

(1) Solve the Kolmogorov backward equation (5.30) with the boundary condition $f(p, s, y, T) = 1_{p<\bar{p}}$. Then, obtain the cumulative distribution of $p_\nu(T)$, $F_{p_\nu(T)}(\bar{p}) = f(1, S(0), Y(0), 0)$.
(2) Solve the Poisson equation (5.30) for f_ν, and obtain ϱ_ν by (5.31).
(3) With this ϱ_ν as ϱ_{ν_*} and ν as ν_*, solve the Poisson equation (5.33) for g_{ν_*}.
(4) If (5.35) holds, then ν_* is the solution to (5.27).

REMARK 5.11. From (5.29), monolinearity implies that the first-order condition under distorted utility coincides with that under EU using the changed probability measure $\mathcal{Q}_*$.

Next, we provide a solvable example in incomplete market similar to that in Detemple and Rindisbacher (2005) under EU. Our example includes probability distortions.

EXAMPLE 5.12. (A solvable incomplete market model with a stochastic interest rate): In (4.1), suppose that $L = K + 1$ and that the stock volatility matrix $\sigma(t)$ has the form $\sigma(t) = [\check{\sigma}(t), 0]$, where $\check{\sigma}(t)$ is a $K \times K$, nonsingular matrix for all $t \in [0, T]$ a.s. Then, in (5.16), the last component of θ_0 and the first K components of ν are both 0, i.e., $\theta_0(t) = [\check{\theta}_0(t)^\top, 0]$ and $\nu(t) = [0, \hat{\nu}(t)]$, where $\check{\theta}_0(t) = \check{\sigma}(t)^{-1}(\mu(t) - r(t))$ is a K-dimensional vector and $\hat{\nu}(t) \in \mathbb{R}$. Decompose W into $\check{W}(t) = (W_1(t), \ldots, W_K(t))^\top$ and $\hat{W}(t) = W_{K+1}(t)$. We assume that $\check{\theta}_0(\cdot)$ is deterministic and that the interest rate $r(\cdot)$ follows the Vasicek model,

$$dr(t) = b(\hat{r}(t) - r(t)) + \hat{\sigma}_1(t)^\top d\check{W}(t) + \hat{\sigma}_2(t)d\hat{W}(t), r(0) = r_0,$$

where the speed of mean reversion b is constant and the long-turn value $\hat{r}(t)$ and the volatility $\hat{\sigma}_1(t)$, $\hat{\sigma}_2(t)$ are deterministic.

We conjecture that there is a deterministic $\hat{\nu}_*(t)$, $0 \le t \le T$, solving the dual minimization problem (5.20). First, we compute the distribution of $p_{\nu_*}(T)$ (cf. (5.17)). We first analyze the stochastic term $\int_0^t r(s)ds$. Applying Ito's formula to $e^{bs}r(s)$, we obtain

$$r(s) = r_0e^{-bs} + \int_0^s be^{-b(s-u)}\hat{r}(u)du + \int_0^s e^{-b(s-u)}(\hat{\sigma}_1(u)^\top d\check{W}(u) + \hat{\sigma}_2(u)d\hat{W}(u)).$$

Define $\hat{b}(s, t) := (1 - e^{-b(t-s)})/b$, then

$$\int_0^t r(s)ds = r_0\hat{b}(0, t) + \int_0^t b\hat{b}(u, t)\hat{r}(u)du + \int_0^t \int_0^s e^{-b(s-u)}(\hat{\sigma}_1(u)^\top d\check{W}(u) + \hat{\sigma}_2(u)d\hat{W}(u))ds.$$

Define $\tilde{r}(t) := r_0\hat{b}(0,t) + \int_0^t b\hat{b}(u,t)\hat{r}(u)du$. Applying Ito's formula to $e^{-bs}\int_0^s e^{bu}(\hat{\sigma}_1(u)^\top d\check{W}(u) + \hat{\sigma}_2(u)d\hat{W}(u))$ and integrating both sides from 0 to t, we get

$$\int_0^t r(s)ds = \tilde{r}(t) + \int_0^t \hat{b}(u,t)(\hat{\sigma}_1(u)^\top d\check{W}(u) + \hat{\sigma}_2(u)d\hat{W}(u)).$$

Hence, the price kernel can be represented as (cf. (5.17))

$$p_{\nu_*}(t) = \exp\left\{-\tilde{r}(t) - \frac{1}{2}\int_0^t (|\check{\theta}_0(u)|^2 + \hat{\nu}_*(u)^2)du \right.$$
$$\left. - \int_0^t (\hat{b}(u,t)\hat{\sigma}_1(u) + \check{\theta}_0(u))^\top d\check{W}(u) - \int_0^t (\hat{b}(u,t)\hat{\sigma}_2(u) + \hat{\nu}_*(u))d\hat{W}(u))\right\}, \tag{5.36}$$

and $p_{\nu_*}(T)$ is lognormal distributed with $F_{p_{\nu_*}(T)}(x) = N((\ln(x) - \bar{\mu}_*)/\bar{\sigma}_*)$, where

$$\bar{\mu}_* = -\tilde{r}(T) - \frac{1}{2}\int_0^T (|\check{\theta}_0(u)|^2 + \hat{\nu}_*(u)^2)du,$$
$$\bar{\sigma}_*^2 = \int_0^T |\hat{b}(u,T)\hat{\sigma}_1(u) + \check{\theta}_0(u)|^2 du + \int_0^T (\hat{b}(u,T)\hat{\sigma}_2(u) + \hat{\nu}_*(u))^2 du.$$

Second, as in Example 5.9, we assume that $U(x) = x^\gamma/\gamma, 0 < \gamma < 1$ and $w(z) = N(N^{-1}(z) - a\bar{\sigma}_*)$ for $a < 1, a \neq \gamma$. Through direct computation, we obtain (cf. (5.19))

$$\chi_{\nu_*}^{v} = \frac{v}{E_{\mathcal{P}}\left[(p_{\nu_*}(T))^{\gamma_*}\right]}(p_{\nu_*}(T))^{\gamma_*-1}\text{with}\gamma_* = 1 - \frac{1-a}{1-\gamma}. \tag{5.37}$$

Third, we use perturbation analysis to solve the dual problem (5.20) for the optimal $\hat{\nu}_*$. Following the same process in the proof of "(C) $\Rightarrow$ (A)" in Theorem 5.7, we have (cf. (5.23))

$$\hat{\nu}_*(t) - \hat{\zeta}_*(t) = 0,$$

where $\hat{\zeta}_*$ is decided by the martingale representation (cf. (5.22))

$$M(t) := E_{\mathcal{P}}[p_{\nu_*}(T)\chi_{\nu_*}^{v}|\mathcal{F}_t] = v - \int_0^t M(u)\check{\zeta}_*^\top(u)d\check{W}(u) - \int_0^t M(u)\hat{\zeta}_*(u)d\hat{W}(u).$$

We lastly compute the conditional expectation $E_{\mathcal{P}}[p_{\nu_*}(T)\chi_{\nu_*}^{v}|\mathcal{F}_t]$ directly to obtain $\hat{\zeta}_*$ (i.e. $\hat{\nu}_*$). By (36), we have $(p_{\nu_*}(T))^{\gamma_*} = N(T)m(T)$, where $m(t), 0 \leq t \leq T$ is a martingale, defined by

$$m(t) = \exp\left\{-\frac{\gamma_*^2}{2}\int_0^t |\hat{b}(u,T)\hat{\sigma}_1(u) + \check{\theta}_0(u)|^2 du - \frac{\gamma_*^2}{2}(\hat{b}(u,T)\hat{\sigma}_2(u) + \hat{\nu}_*(u))^2 du \right.$$
$$\left. -\gamma_*\int_0^t (\hat{b}(u,T)\hat{\sigma}_1(u) + \check{\theta}_0(u))^\top d\check{W}(u) - \gamma_*\int_0^t (\hat{b}(u,T)\hat{\sigma}_2(u) + \hat{\nu}_*(u))d\hat{W}(u))\right\},$$

and $N(T) = E_{\mathcal{P}}\left[(p_{\nu_*}(T))^{\gamma_*}\right]$ is deterministic. Then, by (5.37), $M(t) = E_{\mathcal{P}}[p_{\nu_*}(T)\chi^{\nu}_{\nu_*}|\mathcal{F}_t] = v\, E_{\mathcal{P}}[m(T)|\mathcal{F}_t] = vm(t)$. It follows that

$$\begin{cases} \check{\zeta}_* & = \gamma_*(\hat{b}(t,T)\check{\sigma}_1(t) + \check{\theta}_0(t)); \\ \hat{\nu}_* = \hat{\zeta}_* & = \gamma_*(\hat{b}(t,T)\hat{\sigma}_2(t) + \hat{\nu}_*(t)). \end{cases} \tag{5.38}$$

From the second equation, the optimal solution is given by

$$\hat{\nu}_*(t) = \frac{\gamma_*}{1-\gamma_*}\hat{b}(t,T)\hat{\sigma}_2(t) = \frac{a-\gamma}{1-a}\frac{1-e^{-b(T-t)}}{b}\hat{\sigma}_2(t), \quad t \in [0,T]. \tag{5.39}$$

Note that the above solution is deterministic, so our conjecture is proved to be true.

Given $a \neq \gamma$, when there is an unhedgeable risk of interest rate ($\hat{\sigma}_2 \neq 0$), from (5.39), the incomplete market price of risk ($\theta_0 + \nu_*$) is essentially different from the stock market price of risk (θ_0).

By (5.25), we may further obtain the optimal portfolio (cf. (5.38)),

$$\alpha^*(t) = V^{\nu,\alpha^*}\frac{1-a}{1-\gamma}(\check{\sigma}(t)\check{\sigma}(t)^{\top})^{-1}(\mu(t)-r(t)) - V^{\nu,\alpha^*}\frac{a-\gamma}{1-\gamma}\frac{1-e^{-b(T-t)}}{b}(\check{\sigma}(t)^{\top})^{-1}\check{\sigma}_1(t), \tag{5.40}$$

the first term of which corresponds to the portfolio in a stock market and the second term of which is the hedging for the partial interest rate risk ($\int_0^s e^{-b(s-u)}\check{\sigma}_1(u)^{\top} d\check{W}(u)$).

REMARK 5.4. When a closed-form solution does not exist, in general, one may always apply the stochastic approximation approach to the dual minimization problem in Condition (C). See Robbins and Monro (1951), Kiefer and Wolfowitz (1952), and Lewis and Liu (2012) for more research along this line.

6. CONCLUSIONS

This paper studies the local property of utility with distorted probabilities and its application to the portfolio selection problem. Because linearity and time consistency do not hold with distorted utility, the standard dynamic programming method does not apply.

We propose a sensitivity-based approach to the portfolio selection problem. With this approach, we first show that distorted utility satisfies monolinearity; that is, after changing the underlying probability measure to another measure depending on the payment, distorted utility becomes locally linear near the payment. This new property characterizes the nonlinear behavior of distorted utility. We explore the following applications:

1. With monolinearity, the derivative of distorted utility is an expectation of the sample path derivative under the new measure; this property enables us to apply a perturbation method. The first- and second-order derivatives of distorted utility are derived; these derivatives lead to conditions for convexity and concavity of the utility functional. We also provide simulation algorithms for the derivative of distorted utility. These algorithms can be used in the stochastic approximation approach to determine the optimal distorted utility (see Robbins and Monro 1951; Kiefer and Wolfowitz 1952; and Cao 2007).
2. We apply monolinearity to the portfolio selection problem. Efficient gradient-based optimization algorithms can be derived with the simulated derivative for an initial allocation problem. For the continuous time asset allocation problem

with complete markets, a closed-form solution is obtained. For the problem with incomplete markets, the dual method is extended from EU to RDEU. Examples are provided to illustrate the dual problem. Example 5.12 gives an explicit solution to an incomplete markets problem with a stochastic investment opportunity set. Monolinearity provides the basis for an alternative approach to the quantile approach of Jin and Zhou (2008), He and Zhou (2011b), and Xia and Zhou (2016). This approach permits the resolution of the portfolio selection problem for complete markets. It also leads to a solution for specific models with incomplete markets. Further research is needed to extend the analysis of incomplete markets and tackle related problems such as those with transaction costs.

3. Monolinearity is a fundamental property that characterizes the representation of distorted utility. In a forthcoming paper, Cao and Wan (2013) demonstrate that monolinearity can be written as an axiom that replaces Yaari's independence axiom (Yaari 1987) in the dual theory and identifies a unique representation of the form (2.1). An open question emerging from this analysis is how to handle cases where monolinearity fails. This challenging issue is left for future research.

APPENDIX A: MONOLINEARITY

Theorems 3.3 and 3.7 characterize the local property of $\widetilde{E}_{\mathcal{P}}[X_{\vec{\theta}}]$ in the space of $\vec{\theta} \to X_{\vec{\theta}}$. These theorems form the basis of the sensitivity-based approach discussed in this paper. The property revealed in the theorems represents the intrinsic nature of distorted utility and provides a basis for utility optimization. In this Appendix, we formulate this property in a general way.

A.1. Monolinearity for Finite Differences.

Let Γ be the space of all nonnegative random variables defined on $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\Lambda \subseteq \Gamma$ be a subset of Γ.

DEFINITION A.1. Let $\breve{E}_{\mathcal{P}}$ be an operator (functional) defined on Γ with the form $\breve{E}_{\mathcal{P}}[X] = E_{\mathcal{Q}_X}[X]$, $X \in \Gamma$, and let $\mathcal{Q}_X$ be a probability measure on $(\Omega, \mathcal{F})$ depending on X (cf. (2.6)). $\breve{E}_{\mathcal{P}}$ is monolinear for finite differences on set Λ, if for any two random variables $X, Y \in \Lambda$, we have

$$\breve{E}[Y] - \breve{E}[X] = E_{\mathcal{Q}_X}[Y - X].$$

Because $\breve{E}[X] = E_{\mathcal{Q}_X}[X]$, monolinear for finite differences implies $\breve{E}[Y] = E_{\mathcal{Q}_X}[Y]$. Exchanging the role of X and Y leads to $\breve{E}[X] = E_{\mathcal{Q}_Y}[X]$. Therefore, the fact that $\breve{E}$ is monolinear for finite differences on Λ almost requires $\mathcal{Q}_X = \mathcal{Q}_Y$ for all $X, Y \in \Lambda$. If $\breve{E}$ is monolinear for finite difference on Λ, utility optimization may be conducted on this common probability measure $\mathcal{Q}_X$ in the same way as under EU.

Two random variables $X = X(\omega)$ and $Y = Y(\omega)$, $\omega \in \Omega$, are *comonotonic*, if $[X(\omega) - X(\omega')][Y(\omega) - Y(\omega')] \geq 0$, for all $\omega, \omega' \in \Omega$. Let $\Lambda_c \subset \Gamma$ denote a *comonotonic set* in which all of the random variables are comonotonic to each other.

LEMMA A.2. *If X and Y are comonotonic and atomless, then we have $\mathcal{Q}_X = \mathcal{Q}_Y$ for $\widetilde{E}_{\mathcal{P}}$; that is, $\widetilde{E}_{\mathcal{P}}$ is monolinear for finite differences on any comonotonic set Λ_c.*

Proof. By the comonotonicity of X and Y, if $X(\omega) > X(\omega')$, then $Y(\omega) \geq Y(\omega')$. Thus, for any $\omega_0 \in \Omega$, we have $\{\omega : X(\omega) > X(\omega_0)\} \subseteq \{\omega : Y(\omega) > Y(\omega_0)\} \cup \{\omega : Y(\omega) = Y(\omega_0)\}$. By symmetry, $\{\omega : Y(\omega) > Y(\omega_0)\} \subseteq \{\omega : X(\omega) > X(\omega_0)\} \cup \{\omega : X(\omega) = X(\omega_0)\}$. Because X and Y have no atom, $\{\omega : Y(\omega) = Y(\omega_0)\}$ and $\{\omega : X(\omega) = X(\omega_0)\}$ are zero-measure set. Thus, we have

$$\begin{aligned} H_X[X(\omega_0)] &= \mathcal{P}\{\omega : X(\omega) > X(\omega_0)\} = \mathcal{P}\{\omega : Y(\omega) > Y(\omega_0)\} \\ &= H_Y[Y(\omega_0)], \omega_0 \in \Omega. \end{aligned}$$

Therefore, by (2.5), we have

$$\Big[\frac{d\mathcal{Q}_X}{d\mathcal{P}}\Big](\omega) = \frac{dw(z)}{dz}\Big|_{z=H_X[X(\omega)]} = \frac{dw(z)}{dz}\Big|_{z=H_Y[Y(\omega)]} = \Big[\frac{d\mathcal{Q}_Y}{d\mathcal{P}}\Big](\omega).$$

That is, $\mathcal{Q}_X = \mathcal{Q}_Y$. □

Obviously, for any $X \in \Gamma$, $\Lambda_c := \{\alpha X + d, \alpha > 0, d \geq 0\}$ is a comonotonic set. Thus, $\mathcal{Q}_{\alpha X+d} = \mathcal{Q}_X$ for all $\kappa > 0$ and $c \geq 0$. We have (cf. Remark 2.2)

$$\widetilde{E}_{\mathcal{P}}[\kappa X + c] = \kappa \widetilde{E}_{\mathcal{P}}[X] + c.$$

A.2. Local Monolinearity.

Monolinearity for finite differences requires restrictive conditions, which may be slightly relaxed to obtain local monolinearity.

DEFINITION A.3. Let $\breve{E}_{\mathcal{P}}$ be an operator (functional) defined on Γ with the form $\breve{E}_{\mathcal{P}}[X] = E_{\mathcal{Q}_X}[X]$, $X \in \Gamma$ and let $\mathcal{Q}_X$ be a probability measure on $(\Omega, \mathcal{F})$ depending on X. Let $\Lambda := \{X_{\vec{\theta}}, \vec{\theta} \geq 0\} \subseteq \Gamma$ be any set of parameterized random variables.

(a) $\breve{E}_{\mathcal{P}}$ is said to be locally monolinear on Γ, if

$$\breve{E}_{\mathcal{P}}[X_{\vec{\theta}+\Delta\vec{\theta}}] - \breve{E}_{\mathcal{P}}[X_{\vec{\theta}}] = E_{\mathcal{Q}_{X_{\vec{\theta}}}}[X_{\vec{\theta}+\Delta\vec{\theta}} - X_{\vec{\theta}}] + o(\Delta\vec{\theta}).$$

(b) $\breve{E}_{\mathcal{P}}$ is said to be locally monolinear on Γ up to the kth order, $k \geq 2$, if

$$\breve{E}_{\mathcal{P}}[X_{\vec{\theta}+\Delta\vec{\theta}}] - \breve{E}_{\mathcal{P}}[X_{\vec{\theta}}] = E_{\mathcal{Q}_{X_{\vec{\theta}}}}[X_{\vec{\theta}+\Delta\vec{\theta}} - X_{\vec{\theta}}] + o(|\Delta\vec{\theta}|^k).$$

Occasionally, we simply say that a locally monolinear operator is *monolinear*. The terminology *"mono"* has the following connotation: first, it only requires that we change the measure on one side of the equation to obtain the difference of two random variables ($E_{\mathcal{Q}_{X_{\vec{\theta}}}}$ in the above equations may be replaced by $E_{\mathcal{Q}_{X_{\vec{\theta}+\Delta\vec{\theta}}}}$); second, it holds for comonotonic random variables.

If $\breve{E}_{\mathcal{P}}$ is monolinear and $\vec{\theta} = \theta$ is a scalar and the Lipschitz condition holds under the measure $\mathcal{Q}_{X_\theta}$, then by Definition A.3, we have

$$\frac{d}{d\theta}\breve{E}_{\mathcal{P}}[X_\theta] = E_{\mathcal{Q}_{X_\theta}}\Big[\frac{d}{d\theta}\{X_\theta\}\Big].$$

If $\breve{E}_{\mathcal{P}}$ is monolinear on Γ up to the kth order, $k \geq 2$, it must be monolinear on Γ up to the jth order, $j \leq k$; and if the Lipschitz condition holds for $\frac{d^j}{d\theta^j}\{X_\theta\}$, $j < k$, under the measure $\mathcal{Q}_{X_\theta}$, we have

$$\breve{E}_{\mathcal{P}}[X_{\theta+\Delta\theta}] - \breve{E}_{\mathcal{P}}[X_\theta] = \sum_{j=1}^{k} E_{\mathcal{Q}_{X_\theta}}\Big[\frac{d^j}{d\theta^j}\{X_\theta\}\Big]\frac{(\Delta\theta)^j}{j!} + o(|\Delta\theta|^k),$$

or

$$\frac{d^j}{d\theta^j}\breve{E}_{\mathcal{P}}[X_\theta] = E_{\mathcal{Q}_{X_\theta}}\left[\frac{d^j}{d\theta^j}\{X_\theta\}\right], 1 \leq j \leq k.$$

Next, we consider the probability space consisting of all of the sample paths, i.e., set $\Omega = \{\vec{W}\}$, and let Γ be the set of all nonnegative random variables on Ω. A parameterized random variable takes the form $X_{\vec{\theta}} = h(\vec{\theta}, \vec{W})$. The following lemma follows directly from Theorems 3.3 and 3.7.

LEMMA A.4.

(a) The operator $\widetilde{E}_{\mathcal{P}}$ *of* (2.1) *is monolinear on* Γ.

(b) The operator $\widetilde{E}_{\mathcal{P}}$ *of* (2.1) *is monolinear on* Γ *up to the second order, if and only if there is no probability distortion, or the sample path derivatives are the same for different sample paths with the same sample* $X_{\vec{\theta}} = h(\vec{\theta}, \vec{W})$, *i.e.,*

$$E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta_k}\{h(\vec{\theta}, \vec{W})\}\Big| X_{\vec{\theta}}\Big] = \frac{\partial}{\partial\theta_k}\{h(\vec{\theta}, \vec{W})\}, k = 1, 2, \cdots, K.$$

A trivial example is that $\Omega = [0, 1]$ with $\mathcal{F}$ being the Borel algebra, $\mathcal{P}$ being the Lebesgue measure, and $X_\theta = G_\theta(\omega), \omega \in \Omega, G_\theta(\cdot)$ is the inverse of a distribution function, and $\Gamma = \{X_\theta\}$. Taking derivatives w.r.t. θ on both sides of (3.1), we may easily verify that

$$\frac{d^k}{d\theta^k}\widetilde{E}_{\mathcal{P}}[X_\theta] = E_{\mathcal{Q}_{X_\theta}}\Big[\frac{d^k}{d\theta^k}\{X_\theta\}\Big], k \geq 1.$$

Therefore, $\widetilde{E}_{\mathcal{P}}$ is monolinear on Γ up to any order, or is monolinear for finite differences.

The monolinearity of $\widetilde{E}_{\mathcal{P}}$ forms the basis for the gradient-based optimization approach (see Ho and Cao 1991 and Cao 1985; see also Robbins and Monro 1951 and Kiefer and Wolfowitz 1952) for basic principles of the stochastic approximation.

APPENDIX B: OPTIMAL POLICY α^* WITH COMPLETE MARKETS

In this appendix, we derive an equation for the optimal policy α^* in the complete markets case.

We first assume that $r(t), \mu(t)$, and $\sigma(t)$ are deterministic functions. In this case, $\alpha^*(t) = \alpha^*(t, V^{v,\alpha^*}(t))$, and $V^{v,\alpha^*}(t)$ is Markovian (cf. (5.1)), and

$$\xi_t^{\alpha^*} = E_{\mathcal{P}}[\xi_T^{\alpha^*}|\mathcal{F}_t] = E_{\mathcal{P}}[\xi_T| V^{v,\alpha^*}(t)].$$

Let $g(t, x) = E_{\mathcal{P}}[\xi_T| V^{v,\alpha^*}(t) = x]$, then $\xi_t^{\alpha^*} = g(t, V^{v,\alpha^*}(t))$ and g satisfies the Poisson equation

$$\frac{\partial g}{\partial t}(t, x) \quad +\{xr(t) + \alpha^{*\top}(t, x)\sigma(t)\theta_0(t)\}\frac{\partial g}{\partial x}(t, x)$$

$$+\frac{1}{2}\alpha^{*\top}(t,x)\sigma(t)\sigma(t)^{\top}\alpha^{*}(t,x)\frac{\partial^2 g}{\partial x^2}(t,x)=0, \tag{B.1}$$

with the boundary condition (cf. (5.6))

$$g(T,x)=E_{\mathcal{P}}[\xi_T|V^{v,\alpha^*}(T)=x]=\frac{S_0(T)}{\lambda_0(v)}\frac{dU(x)}{dx}\frac{dw(z)}{dz}\Big|_{z=H^{v,\alpha^*}(x)}. \tag{B.2}$$

With (B.1), using Ito's formula to $\xi_t^{\alpha^*}=g(t,V^{v,\alpha^*}(t))$ and comparing with (5.8), we have

$$\alpha^*(t,x)=-(\sigma(t)\sigma^{\top}(t))^{-1}(\mu(t)-r(t))g(t,x)/\frac{\partial}{\partial x}g(t,x). \tag{B.3}$$

Plugging it into (B.1), we get

$$\frac{\partial g}{\partial t}(t,x)+xr(t)\frac{\partial g}{\partial x}(t,x)-\theta_0^{\top}(t)\theta_0(t)g(t,x)+\frac{1}{2}\theta_0^{\top}(t)\theta_0(t)\left(\frac{g(t,x)}{\frac{\partial g}{\partial x}(t,xv)}\right)^2\frac{\partial^2 g}{\partial x^2}(t,x)=0. \tag{B.4}$$

Solving (B.4) with (B.2) to get g, we obtain the optimal policy α^* as given by (B.3).

If $r(t),\mu(t)$, and $\sigma(t)$ depend on $S(t)$, then $\alpha^*(t)=\alpha^*(t,V^{v,\alpha^*}(t),S(t))$, $\xi_t^{\alpha^*}=g(t,V^{v,\alpha^*}(t),S(t))$, and $g(t,x,S)=E_{\mathcal{P}}[\xi_T|V^{v,\alpha^*}(t)=x,S(t)=S]$. If g is continuously differentiable in t and second-order continuously differentiable in x and S,[7] it satisfies the Poisson equation

$$\begin{aligned}&\frac{\partial g}{\partial t}(t,x,S)+\{xr(t,S)+\alpha^{*\top}(t,x,S)\sigma(t,S)\theta_0(t,S)\}\frac{\partial g}{\partial x}(t,x,S)\\&+\sum_{i=1}^{K}\mu_k(t,S)S_k\frac{\partial g}{\partial S_k}(t,x,S)+\frac{1}{2}|\alpha^{*\top}(t,x,S)\sigma(t,S)|^2\frac{\partial^2 g}{\partial x^2}(t,x,S)\\&+\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{l=1}^{L}\sigma_{il}(t,S)\sigma_{jl}(t,S)S_iS_j\frac{\partial^2 g}{\partial S_i\partial S_j}(t,x,S)\\&+\sum_{k=1}^{K}\sum_{l=1}^{L}\sigma_{kl}^2(t,S)\alpha_k^*(t,x,S)S_k\frac{\partial^2 g}{\partial x\partial S_k}(t,x,S)=0,\end{aligned} \tag{B.5}$$

with the boundary condition (B.2). Comparing the volatility of ξ_t, for $l=1,\dots,L$, we have

[7] A sufficient condition to guarantee the differentiability of g is as follows (cf. He and Pearson 1991): (1) The coefficients of the logarithmic asset processes satisfy a linear growth condition and a uniform Lipschitz condition. (2) The derivatives (up to the second order) of the function $e^{-x}g(T,e^{-x})$ (cf. (B.2)) are continuous and satisfy a polynomial growth condition.

$$\frac{\partial g}{\partial x}(t,x,S)\sum_{k=1}^{K}\sigma_{kl}(t,S)\alpha_k^*(t,x,S) = -\sum_{k=1}^{K}\frac{\partial g}{\partial S_k}(t,x,S)S_k\sigma_{kl}(t,S) \tag{B.6}$$
$$- g(t,x,S)\theta_{0,l}(t,S).$$

The optimal policy α^* may be obtained by (B.5), (B.2), and (B.6).

APPENDIX C: PROOF OF THEOREM 5.4

From Subsection 5.1, $V^{v,\alpha(\epsilon)}(T)$ is of form $X + \epsilon^\top Y$. Denote the distorted utility (5.2) as $\eta(\epsilon)$ ($\eta(\epsilon) := \eta(v, \alpha(\epsilon))$). With Assumption 3.10, from the discussions in footnotes 1 and 2, $\eta(\epsilon)$ is continuously differentiable. In addition, based on Subsection 5.2.1, the first-order derivative of $\eta(\epsilon)$ has a unique zero point at $\epsilon^* = 0$ (χ_0^v is the corresponding terminal wealth). Therefore, to ensure the global maximum of χ_0^v, we must only check that the second-order derivative of $\eta(\epsilon)$ at $\epsilon^* = 0$ is strictly negative semidefinite. To use the second-order derivative formula in Section 3, we use the notations $X_{\vec{\theta}} = V^{v,\alpha(\epsilon)}(T)$ with $\vec{\theta} = \epsilon$. By (3.9) and (3.10), for any $\Delta\vec{\theta} \neq 0$, we have

$$(\Delta\vec{\theta})^\top D^2_{\vec{\theta}}\eta(\epsilon)\Delta\vec{\theta} = E_{\mathcal{Q}}[\Upsilon] = E_{\mathcal{Q}}\left[1_{\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}})\le 0}\Upsilon\right] + E_{\mathcal{Q}}\left[1_{\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}})>0}\Upsilon\right], \tag{C.1}$$

where

$$\Upsilon := \frac{d^2U(y)}{dy^2}\Big|_{y=X_{\vec{\theta}}}\left((D_{\vec{\theta}}X_{\vec{\theta}})^\top\Delta\vec{\theta}\right)^2$$
$$+ f_{\vec{\theta}}(X_{\vec{\theta}})\frac{dU(y)}{dy}\Big|_{y=X_{\vec{\theta}}}\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}}))(\Delta\vec{\theta})^\top Cov\Big[D_{\vec{\theta}}X_{\vec{\theta}}, D_{\vec{\theta}}X_{\vec{\theta}}\Big|X_{\vec{\theta}}\Big]\Delta\vec{\theta}.$$

The first term in (C.1) is negative. For the second term, we have

$$E_{\mathcal{Q}}\Big[1_{\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}})>0}\Upsilon\Big] = E_{\mathcal{P}}\Big[1_{\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}})>0}\Big\{\frac{\partial\psi_{\vec{\theta}}(x)}{\partial x}\Big|_{x=X_{\vec{\theta}}}\left((D_{\vec{\theta}}X_{\vec{\theta}})^\top\Delta\vec{\theta}\right)^2$$
$$-f_{\vec{\theta}}(X_{\vec{\theta}})\frac{dU(y)}{dy}\Big|_{y=X_{\vec{\theta}}}\frac{\partial w(z)}{\partial z}\Big|_{z=H_{\vec{\theta}}(X_{\vec{\theta}})}\mathcal{I}_w(H_{\vec{\theta}}(X_{\vec{\theta}}))\left(E_{\mathcal{P}}\Big[(D_{\vec{\theta}}X_{\vec{\theta}})^\top\Delta\vec{\theta}\Big|X_{\vec{\theta}}\Big]\right)^2\Big\}\Big], \tag{C.2}$$

where $\psi_{\vec{\theta}}(x) := \frac{\partial U(x)}{\partial x}\frac{\partial w(z)}{\partial z}\big|_{z=H_{\vec{\theta}}(x)}$. The second term of (C.2) is negative. For the first term, setting $\vec{\theta} = \vec{\theta}^*(\epsilon^* = 0)$, we have

$$\psi_{\vec{\theta}^*}(x) = \frac{\partial U(x)}{\partial x}\frac{\partial w(z)}{\partial z}\Big|_{z=H_{\vec{\theta}^*}(x)} = \frac{\partial U(x)}{\partial x}\Big|_{x=H^{-1}_{\vec{\theta}^*}(z)}\frac{\partial w(z)}{\partial z} = \lambda_0(v)F^{-1}_{p_0(T)}(H_{\vec{\theta}^*}(x)),$$

where $z = H_{\vec{\theta}^*}(x)$ and the third equation follows from (5.15). We then know that $\psi_{\vec{\theta}^*}(x)$ is a nonincreasing function and that the first term of (C.2) is negative when $\vec{\theta}$ takes the optimal. In the end, the Hessian matrix at $\epsilon^* = 0$ is strictly negative semidefinite.

REFERENCES

BERNARD, C., and M. GHOSSOUB (2010): Static Portfolio Choice under Cumulative Prospect Theory, *Math. Financ. Econ.* 2, 277–306.

BROADIE, M., and P. GLASSERMAN (1996): Estimating Security Price Derivatives Using Simulation, *Manage. Sci.* 42(2), 269–285.

CARLIER, G., and R. A. DANA (2006): Law Invariant Concave Utility Functions and Optimization Problems with Monotonicity and Comonotonicity Constraints, *Stat. Decis.* 24, 127–152.

CARLIER, G., and R. A. DANA (2011): Optimal Demand for Contingent Claims When Agents Have Law Invariant Utilities, *Math. Finance* 21(2), 169–201.

CAO, X. R. (1985): Convergence of Parameter Sensitivity Estimates in a Stochastic Experiment, *IEEE Trans. Autom. Control* 30, 845–853.

CAO, X. R. (2007): *Stochastic Learning and Optimization*, New York: Springer.

CAO, X. R. (2009): Stochastic Learning and Optimization—A Sensitivity-Based Approach, *(IFAC) Ann. Rev. Control* 33, 11–24.

CAO, X. R., and X. W. WAN (2013): Mono-Linearity-Based Axiomatic Approach to Non-Linear Expected Utility, Available at SSRN: http://ssrn.com/abstract=2353740.

CVITANIC, J., and I. KARATZAS (1992): Covex Duality in Constrained Portfolio Optimization, *Ann. Appl. Prob.* 2(4), 767–818.

DETEMPLE, J. B., and M. RINDISBACHER (2005): Closed-Form Solutions for Optimal Portfolio Selection with Stochastic Interest Rate and Investment Constraints, *Math. Finance* 15(4), 539–568.

DETEMPLE, J. B., R. GARCIA, and M. RINDISBACHER (2003): A Monte Carlo Method for Optimal Portfolios, *J. Finance* 58(1), 401–446.

ELLSBERG, D. (1961): Risk, Ambiguity and the Savage Axioms, *Quart. J. Econom.* 75, 643–669.

FRIEDMAN, M., and L. J. SAVAGE (1948): The Utility Analysis of Choices Involving Risk, *J. Polit. Econ.* 56, 279–304.

GLASSERMAN, P. (1991): *Gradient Estimation via Perturbation Analysis*, Boston: Kluwer Academic Publishers.

HE, H., and N. PEARSON (1991): Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case, *J. Econ. Theory* 54, 259–304.

HE, X. D., and X. Y. ZHOU (2011a): Portfolio Choice under Cumulative Prospect Theory: An Analytical Treatment, *Manage. Sci.* 57(2): 315–331.

HE, X. D., and X. Y. ZHOU (2011b): Portfolio Choice via Quantiles, *Math. Finance* 21(2), 203–231.

HEIDELBERGER, P., X. R. CAO, M. A. ZAZANIS, and R. SURI (1988): Convergence Properties of Infinitesimal Perturbation Analysis Estimates, *Manage. Sci.* 34, 1281–1302.

HO, Y. C., and X. R. CAO (1983): Perturbation Analysis and Optimization of Queueing Networks, *J. Optim. Theory Appl.* 40(4), 559–582.

HO, Y. C., and X. R. CAO (1991): *Perturbation Analysis of Discrete-Event Dynamic Systems*, Boston: Kluwer Academic Publisher.

HONG, L. (2009): Estimating Quantile Sensitivities, *Oper. Res.* 57(1), 118–130.

JIN, H., and X. Y. ZHOU (2008): Behavioral Portfolio Selection in Continuous Time, *Math. Finance* 18, 385–426.

JIN, H., and X. Y. ZHOU (2010): Erratum to Behavioral Portfolio Selection in Continuous Time, *Math. Finance* 20(3), 521–525.

KALLENBERG, O. (2002): *Foundations of Modern Probability*, 2nd ed., New York: Springer.

KARATZAS, I., and S. SHREVE (1991): *Brownian Motion and Stochastic Calculus*, 2nd ed., New York: Springer.

KARATZAS, I., and S. SHREVE (1998): *Methods of Mathematical Finance*, New York: Springer.

KARATZAS, I., J. LEHOCZKY, S. SHREVE, and G. XU (1991): Martingale and Duality Methods for Utility Maximization in an Incomplete Market, *SIAM J. Control Optim.* 29(3), 702–730.

KIEFER, J., and J. WOLFOWITZ (1952): Stochastic Estimation of the Maximum of A Regression Function, *Ann. Math. Stat.* 23, 462–466.

LEWIS, F. L., and D. LIU (2012): *Reinforcement Learning and Approximate Dynamic Programming for Feedback Control*, Hoboken, NJ: Wiley-IEEE Press.

MEHRA, R., and E. C. PRESCOTT (1985): The Equity Premium: A Puzzle, *J. Monetary Econ.* 15, 145–161.

MERTON, R. C. (1969): Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case, *Rev. Econ. Stat.* 51(3), 247–257.

MERTON, R. C. (1971): Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *J. Econ. Theory* 3, 373–413.

OCONE, D., and I. KARATZAS (1991): A Generalized Clark Representation Formula, with Application to Optimal Portfolios, *Stochastics Stochastic Rep.* 34, 187–220.

POLKOVNICHENKO, V., and F. ZHAO (2013): Probability Weighting Functions Implied in Options Prices, *J. Financ. Econ.* 107, 580–609.

PRELEC, D. (1998): The Probability Weighting Function, *Econometrica* 66, 497–527.

QUIGGIN, J. (1982): A Theory of Anticipated Utility, *J. Econ. Behav. Org.* 3, 323–343.

QUIGGIN, J. (1987): Decision Weights in Anticipated Utility Theory, *J. Econ. Behav. Org.* 8, 641–645.

QUIGGIN, J. (1993): *Generalized Expected Utility Theory: The Rank-Dependent Model*, Boston: Kluwer Academic Publishers.

ROBBINS, H., and S. MONRO (1951): A Stochastic Approximation Method, *Ann. Math. Stat.* 22, 400–407.

SAMUELSON, P. A. (1969): Lifetime Portfolio Selection by Dynamic Stochastic Programming, *Rev. Econ. Stat.* 51(3), 239–246.

SCHOEMAKER, P. H. (1982): The Expected Utility Model: Its Variants, Purposes, Evidence and Limitations, *J. Econ. Lit.* 20(2), 529–563.

SERFLING, R. J. (1980): *Approximation Theorems of Mathematical Statistics*, New York: Wiley.

STARMER, C. (2000): Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk, *J. Econ. Lit.* 38(2), 332–382.

TVERSKY, A., and D. KAHNEMAN (1992): Advances in Prospect Theory: Cumulative Representation of Uncertainty, *J. Risk Uncertainty* 5, 297–323.

VON NEUMANN, J., and O. MORGENSTERN (1944): *Theory of Games and Economic Behavior*, Princeton, NJ: Princeton University Press.

WU, G., and R. GONZALEZ (1996): Curvature of the Probability Weighting Function, *Manage. Sci.* 42, 1676–1690.

XIA, J. M., and X. Y. ZHOU (2016): Arrow-Debreu Equilibria for Rank-Dependent Utilities, *Math. Finance* 26(3), 558–588.

XU, G. L., and S. E. SHREVE (1992a): A Duality Method for Optimal Consumption and Investment under Short-Selling Prohibition. I. I. General Market Coefficients, *Ann. Appl. Prob.* 2(1), 87–112.

XU, G. L., and S. E. SHREVE (1992b): A Duality Method for Optimal Consumption and Investment under Short-Selling Prohibition. II. Constant Market Coefficients, *Ann. Appl. Prob.* 2(2), 314–328.

YAARI, M. E. (1987): The Dual Theory of Choice under Risk, *Econometrica* 55(1), 95–115.