A Note on the Pricing of Commodity-Linked Bonds

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THE OWNER OF A commodity-linked bond can exchange the bond's face value for the maturity value of a commodity. In an interesting application of option-pricing theory, Schwartz [7] provides a general framework for valuing commodity-linked bonds. This very general valuation framework allows for commodity price risk, default risk, and interest rate risk, along with interest payments and dividends. A partial differential equation and its associated boundary condition are derived but not solved. Schwartz states that the solution to the general problem is difficult even by numerical methods.

In order to find closed-form solutions, Schwartz considers commodity price risk in conjunction with default risk or interest rate risk but not both. This note presents a closed-form solution for the value of a commodity-linked bond when all three types of risk are present.

I. Assumptions

Our valuation model makes the following assumptions:

(A1) "No Dividends and Coupons": There are no payouts from the firm to shareholders or bondholders before the maturity date of the bond.

Schwartz also makes this assumption in order to derive the solution when there is default risk. In his solution to the complementary problem of interest rate risk but no default risk, Schwartz imposed the following assumption:

(A2) "Bond Price Dynamics": Let $Q(\tau)$ be the price of a default-free unit discount bond with the same time to maturity, τ, as the bond to be valued. Assume that $Q(\tau)$ satisfies

$$\frac{dQ}{Q} = \alpha_q(\tau)dt + \sigma_q(\tau)dz_q(t; \tau), \tag{1}$$

where α_q is the instantaneous expected return on the bond, σ_q^2 is the instantaneous variance, written as a deterministic function of time, and $dz_q(t; \tau)$ is a Gauss-Wiener process for maturity τ.

It should be pointed out that Schwartz allows for coupons when considering his model with interest rate risk but not default risk. In addition, coupons and dividends are allowed in his unsolved general model. Furthermore, a different stochastic process for interest rates is assumed in this general model. We repeat the other assumptions made by Schwartz in his general model:

(A3) "Dynamics of Commodity Price and Firm Value": Let P be the value of the commodity reference bundle and let V be the value of the firm issuing the bonds. The continuous paths that these values follow are described by the stochastic differential equations:

$$\frac{dP}{P} = \alpha_p dt + \sigma_p dz_p, \tag{2}$$

$$\frac{dV}{V} = \alpha_v dt + \sigma_v dz_v, \tag{3}$$

where α_p and α_v are the instantaneous expected returns, and the instantaneous return variances σ_p^2 and σ_v^2 are assumed to be deterministic functions of time. dz_q, dz_p, and dz_v are all Gauss-Wiener processes with

$$dz_q \times dz_p = \rho_{qp} dt; \quad dz_q \times dz_v = \rho_{qv} dt; \quad dz_p \times dz_v = \rho_{pv} dt.$$

(A4) "Perfect Capital Markets": There are no taxes, transactions costs, or other frictions. In particular, there are no costs of carrying or convenience yield for the commodity. Assets are perfectly divisible, and continuous trading is allowed.

II. The General Valuation Formula

Under these assumptions, standard hedging arguments can be used to show that the partial differential equation governing the value of a commodity-linked bond is

$$B_t + \tfrac{1}{2}\sigma_q^2 Q^2 B_{qq} + \tfrac{1}{2}\sigma_p^2 P^2 B_{pp} + \tfrac{1}{2}\sigma_v^2 V^2 B_{vv}$$
$$+ \sigma_{qp} QP B_{qp} + \sigma_{qv} QV B_{qv} + \sigma_{pv} PV B_{pv} = 0, \tag{4}$$

where

$$\sigma_{qp} \equiv \rho_{qp}\sigma_q\sigma_p; \quad \sigma_{qv} \equiv \rho_{qv}\sigma_q\sigma_v; \quad \sigma_{pv} \equiv \rho_{pv}\sigma_p\sigma_v.$$

This equation differs from that presented by Schwartz in his general model because of our different assumption governing interest rate dynamics. The boundary condition is:

$$B(P, V, Q, 0) = \min(\tilde{V}, F + \max(0, \tilde{P} - F)), \tag{5}$$

where F is the face value of the debt and the exercise price of the option implicit in the commodity-linked bond.

The solution to the partial differential equation (4) subject to its boundary

condition (5) is

$$
\begin{aligned}
B = \; & V\!\left[1 - N_2\!\left(h_1\!\left(\frac{V}{FQ},\, \sigma_{v/q}\right),\, h_1\!\left(\frac{V}{P},\, \sigma_{v/p}\right);\, \rho_{v/q,v/p}\right)\right] \\
& + PN_2\!\left(h_3\!\left(\frac{V}{FQ},\, \sigma_{vqp},\, \sigma_{v/q}\right),\, h_2\!\left(\frac{V}{P},\, \sigma_{v/p}\right);\, \rho_{v/q,v/p}\right) \\
& + \left[FQN_2\!\left(h_2\!\left(\frac{V}{FQ},\, \sigma_{v/q}\right),\, -h_2\!\left(\frac{P}{FQ},\, \sigma_{p/q}\right);\, \rho_{v/q,q/p}\right)\right. \\
& \left. - PN_2\!\left(h_3\!\left(\frac{V}{FQ},\, \sigma_{vqp},\, \sigma_{v/q}\right),\, -h_1\!\left(\frac{P}{FQ},\, \sigma_{p/q}\right);\, \rho_{v/q,q/p}\right)\right],
\end{aligned} \tag{6}
$$

where $N_2(a, b; \rho)$ is the standard bivariate normal distribution function evaluated at a and b with correlation coefficient ρ,

$$
h_1(y,\, \sigma) \equiv \frac{\ln y + \frac{1}{2}I(\sigma)}{\sqrt{I(\sigma)}},
$$

$$
h_2(y,\, \sigma) \equiv \frac{\ln y - \frac{1}{2}I(\sigma)}{\sqrt{I(\sigma)}},
$$

$$
h_3(y,\, \sigma_1,\, \sigma_2) \equiv \frac{\ln y - \frac{1}{2}I(\sigma_1)}{\sqrt{I(\sigma_2)}}
$$

$$
I(\sigma) = \int_0^\tau \sigma^2\, ds,
$$

$$
\rho_{v/q,v/p} = \frac{\sigma_v^2 - \sigma_{vp} - \sigma_{qv} + \sigma_{qp}}{\sigma_{v/q}\,\sigma_{v/p}},
$$

$$
\rho_{v/q,q/p} = \frac{\sigma_{vq} - \sigma_{vp} - \sigma_q^2 + \sigma_{qp}}{\sigma_{v/q}\,\sigma_{q/p}},
$$

$$
\sigma_{v/q}^2 = \sigma_v^2 + \sigma_q^2 - 2\sigma_{vq},
$$

$$
\sigma_{v/p}^2 = \sigma_v^2 + \sigma_p^2 - 2\sigma_{vp},
$$

$$
\sigma_{q/p}^2 = \sigma_q^2 + \sigma_p^2 - 2\sigma_{qp},
$$

$$
\sigma_{vqp}^2 = \sigma_v^2 - \sigma_q^2 - 2\sigma_{pv} + 2\sigma_{pq}.
$$

The intuition behind the formula may be seen by dividing the expression into three parts. The first term represents the fact that the bondholders get the value of the firm unless the firm is not bankrupt and the commodity price is less than the firm value. The next term reflects the bond's value if the firm is solvent and if the bondholders always get P. The next two terms correct for the fact that the bondholders get F when P is below F.

III. Special Cases

As a check on the solution, we record the formula for the special case of no default risk. The solution Schwartz gives assumes no default on both the face value and the option. Since the commodity price is not bounded above, this must be achieved by setting $V = \infty$ in the arguments of the bivariate normals. The first term disappears and the remaining terms become:

$$
\begin{aligned}
B = FQ &+ P\left[1 - N_1\left(-h_1\left(\frac{P}{FQ}, \sigma_{p/q}\right)\right)\right] \\
&- FQ\left[1 - N_1\left(-h_2\left(\frac{P}{FQ}, \sigma_{p/q}\right)\right)\right] \\
&= FQ + PN_1\left(h_1\left(\frac{P}{FQ}, \sigma_{p/q}\right)\right) - FQN_1\left(h_2\left(\frac{P}{FQ}, \sigma_{p/q}\right)\right).
\end{aligned} \quad (7)
$$

This is exactly the formula given by Schwartz in his Section V except that the identity

$$
N_1(\sqrt{2}h) = \tfrac{1}{2}erfc(-h) \quad (8)
$$

has been used here to express the result using the more familiar normal distribution function.[1] The intuition behind the result is that when there is no default risk, a commodity-linked bond is just a straight bond plus a call option on the commodity. Since interest rates are assumed stochastic, a formula given by Merton [5] has been used to value the call option.

A second check is to reinstate default risk, but now assume a constant interest rate. When the variance of the return on the default-free bond vanishes and its expected return is set equal to the constant r, then

$$
Q = e^{-r\tau}; \quad \sigma_{v/q} = \sigma_v; \quad \sigma_{p/q} = \sigma_p. \quad (9)
$$

With these substitutions, our general formula (6) becomes

$$
\begin{aligned}
B = V&\left[1 - N_2\left(h_1\left(\frac{V}{Fe^{-r\tau}}, \sigma_v\right), h_1\left(\frac{V}{P}, \sigma_{v/p}\right); \rho_{v,v/p}\right)\right] \\
&+ PN_2\left(h_3\left(\frac{V}{Fe^{-r\tau}}, \hat{\sigma}_{vp}, \sigma_v\right), h_2\left(\frac{V}{P}, \sigma_{v/p}\right); \rho_{v,v/p}\right) \\
&+ \left[Fe^{-r\tau}N_2\left(h_2\left(\frac{V}{Fe^{-r\tau}}, \sigma_v\right), -h_2\left(\frac{P}{Fe^{-r\tau}}, \sigma_p\right); \rho_{vp}\right)\right. \\
&\left.- PN_2\left(h_3\left(\frac{V}{Fe^{-r\tau}}, \hat{\sigma}_{vp}, \sigma_v\right), -h_1\left(\frac{P}{Fe^{-r\tau}}, \sigma_p\right); \rho_{vp}\right)\right],
\end{aligned} \quad (10)
$$

where $\rho_{v,v/p} = (\sigma_v^2 - \sigma_{vp})/\sigma_v\sigma_{v/p}$ and $\hat{\sigma}_{vp}^2 \equiv \sigma_v^2 - 2\sigma_{vp}$.

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[1] Also, a typographical error in Schwartz has been corrected. Equation (33) on p. 534 should read:

$$
W(P, Q, \tau) = [Perfc(-h_1) - EQerfc(-h_2)]/2.
$$

Schwartz's numerical analysis does not reflect this typographical error.

As a final check on our formula, we consider the special case when there is neither default risk nor interest rate risk. When we apply the restrictions given in (9) to (7), the formula reduces to a straight bond plus a call option priced by the Black-Scholes [1] formula:

$$B = Fe^{-r\tau} + PN_1\left(h_1\left(\frac{P}{Fe^{-r\tau}}, \sigma_p\right)\right) - Fe^{-r\tau}N_1\left(h_2\left(\frac{P}{Fe^{-r\tau}}, \sigma_p\right)\right). \qquad (11)$$

This formula is identical to that given by Schwartz in his Section III except that Schwartz also allows coupons.

IV. Extensions and Summary

The model in this note also allows coupons whenever default risk is ignored. Furthermore, our model allows for a continuous dividend on the stock and a continuous convenience yield on the commodity. If we assume that the dividend D is proportional to firm value, i.e., $D = dV$, and that the convenience yield C is proportional to the commodity price, i.e., $C = cP$, then our valuation formulas are valid when V and P are replaced by $Ve^{-d\tau}$ and $Pe^{-c\tau}$, respectively.

In a discussion of Schwartz's article, Ingersoll [3] points out that the perfect-markets and continuous-trading assumptions are subject to more than the usual skepticism when applied to commodities rather than financial assets. He also notes that the convenience yield of a commodity may well depend on stochastic variables other than P. As Ingersoll recognizes, both of these objections may be handled if hedging of the bond is done with commodity forwards[2] rather than the commodity itself. As long as forward contracts of the required maturity exist, the convenience yield will disappear from the formula and thus may be left unspecified. If G is the forward price of a contract expiring with the bond, then our valuation formulas are valid for a general convenience yield when P is replaced by GQ.

In summary, this paper has developed a general valuation formula for commodity-linked bonds. By restricting the values of certain parameters in the formula, results given previously by Schwartz arise as special cases. The formula can be further generalized to accommodate a constant continuous dividend yield and a convenience yield. An interesting unsolved extension would allow for discrete dividends or coupons in the presence of default risk.

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[2] Since interest rates are assumed stochastic, the forward price differs theoretically from the futures price. However, empirical differences are negligible so that futures contracts may be used to hedge the bond.

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