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BEHAVIORAL PORTFOLIO SELECTION: ASYMPTOTICS AND STABILITY ALONG A SEQUENCE OF MODELS

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We consider a sequence of financial markets that converges weakly in a suitable sense and maximize a behavioral preference functional in each market. For expected *concave* utilities, it is well known that the maximal expected utilities and the corresponding final positions converge to the corresponding quantities in the limit model. We prove similar results for nonconcave utilities and distorted expectations as employed in behavioral finance, and we illustrate by a counterexample that these results require a stronger notion of convergence of the underlying models compared to the *concave* utility maximization. We use the results to analyze the stability of behavioral portfolio selection problems and to provide numerically tractable methods to solve such problems in complete continuous-time models.

KEY WORDS: portfolio selection, nonconcave utility, Choquet integral, stability, convergence, behavioral finance.

1. INTRODUCTION

Portfolio optimization constitutes a fundamental problem in economics. For classical preference functionals defined by expected concave utility, this problem and its solution are well known; see Biagini (2010) for an overview. In practical applications, however, these classical functionals are often too restrictive. Nonstandard incentives for risk-averse agents such as option compensation or performance-based salary systems lead to nonconcave demand problems. In addition, the behavioral finance literature suggests using nonlinear expectations to account for the observation that people tend to overweight extreme events with small probabilities; see for instance Tversky and Kahneman (1992) and references therein.

The demand problem for nonconcave preference functionals (with or without nonlinear expectations) is less standard; a rigorous mathematical analysis has started only recently. Jin and Zhou (2008), Carlier and Dana (2011), He and Zhou (2011a), Jin Zhang, and Zhou (2011), and Rásonyi and Rodrigues (2013) analyze demand problems for nonconcave utility functions and nonlinear expectations in continuous-time models. De Giorgi and Hens (2006), Bernard and Ghossoub (2010), He and Zhou (2011b), Carassus and Rásonyi (2015), and Pirvu and Schulze (2012) study similar problems in discrete time. Larsen (2005), Carassus and Pham (2009), Rieger (2012), Bichuch and

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Sturm (2011), and Muraviev and Rogers (2013) consider related problems with linear expectations. All these general results lead to several applications: Sung et al. (2011) and Bernard et al. (2015) analyze the consequences of behavioral demand on the optimal insurance design; Jin and Zhou (2013) quantify the notion of greed in the context of behavioral demand problems; and Xu and Zhou (2013) study optimal stopping for behavioral preference functionals. A detailed overview on recent developments in mathematical behavioral finance can be found in Zhou (2010).

All the work mentioned above studies the demand problem for a fixed underlying model. Since one is never exactly sure of the accuracy of a proposed model, it is important to know whether the behavioral predictions generated by a model change drastically if one slightly perturbs the model. To the best of our knowledge, results on the stability of behavioral portfolio selection problems have not been available in the literature so far, and the main purpose of this paper is to study this issue in detail. Formally, we consider a sequence of models, each represented by some probability space $(\Omega^n, \mathcal{F}^n, P^n)$ and some pricing measure Q^n, and we assume that this sequence converges weakly in a suitable sense (to be made precise later) to a limit model $(\Omega^0, \mathcal{F}^0, P^0, Q^0)$. For each model, we are interested in the demand problem

$$v^n(x) := \sup\{V_n(f) \mid f \in L^0_+(\Omega^n, \mathcal{F}^n, P^n),\ E_{Q^n}[f] \le x\}, \tag{1.1}$$

where the functional $V_n : (\Omega^n, \mathcal{F}^n, P^n) \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$V_n(f) := \int U(f)\, d(w \circ P^n), \tag{1.2}$$

for a *nonconcave* and *nonsmooth* utility function U on $\mathbb{R}_+$ and a strictly increasing function $w : [0, 1] \to [0, 1]$ representing the *probability distortion* of the beliefs. We are then interested in the asymptotics of the value (indirect utility) $v^n(x)$ and its maximizer $f^n = \arg\max V_n(f)$, and we want to compare them with the analogous quantities in the limit model.

Functionals of the form (1.2) as well as the demand problem (1.1) have well-established economic interpretations. From a theoretical point of view, (1.2) arises naturally as a representation for preference functionals satisfying a certain comonotonicity condition; see for instance Schmeidler (1986). In applications, they also serve as the main building block for several behavioral theories such as *rank-dependent expected utility* (RDEU); see Quiggin (1993) or *cumulative prospect theory* (CPT); see Tversky and Kahneman (1992). If we set $w(p) = p$, then (1.2) covers the classical expected utility functional.

The problem (1.1) can be seen as portfolio optimization problem in a complete market. More precisely, consider an agent in a complete financial market who is dynamically trading in the underlying (discounted) assets S with filtration $(\mathcal{F}^n_t)_{0\le t\le T}$ and time horizon T. The agent invests the initial capital x in self-financing strategies with nonnegative associated wealth process in order to maximize his/her preference functional V_n that only depends on the terminal wealth. Because the market is complete, any fixed $\mathcal{F}^n_T$-measurable nonnegative random variable f is the terminal wealth associated to a self-financing strategy if and only if $E_{Q^n}[f] = x$, where Q^n denotes the unique martingale measure for S. The agent is thus brought back to solving a static problem of type (1.1).

The main ingredients of the model are described by $(\Omega^n, \mathcal{F}^n, P^n, Q^n)$. The assumption that the sequence of models converges (in a suitable sense) to a limit model means that the economic situation described by the nth model is for sufficiently large n close (in a suitable sense) to the one described by the limit model. Our main contribution is to give easily verifiable assumptions such that similar economic situations also imply similar behavioral predictions for the agent, in the sense that the values $v^n(x)$ as well as (along a subsequence) the optimal final positions f^n converge to the corresponding quantities in the limit model.

In *concave* utility maximization, the (essentially) sufficient condition for these stability results is the weak convergence of the *pricing density* (or *pricing kernel*) dQ^n/dP^n to dQ^0/dP^0; see, for instance, He (1991) and Prigent (2003). However, in our nonconcave setting, we present an example of a sequence of financial markets for which dQ^n/dP^n converges weakly to dQ^0/dP^0, but where the limit $\lim_{n\to\infty} v^n(x)$ and $v^0(x)$ as well as the corresponding final positions differ substantially. We discuss these new effects in detail and give sufficient conditions to prevent such unpleasant phenomena.

In order to illustrate the main results, we provide several applications. First, we consider a sequence of binomial models approximating the Black–Scholes model; this is the typical example for the transition from discrete- to continuous-time models. Apart from its purely theoretical interest, this example is also of practical relevance since the discrete-time analysis provides numerical procedures for the explicit computation of the optimal consumption. This allows one to numerically determine the value function for (computationally difficult) continuous-time models via the value functions for (computationally tractable) discrete-time models. The second application is motivated by the practical difficulties to calibrate an underlying model. As one example, we therefore study whether a (small) misspecification of the drift in the Black–Scholes model significantly influences the optimal behavior of the agent. In both examples above, we use a fixed time horizon T for the portfolio optimization problem. In practical applications, however, the time horizon might be uncertain or changing. In the third application, we therefore analyze whether or not a (marginal) misspecification of the investment horizon significantly influences the optimal behavior of the agent.

These examples show the necessity of our analysis: Our models are at best approximations to the reality, so if we perturb one model slightly in a reasonable way and the behavioral predictions generated by the model change drastically, we may suspect that the model cannot tell us much about the real world behavior. Our convergence results demonstrate that for a fairly broad class of preference functionals and models, the optimal behavior is stable with respect to such small perturbations.

The paper is structured as follows. In Section 2, we abstractly describe the sequence of models, preference functionals, and optimization problems. We also formulate and discuss the main result. In Section 3, we present three applications of the main result. This also allows us to discuss the connections to the existing literature in more detail. In addition, we provide a concrete numerical example to illustrate the results. We prove the main result in Sections 4 and 5.

2. PROBLEM FORMULATION AND MAIN RESULTS

The following notation is used. If $x, y \in \mathbb{R}$, denote $x^{\pm} = \max\{\pm x, 0\}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. For a function G and a random variable X, we write $G(X)^{\pm}$ for the positive/negative parts $(G(X))^{\pm}$. For a sequence (f^n) of random variables, we denote weak convergence of (f^n) to f^0 by $f^n \Rightarrow f^0$. A *quantile function* q_F of a distribution function F is a generalized inverse of F, i.e., a function $q_F : (0, 1) \to \mathbb{R}$ satisfying

$$F(q_F(s)-) \leq s \leq F(q_F(s)) \quad \text{for all } s \in (0, 1).$$

Quantile functions are not unique, but any two for a given F coincide a.e. on (0, 1). Thus, we sometimes blur the distinction between "the" and "a" quantile function. A quantile function q_f of a random variable f is understood to be a quantile function q_F of the distribution F of the random variable f. If the sequence (f^n) converges weakly to f^0, then any corresponding sequence (q_{f^n}) of quantile functions converges a.e. on (0,1) to q_{f^0}; see, for instance, Theorem 25.6 of Billingsley (1986). More properties of quantile functions can be found in Appendix A.3 of Föllmer and Schied (2011).

2.1. Sequence of Models and Optimization Problems

We consider a sequence of probability spaces $(\Omega^n, \mathcal{F}^n, P^n)_{n\in\mathbb{N}_0}$, where the probability space $(\Omega^0, \mathcal{F}^0, P^0)$ is atomless; see Definition A.26 and Proposition A.27 in Föllmer and Schied (2011) for a precise definition and equivalent formulations. On each probability space, there is a probability measure Q^n equivalent to P^n with density $\varphi^n = dQ^n/dP^n \in L^1_+(\Omega^n, \mathcal{F}^n, P^n)$. We refer to Q^n as *pricing measure* and to φ^n as *pricing density* (or *pricing kernel*). We assume that the sequence (φ^n) converges weakly to the pricing density in the atomless model, i.e., $\varphi^n \Rightarrow \varphi^0$. To ensure that the atomic structure tend to the atomless structure, we assume that the atoms disappear in the following sense. Let $\mathcal{G}^n$ be the set of atoms in $\mathcal{F}^n$ (with respect to Q^n).

ASSUMPTION 2.1. $\lim_{n\to\infty} \sup_{A\in\mathcal{G}^n} Q^n[A] = 0.$

We impose the following integrability condition on $(\varphi^n)_{n\in\mathbb{N}}$.

ASSUMPTION 2.2. The family $((\varphi^n)^\xi)_{n\in\mathbb{N}}$ is uniformly integrable for all $\xi < 0$.

Having specified the sequence of models, we turn to the preference functionals. One cornerstone is the concept of a nonconcave utility function.

DEFINITION 2.3. A *nonconcave utility* is a function $U : (0, \infty) \to \mathbb{R}$, which is strictly increasing, continuous and satisfies the growth condition

$$\lim_{x\to\infty} \frac{U(x)}{x} = 0. \tag{2.1}$$

We consider only nonconcave utility functions defined on $\mathbb{R}_+$. To avoid any ambiguity, we set $U(x) = -\infty$ for $x < 0$ and define $U(0) := \lim_{x\searrow 0} U(x)$ and $U(\infty) := \lim_{x\to\infty} U(x)$. Without loss of generality, we may assume that $U(\infty) > 0$. Observe that we do not assume that U is concave. In the concave case, the growth condition (2.1) not only implies, but is even equivalent to, the Inada condition at ∞ that $U'(\infty) = 0$.

DEFINITION 2.4. The *concave envelope* U_c of U is the smallest concave function $U_c : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $U_c(x) \geq U(x)$ holds for all $x \in \mathbb{R}$.

Some properties of U_c as well as of $\{U < U_c\} := \{x \in \mathbb{R}_+ \mid U(x) < U_c(x)\}$ can be found in Lemma 2.11 of Reichlin (2013). A key tool to study the relation between U and U_c is the conjugate of U defined by

$$J(y) := \sup_{x>0}\{U(x) - xy\}.$$

Because of the nonconcavity of U, the concave envelope U_c is not necessarily strictly concave and the latter implies that J is no longer smooth; we therefore work with the subdifferential that is denoted by ∂J for the convex function J and by ∂U_c for the concave function U_c. The right- and left-hand derivatives of J are denoted by J'_+ and J'_-. Our proofs (mainly in the Appendix) use the classical duality relations between U_c, J, ∂J, and ∂U_c. Precise statements and proofs can be found in Lemma 2.12 of Reichlin (2013).

In classical concave utility maximization, the asymptotic elasticity (AE) of the utility function is of importance. In particular, many results impose an upper bound on AE(U). For a nonconcave utility function, we impose a similar condition via the AE of the conjugate J,

$$\mathrm{AE}_0(J) := \limsup_{y\to 0} \sup_{q\in\partial J(y)} \frac{|q|\, y}{J(y)}.$$

In order to define our preference functionals, we introduce an additional function w that represents the distortion of the distribution of the beliefs.

DEFINITION 2.5. A *distortion* is a function $w : [0, 1] \to [0, 1]$ that is strictly increasing and satisfies $w(0) = 0$ and $w(1) = 1$.

In the literature, one can find several explicit functional forms for w. The most prominent example is

$$w(p) = \frac{p^\alpha}{(p^\alpha + (1-p)^\alpha)^{\frac{1}{\alpha}}} \tag{2.2}$$

suggested by Kahneman and Tversky (1979); they use the parameter $\alpha = 0.61$. For each model, we now define a preference functional V_n on $(\Omega^n, \mathcal{F}^n, P^n)$.

DEFINITION 2.6. We consider one of the following cases:
Case 1: The preference functional V_n is defined by

$$V_n(f) := \int_0^\infty w(P^n[U(f) > x])dx, \tag{2.3}$$

for a distortion w and a nonconcave utility U satisfying $U(0) \geq 0$. We refer to this case as *rank-dependent expected utility* (RDEU).
Case 2: The preference functional V_n is defined by

$$V_n(f) := E_n[U(f)], \tag{2.4}$$

for a nonconcave utility U, where we set $E_n[U(f)] := -\infty$ if $U(f)^- \notin L^1$. We refer to this case as *expected nonconcave utility* (ENCU).

The functional V_n defined in (2.3) can be seen as a Choquet integral $\int U(f)d(w \circ P^n)$ with respect to the monotone set function $w \circ P^n$; see Chapter 5 of Denneberg (1994) for an exposition of this concept. In the case $w(p) = p$, the functional V_n in (2.3) coincides with the classical expected utility $E_n[U(f)]$ in (2.4) for a positive nonconcave utility U. We distinguish the two cases since the conditions for their treatments will be different.

Finally, we formulate the sequence of optimization problems. For a fixed (initial capital) $x > 0$, the (budget) set $C^n(x)$ in the nth model is

$$C^n(x) := \{f \in L^0_+(\Omega^n, \mathcal{F}^n, P^n) \mid E_{Q^n}[f] \leq x\}.$$

For each model, we are interested in the *demand problem*

$$v^n(x) := \sup\{V_n(f) \mid f \in C^n(x)\}. \tag{2.5}$$

An element $f \in C^n(x)$ is *optimal* if $V_n(f) = v^n(x)$. By a *maximizer* for $v^n(x)$, we mean an optimal element for the optimization problem (2.5).

2.2 Main Results

Even in the classical case of expected concave utility, the stability of the utility maximization problem is only obtained under suitable growth conditions on U (or its conjugate J). In the case of the RDEU functional in (2.3), the corresponding assumption has to be imposed jointly on U and w.

ASSUMPTION 2.7. We suppose that

$$U(x) \leq k_1 x^\gamma + k_2, \tag{2.6}$$

$$w(p) \leq k_3 p^\alpha, \tag{2.7}$$

$$\gamma < \alpha \tag{2.8}$$

with $\gamma, \alpha \in (0, 1]$ and $k_1, k_2, k_3 > 0$. This allows us to find and fix λ such that $\lambda\alpha > 1$ and $\lambda\gamma < 1$.

This assumption is inspired by Assumption 4.1 in Carassus and Rásonyi (2015). For the example distortion in (2.2), condition (2.7) is satisfied. In the case without distortion, $w(p) = p$, (2.7) is satisfied for $k_3 = \alpha = 1$. A sufficient condition for (2.6) is $\mathrm{AE}_0(J) < \infty$ (see Lemma A.3). For later reference, we summarize the case-dependent assumptions.

ASSUMPTION 2.8. We assume that we have one of the following cases:
RDEU: Let V_n be defined as in (2.3). In this case, we suppose that the distribution of φ^0 is continuous and that Assumption 2.7 is satisfied.
ENCU: Let V_n be defined as in (2.4). In this case, we suppose that Assumption 2.1 and $\mathrm{AE}_0(J) < \infty$ are satisfied.

We are now in a position to formulate the main result of this paper. Note that this covers simultaneously both cases.

THEOREM 2.9. *Let Assumptions 2.2 and 2.8 be satisfied. Then*

$$\lim_{n\to\infty} v^n(x) = v^0(x),$$

and for any sequence of maximizers f^n for $v^n(x)$, there are a subsequence (n_k) and a maximizer $\bar{f}$ for $v^0(x)$ such that $f^{n_k} \Rightarrow \bar{f}$ as $k \to \infty$.

The maximizers for $v^0(x)$ are not necessarily unique; see Example 3.7 of Reichlin (2013). Weak convergence along a subsequence of maximizers is therefore the best we can hope for. Moreover, note that for the second statement in Theorem 2.9, we start with a sequence of maximizers f^n for $v^n(x)$. For the ENCU functional in (2.4), the existence of a maximizer f^n for $v^n(x)$ is guaranteed under the present assumptions; see Theorem 3.4 of Reichlin (2013). For the RDEU functional in (2.3), on the other hand, the existence of a maximizer for $v^n(x)$ has to be verified in any given setting. One sufficient criterion is that $(\Omega^n, \mathcal{F}^n, P^n)$ (or $(\Omega^n, \mathcal{F}^n, Q^n)$ due to the equivalence of Q^n and P^n) is atomless (see Remark 4.5 below). Another sufficient criterion is that $(\Omega^n, \mathcal{F}^n, P^n)$ consists of finitely many atoms. The latter, in fact, implies that any maximizing sequence (f^n) for $v^n(x)$ is bounded; this allows us to extract a subsequence a.s. converging to some limit $\bar{f}$, and arguments similar to the ones in Proposition 4.4 show that $\bar{f}$ is a maximizer for $v^n(x)$. These two criteria cover all the examples discussed in Section 3.

The assumption that U is strictly increasing and continuous is not strictly necessary; it avoids some (more) technical details. Let us shortly discuss a relevant excluded special case.

REMARK 2.10. The ENCU functional defined in (2.4) does not cover the piecewise constant function $U(x) := 1_{\{x \geq 1\}}$ that describes the *goal-reaching problem* initiated by Kulldorff (1993) and investigated extensively by Browne (1999; 2000). But under the assumption that φ^0 has a continuous distribution, one can adapt the arguments in its proof to show that the results of Theorem 2.9 also hold for the goal-reaching problem. We provide a detailed argument at the end of Section 5.

2.3. The Need for Assumption 2.1

For expected *concave* utilities, Assumption 2.1 is not necessary to obtain Theorem 2.9; see Proposition 5.4 below. However, for nonconcave utilities, Assumption 2.1 cannot be dropped. The difference between these cases can be explained as follows. For a risk-averse agent with a concave U, the optimal final position is (essentially) $\sigma(\varphi)$-measurable, and so it is enough to have convergence in distribution of the sequence of pricing densities. For risk-seeking agents, the optimal final position is not necessarily $\sigma(\varphi)$-measurable. Additional information (if available) is used by the agent to avoid the nonconcave part $\{U < U_c\}$ of U. In the atomless limit model, every payoff distribution can be supported, and Assumption 2.1 ensures that also the models along the sequence become sufficiently rich as $n \to \infty$. More concretely, Assumption 2.1 excludes the (pathological) behavior illustrated in the next example.

EXAMPLE 2.11. Consider a nonconcave utility U with $\{U < U_c\} = (a, b)$ that is strictly concave on $(0, a)$ and (b, ∞). The initial capital x is in (a, b), but not exactly in the middle of the interval (a, b). The probability spaces $(\Omega^n, \mathcal{F}^n, P^n)_{n\in\mathbb{N}}$ are all given by the same probability space consisting of two states with $P^n[\{\omega_1\}] = P^n[\{\omega_2\}] = 1/2$; and $(\Omega^0, \mathcal{F}^0, P^0)$ is an arbitrary atomless probability space. Set $\varphi^n \equiv 1$ for all $n \in \mathbb{N}_0$. Jensen's inequality and Theorem 5.1 of Reichlin (2013) give $v^0(x) = U_c(x)$. On the other hand, we have $v^n(x) = v^1(x)$ for every $n \in \mathbb{N}$ and we now show that $v^1(x) < U_c(x)$ for x chosen above. First, note that $v^1(x)$ admits a maximizer $\hat{f}$ since the model consists of two atoms (see the discussion following Theorem 2.9). The maximizer $\hat{f}$ satisfies $E_Q[\hat{f}] = x$ since

U is strictly increasing, so we can replace $\hat{f}(\omega_2)$ by $2x - \hat{f}(\omega_1)$. We therefore get the inequality

$$v^1(x) = \frac{1}{2}U(\hat{f}(\omega_1)) + \frac{1}{2}U(2x - \hat{f}(\omega_1)) \leq \frac{1}{2}U_c(\hat{f}(\omega_1)) + \frac{1}{2}U_c(2x - \hat{f}(\omega_1)) \leq U_c(x).$$

The first inequality is an equality if and only if both values $\hat{f}(\omega_1)$ and $\hat{f}(\omega_2)$ are not in $\{U < U_c\} = (a, b)$; the second inequality is an equality if and only if the two values $\hat{f}(\omega_1)$ and $\hat{f}(\omega_2)$ are in $[a, b]$. But these two conditions cannot be satisfied at the same time by our choice of x. This shows that $v^n(x) = v^1(x) < U_c(x) = v^0(x)$. We conclude that $v^n(x)$ converges as $n \to \infty$ (it is constant), but the limit is not $v^0(x)$.

3. APPLICATIONS

So far, our analysis has been conducted for an abstract sequence of models. We now present three types of application to illustrate the main results. We also provide a numerical example in Section 3.4 to visualize our results.

3.1. (Numerical) Computation of the Value Function

In recent years, there has been remarkable progress in the problem of behavioral portfolio selection. In particular, there are several new results for complete markets in continuous time. Most of them give the existence of a solution and describe the structure of the optimal final position as a decreasing function of the pricing density. While these results are interesting from a theoretical point of view, they are less helpful for explicit computations. In this section, we show how Theorem 2.9 can be used to determine the value function numerically for a complete model in continuous time.

The idea is to approximate the (computationally difficult) continuous-time model by a sequence of (computationally tractable) discrete-time models. We illustrate this for the Black–Scholes model that can be approximated by a sequence of binomial models. This is the typical example for the transition from a discrete- to a continuous-time setting. In this example, the limit model is atomless, while the approximating models are not. He (1991) and Prigent (2003) analyze the stability for expected *concave* utilities for this setting by directly analyzing the sequence of optimal terminal wealths f^n as a function of φ^n. This is possible due to the concavity of their utility function, but cannot be used here.

To fix ideas, we briefly recall the (classical) binomial approximation of the Black–Scholes model to verify that our assumptions in Section 2 are satisfied. We consider a time horizon $T \in (0, \infty)$, a probability space $(\Omega^0, \mathcal{F}^0, P^0)$ on which there is a standard Brownian motion $W = (W_t)_{t \geq 0}$, and a (discounted) market consisting of a savings account $B \equiv 1$ and one stock S described by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_0 = s_0 > 0,\ \sigma > 0,$$

in the filtration generated by W. The pricing density is then given by

$$\varphi^0 := \exp\left(-\frac{\mu}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right).$$

For the construction of the nth approximation, we start with a probability space $(\Omega^n, \tilde{\mathcal{F}}^n, P^n)$ on which we have independent and identically distributed random variables $(\tilde{\epsilon}_k)_{k=1,\dots,n}$ taking values 1 and -1, both with probability $\frac{1}{2}$. For any n, we consider the n-step market consisting of the savings account $B^{(n)} \equiv 1$ and the stock given by $S^{(n)}_t = S_0$ for $t \in [0, \frac{T}{n})$ and

$$S^{(n)}_t = S_0 \prod_{j=1}^{k} \left(1 + \frac{\mu T}{n} + \sqrt{\frac{T}{n}}\sigma \tilde{\epsilon}_j\right), \ \frac{kT}{n} \le t < \frac{(k+1)T}{n}, \ k = 1, \dots, n.$$

This process is right-continuous with left limits. The filtration generated by $S^{(n)}$ is denoted by $\mathbb{F}^n = (\mathcal{F}^n_t)_{0\le t\le T}$ and we take $\mathcal{F}^n := \mathcal{F}^n_T = \sigma(\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_n)$. The market is active at the times $0, \frac{T}{n}, \frac{2T}{n}, \dots, T$. It is well known that this market is complete, and we denote by Q^n the unique martingale measure. The martingale condition implies that

$$Q^n[\tilde{\epsilon}_k = 1]\left(1 + \frac{\mu T}{n} + \frac{\sigma\sqrt{T}}{\sqrt{n}}\right) + (1 - Q^n[\tilde{\epsilon}_k = 1])\left(1 + \frac{\mu T}{n} - \frac{\sigma\sqrt{T}}{\sqrt{n}}\right) = 1,$$

and solving gives $Q^n[\tilde{\epsilon}_k = 1] = \frac{1}{2}(1 - \frac{\mu\sqrt{T}}{\sigma\sqrt{n}})$. This is positive for n large enough and we only consider such n from now on. The measures Q^n and P^n are equivalent on $(\Omega^n, \mathcal{F}^n)$ and we denote the pricing density by $\varphi^n := dQ^n/dP^n$. It is shown in Theorem 1 of He (1990) that $\varphi^n \Rightarrow \varphi^0$; this is a consequence of the central limit theorem. The set $\mathcal{G}^n$ of atoms in $(\Omega^n, \mathcal{F}^n)$ can be identified with the paths of $S^{(n)}$. The Q^n-probability for a path is of the form

$$\left(\frac{1}{2}\right)^n \left(1 - \frac{\mu\sqrt{T}}{\sigma\sqrt{n}}\right)^k \left(1 + \frac{\mu\sqrt{T}}{\sigma\sqrt{n}}\right)^{n-k},$$

for some $k \in \{0, 1, \dots, n\}$ (which is the number of up moves in the path). For n large, we have $|\mu/(\sigma\sqrt{n})| < \frac{1}{2}$ and we see that $\sup_{A\in\mathcal{G}^n} Q^n[A] < (3/4)^n$. Taking the limit $n \to \infty$ gives $\lim_{n\to\infty} \sup_{A\in\mathcal{G}^n} Q^n[A] = 0$, which means that Assumption 2.1 is satisfied. The distribution of φ^0 is continuous if $\mu \neq 0$. Finally, a proof of uniform integrability of $((\varphi^n)^\xi)_{n\in\mathbb{N}}$ for $\xi < 0$ can be found in Prigent (2003, p. 172, Lemma c).

We conclude that all the assumptions of Section 2 are satisfied. We can therefore apply Theorem 2.9 to relate the optimization problem in the Black–Scholes model with the sequence of optimization problems in the sequence of binomial models. More precisely, Theorem 2.9 shows that the sequence of value functions (v^n) in the binomial models converges to the value function v^0 in the Black–Scholes model and that the sequence of maximizers converges along a subsequence. In particular, in the case of the preference functional (2.3), this turns out to be useful for computational purposes: While there are (abstract) results on the existence of a maximizer in the Black–Scholes model in the literature, these results are less helpful to determine a maximizer and the corresponding value explicitly. In the binomial model, however, the (numerical) computation of the value function and its maximizers is straightforward since the model consists of a finite number of atoms. In this context, Theorem 2.9 provides the insight that we can use the value functions in the binomial model to approximate the value function in the Black–Scholes model. This gives a method to determine numerically the value function v_0 in the Black–Scholes model. Note that for *RDEU*, we have no dynamic programming and hence no description of v_0 by a (HJB) PDE we could solve numerically.

3.2. Stability Results

In this section, we use Theorem 2.9 to show, as explained in Section 1, the stability of the portfolio choice results for a fixed model with respect to small perturbations. While Section 3.1 can be seen as perturbation of the underlying model itself, we are interested here in perturbations of a model's parameters.

3.2.1. Misspecifications of the Market Model. The first example is motivated by the practical difficulties one encounters when trying to calibrate an underlying model. In this section, we analyze how the optimal final position and the corresponding value are affected by a (small) misspecification of the underlying market model.

This question is well studied for expected *concave* utilities; see, for instance, Larsen and Žitković (2007) and Kardaras and Žitković (2011). Their sequence of model classes is more general in the sense that they need not restrict the setup to a single pricing density. However, the key to solving their problem is the classical duality theory that can be applied since their utilities are (strictly) concave. In our setting, this is not possible.

As one example in our framework, we can think of the Black–Scholes model where it is generally difficult to measure the drift. To formalize this situation, we fix some time horizon $T \in (0, \infty)$ and a probability space $(\Omega^0, \mathcal{F}^0, P^0)$ on which there is a Brownian motion $W = (W_t)_{t \geq 0}$. We introduce the sequence of probability spaces by $(\Omega^n, \mathcal{F}^n, P^n) := (\Omega^0, \mathcal{F}^0, P^0)$ for $n \geq 1$. In order to define a sequence of price processes, we consider a sequence (μ^n) converging to some drift parameter $\mu^0 \in \mathbb{R}$ in the limit model. For each n, we consider a (discounted) market consisting of a savings account $B \equiv 1$ and one stock S^n described by

$$\frac{dS_t^n}{S_t^n} = \mu^n dt + \sigma dW_t, \quad S_0^n = s_0 > 0,\ \sigma > 0,$$

in the filtration generated by W. The pricing density for the nth model is then given by

$$\varphi^n := \exp\left(-\frac{\mu^n}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu^n}{\sigma}\right)^2 T\right).$$

In this example, each model is atomless. Assumption 2.1 is therefore trivially satisfied. Moreover, looking at the explicit form of φ^n shows that $\varphi^n(\omega) \to \varphi^0(\omega)$ for all $\omega \in \Omega$, so in particular $\varphi^n \Rightarrow \varphi^0$. Finally, it is straightforward to check the uniform integrability of $((\varphi^n)^\xi)$ for any $\xi < 0$. Hence, Assumption 2.2 is satisfied.

We conclude that the assumptions of Section 2 are satisfied and we can apply the results there. Theorem 2.9 tells us that the value functions (as well as the corresponding maximizers along a subsequence) for the model with drift μ^n converge to the corresponding quantities in the model with drift μ^0. The economic interpretation of this result is that the behavioral prediction does not change drastically if we slightly perturb the drift.

It is also worth mentioning that the above arguments only use convergence of the market price of risk μ^n/σ to μ^0/σ. If we consider more generally a stochastic market price of risk $\lambda_t^n = \mu_t^n/\sigma_t^n$, then assuming $E_n[\int_0^T (\lambda_t^n)^2 dt] \to E_0[\int_0^T (\lambda_t^0)^2 dt]$ gives weak convergence of the stochastic exponential $\mathcal{E}(\int_0^\cdot \lambda_t^n dW_t)_T$ to $\mathcal{E}(\int_0^\cdot \lambda_t^0 dW_t)_T$; see Proposition A.1 in Larsen and Žitković (2007). In addition, one then needs some integrability condition on λ^n to ensure that the family $((\mathcal{E}(\int_0^\cdot \lambda^n dW_t)_T)^\xi)_{n \in \mathbb{N}}$ is uniformly integrable for $\xi < 0$; for instance, a nonrandom upper bound for all the $\int_0^T (\lambda_t^n)^2 dt$ is sufficient.

In the present setting, the limit model as well as the approximating sequence are given by atomless models. For this class of models and for the ENCU functional (2.4), the optimization problem v^n can be reduced to the concavified utility maximization problem; see Theorem 5.1 of Reichlin (2013). In this way, the stability result can also be obtained via stability results for expected *concave* utilities. For the RDEU functional (2.3) with distortion, however, the results are new.

3.2.2. Horizon Dependence. In Section 3.1 as well as in the first example in this section, we have started with a fixed time horizon T. In practical applications, however, the time horizon might be uncertain or changing. The goal of this section is to use Theorem 2.9 to study whether a (marginal) misspecification of the investment horizon significantly influences the optimal behavior of the agent. For expected *concave* utilities, Larsen and Yu (2012) analyze this question in an incomplete Brownian setting. The key to solving their problem is again the duality theory that cannot be used in our setup.

In order to formalize a similar situation in our framework, we start again with a probability space $(\Omega^0, \mathcal{F}^0, P^0)$ on which there is a Brownian motion $W = (W_t)_{t\geq 0}$, and we introduce the sequence of probability spaces by setting $(\Omega^n, \mathcal{F}^n, P^n) := (\Omega^0, \mathcal{F}^0, P^0)$ for $n \in \mathbb{N}$. We now fix a sequence $T^n \to T^0 \in (0, \infty)$ representing the time horizons. For each n, we consider the Black–Scholes model with time horizon T^n as described in Section 3.1. The pricing density for the nth model is therefore given by

$$\varphi^n := \exp\left(-\frac{\mu}{\sigma} W_{T^n} - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T^n\right).$$

Assumption 2.1 is again trivially satisfied since each model is atomless. Moreover, adapting the arguments from Section 3.2.1 shows that $\varphi^n \Rightarrow \varphi^0$ and Assumption 2.2 are satisfied as well. As in Section 3.2.1, we can therefore use Theorem 2.9 to conclude that behavioral predictions of the model are stable with respect to small misspecifications in the time horizon.

3.3. Theoretical Applications

For the ENCU functional defined in (2.4) without distortion, the problem in the limit model turns out to be tractable if φ^n has a continuous distribution. In this section, we apply Theorem 2.9 to approximate a pricing density with a general distribution by a pricing density with a continuous distribution. To explain the idea in more detail, we use the notation

$$u^n(x, U) := \sup\{E_n[U(f)] \mid f \in C^n(x)\}.$$

Every maximizer f^n for $u^n(x, U_c)$ satisfies $f^n \in -\partial J(\lambda^n \varphi^n)$ for some λ^n; see Proposition A.1. If φ^n has a continuous distribution, then we have $P^n[f^n \in \{U < U_c\}] = 0$; (see Lemma 5.7 in Reichlin 2013 for details) and it follows that $u^n(x, U) = u^n(x, U_c)$. In this way, the existence of a maximizer as well as several properties of $u^n(x, U)$ can be derived directly via the concavified problem. If the limit model is atomless but the distribution of the pricing density φ^0 is not continuous, then this reduction does not follow directly.

The idea now is to construct a sequence (φ^n) weakly converging to φ^0 for which each φ^n has a continuous distribution. For this approach, we assume that $E_0[(\varphi^0)^\eta] < \infty$ for all $\eta < 0$. Since $(\Omega^0, \mathcal{F}^0, P^0)$ is atomless, we can find a uniformly distributed random variable $\mathcal{U}$ such that $q_{\varphi^0}(\mathcal{U}) = \varphi^0$ P^0-a.s.; see Lemma A.28 in Föllmer and Schied (2011). Moreover, we choose another random variable $Y > 0$ with $E_0[Y] = 1$ having a continuous distribution (e.g., $Y = \mathcal{U} + 1/2$). Now we define the sequence (φ^n) by

$$\varphi^n := \left(1 - \frac{1}{n}\right) q_{\varphi^0}(\mathcal{U}) + \frac{1}{n} q_Y(\mathcal{U}) = \left(1 - \frac{1}{n}\right) \varphi^0 + \frac{1}{n} q_Y(\mathcal{U}).$$

Every element satisfies $E_0[\varphi^n] = (1 - \frac{1}{n}) E_0[q_{\varphi^0}(\mathcal{U})] + \frac{1}{n} E_0[q_Y(\mathcal{U})] = 1$ by construction. Moreover, the function $h_n(x) := (1 - \frac{1}{n}) q_{\varphi^0}(x) + \frac{1}{n} q_Y(x)$ converges pointwise to the function $h(x) := q_{\varphi^0}(x)$. The set D_h of all points where h is not continuous is at most countable since h is increasing; and $\mathcal{U}$ has a continuous distribution. So, it follows that $P^0[\mathcal{U} \in D_h] = 0$ and we obtain $\varphi^n = h_n(\mathcal{U}) \Rightarrow h(\mathcal{U}) = \varphi^0$; see Theorem 5.1 of Billingsley (1968).

With the arguments so far, we have a sequence of probability spaces defined by $(\Omega^n, \mathcal{F}^n, P^n) := (\Omega^0, \mathcal{F}^0, P^0)$ for $n \in \mathbb{N}$ together with a sequence of pricing measures (φ^n) weakly converging to φ^0. To verify Assumption 2.2, note that $\varphi^n \geq \varphi^0(1 - \frac{1}{n})$ gives $(\varphi^n)^{\eta(1+\epsilon)} \leq ((1 - \frac{1}{n})\varphi^0)^{\eta(1+\epsilon)}$ for every $\eta < 0$, which yields a uniformly integrable upper bound due to our assumption that $E_0[(\varphi^0)^\eta] < \infty$ for all $\eta < 0$.

It remains to show that the distribution of $\varphi^n = h_n(\mathcal{U})$ is continuous. Since the function h_n is increasing, it follows that $h_n(x_1) = h_n(x_2) = k$ if and only if $q_{\varphi^0}(x_1) = q_{\varphi^0}(x_2)$ and $q_Y(x_1) = q_Y(x_2)$. But $q_Y(\cdot)$ is strictly increasing since Y has a continuous distribution, so we infer that $P^0[h_n(\mathcal{U}) = k] = 0$ for $k \in \mathbb{R}$.

Theorem 2.9 now gives $u^n(x, U) \to u^0(x, U)$ as $n \to \infty$. Since the distribution of φ^n is continuous for each n, we have that $u^n(x, U) = u^n(x, U_c)$ for all n and we also get $u^0(x, U) = u^0(x, U_c)$ in the limit. In this way, we recover Theorem 5.1 in Reichlin (2013) under less general assumptions, but with completely different techniques. Instead of rearrangement techniques as in Reichlin (2013), we here approximate the mass points in the distribution of φ^0 by continuous distributions and apply Theorem 2.9.

3.4. A Numerical Illustration

The goal of this section is to illustrate the convergence result numerically. We consider the functional $V_n(f) := E_n[U(f)]$ (for a specific nonconcave utility) in the framework presented in Section 3.1 where we can derive $v^0(x)$ explicitly so that we can compare $v^0(x)$ with the value functions $v^n(x)$ in the approximating models. As in Section 3.3, we use the notation $u^n(x, U) := v^n(x)$.

The utility function in this example is given by

$$U(x) := \begin{cases} \ln x, & x < 1, \\ x - \dfrac{4 + \cos(x-1)}{3}, & 1 \leq x < 2\pi + 1, \\ 2\pi + \ln(x - 2\pi), & 2\pi + 1 \leq x. \end{cases}$$

This function is strictly increasing, continuous, in C^1 and satisfies the Inada conditions at 0 and ∞. Its concave envelope is given by

$$U_c(x) = \begin{cases} \ln x, & x < 1, \\ x - 1, & 1 \leq x < 2\pi + 1, \\ 2\pi + \ln(x - 2\pi), & 2\pi + 1 \leq x, \end{cases}$$

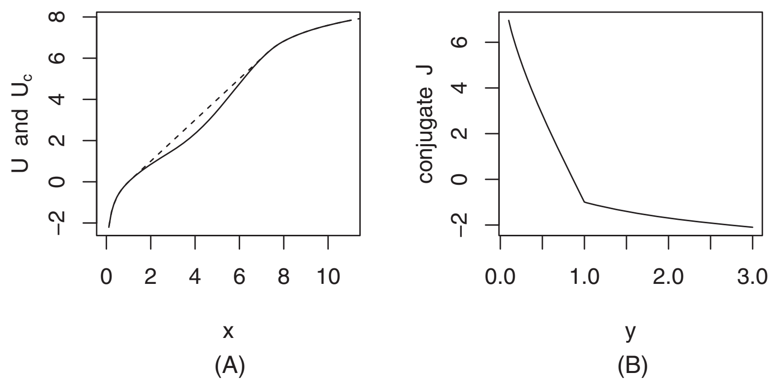

FIGURE 3.1. Panel A shows the nonconcave utility U and its concave envelope U_c (the dotted line). Panel B shows the conjugate J of U and U_c. The conjugate J has a kink in 1 that is the slope of the affine part in U_c in Panel A.

and the conjugate of U (and U_c) is

$$J(y) = \begin{cases} -\ln y - 1 + 2\pi(1-y), & y < 1, \\ -\ln y - 1, & y \geq 1. \end{cases}$$

The conjugate satisfies $\mathrm{AE}_0(J) < \infty$. On $(0,1) \cup (1,\infty)$, the conjugate is differentiable and ∂J is a singleton. More precisely, we have

$$-\partial J(y) = \begin{cases} 2\pi + \dfrac{1}{y}, & y < 1, \\ (1, 1+2\pi), & y = 1, \\ \dfrac{1}{y}, & y > 1. \end{cases}$$

Figure 3.1 shows U and U_c as well as the conjugate J.

Let us now determine $u^0(x, U)$. Recall from Section 3.1 that

$$\varphi^0 = \exp\left(-\zeta W_T - \zeta^2 T/2\right),$$

where $\zeta = \mu/\sigma$ and T are fixed. For simplicity, we assume that $\mu \neq 0$. We now consider some $f \in -\partial J(\lambda\varphi^0)$ for some $\lambda > 0$. Plugging in the above particular form of ∂J, using the fact that $\{\lambda\varphi^0 = 1\}$ has P^0-measure 0 for any $\lambda > 0$, and doing some elementary calculations gives

$$E_0[\varphi^0 f] = \frac{1}{\lambda} + 2\pi\, E_0\left[\varphi^0 1_{\{\lambda\varphi^0 < 1\}}\right].$$

In the next step, we rewrite the set $\{\lambda\varphi^0 < 1\}$ in a suitable way and use that $(W_t + \zeta t)_{t \geq 0}$ is a Q^0-Brownian motion (by Girsanov's theorem) to obtain

$$E_0\left[\varphi^0 1_{\{\lambda\varphi^0 < 1\}}\right] = 1 - \Phi\left(-\frac{1}{\zeta\sqrt{T}}\left(\ln\frac{1}{\lambda} + \frac{\zeta^2 T}{2}\right) + \zeta\sqrt{T}\right),$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution. From this explicit form, we see that $E_0[\varphi^0 1_{\{\lambda\varphi^0<1\}}]$ is a continuous and decreasing function of λ with limits 1 and 0 at 0 and ∞, respectively. The equation

$$x = E_0[\varphi^0 f] = \frac{1}{\lambda} + 2\pi\left(1 - \Phi\left(-\frac{1}{\zeta\sqrt{T}}\left(\ln\frac{1}{\lambda} + \frac{\zeta^2 T}{2}\right) + \zeta\sqrt{T}\right)\right)$$

therefore has a unique solution λ^*. Fix $\hat{f} \in -\partial J(\lambda^*\varphi^0)$. By definition of λ^*, $\hat{f}$ satisfies $E_0[\varphi^0\hat{f}] = x$, which means that $\hat{f} \in C^0(x)$. Moreover, $\hat{f}$ satisfies

$$P^0[\hat{f} \in \{U < U_c\}] = P^0[\hat{f} \in (1, 1+2\pi)] = P^0[\lambda^*\varphi^0 = 1] = 0,$$

and this gives $E_0[U_c(\hat{f})] = E_0[U(\hat{f})]$. The conjugacy relation between U and J and the explicit form of $\hat{f}$ give

$$E_0[U(f)] \le E_0[J(\lambda^*\varphi^0)] + x\lambda^* = E_0[U_c(\hat{f})] = E_0[U(\hat{f})]$$

for all $f \in C^0(x)$ which, together with $\hat{f} \in C^0(x)$, gives optimality of $\hat{f}$ for $u^0(x, U)$. In order to determine $u^0(x, U) = E_0[U(\hat{f})]$, we recall the explicit expression for $\partial J(y)$ and use the fact that $\{\lambda^*\varphi^0 = 1\}$ has measure 0 to get

$$E_0[U(\hat{f})] = E_0[-\ln(\lambda^*\varphi^0)] + 2\pi P^0[\lambda^*\varphi^0 < 1].$$

Elementary calculations show that $E_0[-\ln(\lambda^*\varphi^0)] = -\ln\lambda^* + \zeta^2 T/2$ and

$$P^0[\lambda^*\varphi^0 < 1] = 1 - \Phi\left(-\frac{1}{\zeta\sqrt{T}}\left(\ln\frac{1}{\lambda^*} + \frac{\zeta^2 T}{2}\right)\right).$$

We conclude that

$$\begin{aligned} u^0(x, U) &= E_0[U(\hat{f})] \\ &= -\ln\lambda^* + \frac{\zeta^2 T}{2} + 2\pi\left(1 - \Phi\left(-\frac{1}{\zeta\sqrt{T}}\left(\ln\frac{1}{\lambda^*} + \frac{\zeta^2 T}{2}\right)\right)\right). \end{aligned}$$

In order to illustrate the convergence result, we determine the parameter λ^* for $u^0(x, U)$. For comparison purposes, we compute $u^n(x, U)$ numerically for particular $n \in \mathbb{N}$ by backward recursion. Figure 3.2 shows the value functions for some approximations as well as the value function for the Black–Scholes model.

4. STABILITY OF THE DEMAND PROBLEM FOR RDEU

In this section, we analyze the stability for the *RDEU* for which the functional V_n is defined in (2.3) by

$$V_n(f) := \int_0^\infty w\big(P^n[U(f) > x]\big)dx.$$

The goal is to prove Theorem 2.9 for this case, that is, to prove that

$$\lim_{n\to\infty} v^n(x) = v^0(x) \tag{4.1}$$

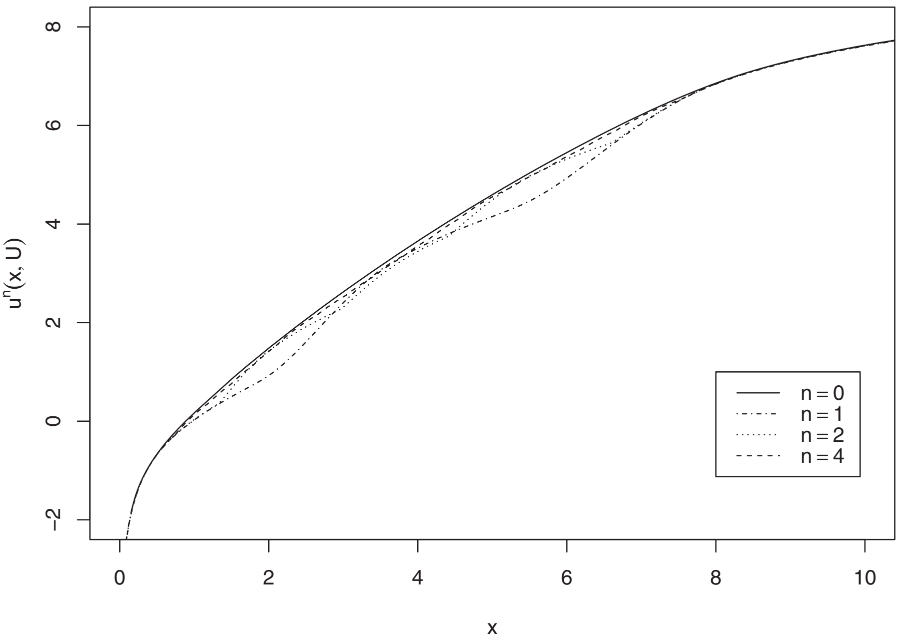

FIGURE 3.2. Value functions $u^n(x, U)$ for parameters $T = 1$, $\mu = 5\%$, $\sigma = 20\%$, and $n = 0, 1, 2, 4$. Recall that $n = 0$ is the limit case.

and to show that any given sequence of maximizers f^n for $v^n(x)$ contains a subsequence that converges weakly to a maximizer for $v^0(x)$.

4.1. Weak Convergence of Maximizers

We start with a convergence result for the maximizers. For later purposes, we prove a slightly more general statement; in particular, our proof needs no assumption on the distribution of φ^0 so that we can use Proposition 4.1 also in Section 5.

PROPOSITION 4.1. *For every sequence* (f^n) *with* $f^n \in C^n(x)$*, there are a subsequence* (n_k) *and some* $\bar{f} \in C^0(x)$ *such that* $f^{n_k} \Rightarrow \bar{f}$ *as* $k \to \infty$.

Let us outline the main ideas of the proof. We first use Helly's selection principle to get a limit distribution $\bar{F}$. In order to find a final position with distribution $\bar{F}$, we then follow the path of Jin and Zhou (2008), He and Zhou (2011a), and Carlier and Dana (2011) and define the candidate payoff in the limit model as a quantile function $q_{\bar{F}}$ applied to a uniformly distributed random variable. This ensures that the distribution of this final position is $\bar{F}$. In order to find the cheapest final position with the given distribution $\bar{F}$, one has to choose the "right" uniformly distributed random variable. If the pricing density φ^0 has a continuous distribution (as assumed in Jin and Zhou 2008, He and Zhou 2011a, and Carlier and Dana 2011 mentioned above), then $1 - F_{\varphi^0}(\varphi^0)$ turns out to be the good choice. In the general case where the distribution of φ^0 is not necessarily continuous, one can work with a uniformly distributed random variable $\mathcal{U}$ satisfying $\varphi^0 = q_{\varphi^0}(\mathcal{U})$ P^0-a.s. and then proceed similarly as in the first case. We also make use of the Hardy–Littlewood inequality, which states that any two random variables $f, g \in L^0_+(\Omega, \mathcal{F}, P)$ satisfy

$$E[fg] \geq \int_0^1 q_f(s) q_g(1-s) ds, \tag{4.2}$$

see Theorem A.24 of Föllmer and Schied (2011) for a proof.

In the proof of Proposition 4.1, we use the following tightness result to apply Helly's selection principle. Its proof is given at the end of this subsection.

LEMMA 4.2. *Let F^n be the distribution of f^n. Then, $(F^n)_{n\in\mathbb{N}}$ is tight, i.e.,*

$$\lim_{c\to\infty} \sup_{n\in\mathbb{N}} P^n[f^n > c] = 0.$$

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Let F^n be the distribution function of f^n. Since the sequence (F^n) is tight (Lemma 4.2), we may apply Helly's selection theorem (Billingsley 1968, Theorem 6.1 and p. 227) to get a subsequence (n_k) and a distribution function $\bar{F}$ such that $\lim_{k\to\infty} F^{n_k}(a) = \bar{F}(a)$ holds for all continuity points a of $\bar{F}$.

Since $(\Omega^0, \mathcal{F}^0, P^0)$ is atomless, it is possible to find on $(\Omega^0, \mathcal{F}^0, P^0)$ a random variable $\mathcal{U}$ uniformly distributed on $(0, 1)$ such that $\varphi^0 = q_{\varphi^0}(\mathcal{U}) P^0$-a.s.; see Lemma A.28 in Föllmer and Schied (2011). Define $\bar{f} := q_{\bar{F}}(1-\mathcal{U})$. Since $1-\mathcal{U}$ is again uniformly distributed on $(0, 1)$, the candidate $\bar{f}$ has distribution $\bar{F}$; see Lemma A.19 in Föllmer and Schied (2011). This gives $f^{n_k} \Rightarrow \bar{f}$ as $k \to \infty$.

The proof is completed by showing that $\bar{f} \in C^0(x)$, as follows. We rewrite φ^0 and $\bar{f}$ in terms of $\mathcal{U}$, and combine Fatou's lemma and the fact that weak convergence implies convergence of any quantile functions to get a first inequality. A second one follows by applying the Hardy–Littlewood inequality (4.2). Finally, we make use of $f^{n_k} \in C^{n_k}(x)$. These steps together give

$$\begin{aligned} E_0[\varphi^0 \bar{f}] &= E_0[q_{\varphi^0}(\mathcal{U}) q_{\bar{F}}(1-\mathcal{U})] = \int_0^1 q_{\varphi^0}(s) q_{\bar{F}}(1-s) ds \\ &\leq \liminf_{k\to\infty} \int_0^1 q_{\varphi^{n_k}}(s) q_{F^{n_k}}(1-s) ds \leq \liminf_{k\to\infty} E_{n_k}[\varphi^{n_k} f^{n_k}] \leq x, \end{aligned}$$

which proves that $\bar{f} \in C^0(x)$. □

It remains to give the

Proof of Lemma 4.2. We show below that

$$\alpha := \lim_{c\to\infty} \limsup_{n\to\infty} P^n[f^n > c] = 0. \tag{4.3}$$

This allows us for every $\epsilon > 0$ to choose c_0 and $n_0(c_0)$ in such a way that $P^n[f^n > c_0] < \epsilon$ for $n > n_0(c_0)$. For any $c \geq c_0$ and any $n \geq n_0(c_0)$, we then obtain the inequality $0 \leq P^n[f^n > c] \leq P^n[f^n > c_0] \leq \epsilon$, and therefore

$$0 \leq \sup_{n\geq n_0(c_0)} P^n[f^n > c] \leq \epsilon \quad \text{for } c \geq c_0.$$

By increasing c_0 to c_1 to account for the finitely many $n < n_0(c_0)$, we get

$$\sup_{n\in\mathbb{N}} P^n[f^n > c] \le \epsilon \quad \text{for } c \ge c_1.$$

Because $\epsilon > 0$ was arbitrary, this means that the family $(F^n)_{n\in\mathbb{N}}$ is tight.

We now show (4.3). First, note that the assumption $\varphi^0 > 0$ implies $F_{\varphi^0}(0) = 0$ and by the definition of a quantile function, that q_{φ^0} is positive and satisfies $F_{\varphi^0}(q_{\varphi^0}(\epsilon)) \ge \epsilon > 0$ for every $\epsilon > 0$. Thus, q_{φ^0} must be strictly positive on $(0, \epsilon)$ for $\epsilon > 0$ which implies that $\int_0^\epsilon q_{\varphi^0}(t)dt$ is strictly positive for $\epsilon > 0$. Assume by way of contradiction that $\alpha > 0$. For $\epsilon > 0$ small enough, choose a constant c_0 in such a way that $\limsup_n P^n[f^n > c_0] \ge \alpha - \epsilon$ and $c_0 \int_0^{\alpha-2\epsilon} q_{\varphi^0}(t)dt > x + 1$. Weak convergence gives convergence of the quantile functions and so we have $q_{\varphi^n}(t) \le q_{\varphi^0}(\alpha - 2\epsilon)+\epsilon$ on $(0, \alpha - 2\epsilon)$ for sufficiently large n, so dominated convergence gives $\int_0^{\alpha-2\epsilon} q_{\varphi^n}(t)dt \to \int_0^{\alpha-2\epsilon} q_{\varphi^0}(t)dt$. Because the limit is strictly positive, this and the choice of c_0 allow us to choose n_0 in such a way that

$$c_0 \int_0^{\alpha-2\epsilon} q_{\varphi^{n_0}}(t)dt > x, \tag{4.4}$$

and $P^{n_0}[f^{n_0} > c_0] \ge \alpha - 2\epsilon$. The latter implies $P^{n_0}[f^{n_0} \le c_0] \le 1 - \alpha + 2\epsilon$ that can be used to control $q_{f^{n_0}}$ on $(1 - \alpha + 2\epsilon, 1)$. Indeed, the last inequality and the definition of a quantile give $F^{n_0}(c_0) < t \le F^{n_0}(q_{f^{n_0}}(t))$ for any $t \in (1 - \alpha + 2\epsilon, 1)$ that implies that

$$q_{f^{n_0}}(t) > c_0 \tag{4.5}$$

on $(1-\alpha+2\epsilon, 1)$. Finally, we use the Hardy–Littlewood inequality (4.2) to rewrite $E_{n_0}[\varphi^{n_0} f^{n_0}]$ in terms of quantiles, plug in (4.5) and use (4.4) to obtain

$$E_{n_0}[\varphi^{n_0} f^{n_0}] \ge \int_0^1 q_{\varphi^{n_0}}(t)q_{f^{n_0}}(1 - t)dt \ge c_0 \int_0^{\alpha-2\epsilon} q_{\varphi^{n_0}}(t)dt > x,$$

which contradicts $f^{n_0} \in C^{n_0}(x)$. □

4.2. Upper Semicontinuity of $v^n(x)$

In this section, we prove the first inequality of (4.1), namely, that

$$\limsup_{n\to\infty} v^n(x) \le v^0(x). \tag{4.6}$$

Having proved weak convergence along a subsequence for any sequence $(f^n)_{n\in\mathbb{N}}$ with $f^n \in C^n(x)$, the remaining step is to show that the corresponding sequence of values $V_n(f^n)$ converges as well. For this, we use the growth condition imposed on U and w as well as of the integrability condition imposed on $(\varphi^n)_{n\in\mathbb{N}}$.

Throughout this section, we assume that Assumptions 2.2 and 2.7 hold true.

LEMMA 4.3. *Let $f^n \in C^n(x)$. Then, the family $(w(P^n[U(f^n) > y]))_{n\in\mathbb{N}}$ is uniformly integrable.*

Proof. Since $w(P^n[U(f^n) > y])$ is nonnegative for every $n \in \mathbb{N}$, it is sufficient to find an integrable upper bound independent of n. We first apply (2.7), the Chebyshev inequality, and (2.6), and then use that $|x+y|^\eta \le c(\eta)(|x|^\eta + |y|^\eta)$ for some constant $c(\eta)$ to obtain

$$\begin{aligned}
w\big(P^n[U(f^n) > y]\big) &\le k_3\big(P^n[U(f^n) > y]\big)^\alpha \\
&\le k_3 \frac{E_n\big[U(f^n)^\lambda\big]^\alpha}{y^{\lambda\alpha}} \\
&\le \frac{k_3}{y^{\lambda\alpha}} E_n\big[\big(k_1(f^n)^\gamma + k_2\big)^\lambda\big]^\alpha \\
&\le \frac{k_3}{y^{\lambda\alpha}} E_n\big[c(\lambda)\big(k_1^\lambda(f^n)^{\gamma\lambda} + k_2^\lambda\big)\big]^\alpha \\
&\le \frac{k_3}{y^{\lambda\alpha}} (c(\lambda))^\alpha \left(k_1^\lambda E_n\big[(f^n)^{\gamma\lambda}\big] + k_2^\lambda\right)^\alpha, \qquad (4.7)
\end{aligned}$$

where λ is the one fixed in Assumption 2.7. In the next step, we estimate the term $E_n\big[(f^n)^{\gamma\lambda}\big]$. Recall that $\gamma\lambda < 1$ by Assumption 2.7, so the conjugate of the function $x \mapsto x^{\gamma\lambda}$ is $y \mapsto c_1 y^{\gamma\lambda/(\gamma\lambda-1)}$ for some constant c_1. Since $f^n \in \mathcal{C}^n(x)$, this gives

$$E_n\big[(f^n)^{\gamma\lambda}\big] \le E_n\big[(f^n)^{\gamma\lambda} - f^n\varphi^n\big] + x \le c_1 E_n\big[(\varphi^n)^{\gamma\lambda/(\gamma\lambda-1)}\big] + x. \qquad (4.8)$$

Recall that $\varphi^n \Rightarrow \varphi^0$ by assumption, so also $(\varphi^n)^{\gamma\lambda/(\gamma\lambda-1)} \Rightarrow (\varphi^0)^{\gamma\lambda/(\gamma\lambda-1)}$. Since the family $(\varphi^n)^{\gamma\lambda/(\gamma\lambda-1)}$ is uniformly integrable by Assumption 2.2, we therefore obtain $E_n\big[(\varphi^n)^{\gamma\lambda/(\gamma\lambda-1)}\big] \to E_0\big[(\varphi^0)^{\gamma\lambda/(\gamma\lambda-1)}\big]$ as $n \to \infty$. Together with (4.8), this gives

$$E_n\big[(f^n)^{\gamma\lambda}\big] \le c_1 E_n\Big[(\varphi^n)^{\frac{\gamma\lambda}{(\gamma\lambda-1)}}\Big] + x \le c_1 E_0\big[(\varphi^0)^{\frac{\gamma\lambda}{(\gamma\lambda-1)}}\big] + x + 1 =: k_4 \qquad (4.9)$$

for sufficiently large n. Combining (4.7) and (4.9) finally yields

$$0 \le w\big(P^n[U(f^n) > y]\big) \le \frac{k_3}{y^{\lambda\alpha}} c(\lambda)\left(k_1^\lambda k_4 + k_2^\lambda\right)^\alpha 1_{\{y\ge1\}} + 1_{\{y<1\}},$$

which gives an integrable upper bound since $\lambda\alpha > 1$ by Assumption 2.7. □

We now combine Proposition 4.1 and Lemma 4.3 to prove the upper-semicontinuity of $v^n(x)$.

PROPOSITION 4.4. *Let (f^n) be a sequence with $f^n \in \mathcal{C}^n(x)$ and (f^{n_k}), and $\bar{f}$ the subsequence and its limit constructed in Proposition 4.1. Then, we have*

$$\lim_{k\to\infty} V_{n_k}(f^{n_k}) = V_0(f^0). \qquad (4.10)$$

Consequently, we have $\limsup_{n\to\infty} v^n(x) \le v^0(x)$.

Proof. Starting from an arbitrary sequence (f^n) with $f^n \in \mathcal{C}^n(x)$, Proposition 4.1 gives a subsequence (n_k) and a weak limit $\bar{f}$ such that $f^{n_k} \Rightarrow \bar{f}$. The function U is continuous, hence $U(f^{n_k}) \Rightarrow U(f^0)$, and therefore we get $P^{n_k}[U(f^{n_k}) > y] \to P^0[U(f^0) > y]$ for all points y where $P^0[U(f^0) > y]$ is continuous. But $P^0[U(f^0) > y]$ is a decreasing function of y; hence, there are at most countably many points where $P^0[U(f^0) > y]$ is not continuous, and we deduce that $P^{n_k}[U(f^{n_k}) > y] \to P^0[U(f^0) > y]$ for a.e. y. Moreover, w is increasing; hence, it is continuous a.e. and we infer that we have $w(P^{n_k}[U(f^{n_k})>y]) \to w(P^0[U(f^0)>y])$ for a.e. y. By Lemma 4.3, the family $(w(P^n[U(f^n)>y]))_{n\in\mathbb{N}}$ is uniformly integrable and we arrive at

$$\begin{aligned}\lim_{k\to\infty} V_{n_k}(f^{n_k}) &= \lim_{k\to\infty}\int_0^\infty w\big(P^{n_k}[U(f^{n_k})>y]\big)dy\\ &= \int_0^\infty w\big(P^0[U(f^0)>y]\big)dy\\ &= V_0(f^0).\end{aligned}$$

For the proof of upper semicontinuity of v^n (in n), assume by way of contradiction that $\limsup_n v^n(x) > v^0(x)$. This allows us to choose a sequence (f^n) with $f^n \in C^n(x)$ and $\limsup_n V_n(f^n) > v^0(x)$. We can then pass to a subsequence realizing the lim sup and apply the first part of proof to the subsequence to get a further subsequence (f^{n_k}) and a weak limit $\bar{f} \in C^0(x)$ with

$$v^0(x) < \limsup_{n\to\infty} V_n(f^n) = \lim_{k\to\infty} V_{n_k}(f^{n_k}) = V_0(\bar{f}),$$

which gives the required contradiction. □

REMARK 4.5. Proposition 4.4 can also be used to prove the existence of a maximizer for $v^0(x)$, as follows. We formally introduce a sequence of models by setting $(\Omega^n, \mathcal{F}^n, P^n) := (\Omega^0, \mathcal{F}^0, P^0)$ for all $n \in \mathbb{N}$ and fix a maximizing sequence (f^n) for $v^0(x)$. Proposition 4.4 then shows that the limit $\bar{f}$ constructed in Proposition 4.1 is a maximizer. As a by-product, we also see that $v^0(x) < \infty$. Note that so far, we have not used the assumption that φ^0 has a continuous distribution. Jin and Zhou (2008) and Carlier and Dana (2011) prove the existence of a maximizer for $v^0(x)$ under the assumption that φ^0 has a continuous distribution. Proposition 4.4 (together with Proposition 4.1) shows how to extend their results to an atomless underlying model with a pricing density that is not necessarily continuous. □

4.3. Lower Semicontinuity of $v^n(x)$

The purpose of this section is to show the second inequality "≥" of (4.1). The natural idea is to approximate payoffs in the limit model by a sequence of payoffs in the approximating models. For a generic payoff, this might be difficult; but we argue in the first step that it suffices to consider payoffs of the form $h(\varphi^0)$ for a bounded function h. Those elements can be approximated by the sequence $(h(\varphi^n))$. Since h is bounded, the sequence $w(P[U(h(\varphi^n))>y])$ as well as $(\varphi^n h(\varphi^n))$ have nice integrability properties so that one obtains the desired convergence results for $(V_n(h(\varphi^n)))$ as well as for $(E_n[\varphi^n h(\varphi^n)])$.

PROPOSITION 4.6. *Suppose that φ^0 has a continuous distribution. Then,*

$$\liminf_{n\to\infty} v^n(x) \geq v^0(x).$$

Proof. 1) *Reduction to a bounded payoff*: Suppose by way of contradiction that $\liminf_n v^n(x) < v^0(x)$. In Lemma 4.7 below, we show lower semicontinuity of the function $v^0(x)$. This allows us to choose $\epsilon > 0$ such that $\liminf_n v^n(x) < v^0(x-\epsilon)$. Therefore, we can find and fix $\tilde{f} \in C^0(x-\epsilon)$ satisfying $\liminf_n v^n(x) < V_0(\tilde{f})$. Next, we define an additional sequence (f^m) by $f^m := \tilde{f} \wedge m$. By construction, this sequence is increasing to

$\tilde{f}$, and this gives $P^0[U(f^m) > y] \nearrow P^0[U(\tilde{f}) > y]$ for all y. The function w is increasing, hence continuous a.e., and we thus have $w(P^0[U(f^m) > y]) \nearrow w(P^0[U(\tilde{f}) > y])$ for a.e. y. Monotone convergence then yields $V_0(f^m) \to V_0(\tilde{f})$. This allows us to find and fix m_0 such that $\liminf_n v^n(x) < V_0(\tilde{f}^{m_0})$.

2) *Reduction to a payoff* $h(\varphi^0)$: We define $h(s) := q_{\tilde{f}^{m_0}}(1 - F_{\varphi^0}(s))$. By the definition of a quantile, $q_{\tilde{f}^{m_0}}$ is increasing, so h is decreasing. Moreover, since $\tilde{f}^{m_0}$ is bounded by m_0, the quantile $q_{\tilde{f}^{m_0}}$ is bounded by m_0 as well. Recall now that the distribution of φ^0 is assumed to be continuous. Thus, $F_{\varphi^0}(\varphi^0)$ as well as $1 - F_{\varphi^0}(\varphi^0)$ are uniformly distributed on $(0, 1)$ and $h(\varphi^0)$ therefore has the same distribution as $\tilde{f}^{m_0}$. But the preference functional V_0 only depends on the distribution of its argument, and so we get

$$V_0(h(\varphi^0)) = V_0(\tilde{f}^{m_0}) > \liminf_{n\to\infty} v^n(x). \tag{4.11}$$

Finally, we use the monotonicity of h together with (4.2), $\tilde{f}^{m_0} \le \tilde{f}$ and $\tilde{f} \in \mathcal{C}^0(x - \epsilon)$ to obtain

$$E_0[\varphi^0 h(\varphi^0)] \le E_0[\varphi^0 \tilde{f}^{m_0}] \le E_0[\varphi^0 \tilde{f}] \le x - \epsilon.$$

This gives $h(\varphi^0) \in \mathcal{C}^0(x - \epsilon)$.

3) *Convergence of* $(V_n(h(\varphi^n)))$: Let D_h denote the set of all points where h is not continuous. The function h is decreasing and so D_h is at most countable; but φ^0 has a continuous distribution and it follows that $P^0[\varphi^0 \in D_h] = 0$. Hence, we get $h(\varphi^n) \Rightarrow h(\varphi^0)$ that then implies $w(P^n[h(\varphi^n) > y]) \to w(P^0[h(\varphi^0) > y])$ for every y. By construction, h is positive and bounded by m_0; hence, $1_{\{y \le m_0\}}$ is an integrable upper bound for $(w(P^n[h(\varphi^n) > y]))_{n\in\mathbb{N}}$ and dominated convergence gives $V_n(h(\varphi^n)) \to V_0(h(\varphi^0)) = V_0(\tilde{f}^{m_0})$.

4) *Convergence of* $(E_n[\varphi^n h(\varphi^n)])$: We define $g(x) := xh(x)$ and note that the set of points where g is not continuous is again D_h. As in step 3), we therefore have $g(\varphi^n) \Rightarrow g(\varphi^0)$. Recall now that $E_n[\varphi^n] = 1$ and $\varphi^n \Rightarrow \varphi^0$. This gives uniform integrability of $(\varphi^n)_{n\in\mathbb{N}}$ that, in turn, implies uniform integrability of $(g(\varphi^n))_{n\in\mathbb{N}}$ since $0 \le g(\varphi^n) \le m_0\varphi^n$. Together with $h(\varphi^0) \in \mathcal{C}^0(x - \epsilon)$ as proved in part 3), we obtain

$$E_n[\varphi^n h(\varphi^n)] = E_n[g(\varphi^n)] \longrightarrow E_0[g(\varphi^0)] = E_0[\varphi^0 h(\varphi^0)] \le x - \epsilon,$$

as $n \to \infty$. This implies $h(\varphi^n) \in \mathcal{C}^n(x)$ (for sufficiently large n). But on the other hand, (4.11) and part 3) give

$$\lim_{n\to\infty} V_n(h(\varphi^n)) = V_0(\tilde{f}^{m_0}) > \liminf_{n\to\infty} v^n(x),$$

which gives the required contradiction. □

LEMMA 4.7. *Suppose that* φ^0 *has a continuous distribution. Then,*

$$\liminf_{x\nearrow x_0} v^0(x) \ge v^0(x_0).$$

Proof. We assume to the contrary that $\liminf_{x\to x_0} v^0(x) < v^0(x_0)$. Fix $f^0 \in \mathcal{C}^0(x_0)$ with $\liminf_{x\to x_0} v^0(x) < V_0(f^0)$ and define $f^\epsilon := (1 - \epsilon)f^0$. It follows that $f^\epsilon \in \mathcal{C}^0((1 - \epsilon)x_0)$ and $f^\epsilon \nearrow f^0$ a.s. The latter implies $P^0[U(f^\epsilon) > y] \nearrow P^0[U(f^0) > y]$ for a.e. y as $\epsilon \to 0$, and since w is increasing, we have $w(P^0[U(f^\epsilon) > y]) \nearrow w(P^0[U(f^0) > y])$ a.e. Monotone convergence then yields $V_0(f^\epsilon) \nearrow V_0(f^0)$ as $\epsilon \to 0$, which gives a contradiction. □

Proof of Theorem 2.9 for RDEU. It follows from Propositions 4.4 and 4.6 that $\lim_n v^n(x) = v^0(x)$. For the second part, we apply Proposition 4.1 to (f^n). This gives a subsequence (f^{n_k}) weakly converging to some $\bar{f} \in C^0(x)$, and Proposition 4.4 implies that $\bar{f}$ is a maximizer for $v^0(x)$. □

5. STABILITY OF THE DEMAND PROBLEM FOR ENCU

In this section, we analyze the case that the functional V_n is defined by

$$V_n(f) := E_n[U(f)]$$

for a nonconcave utility function U. Except for nonconcavity, this coincides with the classical expected utility where the value function is usually denoted by u^n instead of v^n. We follow that tradition and switch to u^n from now on. Moreover, the analysis in Reichlin (2013), in particular sections 4 and 5, shows that the optimization problem for the nonconcave utility function U is closely linked to the optimization problem for its concave envelope U_c, and both of them are useful for the analysis in this section. Therefore, we use in this section the notation

$$u^n(x, U) := v^n(x),$$

for the value function. The goal is to prove Theorem 2.9 for the present case, that is, to prove

$$\lim_{n\to\infty} u^n(x, U) = u^0(x, U), \tag{5.1}$$

and to show that a given sequence of maximizers f^n for $u^n(x, U)$ contains a subsequence that converges weakly to a maximizer for $u^0(x, U)$. In contrast to Section 4, we prove the stability results here without further assumptions on the distribution of φ^0. This is necessary for the theoretical applications described in Section 3.3. An example of an atomless limit model with a unique (nontrivial) pricing density that does not have a continuous distribution can be found in Example 5.10 of Reichlin (2013).

Throughout this section, we assume that $\mathrm{AE}_0(J) < \infty$ **and that Assumption** 2.2 **is satisfied.**

5.1. Upper Semicontinuity of $u^n(x, U)$

The main idea is similar to the proof of upper semicontinuity of $v^n(x)$ in Section 4.2. Starting from a sequence (f^n) with $f^n \in C^n(x)$, we use the results of Section 4.1 to obtain weak convergence along a subsequence (n_k) to an element $\bar{f} \in C^0(x)$. Using Fatou's lemma for $(E_{n_k}[U(f^{n_k})^-])$, the remaining step is then to show that the corresponding sequence $(E_{n_k}[U(f^{n_k})^+])$ converges as well. This requires uniform integrability of the family $(U(f^n)^+))_{n\in\mathbb{N}}$ that can be proved with the help of the following lemma.

LEMMA 5.1. *Assumption 2.2 and $AE_0(J) < \infty$ imply the uniform integrability of the family $(J(\lambda\varphi^n)^+)_{n\in\mathbb{N}}$ for any $\lambda > 0$.*

Proof. We show below that Assumption 2.2 and $\mathrm{AE}_0(J) < \infty$ ensure that

(5.2) the family $(J(\varphi^n)^+)_{n\in\mathbb{N}}$ is uniformly integrable.

The statement is then clear for $\lambda \geq 1$ since J is decreasing. For $\lambda < 1$, the assumption that $\mathrm{AE}_0(J) < \infty$ (in combination with Lemma A.3) can be used to obtain constants $\gamma > 0$ and $y_0 > 0$ such that $J(\mu y) \leq \mu^{-\gamma} J(y)$ for all $\mu \in (0, 1]$ and $y \in (0, y_0]$. Applying this estimate on the set $\{\varphi^n \leq y_0\}$ and using monotonicity of J on the complement gives

$$J(\lambda\varphi^n)^+ \leq \lambda^{-\gamma} J(\varphi^n)^+ 1_{\{\varphi^n \leq y_0\}} + J(\lambda y_0)^+ 1_{\{\varphi^n > y_0\}} \leq \lambda^{-\gamma} J(\varphi^n)^+ + J(\lambda y_0)^+.$$

The second term in the last line is constant, and uniform integrability of the first one is due to (5.2).

It remains to prove (5.2). The second part of Lemma A.3 shows that $\mathrm{AE}_0(J) < \infty$ implies the existence of constants k_1, k_2 and $\gamma < 1$ such that $U(x) \leq k_1 + k_2 x^\gamma$ for $x \geq 0$. Plugging this inequality into the definition of J and doing some elementary computations gives

$$J(y) \leq k_1 + \sup_{x \geq 0}(k_2 x^\gamma - xy) = k_1 + C y^{\gamma/(\gamma-1)},$$

for some constant C. But then, it follows that $0 \leq J(\varphi^n)^+ \leq k_1 + C(\varphi^n)^{\gamma/(\gamma-1)}$, and Assumption 2.2 yields the uniform integrability of $(J(\varphi^n)^+)_{n\in\mathbb{N}}$. □

We now describe, as outlined above, the limit behavior of $(E_n[U(f^n)])_n$. This gives upper semicontinuity in n for $u^n(x, U)$ and it can also be used (later) to deduce the optimality of $\bar{f}$ constructed in Proposition 4.1.

PROPOSITION 5.2. *The sequence (f^n) of maximizers for $u^n(x, U)$ contains a subsequence (f^{n_k}) weakly converging to some limit $\bar{f} \in C^0(x)$, and it satisfies* $\limsup_{n\to\infty} E_n[U(f^n)] \leq E_0[U(\bar{f})]$. *Consequently, we have*

$$\limsup_{n\to\infty} u^n(x, U) \leq u^0(x, U).$$

Proof. We consider a (relabeled) subsequence (f^n) realizing the lim sup γ, say, for $E_n[U(f^n)]$ (or equivalently for $u^n(x, U)$, since the f^n are maximizers). Proposition 4.1 gives a further subsequence (f^{n_k}) with weak limit $\bar{f}$. Since U, $\max(\cdot, 0)$, and $\min(\cdot, 0)$ are continuous, we infer that $U(f^{n_k})^\pm \Rightarrow U(\bar{f})^\pm$. Fatou's lemma then gives

$$E_0[U(\bar{f})^-] \leq \liminf_{k\to\infty} E_{n_k}[U(f^{n_k})^-].$$

We show below that $(U(f^n)^+)_{n\in\mathbb{N}}$ is uniformly integrable, which implies that $(E_{n_k}[U(f^{n_k})^+])$ converges to $E_0[U(\bar{f})^+]$ as $k \to \infty$. Combining this with the inequality for the negative parts yields

$$\begin{aligned} E_0[U(\bar{f})] &\geq \lim_{k\to\infty} E_{n_k}[U(f^{n_k})^+] - \liminf_{k\to\infty} E_{n_k}[U(f^{n_k})^-] \\ &\geq \limsup_{k\to\infty} E_{n_k}[U(f^{n_k})] \\ &= \limsup_{n\to\infty} u^n(x, U), \end{aligned}$$

where we use in the last step that the (f^{n_k}) form a subsequence of the sequence (f^n) for which we have $\lim_n E_n[U(f^n)] = \gamma$ from above.

It remains to show uniform integrability of $(U(f^n)^+)_{n\in\mathbb{N}}$. This family is, by the definition of J, dominated by $(J(\epsilon\varphi^n)^+ + \epsilon\varphi^n f^n)_{n\in\mathbb{N}}$. Uniform integrability of the first summand family follows from Lemma 5.1, and since $(\varphi^n f^n)_{n\in\mathbb{N}}$ is bounded in L^1, the sequence $(\epsilon\varphi^n f^n)$ can be made arbitrarily small in expectation by choosing ϵ small. So, uniform integrability of $(U(f^n)^+)_{n\in\mathbb{N}}$ follows and the proof is complete. □

5.2. Lower Semicontinuity of $u^n(x, U)$

The goal of this section is to show lower semicontinuity in n for $u^n(x, U)$.

THEOREM 5.3. *Suppose that Assumption 2.1 holds true. Then,*

$$\liminf_{n\to\infty} u^n(x, U) \geq u^0(x, U).$$

The approach to prove this statement is as follows. Observe that

$$\liminf_{n\to\infty} u^n(x, U) \geq \liminf_{n\to\infty} \big(u^n(x, U) - u^n(x, U_c)\big) + \liminf_{n\to\infty} u^n(x, U_c).$$

If one shows (as we do below in Section 5.2.1) that

$$\liminf_{n\to\infty} u^n(x, U_c) = \lim_{n\to\infty} u^n(x, U_c) = u^0(x, U_c), \tag{5.3}$$

it only remains to show that

$$u^n(x, U_c) - u^n(x, U) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5.4}$$

While the proof of (5.3) follows (essentially) from nonsmooth versions of known stability results on *concave* utility maximization, the proof of (5.4) requires a careful analysis of the *nonconcave* problem that will be explained in detail in Section 5.2.2. Note that the additional Assumption 2.1 is only used to prove (5.4). We start with the proof of (5.3).

5.2.1. Continuity in n of $u^n(x, U_c)$. Instead of lower semicontinuity, we prove more than needed, namely,

PROPOSITION 5.4. $\lim_{n\to\infty} u^n(x, U_c) = u^0(x, U_c)$.

In the case of strictly concave utility functions, this result follows by directly analyzing the sequence of optimal terminal wealths f^n as a function of φ^n. In the nonconcave framework, U_c is not strictly concave; hence, its conjugate J is nonsmooth and f^n cannot be written as a function of φ^n (f^n only lies in the subgradient of $-J$ at φ^n). Instead, we use the fact that $u^n(x, U_c)$ can be written (see Lemma 5.7 below) in a dual form as

$$u^n(x, U_c) = \inf_{\lambda>0} E_n[J(\lambda\varphi^n) + x\lambda] = E_n[J((\lambda(n)\varphi^n) + x\lambda(n)]$$

for some dual minimizer $\lambda(n) \geq 0$. Continuity in n of $u^n(x, U_c)$ can then be shown by proving that the sequence $(\lambda(n))$ converges (along a subsequence) to a dual minimizer in the limit model and that the sequence $(E_n[J(\lambda(n)\varphi^n)])$ converges to the corresponding value in the limit model. The latter requires uniform integrability of the family $(J(\lambda(n)\varphi^n)_{n\in\mathbb{N}}$.

For the positive parts, this can be proved via Lemma 5.1. We now show that the family of negative parts is uniformly integrable as well.

LEMMA 5.5. *For each $s > 0$, the family $\{J(\lambda\varphi^n)^- \mid n \in \mathbb{N}, \lambda \in [0, s]\}$ is uniformly integrable.*

Proof. The idea for this result goes back to Kramkov and Schachermayer (1999); the extension to the nonsmooth case is proved in Lemma 6.1 of Bouchard, Touzi, and Zeghal (2004). A modified version of their proof works for our setup, as follows.

Since the conjugate J is decreasing, it is enough to check uniform integrability of $(J(s\varphi^n)^-)_{n\in\mathbb{N}}$. If $J(\infty) > -\infty$, all the $J(s\varphi^n)^-$ are bounded by a uniform constant and the statement is clear. So, assume $J(\infty) = -\infty$. To use the de la Vallée–Poussin characterization of uniform integrability, we need to find a convex increasing function $\Phi : [0, \infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} \dfrac{\Phi(x)}{x} = \infty$ and $\sup_n E_n[\Phi(J(s\varphi^n)^-)] < \infty$. The function J is convex, decreasing and finite on $(0, \infty)$; see Lemma 2.12 of Reichlin (2013). So for $J(\infty) = -\infty$, J is strictly decreasing and J as well as $-J$ have a classical inverse. Let $\Phi : (-J(0), +\infty) \to (0, \infty)$ be the inverse of $-J$. Since $-J$ is increasing and concave, its inverse Φ is increasing and convex. In order to prove that

$$\infty = \lim_{x\to\infty} \frac{\Phi(x)}{x} = \lim_{y\to\infty} \frac{y}{-J(y)}, \tag{5.5}$$

note first that $\lim_{y\to\infty} \sup_{q\in-\partial J(y)} q = 0$ (see Lemma A.3 in Reichlin 2013) implies

$$\liminf_{y\to\infty} \inf_{q\in-\partial J(y)} \frac{1}{q} = \liminf_{y\to\infty} \left(\sup_{q\in-\partial J(y)} q \right)^{-1} = \infty.$$

Hence, for all M, there is y_0 such that $\inf_{q\in-\partial J(y)} \frac{1}{q} \geq M$ for all $y \geq y_0$. Fix some y_1 and y_2 satisfying $y_0 < y_1 < y_2$ and set $z := (J(y_2) - J(y_1))/(y_2 - y_1)$. The mean value theorem gives $\tau \in [y_1, y_2]$ such that $z \in \partial J(\tau)$. This implies by the definition of the subdifferential ∂J that

$$M \leq \inf_{q\in-\partial J(y_0)} \frac{1}{q} \leq \frac{y_2 - y_1}{-J(y_2) + J(y_1)} \leq \frac{\dfrac{y_2}{J(y_2)} - \dfrac{y_1}{J(y_2)}}{-1 + \dfrac{J(y_1)}{J(y_2)}}.$$

Taking the lim inf as $y_2 \to \infty$ gives $M \leq \liminf_{y_2\to\infty} -y_2/J(y_2)$. The proof of (5.5) is complete since the constant M is arbitrary.

It remains to prove that $\sup_n E_n[\Phi(J(s\varphi^n)^-)] < \infty$. Recall that J is convex and finite on $(0, \infty)$ and hence continuous, and that $J(0) = U(\infty) > 0$ by the assumption on U. Moreover, $J(\infty) = -\infty$ in the present case, so there is $a \in (0, \infty)$ with $J(a) = 0$ and this implies $\Phi(0) = a < \infty$. By a direct computation, we see that for $s > 0$,

$$\begin{aligned} E_n[\Phi(J(s\varphi^n)^-)] &= E_n[\Phi(\max\{0, -J(s\varphi^n)\})] \\ &\leq E_n[\max\{\Phi(0), s\varphi^n\}] \\ &\leq \Phi(0) + E_n[s\varphi^n] = \Phi(0) + s, \end{aligned}$$

which completes the proof. ☐

We now show that weak convergence of $\lambda_n\varphi^n$ to $\lambda\varphi^0$ indeed implies convergence of $E_n[J(\lambda_n\varphi^n)]$ to $E_0[J(\lambda\varphi^0)]$.

LEMMA 5.6. *Let $\lambda_n \to \lambda \in (0,\infty)$ be given. Then, it holds that*

$$E_n[J(\lambda_n\varphi^n)] \longrightarrow E_0[J(\lambda\varphi^0)] \quad \textit{as } n\to\infty.$$

Proof. The continuity of J together with $\lambda_n \to \lambda \in (0,\infty)$ and $\varphi^n \Rightarrow \varphi^0$ implies $J(\lambda_n\varphi^n) \Rightarrow J(\lambda\varphi^0)$ as $n\to\infty$. Since the limit λ is in $(0,\infty)$, the λ_n lie eventually in a compact set B of the form $[\epsilon, \frac{1}{\epsilon}]$ with $0<\epsilon<1$, and so it is enough to show the uniform integrability of $\{J(\mu\varphi^n) \mid n\in\mathbb{N}, \mu\in B\}$. For the negative parts $\{J(\mu\varphi^n)^- \mid n\in\mathbb{N}, \mu\in B\}$, this is a consequence of Lemma 5.5, and for the positive parts, it follows by Lemma 5.1. □

For the nth model, the classical dual representation of $u^n(x, U_c)$ for our setting with a fixed pricing density gives a dual minimizer $\lambda(n)$. The sequence $(\lambda(n))$ does not necessarily converge; however, every cluster point yields a dual minimizer in the limit model.

LEMMA 5.7. *Given any $n\in\mathbb{N}_0$, the problem $u^n(x,U_c)$ admits a maximizer $f^n \in -\partial J(\lambda(n)\varphi^n)$, where $\lambda(n)\in(0,\infty)$ is a minimizer of*

$$\inf_{\lambda>0} E_n[J(\lambda\varphi^n) + x\lambda]. \tag{5.6}$$

Any cluster point $\bar\lambda$ of the sequence $(\lambda(n))$ is a minimizer of $\inf_{\lambda>0} E_0[J(\lambda\varphi^0)+x\lambda]$ *and satisfies $\bar\lambda\in(0,\infty)$.*

Proof. Lemmas 5.5 and 5.1 give $E_n[J(\lambda\varphi^n)]<\infty$ for all $n\in\mathbb{N}_0$ and all $\lambda>0$. Existence and structure of the solution for $u^n(x,U_c)$ and the dual representation then follow by Proposition A.1.

For the second part, we use the notation

$$H^n(\lambda) := E_n[J(\lambda\varphi^n)] + \lambda x,$$

for $n\in\mathbb{N}_0$. Convexity of J implies convexity of H^n. Fix a minimizer $\lambda(0)$ for $\inf_{\lambda>0} H^0(\lambda)$ and a cluster point $\bar\lambda$ of $(\lambda(n))$. We show below that any values between $\bar\lambda$ and $\lambda(0)$ are minimizers for $\inf_{\lambda>0} H^0(\lambda)$. Since by Proposition A.1, the minimizers of $\inf_{\lambda>0} H^0(\lambda)$ are bounded away from 0 and ∞, we therefore must have $\bar\lambda\in(0,\infty)$, and continuity of H^0 then implies that $\bar\lambda$ is also a minimizer.

We now argue that $\lambda(n_k)\to\bar\lambda$ implies

$$H^0(\lambda) = H^0\big(\lambda(0)\big) \quad \text{for any } \lambda \in (\lambda(0)\wedge\bar\lambda, \lambda(0)\vee\bar\lambda).$$

By way of contradiction, we assume that $H^0(\lambda) > H^0(\lambda(0)) + 2\epsilon$ holds for some $\lambda \in (\lambda(0)\wedge\bar\lambda, \lambda(0)\vee\bar\lambda)$. Lemma 5.6 with $\lambda_n\equiv\lambda(0)$ implies that $H^{n_k}(\lambda(0)) \to H^0(\lambda(0))$ as $k\to\infty$. Thus, for ϵ small enough, there is a constant k_0 such that

$$H^{n_k}(\lambda(0)) \le H^0(\lambda(0)) + \epsilon < H^0(\lambda) - \epsilon \le H^{n_k}(\lambda)$$

for all $k>k_0$. From the definition of the minimizer $\lambda(n_k)$, it holds that $H^{n_k}(\lambda(n_k)) \le H^{n_k}(\lambda(0))$. Putting the two inequalities together gives

$$H^{n_k}(\lambda(n_k)) \le H^{n_k}(\lambda(0)) < H^{n_k}(\lambda), \tag{5.7}$$

for $k > k_0$. Since $\lambda(n_k) \to \bar{\lambda}$, the number λ is between $\lambda(n_k)$ and $\lambda(0)$ for large enough values of k. Thus, (5.7) contradicts the convexity of H^{n_k}. $\square$

We finally have all the ingredients to prove the convergence of $u^n(x, U_c)$.

Proof of Proposition 5.4. To obtain $\limsup_n u^n(x, U_c) \le u^0(x, U_c)$, we apply Proposition 5.2 to U_c. For the other inequality, fix a relabeled sequence of maximizers (f^n) with $\gamma := \liminf_m u^m(x, U_c) = \lim_n E_n[U_c(f^n)]$. We use Lemma 5.7 to fix for each $n \in \mathbb{N}$ a corresponding dual minimizer $\lambda(n) \in (0, \infty)$ of (5.6). By classical duality theory and Lemma 5.7, any cluster point $\bar{\lambda}$ of $(\lambda(n))$ satisfies

$$u^0(x, U_c) = \inf_{\lambda > 0} E_0[J(\lambda\varphi^0) + x\lambda] = E_0[J(\bar{\lambda}\varphi^0)] + \bar{\lambda}x,$$

and $\bar{\lambda} \in (0, \infty)$. Fix one cluster point $\bar{\lambda}$ and a converging subsequence $\lambda(n_k) \to \bar{\lambda}$. It follows from Lemma 5.6 that $E_{n_k}[J(\lambda(n_k)\varphi^{n_k})] \to E_0[J(\bar{\lambda}\varphi^0)]$ and we conclude again from the dual representation for $u^{n_k}(x, U_c)$ that $E_{n_k}[U_c(f^{n_k})] = u^{n_k}(x, U_c) \to u^0(x, U_c)$. But the full sequence $(E_n[U_c(f^n)])$ converges to γ; so we finally obtain $u^0(x, U_c) = \gamma = \liminf_m u^m(x, U_c)$. This completes the proof. $\square$

5.2.2. Controlling the Difference $u^n(x, U_c) - u^n(x, U)$. Let us now turn to (5.4) and prove that $u^n(x, U_c) - u^n(x, U) \to 0$. The idea here is as follows. In general, $u^n(x, U)$ is smaller than $u^n(x, U_c)$ since U_c dominates U. For some initial values x, however, the maximizer for $u^n(x, U_c)$ does not have probability mass in $\{U < U_c\}$, i.e., $P[f^* \in \{U < U_c\}] = 0$, and thus also maximizes $u^n(x, U)$. Consequently, the values $u^n(x, U_c)$ and $u^n(x, U)$ coincide for such "good" initial values, and the key is to analyze the complement of these x more carefully. For the nth model, the "good" initial values induce a (n-dependent) partition of $(0, \infty)$ and its (n-dependent) mesh size, and the maximal distance between two successive partition points goes to 0 as $n \to \infty$ due to Assumption 2.1. The next result formalizes this idea.

PROPOSITION 5.8. *Let Assumption 2.1 be satisfied and let $x_0 > 0$ and $\delta > 0$ be fixed. For every $n \in \mathbb{N}_0$, there is a set $\mathcal{B}^n \subseteq (0, \infty)$ such that*

i) $u^n(x, U) = u^n(x, U_c)$ for $x \in \mathcal{B}^n$ and
ii) there is n_0 such that $\mathcal{B}^n \cap [x_0 - \delta, x_0]$ is nonempty for $n \ge n_0$.

As a consequence, we have $\lim_{n\to\infty} \big(u^n(x, U_c) - u^n(x, U)\big) = 0$.

Let us first outline the two main ideas. The problem $u^n(x, U_c)$ admits (under our conditions) a maximizer $f^n \in -\partial J(\lambda(n)\varphi^n)$ for some $\lambda(n)$. The right- and left-hand derivatives $-J'_\pm$ satisfy $-J'_\pm \notin \{U < U_c\}$; see Lemma A.2 of Reichlin (2013). So, in order to have no probability mass in the area $\{U < U_c\}$, it is sufficient if the maximizer value $f^n(\omega)$ is equal to $-J'_-(\lambda(n)\varphi^n(\omega))$ or $-J'_+(\lambda(n)\varphi^n(\omega))$. Therefore, the initial values given by

$$E_{Q^n}\big[-J'_+\big(\lambda(n)\varphi^n\big)1_{D^c} - J'_-\big(\lambda(n)\varphi^n\big)1_D\big] \tag{5.8}$$

for $D \in \mathcal{F}^n$ are good candidates for initial values satisfying property i).

In order to also have property ii), we need to control the distance between any two points defined by (5.8). This boils down to controlling terms of the form $-J'_-(y) + J'_+(y)$. These are nonzero if y is the slope of an affine part of U_c. The distance between the points defined by (5.8) is therefore dominated by the product of the length of the longest affine part and the Q^n-probability of the biggest atom in $\mathcal{F}^n$. In the case of a single affine part in U_c, this goes to 0 by Assumption 2.1. In general, there is no upper bound for the length of the affine parts, but we can estimate the tails with Lemma 5.9 below. Recall that $\mathcal{G}^n$ is the set of Q^n-atoms in $\mathcal{F}^n$ and that Assumption 2.1 ensures that the maximal Q^n-probability of all elements in $\mathcal{G}^n$ goes to 0.

Proof of Proposition 5.8. In order to define the set $\mathcal{B}^n$ for Proposition 5.8, we start with some preliminary definitions and remarks. For all $n \in \mathbb{N}$, fix a maximizer $f^n_{x_0}$ for $u^n(x_0, U_c)$ and the corresponding minimizer $\lambda(n) \in (0, \infty)$ given in Lemma 5.7. This lemma also yields $\liminf_{n\to\infty} \lambda(n) > 0$. So, fix $\epsilon > 0$ such that $\lambda(n) \geq \epsilon > 0$ for all n. Using Lemma 5.9 below, we obtain

$$
\begin{aligned}
0 &\leq \lim_{\alpha\to\infty} \sup_{n\in\mathbb{N}} E_n\big[\varphi^n \left(-J'_-(\lambda(n)\varphi^n)\right) 1_{\{-J'_-(\lambda(n)\varphi^n)\geq\alpha\}}\big] \\
&= \lim_{\alpha\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{\lambda(n)} E_n\big[\varphi^n \lambda(n) \left(-J'_-(\lambda(n)\varphi^n)\right) 1_{\{-J'_-(\lambda(n)\varphi^n)\geq\alpha\}}\big] \\
&\leq \frac{1}{\epsilon} \lim_{\alpha\to\infty} \sup_{n\in\mathbb{N}} E_n\big[\varphi^n \lambda(n) \left(-J'_-(\lambda(n)\varphi^n)\right) 1_{\{-J'_-(\lambda(n)\varphi^n)\geq\alpha\}}\big] = 0.
\end{aligned}
$$

Hence, we may choose α_0 such that

$$
\sup_{n\in\mathbb{N}} E_n\big[\varphi^n \left(-J'_-(\lambda(n)\varphi^n)\right) 1_{\{-J'_-(\lambda(n)\varphi^n)>\alpha_0\}}\big] < \delta. \tag{5.9}
$$

Define the set

$$
\mathcal{D}^n := \Big\{\omega \in \Omega^n \,\Big|\, -J'_+(\lambda(n)\varphi^n(\omega)) < -J'_-(\lambda(n)\varphi^n(\omega)) \leq \alpha_0\Big\} \in \mathcal{F}^n.
$$

Now we are in a position to define the set $\mathcal{B}^n$ by

$$
\mathcal{B}^n := \Big\{E_{Q^n}\big[-J'_+(\lambda(n)\varphi^n)1_{D^c} - J'_-(\lambda(n)\varphi^n)1_D\big] \,\Big|\, D \in \mathcal{F}^n,\, D \subset \mathcal{D}^n\Big\}.
$$

We claim that this $\mathcal{B}^n$ satisfies the assumptions of Proposition 5.8.

1) *Property i)*: For any $n \in \mathbb{N}$ and $x \in \mathcal{B}^n$, there is some $D \in \mathcal{F}^n$ such that

$$
g^n := -J'_+(\lambda(n)\varphi^n)1_{D^c} - J'_-(\lambda(n)\varphi^n)1_D \in C^n(x).
$$

Note that $g^n \in -\partial J(\lambda(n)\varphi^n)$ by definition and fix some $f \in C^n(x)$. Applying the definition of J together with $E_n[\varphi^n f] \leq x$ gives

$$
E_n[U_c(f)] \leq E_n\big[J\big(\lambda(n)\varphi^n\big)\big] + \lambda(n)x = E_n\big[U_c\big(g^n\big)\big],
$$

where the equality follows from the classical duality relation between U_c and J. Taking the sup over all $f \in C^n(x)$ gives optimality of g^n for $u^n(x, U_c)$. Since $J'_\pm$ do not take values in $\{U < U_c\}$ (see Lemma A.2 of Reichlin 2013), g^n satisfies $P^n[g^n \in \{U < U_c\}] = 0$ and it follows that

$$
u^n(x, U_c) = E_n[U_c(g^n)] = E_n[U(g^n)] = u^n(x, U),
$$

because $E_n[U(g^n)] \leq u^n(x, U) \leq u^n(x, U_c)$.

2) *Property ii)*: For this part, we use Assumption 2.1 to choose n_0 large enough such that $\sup_{A\in\mathcal{G}^n} Q^n[A] \le \delta/\alpha_0$ for $n \ge n_0$. Fix some $n \ge n_0$ and define the map $x : \mathcal{F}^n \to \mathbb{R}_+$ by

$$x(D) := E_{Q^n}\big[-J'_+(\lambda(n)\varphi^n)1_{D^c} - J'_-(\lambda(n)\varphi^n)1_D\big].$$

Monotonicity of ∂J (see Lemma A.1 of Reichlin 2013) implies $x(\emptyset) \le x(\mathcal{D}^n) \le x(\Omega^n)$. Moreover, recall that $f^n_{x_0}$ and $\lambda(n)$ are fixed in such a way that

$$f^n_{x_0} \in -\partial J(\lambda(n)\varphi^n) = [-J'_+(\lambda(n)\varphi^n), -J'_-(\lambda(n)\varphi^n)]$$

satisfies $E_{Q^n}[f^n_{x_0}] = x_0$. This gives $x(\emptyset) \le x_0 \le x(\Omega^n)$.

We first consider the case $x_0 < x(\mathcal{D}^n)$. In order to construct a grid contained in $\mathcal{B}^n \cap [x(\emptyset), x(\mathcal{D}^n)]$, we decompose $\mathcal{D}^n$ into disjoint subsets $D_1, \dots, D_m$ such that $Q^n[D_i] \le \delta/\alpha_0$ and $\bigcup_{i=1}^m D_i = \mathcal{D}^n$; this uses that for $n \ge n_0$, the largest atom in $\mathcal{F}_n$ has Q^n-probability at most δ/α_0. The values $x(\bigcup_{i=1}^k D_i)$, $k = 1, \dots, m$, are contained in $\mathcal{B}^n$, and since $-J'_-(\lambda(n)\varphi^n) \le \alpha_0$ on $D_k \subset \mathcal{D}^n$ and $J'_+ \le 0$, these values satisfy

$$\begin{aligned} x\left(\bigcup_{i=1}^{k} D_i\right) - x\left(\bigcup_{i=1}^{k-1} D_i\right) &= E_{Q^n}\big[\big(J'_+(\lambda(n)\varphi^n) - J'_-(\lambda(n)\varphi^n)\big)\, 1_{D_k}\big] \\ &\le \alpha_0\, Q^n[D_k] \le \alpha_0\delta/\alpha_0 = \delta, \end{aligned}$$

for $k = 1, \dots, m$. We deduce that $x(\emptyset)$ and $x(\bigcup_{i=1}^k D_i)$, $k = 1, \dots, m$, form a grid with starting point $x(\emptyset)$ and endpoint $x(\bigcup_{i=1}^m D_i) = x(\mathcal{D}^n)$ whose mesh size is smaller than δ.

It remains to consider the case $x_0 \in [x(\mathcal{D}^n), x(\Omega^n)]$. Since $x(\mathcal{D}^n) \in \mathcal{B}^n$, it is sufficient to show $x(\Omega^n) - x(\mathcal{D}^n) \le \delta$. Observe first that

$$(\mathcal{D}^n)^c = \big\{-J'_+(\lambda(n)\varphi^n) = -J'_-(\lambda(n)\varphi^n)\big\} \cup \{-J'_-(\lambda(n)\varphi^n) > \alpha_0\}.$$

We rewrite $x(\Omega^n) - x(\mathcal{D}^n)$ in terms of $J'_\pm$ and $(\mathcal{D}^n)^c$ and use $0 \le -J'_+ \le -J'_-$ to obtain

$$\begin{aligned} &x(\Omega^n) - x(\mathcal{D}^n) \\ &\quad = E_{Q^n}\big[\big(-J'_-(\lambda(n)\varphi^n) + J'_+(\lambda(n)\varphi^n)\big)\, 1_{(\mathcal{D}^n)^c}\big] \\ &\quad \le E_{Q^n}\big[\big(-J'_-(\lambda(n)\varphi^n) + J'_+(\lambda(n)\varphi^n)\big)\, 1_{\{-J'_+(\lambda(n)\varphi^n) = -J'_-(\lambda(n)\varphi^n)\}}\big] \\ &\qquad + E_{Q^n}\big[\big(-J'_-(\lambda(n)\varphi^n) + J'_+(\lambda(n)\varphi^n)\big)\, 1_{\{-J'_-(\lambda(n)\varphi^n) > \alpha_0\}}\big] \\ &\quad \le E_n\big[\varphi^n\big(-J'_-(\lambda(n)\varphi^n)\big)\, 1_{\{-J'_-(\lambda(n)\varphi^n) > \alpha_0\}}\big] < \delta, \end{aligned}$$

where the definition of α_0 in (5.9) is used in the last step.

3) *Proof of* $\lim_{n\to\infty}(u^n(x, U_c) - u^n(x, U)) = 0$: Fix $\epsilon > 0$. Because of the continuity of $u^0(x, U_c)$ in x and Proposition 5.4, we can fix $\delta > 0$ and n_1 such that $|x - x_0| \le \delta$ implies $|u^n(x, U_c) - u^n(x_0, U_c)| < \epsilon$ for all $n \ge n_1$. Applying the first part of this proof for δ gives n_0 such that for all $n \ge n_0$, there is some set $\mathcal{B}^n$ with properties i) and ii). So, for each $n \ge n_0$, there is some $x(n) \in \mathcal{B}^n \cap [x_0 - \delta, x_0]$. By definition of $\mathcal{B}^n$, the relation $u^n(x(n), U_c) = u^n(x(n), U)$ holds for all $n \ge n_0$. Moreover, $u^n(x, U)$ is increasing in x, so adding and subtracting $u^n(x(n), U_c) = u^n(x(n), U)$ and using that $x(n) \in [x_0 - \delta, x_0]$ yields

$$u^n(x_0, U_c) - u^n(x_0, U) \le u^n(x_0, U_c) - u^n(x(n), U_c) < \epsilon.$$

With the arguments so far, we have shown that for every $x_0 > 0$, we have

$$\limsup_{n\to\infty} \big(u^n(x_0, U_c) - u^n(x_0, U)\big) \le 0.$$

The result follows since $u^n(x_0, U_c) \ge u^n(x_0, U)$ for each $n \in \mathbb{N}$. □

It remains to state and prove

LEMMA 5.9. *Let B be a compact set of the form $[\epsilon, \frac{1}{\epsilon}]$ for $\epsilon \in (0, 1)$. Then, $\big\{-J'_-(\lambda\varphi^n)\varphi^n\lambda \mid n \in \mathbb{N}, \lambda \in B\big\}$ is uniformly integrable.*

Proof. $\mathrm{AE}_0(J) < \infty$ implies by the definition of $\mathrm{AE}_0(J)$ that there are a constant $M \in (0, \infty)$ and $y_0 > 0$ such that we have

$$\sup_{q\in\partial J(y)} |q|y \le MJ(y),$$

for $0 < y \le y_0$. An application of this inequality for $y = \lambda\varphi^n$ and $q = J'_-(\lambda\varphi^n)$ on the set $\{\varphi^n\lambda \le y_0\}$, some elementary calculations and $\lambda \in [\epsilon, \frac{1}{\epsilon}]$ yield

$$\begin{aligned} 0 &\le -J'_-(\lambda\varphi^n)\varphi^n\lambda 1_{\{\lambda\varphi^n\le y_0\}} \\ &\le M\big|J(\lambda\varphi^n)\big|\,1_{\{\lambda\varphi^n\le y_0\}} \\ &\le MJ(\varphi^n\epsilon)^+ + MJ(\varphi^n/\epsilon)^- . \end{aligned}$$

The family $(J(\varphi^n\epsilon)^+)_{n\in\mathbb{N}}$ is uniformly integrable by Lemma 5.1, and so is the family $(J(\varphi^n/\epsilon)^-)_{n\in\mathbb{N}}$ by Lemma 5.5. With the arguments so far, we have shown that the family $\{-J'_-(\lambda\varphi^n)\lambda\varphi^n 1_{\{\lambda\varphi^n\le y_0\}} \mid n \in \mathbb{N}, \lambda \in B\}$ is uniformly integrable. Now fix some $x_0 \in -\partial J(y_0)$ and recall that any $x \in -\partial J(y)$ for $y \ge y_0$ satisfies $x \le x_0$ and thus also $U_c(x) \le U_c(x_0)$. The classical conjugacy relation between ∂J and ∂U_c gives

$$xy = U_c(x) - J(y) \le U_c(x_0) + J(y)^-$$

for $y \ge y_0$. Applying this inequality for $y = \lambda\varphi^n$ and $x = -J'_-(\lambda\varphi^n)$ on the set $\{\lambda\varphi^n \ge y_0\}$ shows that $\{-J'_-(\lambda\varphi^n)\lambda\varphi^n 1_{\{\lambda\varphi^n\ge y_0\}} \mid n \in \mathbb{N}, \lambda \in B\}$ is dominated by $\{(U_c(x_0) + J(\lambda\varphi^n)^-)1_{\{\lambda\varphi^n\ge y_0\}} \mid n \in \mathbb{N}, \lambda \in B\}$. This completes the proof since the latter family is uniformly integrable by Lemma 5.5. □

The lower semicontinuity in n of $u^n(x, U)$ stated in Theorem 5.3 is now a straightforward consequence of Propositions 5.4 and 5.8. For completeness, we formally carry out the argument.

Proof of Theorem 5.3. Since $u^n(x, U_c) - u^n(x, U)$ converges to 0 by Proposition 5.8, since $u^n(x, U_c)$ converges to $u^0(x, U_c)$ by Proposition 5.4 and because the inequality $u^n(x, U) \le u^n(x, U_c)$ holds true for all $n \in \mathbb{N}_0$, we deduce from $\liminf(a_n + b_n) \ge \liminf a_n + \liminf b_n$ that

$$\begin{aligned} \liminf_{n\to\infty} u^n(x, U) &\ge \liminf_{n\to\infty} \big(u^n(x, U) - u^n(x, U_c)\big) + \lim_{n\to\infty} u^n(x, U_c) \\ &\ge 0 + u^0(x, U_c) \ge u^0(x, U). \end{aligned}$$

This completes the proof. □

5.3. Putting Everything Together

On the way, we have separately proved the second case of Theorem 2.9. For completeness, we summarize the main steps.

Proof of Theorem 2.9 for ENCU. Theorem 5.3 and Proposition 5.2 give the convergence $\lim_n u^n(x, U) = u^0(x, U)$. For the second part, fix a maximizer f_x^n for $u^n(x, U)$ for every n. Proposition 4.1 shows that the sequence (f_x^n) contains a subsequence weakly converging to some $\bar{f} \in C^0(x)$. It then follows from Proposition 5.2, the optimality of f_x^n and $\lim_n u^n(x, U) = u^0(x, U)$ that

$$E_0[U(\bar{f})] \geq \limsup_{n\to\infty} E_n[U(f_x^n)] = \limsup_{n\to\infty} u^n(x, U) = u^0(x, U).$$

This shows that $\bar{f}$ is a maximizer for $u^0(x, U)$ since $\bar{f} \in C^0(x)$. □

It remains to give the proof for the stability of the goal-reaching problem. Recall from Remark 2.10 that this is the case where $U(x) = 1_{\{x\geq 1\}}$ so that $U_c(x) = x \wedge 1$ for $x \in (0, \infty)$. In particular, U_c is strictly increasing on $(0, 1)$ and uniformly bounded by 1.

Proof of Remark 2.10. The statement is clear for $x \geq 1$ since $u^n(x, U) = 1$ there for each $n \in \mathbb{N}$; so, we assume that $x \in (0, 1)$. In Section 5.2.1, strict monotonicity of U_c is only used via Proposition A.1 to show the existence of the lower bound c_1^n. A closer inspection of the argument there shows that we only need strict monotonicity of $u^n(x, U_c)$. But $u^n(x, U_c)$ admits a maximizer f^n (see the discussion following Theorem 2.9) and the constraint $E_n[\varphi^n f^n] \leq x < 1$ implies $P^n[f^n \in [0, 1)] > 0$. This yields strict monotonicity of $u^n(x, U_c)$ for $x \in (0, 1)$ since U_c is strictly increasing on $[0, 1)$ and so we can prove $\lim_n u^n(x, U_c) = u^0(x, U_c)$ for $x \in (0, 1)$ as in Proposition 5.4. This implies $\limsup_n u^n(x, U) \leq \limsup_n u^n(x, U_c) = u^0(x, U_c)$. For the lim inf, we first fix for each $n \in \mathbb{N}$ a maximizer $f^n \in -\partial J(\lambda(n)\varphi^n)$ for $u^n(x, U_c)$ and recall that $f^n \in -\partial J(\lambda(n)\varphi^n)$ implies $-J'_+(\lambda(n)\varphi^n) \leq f^n \leq -J'_-(\lambda(n)\varphi^n)$ and that $U(-J'_\pm(y)) = U_c(-J'_\pm(y))$ holds for $y > 0$; see Lemma A.2 of Reichlin (2013). This gives

$$\liminf_{n\to\infty} u^n(x, U) \geq \liminf_{n\to\infty} E_n[U(f^n)] \geq \liminf_{n\to\infty} E_n\big[U_c\big(-J'_+(\lambda(n)\varphi^n)\big)\big]. \tag{5.10}$$

We now fix a subsequence (n_k) realizing the $\liminf_n u^n(x, U)$ and such that the associated sequence $(\lambda(n_k))$ converges to $\bar{\lambda}$. As in Lemma 5.7 (and again using the modified version of Proposition A.1), this gives $\bar{\lambda} > 0$. The assumptions that φ^0 has a continuous distribution and $\varphi^n \Rightarrow \varphi^0$ imply then that $U_c(-J'_\pm(\lambda(n_k)\varphi^{n_k})) \Rightarrow U_c(-J'_\pm(\bar{\lambda}\varphi^0))$. Moreover, as the function U_c is uniformly bounded, the right-hand side of (5.10) converges to $E_0[U_c(-J'_+(\bar{\lambda}\varphi^0))]$. But since φ^0 has a continuous distribution, it follows that $-J'_+(\bar{\lambda}\varphi^0) = -J'_-(\bar{\lambda}\varphi^0)$ P-a.s. and applying similar arguments in the reverse order, we find that

$$\liminf_{n\to\infty} u^n(x, U) \geq \liminf_{n\to\infty} E_n\big[U_c\big(-J'_-(\lambda(n)\varphi^n)\big)\big] \geq \liminf_{n\to\infty} u^n(x, U_c).$$

With the arguments so far, we have proved that

$$\limsup_{n\to\infty} u^n(x, U) \leq u^0(x, U_c) = \limsup_{n\to\infty} u^n(x, U_c) \leq \liminf_{n\to\infty} u^n(x, U),$$

which gives $\lim_n u^n(x, U) = u^0(x, U_c)$. But the limit model is atomless, so we have $u^0(x, U) = u^0(x, U_c)$ by Theorem 5.1 of Reichlin (2013) and the result follows. Finally, the convergence of the maximizers along a subsequence follows as in Proposition 5.2. □

6. CONCLUSION

In this paper, we study the stability along a sequence of models for a class of behavioral portfolio selection problems. The analyzed preference functionals allow for nonconcave and nonsmooth utility functions as well as for probability distortions. These features are motivated by several applications such as manager compensation, portfolio delegation, and behavioral finance. While there are several explicit results in the literature for behavioral portfolio selection problems in complete continuous-time markets, there are no comparable results for the discrete-time analog.

Our convergence results demonstrate that the explicit results from the continuous-time model are approximately valid also for the discrete-time setting if the latter is *sufficiently close* to the continuous-time setting. As illustrated by a counterexample, the required notion of *sufficiently close* is slightly but strictly stronger compared to the stability results for *concave* utility maximization problems. The convergence results can also be applied to other situations such as (marginal) drift misspecification or changing time horizons.

APPENDIX

A.1. Nonsmooth Utility Maximization

This appendix contains the results on nonsmooth (concave) utility maximization, which are relevant for the proofs in Section 5. Following the notation there, we use $u^n(x, U) := v^n(x)$ to denote the value function. Recall that J is the conjugate of U (as well as U_c).

PROPOSITION A.1. *Fix $n \in \mathbb{N}_0$. Suppose that $E_n[J(\lambda\varphi^n)] < \infty$ for all $\lambda > 0$. Then, the concave problem $u^n(x, U_c)$ has a solution $f^n \in C^n(x)$ for every $x > 0$. Every solution satisfies $f^n \in -\partial J(\lambda^n \varphi)$, where $\lambda^n \in (c_1^n, c_2^n)$ is a minimizer for*

$$\inf_{\lambda>0} E_n[J(\lambda\varphi^n) + x\lambda], \tag{A.1}$$

and c_1^n and c_2^n are strictly positive constants.

Most of the statements contained in Proposition A.1 are proved in greater generality in Bouchard et al. (2004) and Westray and Zheng (2009). For completeness, we include a proof. We make use of Lemma 6.1 in Bouchard et al. (2004) which reads in our setup as follows.

LEMMA A.2. *There is a function $\Phi^n : (-J(0), +\infty) \to (0, \infty)$ that is convex and increasing with $\lim_{x\to\infty} \Phi^n(x)/x = \infty$ and*

$$E_n\left[\Phi^n\left(J(y\varphi^n)^-\right)\right] \le C^n + y \quad \textit{for all } y > 0. \tag{A.2}$$

Proof of Proposition A.1. 1) The existence of a maximizer $f^n \in C^n(x)$ in the present setting is shown in Theorem 3.4 of Reichlin (2013). Remark 3.3 there also shows that

$E_n[J(\lambda\varphi^n)] < \infty$ for all λ implies $u^n(x, U_c) < \infty$ for some $x > 0$ so that we can use Theorem 4.1 there to get

$$E_n[J(\lambda\varphi^n)] = \sup_{x>0}\{u^n(x, U_c) - \lambda x\} \quad \text{for all } \lambda > 0.$$

Moreover, $u^n(x, U_c)$ is on $(0, \infty)$ finite and concave, hence continuous. This implies that we also have $u^n(x, U_c) = \inf_{\lambda>0}\{E_n[J(\lambda\varphi^n)] + x\lambda\}$. In order to find the upper bound c_2^n, we consider a minimizing sequence (λ_k) for (A.1) and show that it is bounded by some constant. Since (λ_k) is minimizing, it holds that

$$\text{(A.3)} \qquad - E_n[J(\lambda_k\varphi^n)^-] + x\lambda_k \leq E_n[J(\lambda_k\varphi^n)] + x\lambda_k \leq u^n(x, U_c) + 1,$$

for k large enough. We use the function Φ^n introduced in Lemma A.2. Then for all $\epsilon > 0$, there is some $x_0 > 0$ such that $\Phi^n(x)/x \geq 1/\epsilon$ for $x \geq x_0$, and then $x \leq x_0 + \epsilon\Phi^n(x)1_{\{x\geq x_0\}} \leq x_0 + \epsilon\Phi^n(x)$ for all $x \geq 0$. Using (A.2), we compute that for some $C^n > 0$,

$$E_n[J(\lambda_k\varphi^n)^-] \leq x_0 + \epsilon E_n\left[\Phi^n\left(J(\lambda_k\varphi^n)^-\right)\right] \leq x_0 + \epsilon(C^n + \lambda_k).$$

Combining this inequality and (A.3) gives $(x - \epsilon)\lambda_k \leq u^n(x, U_c) + 1 + x_0 + \epsilon C^n$. Choosing $\epsilon = x/2 > 0$ shows that (λ_k) is bounded by some constant.

In order to find the lower bound c_1^n, we start with the case $J(0) < \infty$. Due to the existence of a maximizer for $u^n(x, U_c)$ and the strict monotonicity of U_c, we also deduce strict monotonicity of $u^n(x, U_c)$ and we infer $J(0) = U_c(\infty) > u(x, U_c)$. Together with the continuity of the function $H^n(\lambda) := E_n[J(\lambda\varphi^n)] + x\lambda$ in 0, we can find c_1^n such that the minimization in (A.1) can be reduced to $\lambda > c_1^n$. In the case $J(0) = \infty$, we can again find c_1^n since $E_n[J(\lambda\varphi^n)] \to \infty$ for $\lambda \to 0$.

2) With the arguments so far, we find a maximizer $f^n \in C^n(x)$ and some parameter $\lambda^n \in (0, \infty)$ satisfying

$$E_n[U_c(f^n)] = u^n(x, U_c) = \inf_{\lambda>0}\{E_n[J(\lambda\varphi^n)] + x\lambda\} = E_n[J(\lambda^n\varphi^n)] + x\lambda^n.$$

Suppose by way of contradiction that there exists a set $A \in \mathcal{F}^n$ satisfying $P^n[A] > 0$ and $f^n \notin \partial J(\lambda^n\varphi^n)$ on the set A. The conjugacy relation between U_c and J then implies $u^n(x, U_c) = E_n[U_c(f^n)] < E_n[J(\lambda^n\varphi^n)] + x\lambda^n$, which is the required contradiction. $\square$

A.2. Auxiliary Results

LEMMA A.3. *The AE condition $AE_0(J) < \infty$ is equivalent to the existence of two constants $\gamma > 0$ and $y_0 > 0$ such that*

$$J(\mu y) \leq \mu^{-\gamma} J(y) \quad \textit{for all } \mu \in (0, 1] \textit{ and } y \in (0, y_0].$$

Moreover, if $AE_0(J) < \infty$ is satisfied, then there are constants k_1 and k_2 and $\gamma < 1$ such that $U(x) \leq k_1 + k_2x^\gamma$ for $x \geq 0$.

Proof of Lemma A.3. The equivalence is proved in Lemma 4.1 of Deelstra, Pham, and Touzi (2001). We only prove the last implication. Similarly to Lemma 4.1 of Deelstra et al. (2001), we argue that $U_c(\lambda x) \le \lambda^\gamma U_c(x)$ holds for $x \ge x_0$ and $\lambda > 1$. Recall that

$$yx - \gamma U_c(x) < 0 \quad \text{for } x \ge x_0 \text{ and } y \in \partial U_c(x), \tag{A.4}$$

and $0 \le U(x_0) \le U_c(x_0)$. Now choose some $x > x_0$ and observe that $\lambda x \ge x_0$ for all $\lambda \ge 1$. We want to compare the functions $U_c(\lambda x)$ and $\lambda^\gamma U_c(x)$ for $\lambda > 1$. Let F be the concave function on $[1, \infty)$ defined by $F(\lambda) := U_c(\lambda x)$. Fix some $q \in \partial F(\lambda)$. By definition, this implies that b$U_c(zx) \le U_c(\lambda x) + q(z - \lambda)$ and therefore that $\frac{q}{x} \in \partial U_c(\lambda x)$. Thus, it follows from (A.4) that

$$\lambda q - \gamma F(\lambda) < 0 \quad \text{for all } \lambda \ge 1 \text{ and } q \in \partial F(\lambda). \tag{A.5}$$

Set $G(\lambda) := \lambda^\gamma U_c(x)$. In order to complete the proof, we have to check that $(F - G)(\lambda) \le 0$ for all $\lambda \ge 1$. Clearly, the function G satisfies the equation

$$\lambda G'(\lambda) - \gamma G(\lambda) = 0 \tag{A.6}$$

for all $\lambda \ge 1$. Since $G(1) = F(1)$, it follows from (A.5) and (A.6) that $0 > q - \gamma F(1) = q - \gamma G(1) = q - G'(1)$. Hence, we have $q < G'(1)$ for all $q \in \partial F(1)$. Since G is continuously differentiable, there exists $\epsilon > 0$ such that $q < G'(\lambda)$ for all $q \in \partial F(1)$ and $\lambda \in [1, 1 + \epsilon)$. This gives

$$F(\lambda) \le F(1) + q(\lambda - 1) < G(1) + G'(\lambda)(\lambda - 1) \le G(\lambda) \tag{A.7}$$

for all $q \in \partial F(1)$ and $\lambda \in [1, 1 + \epsilon)$. To show that $F(\lambda) < G(\lambda)$ holds for all $\lambda > 1$, let $\bar\lambda := \inf\{\lambda > 1 : F(\lambda) = G(\lambda)\}$ and suppose that $\bar\lambda < \infty$. By the definition of $\bar\lambda$ and (A.7), we have $(F - G) < 0$ on $[1, \bar\lambda)$ and $(F - G)(\bar\lambda) = 0$. This implies that

$$q_0 \ge G'(\bar\lambda) \tag{A.8}$$

for some $q_0 \in \partial F(\bar\lambda)$. On the other hand, combining (A.5) and (A.6) implies $0 > \bar\lambda q - \gamma F(\bar\lambda) = \bar\lambda q - \gamma G(\bar\lambda) = \bar\lambda q - \bar\lambda G'(\bar\lambda)$ for all $q \in \partial F(\bar\lambda)$. The latter is equivalent to $G'(\bar\lambda) > q$ and gives the required contradiction to (A.8).

Above, it is proved that there exist constants $\gamma < 1$ and $x_0 > 0$ such that we have $U_c(\lambda x) \le \lambda^\gamma U_c(x)$ for $x \ge x_0$ and $\lambda > 1$. This gives

$$U(x) \le U_c\left(\frac{x}{x_0} x_0\right) \le \left(\frac{x}{x_0}\right)^\gamma U_c(x_0) = U_c(x_0)\left(\frac{1}{x_0}\right)^\gamma x^\gamma$$

for $x \ge x_0$. Thus, choosing $k_1 := U(x_0)$ and $k_2 := U_c(x_0)\,(1/x_0)^\gamma$ gives $U(x) \le k_1 + k_2 x^\gamma$ that is the desired result. □

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