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REAL OPTIONS WITH COMPETITION AND REGIME SWITCHING

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In this paper, we examine irreversible investment decisions in duopoly games with a variable economic climate. Integrating timing flexibility, competition, and changes in the economic environment in the form of a cash flow process with regime switching, the problem is formulated as a stopping-time game under Stackelberg leader-follower competition, in which both players determine their respective optimal market entry time. By extending the variational inequality approach, we solve for the free boundaries and obtain optimal investment strategies for each player. Despite the lack of regularity in the leader's obstacle and the cash flow regime uncertainty, the regime-dependent optimal policies for both the leader and the follower are obtained. In addition, we perform comprehensive numerical experiments to demonstrate the properties of solutions and to gain insights into the implications of regime switching.

KEY WORDS: variational inequality, irreversible investment, real option, regime switching, game theory, optimal stopping problem.

1. INTRODUCTION

By combining real options, game theory, and a regime-switching formulation, this paper focuses on an irreversible investment problem with Stackelberg leader-follower competition and market regime changes. The duopoly game imparts our analysis with the real-world characteristic that actions by one firm may affect the decisions of its competitors. We introduce a Markov regime-switching process designed to incorporate the impact of changes in market conditions.

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The term real options was coined by Myers (1977) in a model of firm debt capacity that accounted for future capital investment opportunities available to the firm, but yet to be realized. His idea led to a growing literature extending the concept of financial options into the realm of capital budgeting under uncertainty. One important offshoot of this modeling deals with investment opportunities that occur in a competitive environment. Valuation models developed along this line are often referred to as real options game models. These models have been explored at length by Smets (1991), Grenadier (1996), Weeds (2002), Lambrecht and Perraudin (2003), Paxson and Pinto (2005), Kong and Kwok (2007), Bensoussan, Diltz, and Hoe (2010), among others. Extensive reviews can be found in Smit and Trigeorgis (2004, 2006), Dias and Teixeira (2010), and Azevedo and Paxson (2011).

Classical real options models and real options game models have improved decision making in capital investment projects and scenarios. However, they are also limited in scope, primarily by assuming that key factors governing the underlying resource dynamics are constant. The assumption makes them unable to account for changes in market conditions and the overall economic environment. Extant empirical research suggests that market conditions affect operating cash flows. It is therefore important to develop models that incorporate market cycles and the overall economic climate. Similar to Sotomayor and Cadenillas (2009) and Bensoussan, Yan, and Yin (2012), we incorporate a regime-switching model to capture market and economic effects.

Despite the extensive literature in both real options and game theory, regime-switching models in these contexts have received only limited attention. We attempt to fill this gap by integrating real options, Stackelberg competition, and regime switching in a unified framework. Our primary interest is to determine whether investment timing decisions are affected by this integration. Both players decide optimally to invest over an infinite horizon subject to the constraint that the follower is forbidden to enter until the leader has done so. While both firms attempt to maximize their profits, each firm's decision is informed by rational anticipation of the other's optimal response. The follower's problem is similar to the single decision maker (i.e., monopoly) situation; see Bensoussan et al. (2012). The leader's problem is complicated by the fact that he must integrate the consequence of the rational follower's cut into the leader's payoff, once the follower has entered. The entire problem reduces to two distinct optimal stopping problems to be solved by using the variational inequality (V.I.) approach.

We encounter two technical challenges. First, it is the nature of Stackelberg competition that the leader's reward function (i.e., the obstacle) is nonsmooth at the point of the follower's entry, regardless of the regime. This presents a challenge because the regularity of the obstacle is a major element in defining the optimal strategy. The second challenge is to characterize the shape of the optimal stopping region. Numerous studies have explored characteristics of the optimal stopping region in the absence of regime switching; see, for example, Dai and Kwok (2006), Villeneuve (2007), Carmona and Touzi (2008), and Liang, Yang, and Jiang (2013), among others. Unfortunately, regime switching substantially complicates matters because a system of equations (as opposed to a single equation) must be solved in the optimization problem.

Our work makes four significant contributions. First, our work explicitly formulates the leader's payoff function in Stackelberg competition under regime-switching uncertainty. We face technical complexities not present in either Bensoussan et al. (2012) due to the lack of competition, or in Bensoussan et al. (2010) because of the lack of regime changes.

Second, we rigorously prove the existence and uniqueness of the leader's optimal strategy by overcoming technical difficulties involving the nonsmooth obstacle in the leader's optimal stopping problem. Because the leader's payoff function is always nonsmooth at the follower's optimal entry, the leader's V.I. contains an obstacle function that is in C^0, but not in C^1. Proofs of existence and uniqueness of a pair of optimal solutions are particular challenges, because a system of equations has to be dealt with. The challenge here is to find suitable *a priori* estimates. We are able to prove that the leader's solution is indeed C^1. Thus, the V.I. admits a strong solution.

Third, we prove that the optimal strategy (i.e., the V.I. solution) is not of the threshold type, but is rather a two-interval solution characterized by a unique regime-dependent triple. This characteristic is driven by the nonsmooth obstacle. The interval solution is a consequence of the Stackelberg leader's role, in which he must incorporate the follower's actions in his optimal strategy. The two distinct interval solutions correspond to the two states of nature. The properties of solutions are demonstrated in our numerical example. The form of two-interval solution has appeared in Dixit and Pindyck (1994); it was proved in Bensoussan et al. (2010) for simpler models.

Finally, we demonstrate the significance of regime switching on optimal investment policies. Our numerical results demonstrate the myopathy of decisions based on models with isolated states. Despite the myopathy of decisions made using simpler models, two extant results are not affected by regime switching. That is, increased volatility shortens the leader's second investment interval, and both the leader's and the follower's thresholds are linearly dependent on the investment cost, K.

The remainder of the paper is organized as follows. Section 2 presents an overview of the model. Section 3 considers the follower's problem and solution. Section 4 deals with the leader's problem and solution. Our focus is on proving the existence and uniqueness of the optimal investment strategies, and on delineating the shape of the optimal rules. Section 5 reports the results from extensive numerical examples, and Section 6 provides concluding remarks. A technical appendix is included at the end of the paper.

2. MODEL OF STACKELBERG COMPETITION IN MARKET ENTRY

Two firms contemplate the same irreversible capital investment proposal under uncertainty. The proposal costs K when undertaken at the "market entry" time, and it generates an operating cash flow stream. One firm, the leader, invests strictly before its rival, the follower. The follower has no means of adopting a non-Stackelberg follower action.

The firms' roles are exogenously determined. After entry, the leader receives δ_1 portion of market share until the follower enters. At that point, each firm receives δ_2 portion of the market. We assume $\delta_1 > \delta_2$ so that a "first mover" advantage exists and subsequently vanishes, along with the corresponding profit advantage, when the follower enters.

Project cash flows depend on market conditions, assumed to be either "good" or "bad" depending on the state of nature. To reflect such a state-dependent cash flow, we let the project cash flow $Y(t)$ follow a geometric Brownian motion modulated by a two-state Markov chain:

$$\mathrm{d}Y_t = \alpha(z(t))Y_t\mathrm{d}t + \sigma(z(t))Y_t\mathrm{d}W_t, \quad Y(0) = y, \quad z(0) = i, \tag{2.1}$$

where W_t is a standard Brownian motion, $z(t) \in \mathcal{M} = \{0, 1\}$ is a two-state continuous-time Markov chain with 0 identifying the "good" state and 1 identifying the "bad" one. For simplicity, we use the notation $\alpha(i) = \alpha_i$ and $\sigma(i) = \sigma_i$, respectively. Here, W_t and $z(t)$ are defined on $(\Omega, \mathcal{F}, P)$ and are independent. The generator of the Markov chain takes the form

$$Q = (q_{ij}) = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix},$$

with $\lambda_0, \lambda_1 > 0$.

The leader and the follower determine their respective optimal entry times by anticipating their rival's optimal response. Let θ and τ denote the leader's and the follower's stopping times with respect to the filtration $\mathcal{F}_t = \sigma\{W(s), z(s) : s \leq t\}$, respectively, with $\tau > \theta$.

REMARK 2.1. In our model, the state of nature is completely observable to entrepreneurs. Under certain circumstances, the state of economy can only be partially revealed to entrepreneurs. However, to understand the partially observed systems, one must have a thorough understanding on the fully observable case. Thus, this paper is a necessary step prior to approaching partially observable cases. We will consider the partial information case in our future works.

3. FOLLOWER'S PROBLEM AND OPTIMAL POLICY

As usual in control systems context, we solve the stopping time backwards. After the leader has already invested, the follower can choose his optimal stopping time as if he were a monopolist. Therefore, the follower's problem closely resembles a single firm's investment problem, which is a classical real options problem.

Suppose that the follower makes an investment at time τ by paying cost K, and he will receive a perpetual cash flow $\delta_2 Y(t)$ per unit time. This cash flow will depend on the condition at time τ; namely, the project value from the operation is the expected discounted cash flow stream given by

$$\delta_2 E\left[\int_\tau^\infty e^{-\mu(s-\tau)} Y_y(s)\mathrm{d}s|\mathcal{F}_\tau\right] = \delta_2 a_{z(\tau)} Y(\tau), \tag{3.1}$$

where $a_i(\mu + \lambda_i - \alpha_i) - \lambda_i a_{1-i} = 1$, for $i = 0, 1$; see Bensoussan et al. (2012) for details.

In (3.1), the discount factor $\mu > \alpha_i$ is exogenous as a part of the objective function. The expected discounted payoff from the capital investment project undertaken at time τ is

$$J_{y,i}(\tau) = E\left[e^{-\mu\tau}(\delta_2 a_{z(\tau)} Y(\tau)) - K)\mathbb{I}_{\tau<\infty} | Y(0) = y, z(0) = i\right]. \tag{3.2}$$

The firm's objective is to find an optimal stopping time to maximize the expected discounted payoff

$$g_i(y) = \sup_{\tau \geq 0} J_{y,i}(\tau). \tag{3.3}$$

Setting $v_i(y) = g_i(y) - \delta_2 a_i y + K$, as a consequence of dynamic programming, for $i \in \mathcal{M}$, we write the V.I. in the strong sense that $v_i(y)$ must satisfy

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2 v_i'' - \alpha_i y v_i' + (\lambda_i + \mu)v_i - \lambda_i v_{1-i} + \delta_2 y - \mu K \geq 0, \\ v_i \geq 0, \\ v_i\Big(-\frac{1}{2}y^2\sigma_i^2 v_i'' - \alpha_i y v_i' + (\lambda_i + \mu)v_i - \lambda_i v_{1-i} + \delta_2 y - \mu K\Big) = 0, \\ v_i(0) = K. \end{cases} \tag{3.4}$$

The V.I. (3.4) has the following financial meanings (see Shreve 2004): (1) When the first (second) inequality holds as an equality, the firm waits (invests). (2) The complementary slackness condition (the third equality) states that at least one of the inequalities holds as an equality. For more information about the connection between the optimal stopping problem and V.I., we refer to Bensoussan and Lions (1982), and Øksendal (2005).

From Bensoussan et al. (2012), for each $i \in \mathcal{M}$, there exists a unique $v_i(y) \in C^1(0, \infty)$ and $v_i'' \in L^\infty(0, \infty)$, the solution of (3.4), satisfies

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2 v_i'' - \alpha_i y v_i' + (\lambda_i + \mu)v_i - \lambda_i v_{1-i} + \delta_2 y - \mu K = 0 \quad 0 < y < \widehat{y}_i, \\ v_i(0) = K, \quad v_i(y) = 0 \quad y \geq \widehat{y}_i, \end{cases} \tag{3.5}$$

with $\hat{y}_i$ uniquely defined. Let $\widehat{\tau}_{y,i}$ be the optimal stopping time of the follower which corresponds to the thresholds $\widehat{y}_0$ and $\widehat{y}_1$, that is

$$\widehat{\tau}_{y,i} = \tau_{y,i}(\widehat{y}_0, \widehat{y}_1).$$

Therefore, the follower will undertake the investment (i.e., enter into the market) when the cash flow exceeds a certain threshold level. As the follower can enter only after the leader's entry time θ, the follower will enter at time

$$\widehat{\tau}_\theta = \theta + \widehat{\tau}_{Y(\theta),z(\theta)}. \tag{3.6}$$

4. THE LEADER'S PROBLEM AND OPTIMAL POLICY

We now turn to the leader's problem and optimal market entry policy after obtaining the follower's best response.

4.1. The Leader's Problem

The leader makes his optimal entry decision in anticipation of the rational follower's optimal entry at time $\widehat{\tau}_\theta$. When the leader enters at time $\theta < \infty$, by paying cost K, he receives a continuous cash flow $\delta_1 Y(t)$ per unit time prior to the follower's entry and gets a continuous cash flow $\delta_2 Y(t)$ per unit time after the follower's entry.

The leader's expected discounted payoff at θ is

$$
\begin{aligned}
&-K + \delta_1 E\left[\int_\theta^{\hat{\tau}_\theta} e^{-\mu(s-\theta)} Y(s)\mathrm{d}s|\mathcal{F}_\theta\right] + \delta_2 E\left[\mathbb{I}_{\hat{\tau}_\theta<\infty}\int_{\hat{\tau}_\theta}^\infty e^{-\mu(s-\theta)} Y(s)ds|\mathcal{F}_\theta\right]\\
&= -K + \delta_1 E\left[\textstyle\int_\theta^\infty e^{-\mu(s-\theta)} Y(s)ds|\mathcal{F}_\theta\right] + (\delta_2-\delta_1)E\left[\mathbb{I}_{\hat{\tau}_\theta<\infty}\textstyle\int_{\hat{\tau}_\theta}^\infty e^{-\mu(s-\theta)} Y(s)ds|\mathcal{F}_\theta\right]\\
&= -K + \delta_1 a_{z(\theta)} Y(\theta) + (\delta_2-\delta_1)\mathbb{I}_{Y(\theta)\ge\widehat{y}_{z(\theta)}} a_{z(\theta)} Y(\theta)\\
&\quad + (\delta_2-\delta_1)\mathbb{I}_{Y(\theta)<\widehat{y}_{z(\theta)}} E\left[e^{-\mu(\hat{\tau}_\theta-\theta)} a_{z(\hat{\tau}_\theta)} Y(\hat{\tau}_\theta)|\mathcal{F}_\theta\right].
\end{aligned}
$$

We can calculate the conditional expectation in the last equality explicitly; that is, we can write

$$
E\left[e^{-\mu(\hat{\tau}_\theta-\theta)} a_{z(\hat{\tau}_\theta)} Y(\hat{\tau}_\theta)|\mathcal{F}_\theta\right] = \varphi_{z(\theta)}(Y(\theta)). \tag{4.1}
$$

where $\varphi_i \in C^1(0,\widehat{y}_i)$, $\varphi_i'' \in L^\infty(0,\widehat{y}_i)$. The details are provided in the Appendix, in which we define explicitly the function $\varphi_i(y)$.

Using (4.1), the leader's expected discounted payoff at θ can then be written as

$$
\begin{aligned}
&-K + \delta_1 a_{z(\theta)} Y(\theta) + (\delta_2-\delta_1)\mathbb{I}_{Y(\theta)\ge\widehat{y}_{z(\theta)}} a_{z(\theta)} Y(\theta) + (\delta_2-\delta_1)\mathbb{I}_{Y(\theta)<\widehat{y}_{z(\theta)}}\varphi_{z(\theta)}(Y(\theta))\\
&\quad = -K + \delta_2 a_{z(\theta)} Y(\theta) + (\delta_1-\delta_2)\mathbb{I}_{Y(\theta)<\widehat{y}_{z(\theta)}}\left(a_{z(\theta)} Y(\theta) - \varphi_{z(\theta)}(Y(\theta))\right).
\end{aligned}
$$

To facilitate presentation, define

$$
\Psi_i(y) = -K + \delta_2 a_i y + (\delta_1-\delta_2)\mathbb{I}_{y<\widehat{y}_i}(a_i y - \varphi_i(y)), \tag{4.2}
$$

Using the above notations, the leader's objective of finding a stopping time, θ, to maximize his expected discounted payoffs can be written as

$$
\Gamma_i(y) = \sup_\theta E_{y,i}\left[e^{-\mu\theta}\Psi_{z(\theta)}(Y(\theta))\right]. \tag{4.3}
$$

Clearly, the leader's obstacle, $\Psi_i(y)$, is continuous but its derivative is discontinuous in $\widehat{y}_i$.

4.2. The Leader's Optimal Policy

Once successfully formulating the leader's objective function given the rational follower's optimal entry, we proceed to study the leader's optimal solution in two steps. We first establish the uniqueness and existence of the leader's optimal solution in Section 4.2.1, and then obtain the leader's optimal policy as a two-interval solution characterized by a unique triple in Section 4.2.2.

4.2.1. Uniqueness and existence of leader's optimal policy. As, from (4.2), the obstacle is not continuously differentiable, we may expect some similar lack of smoothness for the solution, thus encountering difficulties in writing the V.I. in a strong sense. However, thanks to the fact that the lack of continuity of the derivative occurs only at one point, it will turn out that the value function enjoys more smoothness than the obstacle. So we will write the V.I. in the strong sense as follows:

$$
\begin{cases}
-\frac{1}{2}y^2\sigma_i^2\Gamma_i'' - \alpha_i y\Gamma_i' + (\lambda_i + \mu)\Gamma_i - \lambda_i\Gamma_{1-i} \geq 0, \\
\Gamma_i \geq \Psi_i(y), \\
(\Gamma_i - \Psi_i(y))\left(-\frac{1}{2}y^2\sigma_i^2\Gamma_i'' - \alpha_i y\Gamma_i' + (\lambda_i + \mu)\Gamma_i - \lambda_i\Gamma_{1-i}\right) = 0, \\
\Gamma_i(0) = 0, \\
\Gamma_i \in C^1, \quad \Gamma_i'' \in L^\infty_{loc}.
\end{cases} \tag{4.4}
$$

Let

$$
\begin{aligned}
u_i &= \Gamma_i(y) - \delta_2 a_i y + K, \\
m_i &= \Psi_i(y) - \delta_2 a_i y + K.
\end{aligned}
$$

Then (4.4) can be rewritten as

$$
\begin{cases}
-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta_2 y - \mu K \geq 0, \\
u_i \geq m_i, \\
(u_i - m_i)\left(-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta_2 y - \mu K\right) = 0, \\
u_i(0) = K, \quad u_i \in C^1,
\end{cases} \tag{4.5}
$$

where the function $m_i = (\delta_1 - \delta_2)\mathbb{I}_{y<\widehat{y}_i}(a_i y - \varphi_i(y))$ satisfies

$$
\begin{cases}
-\frac{1}{2}y^2\sigma_i^2 m_i'' - \alpha_i y m_i' + (\lambda_i + \mu)m_i - \lambda_i m_{1-i} = (\delta_1 - \delta_2)y \quad 0 < y < \widehat{y}_i, \\
m_i(0) = 0, \quad m_i(\widehat{y}_i) = 0 \\
m_i(y) = 0, \quad y \geq \widehat{y}_i.
\end{cases} \tag{4.6}
$$

THEOREM 4.1. *There exists a unique pair* (u_0, u_1) *such that* $u_i \in C^1(0, \infty)$ *and* $u_i'' \in L^\infty(0, \infty)$ *for* $i = 0, 1$ *and that the pair is the solution of system* (4.5).

Proof. The proof is divided into three steps.

Step 1: We begin by considering (4.5) on a finite interval $(0, L)$, where $L > \frac{\mu K}{\delta_1}$. Our goal is to show that (4.5) has a unique solution on this interval. □

LEMMA 4.2. *For* $L > 0$ *and* $y \in (0, L)$*, consider the system of variational inequalities*

$$
\begin{cases}
-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta_2 y - \mu K \geq 0, \\
u_i \geq m_i, \\
(u_i - m_i)\left(-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta_2 y - \mu K\right) = 0, \\
u_i(0) = K, \quad u_i(L) = 0, \quad u_i \in C^1.
\end{cases} \tag{4.7}
$$

where m_i satisfies (4.6). *Then, there exists a unique solution of* (4.7). *Moreover,* $0 \le u_i \le M$, *where M is a constant given by*

$$M = \max(\|m_0\|_\infty, \|m_1\|_\infty, K), \tag{4.8}$$

with the norm $\|h\|_\infty = \sup_x |h(x)|$.

Proof of Lemma 4.2. We start proving the existence of the solution based on the idea of penalization method; see Bensoussan and Lions (1982), and Kinderlehrer and Stampacchia (1980). That is, we first approximate the V.I. by penalized equations. Then we show that their solutions converge to the solution of the V.I. For $\varepsilon > 0$, we seek $u_i^\varepsilon \in C^2(0, L)$ such that

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2\left(u_i^\varepsilon\right)'' - \alpha_i y\left(u_i^\varepsilon\right)' + (\lambda_i + \mu)u_i^\varepsilon = \lambda_i u_{1-i}^\varepsilon - \delta_2 y + \mu K + \dfrac{1}{\varepsilon}\left(u_i^\varepsilon - m_i\right)^-, \\ u_i^\varepsilon(0) = K, \quad u_i^\varepsilon(L) = 0. \end{cases} \tag{4.9}$$

Note that $K - \frac{\delta_2 L}{\mu}$ and M are subsolution and supersolution of problem (4.9), respectively. Hence,

$$K - \frac{\delta_2 L}{\mu} \le u_i^\varepsilon \le M. \tag{4.10}$$

We apply the fixed point theorem to prove the existence of the solution of problem (4.9). Denote

$$D = \{v \in C^2(0, L) | K - \frac{\delta_2 L}{\mu} \le v \le M\}.$$

For any $v_i \in D$, define a mapping $\eta_i = T(v_i)$, where η_i is the solution of the following nonlinear problem

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2\eta_i'' - \alpha_i y\eta_i' + \left(\lambda_i + \mu + \dfrac{1}{\varepsilon}\right)\eta_i = \lambda_i\eta_{1-i} - \delta_2 y + \mu K + \dfrac{1}{\varepsilon}v_i + \dfrac{1}{\varepsilon}(v_i - m_i)^-, \\ \eta_i(0) = K, \quad \eta_i(L) = 0. \end{cases} \tag{4.11}$$

For any v_i^1 and v_i^2, we define $\widetilde{v}_i = v_i^1 - v_i^2$ and $\widetilde{\eta}_i = \eta_i^1 - \eta_i^2$. Then we have

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2\widetilde{\eta}_i'' - \alpha_i y\widetilde{\eta}_i' + \left(\lambda_i + \mu + \dfrac{1}{\varepsilon}\right)\widetilde{\eta}_i \\ \quad = \lambda_i\widetilde{\eta}_{1-i} + \dfrac{1}{\varepsilon}\widetilde{v}_i + \dfrac{1}{\varepsilon}\left(v_i^1 - m_i\right)^- \dfrac{1}{\varepsilon}\left(v_i^2 - m_i\right)^-, \\ \widetilde{\eta}_i(0) = K, \quad \widetilde{\eta}_i(L) = 0. \end{cases} \tag{4.12}$$

The first equality in (4.12) satisfies

$$\begin{aligned} &-\frac{1}{2}y^2\sigma_i^2\widetilde{\eta}_i'' - \alpha_i y\widetilde{\eta}_i' + (\lambda_i + \mu)\widetilde{\eta}_i + \frac{1}{\varepsilon}\widetilde{\eta}_i \\ &= \lambda_i\widetilde{\eta}_{1-i} + \frac{1}{\varepsilon}\widetilde{v}_i + \frac{1}{\varepsilon}\left(v_i^1 - m_i\right)^- \frac{1}{\varepsilon}\left(v_i^2 - m_i\right)^- \\ &\le \lambda_i\widetilde{\eta}_{1-i} + \frac{1}{\varepsilon}\widetilde{v}_i + \frac{1}{\varepsilon}\widetilde{v}_i^- = \lambda_i\widetilde{\eta}_{1-i} + \frac{1}{\varepsilon}\widetilde{v}_i^+ \\ &\le \lambda_i\widetilde{\eta}_{1-i} + \frac{1}{\varepsilon}\max_i \|\widetilde{v}_i\|_\infty, \end{aligned} \tag{4.13}$$

and

$$
\begin{aligned}
(4.14)\qquad & -\frac{1}{2}y^2\sigma_i^2\widetilde{\eta}_i'' - \alpha_i y\widetilde{\eta}_i' + (\lambda_i+\mu)\widetilde{\eta}_i + \frac{1}{\varepsilon}\widetilde{\eta}_i \\
&= \lambda_i\widetilde{\eta}_{1-i} + \frac{1}{\varepsilon}\widetilde{v}_i + \frac{1}{\varepsilon}\left(v_i^1 - m_i\right)^- - \frac{1}{\varepsilon}\left(v_i^2 - m_i\right)^- \\
&= \lambda_i\widetilde{\eta}_{1-i} + \frac{1}{\varepsilon}\left(v_i^1 - m_i\right)^+ - \frac{1}{\varepsilon}\left(v_i^2 - m_i\right)^+ \\
&\geq \lambda_i\widetilde{\eta}_{1-i} - \frac{1}{\varepsilon}\widetilde{v}_i^- \geq \lambda_i\widetilde{\eta}_{1-i} - \frac{1}{\varepsilon}\max_i\|\widetilde{v}_i\|_\infty.
\end{aligned}
$$

Consider a point y^* such that

$$\max_{i=0,1}\max_{0\leq y\leq L}\widetilde{\eta}_i(y) = \widetilde{\eta}_{i^*}(y^*) > 0.$$

Rewriting (4.13) with $i = i^*$ and $y = y^*$ yields

$$-\frac{1}{2}(y^*)^2\sigma_{i^*}^2\widetilde{\eta}_{i^*}'' - \alpha_i y^*\widetilde{\eta}_{i^*}' + \left(\lambda_{i^*} + \mu + \frac{1}{\varepsilon}\right)\widetilde{\eta}_{i^*} \leq \lambda_{i^*}\widetilde{\eta}_{1-i^*} + \frac{1}{\varepsilon}\max_i\|\widetilde{v}_i\|_\infty,$$

and we have

$$\left(\mu + \frac{1}{\varepsilon}\right)\widetilde{\eta}_{i^*}(y^*) \leq \frac{1}{\varepsilon}\max_i\|\widetilde{v}_i\|_\infty \tag{4.15}$$

because $y^* \neq 0, L$, $\widetilde{\eta}_{i^*}(y^*) \geq \widetilde{\eta}_{1-i^*}(y^*)$, $\widetilde{\eta}_{i^*}'(y^*) = 0$, and $\widetilde{\eta}_{i^*}''(y^*) \leq 0$. Similarly, consider a point y_* such that

$$\min_{i=0,1}\min_{0\leq y\leq L}\widetilde{\eta}_i(y) = \widetilde{\eta}_{i_*}(y_*) < 0.$$

Then we get

$$\left(\mu + \frac{1}{\varepsilon}\right)\widetilde{\eta}_{i_*}(y_*) \geq -\frac{1}{\varepsilon}\max_i\|\widetilde{v}_i\|_\infty. \tag{4.16}$$

From (4.15) and (4.16), we arrive at

$$\max_i\|\widetilde{\eta}_i\|_\infty \leq \frac{1}{1+\varepsilon\mu}\max_i\|\widetilde{v}_i\|_\infty. \tag{4.17}$$

Let

$$v^j = \begin{pmatrix} v_0^j(y) \\ v_1^j(y) \end{pmatrix} \qquad j = 1, 2,$$

with norm $\|v^j\| = \max_i\|v_i^j\|_\infty$. Then (4.17) can be rewritten as

$$\|T(v^1) - T(v^2)\| \leq \frac{1}{1+\varepsilon\mu}\|v^1 - v^2\| = k\|v^1 - v^2\|,$$

where $k = \frac{1}{1+\varepsilon\mu} < 1$. By the contraction mapping theorem, there exists a unique pair u_i^ε which satisfies (4.9).

We next show that the limit of u_i^ε is the solution of (4.7) as $\varepsilon \to 0$. Denote $w_i^\varepsilon = u_i^\varepsilon - m_i$. Then (4.9) can be written as

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2 \left(w_i^\varepsilon\right)'' - \alpha_i y \left(w_i^\varepsilon\right)' + (\lambda_i + \mu)w_i^\varepsilon - \lambda_i w_{1-i}^\varepsilon = -\delta_1 y + \mu K + \frac{1}{\varepsilon}\left(w_i^\varepsilon\right)^-, \\ w_i^\varepsilon(0) = K, \quad w_i^\varepsilon(L) = 0. \end{cases} \tag{4.18}$$

Consider a point y^* such that

$$\min_{0\le y\le L}\min_{i=0,1} w_i^\varepsilon(y) = \min_{i=0,1} w_i^\varepsilon(y^*) = w_{i^*}^\varepsilon(y^*) < 0. \tag{4.19}$$

Then writing (4.18) with $i = i^*$, $y = y^*$, and noticing that $y^* \neq 0$, L, we have

$$\mu w_{i^*}^\varepsilon(y^*) \ge \mu K - \delta_1 y^* - \frac{1}{\varepsilon} w_{i^*}^\varepsilon(y^*) \ge \mu K - \delta_1 L - \frac{1}{\varepsilon} w_{i^*}^\varepsilon(y^*), \tag{4.20}$$

because

$$w_{i^*}^\varepsilon(y^*) \le w_{1-i^*}^\varepsilon(y^*), \quad \left(w_{i^*}^\varepsilon\right)'(y^*) = 0, \quad \left(w_{i^*}^\varepsilon\right)''(y^*) > 0.$$

From (4.19), (4.20) is equivalent to

$$-(w_{i^*}^\varepsilon)^-(y^*) \ge \frac{\varepsilon}{1+\mu\varepsilon}(\mu K - \delta_1 L).$$

Therefore,

$$\frac{1}{\varepsilon}\left(u_i^\varepsilon - m_i\right)^- \le \delta_1 L - \mu K. \tag{4.21}$$

From (4.9) and the estimates of (4.10) and (4.21), we obtain

$$\int_0^L y^2 \left(u_i^{\varepsilon\,\prime}(y)\right)^2 \mathrm{d}y \le C, \quad \int_0^L y^4 \left(u_i^{\varepsilon\,\prime\prime}(y)\right)^2 \mathrm{d}y \le C, \tag{4.22}$$

where C is a constant. Thus, we can extract from $u_i^\varepsilon(y)$ a subsequence using the same notation, such that

$$\begin{aligned} &u_i^\varepsilon \to u_i, && \text{in} \quad L^\infty(0, L) \text{ weak star}, \\ &y\left(u_i^\varepsilon\right)' \to y u_i', && \text{in} \quad L^2(0, L) \text{ weakly}, \\ &y^2\left(u_i^\varepsilon\right)'' \to y^2 u_i'', && \text{in} \quad L^2(0, L) \text{ weakly}. \end{aligned}$$

From the compactness of $H^1(0, L)$ in $L^2(0, L)$ and by the Rellich theorem (see Adams and Fournier 2003), we have

$$\begin{aligned} &y u_i^\varepsilon \to y u_i, && \text{in} \quad L^2(0, L) \text{ strongly}, \\ &y^2\left(u_i^\varepsilon\right)' \to y^2 u_i', && \text{in} \quad L^2(0, L) \text{ strongly}. \\ &u_i^\varepsilon \to u_i && \text{in} \quad L^2(0, L) \text{ strongly}. \end{aligned}$$

Passing to the limit in (4.21), we have $(u_i - m_i)^- \to 0$. therefore,

$$u_i \ge m_i, \tag{4.23}$$

which is the second inequality in (4.7). In addition, from (4.9) and passing to the limit, we obtain the first inequality in (4.7). Finally, from (4.9) and (4.23), we obtain

$$\left(-\frac{1}{2}y^2\sigma_i^2\left(u_i^\varepsilon\right)'' - \alpha_i y\left(u_i^\varepsilon\right)' + (\lambda_i+\mu)u_i^\varepsilon - \lambda_i u_{1-i}^\varepsilon + \delta_2 y - \mu K\right)\left(u_i^\varepsilon - m_i\right)$$
$$= \frac{1}{\varepsilon}\left(u_i^\varepsilon - m_i\right)^-\left(u_i^\varepsilon - m_i\right) \le 0,$$

and also

$$\left(-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i+\mu)u_i - \lambda_i u_{1-i} + \delta_2 y - \mu K\right)(u_i - m_i) \ge 0.$$

Therefore, we get the equality in (4.7). We can define $u_i(0) = K$, so $u_i^\varepsilon(y) \to u_i(y)$, for $y \in [0, L]$. Therefore, we have proved the existence of a solution of (4.7).

We now prove the uniqueness of the solution of (4.7). Let $v_i \in C^1$ with $0 \le v_i \le M$ (with M defined in (4.8)), $v_i \ge m_i$, $v_i(0) = K$, and $v_i(L) = 0$. Suppose u_i is a solution of (4.7), then we have

$$u_i - m_i + \xi(v_i - m_i - (u_i - m_i)) \ge 0, \quad \forall 0 < \xi < 1.$$

By the first inequality and the complementary slackness condition of (4.7), we have

$$\left(-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i+\mu)u_i - \lambda_i u_{1-i} + \delta_2 y - \mu K\right)(v_i - u_i) \ge 0. \tag{4.24}$$

Assuming there are two solutions of (4.7), u_i^1 and u_i^2, by taking $u_i = u_i^1$, $v_i = u_i^2$ and $u_i = u_i^2$, $v_i = u_i^1$ in (4.24), we obtain

$$\left(-\frac{1}{2}y^2\sigma_i^2\left(u_i^1\right)'' - \alpha_i y\left(u_i^1\right)' + (\lambda_i+\mu)u_i^1 - \lambda_i u_{1-i}^1 + \delta_2 y - \mu K\right)\left(u_i^2 - u_i^1\right) \ge 0,$$

and

$$\left(-\frac{1}{2}y^2\sigma_i^2\left(u_i^2\right)'' - \alpha_i y\left(u_i^2\right)' + (\lambda_i+\mu)u_i^2 - \lambda_i u_{1-i}^2 + \delta_2 y - \mu K\right)\left(u_i^1 - u_i^2\right) \ge 0.$$

Adding the two equations above and using notation $h_i = u_i^1 - u_i^2$, we arrive at

$$-\frac{1}{2}y^2\sigma_i^2 h_i'' h_i - \alpha_i y h_i' h_i + (\lambda_i+\mu)h_i^2 - \lambda_i h_{1-i} h_i \le 0. \tag{4.25}$$

Consider a point y^* such that

$$\max_{i=0,1}\max_{0\le y\le L} h_i(y) = h_{i^*}(y^*).$$

Suppose $h_{i^*}(y^*) > 0$. Rewriting (4.25) with $i = i^*$ and $y = y^*$ yields

$$-\frac{1}{2}(y^*)^2\sigma_{i^*}^2(h_{i^*})'' h_{i^*} - \alpha_{i^*} y^* h_{i^*}' h_{i^*} + (\lambda_{i^*}+\mu)h_{i^*}^2 - \lambda_{i^*} h_{1-i^*} h_{i^*} \ge 0 \tag{4.26}$$

because $y^* \ne 0, L$, $h_{i^*}(y^*) \ge h_{1-i^*}(y^*)$, $h_{i^*}'(y^*) = 0$, and $h_{i^*}''(y^*) \le 0$. Equation (4.26) contradicts (4.25), and thus $h_i(y) \le 0$, implying $u_i^1 \le u_i^2$. Similarly, we can rewrite (4.7) by taking $u_i = u_i^1$, $v_i = u_i^2$, and $u_i = u_i^1$, $v_i = u_i^2$, and we obtain $u_i^2 \le u_i^1$. Therefore, we can conclude that $u_i^1 = u_i^2$. This completes the proof of uniqueness of the solution.

Step 2: We prove $u_i(y)$ vanishes for y sufficiently large on a finite interval $(0, L)$, where L is large enough such that $\overline{y}_i < L$. This result can be stated as the following Lemma. □

LEMMA 4.3. *There exist a $\overline{y}_i$ such that $u_i(y)$ vanishes for $y \ge \overline{y}_i$, where $\overline{y}_i > \widehat{y}_i$ satisfying*

$$\frac{\delta_2}{\lambda_i + \mu - \alpha_i}(\lambda_i + \mu)\left(1 - \frac{1}{\beta_i}\right)\overline{y}_i \ge \mu K + \lambda_i M_1, \tag{4.27}$$

$$\frac{\delta_2}{\lambda_i + \mu - \alpha_i}(\lambda_i + \mu)\left(1 - \frac{1}{\beta_i}\right)\overline{y}_i \ge \delta_1 \widehat{y}_i + \lambda_i M_1, \tag{4.28}$$

$$\frac{\delta_2}{\lambda_i + \mu - \alpha_i}\left(1 - \frac{1}{\beta_i}\right)\overline{y}_i \ge K, \tag{4.29}$$

$$\mu K - \delta_2 \overline{y}_i < 0, \tag{4.30}$$

where M_1 is given by

$$M_1 = \max_i \max_{0 \le y \le L} \{\overline{u}_i(y)\}. \tag{4.31}$$

Proof of Lemma 4.3. Define

$$\overline{u}_i(y) = \begin{cases} -\dfrac{\delta_2}{\lambda_i + \mu - \alpha_i}\left[y - \overline{y}_i - \dfrac{\overline{y}_i}{\beta_i}\left(\left(\dfrac{y}{\overline{y}_i}\right)^{\beta_i} - 1\right)\right] & 0 \le y < \overline{y}_i 0 \quad y \ge \overline{y}_i \end{cases} \tag{4.32}$$

where β_i is given in the Appendix. The function $\overline{u}_i(y)$ is $C^1(0, L)$ and vanishes for $y \ge \overline{y}_i$.

In what follows, we check that $\overline{u}_i(y)$ is a ceiling function of $u_i(y)$. We can choose $\overline{y}_i > \widehat{y}_i$ and $\overline{y}_i > \overline{y}_{1-i}$ satisfying (4.27) and (4.30). Then for $0 \le y < \overline{y}_i$, we have

$$\begin{aligned}
&-\frac{1}{2}y^2\sigma_i^2\overline{u}_i'' - \alpha_i y \overline{u}_i' + (\lambda_i + \mu)\overline{u}_i \\
&\quad = -\delta_2 y + \frac{\delta_2}{\lambda_i + \mu - \alpha_i}(\lambda_i + \mu)\left(1 - \frac{1}{\beta_i}\right)\overline{y}_i \\
&\quad \ge -\delta_2 y + \mu K + \lambda_i M_1 \\
&\quad \ge -\delta_2 y + \mu K + \lambda_i \overline{u}_{1-i},
\end{aligned}$$

and for $\overline{y}_i < y < L$,

$$\begin{aligned}
&-\frac{1}{2}y^2\sigma_i^2\overline{u}_i'' - \alpha_i y \overline{u}_i' + (\lambda_i + \mu)\overline{u}_i - \lambda_i \overline{u}_{1-i} \\
&\quad = 0 \ge -\delta_2 \overline{y}_i + \mu K \\
&\quad \ge -\delta_2 y + \mu K.
\end{aligned}$$

From (4.28), for $0 < y < \widehat{y}_i$, we obtain

$$\begin{aligned}
-\frac{1}{2}&y^2\sigma_i^2\overline{u}_i'' - \alpha_i y\overline{u}_i' + (\lambda_i+\mu)\overline{u}_i \\
&= -\delta_2 y + \frac{\delta_2}{\lambda_i+\mu-\alpha_i}(\lambda_i+\mu)\left(1-\frac{1}{\beta_i}\right)\overline{y}_i \\
&\geq -\delta_2 y + \delta_1\widehat{y}_i + \lambda_i M_1 \\
&\geq -\delta_2 y + \delta_1 y + \lambda_i\overline{u}_{1-i}.
\end{aligned}$$

That is, $\overline{u}_i(y)$ is the supersolution of (4.6) on the interval $(0, \widehat{y}_i)$. Therefore $\overline{u}_i(y) \geq m_i(y)$ on $(0, \widehat{y}_i)$. Clearly, this is also true for $y \in [\widehat{y}_i, L]$. Hence, we arrived at $\overline{u}_i(y)$ satisfying all the inequalities in (4.7) on the interval $(0, L)$, and thus $\overline{u}_i(y) \geq u_i(y)$. From (4.29), $\overline{u}_i(0) = \frac{\delta_2}{\lambda_i+\mu-\alpha_i}(1-\frac{1}{\beta_i})\overline{y}_i \geq K = u_i(0)$. Therefore, $\overline{u}_i(y)$ is a ceiling function, and Lemma 4.3 has been proved.

Step 3: Consider $L \in (\overline{y}_i, \infty)$. As the function vanishes after $\overline{y}_i$, the infinite horizon problem becomes the same as the finite problem. The solution of (4.7) holds. Hence, (4.5) has a unique solution. This completes the proof of Theorem 4.1. □

4.2.2. Leader's two-interval solution. Due to the lack of smoothness of the obstacle, the leader's optimal stopping rule is characterized as a two-interval solution that is stated in the following theorem:

THEOREM 4.4. *The solution of V.I.* (4.5) *satisfies*

$$(4.33)\qquad \begin{cases}
-\dfrac{1}{2}y^2\sigma_i^2u_i'' - \alpha_i yu_i' + (\lambda_i+\mu)u_i - \lambda_i u_1 - i = -\delta_2 y + \mu K \quad y < y_i^1 \quad \textit{and} \\
\quad y_i^2 < y < y_i^3, \\
u_i(y) = m_i(y) \quad y_i^1 \leq y \leq y_i^2, \\
u_i(y) = 0 \quad y \geq y_i^3, \\
u_i'(y_i^1) = m_i'\left(y_i^1\right), \quad u_i'\left(y_i^2\right) = m_i'\left(y_i^2\right), \quad u_i'\left(y_i^3\right) = 0,
\end{cases}$$

where $m_i(y)$ satisfies (4.6)*, and $y_i^1 < y_i^2 < \hat{y}_i < y_i^3$ are uniquely determined.*

Proof. As $u_i(0) > m_i(0)$ and $u_i(\overline{y}_i) = m_i(\overline{y}_i) = 0$, there exists a y_i^1 such that y_i^1 is the first point satisfying $y_i^1 \leq \overline{y}_i$ so that $u_i(y_i^1) = m_i(y_i^1)$. We claim that the following lemma holds. □

LEMMA 4.5. *We must have $y_i^1 < \hat{y}_i$.*

Proof of Lemma 4.5. Suppose it were not true. We consider first $y_i^1 > \hat{y}_i$. Then, we have $y_i^1 = \overline{y}_i$, but then u coincides with the solution of (3.4), it follows that $y_i^1 = \hat{y}_i$ which is a contradiction. Next, if $y_i^1 = \hat{y}_i$, we denote $\widetilde{u}_i = u_i - m_i$. Then $\widetilde{u}_i$ satisfies

$$(4.34)\qquad \begin{cases}
-\dfrac{1}{2}y^2\sigma_i^2\widetilde{u}_i'' - \alpha_i y\widetilde{u}_i' + (\lambda_i+\mu)\widetilde{u}_i - \lambda_i\widetilde{u}_{1-i} = -\delta_1 y + \mu K, \\
\widetilde{u}_i(0) = K, \widetilde{u}_i(\hat{y}_i) = 0.
\end{cases}$$

From $u_i \in C^1$ and $u_i(\widehat{y}_i) = 0$, we have

$$\widetilde{u}_i'(\widehat{y}_i - 0) = -m_i'(\widehat{y}_i - 0) > 0,$$

hence, $\widetilde{u}_i'(y) < 0$ for $y < \widehat{y}_i$ close to $\widehat{y}_i$, which is impossible because it must be positive. Therefore, $y_i^1 < \widehat{y}_i$.

We also have $\delta_1 y_i^1 \geq \mu K$, otherwise, it would lead to a contradiction for $y < y_i^1$ close to y_i^1 by (4.34). Moreover, we note that

$$\widetilde{u}_i(0) = K, \quad \widetilde{u}_i\left(y_i^1\right) = 0, \quad \widetilde{u}_i'\left(y_i^1\right) = 0.$$

The matching of derivatives comes from the fact that $\widetilde{u}_i \in C^1$, $\widetilde{u}_i > 0$, and $\widetilde{u}_i(y_i^1) = 0$. Thus y_i^1 is a local minimum and hence $\widetilde{u}_i'(y_i^1) = 0$. □

Because $u_i(\hat{y}_i) > m_i(\hat{y}_i) = 0$, there exists an interval containing $\hat{y}_i$ such that the equation holds. We take y_i^2 the left end of this interval such that $u_i(y_i^2) = m_i(y_i^2)$ and y_i^3 the right end of this interval such that $y_i^3 = \overline{y}_i$. We have $y_i^1 < y_i^2 < \hat{y}_i$, otherwise, u_i would be the solution on the interval $(0, y_i^3)$, which is the case excluded from the beginning of the proof. On the interval $y \in (y_i^1, y_i^2)$, it necessarily holds that $u_i(y) = m_i(y)$. Indeed, m_i satisfies the second and third conditions in (4.5). Because $\delta_1 y_i^1 > \mu K$ and $y > y_i^1$, we have $\delta_1 y > \mu K$. It follows that $(\delta_1 - \delta_2)y > \mu K - \delta_2 y$. Hence, m_i also satisfies the first inequality. Therefore $u_i = m_i$ and $u_i' = m_i'$ on the interval of (y_i^1, y_i^2). By the uniqueness of u_i, it is necessary that y_i^1, y_i^2, y_i^3 are uniquely determined.

5. NUMERICAL EXAMPLES

A primary contribution of this research is to solve for optimal investment policies under Stackelberg competition with regime switching. In the numerical examples discussed in this section, we examine the case with two regimes. Regime 0 corresponds to a bull market, and regime 1 corresponds to a bear market. By using smooth-fit techniques, the optimal investment policies can be obtained by solving systems of nonlinear equations. The calculation for the follower's solution is similar to what was done in Bensoussan et al. (2012). However, the calculation for the leader's regime-dependent two-interval solution involves a much larger task because eight, instead of two, thresholds are used in calculation. As in the follower's case, the calculation requires specifying the order of trigger thresholds at first as systems of nonlinear equations solving the optimization problem are derived according to the threshold order. Therefore, for a complete calculation for the leader's optimal policies, one requires deriving systems of nonlinear equations arising from all possible orders of eight trigger thresholds, resulting in huge tasks. In addition, we encounter a major hurdle of implementing the aforementioned calculation in the numerical study. Due to the uniqueness property of solutions, for a given set of parameter values, we have to correctly conjecture the threshold order in order to derive accurate systems of nonlinear equations. To avoid this difficulty, we perform our numerical study through value iteration.[1] To illustrate the solution method using the smooth-fit techniques, we provide calculations and report results for the leader's value functions and investment thresholds for our "base case" scenario in the Appendix.

In the next subsection, we report numerical results for a "base case" scenario to illustrate the properties of solutions presented in Section 4.2.2. We then conclude this section with sensitivity analysis on volatility, drift rate, and investment cost.

[1] Comparing the results obtained from solving systems of nonlinear equations using smooth-fit techniques with those from value iterations shows that the differences between the two methods are all under 10^{-3}.

TABLE 5.1
Leader's and Follower's Thresholds under Different Regimes —Base Case

Regimes	Leader's Thresholds and Investment Decision	Follower's Threshold and Investment Decision
Regime 0	$(y_0^1, y_0^2, y_0^3) = (0.2934, 0.5426, 0.7142)$ Invest: $y \in [0.2934, 0.5426]$ or $y \in [0.7142, \infty)$	$\widehat{y}_0 = 0.5827$ Invest: $y \geq 0.5827$ when leader has invested
Regime 1	$(y_1^1, y_1^2, y_1^3) = (0.4863, 0.7806, 1.3343)$ Invest: $y \in [0.4863, 0.7806]$ or $y \in [1.3343, \infty)$	$\widehat{y}_1 = 0.9772$ Invest: $y \geq 0.9772$ when leader has invested

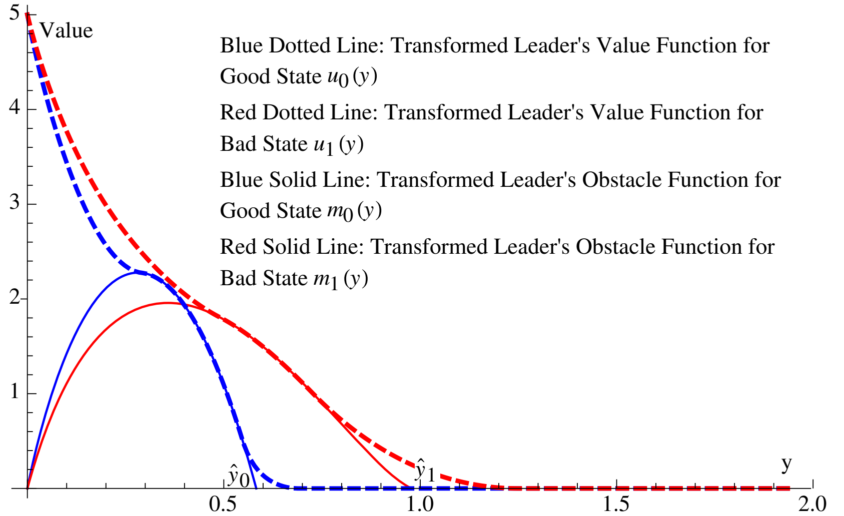

FIGURE 5.1. Leader's Value Function and Threshold—Base Case.

5.1. Base Case

Asset growth rates tend to be higher in a good state of the economy, while volatilities tend to run counter-cyclically. Accordingly, we set our base case parameters as follows: $\delta_1 = 2$, $\delta_2 = 1$, $\mu = 0.09$, $K = 5$, $(\alpha_0, \alpha_1) = (0.08, 0.01)$, $(\sigma_0, \sigma_1) = (0.1, 0.4)$, and $(\lambda_0, \lambda_1) = (0.15, 0.1)$. Table 5.1 reports the threshold values and investment decisions under different regimes. It reveals that the bull market thresholds are consistently smaller than those of the bear market.

Figure 5.1 illustrates the leader's value functions and obstacle functions under the two regimes. It confirms Theorem 4.4 and indicates that the leader's optimal investment policies are regime dependent two-interval solutions. The leader only invests in the interval $[y_i^1, y_i^2) \cup [y_i^3, \infty)$. Otherwise, the leader's transformed option value, $u_i(y)$, exceeds his transformed expected discounted payoff from investing immediately, $m_i(y)$. The two-interval solution results from the kink in the leader's obstacle at the point of the follower's entry. Figure 5.1 shows that bear market option valuation dominates bull

TABLE 5.2
Leader's and Follower's Thresholds under Different Regimes—$\sigma_0 = 0.10$

Volatility	Regime 0				Regime 1			
	Leader's and Follower's Thresholds				Leader's and Follower's Thresholds			
σ_1	y_0^1	y_0^2	y_0^3	$\widehat{y_0}$	y_1^1	y_1^2	y_1^3	$\widehat{y_1}$
0.10	0.2446	0.4594	0.5608	0.4920	0.2625	0.4803	0.6205	0.5265
0.15	0.2535	0.4674	0.5892	0.5041	0.2897	0.5102	0.7143	0.5810
0.25	0.2670	0.4935	0.6432	0.5330	0.3560	0.5987	0.9304	0.7140
0.35	0.2844	0.5238	0.6935	0.5656	0.4390	0.7084	1.1874	0.8805
0.40	0.2934	0.5426	0.7142	0.5827	0.4872	0.7806	1.3343	0.9772
0.45	0.3007	0.5585	0.7365	0.6000	0.5401	0.8531	1.4975	1.0832
0.55	0.3196	0.5966	0.7599	0.6351	0.6605	1.0298	1.8536	1.3245

TABLE 5.3
Leader's and Follower's Thresholds under Different Regimes—$\sigma_1 = 0.40$

Volatility	Regime 0				Regime 1			
	Leader's and Follower's Thresholds				Leader's and Follower's Thresholds			
σ_1	y_0^1	y_0^2	y_0^3	$\widehat{y_0}$	y_1^1	y_1^2	y_1^3	$\widehat{y_1}$
0.10	0.2934	0.5426	0.7142	0.5827	0.4872	0.7806	1.3343	0.9772
0.15	0.3118	0.5513	0.7796	0.6194	0.4900	0.7852	1.3420	0.9828
0.20	0.3310	0.5723	0.8528	0.6660	0.4937	0.7855	1.3521	0.9902
0.25	0.3623	0.6058	0.9454	0.7215	0.4982	0.7926	1.3700	0.9992
0.30	0.3947	0.6407	1.0411	0.7855	0.5034	0.8008	1.3843	1.0096
0.35	0.4282	0.6887	1.1516	0.8582	0.5093	0.8044	1.4005	1.0214
0.40	0.4689	0.7444	1.2779	0.9399	0.5158	0.8148	1.4244	1.0346

market option valuation everywhere. This result is expected for two reasons. First, the bear state has lower growth and higher volatility. Second, the probability that the state changes increases (decreases) the option value to wait for the bear (bull) state.

5.2. Sensitivity Analysis

Drift and volatility are the two variables most sensitive to market conditions. Therefore, we alter base case values for these variables to study their impacts on the optimal investment policies. We also allow investment cost to vary.

5.2.1. Volatility. Tables 5.2 and 5.3 report the effects of changing volatility one state at a time on the investment thresholds. The leader's investment thresholds in both states increase with the volatility of any state. Changes in volatility in a given state have a major impact on investment thresholds in the same state, and a lesser impact on the other state.

TABLE 5.4
Leader's First Investment Interval under Different Regimes—$\sigma_0 = 0.10$

Volatility	Regime 0	Regime 1
σ_1	Length of First Investment Interval	Length of First Investment Interval
0.10	0.2148	0.2178
0.15	0.2140	0.2205
0.25	0.2265	0.2427
0.35	0.2395	0.2694
0.40	0.2491	0.2934

TABLE 5.5
Leader's First Investment Interval under Different Regimes—$\sigma_1 = 0.40$

Volatility	Regime 0	Regime 1
σ_0	Length of First Investment Interval	Length of First Investment Interval
0.10	0.2491	0.2934
0.15	0.2394	0.2951
0.25	0.2434	0.2944
0.35	0.2604	0.2951
0.40	0.2755	0.2989

Tables 5.4 and 5.5 show how volatility changes affect the investment interval length. The length of the second interval $[y_i^3, \infty)$ decreases unambiguously. This observation is consistent with models without regime switching. The length of the first investment interval $[y_i^1, y_i^2]$ does not show a monotone increasing or decreasing relations as volatility increases.

5.2.2. Drift. Tables 5.6 and 5.7 report the results of changing drift rates in regime 1 while fixing the same volatility in both states. For comparisons, we study the case of a high volatility (Table 5.6) versus that of a low volatility (Table 5.7). Increases in the drift rate lower investment thresholds in both states, due to the possibility of shifts in market regimes. Interestingly, drift rate changes affect investment thresholds more significantly when the underlying volatility is large.

From the discussion of Section 5.2.1 and Section 5.2.2, the impact of input parameters on investment thresholds has an interactive effect as regimes switch. The ultimate impact is a combination of direct parameter impacts and the channeling effect from regime switching. This result highlights the importance of regime switching models.

5.2.3. Investment cost. Table 5.8 reports the impact of the investment cost on the investment thresholds. The leader's and the follower's investment thresholds increase

TABLE 5.6
Leader's and Follower's Thresholds under Different Regimes—$\sigma_0 = \sigma_1 = 0.40$, $\alpha_0 = 0.08$

Drift Rate	Regime 0: Leader's and Follower's Thresholds				Regime 1: Leader's and Follower's Thresholds			
α_1	y_0^1	y_0^2	y_0^3	$\widehat{y_0}$	y_1^1	y_1^2	y_1^3	$\widehat{y_1}$
0.01	0.4689	0.7444	1.2779	0.9399	0.5158	0.8148	1.4244	1.0346
0.02	0.4630	0.7374	1.2633	0.9296	0.5030	0.8003	1.3833	1.0089
0.03	0.4572	0.7304	1.2434	0.9195	0.4907	0.7806	1.3438	0.9841
0.04	0.4570	0.7236	1.2296	0.9098	0.4788	0.7617	1.3057	0.9603
0.05	0.4514	0.7169	1.2161	0.9004	0.4673	0.7435	1.2745	0.9373

TABLE 5.7
Leader's and Follower's Thresholds under Different Regimes—$\sigma_0 = \sigma_1 = 0.10$, $\alpha_0 = 0.08$

Drift Rate	Regime 0: Leader's and Follower's Thresholds				Regime 1: Leader's and Follower's Thresholds			
α_1	y_0^1	y_0^2	y_0^3	$\widehat{y_0}$	y_1^1	y_1^2	y_1^3	$\widehat{y_1}$
0.01	0.2446	0.4594	0.5608	0.4920	0.2625	0.4803	0.6205	0.5265
0.02	0.2446	0.4572	0.5475	0.4874	0.2563	0.4718	0.5970	0.5140
0.03	0.2428	0.4570	0.5398	0.4841	0.2513	0.4684	0.5770	0.5041
0.04	0.2418	0.4555	0.5342	0.4819	0.2474	0.4639	0.5624	0.4963
0.05	0.2388	0.4554	0.5303	0.4803	0.2443	0.4609	0.5498	0.4901

almost linearly with respect to investment cost. This observation is supported by Figure 5.2 where the leader's obstacles and value functions increase nearly linearly with investment cost.[2]

6. FURTHER REMARKS

We have developed optimal market entry decisions for the Stackelberg leader and follower competition, taking into consideration random shifts of underlying operating cash flows due to time-varying macroeconomic conditions. We have shown that the optimal policy of the follower is of the threshold type and that of the leader is the

[2]Under models without regime switching, the closed-form solution of the first threshold is given as $\frac{\delta_2}{\delta_1}\widehat{y}$ where δ_1, δ_2, and $\widehat{y}$ represent the market share before the follower's entry, the market share after the follower's entry, and the follower's optimal entry that is a multiple of the investment cost K. Therefore, the leader's first threshold increases linearly with respect to K.

TABLE 5.8
Leader's and Follower's Thresholds under Different Investment Cost

Investment Cost	Regime 0				Regime 1			
	Leader's and Follower's Thresholds				Leader's and Follower's Thresholds			
K	y_0^1	y_0^2	y_0^3	$\widehat{y}_0$	y_1^1	y_1^2	y_1^3	$\widehat{y}_1$
5	0.2934	0.5426	0.7142	0.5827	0.4872	0.7806	1.3343	0.9772
100	5.8686	10.8513	14.2839	11.6534	9.7441	15.6127	26.6854	19.5435
500	29.3429	54.2567	71.4196	58.2669	48.7203	78.0633	133.4270	97.7175
1000	58.6858	108.5134	142.8392	116.5339	97.4406	156.1266	266.8540	195.4350

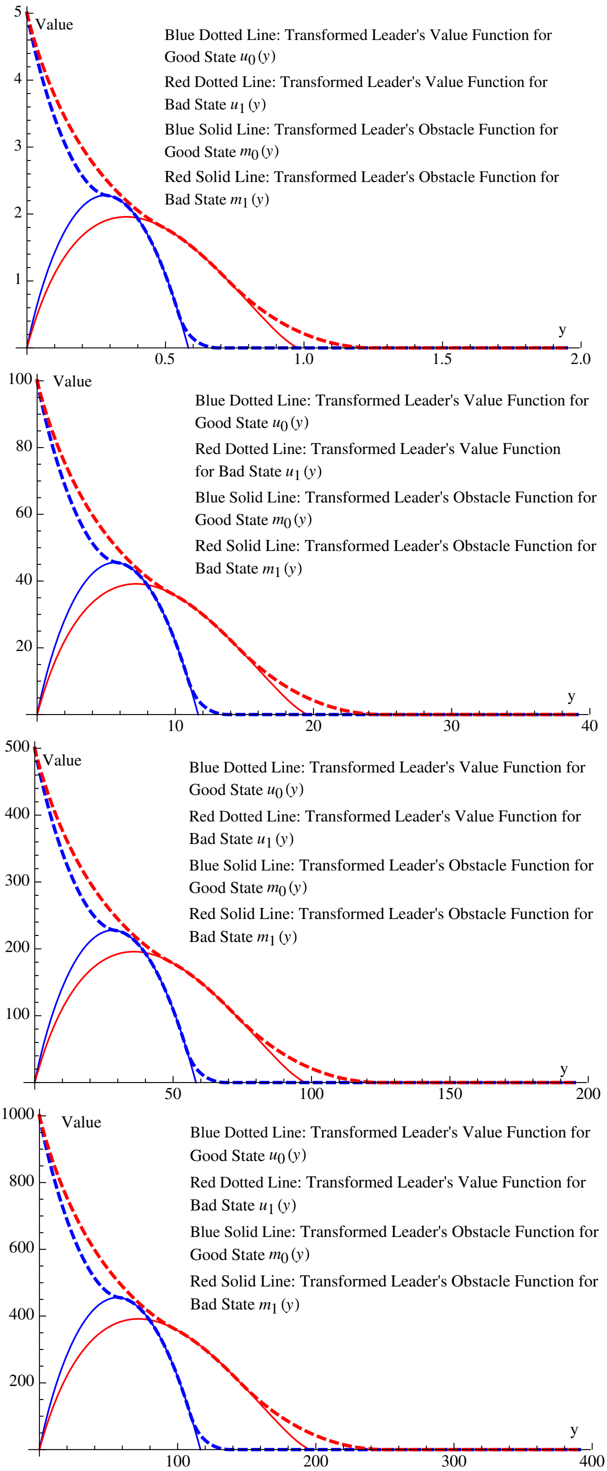

FIGURE 5.2. Leader's Value Function and Threshold—Investment Cost 5, 100, 500, 1000.

two-interval strategy characterized by a uniquely determined triple. When the cash flow is close to the follower's threshold, the leader's option value dominates his monopoly rent from earlier entry. Therefore, the leader will not enter into the market until the cash flow process achieves a sufficiently high level such that the immediate investment payoff from almost simultaneous entry of the follower dominates the option value to invest.

In the current model, only timing flexibility is considered, and the regime-switching intensities are constant. Our framework can be generalized and refined in the following ways.

- First, the current model can be generalized to accommodate different types of managerial flexibility to minimize the potential myopic decisions arising from the ignorance of potential sources of managerial flexibility.
- Second, it would be interesting to examine the case where the intensities of switching from one market mode to the other are affected by other factors. For example, Busch, Korn, and Seifried (2013) developed a large investor model to study the investor's consumption and investment decision where the regime switching intensities depend on the investor's decision. Elliott, Siu, and Badescu (2011) studied the pricing and hedging of European options where the asset price dynamics has a feedback effect on the jump rate. The characterization of the investment region in such a generalized regime-switching situation will be in our future work.
- Moreover, our framework and its generalization mentioned above can also be applied to other areas, where the system has different kinds of states. One typical example is the application to wind forecasting where the speed of wind has seasonal behavior.

APPENDIX: TECHNICAL COMPLEMENTS

Definition of function $\varphi_i(y)$ in (4.1). Let η_0, η_1 be given. We consider

$$\tau_{y,i}(\eta_0, \eta_1) = \inf\{t : Y(t) \geq \eta_{z(t)}\}.$$

Clearly, if $y \geq \eta_i$, then $\tau_{y,i}(\eta_0, \eta_1) = 0$. Denote

$$\varphi_i(y) = E[e^{-\mu\tau_{y,i}} G_{z(\tau_{y,i})}(Y_{\tau_{y,i}})\mathbb{I}_{\tau_{y,i}<\infty}], \tag{A.1}$$

if $y \geq \eta_i$, $\varphi_i(y) = G_i(y) = a_i y$. The $\varphi_i(y)$ satisfies

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2\varphi_i'' - \alpha_i y\varphi_i' + (\lambda_i + \mu)\varphi_i - \lambda_i\varphi_{1-i} = 0, \quad 0 < y < \eta_i \\ \varphi_i(y) = G_i(y), \quad \eta_i \leq y \leq \max(\eta_0, \eta_1) \\ \varphi_i \in C^1(0, \eta_i), \quad \varphi_i'' \in L^\infty(0, \eta_i). \end{cases} \tag{A.2}$$

Assume $\eta_0 < \eta_1$, then

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2\varphi_i'' - \alpha_i y\varphi_i' + (\lambda_i + \mu)\varphi_i - \lambda_i\varphi_{1-i} = 0, \quad 0 < y < \eta_i, \\ \varphi_0(\eta_0) = G_0(\eta_0), \quad \varphi_0(\eta_1) = G_1(\eta_1), \\ \varphi_0 \in C^1(0, \eta_1), \quad \varphi_1 \in C^2(0, \eta_1), \quad \varphi_0'' \in L^\infty(0, \eta_1). \end{cases} \tag{A.3}$$

Set

$$K_i(\beta) = -\frac{1}{2}\sigma_i^2\beta(\beta-1) - \alpha_i\beta + \lambda_i + \mu, \quad u > \alpha_i$$

and let $\beta_i > 1$ and $\beta_i^* < 0$ be the roots of the equation $K_i(\beta) = 0$.

$$K_0(\beta)K_1(\beta) = \lambda_0\lambda_1 \tag{A.4}$$

Consider the positive roots $\overline{\beta}_0, \overline{\beta_1}$ of the function (A.4), with $\overline{\beta}_0 < \overline{\beta}_1$. As

$$\begin{aligned}
&K_0(\beta_i)K_1(\beta_i) = 0 < \lambda_0\lambda_1,\\
&K_0(0)K_1(0) = (\lambda_0+\mu)(\lambda_1+\mu) > \lambda_0\lambda_1,\\
&K_0(1)K_1(1) = (\lambda_0+\mu-\alpha_0)(\lambda_1+\mu-\alpha_1) > \lambda_0\lambda_1,
\end{aligned}$$

we have

$$1 < \overline{\beta}_0 < \beta_i < \overline{\beta}_1, i = 0, 1. \tag{A.5}$$

We write

$$\varphi_0(y) = A_0 y^{\overline{\beta}_0} + A_1 y^{\overline{\beta}_1} \quad 0 < y < \eta_0, \tag{A.6}$$

$$\varphi_1(y) = B_0 y^{\overline{\beta}_0} + B_1 y^{\overline{\beta}_1} \quad 0 < y < \eta_0, \tag{A.7}$$

$$\varphi_0(y) = G_0(y) = a_0 y \quad y \geq \eta_0, \tag{A.8}$$

$$\varphi_1(y) = C_1 y^{\beta_1} + C_1^* y^{\beta_1^*} + \frac{\lambda_1 G_0(y)}{\lambda_1+\mu-\alpha_1} \quad \eta_0 < y < \eta_1, \tag{A.9}$$

$$\varphi_1(\eta_1) = G_1(\eta_1). \tag{A.10}$$

In what follows, we solve the unknown coefficients. From (A.6) and (A.8), applying value matching at η_0, we have

$$A_0\eta_0^{\overline{\beta}_0} + A_1\eta_0^{\overline{\beta}_1} = G_0(\eta_0). \tag{A.11}$$

By (A.6), (A.7), and (A.3), we have

$$B_i = \frac{\lambda_1}{K_1(\overline{\beta}_i)}A_i = \frac{K_0(\overline{\beta}_i)}{\lambda_0}A_i \quad for \quad i = 0, 1. \tag{A.12}$$

From (A.7), (A.9), (A.10), and $\varphi_1(y) \in C^2(0, \eta_1)$, we have

$$\begin{cases}
B_0\eta_0^{\overline{\beta}_0} + B_1\eta_0^{\overline{\beta}_1} = C_1\eta_0^{\beta_1} + C_1^*\eta_0^{\beta_1^*} + \dfrac{\lambda_1 G_0(\eta_0)}{\lambda_1+\mu-\alpha_1},\\
\overline{\beta}_0 B_0\eta_0^{\overline{\beta}_0-1} + \overline{\beta}_1 B_1\eta_0^{\overline{\beta}_1-1} = \beta_1 C_1\eta_0^{\beta_1-1} + \beta_1^* C_1^*\eta_0^{\beta_1^*-1} + \dfrac{\lambda_1 G_0'(\eta_0)}{\lambda_1+\mu-\alpha_1},\\
C_1\eta_1^{\beta_1} + C_1^*\eta_1^{\beta_1^*} + \dfrac{\lambda_1 G_0(\eta_1)}{\lambda_1+\mu-\alpha_1} = G_1(\eta_1).
\end{cases} \tag{A.13}$$

Combine (A.11), (A.12), and (A.13), the unknowns can be determined uniquely. Therefore, we can define (4.1) explicitly.

Next, using the verification theorem, we verify $\varphi(y)$ defined in (A.1) is indeed the expected value of the first hitting time given by $\tau_{y.i}(\eta_0, \eta_1) = \inf\{t | Y(t) \geq \eta_{z(t)}\}$. First, we note that the pair $(Y(t), z(t))$ is a Markov process. For a function $f(y, i) = f_i(y)$, $f_i \in C^2$, the generator $\mathcal{L}$ of the Markov process is given by

$$(\mathcal{L}f)_i(y) = \frac{1}{2} y^2 \sigma_i^2 f_i''(y) + \alpha_i y f_i' - \lambda_i f_i(y) + \lambda_i f_{1-i}(y).$$

By Itô's formula,

$$f_{z(t)}(Y(t)) - \int_0^t (\mathcal{L}f)_{z(s)}(Y(s)) ds$$

is an $\mathcal{F}_t$ martingale with $\mathcal{F}_t = \sigma\{W(s),\ z(s) :\ s \leq t\}$. Therefore,

$$e^{-\mu t} f_{z(t)}(Y(t)) - \int_0^t e^{-\mu s} \left[(\mathcal{L}f)_{z(s)}(Y(s)) - \mu f_{z(s)}(Y(s))\right] ds$$

is an $\mathcal{F}_t$ martingale. We know $\varphi_i \in C^1$, and $\varphi_i'' \in L^\infty$. Taking $f_i(y) = \varphi_i(y)$, we obtain

$$e^{-\mu t} \varphi_{z(t)}(Y(t)) - \int_0^t e^{-\mu s} \left[(\mathcal{L}\varphi)_{z(s)}(Y(s)) - \mu \varphi_{z(s)}(Y(s))\right] ds$$

is an $\mathcal{F}_t$ martingale.

Now let τ be any $\mathcal{F}_t$ stopping time. Because $(Y(t), z(t))$ is a strong Markov process, we have

$$E_{y,i}\left[\mathbb{I}_{\tau<\infty} e^{-\mu\tau} \varphi_{z(\tau)}(Y(\tau))\right] - E_{y,i}\left[\mathbb{I}_{\tau<\infty} \int_0^\tau e^{-\mu s} \left[(\mathcal{L}\varphi)_{z(s)}(Y(s)) - \mu\varphi_{z(s)}(Y(s))\right] ds\right] = \varphi_i(y).$$

(A.14)

Let $\tau_{y,i}$ be the optimal stopping time defined by $\tau_{y,i}(\eta_0, \eta_1) = \inf\{t | Y(t) \geq \eta_{z(t)}\}$. By (A.2), $(\mathcal{L}\varphi)_{z(s)}(Y(s)) - \mu\varphi_{z(s)}(Y(s)) = 0$ for $s < \tau_{y,i}$ because $Y(s) < \eta_{z(s)}$. Taking $\tau = \tau_{y,i}$ in (A.14), we obtain

$$\varphi_i(y) = E[e^{-\mu\tau_{y,i}} G_{z(\tau_{y,i})}(Y_{\tau_{y,i}}) \mathbb{I}_{\tau_{y,i}<\infty}].$$

Leader's investment threshold. Assume $y_0^1 < y_1^1$, we write

(A.15) $$\Gamma_0(y) = LA_0 y^{\overline{\beta}_0} + LA_1 y^{\overline{\beta}_1} \quad 0 < y < y_0^1,$$

(A.16) $$\Gamma_0(y) = m_0 + \delta_2 a_0 y - K = \delta_1 a_0 y - (\delta_1 - \delta_2)\varphi_0(y) - K \quad y_0^1 \leq y \leq y_0^2,$$

(A.17) $$\Gamma_1(y) = LB_0 y^{\overline{\beta}_0} + LB_1 y^{\overline{\beta}_1} \quad 0 < y < y_0^1,$$

(A.18) $$\Gamma_1(y) = LC_1 y^{\beta_1} + LC_1^* y^{\beta_1^*} + \phi(y) \quad y_0^1 < y < y_1^1,$$

where

$$\phi(y) = -\frac{\lambda_1(\delta_1-\delta_2)A_0}{K_1(\overline{\beta}_0)} y^{\overline{\beta}_0} - \frac{\lambda_1(\delta_1-\delta_2)A_1}{K_1(\overline{\beta}_1)} y^{\overline{\beta}_1} + \frac{\lambda_1\delta_1 a_0}{\lambda_1+\mu-\alpha_1} y - \frac{\lambda_1 K}{\lambda_1+\mu}$$
$$= -(\delta_1-\delta_2)\varphi_1(y) + \frac{\lambda_1\delta_1 a_0}{\lambda_1+\mu-\alpha_1} y - \frac{\lambda_1 K}{\lambda_1+\mu},$$

(A.19) $\Gamma_1(y) = m_1 + \delta_2 a_1 y - K = \delta_1 a_1 y - (\delta_1-\delta_2)\varphi_1(y) - K \quad y_1^1 \le y \le y_1^2.$

From (4.4), for $y < y_0^1$, we have

$$\text{(A.20)} \qquad \begin{cases} -\frac{1}{2}y^2\sigma_0^2\Gamma_0'' - \alpha_0 y\Gamma_0' + (\lambda_0+\mu)\Gamma_0 - \lambda_0\Gamma_1 = 0, \\ -\frac{1}{2}y^2\sigma_1^2\Gamma_1'' - \alpha_1 y\Gamma_1' + (\lambda_1+\mu)\Gamma_1 - \lambda_1\Gamma_0 = 0, \end{cases}$$

and for $y_0^1 \le y < y_1^1$, we have

$$\text{(A.21)} \qquad \begin{cases} -\frac{1}{2}y^2\sigma_1^2\Gamma_1'' - \alpha_1 y\Gamma_1' + (\lambda_1+\mu)\Gamma_1 - \lambda_1\Gamma_0 = 0, \\ \Gamma_0(y) = m_0 + \delta_2 a_0 y - K = \delta_1 a_0 y - (\delta_1-\delta_2)\varphi_0(y) - K. \end{cases}$$

Substituting (A.15) and (A.17) into (A.20), we get

$$\text{(A.22)} \qquad LB_i = \frac{\lambda_1}{K_1(\overline{\beta}_i)} LA_i \quad for \quad i = 0, 1.$$

From (A.15), applying value matching and smooth pasting conditions at y_0^1, we obtain

$$\text{(A.23)} \qquad \begin{cases} LA_0\left(y_0^1\right)^{\overline{\beta}_0} + LA_1\left(y_0^1\right)^{\overline{\beta}_1} = \delta_1 a_0 y_0^1 - (\delta_1-\delta_2)\varphi_0\left(y_0^1\right) - K, \\ \overline{\beta}_0 LA_0\left(y_0^1\right)^{\overline{\beta}_0-1} + \overline{\beta}_1 LA_1\left(y_0^1\right)^{\overline{\beta}_1-1} = \delta_1 a_0 - (\delta_1-\delta_2)\varphi_0'\left(y_0^1\right). \end{cases}$$

From (A.17) and (A.18), applying value matching and smooth pasting conditions at y_0^1 and y_1^1, we obtain

$$\text{(A.24)} \begin{cases} LB_0\left(y_0^1\right)^{\overline{\beta}_0} + LB_1\left(y_0^1\right)^{\overline{\beta}_1} = LC_1\left(y_0^1\right)^{\beta_1} + LC_1^*\left(y_0^1\right)^{\beta_1^*} + \phi\left(y_0^1\right), \\ \overline{\beta}_0 LB_0\left(y_0^1\right)^{\overline{\beta}_0-1} + \overline{\beta_1} LB_1\left(y_0^1\right)^{\overline{\beta}_1-1} = \beta_1 LC_1\left(y_0^1\right)^{\beta_1} + \beta_1^* LC_1^*\left(y_0^1\right)^{\beta_1^*-1} + \phi'\left(y_0^1\right), \end{cases}$$

$$\text{(A.25)} \quad \begin{cases} LC_1\left(y_1^1\right)^{\beta_1} + LC_1^*\left(y_1^1\right)^{\beta_1^*} + \phi\left(y_1^1\right) = \delta_1 a_1 y_1^1 - (\delta_1-\delta_2)\varphi_1\left(y_1^1\right) - K, \\ \beta_1 LC_1\left(y_1^1\right)^{\beta_1} + \beta_1^* LC_1^*\left(y_1^1\right)^{\beta_1^*-1} + \phi'\left(y_1^1\right) = \delta_1 a_1 - (\delta_1-\delta_2)\varphi_1'\left(y_1^1\right). \end{cases}$$

Using eight equations from (A.22), (A.23), (A.24), and (A.25), we can solve for the eight unknowns LA_0, LA_1, LB_0, LB_1, LC_1, LC_1^*, y_0^1, and y_1^1. In what follows, we give the leader's value function and investment thresholds, y_i^2 and y_i^3. In the base case,

$$y_0^1 < y_1^1 < y_0^2 < \hat{y}_0 < y_0^3 < y_1^2 < \hat{y}_1 < y_1^3.$$

We write

$$\Gamma_0(y) = LC_0 y^{\beta_0} + LC_0^* y^{\beta_0^*} + \phi_0(y) \quad y_0^2 < y < \widehat{y}_0, \tag{A.26}$$

where

$$\begin{aligned}\phi_0(y) &= -\frac{\lambda_0(\delta_1-\delta_2)B_0}{K_0(\overline{\beta}_0)} y^{\overline{\beta}_0} - \frac{\lambda_0(\delta_1-\delta_2)B_1}{K_0(\overline{\beta}_1)} y^{\overline{\beta}_1} + \frac{\lambda_0\delta_1 a_1}{\lambda_0+\mu-\alpha_0} y - \frac{\lambda_0 K}{\lambda_0+\mu}\\ &= -(\delta_1-\delta_2)\varphi_0(y) + \frac{\lambda_0\delta_1 a_1}{\lambda_0+\mu-\alpha_0} y - \frac{\lambda_0 K}{\lambda_0+\mu}.\end{aligned}$$

$$\begin{aligned}\Gamma_0(y) &= LC_3 y^{\beta_0} + LC_3^* y^{\beta_0^*} - \frac{\lambda_0(\delta_1-\delta_2)C_1}{K_0(\beta_1)} y^{\beta_1} - \frac{\lambda_0(\delta_1-\delta_2)C_1^*}{K_0(\beta_1^*)} y^{\beta_1^*}\\ &\quad -\frac{\lambda_0\lambda_1 a_0(\delta_1-\delta_2)y}{(\lambda_0+\mu-\alpha_0)(\lambda_1+\mu-\alpha_1)} + \frac{\lambda_0\delta_1 a_1 y}{\lambda_0+\mu-\alpha_0} - \frac{\lambda_0 K}{\lambda_0+\mu} \quad \widehat{y}_0 \le y < y_0^3,\end{aligned} \tag{A.27}$$

$$\Gamma_0(y) = \delta_2 a_0 y - K \quad y \ge y_0^3, \tag{A.28}$$

$$\Gamma_1(y) = LC_2 y^{\beta_1} + LC_2^* y^{\beta_1^*} + \frac{a_0\delta_2\lambda_1 y}{\mu+\lambda_1-\alpha_1} - \frac{\lambda_1 K}{\mu+\lambda_1} \quad y_1^2 \le y < y_1^3, \tag{A.29}$$

$$\Gamma_1(y) = \delta_2 a_1 y - K \quad y \ge y_1^3. \tag{A.30}$$

Applying value matching and smooth pasting conditions at $\widehat{y}_0(\Gamma_0(y) \in C^1(0,\infty))$, y_0^2 and y_0^3 for $\Gamma_0(y)$, we have

$$\left\{\begin{aligned} &LC_0(\widehat{y}_0)^{\beta_0} + LC_0^*(\widehat{y}_0)^{\beta_0^*} + \phi_0(\widehat{y}_0) = LC_3(\widehat{y}_0)^{\beta_0} + LC_3^*(\widehat{y}_0)^{\beta_0^*}\\ &-\frac{\lambda_0(\delta_1-\delta_2)C_1}{K_0(\beta_1)}(\widehat{y}_0)^{\beta_1} - \frac{\lambda_0(\delta_1-\delta_2)C_1^*}{K_0(\beta_1^*)}(\widehat{y}_0)^{\beta_1^*}\\ &-\frac{\lambda_0\lambda_1 a_0(\delta_1-\delta_2)\widehat{y}_0}{(\lambda_0+\mu-\alpha_0)(\lambda_1+\mu-\alpha_1)} + \frac{\lambda_0\delta_1 a_1\widehat{y}_0}{\lambda_0+\mu-\alpha_0} - \frac{\lambda_0 K}{\lambda_0+\mu},\\ &\beta_0 LC_0(\widehat{y}_0)^{\beta_0-1} + \beta_0^* LC_0^*(\widehat{y}_0)^{\beta_0^*-1} + \phi_0'(\widehat{y}_0) = \beta_0 LC_3(\widehat{y}_0)^{\beta_0-1} + \beta_0^* LC_3^*(\widehat{y}_0)^{\beta_0^*-1}\\ &-\beta_1\frac{\lambda_0(\delta_1-\delta_2)C_1}{K_0(\beta_1)}(\widehat{y}_0)^{\beta_1-1} - \beta_1^*\frac{\lambda_0(\delta_1-\delta_2)C_1^*}{K_0(\beta_1^*)}(\widehat{y}_0)^{\beta_1^*-1}\\ &-\frac{\lambda_0\lambda_1 a_0(\delta_1-\delta_2)}{(\lambda_0+\mu-\alpha_0)(\lambda_1+\mu-\alpha_1)} + \frac{\lambda_0\delta_1 a_1}{\lambda_0+\mu-\alpha_0}, \end{aligned}\right. \tag{A.31}$$

$$\left\{\begin{aligned} &LC_0\left(y_0^2\right)^{\beta_0} + LC_0^*\left(y_0^2\right)^{\beta_0^*} + \phi_0\left(y_0^2\right) = \delta_1 a_0 y_0^2 - (\delta_1-\delta_2)\varphi_0\left(y_0^2\right) - K,\\ &\beta_0 LC_0\left(y_0^2\right)^{\beta_0-1} + \beta_0^* LC_0^*\left(y_0^2\right)^{\beta_0^*-1} + \phi_0'\left(y_0^2\right) = \delta_1 a_0 - (\delta_1-\delta_2)\varphi_0'\left(y_0^2\right), \end{aligned}\right. \tag{A.32}$$

$$
\text{(A.33)} \quad \begin{cases} LC_3\left(y_0^3\right)^{\beta_0} + LC_3^*\left(y_0^3\right)^{\beta_0^*} \\ -\dfrac{\lambda_0(\delta_1-\delta_2)C_1}{K_0(\beta_1)}\left(y_0^3\right)^{\beta_1} - \dfrac{\lambda_0(\delta_1-\delta_2)C_1^*}{K_0(\beta_1^*)}\left(y_0^3\right)^{\beta_1^*} \\ -\dfrac{\lambda_0\lambda_1 a_0(\delta_1-\delta_2)y_0^3}{(\lambda_0+\mu-\alpha_0)(\lambda_1+\mu-\alpha_1)} + \dfrac{\lambda_0\delta_1 a_1 y_0^3}{\lambda_0+\mu-\alpha_0} - \dfrac{\lambda_0 K}{\lambda_0+\mu} = \delta_2 a_0 y_0^3 - K, \\ \beta_0 LC_3\left(y_0^3\right)^{\beta_0-1} + \beta_0^* LC_3^*\left(y_0^3\right)^{\beta_0^*-1} \\ -\beta_1\dfrac{\lambda_0(\delta_1-\delta_2)C_1}{K_0(\beta_1)}\left(y_0^3\right)^{\beta_1-1} - \beta_1^*\dfrac{\lambda_0(\delta_1-\delta_2)C_1^*}{K_0(\beta_1^*)}\left(y_0^3\right)^{\beta_1^*-1} \\ -\dfrac{\lambda_0\lambda_1 a_0(\delta_1-\delta_2)}{(\lambda_0+\mu-\alpha_0)(\lambda_1+\mu-\alpha_1)} + \dfrac{\lambda_0\delta_1 a_1}{\lambda_0+\mu-\alpha_0} = \delta_2 a_0. \end{cases}
$$

Applying value matching and smooth pasting conditions at y_1^2 and y_1^3 for $\Gamma_1(y)$ yields

$$
\begin{cases} LC_2\left(y_1^2\right)^{\beta_1} + LC_2^*\left(y_1^2\right)^{\beta_1^*} + \dfrac{a_0\delta_2\lambda_1 y_1^2}{\mu+\lambda_1-\alpha_1} - \dfrac{\lambda_1 K}{\mu+\lambda_1} = \delta_1 a_1 y_1^2 - (\delta_1-\delta_2)\varphi_1\left(y_1^2\right) - K, \\ \beta_1 LC_2\left(y_1^2\right)^{\beta_1-1} + \beta_1^* LC_2^*\left(y_1^2\right)^{\beta_1^*-1} + \dfrac{a_0\delta_2\lambda_1}{\mu+\lambda_1-\alpha_1} \\ = \delta_1 a_1 - (\delta_1-\delta_2)\varphi_1'\left(y_1^2\right), \end{cases}
$$

(A.34)

$$
\text{(A.35)} \quad \begin{cases} LC_2\left(y_1^3\right)^{\beta_1} + LC_2^*\left(y_1^3\right)^{\beta_1^*} + \dfrac{a_0\delta_2\lambda_1 y_1^3}{\mu+\lambda_1-\alpha_1} - \dfrac{\lambda_1 K}{\mu+\lambda_1} = \delta_2 a_1 y_1^3 - K, \\ \beta_1 LC_2\left(y_1^3\right)^{\beta_1-1} + \beta_1^* LC_2^*\left(y_1^3\right)^{\beta_1^*-1} + \dfrac{a_0\delta_2\lambda_1}{\mu+\lambda_1-\alpha_1} = \delta_2 a_1. \end{cases}
$$

Now we get ten equations (A.31)$\sim$(A.35) and ten unknowns LC_0, LC_0^*, LC_2, LC_2^*, LC_3, LC_3^*, y_0^2, y_1^2, y_0^3, y_1^3. By solving these nonlinear equations, we can obtain the value function and the investment thresholds of the leader.

We have solved the system equations stated above. Given the parameter values in the base case, solutions of LA_0, LA_0^*, LC_1, LC_1^*, LC_0, LC_0^*, LC_2, LC_2^*, LC_3, LC_3^*, y_0^1, y_1^1, y_0^2, y_1^2, y_0^3, y_1^3 are 23.2461, 17.9807, 12.0946, 0.1102, 24.131, 0, 2.3017, 1.2585, 13.7115, 0, 0.2913, 0.4886, 0.5412, 0.7786, 0.7166, and 1.3351, respectively.

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