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OPTIMAL INVESTMENT IN CREDIT DERIVATIVES PORTFOLIO UNDER CONTAGION RISK

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We consider the optimal portfolio problem of a power investor who wishes to allocate her wealth between several credit default swaps (CDSs) and a money market account. We model contagion risk among the reference entities in the portfolio using a reduced-form Markovian model with interacting default intensities. Using the dynamic programming principle, we establish a lattice dependence structure between the Hamilton–Jacobi–Bellman equations associated with the default states of the portfolio. We show existence and uniqueness of a classical solution to each equation and characterize them in terms of solutions to inhomogeneous Bernoulli type ordinary differential equations. We provide a precise characterization for the directionality of the CDS investment strategy and perform a numerical analysis to assess the impact of default contagion. We find that the increased intensity triggered by default of a very risky entity strongly impacts size and directionality of the investor strategy. Such findings outline the key role played by default contagion when investing in portfolios subject to multiple sources of default risk.

KEY WORDS: dynamic portfolio optimization, credit default swaps, contagion risk, interacting default intensities.

1. INTRODUCTION

Since the seminal work of Merton (1969), who pioneered the use of optimal stochastic control techniques to solve continuous-time portfolio optimization problems, there has been enormous interest in continuous-time utility maximization problems (see Karatzas et al. 1996; Fleming and Pang 2004). Most of the proposed models have dealt with markets consisting of default-free securities such as stocks, where the uncertainty in price is typically governed by a continuous process, chosen to be a Brownian motion. The economy-wide crisis initiated at the end of 1990s in Latin American countries, as well as the global financial crisis of 2007–2008 resulting in the failure of systemically important financial institutions, have outlined the importance of including the default

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feature into the models. As a consequence, many researchers have started to analyze portfolio optimization problems in defaultable markets, as discussed next. Korn and Kraft (2003) studied optimal portfolio problems with defaultable assets within a Merton structural default framework. Kraft and Steffensen (2005) extended the analysis to deal with a Black–Cox framework, where default is defined as the first passage time of an economic state process below a given threshold. A different branch of the literature has modeled default events using a reduced-form approach. Starting from the seminal paper of Merton (1971) who assumes constant interest rate and default intensity, many other extensions have been proposed, some of which are surveyed next. Bielecki and Jang (2006) derived optimal investment strategies for a constant relative risk aversion (CRRA) investor, allocating her wealth among a defaultable bond, risk-free bank account, and a stock. Bo, Wang, and Yang (2010) considered an infinite horizon portfolio optimization problem, where a logarithmic investor can choose a consumption rate, and invest her wealth across a defaultable perpetual bond, a stock, and a money market account. Lakner and Liang (2008) employed duality theory to obtain the optimal investment strategy in a market consisting of a defaultable bond and a money market account. Capponi and Figueroa-López (2014) considered a Markov modulated economy driving the prices of stock and defaultable bond securities. Using the Hamilton-Jacobi-Bellman (HJB) approach, they recovered the optimal strategies as the unique solution to a coupled system of partial differential equations. Bielecki et al. (2008a) considered a Markov modulated default intensity framework for pricing and hedging defaultable game options using forward–backward stochastic differential equations (SDEs). Jiao and Pham (2011) combined duality theory and dynamic programming to optimize the utility of a CRRA investor, in a market consisting of a riskless bond and a stock subject to counterparty risk.

The literature surveyed above has considered markets consisting of one defaultable security. This prevents the analysis of default contagion effects on optimal portfolio allocations, despite the fact that contagion risk plays a fundamental role in financially distressed situations. As demonstrated during the crisis, contagion phenomena may be responsible for high mark-to-market variations in prices of credit sensitive securities. There are only few studies, discussed next, which have studied portfolio frameworks consisting of multiple defaultable securities. Giesecke et al. (2014) studied a static selection problem where an investor maximizes the mark-to-market value of a fixed income portfolio. Kraft and Steffensen (2008) considered an investor who can allocate her wealth across multiple defaultable bonds, in a model with constant default intensity and risk premium, but where simultaneous defaults are allowed. In the same market model, Kraft and Steffensen (2009) defined default as the beginning of financial distress, and discussed contagion effects on defaultable bond prices. Callegaro, Jeanblanc, and Runggaldier (2010) considered an investor who can allocate her wealth across several defaultable assets, whose discrete dynamics depends on a partially observed exogenous factor process. Most recently, Jiao, Kharroubi, and Pham (2013) analyzed a portfolio framework under multiple jumps and default events. Using the density hypothesis, they separate before and after default scenarios, and establish existence and uniqueness of the value function via recursive systems of backward stochastic differential equations.

We next summarize our main contributions. First, we consider a market model consisting of credit derivatives, namely, credit default swaps (CDSs). To the best of our knowledge, our work represents the first attempt to include credit derivatives in a dynamic portfolio optimization framework. Earlier contributions discussed above consider defaultable assets to be primary securities, such as corporate bonds. From an empirical point of view, the continuous-time framework is much better suited for CDSs, rather than corporate bonds. As described in Currie and Morris (2002), credit has become much more tradable due to the development of the credit default swap market, and consequently trading strategies based both on equity and CDSs have become quite popular. Second, we develop our analysis within a contagion credit risk model with interacting default intensities. In our model, the default of one name may trigger an increase in default intensities of other names in the portfolio. This in turn leads to jumps in the market valuation of CDSs referencing the surviving names, and consequently to jumps in the optimal wealth proportion allocated to these CDSs. We refer the reader to Frey and Backhaus (2008) for a theoretical illustration, and to Jarrow and Yu (2001) for more explicit examples of default contagion effects in interacting intensity models. The Markovian nature of the default intensity process leads to explicit expressions for the market value of the CDSs obtained as solutions to Feynman–Kac equations. Moreover, the interacting intensity feature introduces an explicit default contagion term in the market value dynamics of each CDS (the precise statement is in Theorem 2.4). Clearly, when there is only one name in portfolio the contagion effect disappears, and our framework reduces to Bielecki, Jeanblanc, and Rutkowski (2008b) (see Remark 2.8).

We consider a power investor who maximizes expected utility from terminal wealth in a finite horizon, and obtain the optimal investment strategies using a HJB dynamic programming approach. The interacting nature of the default intensity process leads to a lattice dependence structure, where the partial order is induced by the default state of the portfolio. As we demonstrate later in the paper, the value function corresponding to the state with all names defaulted acts as the maximum, while the one associated with the state where all names are alive acts as the minimum. Following the dynamic programming principle, the value function corresponding to the "all-default" configuration is independent of all others, while the value function corresponding to the "all-alive" configuration depends on the value functions associated with all possible default states in the lattice. We show that for each default state of the portfolio, the value function and the corresponding optimal strategy satisfy a coupled system of inhomogeneous Bernoulli type ordinary differential equations (ODEs) and nonlinear equations. We characterize the directionality of the CDS investment strategy in terms of a relation between the expected instantaneous price change computed under an adjusted historical measure, and the same expectation computed under an adjusted equivalent measure.

We provide a numerical analysis to assess the impact of default contagion on the optimal CDS strategy. Considering a portfolio consisting of two CDSs, we illustrate how contagion impacts optimal portfolio allocations. We find that the default of a very risky name significantly alters the amount of wealth allocated to the CDSs referencing the surviving name, and may even induce a change in the directionality of the strategy (from long credit to short credit). Moreover, if default risk premium, risk neutral and historical default intensities are time invariant, the investor trades more aggressively if the planning horizon is higher. Furthermore, we find that default contagion effects may be the most dominating in situations where one name is very risky. In this case, the amount invested on the CDS referencing the safest name becomes highly dependent on the postdefault intensity of the safest name (i.e., the one after default of the risky name) and only mildly dependent on the predefault intensity of the safest name (i.e., the one before default of the risky name).

The rest of the paper is organized as follows. Section 2 develops the market and default model. Section 3 formulates the utility maximization problem. Section 4 analyzes the optimal investment strategy and proves existence and uniqueness of a positive solution to the resulting HJB equation. Section 5 presents a rigorous proof that the solutions of the HJB equations associated with the different default states correspond with the value functions. Section 6 develops a numerical analysis to assess the impact of default contagion on the optimal strategies. Additional technical proofs are delegated to an Appendix.

2. THE MODEL

We present the default model in Section 2.1, and introduce the market model in Section 2.2.

2.1. The Default Model

We consider $M \geq 2$ entities referencing the CDSs in the portfolio. The default state is described by an M-dimensional default indicator process $\mathbf{H}(t) = (H_1(t), \ldots, H_M(t))$ with $t \geq 0$, supported by a filtered probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. Here, $\mathbb{Q}$ denotes the risk-neutral probability measure and $\mathbb{E}$ the expectation operator with respect to $\mathbb{Q}$. The state space of the default indicator process $\mathbf{H} = (\mathbf{H}(t);\ t \geq 0)$ is given by $\mathcal{S} = \{0, 1\}^M$, where $H_i(t) = 1$ if the name i has defaulted by time t and $H_i(t) = 0$ otherwise. The default time of the ith name in the portfolio is given by

$$\tau_i = \inf\{t \geq 0;\ H_i(t) = 1\}, \qquad i = 1, \ldots, M.$$

Hence, we have $H_i(t) = \mathbf{1}_{\{\tau_i \leq t\}}$, where $t \geq 0$. Here, $\mathbf{1}_A$ denotes the indicator of the event A.

We model default contagion through a Markovian model with interacting intensities defined on the risk-neutral probability space. The default indicator process $\mathbf{H}$ is assumed to follow a continuous-time Markov chain on $\mathcal{S} = \{0, 1\}^M$, where $\mathbf{H}(t)$ transits to a neighboring state $\mathbf{H}^i(t) := (H_1(t), \ldots, H_{i-1}(t), 1 - H_i(t), H_{i+1}(t), \ldots, H_M(t))$ at rate $\mathbf{1}_{\{H_i(t)=0\}} h_i(\mathbf{H}(t))$. Here, $h_i(\mathbf{z})$ $(i = 1, \ldots, M)$ are positive measurable functions defined on $\mathbf{z} \in \mathcal{S}$. We refer the reader to Frey and Backhaus (2008) for explicit probabilistic models under this setup. Hence, the M-dimensional default indicator process $\mathbf{H}$ admits the following $\mathbb{Q}$-infinitesimal generator given by

$$\mathcal{A}g(\mathbf{z}) = \sum_{j=1}^{M} (1 - z_j) h_j(\mathbf{z}) \left[g(\mathbf{z}^j) - g(\mathbf{z}) \right], \quad \mathbf{z} = (z_1, \ldots, z_M) \in \mathcal{S}, \tag{2.1}$$

where $g(\mathbf{z})$ is an arbitrary measurable function defined on $\mathbf{z} \in \mathcal{S}$, and the vector

$$\mathbf{z}^j := (z_1, \ldots, z_{j-1}, 1 - z_j, z_{j+1}, \ldots, z_M), \quad j = 1, \ldots, M. \tag{2.2}$$

The market filtration is given by $\mathcal{G}_t = \sigma(\mathbf{H}(s);\ s \leq t)$, after completion and regularization on the right (see Belanger, Shreve, and Wong 2004). Using the Dynkin's formula (see (10.13) in Rogers and Williams 2000, p. 254), and choosing $g(\mathbf{z}) = z_i$, $i \in \{1, \ldots, M\}$, we have that

$$\xi_i(t) := H_i(t) - \int_0^t (1 - H_i(s)) h_i(\mathbf{H}(s)) \mathrm{d}s, \quad t \geq 0 \tag{2.3}$$

is a $(\mathbb{Q}, \mathcal{G}_t)$-martingale.

Interacting intensity models have been first proposed by Jarrow and Yu (2001). They consider a specific type of interaction between defaults, referred to as primary secondary framework. In this framework, secondary banks lend to primary banks and are exposed to their default risk, while primary banks do not have any debt exposure to secondary banks. A more general analysis with cyclical dependence among defaults is presented in Yu (2007), where firms hold the other firm's debt and are affected by their default. Yu (2007) develops a simulation method for generating the correlated default times under various forms of default interactions. We remark that models of this type also provide a very natural framework for studying systemic risk via mean field interaction models, and are especially suitable in cases when the number of defaulting entities grows large. We refer to Frey and Backhaus (2008) for a related discussion on this aspect. Next, we provide two meaningful specializations of this model, drawing from results in Yu (2007).

EXAMPLE 2.1 Yu (2007) proposes the following specification for the default intensity of the ith name

$$h_i(\mathbf{H}(t)) = \left(a_1 + a_2 \mathbf{1}_{\{\tau_F \le t\}}\right) \mathbf{1}_{\{\tau_i > t\}}, \tag{2.4}$$

where $\tau_F = \min(\tau_1, \tau_2, \ldots, \tau_M)$. Hence, defaults are correlated through the first-to-default time, a reasonable choice in case of populations consisting of names with high credit quality and hence low default frequency.

EXAMPLE 2.2 This specification is well suited to price CDSs inclusive of counterparty risk, i.e., allowing for both protection buyer and seller to default in addition to the reference entity. The default of the three names "1," "2," and "3," representing protection buyer, reference entity, and protection seller, is specified as follows:

$$\begin{aligned}
h_1(\mathbf{H}(t)) &= \left(a_{10} + a_{12}\mathbf{1}_{\{\tau_2 \le t\}} + a_{13}\mathbf{1}_{\{\tau_3 \le t\}}\right) \mathbf{1}_{\tau_1 > t},\\
h_2(\mathbf{H}(t)) &= \left(a_{20} + a_{21}\mathbf{1}_{\{\tau_1 \le t\}} + a_{23}\mathbf{1}_{\{\tau_3 \le t\}}\right) \mathbf{1}_{\tau_2 > t},\\
h_3(\mathbf{H}(t)) &= \left(a_{30} + a_{31}\mathbf{1}_{\{\tau_1 \le t\}} + a_{32}\mathbf{1}_{\{\tau_3 \le t\}}\right) \mathbf{1}_{\tau_3 > t}.
\end{aligned}$$

As argued in Yu (2007), the above specification allows for analytical representations of CDS prices, which can also be efficiently simulated under this model. We remark that by suitably specifying the a_{jk} coefficients in the above model, it is possible to produce rich patterns of default correlation hence capturing highly relevant effects such as wrong way risk. The latter plays a fundamental role in valuation and risk management of over-the-counter credit derivatives, whose importance has also been stressed in the new Basel III accords reviewed in Basel III (2010).

2.2. The Market Model

We consider a frictionless financial market consisting of a risk-free money market account and M CDSs. We have that $B_t = e^{rt}$ is the money market account at t, with $r > 0$ being the constant interest rate.

$$C_t^i = (1 - H_i(t))\Phi_i(t, \mathbf{H}(t)), \qquad 0 \le t \le T_i, \tag{2.5}$$

where T_i denotes the maturity of the ith CDS, and $\Phi_i(t, \mathbf{H}(t))$ denotes the *predefault* price of the ith CDS contract. On $\tau_i > t$ this is given by

$$\begin{aligned}\Phi_i(t, \mathbf{H}(t)) &:= \mathbb{E}\left[L_i \int_t^{T_i} e^{-\int_t^u r\,\mathrm{d}s}\mathrm{d}H_i(u) - \nu_i \int_t^{T_i} e^{-\int_t^u r\,\mathrm{d}s}(1 - H_i(u))\mathrm{d}u \middle| \mathcal{G}_t\right] \\ &= \mathbb{E}\left[L_i H_i(T_i) e^{-\int_t^{T_i} r(1-H_i(u))\mathrm{d}u} - \nu_i \int_t^{T_i} e^{-\int_t^u r\,\mathrm{d}s}(1 - H_i(u))\mathrm{d}u \middle| \mathcal{G}_t\right].\end{aligned} \tag{2.6}$$

The Markov property of the default framework allows obtaining explicit expressions for the predefault price of each CDS contract. Concretely, for $i = 1, \dots, M$, define the functions

$$\begin{aligned}\Phi_i^{(1)}(t, \mathbf{z}) &:= \mathbb{E}\left[\int_t^{T_i} e^{-\int_t^u r\,\mathrm{d}s}(1 - H_i(u))\mathrm{d}u \middle| \mathbf{H}(t) = \mathbf{z}\right], \\ \Phi_i^{(2)}(t, \mathbf{z}) &:= \mathbb{E}\left[H_i(T_i) e^{-\int_t^{T_i} r(1-H_i(u))\mathrm{d}u} \middle| \mathbf{H}(t) = \mathbf{z}\right]\end{aligned} \tag{2.7}$$

and recall the definition of $\mathbf{z}^i$ given in equation (2.2). We then have the following result, whose proof is reported in the Appendix.

LEMMA 2.3. *For $\mathbf{z} \in \{0, 1\}^M$, let*

$$\Phi_i(t, \mathbf{z}) := L_i \Phi_i^{(2)}(t, \mathbf{z}) - \nu_i \Phi_i^{(1)}(t, \mathbf{z}), \qquad i = 1, \dots, M. \tag{2.8}$$

It holds that, for each $i = 1, \dots, M$,

$$\begin{aligned}\Phi_i(t, \mathbf{H}(t)) &= \Phi_i(0, \mathbf{H}(0)) \\ &\quad + \int_0^t \Big(r(1 - H_i(s))\Phi_i(s, \mathbf{H}(s)) - r\nu_i H_i(s)\Phi_i^{(1)}(s, \mathbf{H}(s)) + \nu_i(1 - H_i(s))\Big)\,\mathrm{d}s \\ &\quad + \sum_{j=1}^M \int_0^t \big[\Phi_i(s, \mathbf{H}^j(s-)) - \Phi_i(s, \mathbf{H}(s-))\big]\,\mathrm{d}\xi_j(s),\end{aligned} \tag{2.9}$$

where $\xi_j(t) := H_j(t) - \int_0^t (1 - H_j(s)) h_j(\mathbf{H}(s))\mathrm{d}s$ is a $(\mathbb{Q}, \mathcal{G}_t)$-martingale for each $j = 1, \dots, M$.

Explicit expressions for $\Phi_i(t, \mathbf{z})$, obtained by solving the associated Feynman–Kac equations, are provided in Lemma A.1 and Corollary A.2 in the Appendix.

The $\mathbb{Q}$-dynamics followed by the price process of the ith CDS is given in the following proposition, whose proof is reported in the Appendix.

PROPOSITION 2.4. *The $\mathbb{Q}$-dynamics of the ith CDS price is given by*

$$\begin{aligned}\mathrm{d}C_t^i &= (1 - H_i(t))[r\Phi_i(t, \mathbf{H}(t)) + (\nu_i - h_i(\mathbf{H}(t))L_i)]\,\mathrm{d}t - \Phi_i(t, \mathbf{H}(t-))\mathrm{d}\xi_i(t) \\ &\quad + (1 - H_i(t-)) \sum_{j \neq i} \big[\Phi_i(t, \mathbf{H}^j(t-)) - \Phi_i(t, \mathbf{H}(t-))\big]\,\mathrm{d}\xi_j(t), \\ C_0^i &= (1 - H_i(0))\Phi_i(0, \mathbf{H}(0)) \in [-\nu_i, L_i].\end{aligned} \tag{2.10}$$

Although the price process observed in the market is given under the risk-neutral measure $\mathbb{Q}$, the investor wishes to optimize his utility from terminal wealth under the real-world measure, i.e., under the objective measure $\mathbb{P}$. To this purpose, we next provide a formula which allows identifying the objective probability measure $\mathbb{P}$ from the risk-neutral probability measure $\mathbb{Q}$. Let $\lambda_i(\mathbf{z})$ be an arbitrary bounded measurable function defined on $\mathbf{z} \in \mathcal{S}$, which takes values on $(-1, \infty)$, where $i \in \{1, \dots, M\}$. Assume that the process $X = (X_t; t \geq 0)$ satisfies the following SDE given by

$$\text{(2.11)} \qquad \frac{\mathrm{d}X_t}{X_{t-}} = \sum_{i=1}^{M} \lambda_i(\mathbf{H}(t-))\mathrm{d}\xi_i(t), \qquad X_0 = 1,$$

where the $(\mathbb{Q}, \mathcal{G}_t)$-default martingale process $\xi_i = (\xi_i(t);\ t \geq 0)$ is defined by (2.3). Then we have

LEMMA 2.5. *For $T > 0$, define a new probability measure $\mathbb{P} \ll \mathbb{Q}$ on $\mathcal{G}_T$ by*

$$\mathrm{d}\mathbb{P} = X_T \mathrm{d}\mathbb{Q}.$$

Then, for each $i = 1, \dots, M$,

$$\text{(2.12)} \qquad \xi_i^{\mathbb{P}}(t) := H_i(t) - \int_0^t (1 - H_i(s))h_i^{\mathbb{P}}(\mathbf{H}(s))\mathrm{d}s, \qquad t \geq 0$$

is a $(\mathbb{P}, \mathcal{G}_t)$-martingale, where the relation between the $\mathbb{P}$-default intensity of the ith default indicator $H_i(t)$ and its $\mathbb{Q}$-default intensity is given by

$$h_i^{\mathbb{P}}(\mathbf{z}) = h_i(\mathbf{z})(1 + \lambda_i(\mathbf{z})), \qquad \mathbf{z} \in \mathcal{S}.$$

The proof of Lemma 2.5 is reported in the Appendix. The term $\frac{1}{1+\lambda_i(\mathbf{z})}$ is referred to as the default risk premium, and captures the compensation required by market investors for bearing the default risk of entity i. Before proceeding further, we describe in the form of a remark a procedure proposed by Berndt et al. (2005) to estimate the default risk premium from a CDS portfolio.

REMARK 2.6. Berndt et al. (2005) estimate the default risk premium by means of a sectional regression using a log-linear model of the following form:

$$\text{(2.13)} \qquad \log h_i(\mathbf{H}(t)) = \beta_0 + \beta_1 \log h_i^{\mathbb{P}}(\mathbf{H}(t)) + \beta_2 \log(v(\mathbf{H}(t))) + u_t^i.$$

Here, u_t^i are the residual errors, β_0, β_1, and β_2 are constants, and $v(\mathbf{H}(t))$ is the geometric average of the default intensities of the names in the portfolio, i.e.,

$$\log(v(\mathbf{H}(t))) = \frac{1}{M}\sum_{i=1}^{M} \log h_i^{\mathbb{P}}(\mathbf{H}(t)).$$

Using market quotes of CDS spread premiums referencing name "i" as response variables, and Moody's KMV firm-by-firm estimates of conditional probabilities of default as predictors, they recover the least square estimates $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$ in the regression model (2.13). Using such procedure, our model can then be calibrated to recover the default risk premium

$$\frac{1}{1 + \lambda_i(\mathbf{z})} = \frac{h_i(\mathbf{z})}{h_i^{\mathbb{P}}(\mathbf{z})}.$$

In order to proceed with our utility maximization problem, we need to provide the dynamics of each CDS price under the historical measure. This is done in the following.

COROLLARY 2.7. *The $\mathbb{P}$-dynamics of the ith CDS price is given by*

$$\begin{aligned} \mathrm{d}C_t^i &= (1-H_i(t))\left[r\Phi_i(t,\mathbf{H}(t))+\nu_i - h_i(\mathbf{H}(t))L_i\right]\mathrm{d}t + \Phi_i(t,\mathbf{H}(t))\left(h_i(\mathbf{H}(t)) - h_i^{\mathbb{P}}(\mathbf{H}(t))\right)\mathrm{d}t \\ &\quad + (1-H_i(t))\sum_{j\neq i}\left[\Phi_i(t,\mathbf{H}^j(t)) - \Phi_i(t,\mathbf{H}(t))\right](1-H_j(t))\left(h_j^{\mathbb{P}}(\mathbf{H}(t)) - h_j(\mathbf{H}(t))\right)\mathrm{d}t \\ &\quad - \Phi_i(t,\mathbf{H}(t-))\mathrm{d}\xi_i^{\mathbb{P}}(t) + (1-H_i(t-))\sum_{j\neq i}\left[\Phi_i(t,\mathbf{H}^j(t-)) - \Phi_i(t,\mathbf{H}(t-))\right]\mathrm{d}\xi_j^{\mathbb{P}}(t) \end{aligned}$$

$$C_0^i = (1-H_i(0))\Phi_i(0,\mathbf{H}(0)) \in [-\nu_i, L_i]. \tag{2.14}$$

Using the above $\mathbb{P}$ dynamics, it follows that the jump of the ith CDS price is given as follows. If the ith name defaults at t, $\Delta C_t^i = -\Phi_i(t,\mathbf{H}(t))$. If the ℓth $\neq i$th name defaults at t, then $\Delta C_t^i = \Phi_i(t,\mathbf{H}^\ell(t-)) - \Phi_i(t,\mathbf{H}(t-))$.

REMARK 2.8. In the case of a single CDS, the default contagion term

$$(1-H_i(t-))\sum_{j\neq i}\left[\Phi_i(t,\mathbf{H}^j(t-)) - \Phi_i(t,\mathbf{H}(t-))\right]\mathrm{d}\xi_j(t)$$

in (2.10) disappears from the dynamics. In such a case, the dynamics of the CDS reduces to the one in Bielecki et al. (2008b, see equation (32), p. 2506, therein).

3. THE UTILITY MAXIMIZATION PROBLEM

We consider an investor who wants to maximize her power utility from terminal wealth at time T ($\leq \min(T_1,\ldots,T_M)$) by dynamically allocating her wealth into the money market account and M CDSs. The investor does not have intermediate consumption nor capital income to support her purchase of financial assets. For each $i = 1,\ldots,M$, denote by $\phi_i(t)$ the number of shares of the ith CDS that the investor buys ($\phi_i(t) > 0$) or sells ($\phi_i(t) < 0$) at time t. Similarly, $\phi^B(t)$ denote the number of shares invested in the money market account at t. The process $\bar{\boldsymbol{\phi}} = (\boldsymbol{\phi}(t), \phi^B(t);\ t \geq 0)$ with $\boldsymbol{\phi}(t) = (\phi_1(t),\ldots,\phi_M(t))$ is called a portfolio process. The wealth process associated with the portfolio process $\bar{\boldsymbol{\phi}} = (\boldsymbol{\phi}(t), \phi^B(t);\ t \geq 0)$, denoted by $V_t(\bar{\boldsymbol{\phi}})$, is given by

$$V_t(\bar{\boldsymbol{\phi}}) = \sum_{i=1}^M \phi_i(t)C_t^i + \phi^B(t)B_t, \qquad t \geq 0. \tag{3.1}$$

As usual, we require the portfolio process $\bar{\boldsymbol{\phi}}$ to be $\mathbb{G}$-adapted. Following Bielecki et al. (2008b), see also Bielecki et al. (2008a), a $\mathbb{G}$-adapted portfolio process $\bar{\boldsymbol{\phi}} = (\boldsymbol{\phi}(t), \phi^B(t);\ t \geq 0)$ is said to be self-financing if $V_t(\bar{\boldsymbol{\phi}}) = V_0(\bar{\boldsymbol{\phi}}) + G_t(\bar{\boldsymbol{\phi}})$, where the gains process $G(\bar{\boldsymbol{\phi}})$ is given by

$$G_t(\bar{\boldsymbol{\phi}}) = \sum_{i=1}^M \int_0^t \phi_i(s-)\mathrm{d}(C_s^i + D_s^i) + \int_0^t \phi^B(s)\mathrm{d}B_s, \quad t \geq 0, \tag{3.2}$$

with $D^i = (D^i_t;\ t \geq 0)$ being the dividend process corresponding to the ith CDS. The latter is a finite variation process given by

$$D^i_t = L_i H_i(t) - \nu_i \int_0^t (1 - H_i(s))\mathrm{d}s, \quad t \geq 0, \tag{3.3}$$

where $D^i_0 = 0$, for $i = 1, \ldots, M$. For $0 \leq t \leq T$, we define $\pi^B_t := \frac{\phi^B(t)B_t}{V_t(\bar{\boldsymbol{\phi}})}$ to denote the proportion of wealth invested in the money market account at t. For the CDS investment strategy, it turns out more convenient to develop the analysis using the number of shares. Next, we provide a definition.

DEFINITION 3.1. The admissible control set $\mathcal{U}_t(v, \mathbf{z})$, $(t, v, \mathbf{z}) \in [0, T] \times R_+ \times \mathcal{S}$, is a class of $\mathbb{G}$-predictable locally bounded feedback trading strategies given by

$$\boldsymbol{\phi}(u) = \boldsymbol{\phi}\left(u, V^{t,v}_{u-}(\bar{\boldsymbol{\phi}}), \mathbf{H}(u-)\right), \quad u \in [t, T],$$

where $\boldsymbol{\phi}(u) = (\phi_1(u), \ldots, \phi_M(u))$, and $\phi_i(u)$ denotes the predefault number of shares of the ith CDS contract

$$\phi_i(u) = (1 - H_i(u))\phi_i(u) = (1 - H_i(u))\phi_i(u, V_{u-}(\bar{\boldsymbol{\phi}})).$$

Here $(V^{t,v}_u(\bar{\boldsymbol{\phi}});\ u \in [t, T])$ denotes the positive wealth process associated with the strategy $\bar{\boldsymbol{\phi}} = (\boldsymbol{\phi}, \pi^B)$ when $V_t(\bar{\boldsymbol{\phi}}) = v$ and $\mathbf{H}(t) = \mathbf{z}$. Throughout the paper, a trading strategy satisfying the above conditions is said to be t-admissible with respect to the initial conditions $V_t(\bar{\boldsymbol{\phi}}) = v$ and $\mathbf{H}(t) = \mathbf{z}$.

Let $\boldsymbol{\phi} \in \mathcal{U}_0 = \mathcal{U}_0(v, \mathbf{z})$ and $\bar{\boldsymbol{\phi}} = (\boldsymbol{\phi}, \pi^B)$. Then the dynamics of the wealth process is given by

$$\begin{aligned}
\mathrm{d}V_t(\bar{\boldsymbol{\phi}}) &= \sum_{i=1}^M \phi_i(t-)\mathrm{d}C^i_t + \sum_{i=1}^M \phi_i(t-)\mathrm{d}D^i_t + V_{t-}(\boldsymbol{\phi})\pi^B_t \frac{\mathrm{d}B_t}{B_t} \\
&= \sum_{i=1}^M \phi_i(t)\Bigg\{ r C^i_t + (C^i_t - L_i)h_i(\mathbf{H}(t)) \\
&\quad - \sum_{j\neq i} \left[\Phi_i(t, \mathbf{H}^j(t)) - C^i_t\right](1 - H_j(t))h_j(\mathbf{H}(t))\Bigg\}\mathrm{d}t \\
&\quad + \sum_{i=1}^M \phi_i(t-)\left(L_i - C^i_{t-}\right)\mathrm{d}H_i(t) \\
&\quad + \sum_{i=1}^M \phi_i(t-)\sum_{j\neq i}\left[\Phi_i(t, \mathbf{H}^j(t-)) - C^i_{t-}\right]\mathrm{d}H_j(t) + V_t(\bar{\boldsymbol{\phi}})r\pi^B_t\mathrm{d}t \\
&= \Bigg\{ r V_t(\bar{\boldsymbol{\phi}}) + \sum_{i=1}^M \phi_i(t)\Big[(C^i_t - L_i)h_i(\mathbf{H}(t)) \\
&\quad - \sum_{j\neq i}\left(\Phi_i(t, \mathbf{H}^j(t)) - C^i_t\right)(1 - H_j(t))h_j(\mathbf{H}(t))\Big]\Bigg\}\mathrm{d}t
\end{aligned}$$

$$+ \sum_{i=1}^{M} \phi_i(t-) \left(L_i - C^i_{t-}\right) \mathrm{d} H_i(t)$$

$$+ \sum_{i=1}^{M} \phi_i(t-) \sum_{j \neq i} \left(\Phi_i(t, \mathbf{H}^j(t-)) - C^i_{t-}\right) \mathrm{d} H_j(t), \tag{3.4}$$

with $V_0(\bar{\boldsymbol{\phi}}) = v$. Above, we have used the equalities $C^i_t = (1 - H_i(t))\Phi_i(t, \mathbf{H}(t))$ (see (2.5)), $\phi_i(t) = \phi_i(t)(1 - H_i(t))$. Moreover, we have used the self-financing condition

$$\sum_{i=1}^{M} \phi_i(t) C^i_t = V_t(\bar{\boldsymbol{\phi}}) \left(1 - \pi^B_t\right),$$

so that the dynamics of the wealth process $V_t(\bar{\boldsymbol{\phi}})$ only depends on the CDS investment strategy $\boldsymbol{\phi}$. For this reason, we use $V_t(\boldsymbol{\phi})$ in place of $V_t(\bar{\boldsymbol{\phi}})$ hereafter. Then, we have the following

LEMMA 3.2. *For any admissible strategy $\boldsymbol{\phi} \in \mathcal{U}_0$, the $\mathbb{P}$-dynamics of the wealth process $V = (V_t(\boldsymbol{\phi});\ t \geq 0)$ is given by*

$$\begin{aligned}
\mathrm{d} V_t(\boldsymbol{\phi}) = &\Bigg\{ r V_t(\boldsymbol{\phi}) + \sum_{i=1}^{M} \phi_i(t) \Big[(\Phi_i(t, \mathbf{H}(t)) - L_i)\, h_i(\mathbf{H}(t)) \\
&- \sum_{j \neq i} \left(\Phi_i(t, \mathbf{H}^j(t)) - \Phi_i(t, \mathbf{H}(t))\right) (1 - H_j(t)) h_j(\mathbf{H}(t)) \Big] \Bigg\} \mathrm{d}t \\
&+ \sum_{i=1}^{M} \phi_i(t-)\, (L_i - \Phi_i(t, \mathbf{H}(t-)))\, \mathrm{d} H_i(t) \\
&+ \sum_{i=1}^{M} \phi_i(t-) \sum_{j \neq i} \left(\Phi_i(t, \mathbf{H}^j(t-)) - \Phi_i(t, \mathbf{H}(t-))\right) \mathrm{d} H_j(t).
\end{aligned} \tag{3.5}$$

REMARK 3.3. From the wealth dynamics given by (3.5), it can be seen that the jump $\Delta V_u = V_u - V_{u-}$ at time u is given by

$$\begin{aligned}
\Delta V_u = &\sum_{i=1}^{M} \phi_i(u, V_{u-}, \mathbf{H}(u-))\, (L_i - \Phi_i(u, \mathbf{H}(u-)))\, \Delta H_i(u) \\
&+ \sum_{i=1}^{M} \phi_i(u, V_{u-}, \mathbf{H}(u-)) \sum_{j \neq i} \left(\Phi_i(t, \mathbf{H}^j(u-)) - \Phi_i(u, \mathbf{H}(u-))\right) \Delta H_j(u).
\end{aligned} \tag{3.6}$$

For $(V_u;\ u \in [t, T])$ to be strictly positive, it is necessary and sufficient that $\Delta V_u > -V_{u-}$ a.s. for any $u \in [t, T]$ (see, cf. Jacod and Shiryaev 2003, theorem 4.61). We will account for this condition in Section ,4.1 when we characterize the optimal CDS investment strategy under the different default states.

For $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_M) \in \mathcal{U}_0$, define the operators, depending on $\boldsymbol{\phi}$, acting on the smooth function $w(t, v, \mathbf{z})$ as

$$
\begin{aligned}
\mathcal{L}_v w(t, v, \mathbf{z}) = \Bigg\{ \sum_{i=1}^{M} \phi_i (1 - z_i) \Bigg[(\Phi_i(t, \mathbf{z}) - L_i) h_i(\mathbf{z}) \\
- \sum_{j \neq i} (\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z}))(1 - z_j) h_j(\mathbf{z}) \Bigg] + r v \Bigg\} w_v(t, v, \mathbf{z}), \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_J w(t, v, \mathbf{z}) = \sum_{i=1}^{M} \Big[w\left(t, v + \phi_i (1 - z_i)(L_i - \Phi_i(t, \mathbf{z})), \mathbf{z}^i\right) - w(t, v, \mathbf{z}) \Big] (1 - z_i) h_i^{\mathbb{P}}(\mathbf{z}) \\
+ \sum_{i=1}^{M} \Bigg\{ \sum_{j \neq i} \Big[w\left(t, v + \phi_i (1 - z_i)(\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z})), \mathbf{z}^j\right) - w(t, v, \mathbf{z}) \Big] \\
\times (1 - z_j) h_j^{\mathbb{P}}(\mathbf{z}) \Bigg\}.
\end{aligned}
$$

REMARK 3.4. Let $\varphi^R(t, v, \mathbf{z})$ be a smooth function, where $(t, v, \mathbf{z}) \in [0, T] \times R_+ \times \{0, 1\}^M$. Then using Itô's formula, it follows that

$$
\begin{aligned}
\varphi^R(t, V_t, \mathbf{H}(t)) = \varphi^R(t, v, \mathbf{z}) + \int_0^t \varphi_s^R(s, V_s, \mathbf{H}(s)) \mathrm{d}s + \int_0^t \varphi_v^R(s, V_s, \mathbf{H}(s)) \mathrm{d} V_s^c \\
+ \sum_{i=1}^{M} \Bigg\{ \int_0^t \Big[\varphi^R\left(s, V_{s-} + \phi_i(s-)(L_i - \Phi_i(s, \mathbf{H}(s-)))\right), \mathbf{H}^i(s-)) \\
- \varphi^R(s, V_{s-}, \mathbf{H}(s-)) \Big] \mathrm{d} H_i(s) \Bigg\} \\
+ \sum_{i=1}^{M} \Bigg\{ \sum_{j \neq i} \int_0^t \Big[\varphi^R\big(s, V_{s-} + \phi_i(s-)(\Phi_i(s, \mathbf{H}^j(s-)) \\
- \Phi_i(s, \mathbf{H}(s-))), \mathbf{H}^j(s-)\big) - \varphi^R(s, V_{s-}, \mathbf{H}(s-)) \Big] \mathrm{d} H_j(s) \Big] \Bigg\},
\end{aligned}
$$

where $V_t = V_t(\boldsymbol{\phi})$ is the wealth process given by (3.5) and V^c denotes the continuous part of the wealth process.

Next, we formulate the portfolio optimization problem. To this purpose, we define the objective functional

$$
J^T(\boldsymbol{\phi}; v, \mathbf{z}) := \mathbb{E}^{\mathbb{P}} \left[U(V_T(\boldsymbol{\phi})) \Big| V_0 = v, \ \mathbf{H}(0) = \mathbf{z} \right],
$$

where $T \in (0, \min(T_1, \ldots, T_M))$ is the time horizon. The utility function $U : [0, \infty) \longrightarrow [0, \infty)$ is chosen to be the power utility

$$
U(v) = \frac{v^\gamma}{\gamma},
$$

where $\gamma \in (0, 1)$ is the risk-aversion parameter.

Our goal is to maximize the objective functional $J^T(\boldsymbol{\phi}; v, \mathbf{z})$ across the class of admissible strategies $\boldsymbol{\phi} \in \mathcal{U}_t$ defined in (3.1). Hence, we consider the following dynamic optimization problem:

$$w^T(t, v, \mathbf{z}) := \sup_{\boldsymbol{\phi}\in\mathcal{U}_t=\mathcal{U}_t(v,\mathbf{z})} \mathbb{E}^{\mathbb{P}}\left[U(V_T(\boldsymbol{\phi}))\Big| V_t(\boldsymbol{\phi}) = v,\ \mathbf{H}(t) = \mathbf{z}\right], \tag{3.8}$$

where $V_t(\boldsymbol{\phi})$ is given by Lemma 3.2. Here, we use $\mathbb{E}^{\mathbb{P}}$ to denote the expectation with respect to the historical measure $\mathbb{P}$.

To start with, let us assume that $w^T(t, v, \mathbf{z})$ is C^1 in t, and C^2 in v for each $\mathbf{z} \in \mathcal{S}$. Then using Itô's formula along the lines of Remark 3.4, we have that for any $u > t \geq 0$,

$$\begin{aligned} w^T(u, V_u(\boldsymbol{\phi}), \mathbf{H}(u)) &= w^T(t, V_t(\boldsymbol{\phi}), \mathbf{H}(t)) + \\ &\quad + \int_t^u \left(\frac{\partial}{\partial s} + \mathcal{L}\right) w^T(s, V_s(\boldsymbol{\phi}), \mathbf{H}(s))\, \mathrm{d}s + \mathcal{M}(u) - \mathcal{M}(t), \end{aligned}$$

where the operator $\mathcal{L} = \mathcal{L}_v + \mathcal{L}_J$ and the process $\mathcal{M} = (\mathcal{M}(t);\ t \geq 0)$ is a $\mathbb{P}$-(local) martingale. Next, for $0 \leq t < u \leq T$, by virtue of the dynamic programming principle we expect that

$$w^T(t, v, \mathbf{z}) = \sup_{\boldsymbol{\phi}\in\mathcal{U}_t} \mathbb{E}^{\mathbb{P}}\left[w^T(u, V_u(\boldsymbol{\phi}), \mathbf{H}(u))\,\Big|\, V_t(\boldsymbol{\phi}) = v, \mathbf{H}(t) = \mathbf{z}\right]. \tag{3.9}$$

Therefore, we obtain $\mathbb{E}_t^{\mathbb{P}}\left[\int_t^u \left(\frac{\partial}{\partial s} + \mathcal{L}\right) w^T(s, V_s(\boldsymbol{\phi}), \mathbf{H}(s))\, \mathrm{d}s\right] \leq 0$, with the inequality becoming an equality if $\boldsymbol{\phi} = \boldsymbol{\phi}^*$, where $\boldsymbol{\phi}^*$ denotes the optimum. This leads to the following HJB equation: for each $\mathbf{z} \in \mathcal{S}$, on $(t, v) \in [0, T) \times R_+$,

$$\sup_{\boldsymbol{\phi}\in\mathcal{U}_t} \left(\frac{\partial}{\partial t} + \mathcal{L}\right) w^T(t, v, \mathbf{z}) = 0, \tag{3.10}$$

with terminal condition $w^T(T, v, \mathbf{z}) = U(v)$ for all $(v, \mathbf{z}) \in R_+ \times \mathcal{S}$.

4. OPTIMAL INVESTMENT STRATEGIES AND HJB EQUATIONS

We analyze the optimal investment strategy in CDSs in Section 4.1, assuming that a smooth solution to the HJB equation exists. We then prove existence and uniqueness of a positive solution to the HJB equation (3.10) in Section 4.2.

4.1. Optimal Strategies

We start analyzing the optimal strategy, denoted by $\boldsymbol{\phi}^*$, using the first-order condition. For future purposes, it is convenient to introduce the notation ψ_i, defined by $\phi_i := v\psi_i$, for $i = 1, \ldots, M$. The optimum can be written as $\boldsymbol{\phi}^* = v\boldsymbol{\psi}^* = v(\psi_1^*, \ldots, \psi_M^*)$. Define

$$f^T(\boldsymbol{\phi}; t, v, \mathbf{z}) := (\mathcal{L}_v + \mathcal{L}_J)\, w^T(t, v, \mathbf{z}).$$

For each $i = 1, \ldots, M$, the first-order condition given by

$$f_i^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) := \frac{\partial f^T(\boldsymbol{\phi}; t, v, \mathbf{z})}{\partial \psi_i} = 0 \tag{4.1}$$

yields the optimum ψ_i^*, i.e., $f_i^{T,'}(\boldsymbol{\phi}^*; t, v, \mathbf{z}) = 0$. It can be seen that

$$
\begin{aligned}
& f_i^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) \\
&= v w_v^T(t, v, \mathbf{z})(1 - z_i)\Big[(\Phi_i(t, \mathbf{z}) - L_i)\, h_i(\mathbf{z}) - \sum_{j \neq i} \left(\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z})\right)(1 - z_j) h_j(\mathbf{z}) \Big] \\
&\quad + v w_v^T\left(t, v + v\psi_i(1 - z_i)(L_i - \Phi_i(t, \mathbf{z})), \mathbf{z}^i\right)(L_i - \Phi_i(t, \mathbf{z}))(1 - z_i) h_i^{\mathbb{P}}(\mathbf{z}) \\
&\quad + \sum_{j \neq i} v w_v^T\left(t, v + v\psi_i(1 - z_i)\left(\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z})\right), \mathbf{z}^j\right)(1 - z_j) h_j^{\mathbb{P}}(\mathbf{z})(1 - z_i) \\
&\quad \times \left(\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z})\right). \qquad (4.2)
\end{aligned}
$$

Using the first-order condition (4.1), we can write the optimum ψ_i^* as

$$\psi_i^* = \psi_i^*(t, v, \mathbf{z}).$$

We postulate (and later show) that the value function w^T is separable, and given by

$$w^T(t, v, \mathbf{z}) = v^\gamma B(t, \mathbf{z}), \qquad (4.3)$$

where $B(t, \mathbf{z})$ satisfies an ODE analyzed in the next section. We now discuss the existence and uniqueness of the optimal strategy $\boldsymbol{\psi}^*(t, \mathbf{z}) = (\psi_1^*(t, \mathbf{z}), \ldots, \psi_M^*(t, \mathbf{z}))$. We separate the analysis into three cases, namely (1) all names alive, (2) some names defaulted, and (3) all names defaulted. We have

(1) All names are alive, i.e., $\mathbf{z} = \mathbf{0}$. From (4.2), for $i = 1, \ldots, M$, the optimum $\psi_i^*(t, \mathbf{0})$ satisfies

$$
\begin{aligned}
0 &= v w_v^T(t, v, \mathbf{0})\Big[(\Phi_i(t, \mathbf{0}) - L_i)\, h_i(\mathbf{0}) - \sum_{j \neq i} \left(\Phi_i(t, \mathbf{0}^j) - \Phi_i(t, \mathbf{0})\right) h_j(\mathbf{0}) \Big] \\
&\quad + v w_v^T\left(t, v + v\psi_i^*(t, \mathbf{0})(L_i - \Phi_i(t, \mathbf{0})), \mathbf{0}^i\right)(L_i - \Phi_i(t, \mathbf{0})) h_i^{\mathbb{P}}(\mathbf{0}) + \sum_{j \neq i} v w_v^T \\
&\quad \times \left(t, v + v\psi_i^*(t, \mathbf{0})\left(\Phi_i(t, \mathbf{0}^j) - \Phi_i(t, \mathbf{0})\right), \mathbf{0}^j\right) h_j^{\mathbb{P}}(\mathbf{0})\left(\Phi_i(t, \mathbf{0}^j) - \Phi_i(t, \mathbf{0})\right).
\end{aligned}
$$

Using the representation $w^T(t, v, \mathbf{0}) = v^\gamma B(t, \mathbf{0})$, the optimum $\psi_i^*(t, \mathbf{0})$ satisfies

$$
\begin{aligned}
0 &= V_i(t) B(t, \mathbf{0}) + B(t, \mathbf{0}^i)(L_i - \Phi_i(t, \mathbf{0})) h_i^{\mathbb{P}}(\mathbf{0})\left(1 + \psi_i^*(t, \mathbf{0})(L_i - \Phi_i(t, \mathbf{0}))\right)^{\gamma - 1} \\
&\quad + \sum_{j \neq i} B(t, \mathbf{0}^j)\left(\Phi_i(t, \mathbf{0}^j) - \Phi_i(t, \mathbf{0})\right) h_j^{\mathbb{P}}(\mathbf{0})\left[1 + \psi_i^*(t, \mathbf{0})\left(\Phi_i(t, \mathbf{0}^j)\right.\right. \\
&\qquad \left.\left. -\ \Phi_i(t, \mathbf{0})\right)\right]^{\gamma - 1}, \qquad (4.4)
\end{aligned}
$$

where the coefficient

$$V_i(t) := (\Phi_i(t, \mathbf{0}) - L_i)\, h_i(\mathbf{0}) - \sum_{j \neq i} \left(\Phi_i(t, \mathbf{0}^j) - \Phi_i(t, \mathbf{0})\right) h_j(\mathbf{0}).$$

Define the $M + 1$-dimensional vector

$$\mathbf{B}^{(i)}(t) := \left[B(t, \mathbf{0}),\ B(t, \mathbf{0}^i),\ B(t, \mathbf{0}^j);\ j \in \{1, \ldots, M\}, j \neq i\right]^\top.$$

Moreover, for $i = 1, \dots, M$, define

$$g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right) = V_i(t)\mathbf{B}_1^{(i)} + \mathbf{B}_2^{(i)}(L_i - \Phi_i(t,\mathbf{0}))h_i^{\mathbb{P}}(\mathbf{0})\left[1 + \psi_i(L_i - \Phi_i(t,\mathbf{0}))\right]^{\gamma-1} + \sum_{j\neq i}\mathbf{B}_j^{(i)}\left(\Phi_i(t,\mathbf{0}^j) - \Phi_i(t,\mathbf{0})\right)h_j^{\mathbb{P}}(\mathbf{0})\left[1 + \psi_i\left(\Phi_i(t,\mathbf{0}^j) - \Phi_i(t,\mathbf{0})\right)\right]^{\gamma-1}. \tag{4.5}$$

Our goal is to find the optimum $\psi_i^*(t,\mathbf{0})$ satisfying the following conditions. For each $i = 1, \dots, M$,

$$\psi_i^*(t,\mathbf{0}) > -\frac{1}{L_i - \Phi_i(t,\mathbf{0})} =: M_1^{(i)}(t), \quad \text{and}$$
$$\psi_i^*(t,\mathbf{0})\left(\Phi_i(t,\mathbf{0}^j) - \Phi_i(t,\mathbf{0})\right) > -1, \quad \forall\, j \neq i. \tag{4.6}$$

Without loss of generality, assume that for some $m = 1, \dots, M$, there exist $j_1, \dots, j_m$, $j_1 \neq \dots \neq j_m$, belonging to $\{1, \dots, M\} \setminus \{i\}$ such that $\Phi_i(t,\mathbf{0}^{j_k}) < \Phi_i(t,\mathbf{0})$ for $k = 1, \dots, m$. Then, for $k = m+1, \dots, M$, it holds that $\Phi_i(t,\mathbf{0}^{j_k}) > \Phi_i(t,\mathbf{0})$.

Note that

$$\max\left\{M_1^{(i)}(t), \frac{1}{\Phi_i(t,\mathbf{0}) - \Phi_i(t,\mathbf{0}^{j_k})}; \; k = m+1, \dots, M\right\} = M_1^{(i)}(t),$$

since $\Phi_i(t,\mathbf{0}^j) < L_i$ for all $j = 1, \dots, M$, $j \neq i$, using Lemma A.1. Define

$$M_2^{(i)}(t) := \min\left\{\frac{1}{\Phi_i(t,\mathbf{0}) - \Phi_i(t,\mathbf{0}^{j_k})}; \; k = 1, \dots, m\right\},$$

where $M_2^{(i)}(t) = +\infty$ if the set $\{\cdot\}$ on the right-hand side of the above equality is empty (i.e., $m = 0$). Hence, for each $i = 1, \dots, M$, the optimum $\psi_i^*(t,\mathbf{0})$ should satisfy

$$M_1^{(i)}(t) < \psi_i^*(t,\mathbf{0}) < M_2^{(i)}(t). \tag{4.7}$$

The following lemma, whose proof is reported in the Appendix, establishes existence and uniqueness of a unique solution ψ_i satisfying the first-order condition (4.4), under the constraints specified by (4.7).

LEMMA 4.1. *For each $i = 1, \dots, M$, there exists a unique optimum $\psi_i^*(t,\mathbf{0})$ satisfying the equation*

$$g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right) = 0, \tag{4.8}$$

for each $(t, \mathbf{B}^{(i)}) \in [0,T] \times R_+^{M+1}$, subject to the condition (4.7). Moreover, the optimum $\psi_i^*(t,\mathbf{0})$, viewed as a function of $(t, \mathbf{B}^{(i)}) \in [0,T] \times R_+^{M+1}$, is continuous in $(t, \mathbf{B}^{(i)})$ for each $i = 1, \dots, M$.

(2) Some names are defaulted, i.e., $\mathbf{z} = \left((\mathbf{0}^{j_1})^{\cdots}\right)^{j_m}$, for some $j_1 \neq \dots \neq j_m$ belonging to $\{1, \dots, M\}$, where $1 \le m \le M-1$. Recall here the definition of $\mathbf{z}^i$ given in (2.2). For brevity, we use the shorthand notation $\mathbf{0}^{j_1,\dots,j_m} := ((\mathbf{0}^{j_1})^{\cdots})^{j_m}$. Clearly, $\mathbf{0}^{j_1,\dots,j_M} = \mathbf{1}$. As in the previous case, we consider existence and uniqueness

of the optimum $\psi_i^*(t, \mathbf{z})$ implied by the first-order condition (4.1). This is done via a sequential procedure. Before proceeding further, we also introduce the notation $\psi_{i,j_1,\dots,j_m}(t) := \psi_i(t, \mathbf{0}^{j_1,\dots,j_m})$, $B_{j_1,\dots,j_m}(t) := B(t, \mathbf{0}^{j_1,\dots,j_m})$, $h_{i,j_1,\dots,j_m} := h_i(\mathbf{0}^{j_1,\dots,j_m})$ and $\Phi_{i,j_1,\dots,j_m}(t) := \Phi_i(t, \mathbf{0}^{j_1,\dots,j_m})$. For $i = 1, \dots, M$, we consider separate cases, using again the separable form $w^T(t, v, \mathbf{0}^{j_1,\dots,j_m}) = v^\gamma B_{j_1,\dots,j_m}(t)$.

Case 1. If $i \neq j_1, \dots, j_m$, then the first-order condition (4.1) for $\mathbf{z} = \mathbf{0}^{j_1,\dots,j_m}$ is reduced to

$$
\begin{aligned}
0 = {} & B_{j_1,\dots,j_m}(t)\Big[\left(\Phi_{i,j_1,\dots,j_m}(t) - L_i\right) h_{i,j_1,\dots,j_m} \\
& - \sum_{\ell\neq i, \ell\neq j_1,\dots,j_m} \left(\Phi_{i,j_1,\dots,j_m,\ell}(t) - \Phi_{i,j_1,\dots,j_m}(t)\right) h_{\ell,j_1,\dots,j_m}\Big] \\
& + B_{j_1,\dots,j_m,i}(t)\left[1 + \psi_{i,j_1,\dots,j_m}(L_i - \Phi_{i,j_1,\dots,j_m}(t))\right]^{\gamma-1} (L_i - \Phi_{i,j_1,\dots,j_m}(t)) h^{\mathbb{P}}_{i,j_1,\dots,j_m} \\
& + \sum_{\ell\neq i, \ell\neq j_1,\dots,j_m} B_{j_1,\dots,j_m,\ell}(t)\left[1 + \psi_{i,j_1,\dots,j_m}\left(\Phi_{i,j_1,\dots,j_m,\ell}(t) - \Phi_{i,j_1,\dots,j_m}(t)\right)\right]^{\gamma-1} \\
& \times h^{\mathbb{P}}_{\ell,j_1,\dots,j_m}\left(\Phi_{i,j_1,\dots,j_m,\ell}(t) - \Phi_{i,j_1,\dots,j_m}(t)\right).
\end{aligned} \tag{4.9}
$$

Next, we distinguish two subcases:

- $m = M - 1$. Then, it holds that for $i \neq j_1, \dots, j_{M-1}$,

$$
\begin{aligned}
& \sum_{\ell\neq i, \ell\neq j_1,\dots,j_{M-1}} \left(\Phi_{i,j_1,\dots,j_{M-1},\ell}(t) - \Phi_{i,j_1,\dots,j_{M-1}}(t)\right) h_{\ell,j_1,\dots,j_{M-1}} = 0, \quad \text{and} \\
& \sum_{\ell\neq i, \ell\neq j_1,\dots,j_{M-1}} B_{j_1,\dots,j_{M-1},\ell}(t)\left[1 + \psi_{i,j_1,\dots,j_{M-1}}\left(\Phi_{i,j_1,\dots,j_{M-1},\ell}(t) - \Phi_{i,j_1,\dots,j_{M-1}}(t)\right)\right]^{\gamma-1} \\
& \qquad \times h^{\mathbb{P}}_{\ell,j_1,\dots,j_{M-1}}\left(\Phi_{i,j_1,\dots,j_{M-1},\ell}(t) - \Phi_{i,j_1,\dots,j_{M-1}}(t)\right) = 0.
\end{aligned}
$$

  Consequently, we obtain that equation (4.9) simplifies to

$$
\begin{aligned}
0 = {} & B_{j_1,\dots,j_{M-1}}(t)\left(\Phi_{i,j_1,\dots,j_{M-1}}(t) - L_i\right) h_{i,j_1,\dots,j_{M-1}} \\
& + B_{j_1,\dots,j_{M-1},i}(t)\left[1 + \psi_{i,j_1,\dots,j_{M-1}}(L_i - \Phi_{i,j_1,\dots,j_{M-1}}(t))\right]^{\gamma-1} \\
& \times (L_i - \Phi_{i,j_1,\dots,j_{M-1}}(t)) h^{\mathbb{P}}_{i,j_1,\dots,j_{M-1}},
\end{aligned}
$$

  where we used $\Phi_{i,j_1,\dots,j_{M-1},i}(t) = \Phi_{i,\mathbf{1}}(t) = L_i$, which leads, for $i \neq j_1, \dots, j_{M-1}$, to

$$
\begin{aligned}
\psi^*_{i,j_1,\dots,j_{M-1}}(t) &= \frac{1}{L_i - \Phi_{i,j_1,\dots,j_{M-1}}(t)} \left\{ \left[\frac{B_{j_1,\dots,j_{M-1}}(t) h_{i,j_1,\dots,j_{M-1}}}{B_{j_1,\dots,j_{M-1},i}(t) h^{\mathbb{P}}_{i,j_1,\dots,j_{M-1}}} \right]^{\frac{1}{\gamma-1}} - 1 \right\} \\
&= \frac{1}{L_i - \Phi_{i,j_1,\dots,j_{M-1}}(t)} \left\{ \left[\frac{B_{j_1,\dots,j_{M-1}}(t) h_{i,j_1,\dots,j_{M-1}}}{B(t, \mathbf{1}) h^{\mathbb{P}}_{i,j_1,\dots,j_{M-1}}} \right]^{\frac{1}{\gamma-1}} - 1 \right\}.
\end{aligned} \tag{4.10}
$$

- $m < M - 1$. Define the set

$$
N_{i,j_1,\dots,j_m} := \{\ell \in \{1, \dots, M\};\ \ell \neq i,\ \ell \neq j_1, \dots, j_m\}. \tag{4.11}
$$

Moreover, define

$$\mathbf{B}^{(j_1,\dots,j_m,i)}(t) := \big[B_{j_1,\dots,j_m}(t),\ B_{j_1,\dots,j_m,i}(t),\ B_{j_1,\dots,j_m,\ell}(t);\ \ell \in N_{i,j_1,\dots,j_m}\big]^\top, \tag{4.12}$$

and, on $(t, \mathbf{B}^{(j_1,\dots,j_m,i)}) \in [0, T] \times R_+^{2+|N_{i,j_1,\dots,j_m}|}$, define the function

$$\begin{aligned} & g_{i,j_1,\dots,j_m}\left(\psi_{i,j_1,\dots,j_m}, t, \mathbf{B}^{(j_1,\dots,j_m,i)}\right) \\ &:= V_{i,j_1,\dots,j_m}(t)\mathbf{B}_1^{(j_1,\dots,j_m,i)} \\ &\quad + \mathbf{B}_2^{(j_1,\dots,j_m,i)}\left[1 + \psi_{i,j_1,\dots,j_m}(L_i - \Phi_{i,j_1,\dots,j_m}(t)\right]^{\gamma-1}(L_i - \Phi_{i,j_1,\dots,j_m}(t))h^{\mathbb{P}}_{i,j_1,\dots,j_m} \\ &\quad + \sum_{\ell\in N_{i,j_1,\dots,j_m}} \mathbf{B}_\ell^{(j_1,\dots,j_m,i)}\left[1 + \psi_{i,j_1,\dots,j_m}\left(\Phi_{i,j_1,\dots,j_m,\ell}(t) - \Phi_{i,j_1,\dots,j_m}(t)\right)\right]^{\gamma-1} \\ &\quad \times h^{\mathbb{P}}_{\ell,j_1,\dots,j_m}\left(\Phi_{i,j_1,\dots,j_m,\ell}(t) - \Phi_{i,j_1,\dots,j_m}(t)\right), \end{aligned} \tag{4.13}$$

where the coefficient

$$\begin{aligned} V_{i,j_1,\dots,j_m}(t) &:= \left(\Phi_{i,j_1,\dots,j_m}(t) - L_i\right) h_{i,j_1,\dots,j_m} \\ &\quad - \sum_{\ell\in N_{i,j_1,\dots,j_m}} \left(\Phi_{i,j_1,\dots,j_m,\ell}(t) - \Phi_{i,j_1,\dots,j_m}(t)\right) h_{\ell,j_1,\dots,j_m}. \end{aligned}$$

Then, the first-order condition in equation (4.9) simplifies to

$$g_{i,j_1,\dots,j_m}\left(\psi_{i,j_1,\dots,j_m}, t, \mathbf{B}^{(j_1,\dots,j_m,i)}\right) = 0.$$

Altogether, we obtain the following lemma whose proof is reported in the Appendix.

LEMMA 4.2. *For each $i \neq j_1, \dots, j_m$, there exists a unique optimum $\psi^*_{i,j_1,\dots,j_m}(t)$ satisfying the first-order condition* (4.9) *subject to*

$$M_1^{(i,j_1,\dots,j_m)}(t) < \psi^*_{i,j_1,\dots,j_m}(t) < M_2^{(i,j_1,\dots,j_m)}(t), \tag{4.15}$$

where

$$\begin{aligned} M_1^{(i,j_1,\dots,j_m)}(t) &= -\frac{1}{L_i - \Phi_{i,j_1,\dots,j_m}(t)}, \\ M_2^{(i,j_1,\dots,j_m)}(t) &= \min\left\{\frac{1}{\Phi_{i,j_1,\dots,j_m}(t) - \Phi_{i,j_1,\dots,j_m,\ell_n}(t)};\ n = 1, \dots, k\right\}. \end{aligned}$$

Moreover, the optimum $\psi^*_{i,j_1,\dots,j_m}(t)$, viewed as a function of $(t, \mathbf{B}^{(j_1,\dots,j_m,i)}) \in [0, T] \times R_+^{2+|N_{i,j_1,\dots,j_m}|}$, is continuous in $(t, \mathbf{B}^{(j_1,\dots,j_m,i)})$ for each $i = 1, \dots, M$.

Case 2. If $i = j_k$ for some $k = 1, \dots, m$, then $f_i^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{0}^{j_1,\dots,j_m}) = 0$. We have that $\psi^*_{i,j_1,\dots,j_m}(t) = 0$, in this case.

(3) All names are defaulted, i.e., $\mathbf{z} = \mathbf{1}$. Then the optimum $\psi^*_{i,\mathbf{1}}(t) = 0$.

4.2. Analysis of HJB Equation

We analyze the HJB equation (3.10). Using the separable form (4.3) of w^T, the HJB equation (3.10) may be rewritten as

$$
\begin{aligned}
0 = {} & \frac{\mathrm{d}B(t,\mathbf{z})}{\mathrm{d}t} + \gamma\Bigg\{ \sum_{i=1}^{M} \psi_i^*(1-z_i)\Bigg[(\Phi_i(t,\mathbf{z}) - L_i)\, h_i(\mathbf{z}) \\
& - \sum_{j\neq i} \big(\Phi_i(t,\mathbf{z}^j) - \Phi_i(t,\mathbf{z})\big)\,(1-z_j)h_j(\mathbf{z})\Bigg] + r\Bigg\} B(t,\mathbf{z}) \\
& + \sum_{i=1}^{M} \big[\big(1+\psi_i^*(1-z_i)(L_i - \Phi_i(t,\mathbf{z}))\big)^{\gamma}\, B(t,\mathbf{z}^i) - B(t,\mathbf{z})\big](1-z_i)h_i^{\mathbb{P}}(\mathbf{z}) \\
& + \sum_{i=1}^{M}\Bigg\{\sum_{j\neq i}\Bigg[\Big(1+\psi_i^*(1-z_i)\Big(\Phi_i(t,\mathbf{z}^j) - \Phi_i(t,\mathbf{z})\Big)\Big)^{\gamma} B(t,\mathbf{z}^j) - B(t,\mathbf{z})\Bigg](1-z_j)h_j^{\mathbb{P}}(\mathbf{z})\Bigg\}.
\end{aligned} \tag{4.16}
$$

We first consider the case when $\mathbf{z} = \mathbf{0}$. Then, we obtain

$$
\begin{aligned}
0 = {} & \frac{\mathrm{d}B(t,\mathbf{0})}{\mathrm{d}t} + \gamma\Bigg\{ \sum_{i=1}^{M} \psi_{i,\mathbf{0}}^*(t)\Bigg[(\Phi_i(t,\mathbf{0}) - L_i)h_i(\mathbf{0}) \\
& - \sum_{j\neq i} \big(\Phi_i(t,\mathbf{0}^j) - \Phi_i(t,\mathbf{0})\big)\, h_j(\mathbf{0})\Bigg]\Bigg\} B(t,\mathbf{0}) + a(\gamma)B(t,\mathbf{0}) \\
& + \sum_{i=1}^{M} \Bigg[\big(1+\psi_{i,\mathbf{0}}^*(t)(L_i - \Phi_i(t,\mathbf{0}))\big)^{\gamma}\, B(t,\mathbf{0}^i) - B(t,\mathbf{0})\Bigg] h_i^{\mathbb{P}}(\mathbf{0}) \\
& + \sum_{i=1}^{M}\Bigg\{\sum_{j\neq i}\Bigg[\Big(1+\psi_{i,\mathbf{0}}(t)\Big(\Phi_i(t,\mathbf{0}^j) - \Phi_i(t,\mathbf{0})\Big)\Big)^{\gamma}\, B(t,\mathbf{0}^j) - B(t,\mathbf{0})\Bigg] h_j^{\mathbb{P}}(\mathbf{0})\Bigg\},
\end{aligned} \tag{4.17}
$$

where $a(\gamma) = \gamma r$. The above equation indicates that $B(t,\mathbf{0})$, associated with the state where all names are alive, depends on $B(t,\mathbf{0}^j)$, $j = 1, \ldots, M$, i.e., on the solutions of the ODEs associated with states where one entity defaults. This reflects the contagion effect in the control problem. Iterating this procedure, it is immediate that we need to consider the equations associated with $B_{j_1,\ldots,j_m}(t)$ for $j_1 \neq \cdots \neq j_m$ belonging to $\{1, \ldots, M\}$, where $1 \leq m \leq M$. Hence, from equation (4.16), we can establish the following inductive relationship between $B_{j_1,\cdots,j_m,j_{m+1}}(t)$ and $B_{j_1,\cdots,j_m}(t)$. For $1 \leq m \leq M-1$, we have

$$
\begin{aligned}
0 = {} & \frac{\mathrm{d}B_{j_1,\cdots,j_m}(t)}{\mathrm{d}t} + \gamma\Bigg\{ \sum_{i\neq j_1,\cdots,j_m} \psi_{i,j_1,\ldots,j_m}^*(t)\Bigg[\big(\Phi_{i,j_1,\ldots,j_m}(t) - L_i\big)\, h_{i,j_1,\cdots,j_m} \\
& - \sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m} \big(\Phi_{i,j_1,\cdots,j_m,\ell}(t) - \Phi_{i,j_1,\ldots,j_m}(t)\big)\, h_{\ell,j_1,\cdots,j_m}\Bigg]\Bigg\} B_{j_1,\cdots,j_m}(t) \\
& + a(\gamma)B_{j_1,\ldots,j_m}(t) \\
& + \sum_{i\neq j_1,\cdots,j_m} \Bigg[\big(1+\psi_{i,j_1,\ldots,j_m}^*(t)(L_i - \Phi_{i,j_1,\ldots,j_m}(t))\big)^{\gamma}\, B_{j_1,\cdots,j_m,i}(t) - B_{j_1,\cdots,j_m}(t)\Bigg] h_{i,j_1,\cdots,j_m}^{\mathbb{P}}
\end{aligned}
$$

$$+\sum_{i=1}^{M}\Bigg\{\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\Bigg[\Big(1+\psi^*_{i,j_1,\ldots,j_m}(t)(1-z_i)\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\ldots,j_m}(t)\Big)\Big)^{\gamma}$$

$$\times B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}. \tag{4.18}$$

The last term of equation (4.18) may be expanded further as

$$\begin{aligned}
&\sum_{i=1}^{M}\Bigg\{\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\Bigg[\Big(1+\psi^*_{i,j_1,\ldots,j_m}(t)(1-z_i)\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\Big)\Big)^{\gamma}\\
&\quad\times B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}\\
&=\sum_{i=j_1,\ldots,j_m}\Bigg\{\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\Bigg[\Big(1+\psi^*_{i,j_1,\ldots,j_m}(t)(1-z_i)\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\Big)\Big)^{\gamma}\\
&\quad\times B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}\\
&\quad+\sum_{i\neq j_1,\ldots,j_m}\Bigg\{\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\Bigg[\Big(1+\psi^*_{i,j_1,\ldots,j_m}(t)(1-z_i)\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\Big)\Big)^{\gamma}\\
&\quad\times B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}\\
&=\sum_{i=j_1,\ldots,j_m}\Bigg\{\sum_{\ell\neq j_1,\cdots,j_m}\Bigg[B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}\\
&\quad+\sum_{i\neq j_1,\ldots,j_m}\Bigg\{\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\Bigg[\Big(1+\psi^*_{i,j_1,\ldots,j_m}(t)\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\Big)\Big)^{\gamma}\\
&\quad\times B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}.
\end{aligned}\tag{4.19}$$

Hence, equation (4.18) reduces to

$$\begin{aligned}
0&=\frac{\mathrm{d}B_{j_1,\cdots,j_m}(t)}{\mathrm{d}t}+\gamma\Bigg\{\sum_{i\neq j_1,\cdots,j_m}\psi^*_{i,j_1,\ldots,j_m}(t)\Bigg[\big(\Phi_{i,j_1,\ldots,j_m}(t)-L_i\big)h_{i,j_1,\cdots,j_m}\\
&\quad-\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\ldots,j_m}(t)\big)h_{\ell,j_1,\cdots,j_m}\Bigg]\Bigg\}B_{j_1,\cdots,j_m}(t)\\
&\quad+a(\gamma)B_{j_1,\ldots,j_m}(t)\\
&\quad+\sum_{i\neq j_1,\cdots,j_m}\Bigg[\big(1+\psi^*_{i,j_1,\ldots,j_m}(t)(L_i-\Phi_{i,j_1,\ldots,j_m}(t))\big)^{\gamma}\,B_{j_1,\cdots,j_m,i}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{i,j_1,\cdots,j_m}\\
&\quad+\sum_{i=j_1,\ldots,j_m}\Bigg\{\sum_{\ell\neq j_1,\cdots,j_m}\Bigg[B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\Bigg]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\}
\end{aligned}$$

$$+\sum_{i\neq j_1,\ldots,j_m}\left\{\sum_{\ell\neq i,\ell\neq j_1,\cdots,j_m}\left[\left(1+\psi^*_{i,j_1,\ldots,j_m}(t)\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\Big)\right)^\gamma\right.\right.$$

$$(4.20)\quad \left.\left.\times\ B_{j_1,\cdots,j_m,\ell}(t)-B_{j_1,\cdots,j_m}(t)\right]h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\right\}.$$

We next show via a backward procedure that a unique positive solution to equation (4.20) exists.

(1) $m=M$. This means that $\mathbf{z}=\mathbf{1}$, hence $B(t,\mathbf{1})$ satisfies

$$\frac{\mathrm{d}B(t,\mathbf{1})}{\mathrm{d}t}=-a(\gamma)B(t,\mathbf{1}),\qquad B(T,\mathbf{1})=\frac{1}{\gamma}$$

yielding the positive solution:

$$B(t,\mathbf{1})=\frac{1}{\gamma}e^{a(\gamma)(T-t)},\quad 0\le t\le T. \tag{4.21}$$

(2) $m=M-1$. Then we have that $\mathbf{0}^{j_1,\cdots,j_{M-1},i}=\mathbf{1}$, for $i\neq j_1,\ldots,j_{M-1}$. From equation (4.20), it follows that (setting $j_M:=\{1,\ldots,M\}\setminus\{j_1,\ldots,j_{M-1}\}$)

$$\begin{aligned}
0=&\ \frac{\mathrm{d}B_{j_1,\ldots,j_{M-1}}(t)}{\mathrm{d}t}+a(\gamma)B_{j_1,\ldots,j_{M-1}}(t)\\
&+\gamma\psi^*_{j_M,j_1,\ldots,j_{M-1}}(t)\left(\Phi_{j_M,j_1,\cdots,j_{M-1}}(t)-L_{j_M}\right)h_{j_M,j_1,\cdots,j_{M-1}}B_{j_1,\ldots,j_{M-1}}(t)\\
&+\Big[\left(1+\psi^*_{j_M,j_1,\ldots,j_{M-1}}(t)(L_{j_M}-\Phi_{j_M,j_1,\cdots,j_{M-1}}(t)\right)^\gamma B(t,\mathbf{1})-B_{j_1,\ldots,j_{M-1}}(t)\Big]\\
&\times h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}+\sum_{i=j_1,\ldots,j_{M-1}}\Big[B(t,\mathbf{1})-B_{j_1,\ldots,j_{M-1}}(t)\Big]h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}.
\end{aligned}\tag{4.22}$$

Plugging the optimum $\psi^*_{j_M,j_1,\ldots,j_{M-1}}(t)$ given by (4.10) and the explicit solution $B(t,\mathbf{1})$ given by (4.21) into (4.22), we conclude that equation (4.22) may be reduced to

$$\frac{\mathrm{d}B_{j_1,\ldots,j_{M-1}}(t)}{\mathrm{d}t}=D_{j_M}\left(t,\,B_{j_1,\ldots,j_{M-1}}(t)\right),\quad B_{j_1,\ldots,j_{M-1}}(T)=\frac{1}{\gamma}, \tag{4.23}$$

where

$$\begin{aligned}
D_{j_M}(t,B):=&-\left[\gamma\psi^*_{j_M,j_1,\ldots,j_{M-1}}(t)\left(\Phi_{j_M,j_1,\cdots,j_{M-1}}(t)-L_{j_M}\right)h_{j_M,j_1,\cdots,j_{M-1}}+a(\gamma)\right]B\\
&-\Big[\left(1+\psi^*_{j_M,j_1,\ldots,j_{M-1}}(t)(L_{j_M}-\Phi_{j_M,j_1,\cdots,j_{M-1}}(t))\right)^\gamma B(t,\mathbf{1})-B\Big]h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}\\
&-\sum_{i=j_1,\ldots,j_{M-1}}(B(t,\mathbf{1})-B)\,h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}\\
=&\ K_{j_M}B+U_{j_M}(t)B^{\frac{\gamma}{\gamma-1}}+A_{j_M}(t).
\end{aligned}$$

Above, the coefficients are given by

$$\begin{aligned}
K_{j_M}&:=Mh^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}-\gamma h_{j_M,j_1,\cdots,j_{M-1}}-a(\gamma),\\
A_{j_M}(t)&:=-(M-1)B(t,\mathbf{1})h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}},
\end{aligned}$$

$$U_{j_M}(t) := \gamma h_{j_M,j_1,\cdots,j_{M-1}} \left(\frac{h_{j_M,j_1,\cdots,j_{M-1}}}{B(t,\mathbf{1}) h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}} \right)^{\frac{1}{\gamma-1}}$$
$$- B(t,\mathbf{1}) h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}} \left(\frac{h_{j_M,j_1,\cdots,j_{M-1}}}{B(t,\mathbf{1}) h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}} \right)^{\frac{\gamma}{\gamma-1}}$$
$$= (\gamma-1) h_{j_M,j_1,\cdots,j_{M-1}} \eta_{j_M,j_1,\cdots,j_{M-1}}^{\frac{1}{\gamma-1}} B(t,\mathbf{1})^{\frac{1}{1-\gamma}},$$

with $\eta_{j_M,j_1,\cdots,j_{M-1}} := \frac{h_{j_M,j_1,\cdots,j_{M-1}}}{h^{\mathbb{P}}_{j_M,j_1,\cdots,j_{M-1}}}$.
Hence, we can conclude that the function $B_{j_1,\dots,j_{M-1}}(t)$ satisfies the following ODE:

$$u'(t) = K_{j_M} u(t) + U_{j_M}(t) u(t)^{\frac{\gamma}{\gamma-1}} + A_{j_M}(t), \qquad u(T) = \frac{1}{\gamma}.$$

This implies that

$$u(t)^{\frac{\gamma}{1-\gamma}} u'(t) = K_{j_M} u(t)^{\frac{1}{1-\gamma}} + U_{j_M}(t) + A_{j_M}(t) u(t)^{\frac{\gamma}{1-\gamma}}.$$

Let $\hat{u}(t) = u(t)^{\frac{1}{1-\gamma}}$. Then, $\hat{u}'(t) = \frac{1}{1-\gamma} u(t)^{\frac{\gamma}{1-\gamma}} u'(t)$ and hence we have that

$$\hat{u}'(t) = K_{j_M}(\gamma)\hat{u}(t) + A_{j_M}(t,\gamma)\hat{u}(t)^{\gamma} + U_{j_M}(t,\gamma), \quad \hat{u}(T) = \gamma^{\frac{1}{\gamma-1}} > 0, \tag{4.24}$$

where $K_{j_M}(\gamma) := \frac{K_{j_M}}{1-\gamma}$, $A_{j_M}(t,\gamma) := \frac{A_{j_M}(t)}{1-\gamma}$, and $U_{j_M}(t,\gamma) := \frac{U_{j_M}(t)}{1-\gamma}$.

The ODE (4.24) is an inhomogeneous Bernoulli type ODE (also referred to as Chini's equation). We then have the following lemma whose proof is reported in the Appendix.

LEMMA 4.3. *There exists a unique positive solution to the ODE* (4.24).

Recall that $B_{j_1,\dots,j_{M-1}}(t)^{\frac{1}{1-\gamma}} = \hat{u}(t) > 0$ for all $t \in [0,T]$. Then using Lemma 4.3, we deduce a unique positive solution $B_{j_1,\dots,j_{M-1}}(t)$ to equation (4.23).

REMARK 4.4. If $\gamma = \frac{1}{2}$, then we have that $\hat{u}(t) = (B_{j_1,\dots,j_{M-1}}(t))^2 > 0$. Thus we have a unique positive solution to equation (4.23), which is given by $B_{j_1,\dots,j_{M-1}}(t) = \sqrt{\hat{u}(t)}$.
$m \le M-2$. We show existence and uniqueness of a positive solution $B_{j_1,\dots,j_m}(t)$ to equation (4.20). Notice that when $m = 0$ we have $B_{j_1,\dots,j_m}(t) = B(t,\mathbf{0})$. We first give the following result.

LEMMA 4.5. *Let* $0 \le m < M-1$. *Assume that there exists a unique positive solution* $B_{j_1,\dots,j_{m+1}}(t)$ *to the ODE* (4.20) *in the domain* $[0,T]$. *Then, there exists a unique positive solution* $B_{j_1,\dots,j_m}(t)$ *to the ODE* (4.20) *in the same domain.*

The proof of this lemma is reported in the Appendix. From Case (2), when $m = M-1$ we know that a unique positive solution $B_{j_1,\dots,j_{M-1}}(t)$ exists for any subset $\{j_1, j_2, \dots, j_{M-1}\} \subset \{1,\dots,M\}$. Hence, applying Lemma 4.5 we obtain the existence of a unique positive solution $B_{j_1,\dots,j_m}(t)$ for any $\{j_1, j_2, \dots, j_m\} \subset \{1,\dots,M\}$. This concludes the analysis.

4.3. Directionality of CDS Investment Strategy

We provide conditions under which the investor goes long or short in his CDS investment strategy. First, we provide some useful notation and definitions. Let

$$\mathcal{A}^{\Upsilon} g(\mathbf{z}) = \sum_{j=1}^{M} (1 - z_j) h_j^{\Upsilon}(t, \mathbf{z}) \left[g(\mathbf{z}^j) - g(\mathbf{z}) \right], \quad \mathbf{z} = (z_1, \dots, z_M) \in \mathcal{S}, \tag{4.25}$$

be the infinitesimal generator of the M-dimensional default indicator process $\mathbf{H}(t)$ under an equivalent probability measure Υ. For any $s > t$, we introduce the following terminology:

- The *expected per unit time price change* of the ith CDS under the measure Υ and default state $\mathbf{z}$ during the interval $[t, s]$ is denoted by $E_{i,\mathbf{z}}^{\Upsilon}(t, s)$, and defined as

$$E_{i,\mathbf{z}}^{\Upsilon}(t, s) := \frac{1}{s - t} \mathbb{E}^{\Upsilon} \left[\Phi_i(s, \mathbf{H}(s)) - \Phi_i(t, \mathbf{z}) \middle| \mathbf{H}(t) = \mathbf{z} \right].$$

- The *expected instantaneous price change* of the ith CDS under the measure Υ and default state $\mathbf{z}$ is defined as

$$E_{i,\mathbf{z}}^{\Upsilon}(t) := \lim_{s \to t^+} E_{i,\mathbf{z}}^{\Upsilon}(t, s).$$

A straightforward calculation shows that

$$E_{i,\mathbf{z}}^{\Upsilon}(t) = \lim_{s \to t^+} \frac{1}{s - t} \mathbb{E}^{\Upsilon} \left[\Phi_i(s, \mathbf{H}(s)) - \Phi_i(t, \mathbf{z}) \middle| \mathbf{H}(t) = \mathbf{z} \right] = \sum_{j=1}^{M} (1 - z_j) h_j^{\Upsilon}(\mathbf{z}) \big(\Phi_i(s, \mathbf{z}^j) - \Phi_i(t, \mathbf{z}) \big).$$

Then we have the following

LEMMA 4.6. *Let $m \leq M - 1$ and $\{j_1, \dots, j_m\} \subseteq \{1, \dots, M\}$. Let $i \in \{1, \dots, M\} \setminus \{j_1, \dots, j_m\}$. Assume that for any $l \notin \{i, j_1, \dots, j_m\}$, we have that*

$$\Phi_{i, j_1, \dots, j_m, l}(t) > \Phi_{i, j_1, \dots, j_m}(t). \tag{4.26}$$

Set $\mathbf{z} = \mathbf{0}^{j_1, \dots, j_m}$. Then, the optimum $\psi_{i, j_1, \dots, j_m}^*(t) < 0$ if and only if

$$E_{i,\mathbf{z}}^{\Upsilon}(t) > E_{i,\mathbf{z}}^{\tilde{\Upsilon}}(t), \tag{4.27}$$

where the time-varying generators of the default indicator processes are given by

$$h_l^{\Upsilon}(t, \mathbf{0}^{j_1, \dots, j_m}) = B_{j_1, \dots, j_m}(t) h_{l, j_1, \dots, j_m}, \qquad l \notin \{j_1, \dots, j_m\}, \tag{4.28}$$

and

$$h_l^{\tilde{\Upsilon}}(t, \mathbf{0}^{j_1, \dots, j_m}) = B_{j_1, \dots, j_m, l}(t) h_{l, j_1, \dots, j_m}^{\mathbb{P}}, \qquad l \notin \{j_1, \dots, j_m\}. \tag{4.29}$$

Proof. First, we establish that the optimum $\psi^*_{i,j_1,\dots,j_m} < 0$ if and only if the following relationship holds:

$$-B_{j_1,\dots,j_m}(t)\Big[\big(\Phi_{i,j_1,\dots,j_m}(t)-L_i\big)\,h_{i,j_1,\dots,j_m}-\sum_{\ell\neq i,\ell\neq j_1,\dots,j_m}\big(\Phi_{i,j_1,\dots,j_m,\ell}(t)-\Phi_{i,j_1,\dots,j_m}(t)\big)\,h_{\ell,j_1,\dots,j_m}\Big]$$
$$> B_{j_1,\dots,j_m,i}(t)(L_i-\Phi_{i,j_1,\dots,j_m}(t))h^{\mathbb{P}}_{i,j_1,\dots,j_m}$$
$$+\sum_{\ell\neq i,\ell\neq j_1,\dots,j_m} B_{j_1,\dots,j_m,\ell}(t)h^{\mathbb{P}}_{\ell,j_1,\dots,j_m}\big(\Phi_{i,j_1,\dots,j_m,\ell}(t)-\Phi_{i,j_1,\dots,j_m}(t)\big). \tag{4.30}$$

From Lemma 4.2, we know that the optimal strategy $\psi^*_{i,j_1,\dots,j_m}$ corresponding to the ith CDS satisfies the following equation:

$$-B_{j_1,\dots,j_m}(t)\Big[\big(\Phi_{i,j_1,\dots,j_m}(t)-L_i\big)\,h_{i,j_1,\dots,j_m}-\sum_{\ell\neq i,\ell\neq j_1,\dots,j_m}\big(\Phi_{i,j_1,\dots,j_m,\ell}(t)-\Phi_{i,j_1,\dots,j_m}(t)\big)\,h_{\ell,j_1,\dots,j_m}\Big]$$
$$= B_{j_1,\dots,j_m,i}(t)\left[1+\psi^*_{i,j_1,\dots,j_m}(L_i-\Phi_{i,j_1,\dots,j_m}(t))\right]^{\gamma-1}(L_i-\Phi_{i,j_1,\dots,j_m}(t))h^{\mathbb{P}}_{i,j_1,\dots,j_m}$$
$$+\sum_{\ell\neq i,\ell\neq j_1,\dots,j_m} B_{j_1,\dots,j_m,\ell}(t)\left[1+\psi^*_{i,j_1,\dots,j_m}\big(\Phi_{i,j_1,\dots,j_m,\ell}(t)-\Phi_{i,j_1,\dots,j_m}(t)\big)\right]^{\gamma-1}$$
$$\times h^{\mathbb{P}}_{\ell,j_1,\dots,j_m}\big(\Phi_{i,j_1,\dots,j_m,\ell}(t)-\Phi_{i,j_1,\dots,j_m}(t)\big). \tag{4.31}$$

Using that $L_i > \Phi_{i,j_1,\dots,j_m}(t)$ and $\Phi_{i,j_1,\dots,j_m,\ell}(t) > \Phi_{i,j_1,\dots,j_m}(t)$ by the assumption of the lemma, along with the fact that $B(t,\mathbf{z})$ is strictly positive in light of equation (4.21), Lemmas (4.3) and (4.5), we obtain that the right-hand side of equation (4.31) is a decreasing function of $\psi_{i,j_1,\dots,j_m}$. Moreover, from Lemma 4.2 we know that there exists a unique optimal strategy $\psi^*_{i,j_1,\dots,j_m}(t)$ such that equation (4.31) holds. Evaluating the right-hand side of equation (4.31) at $\psi_{i,j_1,\dots,j_m}=0$, we obtain that $\psi^*_{i,j_1,\dots,j_m} < 0$ if and only if the inequality (4.30) holds. From the above definition of expected instantaneous price change, we obtain that the condition (4.30) may be rewritten as in equation (4.27), which completes the proof. □

The long–short condition in (4.27) has an interesting financial interpretation. It states that under the *contagion condition* given in equation (4.26), the investor sells the credit default swap only if the expected instantaneous price change computed under the adjusted risk-neutral measure given by (4.28) exceeds the analogous quantity computed under the adjusted historical measure given by (4.29). Such a contagion condition is natural and captures realistic market scenarios where default of a name has a positive upward impact on the default risk of other names. The adjustments account for the risk averse nature of the power investor through the value function. Hence, the investor sells the CDS security (receiving spread premium and paying loss at default) when the expected price change perceived by a "risk averse market" exceeds the expected price change computed under the objective measure adjusted for the investor risk aversion. Broadly speaking, the power investor sells CDS protection when the market prices the contagion effect induced by the defaulted names higher than the history.

We also notice from equation (4.28) that the adjusted risk-neutral measure is obtained by multiplying the risk-neutral default intensity of each surviving name by the value function associated with the current default state of the portfolio. The adjusted historical measure is instead obtained by multiplying the historical default intensity of each alive name l by the value function associated with the default state where the additional name l defaults, see equation (4.29) for details.

4.4. An Illustrative Case When $M = 2$

We discuss investor strategies, value functions, and directionality of CDS strategy for the case of a portfolio consisting of two entities, i.e., $M = 2$. In such a case, we are able to provide more explicit characterizations for the value functions and CDS strategies. When $M = 2$, we have the first-order derivative functions $f_1^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z})$ and $f_2^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z})$ given by

$$
\begin{aligned}
f_1^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) = v w_v^T(t, v, \mathbf{z})(1 - z_1)\Big[& (\Phi_1(t, z_1, z_2) - \Phi_1(t, 1 - z_1, z_2))\, h_1(\mathbf{z}) \\
& - (\Phi_1(t, z_1, 1 - z_2) - \Phi_1(t, z_1, z_2))\,(1 - z_2) h_2(\mathbf{z})\Big] \\
& + v w_v^T\,(t, v + v\psi_1(1 - z_1)(L_1 - \Phi_1(t, z_1, z_2)), (1 - z_1, z_2)) \\
& \times (L_1 - \Phi_1(t, z_1, z_2))(1 - z_1) h_1^{\mathbb{P}}(\mathbf{z}) \\
& + v w_v^T\,(t, v + v\psi_1(1 - z_1)\,(\Phi_1(t, z_1, 1 - z_2) - \Phi_1(t, z_1, z_2))\,, (z_1, 1 - z_2)) \\
& \times (1 - z_2) h_2^{\mathbb{P}}(\mathbf{z})(1 - z_1)\,(\Phi_1(t, z_1, 1 - z_2) - \Phi_1(t, z_1, z_2))\,,
\end{aligned}
$$

and

$$
\begin{aligned}
f_2^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) = v w_v^T(t, v, \mathbf{z})(1 - z_2)\Big[& (\Phi_2(t, z_1, z_2) - \Phi_2(t, z_1, 1 - z_2))\, h_2(\mathbf{z}) \\
& - (\Phi_2(t, 1 - z_1, z_2) - \Phi_2(t, z_1, z_2))\,(1 - z_1) h_1(\mathbf{z})\Big] \\
& + v w_v^T\,(t, v + v\psi_2(1 - z_2)(L_2 - \Phi_2(t, z_1, z_2)), (z_1, 1 - z_2)) \\
& \times (L_2 - \Phi_2(t, z_1, z_2))(1 - z_2) h_2^{\mathbb{P}}(\mathbf{z}) \\
& + v w_v^T\,(t, v + v\psi_2(1 - z_2)\,(\Phi_2(t, 1 - z_1, z_2) - \Phi_2(t, z_1, z_2))\,, (1 - z_1, z_2)) \\
& \times (1 - z_1) h_1^{\mathbb{P}}(\mathbf{z})(1 - z_2)\,(\Phi_2(t, 1 - z_1, z_2) - \Phi_2(t, z_1, z_2))\,.
\end{aligned}
$$

For notational convenience, we write $w^{T,(i,j)}(t, v) := w^T(t, v, (i, j))$, where $(i, j) \in \{0, 1\}^2$. Then, we distinguish the following cases:

- For $\mathbf{z} = (z_1, z_2) = (0, 0)$, we have that

$$
\begin{aligned}
& f_1^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) \\
&= v w_v^{T,(0,0)}(t, v)\Big[(\Phi_1(t, 0, 0) - \Phi_1(t, 1, 0))\, h_1(0, 0) - (\Phi_1(t, 0, 1) - \Phi_1(t, 0, 0)) h_2(0, 0)\Big] \\
&\quad + v w_v^{T,(1,0)}\,(t, v + v\psi_1(L_1 - \Phi_1(t, 0, 0)))\,(L_1 - \Phi_1(t, 0, 0)) h_1^{\mathbb{P}}(0, 0) \\
&\quad + v w_v^{T,(0,1)}\,(t, v + v\psi_1\,(\Phi_1(t, 0, 1) - \Phi_1(t, 0, 0))) \\
&\quad \times (\Phi_1(t, 0, 1) - \Phi_1(t, 0, 0))\, h_2^{\mathbb{P}}(0, 0),
\end{aligned}
$$

and

$$
\begin{aligned}
&f_2^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) \\
&= v w_v^{T,(0,0)}(t, v)\Big[(\Phi_2(t,0,0) - \Phi_2(t,0,1))\, h_2(0,0) - (\Phi_2(t,1,0) - \Phi_2(t,0,0))\, h_1(0,0)\Big] \\
&\quad + v w_v^{T,(0,1)}(t, v + v\psi_2(L_2 - \Phi_2(t,0,0)))\,(L_2 - \Phi_2(t,0,0)) h_2^{\mathbb{P}}(0,0) \\
&\quad + v w_v^{T,(1,0)}(t, v + v\psi_2(\Phi_2(t,1,0) - \Phi_2(t,0,0))) \\
&\quad \times (\Phi_2(t,1,0) - \Phi_2(t,0,0))\, h_1^{\mathbb{P}}(0,0).
\end{aligned}
$$

- For $\mathbf{z} = (z_1, z_2) = (1, 0)$, we have that $f_1^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) = 0$, and

$$
\begin{aligned}
f_2^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) &= v w_v^{T,(1,0)}(t, v)\,(\Phi_2(t,1,0) - L_2)\, h_2(1,0) \\
&\quad + v w_v^{T,(1,1)}(t, v + v\psi_2(L_2 - \Phi_2(t,1,0)))\,(L_2 - \Phi_2(t,1,0)) h_2^{\mathbb{P}}(1,0).
\end{aligned}
$$

- For $\mathbf{z} = (z_1, z_2) = (0, 1)$, we have $f_2^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) = 0$, and

$$
\begin{aligned}
f_1^{T,'}(\boldsymbol{\phi}; t, v, \mathbf{z}) &= v w_v^{T,(0,1)}(t, v)\,(\Phi_1(t,0,1) - L_1)\, h_1(0,1) \\
&\quad + v w_v^{T,(1,1)}(t, v + v\psi_1(L_1 - \Phi_1(t,0,1)))\,(L_1 - \Phi_1(t,0,1)) h_1^{\mathbb{P}}(0,1).
\end{aligned}
$$

Consider the form (4.3) of the value function w^T and use the simplified notation $B^{(\mathbf{z})}(t) = B(t, \mathbf{z})$. Note that $\Phi_1(t,1,1) = L_1$ and $\Phi_2(t,1,1) = L_2$ using Lemma A.2. Then we have the following

LEMMA 4.7. *The optimum $\boldsymbol{\psi}^* = (\psi_1^*, \psi_2^*)$ on CDS is given by*

- $\psi_1^*(t,1,1) = \psi_2^*(t,1,1) = 0$;
- $\psi_1^*(t,1,0) = \psi_2^*(t,0,1) = 0$;
- *We have*

$$
\begin{aligned}
\psi_1^*(t,0,1) &= \frac{1}{L_1 - \Phi_1(t,0,1)} \left\{ \left[\frac{B(t,0,1)(L_1 - \Phi_1(t,0,1)) h_1(0,1)}{B(t,1,1)(L_1 - \Phi_1(t,0,1)) h_1^{\mathbb{P}}(0,1)} \right]^{\frac{1}{\gamma-1}} - 1 \right\} \\
&= \frac{1}{L_1 - \Phi_1(t,0,1)} \left\{ \left[\frac{B(t,0,1) h_1(0,1)}{B(t,1,1) h_1^{\mathbb{P}}(0,1)} \right]^{\frac{1}{\gamma-1}} - 1 \right\}, \qquad (4.32)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_2^*(t,1,0) &= \frac{1}{L_2 - \Phi_2(t,1,0)} \left\{ \left[\frac{B(t,1,0)\,(L_2 - \Phi_2(t,1,0))\, h_2(1,0)}{B(t,1,1)(L_2 - \Phi_2(t,1,0)) h_2^{\mathbb{P}}(1,0)} \right]^{\frac{1}{\gamma-1}} - 1 \right\} \\
&= \frac{1}{L_2 - \Phi_2(t,1,0)} \left\{ \left[\frac{B(t,1,0) h_2(1,0)}{B(t,1,1) h_2^{\mathbb{P}}(1,0)} \right]^{\frac{1}{\gamma-1}} - 1 \right\}. \qquad (4.33)
\end{aligned}
$$

Moreover, $\psi_1^*(t,0,0)$ satisfies the equation:

$$
\begin{aligned}
0 = {} & V_1(t) B(t,0,0) + B(t,1,0)(L_1 - \Phi_1(t,0,0)) h_1^{\mathbb{P}}(0,0) \left[1 + \psi_1^*(t,0,0)(L_1 - \Phi_1(t,0,0))\right]^{\gamma-1} \\
& + B(t,0,1)\,(\Phi_1(t,0,1) - \Phi_1(t,0,0))\, h_2^{\mathbb{P}}(0,0) \\
& \times \left[1 + \psi_1^*(t,0,0)\,(\Phi_1(t,0,1) - \Phi_1(t,0,0))\right]^{\gamma-1}, \qquad (4.34)
\end{aligned}
$$

and $\psi_2^*(t, 0, 0)$ *satisfies the equation:*

$$
\begin{aligned}
0 = {} & V_2(t) B(t, 0, 0) + B(t, 0, 1)(L_2 - \Phi_2(t, 0, 0)) h_2^{\mathbb{P}}(0, 0) \left[1 + \psi_2^*(t, 0, 0)(L_2 - \Phi_2(t, 0, 0))\right]^{\gamma-1} \\
& + B(t, 1, 0) (\Phi_2(t, 1, 0) - \Phi_2(t, 0, 0)) \, h_1^{\mathbb{P}}(0, 0) \\
& \times \left[1 + \psi_2^*(t, 0, 0) (\Phi_2(t, 1, 0) - \Phi_2(t, 0, 0))\right]^{\gamma-1},
\end{aligned}
\tag{4.35}
$$

where the coefficients

$$
\begin{aligned}
V_1(t) &:= (\Phi_1(t, 0, 0) - L_1) \, h_1(0, 0) - (\Phi_1(t, 0, 1) - \Phi_1(t, 0, 0)) \, h_2(0, 0), \\
V_2(t) &:= (\Phi_2(t, 0, 0) - L_2) \, h_2(0, 0) - (\Phi_2(t, 1, 0) - \Phi_2(t, 0, 0)) \, h_1(0, 0).
\end{aligned}
$$

Proof. The above expressions for $\psi_i^*(t, \mathbf{z})$, $i = 1, 2$ and $\mathbf{z} \neq \mathbf{0}$, come from a direct calculation. Equations (4.34) and (4.35) are derived from the first-order condition (4.1) using the expressions for $f_1^{T,'}$ and $f_2^{T,'}$ given above. $\square$

In what follows, we define the following vectors:

$$
\mathbf{B}^{(1)}(t) := [B(t, 0, 0), \; B(t, 1, 0), \; B(t, 0, 1)]^\top ,
$$

and

$$
\mathbf{B}^{(2)}(t) := [B(t, 0, 0), \; B(t, 0, 1), \; B(t, 1, 0)]^\top .
$$

Applying Lemma 4.1 in the case when $M = 2$, we obtain the following.

COROLLARY 4.8. *There exists a unique optimum* $\boldsymbol{\psi}^*(t, 0, 0) = (\psi_1^*(t, 0, 0), \psi_2^*(t, 0, 0))$ *satisfying the first-order conditions* (4.34) *and* (4.35)*. Moreover, for each* $i = 1, 2$*, the optimum* $\psi_i^*(t, 0, 0)$ *viewed as function of* $(t, \mathbf{B}^{(i)}) \in [0, T] \times R_+^3$ *is continuous in* $(t, \mathbf{B}^{(i)})$ *for each* $i = 1, 2$.

When $M = 2$, we obtain the HJB equations given in the following.

LEMMA 4.9. *The positive functions* $B(t, 1, 1)$*,* $B(t, 1, 0)$*,* $B(t, 0, 1)$*, and* $B(t, 0, 0)$ *satisfy the following equations:*

$$
\begin{aligned}
0 &= \frac{\mathrm{d}B(t, 1, 1)}{\mathrm{d}t} + a(\gamma) B(t, 1, 1), \\
0 &= \frac{\mathrm{d}B(t, 1, 0)}{\mathrm{d}t} + \gamma \left\{ \psi_2^*(t, 1, 0) (\Phi_2(t, 1, 0) - L_2) \, h_2(1, 0) + r \right\} B(t, 1, 0) \\
&\quad + \left[(1 + \psi_2^*(t, 1, 0)(L_2 - \Phi_2(t, 1, 0)))^\gamma \, B(t, 1, 1) - B(t, 1, 0) \right] h_2^{\mathbb{P}}(1, 0) \\
&\quad + [B(t, 1, 1) - B(t, 1, 0)] \, h_2^{\mathbb{P}}(1, 0),
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}B(t, 0, 1)}{\mathrm{d}t} + \gamma \left\{ \psi_1^*(t, 0, 1) + (\Phi_1(t, 0, 1) - L_1) h_1(0, 1) + r \right\} B(t, 0, 1) \\
&\quad + \left[(1 + \psi_1^*(t, 0, 1)(L_1 - \Phi_1(t, 0, 1)))^\gamma \, B(t, 1, 1) - B(t, 0, 1) \right] h_1^{\mathbb{P}}(0, 1) \\
&\quad + [B(t, 1, 1) - B(t, 0, 1)] \, h_1^{\mathbb{P}}(0, 1),
\end{aligned}
$$

and

$$0 = \frac{\mathrm{d}B(t,0,0)}{\mathrm{d}t} + \gamma\Bigg\{\sum_{i=1}^{2}\psi_i^*(t,0,0)\Bigg[(\Phi_i(t,0,0)-L_i)h_i(0,0)$$

$$-\sum_{j\neq i}\left(\Phi_i(t,(0,0)^j)-\Phi_i(t,0,0)\right)h_j(0,0)\Bigg]+r\Bigg\}B(t,0,0)$$

$$+\left[(1+\psi_1^*(t,0,0)(L_1-\Phi_1(t,0,0)))^\gamma\, B(t,1,0)-B(t,0,0)\right]h_1^{\mathbb{P}}(0,0)$$

$$+\left[(1+\psi_2^*(t,0,0)(L_2-\Phi_2(t,0,0)))^\gamma\, B(t,0,1)-B(t,0,0)\right]h_2^{\mathbb{P}}(0,0)$$

$$+\left[(1+\psi_1^*(t,0,0)\,(\Phi_1(t,0,1)-\Phi_1(t,0,0)))^\gamma\, B(t,0,1)-B(t,0,0)\right]h_2^{\mathbb{P}}(0,0)$$

$$+\left[(1+\psi_2^*(t,0,0)\,(\Phi_2(t,1,0)-\Phi_2(t,0,0)))^\gamma\, B(t,1,0)-B(t,0,0)\right]h_1^{\mathbb{P}}(0,0). \tag{4.37}$$

Proof. The above equations are obtained by specializing (4.16) to the case when $M=2$. □

Applying Lemma 4.6, we can also provide conditions under which the investor goes long or short in his CDS strategy.

COROLLARY 4.10. *Assume the contagion conditions* $\Phi_1(t,0,1)>\Phi_1(t,0,0)$ *and* $\Phi_2(t,1,0)>\Phi_2(t,0,0)$. *Then, it holds that*

$$\psi_1(t,0,0)<0, \qquad \text{iff} \qquad E_{1,\mathbf{0}}^{\Upsilon}(t)>E_{1,\mathbf{0}}^{\tilde{\Upsilon}}(t),$$

and

$$\psi_2(t,0,0)<0, \qquad \text{iff} \qquad E_{2,\mathbf{0}}^{\Upsilon}(t)>E_{2,\mathbf{0}}^{\tilde{\Upsilon}}(t),$$

where

$$h_1^{\Upsilon}(t,0,0)=B(t,0,0)h_1(0,0), \qquad h_2^{\Upsilon}(t,0,0)=B(t,0,0)h_2(0,0),$$
$$h_1^{\tilde{\Upsilon}}(t,0,0)=B(t,1,0)h_1^{\mathbb{P}}(0,0), \qquad h_2^{\tilde{\Upsilon}}(t,0,0)=B(t,0,1)h_2^{\mathbb{P}}(0,0).$$

Hence, the investor considers the expected price change under a historical probability measure adjusted for contagion effects due to default of a name. On the other hand, under the risk-neutral probability measure the adjustment is myopic and does not account for portfolio states associated with default events.

5. MAIN RESULTS

We use the results derived in the previous section and show that the unique solution of the HJB equation coincides with the value function. Furthermore, we identify the optimal admissible investment strategies in the CDS asset.

Recall the operator defined by $\mathcal{L}=\mathcal{L}_v+\mathcal{L}_J$ in equation (3.7). Let $w(t,v,\mathbf{z})$ be C^1 in t and v. Then for $t<u$,

$$w(u,V_u,\mathbf{H}(u))=w(t,V_t,\mathbf{H}(t))+\int_t^u\left(\frac{\partial}{\partial s}+\mathcal{L}\right)w(s,V_s,\mathbf{H}(s))\mathrm{d}s+\mathcal{M}_u-\mathcal{M}_t,$$

where $V = (V_t;\ t \geq 0)$ is the wealth process given in Lemma 3.2 and the $\mathbb{P}$-(local) martingale process is defined by

$$\begin{aligned}\mathcal{M}_t :=& \sum_{i=1}^{M}\Big\{\int_0^t \big[w\big(s, V_{s-} + V_{s-}\psi_i(s-)(L_i - \Phi_i(t, \mathbf{H}(s-))), \mathbf{H}^i(s-)\big) - w(s, V_{s-}, \mathbf{H}(s-))\big] \\ & \times \mathrm{d}\xi_i^{\mathbb{P}}(s)\Big\} \\ & + \sum_{i=1}^{M}\Big\{\sum_{j\neq i}\int_0^t \Big[w\big(s, V_{s-} + V_{s-}\psi_i(s-)(\Phi_i(s, \mathbf{H}^j(s-)) - \Phi_i(t, \mathbf{H}(s-))), \mathbf{H}^j(s-)\big) \\ & - w(s, V_{s-}, \mathbf{H}(s-))\Big]\mathrm{d}\xi_j^{\mathbb{P}}(s)\Big]\Big\}, \quad t \geq 0. \end{aligned} \tag{5.1}$$

THEOREM 5.1. *For each $m \in \{1, \ldots, M\}$, let $j_k \in \{1, \ldots, M\}$ for all $k = 1, \ldots, m$, where $j_1 \neq \cdots \neq j_m$. Define $\psi^*_{i,j_1,\ldots,j_m}(t)$ as follows:*

- *If $i \neq j_1, \ldots, j_m$,*

$$\psi^*_{i,j_1,\ldots,j_m}(t) := \begin{cases} \frac{1}{L_i - \Phi_{i,j_1,\ldots,j_{M-1}}(t)}\left\{\left[\frac{B_{j_1,\ldots,j_{M-1}}(t)h_{i,j_1,\ldots,j_{M-1}}}{B(t,\mathbf{1})h^{\mathbb{P}}_{i,j_1,\ldots,j_{M-1}}}\right]^{\frac{1}{\gamma-1}} - 1\right\}, & m = M-1; \\ 0, & m = M; \\ \textit{satisfies the first-order condition } (33), & 1 \leq m < M-1; \\ \textit{satisfies the first-order condition } (28), & m = 0, \end{cases} \tag{5.2}$$

  where $B_{j_1,\ldots,j_{M-1}}(t)$ is the unique positive solution to equation (4.22) and $B(t, \mathbf{1}) = \frac{1}{\gamma}e^{\gamma r(T-t)}$, $0 \leq t \leq T$.
- *If $i = j_k$ for some $k = 1, \ldots, m$, $\psi^*_{i,j_1,\ldots,j_m}(t) = 0$.*

Let $B^*_{j_1,\ldots,j_m}(t)$ satisfy the ODE (4.20) with the optimum $\psi^*_{i,j_1,\ldots,j_m}(t)$ given by (5.2). Then the optimal fraction of wealth invested in the ith CDS is given by $\Phi_{i,j_1,\ldots,j_m}(t) \cdot \psi^*_{i,j_1,\ldots,j_m}(t)$. Moreover, the value function is given by

$$w^T(t, v, \mathbf{0}^{j_1,\ldots,j_m}) = v^\gamma B^*_{j_1,\ldots,j_m}(t), \qquad m = 1, \ldots, M.$$

Proof. For $\mathbf{z} \in \mathcal{S}$, define $B(t, \mathbf{z})$ so that $B(t, \mathbf{0}^{j_1,\ldots,j_m}) = B_{j_1,\ldots,j_m}(t)$, where $B_{j_1,\ldots,j_m}(t)$ satisfies the ODE (4.20). For any admissible feedback control $\boldsymbol{\phi} = (v\psi_i(t, v, \mathbf{z});\ i = 1, \ldots, M)$, define the process

$$Y_u^{\boldsymbol{\phi}} := (V_u(\boldsymbol{\phi}))^\gamma\ B(u, \mathbf{H}(u)), \qquad u \geq t \geq 0.$$

From Itô's formula, it follows that

$$\begin{aligned} Y_u^{\boldsymbol{\phi}} =& Y_t^{\boldsymbol{\phi}} + \int_t^u F((\psi_1(s, \mathbf{H}(s)), \ldots, \psi_M(s, \mathbf{H}(s))); s, V_s(\boldsymbol{\phi}), \mathbf{H}(s))\,\mathrm{d}s \\ & + \mathcal{M}_u^{\boldsymbol{\phi}} - \mathcal{M}_t^{\boldsymbol{\phi}}, \end{aligned} \tag{5.3}$$

where the $\mathbb{P}$-local martingale process is defined by

$$\begin{aligned}\mathcal{M}_t^{\boldsymbol{\phi}} :=& \sum_{i=1}^{M}\left\{\int_0^t (V_s(\boldsymbol{\phi}))^{\gamma}\left[(1+\psi_i(s,\mathbf{H}(s-))(L_i-\Phi_i(s,\mathbf{H}(s-))))^{\gamma}\, B(s,\mathbf{H}^i(s-))\right.\right.\\ &\left.\left. - B(s,\mathbf{H}(s-))\right]\mathrm{d}\xi_i^{\mathbb{P}}(s)\right\}\\ &+\sum_{i=1}^{M}\left\{\sum_{j\neq i}\int_0^t (V_s(\boldsymbol{\phi}))^{\gamma}\left[\left(1+\psi_i(s,\mathbf{H}(s-))(\Phi_i(s,\mathbf{H}^j(s-))-\Phi_i(s,\mathbf{H}(s-)))\right)^{\gamma}\right.\right.\\ &\left.\left.\times B(s,\mathbf{H}^j(s-)) - B(s,\mathbf{H}(s-))\right]\mathrm{d}\xi_j^{\mathbb{P}}(s)\right\},\end{aligned}\tag{5.4}$$

while the function $F(\boldsymbol{\phi};t,v,\mathbf{z})$ is given by

$$\begin{aligned}F(\boldsymbol{\phi};t,v,\mathbf{z}) =& v^{\gamma}\frac{\partial B(t,\mathbf{z})}{\partial t} + \gamma v^{\gamma}\left\{\sum_{i=1}^{M}\psi_i(1-z_i)\Big[(\Phi_i(t,\mathbf{z})-L_i)h_i(\mathbf{z})\right.\\ &\left.-\sum_{j\neq i}(\Phi_i(t,\mathbf{z}^j)-\Phi_i(t,\mathbf{z}))(1-z_j)h_j(\mathbf{z})\Big]\right\}\\ &\times B(t,\mathbf{z}) + v^{\gamma}a(\gamma)\,B(t,\mathbf{z})\\ &+v^{\gamma}\sum_{i=1}^{M}\Big[(1+\psi_i(1-z_i)(L_i-\Phi_i(t,\mathbf{z})))^{\gamma}\,B(t,\mathbf{z}^i)-B(t,\mathbf{z})\Big](1-z_i)h_i^{\mathbb{P}}(\mathbf{z})\\ &+v^{\gamma}\sum_{i=1}^{M}\left\{\sum_{j\neq i}\Big[\left(1+\psi_i(1-z_i)(\Phi_i(t,\mathbf{z}^j)-\Phi_i(t,\mathbf{z}))\right)^{\gamma}\,B(t,\mathbf{z}^j)-B(t,\mathbf{z})\Big]\right.\\ &\left.\times(1-z_j)h_j^{\mathbb{P}}(\mathbf{z})\right\},\end{aligned}$$

with $a(\gamma)=\gamma r$.

Then, we have that for each $i,j=1,\dots,M$,

$$\begin{aligned}F_{\psi_i,\psi_i} =& \gamma(\gamma-1)v^{\gamma}\,[1+\psi_i(1-z_i)(L_i-\Phi_i(t,\mathbf{z}))]^{\gamma-2}\,(L_i-\Phi_i(t,\mathbf{z}))^2(1-z_i)h_i^{\mathbb{P}}(\mathbf{z})B(t,\mathbf{z}^i)\\ &+\gamma(\gamma-1)v^{\gamma}\sum_{j\neq i}\left[1+\psi_i(1-z_i)(\Phi_i(t,\mathbf{z}^j)-\Phi_i(t,\mathbf{z}))\right]^{\gamma-2}(\Phi_i(t,\mathbf{z}^j)-\Phi_i(t,\mathbf{z}))^2\\ &\times(1-z_i)(1-z_j)h_j^{\mathbb{P}}(\mathbf{z})B(t,\mathbf{z}^j)\le 0,\\ F_{\psi_i,\psi_j} =& 0,\quad \text{for all } i\neq j.\end{aligned}$$

This implies that the critical point is a maximum, and we denote it by $\boldsymbol{\phi}^*(t,\mathbf{z})$. Hence, the vector

$$\boldsymbol{\phi}^*(t,\mathbf{0}^{j_1,\dots,j_m}) = \left(v\psi^*_{i,j_1,\dots,j_m}(t);\ i=1,\dots,M\right)$$

is the optimum when $\mathbf{z}=\mathbf{0}^{j_1,\dots,j_m}$. Therefore, it holds that

$$F(\boldsymbol{\phi};t,v,\mathbf{0}^{j_1,\dots,j_m})\le F(\boldsymbol{\phi}^*(t,\mathbf{0}^{j_1,\dots,j_m});t,v,\mathbf{0}^{j_1,\dots,j_m})=0,$$

where we have used that $B_{j_1,\ldots,j_m}(t)$ satisfies equation (4.20), and $B_{j_1,\ldots,j_m}(t) = B^*_{j_1,\ldots,j_m}(t)$ when

$$\psi_i(t, \mathbf{0}^{j_1,\ldots,j_m}) = \psi^*_{i,j_1,\ldots,j_m}(t).$$

Set $\mathbb{E}^{\mathbb{P}}_t[\cdot] := \mathbb{E}^{\mathbb{P}}_t[\cdot|\mathcal{G}_t]$. Then, for any admissible feedback control $\boldsymbol{\phi} = (v\psi_i(t, v, \mathbf{z});\ i = 1, \ldots, M)$, it holds that for arbitrary $0 \leq t < u \leq T$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}_t\left[Y^{\boldsymbol{\phi}}_u\right] &\leq Y^{\boldsymbol{\phi}}_t + \mathbb{E}^{\mathbb{P}}_t\left[\mathcal{M}^{\boldsymbol{\phi}}_u - \mathcal{M}^{\boldsymbol{\phi}}_t\right] \\ &= (V_t(\boldsymbol{\phi}))^{\gamma}\ B(t, \mathbf{H}(t)) + \mathbb{E}^{\mathbb{P}}_t\left[\mathcal{M}^{\boldsymbol{\phi}}_u - \mathcal{M}^{\boldsymbol{\phi}}_t\right],\end{aligned} \tag{5.5}$$

with equality when $\boldsymbol{\phi} = \boldsymbol{\phi}^*$. Take $u = T \wedge \tau_{a,b}$ in (5.5) above, where $\tau_{a,b} = \inf\{u \geq t;\ V_u \geq b^{-1} \text{or } V_u \leq a\}$ with $0 < a < V_t = v < b^{-1}$. Hence

$$\mathbb{E}^{\mathbb{P}}_t\left[Y^{\boldsymbol{\phi}}_{T\wedge\tau_{a,b}}\right] \leq (V_t(\boldsymbol{\phi}))^{\gamma}\ B(t, \mathbf{H}(t))$$

with the inequality becoming an equality when $\boldsymbol{\phi} = \boldsymbol{\phi}^*$. It remains to prove that

$$\lim_{a,b\to 0} \mathbb{E}^{\mathbb{P}}_t\left[\left(V_{T\wedge\tau_{a,b}}(\boldsymbol{\phi}^*)\right)^{\gamma}\ B(T\wedge\tau_{a,b}, \mathbf{H}(T\wedge\tau_{a,b}))\right] = \mathbb{E}^{\mathbb{P}}_t\left[U\left(V_T(\boldsymbol{\phi}^*)\right)\right]. \tag{5.6}$$

From the continuous differentiability of the function $B(t, \mathbf{0}^{j_1,\ldots,j_m})$ with respect to $t \in [0, T]$, it follows that there exist positive constants C_1 and C_2 so that

$$\mathbb{E}^{\mathbb{P}}_t\left[\left|\left(V_{T\wedge\tau_{a,b}}(\boldsymbol{\phi}^*)\right)^{\gamma}\ B(T\wedge\tau_{a,b}, \mathbf{H}(T\wedge\tau_{a,b}))\right|^2\right] \leq C_1 + C_2\mathbb{E}^{\mathbb{P}}_t\left[\left|V_{T\wedge\tau_{a,b}}(\boldsymbol{\phi}^*)\right|^2\right]. \tag{5.7}$$

Define the functions

$$\begin{aligned}\alpha(t, v, \mathbf{z}) &:= v\Bigg\{r + \sum_{i=1}^{M}\psi^*_i(t, \mathbf{z}, B(t, \mathbf{z}))(1 - z_i)\Big[(\Phi_i(t, \mathbf{z}) - L_i)(h_i(\mathbf{z}) - h^{\mathbb{P}}_i(\mathbf{z})) \\ &\qquad + \sum_{j\neq i}(\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z}))(1 - z_j)(h^{\mathbb{P}}_j(\mathbf{z}) - h_j(\mathbf{z}))\Big]\Bigg\}, \\ \beta_i(t, v, \mathbf{z}) &:= v\psi^*_i(t, \mathbf{z}, B(t, \mathbf{z}))(1 - z_i)\left(L_i - \Phi_i(t, \mathbf{z})\right), \\ \theta_{ij}(t, v, \mathbf{z}) &:= v\psi^*_i(t, \mathbf{z}, B(t, \mathbf{z}))(1 - z_i)\left(\Phi_i(t, \mathbf{z}^j) - \Phi_i(t, \mathbf{z})\right)(1 - z_j), \quad i \neq j.\end{aligned}$$

Then the $\mathbb{P}$-dynamics of the wealth process $V_t(\boldsymbol{\phi}^*)$ may be rewritten as

$$\begin{aligned}\mathrm{d}V_t(\boldsymbol{\phi}^*) &= \alpha(t,\ V_t(\boldsymbol{\phi}^*), \mathbf{H}(t))\mathrm{d}t + \sum_{i=1}^{M}\beta_i(t,\ V_t(\boldsymbol{\phi}^*), \mathbf{H}(t))\mathrm{d}\xi^{\mathbb{P}}_i(t) \\ &\quad + \sum_{i=1}^{M}\sum_{j\neq i}\theta_{ij}(t,\ V_t(\boldsymbol{\phi}^*), \mathbf{H}(t))\mathrm{d}\xi^{\mathbb{P}}_j(t).\end{aligned}$$

In terms of (4.15), we have that for each $i, m = 1, \ldots, M$, it holds that $M^{(i)}_1(t) < \psi^*_{i,j_1,\ldots,j_m}(t) < M^{(i,m)}_2(t)$ where $0 \leq t \leq T$. Next, we assume that $M^{(i,m)}_2(t) < +\infty$. Let $C_{M,T}$ be a generic constant depending on M and T that may be different for each inequality below. For $0 \leq t < u \leq T$, using Hölder's inequality

$$\mathbb{E}_t^{\mathbb{P}}\left[\left|\int_t^u \alpha\left(s,\ V_s(\boldsymbol{\phi}^*), \mathbf{H}(s)\right) \mathrm{d}s\right|^2\right] \leq (T-t) C_{M,T} \mathbb{E}_t^{\mathbb{P}}\left[\int_t^u \left(V_s(\boldsymbol{\phi}^*) - V_t(\boldsymbol{\phi}^*) + V_t(\boldsymbol{\phi}^*)\right)^2 \mathrm{d}s\right]$$
$$\leq C_{M,T}\left((T-t)\left|V_t(\boldsymbol{\phi}^*)\right|^2 + \mathbb{E}_t^{\mathbb{P}}\left[\int_t^u \left|V_s(\boldsymbol{\phi}^*) - V_t(\boldsymbol{\phi}^*)\right|^2 \mathrm{d}s\right]\right).$$

From BDG inequality (see, e.g., theorem IV.48 in Protter 2004, p. 193), it follows that

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}\left|\sum_{i=1}^M \int_t^u \beta_i\left(s,\ V_s(\boldsymbol{\phi}^*), \mathbf{H}(s)\right) \mathrm{d}\xi_i^{\mathbb{P}}(s)\right|^2\right] \\
&\quad \leq C_{M,T} \sum_{i=1}^M \mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}\left|\int_t^u \beta_i\left(s,\ V_s(\boldsymbol{\phi}^*), \mathbf{H}(s)\right) \mathrm{d}\xi_i^{\mathbb{P}}(s)\right|^2\right] \\
&\quad \leq C_{M,T} \sum_{i=1}^M \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T \left|\beta_i\left(s,\ V_s(\boldsymbol{\phi}^*), \mathbf{H}(s)\right)\right|^2 \mathrm{d}H_i(s)\right] \\
&\quad = C_{M,T} \sum_{i=1}^M \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T \left|\beta_i\left(s,\ V_s(\boldsymbol{\phi}^*), \mathbf{H}(s)\right)\right|^2 h_i(\mathbf{H}(s)) \mathrm{d}s\right] \\
&\quad \leq C_{M,T} \mathbb{E}_t^{\mathbb{P}}\left((T-t)\left|V_t(\boldsymbol{\phi}^*)\right|^2 + \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T \left|V_s(\boldsymbol{\phi}^*) - V_t(\boldsymbol{\phi}^*)\right|^2 \mathrm{d}s\right]\right),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}\left|\sum_{i=1}^M \sum_{j\neq i} \int_t^u \theta_{ij}\left(s,\ V_s(\boldsymbol{\phi}^*), \mathbf{H}(s)\right) \mathrm{d}\xi_j^{\mathbb{P}}(s)\right|^2\right] \\
&\quad \leq C_{M,T} \mathbb{E}_t^{\mathbb{P}}\left((T-t)\left|V_t(\boldsymbol{\phi}^*)\right|^2 + \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T \left|V_s(\boldsymbol{\phi}^*) - V_t(\boldsymbol{\phi}^*)\right|^2 \mathrm{d}s\right]\right).
\end{aligned}$$

Thus by Gronwall's inequality, we can conclude that the wealth process satisfies the moment condition:

$$\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}\left|V_u(\boldsymbol{\phi}^*) - V_t(\boldsymbol{\phi}^*)\right|^2\right] \leq C_{M,T}\left|V_t(\boldsymbol{\phi}^*)\right|^2 + D_{M,T}, \tag{5.8}$$

where $D_{M,T} > 0$ is also a positive constant depending on M and on the terminal time T.

For the case that $M_2^{(i,m)}(t) = +\infty$, we notice that

$$\psi_i^*(t, \mathbf{0}^{j_1,\ldots,j_m}, B_{j_1,\ldots,j_m}(t)) = \psi_{i,j_1,\ldots,j_m}^*(t, \mathbf{B}^{(j_1,\ldots,j_m,i)}(t)),$$

for all $i \neq j_1 \neq \cdots \neq j_m$, and

$$\mathbf{B}^{(j_1,\ldots,j_m,i)}(t) = \left[B_{j_1,\ldots,j_m}(t),\ B_{j_1,\ldots,j_m,\ell}(t);\ \ell \in N_{i,j_1,\ldots,j_m}\right]^\top.$$

Since $t \in [0, T]$, which is a bounded closed set, we have that for any $i, m \in \{1,\ldots,m\}$ and $i \neq j_1 \neq \cdots \neq j_m$, the optimum $\psi_i^*(t, \mathbf{0}^{j_1,\ldots,j_m}, B_{j_1,\ldots,j_m}(t))$ is bounded using the continuity of $\psi_{i,j_1,\ldots,j_m}^*(t, \mathbf{B}^{(j_1,\ldots,j_m,i)})$ with respect to $(t, \mathbf{B}^{(j_1,\ldots,j_m,i)})$ (see Lemma 4.2) and the continuity

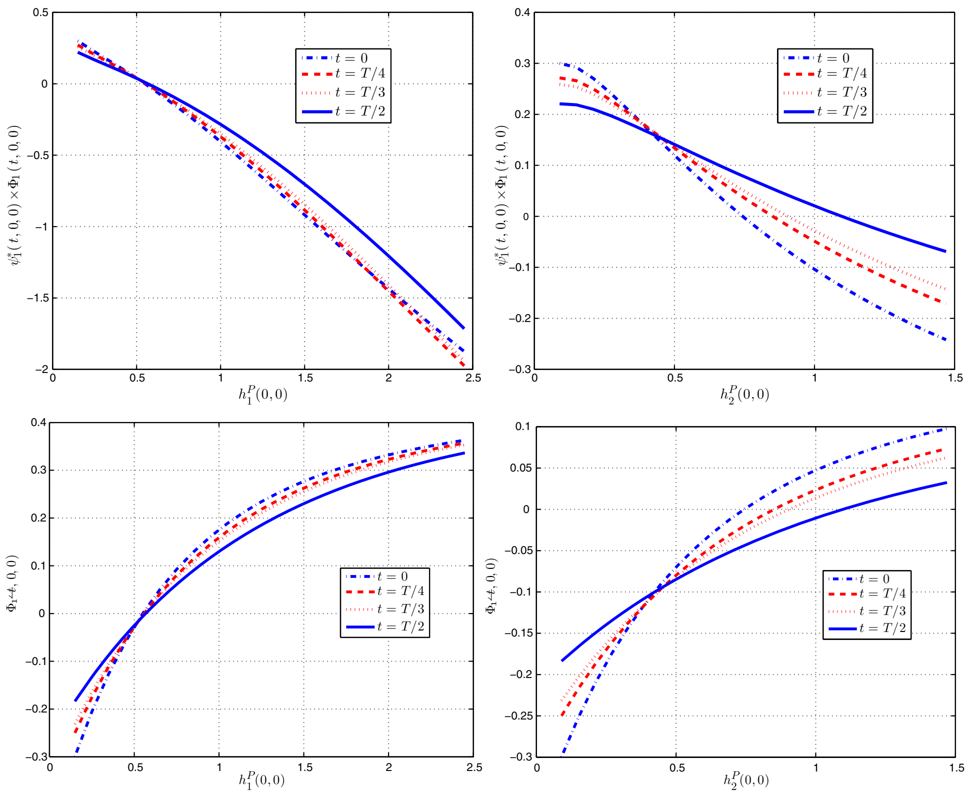


FIGURE 6.1. The top panels report the dependence of the optimal investment strategy in CDS "1" on the default intensity of name "1" and "2" before any default occurs. The bottom panels report the dependence of the CDS market value on both intensities. We fix $h_1^{\mathbb{P}}(0,1) = h_2^{\mathbb{P}}(1,0) = 2.3$. Whenever varying $h_2^{\mathbb{P}}(0,0)$ ($h_1^{\mathbb{P}}(0,0)$), we fix $h_1^{\mathbb{P}}(0,0) = 0.2$ ($h_2^{\mathbb{P}}(0,0) = 0.2$).

of $B_{j_1,\ldots,j_m}(t)$ with respect to time t. Thus we can also obtain the moment estimate (5.8) using a similar argument to the one used in the case where $M_2^{(i,m)}(t) < +\infty$. From (5.8), we obtain

$$\sup_{0<a<v<b^{-1}<+\infty} \mathbb{E}_t^{\mathbb{P}}\left[\left|V_{T\wedge\tau_{a,b}}(\boldsymbol{\phi}^*)\right|^2\right] \leq 2\left|V_t(\boldsymbol{\phi}^*)\right|^2 + 2\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\leq u\leq T}\left|V_u(\boldsymbol{\phi}^*) - V_t(\boldsymbol{\phi}^*)\right|^2\right] < +\infty.$$

Then, the equality (5.6) follows from corollary 7.1.5 in Chow and Teicher (1978). This completes the proof of the theorem. □

6. COMPARATIVE STATICS ANALYSIS

We perform a comparative statics analysis to investigate how the optimal investment strategies depend on default risk. We fix $M = 2$ and use the explicit expressions derived in Section 4.4. The numerical procedure followed to solve the equations in (4.37) and obtain the coupled investment strategies is reported in the Appendix. Throughout the section, we use the following contractual CDS parameters: $\nu_1 = 0.6$, $\nu_2 = 0.7$, $L_1 = 0.5$, $L_2 = 0.4$.

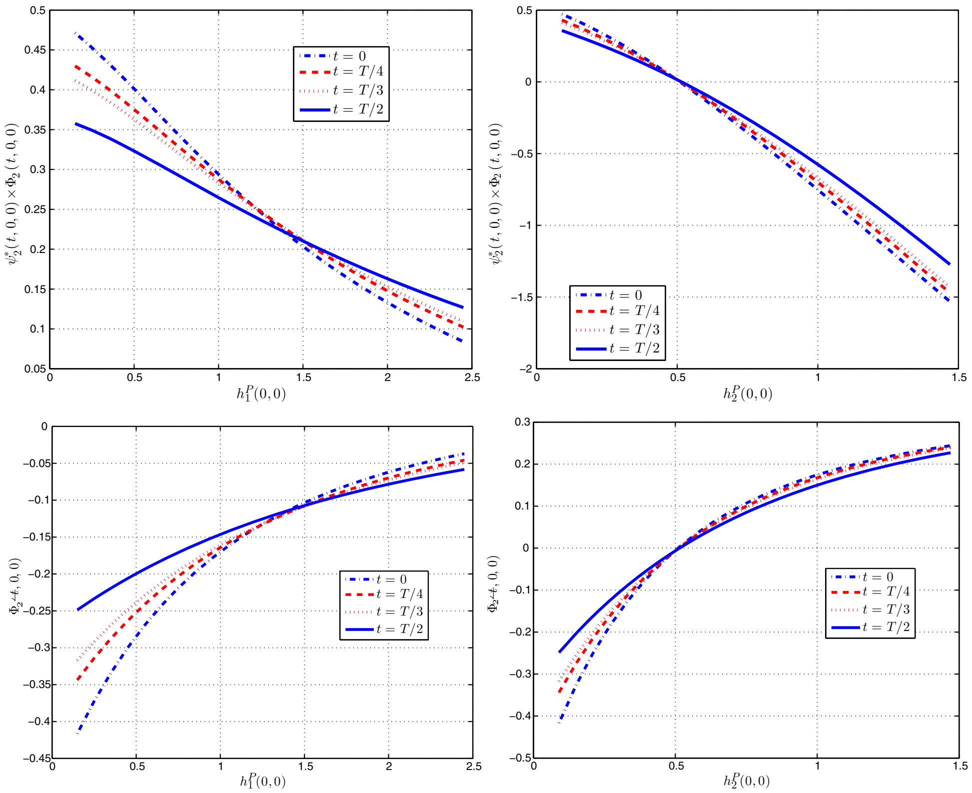


FIGURE 6.2. The top panels report the dependence of the optimal investment strategy in CDS "2" on the default intensity of name "1" and "2" before any default occurs. The bottom panels report the dependence of the CDS market value on both intensities. We fix $h_1^{\mathbb{P}}(0, 1) = h_2^{\mathbb{P}}(1, 0) = 2.3$. Whenever varying $h_2^{\mathbb{P}}(0, 0)$ ($h_1^{\mathbb{P}}(0, 0)$), we fix $h_1^{\mathbb{P}}(0, 0) = 0.2$ ($h_2^{\mathbb{P}}(0, 0) = 0.2$).

We fix $r = 0.03$ and $\gamma = 0.5$. We choose the default risk premiums to be $\frac{h_1(\mathbf{z})}{h_1^{\mathbb{P}}(\mathbf{z})} = 0.5$, and $\frac{h_2(\mathbf{z})}{h_2^{\mathbb{P}}(\mathbf{z})} = 0.3$, for each $\mathbf{z} \in \{0, 1\}^2$. We choose $T = T_1 = T_2 = 1$. Throughout the section, we report the proportions of wealth invested in the CDS assets, given by $\psi_i(t, \mathbf{z}) \cdot \Phi_i(t, \mathbf{z})$, $i = 1, 2$.

Figures 6.1 and 6.2 show that the square root investor sells the CDS asset when its market valuation is positive (hence the protection leg has a higher value than the premium leg), and buys it when it is negative. They also illustrate how default contagion impacts the investment strategy. As it appears from Figure 6.2, when the default intensity of name "1" increases, the investor decreases the proportion of wealth allocated to CDS "2." This happens because at the default of name "1," the default intensity of name "2" will instantaneously increase (jumping upward from $h_2^{\mathbb{P}}(0, 0)$ to $h_2^{\mathbb{P}}(1, 0)$), inducing an upward jump in the market value of CDS "2." Consequently, the risk averse investor will allocate a smaller proportion of wealth to it. The contagion effect becomes more pronounced for the case of the CDS referencing name "1." From the top panels of Figures 6.1 and 6.2, we observe that the investor can change the directionality of her strategy when default

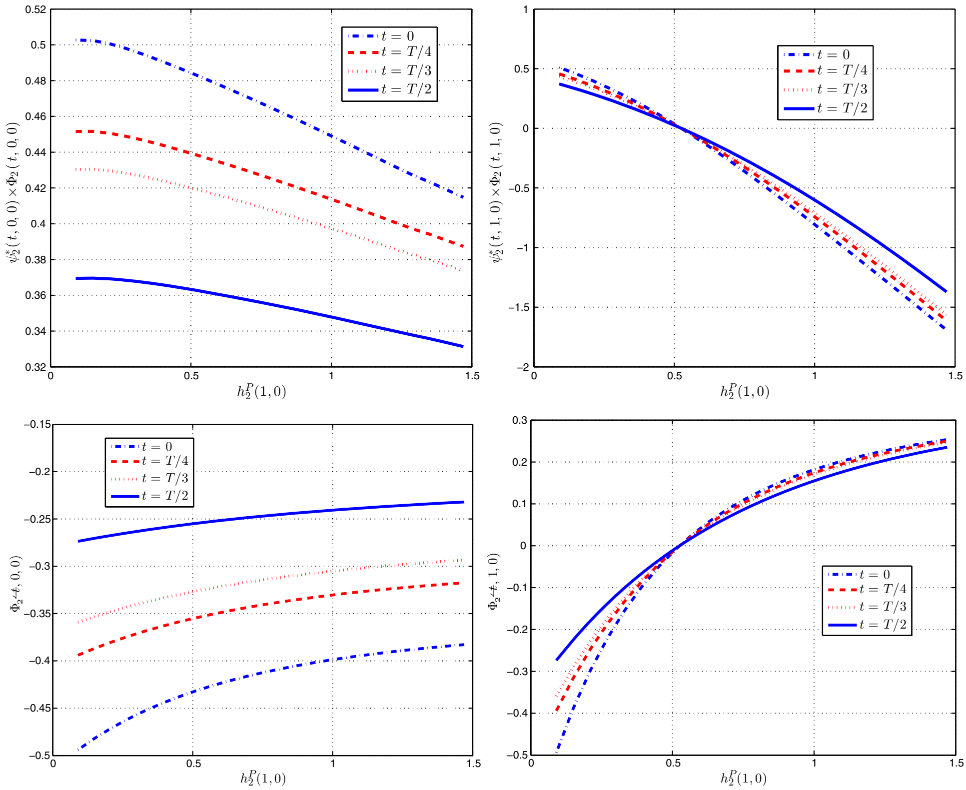


FIGURE 6.3. The top left panel reports the dependence of the optimal investment strategy in CDS "2" on changes in default intensity of name "'2" after name "1" defaults. The top right panel shows the same dependence, but under the scenario when "1" has already defaulted. The bottom panels report the dependence of the corresponding CDS market values on the default intensities. We fix $h_1^{\mathbb{P}}(0,0) = h_2^{\mathbb{P}}(0,0) = 0.2$. Whenever varying $h_2^{\mathbb{P}}(0,0)$ ($h_1^{\mathbb{P}}(0,0)$), we fix $h_1^{\mathbb{P}}(0,1) = 2.3$ ($h_2^{\mathbb{P}}(1,0) = 2.3$).

intensities increase, moving from long credit positions (pays the loss rate) to short credit positions (pays the spread premium).

Figure 6.3 shows that in the predefault scenario (before any name defaults), the market value of the CDS referencing name "2" is not too sensitive to the postdefault intensity, i.e., to the default intensity of name "2" after name "1" defaults. This is because name "1" is unlikely to default ($h_1(0,0) = 0.2$ is small enough), hence the contagion effect of name "1" on name "2" is limited. The top left panel of Figure 6.3 shows that the investor slightly reduces the amount of units purchased as the postdefault intensity of name "2" (i.e., after default of "1") increases. On other hand, if name "1" has already defaulted the investor strategy in CDS "2" will be more sensitive to increases in default intensity of name "2." This is because name "2" has become riskier, hence the investor will find it optimal to sell an increasingly higher amount of CDS units referencing "2" given the increased market valuation of CDS "2."

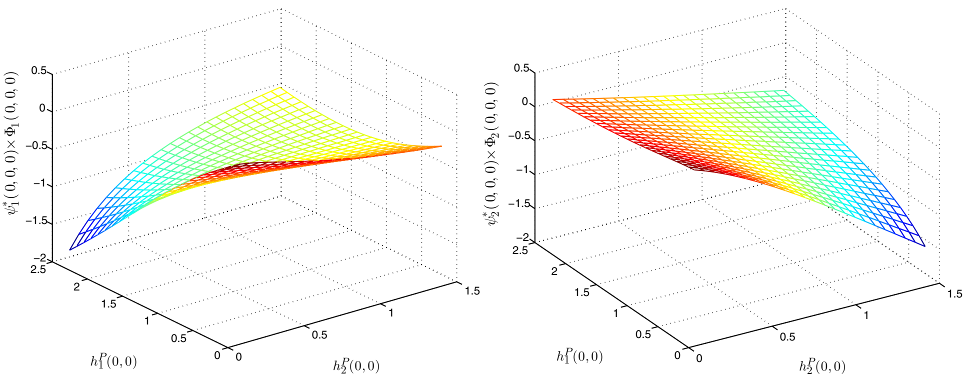
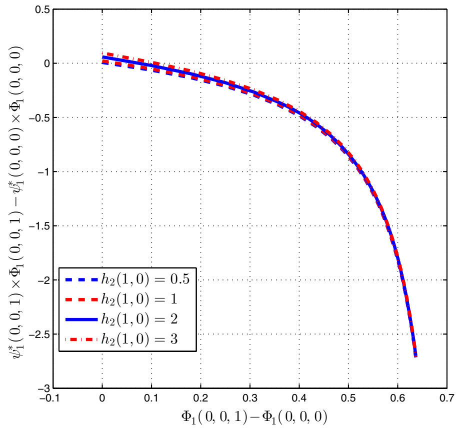


FIGURE 6.4. The graphs describe the dependence of the predefault CDS investment strategy on the joint behavior of the predefault intensities. We fix $t = 0$, $h_1^{\mathbb{P}}(0,1) = h_2^{\mathbb{P}}(1,0) = 2.3$.

From Figures 6.1, 6.2, and 6.3, we find that the investor is more aggressive when the planning horizon is larger. This is justified by the fact that risk premium, historical and risk-neutral default intensities are time invariant in our model. Therefore, the likelihood of a default event happening within a given time interval and the compensation for bearing default risk remains the same as time progresses. Consequently, the investor should buy more (sell less) units of CDS when the market valuation of the CDS is low (high), that is for longer times to maturity, all else being equal. Similar findings are also obtained from Bielecki and Jang (2006) and by Capponi and Figueroa-López (2014) under a different framework.

Next, we investigate how the investment strategies in CDS "1" and "2" depend on the joint behavior of the predefault intensities. The graphs in Figure 6.4 confirm how the contagion effect generated by the default of name "2" (respectively, name "1") affects the proportion of wealth invested in CDS "1" (respectively, CDS "2"). When the default intensity of name "2" is high, the contagion effect on name "1" is stronger, hence the investment strategy in CDS "1" has a smaller sensitivity to changes in the predefault intensity of name "1." This occurs because default of name "2" is very likely to occur, and hence the postdefault intensity of name "1" has a larger impact than its predefault intensity on the market valuation of CDS "1." Vice versa, if the default intensity of name "2" is small, the contagion effect is small, and the investment strategy in CDS "1" mostly depends on the predefault intensity of name "1." Similar patterns occur for the investment strategy in the CDS referencing name "2," as can be confirmed from the bottom right graph.

We conclude with an illustration of how the optimal proportion of wealth allocated to each CDS asset depends on the increased market valuation of the CDS, after accounting for the contagion effect. As expected, when name "2" defaults the market value of CDS "1" increases due to the increased default intensity of "1." As a result, the investor would then sell CDS units referencing "1." Figure 6.5 shows that as the difference between the price of CDS "1" after and before the default of "2" gets larger, the investor sells an increasingly higher number of CDS "1" units. The top right panel of Figure 6.5 confirms a similar trend for CDS "2."



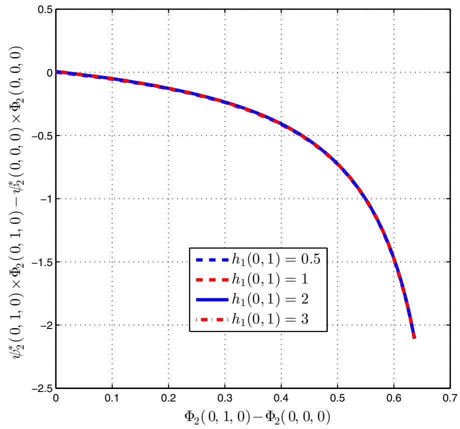


FIGURE 6.5. The graphs describe the dependence of the optimal CDS investment strategy on the before and after default price differences. We fix $t = 0$, $h_1^{\mathbb{P}}(0,0) = h_2^{\mathbb{P}}(0,0) = 0.2$.

7. CONCLUSIONS

We have presented a novel portfolio optimization framework to study the effects generated by default contagion on the optimal investment strategies. We have considered a power investor who can allocate her wealth across multiple CDSs and a money market account. Under a Markovian interacting intensity framework, we have derived the optimal strategies using the HJB approach. We have provided a rigorous analysis showing that the value functions can be characterized as the unique positive solutions of interrelated inhomogeneous Bernoulli type ODEs. Such a dependence reflects the interacting nature of the default process, and establishes a lattice dependence structure. In such a structure, the value function associated with the state where all names are defaulted acts as the maximum, while the one associated with the state where all names are alive acts as the minimum. We have given a precise characterization of the conditions under which the investor goes long or short in each CDS instrument, and expressed them in terms of relations between expectations of instantaneous CDS price changes taken under different probability measures. By means of a numerical analysis, we have demonstrated that default contagion has a strong impact on the optimal allocation decisions, pushing the investor to sell higher number of CDS units if the CDS market valuation increases due to a contagious default event.

APPENDIX: PROOFS OF THEOREMS, LEMMAS, AND PROPOSITIONS

Proof of Lemma 2.3.

Proof. Using Feynman–Kac's formula, we obtain that the functions $\Phi^{(1)}(t, \mathbf{z})$ and $\Phi^{(2)}(t, \mathbf{z})$ satisfy, respectively,

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\right) \Phi_i^{(1)}(t, \mathbf{z}) + (1 - z_i) = r\, \Phi_i^{(1)}(t, \mathbf{z}), \qquad \Phi_i^{(1)}(T_i, \mathbf{z}) = 0, \quad \text{and}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\right) \Phi_i^{(2)}(t, \mathbf{z}) = r(1 - z_i)\Phi_i^{(2)}(t, \mathbf{z}), \qquad \Phi_i^{(2)}(T_i, \mathbf{z}) = z_i, \tag{A.1}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}g(\mathbf{z}) = \sum_{j=1}^{M}(1-z_j)h_j(\mathbf{z})\left[g(\mathbf{z}^j)-g(\mathbf{z})\right], \quad \mathbf{z}\in\mathcal{S}. \tag{A.2}$$

Then the function $\Phi_i(t,\mathbf{z})$ satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\Phi_i(t,\mathbf{z}) &= L_i\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\Phi_i^{(2)}(t,\mathbf{z}) - \nu_i\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\Phi_i^{(1)}(t,\mathbf{z})\\ &= rL_i(1-z_i)\Phi_i^{(2)}(t,\mathbf{z}) - r\nu_i\Phi_i^{(1)}(t,\mathbf{z}) + \nu_i(1-z_i)\\ &= r(1-z_i)\Phi_i(t,\mathbf{z}) + r\nu_i(1-z_i)\Phi_i^{(1)}(t,\mathbf{z}) - r\nu_i\Phi_i^{(1)}(t,\mathbf{z}) + \nu_i(1-z_i)\\ &= r(1-z_i)\Phi_i(t,\mathbf{z}) - r\nu_i z_i\Phi_i^{(1)}(t,\mathbf{z}) + \nu_i(1-z_i).\end{aligned}$$

Using Itô's formula, for all $t\geq 0$, we have

$$\begin{aligned}\Phi_i(t,\mathbf{H}(t)) &= \Phi_i(0,\mathbf{H}(0)) + \int_0^t\left(\frac{\partial}{\partial s}+\mathcal{A}\right)\Phi_i(s,\mathbf{H}(s))\mathrm{d}s\\ &\quad+\sum_{j=1}^{M}\int_0^t\left[\Phi_i(s,\mathbf{H}^j(s-))-\Phi_i(s,\mathbf{H}(s-))\right]\mathrm{d}\xi_j(s)\\ &= \Phi_i(0,\mathbf{H}(0))\\ &\quad+\int_0^t\Big(r(1-H_i(s))\Phi_i(s,\mathbf{H}(s)) - r\nu_i H_i(s)\Phi_i^{(1)}(s,\mathbf{H}(s)) + \nu_i(1-H_i(s))\Big)\mathrm{d}s\\ &\quad+\sum_{j=1}^{M}\int_0^t\left[\Phi_i(s,\mathbf{H}^j(s-))-\Phi_i(s,\mathbf{H}(s-))\right]\mathrm{d}\xi_j(s),\end{aligned}$$

which corresponds to the equality in (2.9). □

Proof of Lemma 2.5.

Proof. It is enough to prove that, for every nonnegative predictable process Y_t, we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t\mathrm{d}H_i(t)\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t(1-H_i(t))h_i^{\mathbb{P}}(\mathbf{H}(t))\mathrm{d}t\right], \quad \text{for any } T>0, \tag{A.3}$$

where we recall that $\mathbb{E}^{\mathbb{P}}$ denotes the expectation with respect to the probability measure $\mathbb{P}$, and $h_i^{\mathbb{P}}(\mathbf{H}(t))$ is the historical default intensity. Consider the right-hand side of equation (A.3). Then we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t(1-H_i(t))h_i^{\mathbb{P}}(\mathbf{H}(t))\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T X_t Y_t(1-H_i(t))h_i^{\mathbb{P}}(\mathbf{H}(t))\mathrm{d}t\right]. \tag{A.4}$$

Furthermore, we may rewrite the left-hand side of (A.3) as

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y_t\mathrm{d}H_i(t)\right] = \mathbb{E}\left[X_T\int_0^T Y_t\mathrm{d}H_i(t)\right]. \tag{A.5}$$

Using Itô's formula, it follows that for any $0 \le t \le T$,

$$\begin{aligned}
\mathrm{d}\left(X_t \int_0^t Y_s \mathrm{d}H_i(s)\right) &= X_{t-} Y_t \mathrm{d}H_i(t) + \left(\int_0^{t-} Y_s \mathrm{d}H_i(s)\right) \mathrm{d}X_t + X_{t-} \left(\sum_{i=1}^{M} \lambda_i(\mathbf{H}(t-))\Delta H_i(t)\right) \\
&\quad \times Y_t \Delta H_i(t) \\
&= X_{t-} Y_t \mathrm{d}H_i(t) + \left(\int_0^{t-} Y_s \mathrm{d}H_i(s)\right) \mathrm{d}X_t + X_{t-} Y_t \lambda_i(\mathbf{H}(t-))\mathrm{d}H_i(t) \\
&= X_{t-} Y_t (1 + \lambda_i(\mathbf{H}(t-)))\, \mathrm{d}H_i(t) + \left(\int_0^{t-} Y_s \mathrm{d}H_i(s)\right) \mathrm{d}X_t,
\end{aligned}$$

where we have used the fact that the default indicator processes $H_1(t), \ldots, H_M(t)$ do not jump simultaneously. Since $X = (X_t;\ t \ge 0)$ is a $(\mathbb{Q}, \mathcal{G}_t)$-martingale, we have

$$\begin{aligned}
\mathbb{E}\left[X_T \int_0^T Y_t \mathrm{d}H_i(t)\right] &= \mathbb{E}\left[\int_0^T X_{t-} Y_t (1 + \lambda_i(\mathbf{H}(t-)))\, \mathrm{d}H_i(t)\right] \\
&= \mathbb{E}\left[\int_0^T X_{t-} Y_t (1 - H_i(t)) h_i^{\mathbb{P}}(\mathbf{H}(t))\mathrm{d}t\right].
\end{aligned}$$

The above expression, along with (A.4) and (A.5), results in the equality (A.3). □

Explicit Solutions to Feynman–Kac Equations (A.1).

LEMMA A.1. *Let $M \ge 2$. For any $i = 1, \ldots, M$, the Feynman–Kac equations in* (A.1) *admit the solutions given next.*

- $\mathbf{z} = \mathbf{1}$. *Then it holds that*

$$\Phi_i^{(1)}(t, \mathbf{1}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, \mathbf{1}) = 1.$$

- $\mathbf{z} = \mathbf{0}^{j_1,\ldots,j_{M-1}}$, *where $j_1, \ldots, j_{M-1} \in \{1, \ldots, M\}$ satisfy $j_1 \neq \cdots \neq j_{M-1}$. Set $j_M := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_{M-1}\}$. Hence, if $i \neq j_M$,*

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\ldots,j_{M-1}}) = 0, \quad \text{and } \Phi_i^{(2)}(t, \mathbf{0}^{j_1,\ldots,j_{M-1}}) = 1.$$

If $i = j_M$, then it holds that

$$\begin{aligned}
\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\ldots,j_{M-1}}) &= \frac{1}{p_{j_1,\ldots,j_{M-1}}} \left(1 - e^{-p_{j_1,\ldots,j_{M-1}}(T_i - t)}\right), \\
\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\ldots,j_{M-1}}) &= \frac{h_{j_M,j_1,\ldots,j_{M-1}}}{p_{j_1,\ldots,j_{M-1}}} \left(1 - e^{-p_{j_1,\ldots,j_{M-1}}(T_i - t)}\right),
\end{aligned}$$

where $h_{j_M,j_1,\ldots,j_{M-1}} := h_{j_M}(\mathbf{0}^{j_1,\ldots,j_{M-1}})$.
- $\mathbf{z} = \mathbf{0}^{j_1,\ldots,j_m}$ *with $0 \le m \le M-2$ (where $\mathbf{0}^{j_1,\ldots,j_m} = \mathbf{0}$ if $m = 0$). Set $\{j_{m+1}, \ldots, j_M\} := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\}$. Hence, if $i \neq j_{m+1}, \ldots, j_M$, it holds that*

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\ldots,j_m}) = 0, \quad \text{and } \Phi_i^{(2)}(t, \mathbf{0}^{j_1,\ldots,j_m}) = 1.$$

If $i \in \{j_{m+1}, \dots, j_M\}$, then it holds that

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m}) = e^{-p_{j_1,\dots,j_m}(T_i-t)} \int_t^{T_i} Q_{i,j_1,\dots,j_m}^{(1)}(s) e^{p_{j_1,\dots,j_m}(T_i-s)} \mathrm{d}s,$$

$$\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m}) = e^{-p_{j_1,\dots,j_m}(T_i-t)} \int_t^{T_i} Q_{i,j_1,\dots,j_m}^{(2)}(s) e^{p_{j_1,\dots,j_m}(T_i-s)} \mathrm{d}s,$$

where the coefficients

$$p_{j_1,\dots,j_m} := r + \sum_{k=j_{m+1},\dots,j_M} h_{k,j_1,\dots,j_m},$$

$$Q_{i,j_1,\dots,j_m}^{(1)}(t) := \sum_{k=j_{m+1},\dots,j_M,\ k\neq i} h_{k,j_1,\dots,j_m} \Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m,k}) + 1,$$

$$Q_{i,j_1,\dots,j_m}^{(2)}(t) := \sum_{k=j_{m+1},\dots,j_M,\ k\neq i} h_{k,j_1,\dots,j_m} \Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m,k}) + h_{i,j_1,\dots,j_m}.$$

Proof. We consider the solution to the above equations iteratively.

- $\mathbf{z} = \mathbf{1}$. For all $i \in \{1, \dots, M\}$, we have that

$$\Phi_i^{(1)}(t, \mathbf{1}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, \mathbf{1}) = 1.$$

- $\mathbf{z} = \mathbf{0}^{j_1,\dots,j_{M-1}}$. Here $j_1, \dots, j_{M-1} \in \{1, \dots, M\}$ satisfying $j_1 \neq \cdots \neq j_{M-1}$. Let $j_M := \{1, \dots, M\} \setminus \{j_1, \dots, j_{M-1}\}$. Hence, if $i \neq j_M$,

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_{M-1}}) = 0, \quad \text{and } \Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_{M-1}}) = 1.$$

If $i = j_M$, then

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_{M-1}}) = \frac{1}{p_{j_1,\dots,j_{M-1}}} \left(1 - e^{-p_{j_1,\dots,j_{M-1}}(T_i-t)}\right),$$

$$\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_{M-1}}) = \frac{h_{j_M,j_1,\dots,j_{M-1}}}{p_{j_1,\dots,j_{M-1}}} \left(1 - e^{-p_{j_1,\dots,j_{M-1}}(T_i-t)}\right),$$

where $p_{j_1,\dots,j_{M-1}} = r + h_{j_M,j_1,\dots,j_{M-1}}$.

- $\mathbf{z} = \mathbf{0}^{j_1,\dots,j_m}$, $0 \leq m \leq M-2$ (where $\mathbf{0}^{j_1,\dots,j_m} = \mathbf{0}$, if $m = 0$). Let $\{j_{m+1}, \dots, j_M\} := \{1, \dots, M\} \setminus \{j_1, \dots, j_m\}$. Hence, if $i \neq j_{m+1}, \dots, j_M$, it holds that

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m}) = 1.$$

If $i \in \{j_{m+1}, \dots, j_M\}$, then the function $\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m})$ should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m}) = p_{j_1,\dots,j_m} \Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m}) - Q_{i,j_1,\dots,j_m}^{(1)}(t),$$

$$\Phi_i^{(1)}(T_i, \mathbf{0}^{j_1,\dots,j_m}) = 0,$$

where

$$p_{j_1,\dots,j_m} := r + \sum_{k=j_{m+1},\dots,j_M} h_{k,j_1,\dots,j_m},$$

$$Q_{i,j_1,\dots,j_m}^{(1)}(t) := \sum_{k=j_{m+1},\dots,j_M,\ k\neq i} h_{k,j_1,\dots,j_m} \Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m,k}) + 1. \tag{A.6}$$

Therefore, we have

$$\Phi_i^{(1)}(t, \mathbf{0}^{j_1,\dots,j_m}) = e^{-p_{j_1,\dots,j_m}(T_i-t)} \int_t^{T_i} Q^{(1)}_{i,j_1,\dots,j_m}(s) e^{p_{j_1,\dots,j_m}(T_i-s)} \mathrm{d}s.$$

If $i \in \{j_{m+1}, \dots, j_M\}$, then the function $\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m})$ should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m}) = p_{j_1,\dots,j_m}\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m}) - Q^{(2)}_{i,j_1,\dots,j_m}(t),$$
$$\Phi_i^{(2)}(T_i, \mathbf{0}^{j_1,\dots,j_m}) = 0,$$

where

$$Q^{(2)}_{i,j_1,\dots,j_m}(t) := \sum_{k=j_{m+1},\dots,j_M,\ k\neq i} h_{k,j_1,\dots,j_m}\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m,k}) + h_{i,j_1,\dots,j_m}. \tag{A.7}$$

Therefore, we have

$$\Phi_i^{(2)}(t, \mathbf{0}^{j_1,\dots,j_m}) = e^{-p_{j_1,\dots,j_m}(T_i-t)} \int_t^{T_i} Q^{(2)}_{i,j_1,\dots,j_m}(s) e^{p_{j_1,\dots,j_m}(T_i-s)} \mathrm{d}s.$$

This completes the proof of the lemma. □

COROLLARY A.2. *Let $M = 2$. Set $p_1 = r + h_1(0,0) + h_2(0,0)$, $p_2 = r + h_2(1,0)$, $p_3 = r + h_1(0,1)$. Then, the Feynman–Kac equations in* (A.1) *admit an explicit representation:*

$\Phi_1^{(1)}(t,1,0) = 0$, $\Phi_1^{(2)}(t,1,0) = 1$, and hence $\Phi_1(t,1,0) := L_1\Phi_1^{(2)}(t,1,0) - \nu_1\Phi_1^{(1)}(t,1,0) = L_1$,

$\Phi_1^{(1)}(t,1,1) = 0$, $\Phi_1^{(2)}(t,1,1) = 1$, and hence $\Phi_1(t,1,1) := L_1\Phi_1^{(2)}(t,1,1) - \nu_1\Phi_1^{(1)}(t,1,1) = L_1$,

$$\Phi_1^{(1)}(t,0,1) = \frac{1}{p_3}\left(1 - e^{-p_3(T_1-t)}\right),\quad \Phi_1^{(2)}(t,0,1) = \frac{h_1(0,1)}{p_3}\left(1 - e^{-p_3(T_1-t)}\right),$$

and hence
$$\Phi_1(t,0,1) := L_1\Phi_1^{(2)}(t,0,1) - \nu_1\Phi_1^{(1)}(t,0,1) = \frac{L_1h_1(0,1) - \nu_1}{p_3}\left(1 - e^{-p_3(T_1-t)}\right). \tag{A.8}$$

- *For the case $p_1 \neq p_3$, it holds that*

$$\begin{aligned}
\Phi_1^{(1)}(t,0,0) &= Ee^{-p_3(T_1-t)} - (E+F)e^{-p_1(T_1-t)} + F,\\
\Phi_1^{(2)}(t,0,0) &= Ge^{-p_3(T_1-t)} - (G+H)e^{-p_1(T_1-t)} + H,\\
\Phi_1(t,0,0) &= (L_1G - \nu_1E)\,e^{-p_3(T_1-t)} - (L_1G - \nu_1E + L_1H - \nu_1F)e^{-p_1(T_1-t)}\\
&\quad + (L_1H - \nu_1F),
\end{aligned} \tag{A.9}$$

where

$$E = \frac{h_2(0,0)}{p_3(p_3-p_1)},\qquad F = \frac{h_2(0,0)+p_3}{p_1p_3},$$
$$G = \frac{h_1(0,1)h_2(0,0)}{p_3(p_3-p_1)},\qquad H = \frac{h_1(0,1)h_2(0,0)+h_1(0,0)p_3}{p_1p_3}.$$

- *For the case $p_1 = p_3$, it holds that*

$$\Phi_1^{(1)}(t,0,0) = F\left(1 - e^{-p_1(T_1-t)}\right) - \frac{h_2(0,0)}{p_1}(T_1 - t)e^{-p_1(T_1-t)},$$
$$\Phi_1^{(2)}(t,0,0) = H\left(1 - e^{-p_1(T_1-t)}\right) - \frac{h_1(0,1)h_2(0,0)}{p_1}(T_1 - t)e^{-p_1(T_1-t)},$$
$$\Phi_1(t,0,0) = (L_1 H - \nu_1 F)\left(1 - e^{-p_1(T-1-t)}\right)$$
$$- \frac{(L_1 h_1(0,1) - \nu_1)h_2(0,0)}{p_1}(T_1 - t)e^{-p_1(T_1-t)}, \tag{A.10}$$

and

$$\Phi_2^{(1)}(t,0,1) = 0,\ \Phi_2^{(2)}(t,0,1) = 1,\ \text{and hence } \Phi_2(t,0,1) = L_2,$$
$$\Phi_2^{(1)}(t,1,1) = 0,\ \Phi_2^{(2)}(t,1,1) = 1,\ \text{and hence } \Phi_2(t,1,1) = L_2,$$
$$\Phi_2^{(1)}(t,1,0) = \frac{1}{p_2}\left(1 - e^{-p_2(T_2-t)}\right),\ \Phi_2^{(2)}(t,1,0) = \frac{h_2(1,0)}{p_2}\left(1 - e^{-p_2(T_2-t)}\right),$$
$$\text{and hence } \Phi_2(t,1,0) = \frac{L_2 h_2(1,0) - \nu_2}{p_2}\left(1 - e^{-p_2(T_2-t)}\right). \tag{A.11}$$

- *For the case $p_1 \neq p_2$, it holds that*

$$\Phi_2^{(1)}(t,0,0) = Ce^{-p_2(T_2-t)} - (C + D)e^{-p_1(T_2-t)} + D,$$
$$\Phi_2^{(2)}(t,0,0) = Ae^{-p_2(T_2-t)} - (A + B)e^{-p_1(T_2-t)} + B,$$
$$\Phi_2(t,0,0) = (L_2 A - \nu_2 C)\, e^{-p_2(T_2-t)} - (L_2 A - \nu_2 C + L_2 B - \nu_2 D)e^{-p_1(T_2-t)}$$
$$+ (L_2 B - \nu_2 D), \tag{A.12}$$

where

$$C = \frac{h_1(0,0)}{p_2(p_2 - p_1)}, \qquad D = \frac{h_1(0,0) + p_2}{p_1 p_2},$$
$$A = \frac{h_1(0,0)h_2(1,0)}{p_2(p_2 - p_1)}, \qquad B = \frac{h_1(0,0)h_2(1,0) + h_2(0,0)p_2}{p_1 p_2}.$$

- *For the case $p_1 = p_2$, it holds that*

$$\Phi_2^{(1)}(t,0,0) = D\left(1 - e^{-p_1(T_2-t)}\right) - \frac{h_1(0,0)}{p_1}(T_2 - t)e^{-p_1(T_2-t)},$$
$$\Phi_2^{(2)}(t,0,0) = B\left(1 - e^{-p_1(T_2-t)}\right) - \frac{h_2(1,0)h_1(0,0)}{p_1}(T_2 - t)e^{-p_2(T_1-t)},$$
$$\Phi_2(t,0,0) = (L_2 B - \nu_2 D)\left(1 - e^{-p_1(T-1-t)}\right)$$
$$- \frac{(L_2 h_2(1,0) - \nu_2)h_1(0,0)}{p_1}(T_2 - t)e^{-p_1(T_2-t)}. \tag{A.13}$$

Proof of Proposition 2.4.

Proof. In light of the representation given in (2.5) and using Itô's formula, we obtain

$$
\begin{aligned}
\mathrm{d}C_t^i &= (1 - H_i(t-))\mathrm{d}\Phi_i(t, \mathbf{H}(t)) - \Phi_i(t, \mathbf{H}(t-))\mathrm{d}H_i(t) + \Delta(1 - H_i(t))\Delta\Phi_i(t, \mathbf{H}(t)) \\
&= (1 - H_i(t-))\mathrm{d}\Phi_i(t, \mathbf{H}(t)) - \Phi_i(t, \mathbf{H}(t-))\mathrm{d}H_i(t) - \Delta\Phi_i(t, \mathbf{H}(t))\mathrm{d}H_i(t) \\
&= (1 - H_i(t-))\mathrm{d}\Phi_i(t, \mathbf{H}(t)) - \Phi_i(t, \mathbf{H}(t-))\mathrm{d}H_i(t) \\
&\quad - \left[\Phi_i(t, \mathbf{H}^i(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}H_i(t),
\end{aligned}
\tag{A.14}
$$

where we used the equality

$$
\Delta\Phi_i(t, \mathbf{H}(t))\mathrm{d}H_i(t) = \left[\Phi_i(t, \mathbf{H}^i(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}H_i(t),
$$

which follows from the fact that our default model excludes the occurrence of simultaneous defaults. Hence, if the ith name defaults, any other name $j \neq i$ would not, and consequently $\Delta\Phi_i(t, \mathbf{H}(t)) = \Phi_i(t, \mathbf{H}^i(t-)) - \Phi_i(t, \mathbf{H}(t-))$. If jth $\neq$ ith name defaults, then $\mathrm{d}H_i(t) = 0$ and hence the above equality still holds.

Using equation (2.9) in Lemma 2.3, we obtain

$$
\begin{aligned}
&(1 - H_i(t-))\mathrm{d}\Phi_i(t, \mathbf{H}(t)) \\
&= (1 - H_i(t))\Big[r(1 - H_i(t))\Phi_i(t, \mathbf{H}(t)) - r\nu_i H_i(t)\Phi_i^{(1)}(t, \mathbf{H}(t)) + \nu_i(1 - H_i(t))\Big]\mathrm{d}t \\
&\quad + (1 - H_i(t-))\sum_{j=1}^{M}\left[\Phi_i(t, \mathbf{H}^j(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}\xi_j(t) \\
&= [r(1 - H_i(t))\Phi_i(t, \mathbf{H}(t)) + \nu_i(1 - H_i(t))]\,\mathrm{d}t \\
&\quad + (1 - H_i(t-))\sum_{j=1}^{M}\left[\Phi_i(t, \mathbf{H}^j(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}\xi_j(t).
\end{aligned}
$$

From (A.14), it follows that

$$
\begin{aligned}
\mathrm{d}C_t^i &= (1 - H_i(t-))\mathrm{d}\Phi_i(t, \mathbf{H}(t)) - \Phi_i(t, \mathbf{H}(t-))\mathrm{d}H_i(t) \\
&\quad - \left[\Phi_i(t, \mathbf{H}^i(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}H_i(t) \\
&= (1 - H_i(t))\,(r\Phi_i(t, \mathbf{H}(t)) + \nu_i)\,\mathrm{d}t - \Phi_i(t, \mathbf{H}(t))\mathrm{d}H_i(t) \\
&\quad + (1 - H_i(t-))\sum_{j=1}^{M}\left[\Phi_i(t, \mathbf{H}^j(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}\xi_j(t) \\
&\quad - \left[\Phi_i(t, \mathbf{H}^i(t-)) - \Phi_i(t, \mathbf{H}(t-))\right]\mathrm{d}H_i(t).
\end{aligned}
\tag{A.15}
$$

Using the equality $\Phi_i(t, \mathbf{z}^i) = L_i$ following from Lemma A.1, we obtain the desired result. □

Proof of Lemma 4.1.

Proof. Using the expression of g_i given in (4.5), we have that it is decreasing with respect to ψ_i. We consider the following subcases:

- $M_2^{(i)}(t) < +\infty$. Then we have

$$\lim_{\psi_i \downarrow M_1^{(i)}(t)} g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right) = +\infty, \quad \text{and} \quad \lim_{\psi_i \uparrow M_2^{(i)}(t)} g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right) = -\infty.$$

  Since $g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right)$ is continuous in $\psi_i \in (M_1^{(i)}(t), M_2^{(i)}(t))$, we can apply the Intermediate Value Theorem, and obtain a unique solution $\psi_i^* \in (M_1^{(i)}(t), M_2^{(i)}(t))$ to the equation (4.8).
- $M_2^{(i)}(t) = +\infty$. Then it holds that $\Phi_i(t, \mathbf{0}^j) > \Phi_i(t, \mathbf{0})$ for all $j \in \{1, \ldots, M\} \setminus \{i\}$. Hence

$$\lim_{\psi_i \downarrow M_1^{(i)}(t)} g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right) = +\infty, \quad \text{and} \quad \lim_{\psi_i \uparrow +\infty} g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right) = V_i(t)\mathbf{B}_1^{(i)}.$$

  Since $\Phi_i(t, \mathbf{0}^j) > \Phi_i(t, \mathbf{0})$ for all $j \in \{1, \ldots, M\} \setminus \{i\}$, it holds that

$$V_i(t) = (\Phi_i(t, \mathbf{0}) - L_i)\, h_i(\mathbf{0}) - \sum_{j \neq i} \left(\Phi_i(t, \mathbf{0}^j) - \Phi_i(t, \mathbf{0})\right) h_j(\mathbf{0}) < 0.$$

  Thus, $V_i(t)\mathbf{B}_1^{(i)} < 0$, since $\mathbf{B}_1^{(i)} > 0$.

Hence, we have that there exists a unique solution ψ_i^* to equation (4.8) in the desired domain of ψ_i. In light of Kumagai (1980)'s implicit function theorem, we also have that $\psi_i^*\left(t, \mathbf{B}^{(i)}\right)$ is continuous if we can prove that $g_i\left(\psi_i, t, \mathbf{B}^{(i)}\right)$ is continuous in $(\psi_i, t, \mathbf{B}^{(i)})$. The latter property follows because $\Phi_i(t, \mathbf{0}^j)$ is continuous for $j \neq i$, $\Phi_i(t, \mathbf{0}^i) = L_i$ is constant, and the function $x^{\gamma-1}$ is continuous in x. Since composition of continuous functions is continuous, we obtain the desired result. □

Proof of Lemma 4.2.

Proof. Denote $\{j_{m+1}, \ldots, j_M\}$ all elements of the set $\{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\}$. We want to find the optimum $\psi^*_{i,j_1,\ldots,j_m}$ satisfying the following conditions. For any $i = j_{m+1}, \ldots, j_M$,

$$\psi^*_{i,j_1,\ldots,j_m}(t) > -\frac{1}{L_i - \Phi_{i,j_1,\ldots,j_m}(t)} := M_1^{(i,j_1,\ldots,j_m)}(t), \quad \text{and}$$

$$\psi^*_{i,j_1,\ldots,j_m}(t)\left(\Phi_{i,j_1,\ldots,j_m,\ell}(t) - \Phi_{i,j_1,\ldots,j_m}(t)\right) > -1, \quad \text{for all } \ell \in N_{i,j_1,\ldots,j_m}. \tag{A.16}$$

We can assume that there exist $\ell_1, \ldots, \ell_k \in N_{i,j_1,\ldots,j_m}$ for some $k \in \{0, \ldots, |N_{i,j_1,\ldots,j_m}|\}$ so that $\Phi_{i,j_1,\ldots,j_m,\ell_n}(t) < \Phi_{i,j_1,\ldots,j_m}(t)$ for $n = 1, \ldots, k$. Then it holds that $\Phi_{i,j_1,\ldots,j_m,\ell_n}(t) > \Phi_{i,j_1,\ldots,j_m}(t)$ for $n = k+1, \ldots, |N_{i,j_1,\ldots,j_m}|$.

Here, $k = 0$ implies that $\Phi_{i,j_1,\ldots,j_m,\ell_n}(t) > \Phi_{i,j_1,\ldots,j_m}(t)$ for $n = 1, \ldots, |N_{i,j_1,\ldots,j_m}|$. If $k = |N_{i,j_1,\ldots,j_m}|$, then it implies that $\Phi_{i,j_1,\ldots,j_m,\ell_n}(t) < \Phi_{i,j_1,\ldots,j_m}(t)$ for $n = 1, \ldots, |N_{i,j_1,\ldots,j_m}|$.

Note that

$$\max\left\{M_1^{(i,j_1,\ldots,j_m)}(t), \frac{1}{\Phi_{i,j_1,\ldots,j_m}(t) - \Phi_{i,j_1,\ldots,j_m,\ell_n}(t)};\ n = k+1, \ldots, |N_{j_1,\ldots,j_m,i}|\right\}$$
$$= M_1^{(i,j_1,\ldots,j_m)}(t),$$

since $\Phi_{i,j_1,\ldots,j_m,\ell_n}(t) < L_i$. Define

$$M_2^{(i,j_1,\ldots,j_m)}(t) := \min\left\{\frac{1}{\Phi_{i,j_1,\ldots,j_m}(t) - \Phi_{i,j_1,\ldots,j_m,\ell_n}(t)};\ n = 1,\ldots,k\right\}, \tag{A.17}$$

where $M_2^{(i,j_1,\ldots,j_m)}(t) = +\infty$ if the set $\{\cdot\}$ on the right-hand side of the above definition is empty (i.e., if $k = 0$). Hence, for each $i \neq j_1,\ldots,j_m$ the optimum must satisfy

$$M_1^{(i,j_1,\ldots,j_m)}(t) < \psi^*_{i,j_1,\ldots,j_m}(t) < M_2^{(i,j_1,\ldots,j_m)}(t). \tag{A.18}$$

Next we consider the existence and uniqueness of the optimum $\psi^*_{i,j_1,\ldots,j_m}$ as a solution to the equation with unknown $\psi_{i,j_1,\ldots,j_m}$ satisfying (4.15), i.e.,

$$g_{i,j_1,\ldots,j_m}\left(\psi_{i,j_1,\ldots,j_m}, t, \mathbf{B}^{(j_1,\ldots,j_m,i)}\right) = 0. \tag{A.19}$$

Using the expression of $g_{i,j_1,\ldots,j_m}$ given in (4.13), we see that it is decreasing with respect to $\psi_{i,j_1,\ldots,j_m}$. We have the following subcases:

- $M_2^{(i,j_1,\ldots,j_m)}(t) < +\infty$. Then we have

$$\lim_{\psi_{i,j_1,\ldots,j_m}\downarrow M_1^{(i,j_1,\ldots,j_m)}(t)} g_{i,j_1,\ldots,j_m}\left(\psi_{i,j_1,\ldots,j_m}, t, \mathbf{B}^{(j_1,\ldots,j_m,i)}\right) = +\infty,$$

  and

$$\lim_{\psi_{i,j_1,\ldots,j_m}\uparrow M_2^{(i,j_1,\ldots,j_m)}(t)} g_{j_1,\ldots,j_m,i}\left(\psi_{i,j_1,\ldots,j_m}, t, \mathbf{B}^{(j_1,\ldots,j_m,i)}\right) = -\infty.$$

  Since $g_{i,j_1,\ldots,j_m}\left(\psi_{i,j_1,\ldots,j_m}, t, \mathbf{B}^{(j_1,\ldots,j_m,i)}\right)$ is continuous in $\psi_{i,j_1,\ldots,j_m} \in (M_1^{(i,j_1,\ldots,j_m)}(t), M_2^{(i,j_1,\ldots,j_m)}(t))$, we can apply the Intermediate Value Theorem, and obtain a unique solution

$$\psi_{i,j_1,\ldots,j_m} \in (M_1^{(i,j_1,\ldots,j_m)}(t), M_2^{(i,j_1,\ldots,j_m)}(t))$$

  to the equation (A.19).
- $M_2^{(i,j_1,\ldots,j_m)}(t) = +\infty$. Then it holds that $\Phi_{i,j_1,\ldots,j_m,\ell}(t) > \Phi_{i,j_1,\ldots,j_m}(t)$ for $\ell \in N_{i,j_1,\ldots,j_m}$. Hence

$$\lim_{\psi_{i,j_1,\ldots,j_m}\downarrow M_1^{(i,j_1,\ldots,j_m)}(t)} g_{i,j_1,\ldots,j_m}\left(\psi_{i,j_1,\ldots,j_m}, t, \mathbf{B}^{(j_1,\ldots,j_m,i)}\right) = +\infty,$$

  and

$$\lim_{\psi_{i,j_1,\ldots,j_m}\uparrow+\infty} g_{i,j_1,\ldots,j_m}\left(\psi_{i,j_1,\ldots,j_m}, t, \mathbf{B}^{(j_1,\ldots,j_m,i)}\right) = V_{i,j_1,\ldots,j_m}(t)\mathbf{B}_1^{(j_1,\ldots,j_m,i)}.$$

  Since it holds that $\Phi_{i,j_1,\ldots,j_m,\ell}(t) > \Phi_{i,j_1,\ldots,j_m}(t)$ for $\ell \in N_{i,j_1,\ldots,j_m}$, we have

$$\begin{aligned} V_{i,j_1,\ldots,j_m}(t) := {} & \left(\Phi_{i,j_1,\ldots,j_m}(t) - L_i\right) h_{i,j_1,\ldots,j_m} \\ & - \sum_{\ell\in N_{i,j_1,\ldots,j_m}} \left(\Phi_{i,j_1,\ldots,j_m,\ell}(t) - \Phi_{i,j_1,\ldots,j_m}(t)\right) h_{\ell,j_1,\ldots,j_m} \\ < {} & 0. \end{aligned}$$

  Thus $V_{i,j_1,\ldots,j_m}(t)\mathbf{B}_1^{(j_1,\ldots,j_m,i)} < 0$, since $\mathbf{B}_1^{(j_1,\ldots,j_m,i)} > 0$.

Hence, we have that there exists a unique solution $\psi^*_{i,j_1,\dots,j_m}$ to equation (A.19) in the desired domain of $\psi_{i,j_1,\dots,j_m}$. In light of Kumagai's (1980) implicit function theorem, we also have that $\psi_{i,j_1,\dots,j_m}\left(t, \mathbf{B}^{(j_1,\dots,j_m,i)}\right)$ is continuous if we can prove that $g_{i,j_1,\dots,j_m}\left(\psi_{i,j_1,\dots,j_m}, t, \mathbf{B}^{(j_1,\dots,j_m,i)}\right)$ is continuous in $(\psi_{i,j_1,\dots,j_m}, t, \mathbf{B}^{(j_1,\dots,j_m,i)})$. The latter property follows because $\Phi_{i,j_1,\dots,j_m,\ell}(t)$ is continuous, $\Phi_{i,j_1,\dots,j_m,i}(t) = L_i$ is constant, and the function $x^{\gamma-1}$ is continuous in x. Since composition of continuous functions is continuous, we obtain the desired result. □

Proof of Lemma 4.3.

Proof. Let us start defining

$$G(t, \hat{u}) := K_{j_M}(\gamma)\hat{u} + A_{j_M}(t, \gamma)\hat{u}^\gamma + U_{j_M}(t, \gamma), \qquad (t, \hat{u}) \in R \times R_+,$$

where we extend the coefficients $A_{j_M}(t, \gamma)$ and $U_{j_M}(t, \gamma)$, defined on $t \in [0, T]$ to R by setting $A_{j_M}(t, \gamma) = A_{j_M}(0, \gamma)$, $U_{j_M}(t, \gamma) = U_{j_M}(0, \gamma)$ for $t < 0$ and $A_{j_M}(t, \gamma) = A_{j_M}(T, \gamma)$, $U_{j_M}(t, \gamma) = U_{j_M}(T, \gamma)$ for $t > T$. Then the functions $G(t, \hat{u})$ and $\frac{\partial G(t,\hat{u})}{\partial \hat{u}}$ are continuous on the domain $(t, \hat{u}) \in \mathcal{O} := R \times R_+$ containing the point $(T, \gamma^{\frac{1}{\gamma-1}})$. By virtue of the fundamental theorem of existence and uniqueness of the solution (see, e.g., Chicone 2006) to the following equation:

$$\hat{u}'(t) = G(t, \hat{u}(t)), \quad \hat{u}(T) = \gamma^{\frac{1}{\gamma-1}} > 0, \tag{A.20}$$

we obtain that there exists a unique solution to equation (A.20) for $t \in [T - \delta_1, T + \delta_1]$, given a suitably chosen constant $\delta_1 > 0$. Let $y(t) = \hat{u}(t)e^{-K_{j_M}(\gamma)t}$. Then we have

$$y'(t) = e^{-K_{j_M}(\gamma)t}\left(A_{j_M}(t, \gamma)e^{\gamma K_{j_M}(\gamma)t}y^\gamma(t) + U_{j_M}(t, \gamma)\right), \quad y(T) = \gamma^{\frac{1}{\gamma-1}}e^{-K_{j_M}(\gamma)T} > 0.$$

Note that $A_{j_M}(t, \gamma) < 0$ and $U_{j_M}(t, \gamma) < 0$ for all $t \in R$. Hence, $y'(T) < 0$, since $y(T) > 0$. Using the continuity of the solution $\hat{u}'(t)$ at T, there exists $\hat{\delta}_1 > 0$ (possibly smaller than δ_1) so that $y'(t) < 0$ for all $t \in [T - \delta_2, T + \delta_2]$. Since $y(T) > 0$, we have that $y(t) > 0$ for all $t \in [T - \hat{\delta}_1, T]$. For the case of $\hat{\delta}_1 < \delta_1$, we can repeat the above argument and show that $y(t) > 0$ for all $t \in [T - \delta_1, T]$. This implies that $\hat{u}(t) > 0$ for all $t \in [T - \delta_1, T]$.

Using the above established existence and uniqueness of a positive local solution $\hat{u}(t)$ for $t \in [T - \delta_1, T + \delta_1]$, we next prove existence and uniqueness of a global solution on $[0, T]$. This is done by repeatedly solving the equation $\hat{u}'(t) = G(t, \hat{u}(t))$ backward. If $\delta_1 \geq T$, then we have obtained the unique positive $\hat{u}(t)$ for $t \in [0, T] \subset [T - \delta_1, T + \delta_1]$. If $\delta_1 < T$, we consider $v'(t) = G(t, v(t))$ on $t \in [0, T - \delta_1]$ with terminal condition $v(T - \delta_1) = u(T - \delta_1) > 0$. Since $G(t, v)$ and $\frac{\partial G(t,v)}{\partial v}$ are continuous on the domain $(t, v) \in \mathcal{O}$ containing the point $(T - \delta_1, u(T - \delta_1))$, we can repeat an argument similar to the one above, and obtain a unique positive solution on $[T - \delta_1 - \delta_2, T - \delta_1]$ for some positive $\delta_2 > 0$. Hence, we have the solution on $[T - \delta_1 - \delta_2, T - \delta_1] \cup [T - \delta_1, T] = [T - \delta_1 - \delta_2, T]$. Iterating such procedure, we obtain the solution on the whole interval $[0, T]$. □

Proof of Lemma 4.5.

Proof. Using that $\Phi_{i,j_1,\cdots,j_m,i}(t) = L_i$, define for $i \in \{j_{m+1}, \ldots, j_M\} := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\}$,

$$
\begin{aligned}
&C_{j_1,\ldots,j_m}(t,u) \\
&:= \Bigg\{\gamma \sum_{i=j_{m+1},\ldots,j_M} \psi^*_{i,j_1,\cdots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}(t)\right)\Bigg[\left(\Phi_{i,j_1,\cdots,j_m}(t)-L_i\right)h_{i,j_1,\cdots,j_m} \\
&\quad - \sum_{\ell\in N_{i,j_1,\ldots,j_m}} \left(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\right)h_{\ell,j_1,\cdots,j_m}\Bigg] + a(\gamma) \\
&\quad -(m+1)\sum_{i=j_{m+1},\ldots,j_M} h^{\mathbb{P}}_{i,j_1,\ldots,j_m} - \sum_{i=j_{m+1},\ldots,j_M}\Bigg(\sum_{\ell\in N_{i,j_1,\ldots,j_m}} h^{\mathbb{P}}_{\ell,j_1,\ldots,j_m}\Bigg)\Bigg\}u \\
&\quad + \sum_{i=j_{m+1},\ldots,j_M}\left[1+\psi^*_{i,j_1,\cdots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}(t)\right)(L_i-\Phi_{i,j_1,\cdots,j_m}(t))\right]^{\gamma}\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}_1(t)h^{\mathbb{P}}_{i,j_1,\ldots,j_m} \\
&\quad + \sum_{i=j_1,\ldots,j_m}\Bigg\{\sum_{\ell=j_{m+1},\ldots,j_M}\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}_{\ell}(t)h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\} \\
&\quad + \sum_{i=j_{m+1},\ldots,j_M}\Bigg\{\sum_{\ell\in N_{i,j_1,\ldots,j_m}}\left[1+\psi^*_{i,j_1,\ldots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}(t)\right)\left(\Phi_{i,j_1,\cdots,j_m,\ell}(t)-\Phi_{i,j_1,\cdots,j_m}(t)\right)\right]^{\gamma} \\
&\quad \times\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}_{\ell}(t)h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\Bigg\},
\end{aligned}
\tag{A.21}
$$

where we have used the notation:

$$\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}(t) := \left[B_{j_1,\ldots,j_m,i}(t),\ B_{j_1,\ldots,j_m,\ell}(t);\ \ell\in N_{i,j_1,\ldots,j_m}\right]^{\top}.$$

We have assumed that for each tuple $(j_1, \ldots, j_{m+1})$ satisfying $j_1 \neq \cdots \neq j_{m+1}$, there exists a unique positive solution $B_{j_1,\ldots,j_{m+1}}(t)$ to equation (4.20). Hence, $B_{j_1,\ldots,j_{m+1}}(t)$ has continuous first-order partial derivatives with respect to $t \in [0, T]$. It remains to prove that the optimal strategies

$$\psi^*_{i,j_1,\ldots,j_m}(t,u) := \psi^*_{i,j_1,\cdots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\ldots,j_m,i)}(t)\right),\quad i=j_{m+1},\ldots,j_M \tag{A.22}$$

have continuous first-order partial derivatives with respect to $(t, u) \in [0, T] \times R_+$. By virtue of Lemma 4.2, we know that the optimum $\psi^*_{i,j_1,\ldots,j_m}(t, \mathbf{B}^{(j_1,\cdots,j_m,i)})$ is continuous with respect to $(t, \mathbf{B}^{(j_1,\cdots,j_m,i)})$ and hence $\psi^*_{i,j_1,\ldots,j_m}(t, u)$ is continuous with respect to $(t, u) \in [0, T] \times R_+$. Recall $g_{i,j_1,\ldots,j_m}(\psi_{i,j_1,\ldots,j_m}, t, (u, \widehat{\mathbf{B}}^{(j_1,\cdots,j_m,i)}))$ defined by (4.13). Then $g_{i,j_1,\ldots,j_m}(\psi_{i,j_1,\ldots,j_m}, t, (u,\cdot))$ is continuously differentiable on the set $(\psi_{i,j_1,\ldots,j_m}, t, u) \in D^1 \times [0, T] \times R_+$, where D^1 is the admissibility domain of the optimum $\psi^*_{i,j_1,\ldots,j_m}$, specified by (4.15). Using Lemma 4.2 and the Implicit Function Theorem, it follows that the optimum $\psi^*_{i,j_1,\ldots,j_m}(t, u)$ defined by (A.22) is continuously differentiable on $[0, T] \times R_+$. By virtue of the fundamental theorem of existence and uniqueness of the local solution to the following equation

$$u'(t) = -C_{j_1,\ldots,j_m}(t, u(t)),\qquad u(T) = \frac{1}{\gamma}, \tag{A.23}$$

we have that there exists a unique solution to equation (A.23) for $t \in [T-\delta_1, T+\delta_1]$ for some constant $\delta_1 > 0$. Note that we can write the function $C_{j_1,\dots,j_m}(t, u(t))$ in the following form:

$$C_{j_1,\dots,j_m}(t,u) = f_{j_1,\dots,j_m}(t,u)u + q_{j_1,\dots,j_m}(t,u) + v_{j_1,\dots,j_m}(t), \tag{A.24}$$

where the coefficients

$$\begin{aligned}
f_{j_1,\dots,j_m}(t,u) &:= \gamma \sum_{i=j_{m+1},\dots,j_M} \psi^*_{i,j_1,\cdots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\dots,j_m,i)}(t)\right)\bigg[\left(\Phi_{i,j_1,\cdots,j_m}(t) - L_i\right) h_{i,j_1,\cdots,j_m} \\
&\quad - \sum_{\ell\in N_{i,j_1,\dots,j_m}} \left(\Phi_{i,j_1,\cdots,j_m,\ell}(t) - \Phi_{i,j_1,\cdots,j_m}(t)\right) h_{\ell,j_1,\cdots,j_m}\bigg] + a(\gamma) \\
&\quad -(m+1)\sum_{i=j_{m+1},\dots,j_M} h^{\mathbb{P}}_{i,j_1,\dots,j_m} - \sum_{i=j_{m+1},\dots,j_M}\Big(\sum_{\ell\in N_{i,j_1,\dots,j_m}} h^{\mathbb{P}}_{\ell,j_1,\dots,j_m}\Big), \\
q_{j_1,\dots,j_m}(t,u) &:= \sum_{i=j_{m+1},\dots,j_M}\left[1+\psi^*_{i,j_1,\cdots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\dots,j_m,i)}(t)\right)(L_i - \Phi_{i,j_1,\cdots,j_m}(t))\right]^{\gamma} \\
&\quad \times \widehat{\mathbf{B}}_1^{(j_1,\dots,j_m,i)}(t) h^{\mathbb{P}}_{i,j_1,\dots,j_m} \\
&\quad + \sum_{i=j_{m+1},\dots,j_M}\bigg\{\sum_{\ell\in N_{i,j_1,\dots,j_m}}\bigg[1+\psi^*_{i,j_1,\dots,j_m}\left(t,u,\widehat{\mathbf{B}}^{(j_1,\dots,j_m,i)}(t)\right) \\
&\quad \times\Big(\Phi_{i,j_1,\cdots,j_m,\ell}(t) - \Phi_{i,j_1,\cdots,j_m}(t)\Big)\bigg]^{\gamma}\widehat{\mathbf{B}}_\ell^{(j_1,\dots,j_m,i)}(t) h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\bigg\}, \\
v_{j_1,\dots,j_m}(t) &:= \sum_{i=j_1,\dots,j_m}\bigg\{\sum_{\ell=j_{m+1},\dots,j_M}\widehat{\mathbf{B}}_\ell^{(j_1,\dots,j_m,i)}(t) h^{\mathbb{P}}_{\ell,j_1,\cdots,j_m}\bigg\}.
\end{aligned} \tag{A.25}$$

Define the function $\hat{u}(t) = u(t)e^{-\int_t^T f_{j_1,\dots,j_m}(s,u(s))\mathrm{d}s}$, where $u(t)$ is the solution to equation (A.23) for $t \in [T-\delta_1, T+\delta]$. Then we have that for $t \in [T-\delta_1, T+\delta]$, it holds that

$$\hat{u}'(t) = e^{-\int_t^T f_{j_1,\dots,j_m}(s,u(s))\mathrm{d}s}\left[-q_{j_1,\dots,j_m}(t,u(t)) - v_{j_1,\dots,j_m}(t)\right], \quad \hat{u}'(T) = \frac{1}{\gamma} > 0. \tag{A.26}$$

Recall that we have assumed that $\widehat{\mathbf{B}}_\ell^{(j_1,\dots,j_m,i)}(t)$ is positive for all $\ell = 1, j_{m+1}, \dots, j_M$ and $i = j_1, \dots, j_m$. This leads to $v_{j_1,\dots,j_m}(t) > 0$. Moreover, since the optimum $\psi^*_{i,j_1,\dots,j_m}$ satisfies $1 + \psi_{i,j_1,\dots,j_m}(t,u,\widehat{\mathbf{B}}^{(j_1,\dots,j_m,i)}(t))(L_i - \Phi_{i,j_1,\cdots,j_m}(t)) > 0$ and $1 + \psi^*_{i,j_1,\dots,j_m}(t,u,\widehat{\mathbf{B}}^{(j_1,\dots,j_m,i)}(t))(\Phi_{i,j_1,\cdots,j_m,\ell}(t) - \Phi_{i,j_1,\cdots,j_m}(t)) > 0$ for each $i = j_{m+1}, \dots, j_M$, using (4.15), we also have $q_{j_1,\dots,j_m}(t,u(t)) > 0$. This results in $\hat{u}'(T) < 0$. Using the continuity of the function $u(t)$ at time $t = T$, we have that there exists $\hat{\delta}_1 > 0$ (possibly smaller than δ_1) so that $\hat{u}'(t) < 0$ for all $t \in [T-\hat{\delta}_1, T]$. Thus we have $\hat{u}(t) > 0$ for all $t \in [T-\hat{\delta}_1, T]$ and hence $u(t) = \hat{u}(t)e^{\int_t^T f_{j_1,\dots,j_m}(s,u(s))\mathrm{d}s} > 0$ for all $t \in [T-\hat{\delta}_1, T]$. If $\hat{\delta}_1 \geq \delta_1$, then we indeed have $u(t) > 0$ for all $t \in [T-\delta_1, T]$. For the case of $\hat{\delta}_1 < \delta_1$, we can repeat the above argument to show the positivity of $u(t)$ when $t \in [T-\delta_1, T-\hat{\delta}_1]$ and thus we still have $u(t) > 0$ for all $t \in [T-\delta_1, T]$.

Based on the existence and uniqueness of a positive local solution $u(t)$ to equation (A.23) proven above, we next show existence and uniqueness of a global solution $[0, T]$

by repeatedly solving equation (A.23) backward. If $\delta_1 \geq T$, then we have obtained the unique positive $u(t)$ for $t \in [0, T] \subset [T - \delta_1, T]$. If $\delta_1 < T$, we consider the equation $v(t) = -C_{j_1,\ldots,j_m}(t, v(t))$ on $t \in [0, T - \delta_1]$ with the terminal condition $v(T - \delta_1) = u(T - \delta_1) > 0$. Since $C_{j_1,\ldots,j_m}(t, u)$ and $\frac{\partial C_{j_1,\ldots,j_m}(t,u)}{\partial u}$ are continuous on the domain $(t, u) \in [0, T] \times R_+$ containing the point $(T - \delta_1, u(T - \delta_1))$, we can repeat an argument similar to the one above, and obtain a unique positive solution on $t \in [T - \delta_1 - \delta_2, T - \delta_1]$ for some constant $\delta_2 > 0$. Hence, we have the positive solution on $t \in [T - \delta_1 - \delta_2, T - \delta_1] \cup [T - \delta_1, T] = [T - \delta_1 - \delta_2, T]$. Iterating such procedure, we obtain the solution on the whole interval $[0, T]$.

Recalling that $B_{j_1,\ldots,j_m}(t)^{\frac{1}{1-\gamma}} = \hat{u}(t) > 0$ for all $t \in [0, T]$, we deduce a unique positive solution $B_{j_1,\ldots,j_m}(t)$ to equation (4.20). □

Procedure Used to Solve Equations (4.36) and (4.37).
We first solve equation (4.36). Recall that $a(\gamma) := \gamma r > 0$. Then (4.36) may be reduced to

$$\frac{\partial B(t, 1, 1)}{\partial t} = -a(\gamma)B(t, 1, 1), \qquad B(T, 1, 1) = \frac{1}{\gamma}.$$

Obviously, we have that $B(t, 1, 1) = \frac{1}{\gamma}e^{a(\gamma)(T-t)}$ for $0 \leq t \leq T$. Then, we substitute the optimum $\psi_2^*(t, 1, 0)$ given by (4.33) into the equation (4.36). We obtain

$$\text{(A.27)} \qquad \frac{\mathrm{d}B(t, 1, 0)}{\mathrm{d}t} = D_2\left(t, B(t, 1, 0)\right), \qquad B(T, 1, 0) = \frac{1}{\gamma},$$

where D_2 is defined on $(t, B) \in [0, T] \times R_+$, and given by

$$\text{(A.28)} \qquad D_2(t, B) := K_2 B + U_2(t)B^{\frac{\gamma}{\gamma-1}} + A_2(t),$$

with coefficients

$$\begin{aligned}
K_2 &:= 2h_2^{\mathbb{P}}(1, 0) - \gamma h_2(1, 0) - a(\gamma),\\
A_2(t) &:= -B(t, 1, 1)h_2^{\mathbb{P}}(1, 0),\\
U_2(t) &:= \gamma h_2(1, 0)\left(\frac{h_2(1, 0)}{B(t, 1, 1)h_2^{\mathbb{P}}(1, 0)}\right)^{\frac{1}{\gamma-1}} - B(t, 1, 1)h_2^{\mathbb{P}}(1, 0)\left(\frac{h_2(1, 0)}{B(t, 1, 1)h_2^{\mathbb{P}}(1, 0)}\right)^{\frac{\gamma}{\gamma-1}}\\
&= (\gamma - 1)h_2(1, 0)\eta_2^{\frac{1}{\gamma-1}} B(t, 1, 1)^{\frac{1}{1-\gamma}}, \qquad \text{where } \eta_2 = \frac{h_2(1, 0)}{h_2^{\mathbb{P}}(1, 0)}.
\end{aligned}$$

We numerically solve the above nonlinear ODE (unique solution guaranteed by Lemma 4.3), and then plug the corresponding solution into (4.33) to explicitly obtain the optimal number of shares invested in the CDS asset referencing name "1" after the default of name "2."

Similarly, after substituting the optimum $\psi_1^*(t, 0, 1)$ given by (4.32) into the equation (4.36), we obtain

$$\text{(A.29)} \qquad \frac{\mathrm{d}B(t, 0, 1)}{\mathrm{d}t} = D_1\left(t, B(t, 0, 1)\right), \qquad B(T, 0, 1) = \frac{1}{\gamma},$$

where the function

$$D_1(t, B) = K_1 B + U_1(t) B^{\frac{\gamma}{\gamma-1}} + A_1(t), \tag{A.30}$$

with the coefficients

$$\begin{aligned}
K_1 &:= 2h_1^{\mathbb{P}}(0,1) - \gamma h_1(0,1) - a(\gamma),\\
A_1(t) &:= -B(t,1,1) h_1^{\mathbb{P}}(0,1),\\
U_1(t) &:= \gamma h_1(0,1)\left(\frac{h_1(0,1)}{B(t,1,1)h_1^{\mathbb{P}}(0,1)}\right)^{\frac{1}{\gamma-1}} - B(t,1,1)h_1^{\mathbb{P}}(0,1)\left(\frac{h_1(0,1)}{B(t,1,1)h_1^{\mathbb{P}}(0,1)}\right)^{\frac{\gamma}{\gamma-1}}\\
&= (\gamma-1)h_1(0,1)\eta_1^{\frac{1}{\gamma-1}} B(t,1,1)^{\frac{1}{1-\gamma}}, \qquad \text{where } \eta_1 = \frac{h_1(0,1)}{h_1^{\mathbb{P}}(0,1)}.
\end{aligned}$$

We numerically solve the above nonlinear ODE (unique solution guaranteed by Lemma 4.3), and then plug the corresponding solution into (4.32) to explicitly obtain the optimal number of shares invested in the CDS asset referencing name "2" after the default of name "1." Next, we rewrite equation (4.37) as the following quasi-linear first-order ODE:

$$u'(t) = -C(t, u(t)), \qquad u(T) = \frac{1}{\gamma}, \tag{A.31}$$

where $t \in [0, T]$, and

$$\begin{aligned}
C(t,u) := \Big\{&\gamma\psi_1^*(t,u)\left[(h_1(0,0)+h_2(0,0))\,\Phi_1(t,0,0) - L_1 h_1(0,0) - \Phi_1(t,0,1)h_2(0,0)\right]\\
&+\gamma\psi_2^*(t,u)\left[(h_1(0,0)+h_2(0,0))\,\Phi_2(t,0,0) - L_2 h_2(0,0) - \Phi_2(t,1,0)h_1(0,0)\right]\\
&+a(\gamma) - 2\left(h_1^{\mathbb{P}}(0,0)+h_2^{\mathbb{P}}(0,0)\right)\Big\}u\\
&+B(t,1,0)h_1^{\mathbb{P}}(0,0)\Big\{\left[1+\psi_1^*(t,u)(L_1-\Phi_1(t,0,0))\right]^{\gamma}\\
&+\left[1+\psi_2^*(t,u)\,(\Phi_2(t,1,0)-\Phi_2(t,0,0))\right]^{\gamma}\Big\}\\
&+B(t,0,1)h_2^{\mathbb{P}}(0,0)\Big\{\left[1+\psi_2^*(t,u)(L_2-\Phi_2(t,0,0))\right]^{\gamma}\\
&+\left[1+\psi_1^*(t,u)\,(\Phi_1(t,0,1)-\Phi_1(t,0,0))\right]^{\gamma}\Big\}.
\end{aligned} \tag{A.32}$$

We use a fixed point algorithm to solve the coupled system consisting of equation (A.31) and of the system of two nonlinear equations given by equations (4.34) and (4.35). Notice also that explicit expressions for $\Phi_i(t, \mathbf{z})$, $i = 1, 2$, $\mathbf{z} \in \{0, 1\}^2$, can be obtained from Corollary A.2. This algorithm initially sets $u(t) = \frac{B(t,1,0)+B(t,0,1)}{2}$. Then, it keeps iterating between solving for $\psi_1^*(t, 0, 0)$ and $\psi_2^*(t, 0, 0)$ given prespecified $u(t)$, and the ODE (A.31) given the latest estimates of $\psi_1^*(t, 0, 0)$ and $\psi_2^*(t, 0, 0)$. It stops when a desired level of convergence is achieved.

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