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# MULTIVARIATE SUBORDINATION OF MARKOV PROCESSES WITH FINANCIAL APPLICATIONS

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This paper develops the procedure of multivariate subordination for a collection of independent Markov processes with killing. Starting from $d$ independent Markov processes $X^i$ with killing and an independent $d$-dimensional time change $\mathcal{T}$, we construct a new process by time, changing each of the Markov processes $X^i$ with a coordinate $\mathcal{T}^i$. When $\mathcal{T}$ is a $d$-dimensional Lévy subordinator, the time changed process $(Y^i := X^i(\mathcal{T}^i(t))$ is a time-homogeneous Markov process with state-dependent jumps and killing in the product of the state spaces of $X^i$. The dependence among jumps of its components is governed by the $d$-dimensional Lévy measure of the subordinator. When $\mathcal{T}$ is a $d$-dimensional additive subordinator, $Y$ is a time-inhomogeneous Markov process. When $\mathcal{T}^i = \int_0^t V_s^i ds$ with $V^i$ forming a multivariate Markov process, $(Y^i, V^i)$ is a Markov process, where each $V^i$ plays a role of stochastic volatility of $Y^i$. This construction provides a rich modeling architecture for building multivariate models in finance with time- and state-dependent jumps, stochastic volatility, and killing (default). The semigroup theory provides powerful analytical and computational tools for securities pricing in this framework. To illustrate, the paper considers applications to multiname unified credit-equity models and correlated commodity models.

KEY WORDS: JDCEV model, multiparameter semigroups, multivariate subordination, subordinators, time-inhomogeneous, multiple commodities, additive subordinators, stochastic volatility.

## 1. INTRODUCTION

Many popular Lévy models in finance are constructed by time-changing Brownian motion with drift with a Lévy subordinator. The Variance Gamma (VG) and the Normal Inverse Gaussian (NIG) models belong to this class (see Madan and Seneta 1990; Madan, Carr, and Chang 1998; Barndorff-Nielsen 1998; Cont and Tankov 2004, for a survey). Carr et al. (2002, 2003) have extended Lévy models by time changing with additive subordinators to introduce time inhomogeneity and with time integrals of activity rates to introduce stochastic volatility, respectively.

Multiasset extensions of Lévy models to multiple dependent assets with jumps have recently been studied by Luciano and Schoutens (2006) and Luciano and Semeraro (2010), who introduced multivariate versions of some of the popular one-dimensional

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(1D) Lévy models based on multivariate subordination of Lévy processes first introduced by Barndorff-Nielsen, Pedersen, and Sato (2001), such as the multivariate VG and NIG models. However, the space homogeneity of Lévy processes imposes limitations on the structure of Lévy-based models. In particular, in Lévy models jumps cannot be mean-reverting (having a higher probability of a jump back towards the long run mean, when the process is away from the mean) and cannot exhibit the leverage effect (increasing arrival rates of larger jumps when the asset price falls).

The purpose of the present paper is to construct multidimensional Markov jump-diffusion and pure-jump processes by the procedure of *multivariate subordination* of Markov processes. *Univariate subordination* of Markov processes has recently been applied to modeling in commodity markets (Li and Linetsky 2014; Li and Mendoza-Arriaga 2013a), fixed-income markets (Boyarchenko and Levendorskii 2007; Lim, Li, and Linetsky 2012), and corporate credit and equity markets (Mendoza-Arriaga, Carr, and Linetsky 2010; Lorig 2011; Lorig, Lozano-Carbassé, and Mendoza-Arriaga 2013; Mendoza-Arriaga and Linetsky 2014). While Lévy processes are state-homogeneous, subordinating diffusion processes with state-dependent drift and volatility allows one to construct processes with state-dependence jumps.

- Subordinating Ornstein–Uhlenbeck (OU) diffusions yields jump-diffusion and pure-jump processes with mean-reverting jumps (higher arrival rates of jumps toward the long-run mean; for example, Li and Linetsky 2014).
- Subordinating Cox-Ingersoll-Ross (CIR) diffusions yields jump-diffusion and pure-jump processes with mean-reverting *two-sided* jumps that nevertheless stay positive (e.g., Mendoza-Arriaga and Linetsky 2014). This is in contrast to affine jump-diffusion models that allow only positive jumps to keep the process positive.
- Subordinating constant elasticity of variance (CEV) diffusions yields jump-diffusion and pure-jump processes with the leverage effect—arrival rates of jumps increase as the stock price falls (e.g., Mendoza-Arriaga et al. 2010).

The present paper focuses on multivariate subordination of Markov processes. Multivariate subordination of a collection of independent Lévy processes with a Lévy subordinator was first studied by Barndorff-Nielsen et al. (2001). They show that the result of the multivariate subordination of Lévy processes is again a Lévy process in the product space. In finance, multivariate subordination of Lévy processes has been applied by Luciano and Semeraro (2010), Semeraro (2008), and Luciano and Schoutens (2006) to construct multivariate Lévy models for the joint dynamics of multiple stocks and by Winkel (2005) for modeling electronic foreign exchange markets. However, the power of multivariate subordination when applied to Lévy processes is limited by the fact that the result of the multivariate subordination of a set of independent Lévy processes is again a Lévy process (in the product space). Hence, one remains in the class of Lévy processes.

In contrast, the present paper studies multivariate subordination of *Markov processes*. Starting from $d$ independent time-homogeneous Markov processes $X^i$ with state spaces $E_i$ and with killing, and a process $\mathcal{T}$ taking values in $\mathbb{R}_+^d$, such that each of its coordinates is an increasing càdlàg process starting from the origin, we construct a new process $Y = (Y^i)$, $Y^i := X^i(\mathcal{T}^i(t))$, by time changing each of the independent Markov processes $X^i$ with a coordinate $\mathcal{T}^i$ of the multivariate time change. The key examples of multivariate time changes are as follows:

- Lévy subordinator: When $\mathcal{T}$ is a Lévy subordinator, the process $Y$ is a time-homogeneous Markov process in the product space $E_1 \times \ldots \times E_d$ with dependence among jumps of its coordinates $Y^i$ governed by the Lévy measure of the subordinator.

- Additive subordinator: When $\mathcal{T}$ is an additive subordinator, the process $Y$ is a *time-inhomogeneous* Markov process in the product space $E_1 \times \ldots \times E_d$ with dependence among jumps of its coordinates $Y^i$ governed by the time-dependent Lévy measure of the additive subordinator.
- Absolutely continuous time change: When $\mathcal{T}^i$ are absolutely continuous with respect to the Lebesgue measure, $\mathcal{T}^i = \int_0^t V_s^i ds$, with nonnegative *activity rates* $V^i$ forming a multivariate Markov process $(V^i)$ independent of $(X^i)$, $(Y^i, V^i)$ is a Markov process in $E_1 \times \ldots \times E_d \times \mathbb{R}_+^d$, where each $V^i$ plays a role of stochastic volatility of $Y^i$.

Multivariate subordination of Markov processes provides a rich architecture for building multivariate models in finance with dependent jumps, stochastic volatility and killing (default). The semigroup theory provides powerful analytical and computational tools for securities pricing in this framework.

The rest of this paper is organized as follows. Section 2 lays out the mathematical foundations for multivariate subordination of a collection of independent Markov processes and their associated transition semigroups with a multivariate Lévy subordinator and develops analytical tools for securities pricing in these models based on semigroup theory. Section 3.1 replaces the multivariate Lévy subordinator with an additive subordinator and considers the resulting time-inhomogeneous Markov process. Sections 3.2 and 3.3 consider multivariate absolutely continuous time changes, as well as compositions of Lévy subordinators and absolutely continuous time changes. Section 4 provides two examples of applications in financial modeling. Section 4.1 briefly sketches applications to multivariate commodity models for closely related and highly correlated commodities. Section 4.2 develops in more detail a multiname unified credit-equity model for the stock prices of $n$ firms, as well as their default times, by subordinating $n$ jump-to-default extended CEV (JDCEV) diffusions introduced by Carr and Linetsky (2006) for unified single-name credit-equity modeling. Each of the stock prices experiences state-dependent jumps with the leverage effect (arrival rates of jumps increase as the stock price falls), including the possibility of a jump to zero (jump-to-default). Some of the jumps are idiosyncratic to each firm, while some are either common to all firms (systematic), or common to a subgroup of firms.

Section 5 presents numerical illustrations of multiname credit-equity models.

## 2. MULTIVARIATE SUBORDINATION OF MARKOV PROCESSES

### 2.1. Multivariate Subordination of Multiparameter Semigroups

A $d$-dimensional subordinator is a Lévy process in $\mathbb{R}_+^d = [0, \infty)^d$ nondecreasing in each of its coordinates. That is, each of its coordinates is a 1D subordinator (a nondecreasing Lévy process). The $d$-dimensional Laplace transform of a $d$-dimensional subordinator $\mathcal{T} = \{\mathcal{T}_t, t \geq 0\}$ is given by (here $u_i \geq 0$ and $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^d u_i v_i$):

$$\mathbb{E}[e^{-\langle \mathbf{u}, \mathcal{T}_t\rangle}] = e^{-t\phi(\mathbf{u})} \tag{2.1}$$

with the Laplace exponent given by the Lévy–Khintchine formula:

$$\phi(\mathbf{u}) = \langle \gamma, \mathbf{u}\rangle + \int_{\mathbb{R}_+^d \setminus\{0\}} (1 - e^{-\langle \mathbf{u}, \mathbf{s}\rangle})\nu(d\mathbf{s}), \tag{2.2}$$

where $\gamma \in \mathbb{R}_+^d$ is the drift vector with nonnegative coordinates and the Lévy measure $\nu$ is a $\sigma$-finite measure on $\mathbb{R}^d$ concentrated on $\mathbb{R}_+^d \backslash \{0\}$ such that $\int_{\mathbb{R}_+^d} (\|\mathbf{s}\| \wedge 1)\nu(d\mathbf{s}) < \infty$. Let $\pi_t(d\mathbf{s})$ denote its transition kernel, so that $\int_{\mathbb{R}_+^d} e^{-\langle \mathbf{u},\mathbf{s}\rangle}\pi_t(d\mathbf{s}) = e^{-t\phi(\mathbf{u})}$. We assume the subordinator is conservative (hence, no constant term in the Laplace exponent (2.2), i.e., $\phi(0) = 0$).

The key result about multiparameter semigroups is that they factorize into the direct products of commuting one-parameter semigroups; see Butzer and Berens (1967), propositions 1.1.8 and 1.1.9, and Baeumer, Kovacs, and Meerschaert (2008), proposition 2.1.

PROPOSITION 2.1. *If $\{\mathcal{P}_\mathbf{t}, \mathbf{t} \in \mathbb{R}_+^d\}$ is a $d$-parameter strongly continuous semigroup on a Banach space $\mathfrak{B}$, then it is the direct product of $d$ one-parameter strongly continuous semigroups $\{\mathcal{P}_t^i, t \geq 0\}$ on $\mathfrak{B}$ with infinitesimal generators $\mathcal{G}_i$ with domains* $\mathrm{Dom}(\mathcal{G}_i) \subset \mathfrak{B}$. *That is, for $\mathbf{t} = (t_1, \dots, t_d)$ we have $\mathcal{P}_\mathbf{t} = \prod_{i=1}^d \mathcal{P}_{t_i}^i$, and the semigroup operators $\mathcal{P}_{t_i}^i$ commute with each other, $t_i \geq 0$, $i = 1, \dots, d$. The set of generators $(\mathcal{G}_i, \mathrm{Dom}(\mathcal{G}_i))_{i=1}^d$ of commuting one-parameter semigroups $\{\mathcal{P}_t^i, t \geq 0\}$ is called the set of generators of the $d$-parameter semigroup $\{\mathcal{P}_\mathbf{t}, \mathbf{t} \in \mathbb{R}_+^d\}$.*

For univariate subordination of one-parameter semigroups, including the classical Phillips theorem that characterizes the subordinate semigroup and its generator, we refer the reader to chapter 12 of Schilling, Song, and Vondracek (2010) and theorem 32.1 in Sato (1999). The multiparameter version of Phillips theorem has recently been proved by Baeumer et al. (2008), who studied multivariate subordination of multiparameter semigroups. Here we present their result in the form convenient for our purposes.[1]

THEOREM 2.2. *Let $\{\mathcal{T}_t, t \geq 0\}$ be a $d$-dimensional subordinator with the Lévy measure $\nu$, drift vector $\gamma$, Laplace exponent $\phi(\lambda)$, and transition kernel $\pi_t(d\mathbf{s})$. Let $\{\mathcal{P}_\mathbf{t}, \mathbf{t} \in \mathbb{R}_+^d\}$ be a strongly continuous $d$-parameter semigroup of linear operators in the Banach space $\mathfrak{B}$ with the set of generators $(\mathcal{G}_i, D(\mathcal{G}_i))$, $i = 1, \dots, d$. Define:*

$$\mathcal{P}_t^\phi f = \int_{\mathbb{R}_+^d} (\mathcal{P}_\mathbf{s} f)\pi_t(d\mathbf{s}), \quad f \in \mathfrak{B}, \quad t \geq 0. \tag{2.3}$$

*Then $\{\mathcal{P}_t^\phi, t \geq 0\}$ is a strongly continuous one-parameter semigroup of linear operators on $\mathfrak{B}$. Denote its infinitesimal generator by $\mathcal{G}^\phi$. Then $\bigcap_{i=1}^d \mathrm{Dom}(\mathcal{G}_i) \subset \mathrm{Dom}(\mathcal{G}^\phi)$ and*

$$\mathcal{G}^\phi f = \sum_{i=1}^d \gamma_i \mathcal{G}_i f + \int_{\mathbb{R}_+^d \backslash \{0\}} (\mathcal{P}_\mathbf{s} f - f)\nu(d\mathbf{s}), \quad f \in \bigcap_{i=1}^d \mathrm{Dom}(\mathcal{G}_i). \tag{2.4}$$

When $\mathfrak{B} = \mathcal{B}_b(E)$, the Banach space of bounded Borel-measurable functions on a locally compact separable metric space $E$, and the $d$-parameter semigroup $\{\mathcal{P}_\mathbf{t}, \mathbf{t} \in \mathbb{R}_+^d\}$ gives rise to the transition function of a $d$-parameter Markov process $\{X_\mathbf{t}, \mathbf{t} \in \mathbb{R}_+^d\}$ on $E$, the multivariate subordination has a probabilistic interpretation. Let $\mathcal{T}$ be a $d$-dimensional subordinator with the Laplace exponent $\phi$ and $X$ be a $d$-parameter Markov process independent of $\mathcal{T}$. Then the time changed (subordinate) process defined by

[1] Baeumer et al. (2008) proved a more general result on subordination of multiparameter groups by Lévy processes. We restrict our attention to subordination of multiparameter semigroups by subordinators. The corresponding result is theorem 2.15 in Baeumer et al. (2008). In our case $c = 0$ in their equation (2.17), as we assume that the subordinator $\mathcal{T}$ is conservative.

$X_t^\phi := X_{\mathcal{T}_t^1,\dots,\mathcal{T}_t^d}$ is a one-parameter Markov process on $E$. Because of independence of $\mathcal{T}$ and $X$, the associated operator semigroup is given by

$$\mathcal{P}_t^\phi f(\mathbf{x}) = \mathbb{E}^{\mathbf{x}}[f(X_t^\phi)] = \int_{\mathbb{R}_+^d} \mathbb{E}^{\mathbf{x}}\left[f(X_{\mathbf{s}})\right]\pi_t(d\mathbf{s}) = \int_{\mathbb{R}_+^d} \mathcal{P}_{\mathbf{s}} f(\mathbf{x})\pi_t(d\mathbf{s}).$$

Thus, the multivariate subordination procedure can be interpreted as a multivariate stochastic time change of a multiparameter Markov process with respect to an independent $d$-dimensional subordinator. For details on multiparameter Markov processes we refer to Khoshnevisan (2002), chapter 11. When $X_{\mathbf{t}}$ is a multiparameter Lévy process, the multivariate subordination procedure has been introduced by Barndorff-Nielsen et al. (2001).

## 2.2. Multivariate Subordination of Markov Processes

We now apply this construction to the multivariate subordination of a collection of independent Markov processes. Our treatment follows the probabilistic treatment of the univariate subordination in Bouleau (1984) (see also Song and Vondracek 2008, pp. 326–327). Because we are interested in credit risk applications, we explicitly deal with killing. Let $X^i = \{X_t^i, t \geq 0\}$, $i = 1, \dots, d$, be $d$ time-homogeneous strong Markov processes with right-continuous with left limits paths taking values in locally compact separable metric spaces $E_i$ and with lifetimes $\zeta_i$. It is assumed that $X^i$ is sent to the cemetery state $\Delta_i$ at its lifetime $\zeta_i$, where it remains for all $t \geq \zeta_i$. The state space is then $E_i^\Delta := E_i \cup \{\Delta_i\}$. In detail, let $\Omega_i$ be the sets of all functions $\omega_i : [0, \infty) \to E_i^\Delta$, which are right continuous and have left limits. For each $t \geq 0$, let $X^i : \Omega_i \to E_i^\Delta$ be defined by $X_t^i(\omega_i) = \omega_i(t)$. Let $\mathbb{F}^{i,0} = (\mathcal{F}_t^{i,0}, t \geq 0)$, $\mathcal{F}_t^{i,0} = \sigma(X_s^i, 0 \leq s \leq t)$, be the natural filtrations generated by the processes $X^i = \{X_t^i, t \geq 0\}$, and also let $\mathcal{F}_i = \sigma(X_t, t \geq 0)$. Let $\mathcal{F}_{t+}^{i,0} = \cap_{s>t}\mathcal{F}_s^{i,0}$. We assume that $(\mathbb{P}_i^{x_i}, x_i \in E_i^\Delta)$ are families of probability measures on $(\Omega_i, \mathcal{F}_i)$ such that $(\Omega_i, \mathcal{F}_i, (\mathcal{F}_{t+}^{i,0})_{t\geq 0}, (X_t^i)_{t\geq 0}, \mathbb{P}_i^{x_i})$ are strong Markov processes.

We denote the transition semigroup of $X^i$ on the Banach space $\mathcal{B}_b(E_i^\Delta)$ of real-valued bounded Borel measurable functions on $E_i^\Delta$ by $\{\tilde{\mathcal{P}}_t^i, t \geq 0\}$, $\tilde{\mathcal{P}}_t^i f(x_i) = \mathbb{E}_i^{x_i}[f(X_t^i)] = \int_{E_i^\Delta} f(y)\tilde{P}_t^i(x_i, dy)$, $f \in \mathcal{B}_b(E_i^\Delta)$, $x_i \in E_i^\Delta$, $t \geq 0$.

We also denote by $\{\mathcal{P}_t^i, t \geq 0\}$ the transition semigroup of $X^i$ on the Banach space $\mathcal{B}_b(E_i)$ obtained by considering only functions $f$ in $\mathcal{B}_b(E_i^\Delta)$ that identically vanish on $\Delta_i$, $f(\Delta_i) = 0$, and restricting them to $E_i$. That is, $\mathcal{P}_t^i f(x_i) = \mathbb{E}_i^{x_i}[\mathbf{1}_{\{t<\zeta_i\}} f(X_t^i)] = \int_{E_i} f(y)P_t^i(x_i, dy)$, $x_i \in E_i$, $f \in \mathcal{B}_b(E_i)$, $t \geq 0$. The semigroups $\{\tilde{\mathcal{P}}_t^i, t \geq 0\}$ are Markovian (conservative, $\tilde{P}_t^i(x_i, E_i^\Delta) = 1$ for all $x \in E_i^\Delta$ and $t \geq 0$), while the semigroups $\{\mathcal{P}_t^i, t \geq 0\}$ are sub-Markovian ($P_t^i(x_i, E_i) \leq 1$ for all $x \in E_i$ and $t \geq 0$). In the context of credit risk, when the cemetery state is interpreted as the default state, $\mathcal{B}_b(E_i)$ can be identified as the space of contingent claims with promised payoffs $f \in \mathcal{B}_b(E_i)$ at maturity if no default occurs and zero recovery in default, while $\mathcal{B}_b(E_i^\Delta)$ as the space of contingent claims with promised payoff $f \in \mathcal{B}_b(E_i)$ at maturity if no default occurs and with recovery payment $f(\Delta_i)$ at maturity in the event of default. The relationship between the two semigroups for each $i$ is: $\tilde{\mathcal{P}}_t^i f(x_i) = \mathcal{P}_t^i f(x_i) + f(\Delta_i)P_t^i(x_i, \{\Delta_i\})$ for $x_i \in E_i$, $f \in \mathcal{B}_b(E_i^\Delta)$, $t \geq 0$, and, abusing notation, in the first term on the left-hand side $f \in \mathcal{B}_b(E_i)$ is the restriction of $f \in \mathcal{B}_b(E_i^\Delta)$ to $\mathcal{B}_b(E_i)$.

Further, let $\Omega_{\mathcal{T}}$ be the set of all functions $\omega_{\mathcal{T}} : [0, \infty) \to \mathbb{R}_+^d$, which are right continuous and have left limits. For each $t \geq 0$, let $\mathcal{T}_t : \Omega_{\mathcal{T}} \to [0, \infty)$ be defined by $\mathcal{T}_t(\omega_{\mathcal{T}}) = \omega_{\mathcal{T}}(t)$.

Let $\mathbb{G}^0 = (\mathcal{G}^0_t, t \geq 0)$, $\mathcal{G}^0_t = \sigma(\mathcal{T}_s, 0 \leq s \leq t)$, be the natural filtration generated by the process $\mathcal{T} = \{\mathcal{T}_t, t \geq 0\}$, and also let $\mathcal{G} = \sigma(\mathcal{T}_t, t \geq 0)$. Let $\mathcal{G}^0_{t+} = \cap_{s>t}\mathcal{G}^0_s$. We assume that $(\mathbb{P}^z_{\mathcal{T}}, z \in \mathbb{R}^d_+)$ is a family of probability measures on $(\Omega_{\mathcal{T}}, \mathcal{G})$ such that $(\mathcal{T}_t, \mathbb{P}^z_{\mathcal{T}})$ is a $d$-dimensional Lévy subordinator. In particular, we assume that under $\mathbb{P}_{\mathcal{T}} := \mathbb{P}^0_{\mathcal{T}}$ the law of $\mathcal{T}$ is given by equations (2.1) and (2.2).

Now let $\Omega = \Omega_1 \times \ldots \times \Omega_d \times \Omega_{\mathcal{T}}$, and, for any $\mathbf{x} = (x_1, \ldots, x_d)$, $x_i \in E^\Delta_i$, $i = 1, \ldots, d$, and $z \in \mathbb{R}^d_+$, let $\mathbb{P}^{\mathbf{x},z} = \mathbb{P}^{x_1}_1 \times \ldots \times \mathbb{P}^{x_d}_d \times \mathbb{P}^z_{\mathcal{T}}$ be the product of probability measures on $\mathcal{H} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_d \times \mathcal{G}$. The probability $\mathbb{P}^{\mathbf{x},0}$ will be denoted by $\mathbb{P}^{\mathbf{x}}$. The elements of $\Omega$ are denoted by $\omega = (\omega_1, \ldots, \omega_d, \omega_{\mathcal{T}})$. The *subordinate process* $\tilde{X}^\phi = \{\tilde{X}^\phi_t, t \geq 0\}$ with the state space $\tilde{E} = E^\Delta_1 \times \ldots \times E^\Delta_d$ is defined on $\Omega$ by:

$$\tilde{X}^\phi_t(\omega) := (X^1_{\mathcal{T}^1_t(\omega_{\mathcal{T}})}(\omega_1), \ldots, X^d_{\mathcal{T}^d_t(\omega_{\mathcal{T}})}(\omega_d)).$$

Extending the definition of the filtration $\mathcal{H}^1$ in Bouleau (1984), p. 64, for the subordinate Markov process (see also Song and Vondracek 2008, p. 327) to multivariate subordination, we introduce the following filtration. For $t \geq 0$ define the following collections of sets:

$$\begin{aligned}\mathcal{S}_t &= \big\{A^1 \times \ldots \times A^d \times \left(A^{\mathcal{T}} \cap \{\mathcal{T}^1_t \geq u_1, \ldots, \mathcal{T}^d_t \geq u_d\}\right),\ A^1 \in \mathcal{F}^{1,0}_{u_1}, \ldots, \\ &\qquad A^d \in \mathcal{F}^{d,0}_{u_d},\ A^{\mathcal{T}} \in \mathcal{G}^0_t,\ u_1, \ldots, u_d \geq 0\big\},\end{aligned}$$

and let $\mathcal{H}_t = \sigma(\mathcal{S}_t)$. Then $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ is a filtration on $(\Omega, \mathcal{H})$ such that $\sigma(\tilde{X}^\phi_s, 0 \leq s \leq t) \subset \mathcal{H}_t$.

THEOREM 2.3. *The subordinate process* $\tilde{X}^\phi = (\Omega, \mathcal{H}, (\mathcal{H}_{t+})_{t\geq 0}, (\tilde{X}^\phi_t)_{t\geq 0}, (\mathbb{P}^{\mathbf{x}})_{\mathbf{x}\in\tilde{E}})$ *is a strong Markov process with the transition semigroup* $\{\tilde{\mathcal{P}}^\phi_t, t \geq 0\}$ *on* $\mathcal{B}_b(\tilde{E})$ *given by the subordinate semigroup:*

$$\tilde{\mathcal{P}}^\phi_t f(\mathbf{x}) = \int_{\mathbb{R}^d_+} (\tilde{\mathcal{P}}^1_{s_1} \ldots \tilde{\mathcal{P}}^d_{s_d} f)(\mathbf{x})\pi_t(d\mathbf{s}) = \int_{\tilde{E}} f(\mathbf{y})\tilde{P}^\phi_t(\mathbf{x}, d\mathbf{y}), \quad \mathbf{x} \in \tilde{E}, \quad t \geq 0, \tag{2.5}$$

$$\tilde{P}^\phi_t(\mathbf{x}, d\mathbf{y}) = \int_{\mathbb{R}^d_+} \tilde{P}^1_{s_1}(x_1, dy_1) \ldots \tilde{P}^d_{s_d}(x_d, dy_d)\pi_t(d\mathbf{s}),$$

*where* $\{\tilde{\mathcal{P}}^i_t, t \geq 0\}$ *are extensions to* $\mathcal{B}_b(\tilde{E})$ *of the transition semigroups of* $X^i$ *on* $\mathcal{B}_b(E^\Delta_i)$ *defined by:*

$$\tilde{\mathcal{P}}^i_t f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d) = \int_{E^\Delta_i} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d)\tilde{P}^i_t(x_i, dy_i),$$
$$f \in \mathcal{B}_b(\tilde{E}),$$

*where we use the same notation for both the transition semigroup of* $X^i$ *on* $\mathcal{B}_b(E^\Delta_i)$ *and its extension to* $\mathcal{B}_b(\tilde{E})$ *defined above.*

We now define a subprocess $X^\phi$ of $\tilde{X}^\phi$ by killing it at the first time *any* of the processes $\tilde{X}^{i,\phi}_t = X^i(\mathcal{T}^i(t))$ enter their respective trap states $\Delta_i$. This corresponds to killing $X^\phi$ at the *first-to-default time* in the credit risk context. Namely, recall that $\zeta_i$ are lifetimes of the original processes $X^i$ (the first time $X^i$ enters the trap state $\Delta_i$), and consider the following random times:

$$\tau_i = \inf\{t \geq 0 : \mathcal{T}_t^i \geq \zeta_i\}, \quad \tau = \tau_1 \wedge \ldots \wedge \tau_d. \tag{2.6}$$

Due to Bouleau (1984), p. 65 (see also proposition 2.2 in Song and Vondracek 2008), $\tau_i$ and $\tau$ are $(\mathcal{H}_{t+})$-stopping times. Define $E := E_1 \times \ldots \times E_d$. Suppose the process $\tilde{X}^\phi$ starts at $\mathbf{x} \in E$, so that all $\tilde{X}_0^{i,\phi} = x_i \in E_i$ are not in $\Delta_i$ (in the credit risk context, no one has defaulted yet at time $t = 0$). Then $\tau$ is the first passage time of $\tilde{X}^\phi$ into the complement of $E$ in $\tilde{E}$. We can then define the process $X^\phi$ by killing the process $\tilde{X}^\phi$ at $\tau$ and sending it into the cemetery state $\Delta$. The state space of the killed process $X^\phi$ is then $E^\Delta = E \cup \{\Delta\}$ in contrast to the state space $\tilde{E} = (E_1 \cup \{\Delta_1\}) \times \ldots \times (E_d \cup \{\Delta_d\})$ of the process $\tilde{X}^\phi$. The (sub-Markovian) semigroup of the process $X^\phi$ killed at $\tau$ is

$$\mathcal{P}_t^\phi f(\mathbf{x}) = \int_{\mathbb{R}_+^d} (\mathcal{P}_{s_1}^1 \ldots \mathcal{P}_{s_d}^d f)(\mathbf{x})\pi_t(d\mathbf{s}) = \int_E f(\mathbf{y}) P_t^\phi(\mathbf{x}, d\mathbf{y}), \quad \mathbf{x} \in E, \quad t \geq 0,$$

$$P_t^\phi(\mathbf{x}, d\mathbf{y}) = \int_{\mathbb{R}_+^d} P_{s_1}^1(x_1, dy_1) \ldots P_{s_d}^d(x_d, dy_d)\pi_t(d\mathbf{s}),$$

where $\{\mathcal{P}_t^i, t \geq 0\}$ are extensions to $\mathcal{B}_b(E)$ of the (sub-Markovian) semigroups of $X^i$ killed at $\zeta_i$ on $\mathcal{B}_b(E_i)$ defined by:

$$\mathcal{P}_t^i f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d) = \int_{E_i} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d) P_t^i(x_i, dy_i), \; f \in \mathcal{B}_b(E).$$

For credit risk applications it is important to establish the relationship between the semigroups $(\tilde{\mathcal{P}}_t^\phi)$ on $\mathcal{B}_b(\tilde{E})$ and $(\mathcal{P}_t^\phi)$ on $\mathcal{B}_b(E)$. To this end, we first observe that for any subset of Markov processes $\{X^i, i \in \Xi\}$ with $\Xi = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, d\}$, $1 \leq k \leq d$, we can define the subordinate process $X^{\Xi,\phi}$ with the $k$-dimensional subordinator $\mathcal{T}^\Xi = (\mathcal{T}^{i_1}, \ldots, \mathcal{T}^{i_k})$. It is a Markov process on $E_\Xi := E_{i_1} \times \ldots \times E_{i_k}$ with lifetime $\tau_\Xi := \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}$, and with the (sub-Markovian) semigroup $\mathcal{P}^{\Xi,\phi}$. This is an immediate corollary of the fact that any subset of $1 \leq k \leq d$ coordinates of a $d$-dimensional subordinator is itself a $k$-dimensional subordinator and, hence, Theorem 2.3 applies. We call the corresponding semigroups $\mathcal{P}^{\Xi,\phi}$ *marginal subordinate semigroups* of the subordinate semigroup $\mathcal{P}^\phi$. We thus have $2^d - 1$ nontrivial marginal subordinate semigroups associated with a set of $d$ independent Markov processes $X^i$ on $E_i$ with lifetimes $\zeta_i$ and a $d$-dimensional subordinator $\mathcal{T}$. When $\Xi = \{i\}$, the marginal subordinate semigroup $\mathcal{P}^{i,\phi} \equiv \mathcal{P}^{\{i\},\phi}$ is the semigroup of the Markov process $X^{i,\phi}$ constructed by subordinating the process $X^i$ with respect to the 1D subordinator $\mathcal{T}_t^i$.

The action of the semigroup $\tilde{\mathcal{P}}^\phi$ on a function from $\mathcal{B}_b(\tilde{E})$ can then be expressed in terms of the $2^d - 1$ marginal subordinate semigroups $\mathcal{P}^{\Xi,\phi}$, with each of the subordinate semigroups acting on $\mathcal{B}_b(E_\Xi)$, where $E_\Xi = E_{i_1} \times \ldots \times E_{i_k}$. In particular, we have the following result.

THEOREM 2.4. *For any $f \in \mathcal{B}_b(\tilde{E})$ and $x_i \in E_i$, $i = 1, \ldots, d$, we have:*

$$\begin{aligned}
\tilde{\mathcal{P}}_t^\phi f(x_1, \ldots, x_d) &= \mathbb{E}^{(x_1,\ldots,x_d)}[f(\tilde{X}_t^{1,\phi}, \ldots, \tilde{X}_t^{d,\phi})] \\
&= \sum_{\Xi \subseteq \Upsilon} \mathbb{E}^{(x_{i_1},\ldots,x_{i_k})}[\mathbf{1}_{\{\tau_\Xi > t\}} f_\Xi(\tilde{X}_t^{i_1,\phi}, \ldots, \tilde{X}_t^{i_k,\phi})] = \sum_{\Xi \subseteq \Upsilon} \mathcal{P}_t^{\Xi,\phi} f_\Xi(x_{i_1}, \ldots, x_{i_k}),
\end{aligned}$$

*where the sum is over all $2^d$ subsets $\Xi = \{i_1, \ldots, i_k\} \subseteq \Upsilon$, $\tau_\Xi := \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}$ is the first time any of the processes $\tilde{X}^{i,\phi}_t = X^i(\mathcal{T}^i(t))$ with $i \in \Xi$ enters its trap states $\Delta_i$ (by convention $\tau_\emptyset = \infty$), $\mathcal{P}^{\Xi,\phi}$ are marginal subordinate semigroups of the subordinate semigroup $\tilde{\mathcal{P}}^\phi$, and the $2^d$ functions $f_\Xi \in \mathcal{B}_b(E_\Xi)$ are constructed from $f \in \mathcal{B}_b(\tilde{E})$ as follows ($|\Xi|$ denotes the cardinality of the set $\Xi$):*

$$f_\Xi(x_{i_1}, \ldots, x_{i_k}) = \sum_{\Theta \subseteq \Xi} (-1)^{|\Xi|-|\Theta|} f(\mathbf{1}_\Theta(1)x_1 + \mathbf{1}_{\Theta^c}(1)\Delta_1, \ldots, \mathbf{1}_\Theta(d)x_d + \mathbf{1}_{\Theta^c}(d)\Delta_d),$$

*where $\mathbf{1}_\Theta(i) = 1\,(0)$ if $i \in \Theta$ ($i \in \Theta^c = \Upsilon \backslash \Theta$), by convention $f_{\{\emptyset\}} = f(\Delta_1, \ldots, \Delta_d)$, and the sum is over all $2^k$ subsets of $\Xi$, where $k = |\Xi|$ is the cardinality of $\Xi$.*

Thus, we have the relationship between the $2^d - 1$ marginal subordinate semigroups on $\mathcal{B}_b(E_\Xi)$, $\Xi \subseteq \Upsilon$, associated with $2^d - 1$ killed subordinate processes $X^{\Xi,\phi}$ on $E_\Xi = E_{i_1} \times \ldots \times E_{i_k}$, where $E_i$ does not include the trap state and each $X^{\Xi,\phi}$ is killed the first time *any* of the processes $X^{i,\phi}_t = X^i(\mathcal{T}^i_t)$ with $i = i_1, \ldots, i_k$, enters its trap state $\Delta_i$, and the semigroup $\tilde{\mathcal{P}}^\phi$ defined on $\mathcal{B}_b(\tilde{E})$, where $\tilde{E}$ is defined as the product of $E^\Delta_i$.

Next we consider an important special case of Feller processes. Suppose each $X^i$ is a Feller process with the (sub-Markovian) Feller semigroup $\mathcal{P}^i$, that is, it leaves $C_0(E_i)$ (the space of continuous functions vanishing at infinity) invariant.

THEOREM 2.5. *Let the process $\tilde{X}^\phi$ be constructed by multivariate subordination of d Feller processes on locally compact separable metric spaces $E_i$ with respect to a d-dimensional subordinator with Laplace exponent $\phi$ as described above. Let $X^\phi$ be the subprocess of $\tilde{X}^\phi$ killed at the first time any of the components $\tilde{X}^{i,\phi}$ enter their trap states $\Delta_i$. Then, $X^\phi$ is a Feller process on $E = E_1 \times \ldots \times E_d$ with the sub-Markovian Feller transition semigroup $(\mathcal{P}^\phi_t)$ on $C_0(E)$.*

Suppose further that each $E_i = \mathbb{R}^{n_i}$, $n_i \geq 1$. In that case, each generator $\mathcal{G}_i$ of a Feller process satisfies the positive maximum principle. We also assume that the domain of each generator $\mathcal{G}_i$ contains $C^\infty_c(\mathbb{R}^{n_i})$ (functions with their derivatives of all orders continuous and with compact supports). In that case, by Courrège's theorem (Courrège 1965), each $\mathcal{G}^i$ has the Lévy–Khintchine representation on $C^\infty_c(\mathbb{R}^{n_i})$ with variable coefficients (cf. theorem 3.5.3 in Applebaum 2004):

$$\mathcal{G}_i f(x) = \frac{1}{2} \sum_{a,b=1}^{n_i} \alpha^i_{ab}(x) \frac{\partial^2 f}{\partial x_a \partial x_b}(x) + \sum_{a=1}^{n_i} \beta^i_a(x) \frac{\partial f}{\partial x_a}(x) - k_i(x) f(x)$$

$$+ \int_{\mathbb{R}^{n_i} \backslash \{x\}} \left( f(y) - f(x) - \mathbf{1}_{\{\|y-x\| \leq 1\}} \sum_{a=1}^{n_i} (y_a - x_a) \frac{\partial f}{\partial x_a}(x) \right) \Pi_i(x, dy), \quad x \in \mathbb{R}^{n_i}, \tag{2.7}$$

where $(\alpha^i_{ab}(x))^{n_i}_{a,b=1}$ is a positive definite symmetric diffusion matrix for each $x$ and the map $x \to (y, \alpha^i(x)y)$ is upper semicontinuous for each $y \in \mathbb{R}^{n_i}$, $(\beta^i_a(x))^{n_i}_{a=1}$ is a drift vector with coordinates continuous in $x$, $k_i(x) \geq 0$ is a continuous nonnegative killing rate, and $\Pi_i(x, dy)$ is a state-dependent Lévy kernel, such that, for each $x$, $\Pi_i(x, \cdot)$ is a Lévy measure on $\mathbb{R}^{n_i} \backslash \{x\}$ satisfying $\int_{\mathbb{R}^{n_i} \backslash \{x\}} (\|y - x\|^2 \wedge 1) \Pi_i(x, dy) < \infty$ and depending measurably on $x \in \mathbb{R}^{n_i}$.

We can write down an explicit Lévy–Khintchine representation for the generator of the subordinate Feller semigroup $\mathcal{P}^\phi$ if we assume that the following four estimates hold for each of the Feller semigroups $\mathcal{P}^i$, $i = 1, \ldots, d$. For any $\epsilon > 0$ there is $C(\epsilon)$ such that for any $t < 1$

$$\int_{\{\|y-x\|>\epsilon\}} P_t(x, dy) \leq C(\epsilon)t. \tag{2.8}$$

There are $C_1$, $C_2$, and $C_3$ such that, for any $t \leq 1$,

$$\int_{\{\|y-x\|\leq 1\}} \|y - x\|^2 P_t(x, dy) \leq C_1 t, \ \left\| \int_{\{\|y-x\|\leq 1\}} (y - x) P_t(x, dy) \right\| \leq C_2 t, \ 1 - P_t(x, \mathbb{R}^n) \leq C_3 t. \tag{2.9}$$

THEOREM 2.6. *In Theorem 2.5, let each $X^i$ be a Feller process on $\mathbb{R}^{n_i}$ such that $C_c^\infty(\mathbb{R}^{n_i})$ is in the domain of its generator $\mathcal{G}_i$ given by* (2.7) *and the four estimates* (2.8)–(2.9) *are satisfied. Then the generator of the subordinate semigroup $\mathcal{G}^\phi$ admits the Lévy-Khintchine representation on $C_c^\infty(\mathbb{R}^n)$ with variable coefficients:*

$$\mathcal{G}^\phi f(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{d} \sum_{a_i, b_i=1}^{n_i} \gamma_i \alpha^i_{a_i b_i}(x_i) \frac{\partial^2 f}{\partial x_{i,a_i} \partial x_{i,b_i}}(\mathbf{x}) + \sum_{i=1}^{d} \sum_{a_i=1}^{n_i} \beta^{i,\phi}_{a_i}(\mathbf{x}) \frac{\partial f}{\partial x_{i,a_i}}(\mathbf{x}) - k^\phi(\mathbf{x}) f(\mathbf{x})$$

$$+ \int_{\mathbb{R}^n \setminus \{\mathbf{x}\}} \Big( f(\mathbf{y}) - f(\mathbf{x}) - \mathbf{1}_{\{\|\mathbf{y}-\mathbf{x}\|\leq 1\}} \sum_{i=1}^{d} \sum_{a_i=1}^{n_i} (y_{i,a_i} - x_{i,a_i}) \frac{\partial f}{\partial x_{i,a_i}}(\mathbf{x}) \Big) \Pi^\phi(\mathbf{x}, d\mathbf{y}), \ \mathbf{x} \in \mathbb{R}^n, \tag{2.10}$$

*where*

$$\beta^{i,\phi}_a(\mathbf{x}) = \gamma_i \beta^i_a(x_i) + \int_{\mathbb{R}^d_+ \setminus \{0\}} \int_{\mathbb{R}^n} \mathbf{1}_{\{\|\mathbf{y}-\mathbf{x}\|\leq 1\}} (y_{i,a} - x_{i,a}) \prod_{i=1}^{d} P^i_{s_i}\big(x_i, dy_i\big) \nu(d\mathbf{s}), \ a = 1, \ldots, n_i, \tag{2.11}$$

$$k^\phi(\mathbf{x}) = \sum_{i=1}^{d} \gamma_i k_i(x_i) + \int_{\mathbb{R}^d_+ \setminus \{0\}} \Big(1 - \prod_{i=1}^{d} P^i_{s_i}\big(x_i, \mathbb{R}^{n_i}\big)\Big) \nu(d\mathbf{s}), \tag{2.12}$$

$$\Pi^\phi(\mathbf{x}, d\mathbf{y}) = \sum_{i=1}^{d} \gamma_i \Pi_i(x_i, dy_i) \prod_{j \neq i} \delta_{x_j}(dy_j) + \int_{\mathbb{R}^d_+ \setminus \{0\}} \prod_{i=1}^{d} P^i_{s_i}\big(x_i, dy_i\big) \nu(d\mathbf{s}), \quad \mathbf{x} \neq \mathbf{y} \tag{2.13}$$

*where $P^i_t(x_i, dy_i)$ are transition kernels of the semigroups $\mathcal{P}^i$, $\mathcal{P}^i_t f(x_i) = \int_{\mathbb{R}^{n_i}} f(y_i) P^i_t(x_i, dy_i)$, $P^i_t\big(x_i, \mathbb{R}^{n_i}\big)$ are the corresponding "survival probabilities" (no killing by time t for the process $X^i$), so that $1 - \prod_{i=1}^d P^i_{s_i}\big(x_i, \mathbb{R}^{n_i}\big)$ is the probability of at least one of the following d events occurring: $X^1$ killed by time $s_1$, ..., $X^d$ killed by time $s_d$, $\delta_{x_i}(dy_i)$ are Dirac measures on $\mathbb{R}^{n_i}$, $\mathbf{x} = (x_1, \ldots, x_i)^\top \in \mathbb{R}^n$, $x_i = (x_{i,1}, \ldots, x_{i,n_i})^\top \in \mathbb{R}^{n_i}$, $i = 1, \ldots, d$.*

Thus, when $X^i$ are Feller jump-diffusion processes on $\mathbb{R}^{n_i}$ with the killing rates $k^i$, the subordinate process $X^\phi$ is a Feller jump-diffusion process on $\mathbb{R}^n$ with $n = n_1 + \ldots + n_d$ with the diffusion matrix in block-diagonal form, where each block is a diffusion matrix of the process $X^i$ scaled with the drift $\gamma_i$ of the $i$th coordinate $\mathcal{T}^i$ of the $d$-dimensional subordinator. The killing rate $k^\phi(\mathbf{x})$ of the process $X^\phi$ given by (2.12) is the sum of the killing rates $k^i$ of the processes $X^i$ scaled with the subordinator drifts $\gamma_i$ plus the additional term $\int_{\mathbb{R}^d_+ \setminus \{0\}} (1 - \prod_{i=1}^d P^i_{s_i}(x_i, \mathbb{R}^{n_i})) \nu(d\mathbf{s})$. Here $\prod_{i=1}^d P^i_{s_i}\big(x_i, \mathbb{R}^{n_i}\big)$ is the joint survival probability of the $d$ independent processes $X^i$ to survive until the corresponding times $s_i$, $i = 1, \ldots, d$. The killing rate $k^\phi(\mathbf{x})$ of the process $X^\phi$ is a function of the $n$-dimensional state vector $\mathbf{x}$. In credit risk applications where the killing time is the time of default, this structure leads to the dependence among defaults of multiple obligors. The state-dependent Lévy measure (2.13) consists of two parts. The sum of the Lévy measures of $X^i$ supported on $\mathbb{R}^{n_i}$ scaled with the drift of the corresponding subordinator $\gamma_i$, as well as the new term $\int_{\mathbb{R}^d_+\setminus\{0\}}\prod_{i=1}^d P^i_{s_i}(x_i, dy_i)\nu(d\mathbf{s})$ induced by the subordination. This new term in the Lévy measure depends on the $n$-dimensional state vector $\mathbf{x}\in\mathbb{R}^n$. The multivariate subordination induces dependence among jumps in each of the "coordinates" $X^{i,\phi}_t$, $i=1,\dots,d$, of the subordinate process. When $\gamma_i=0$ for all $i$, the subordinated process $X^\phi$ is a pure-jump process with killing with dependence among jumps of its components.

If each $X^i$ is a Lévy process on $\mathbb{R}^{n_i}$ with the generator given by (2.7) with state-independent coefficients, then our expressions (2.10)–(2.13) for the generator of the subordinate Feller process $X^\phi$ yields the Lévy characteristics of the Lévy process on $\mathbb{R}^n$, $n=n_1+\ldots+n_d$. These Lévy characteristics coincide with those first given in theorem 3.3 of Barndorff-Nielsen et al. (2001) that generalized the classical theorem 30.1 in Sato (1999) for univariate subordination of Lévy processes to multivariate subordination.

REMARK 2.7 (Linear Factor Models for Multidimensional Subordinators). A class of multidimensional subordinators convenient for applications can be constructed as follows. Let $\mathcal{S}^a_t$ be $m$ independent 1D subordinators and $A$ a $d\times m$ matrix with non-negative entries $A_{i,a}\geq 0$. Define

$$\mathcal{T}^i_t := \sum_{a=1}^m A_{i,a}\mathcal{S}^a_t, \quad i=1,\dots,d. \tag{2.14}$$

Then the $\mathbb{R}^d_+$-valued process $\mathcal{T}_t$ is a $d$-dimensional subordinator. In this structure, each independent subordinator $\mathcal{S}^a$ contributes to each time change $\mathcal{T}^i$ that affects the Markov process $X^i$, $i=1,\dots,d$. The coefficient $A_{i,a}$ is the corresponding factor loading. We call this specification the linear factor model. The Laplace exponent is given by: $\phi(u)=\sum_{a=1}^m\phi_a(v_a)$ with $v_a=\sum_{i=1}^d A_{i,a}u_i$, where $\phi_a(v)$ are the Laplace exponents of the $m$ independent 1D subordinators $\mathcal{S}^a$. The drift vector and Lévy measure of $\mathcal{T}$ are: $\gamma_i=\sum_{a=1}^m A_{i,a}\gamma_a$, $\nu(G)=\sum_{a=1}^m\nu_a(G_a)$, $G\in\mathcal{B}(\mathbb{R}^d_+)$, where $\gamma_a$ and $\nu_a$ are the drift and Lévy measure of $\mathcal{S}^a$, and $G_a=\{s_a\geq 0:\ A(0,\dots,s_a,\dots,0)^\top\in G\}$, where $s_a$ is in the $a$th place. The matrix $A$ defines the covariance: $Cov(\mathcal{T}^i_t,\mathcal{T}^j_t)=t\Sigma_{ij}$, $\Sigma_{ij}=-\sum_{a=1}^m\phi''_a(0)A_{i,a}A_{j,a}$, where $-\phi''_a(0)t$ is the variance of the $a$th independent subordinator component $\mathcal{S}^a_t$.

A family of 1D subordinators important in financial applications that can be conveniently used as basis in the linear factor models is defined by the following three-parameter family of Lévy measures:

$$\nu(ds)=Cs^{-\alpha-1}e^{-\eta s}ds \tag{2.15}$$

with $C>0$, $\eta\geq 0$, and $\alpha<1$. For $\alpha\in(0,1)$ these are the *tempered stable subordinators* (exponentially dampened versions of the stable subordinators with $\nu(ds)=Cs^{-\alpha-1}ds$ with $\eta=0$). The special case with $\alpha=1/2$ is the inverse Gaussian process (Barndorff-Nielsen 1998). The limiting case $\alpha=0$ is the gamma process (Madan and Seneta 1990; Madan et al. 1998). The processes with $\alpha\in[0,1)$ are infinite activity processes. The processes with $\alpha < 0$ are compound Poisson processes with gamma distributed jump sizes (exponential jumps when $\alpha = -1$). The Laplace exponent is given by:

$$\phi(\lambda) = \begin{cases} -C\Gamma(-\alpha)\left((\lambda+\eta)^\alpha - \eta^\alpha\right), & \alpha \neq 0 \\ C\ln(1+\lambda/\eta), & \alpha = 0 \end{cases}, \tag{2.16}$$

where $\Gamma(x)$ is the gamma function. Further information on subordinators can be found in Bertoin (1996) and Sato (1999). An excellent exposition of Bernstein functions (Laplace exponents of subordinators) and related topics is given in Schilling et al. (2010). For applications in finance see Boyarchenko and Levendorskii (2002), Cont and Tankov (2004), and Schoutens (2003).

Another class of examples of multidimensional subordinators can be constructed as follows: Let $\mathcal{S}_t^i$, $i = 1, \ldots, d$, be $d$ independent 1D subordinators and $\mathcal{S}_t^{d+1}$ another independent 1D subordinator. Define $\mathcal{T}_t^i := \mathcal{S}^i_{\mathcal{S}_t^{d+1}}$, $i = 1, \ldots, d$. Then $\mathcal{T}_t$ is a $d$-dimensional subordinator. Further examples of multidimensional Lévy subordinators can be found in Barndorff-Nielsen et al. (2001).

REMARK 2.8 (Dependence and Correlation). To discuss dependence and, in particular, correlations that arise among the components $X^{i,\phi}$ of the subordinate process, assume for simplicity that each $X^i$ is a 1D diffusion. First assume that the drift and the killing rate are equal to zero so each $X^i$ is a local martingale. To further simplify discussion, assume that the volatility functions $\sigma_i(x)$ are such that each $X^i$ is, in fact, a martingale. After the multivariate time change with a $d$-dimensional Lévy subordinator, the diffusion components of $X^{i,\phi}$ remain independent, but each $X^{i,\phi}$ jumps at the times of jumps of the corresponding coordinate of the subordinator $\mathcal{T}^i$. When $\mathcal{T}^i$ and $\mathcal{T}^j$ jump at the same time $t$, $X^{i,\phi}$ and $X^{j,\phi}$ will also jump at the same time. Thus there is dependence in the timing of jumps. The magnitudes of jumps of $X^{i,\phi}$ and $X^{j,\phi}$ at the common time of their jumps are determined as the distances the diffusions $X^i$ and $X^j$ travel during the time intervals $[\mathcal{T}_t^i, \mathcal{T}_t^i + \Delta\mathcal{T}_t^i]$ and $[\mathcal{T}_t^j, \mathcal{T}_t^j + \Delta\mathcal{T}_t^j]$, respectively, where $\Delta\mathcal{T}_t^i$ and $\Delta\mathcal{T}_t^j$ are the jumps experienced by $\mathcal{T}_t^i$ and $\mathcal{T}_t^j$ at time $t$. If the jumps of $\mathcal{T}_t^i$ and $\mathcal{T}_t^j$ are both large, the jumps of $X^{i,\phi}$ and $X^{j,\phi}$ will also tend to be large, as the martingales $X^i$ and $X^j$ have more time to travel. Thus, there is also dependence in the absolute values of jump magnitudes. However, the covariance and, thus, correlation of the martingales $X_t^{i,\phi}$ and $X_t^{j,\phi}$ remains zero. Indeed,

$$\begin{aligned}\mathbb{E}[(X_t^{i,\phi} - x^i)(X_t^{j,\phi} - x^j)] &= \mathbb{E}[(X^i(\mathcal{T}^i(t)) - x^i)(X^j(\mathcal{T}^j(t)) - x^j)] \\ &= \mathbb{E}[\mathbb{E}[(X^i(\mathcal{T}^i(t)) - x^i)(X^j(\mathcal{T}^j(t)) - x^j)|\mathcal{T}^i(t), \mathcal{T}^j(t)]] = 0,\end{aligned}$$

where $X_0^i = x^i$. Roughly speaking, the directions of jumps of $X^{i,\phi}$ and $X^{j,\phi}$ are still independent, since $X^i$ and $X^j$ are martingales. Next assume that $X^i$ and $X^j$ have nonzero drifts. Drifts induce correlation by introducing dependence in the direction of jumps of $X^{i,\phi}$ and $X^{j,\phi}$ as follows. Suppose the drifts $\mu^i$ and $\mu^j$ are constant and such that $\mu^i\mu^j > 0$ ($\mu^i\mu^j < 0$). Then during the time intervals $[\mathcal{T}_t^i, \mathcal{T}_t^i + \Delta\mathcal{T}_t^i]$ and $[\mathcal{T}_t^j, \mathcal{T}_t^j + \Delta\mathcal{T}_t^j]$ diffusions $X^i$ and $X^j$ will drift in the same direction. That will induce positive (negative) correlation among $X^{i,\phi}$ and $X^{j,\phi}$. However, when the drifts of $X^i$ are small relative to their volatilities, the correlation will generally be weak, as the martingale components will dominate the drifts. This limitation has already been observed in models based on multivariate subordination of Brownian motions (Luciano and Schoutens 2006; Semeraro 2008; Luciano and Semeraro 2010). Luciano and Semeraro (2010) proposed to circumvent this limitation on correlation in building financial models by correlating Brownian motions first, and then subordinating. In the context of multivariate subordination of Markov processes in this paper, when modeling highly correlated financial variables, we interpret the coordinates $X^{i,\phi}$ as the underlying stochastic factors in a factor model, and take the financial variables, such as log returns, to be linear combinations of the factors,

$$Y_t^i := \sum_j A_{ij} X_t^{j,\phi}.$$

In Section 4.1 this approach is applied to model highly correlated commodities.

### 2.3. Multivariate Subordination of Symmetric Markov Processes and Eigenfunction Expansions

Next we consider an important special case of *symmetric Markov processes* and show how the spectral theory can be applied in this case under some additional assumptions. A Markov process $X$ is $m$-symmetric if there exists a positive Radon measure $m$ with full support on $E$ such that the transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ of $X$ can be extended to a symmetric semigroup on the Hilbert space $L^2(E, m)$, that is, $(\mathcal{P}_t f, g) = (f, \mathcal{P}_t g)$ for all $f, g \in L^2(E, m)$, where $(f, g) = \int_E f(x)g(x)m(dx)$ is the inner product in $L^2(E, m)$ (see Chen and Fukushima 2011; Fukushima, Oshima, and Takeda 2011, for the theory of SMP).

Here we limit ourselves to the special case where all $X^i$ are symmetric Markov processes whose transition semigroups $(\mathcal{P}_t^i)_{t\geq 0}$ in $L^2(E_i, m_i)$ are *trace-class*, that is, the operators $\mathcal{P}_t^i$ are trace-class for all $t > 0$. Recall that for a positive semidefinite operator $A$ on a separable Hilbert space $\mathcal{H}$, the trace of $A$ is defined by $\mathrm{tr}\, A = \sum_{n=1}^{\infty}(\varphi_n, A\varphi_n) \in [0, \infty]$, where $\varphi_n$ is some orthonormal basis in $\mathcal{H}$. The trace is independent of the orthonormal basis chosen (cf. Reed and Simon 1980, p. 206). A positive semidefinite operator is called *trace-class* if and only if its trace is finite. The transition semigroup operators $\mathcal{P}_t$ are positive semidefinite. Under the assumption that $\mathcal{P}_t$ are trace-class for all $t > 0$, the spectra of each $\mathcal{P}_t$, as well as of the infinitesimal generator $\mathcal{G}$ of the semigroup $(\mathcal{P}_t)$ in $L^2(E, m)$, are purely discrete with eigenvalues $(e^{-\lambda_n t})_{n\geq 1}$ (for $t > 0$) and $(-\lambda_n)_{n\geq 1}$, respectively, and

$$\mathrm{tr}\mathcal{P}_t = \sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty$$

for all $t > 0$ (cf. lemma 7.2.1 of Davies 2007). Here $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ are arranged in increasing order and repeated according to multiplicity. Then the function $\mathcal{P}_t f(x)$ has an eigenfunction expansion of the form:

$$\mathcal{P}_t f(x) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\varphi_n(x),\ \ f_n = (f, \varphi_n) \text{ for any } f \in L^2(E, m) \text{ and all } t \geq 0, \tag{2.17}$$

where $\varphi_n$ is the eigenfunction corresponding to $\lambda_n$ (i.e., $\mathcal{P}_t\varphi_n = e^{-\lambda_n t}\varphi_n$). The eigenfunctions $(\varphi_n)_{n\geq 1}$ form a complete orthonormal basis in $L^2(E, m)$, and $f_n$ is the $n$-th expansion coefficient in this basis. Each $\mathcal{P}_t$ with $t > 0$ admits a symmetric kernel $p(t, x, y) \in L^2(E \times E, m \times m)$ with respect to $m$ (i.e., $p(t, x, y) = p(t, y, x)$, $\mathcal{P}_t f(x) =$

$\int_E p(t, x, y) f(y) m(dy)$ for $f \in L^2(E, m)$, and $\int_{E\times E} p^2(t, x, y) m(dx) m(dy) < \infty$), which has the following bi-linear expansion:

$$p(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y). \tag{2.18}$$

The expansions in (2.17) and (2.18) converge, in general, under the $L^2(E, m)$ and $L^2(E \times E, m \times m)$ norms, respectively. If we also suppose in addition that for each $t > 0$ the kernel $p(t, x, y)$ is jointly continuous in $x$ and $y$, then each eigenfunction $\varphi_n$ is continuous, and satisfies the estimate $|\varphi_n(x)| \le e^{\lambda_n t/2} \sqrt{p(t, x, x)}$ for all $n$, $x$, and $t > 0$. Moreover, for any $f \in L^2(E, m)$, the expansion (2.17) converges uniformly in $x$ on compacts for each $t > 0$, $\mathcal{P}_t f(x)$ is continuous in $x$, and the bilinear expansion (2.18) converges uniformly on compacts (cf. theorem 7.2.5 of Davies 2007).

One-dimensional diffusions are an important example of symmetric Markov processes. In this case, the state space $E = I$ is an interval with endpoints $-\infty \le l < r \le \infty$ that can be finite or infinite, and can be included or excluded from the state space, and the 1D diffusion is characterized by its speed measure $m$, scale function $s$, and killing measure $k$ (see Borodin and Salminen 2002, chapter II for general facts about 1D diffusions). The transition function is absolutely continuous with respect to the speed measure $m$, $P_t(x, A) = \int_A p(t, x, y) m(dy)$. The density $p(t, x, y)$ is positive, symmetric in $x$ and $y$, and jointly continuous in all variables. The semigroup of operators $\mathcal{P}_t f(x) = \int_I p(t, x, y) f(y) m(dy)$ can be defined on $L^2(I, m)$, where each operator is symmetric. The general spectral representation for the density of a 1D diffusion was obtained by McKean (1956). In general the spectrum contains some continuous spectrum, and the spectral representation is in terms of the integral with respect to the spectral measure (see also Langer and Schenk 1990). Nevertheless, many diffusions useful in finance applications have pure discrete spectra with explicitly known eigenvalues and eigenfunctions, including OU, CIR, CEV, and JDCEV diffusions (see surveys Linetsky 2004, 2008, and references therein for finance applications). As long as the spectrum is purely discrete and the eigenvalues and eigenfunctions are known analytically, the spectral expansion reduces to the eigenfunction expansion, and the task of computing the eigenfunction expansion of $\mathcal{P}_t f(x)$ is thus reduced to computing the expansion coefficients and evaluating the expansion. Moreover, the uniform convergence of the eigenfunction expansion of $\mathcal{P}_t f(x)$ allows one to approximate the function by the eigenfunction expansion truncated to a finite sum with a uniform bound on the truncation error in any compact domain of interest. This is useful for applications of eigenfunction expansions in option pricing. The trace class assumption is satisfied for many diffusions of interest in finance, including OU, CIR, CEV, and JDCEV, leading to uniformly convergent eigenfunction expansions for value functions of contingent claims. We now formulate the eigenfunction expansion for the subordinate semigroup.

THEOREM 2.9. *Let $X^i$, $i = 1, \dots d$, be symmetric Markov process on locally compact separable metric spaces $E_i$ with the symmetry measures $m_i$ and with the transition semigroups $(\mathcal{P}^i_t)_{t\ge 0}$ symmetric on $L^2(E_i, m_i)$. Suppose that all $\mathcal{P}^i_t$ are trace-class with eigenvalues and eigenfunctions $e^{-\lambda^i_n t}$ and $\varphi^i_n(x)$, respectively, and possess densities $p_i(t, x_i, y_i)$ with respect to $m_i$ that are continuous in $x$, $y$. Further suppose that the eigenfunctions have bounds $|\varphi^i_n(x)| \le C^i_K$ on each compact set $K \subset E_i$ with $C^i_K$ independent of $n$ but possibly dependent on $K$. Abusing notation, let $(\mathcal{P}^i_t)_{t\ge 0}$ also denote extensions of these semigroups to $L^2(E, m)$, where $E := E_1 \times \dots \times E_d$ is the product space with the product measure*

*$m(d\mathbf{x}) := m_1(dx_1)\dots m_d(dx_d)$. Let $\mathcal{T}$ be a $d$-dimensional subordinator with the Laplace exponent satisfying the following condition:*

$$\sum_{\mathbf{n}\in\mathbb{N}^d} e^{-\phi(\lambda^1_{n_1},\dots,\lambda^d_{n_d})t} < \infty, \tag{2.19}$$

*where $\sum_{\mathbf{n}\in\mathbb{N}^d} = \sum_{n_1=1}^{\infty}\dots\sum_{n_d=1}^{\infty}$, $\mathbb{N} = \{1, 2, \dots\}$.*
*Then the subordinate semigroup $(\mathcal{P}^{\phi}_t)_{t\geq 0}$ is a symmetric semigroup on $L^2(E, m)$, trace-class for all $t > 0$ with the eigenvalues $e^{-\phi(\lambda^1_{n_1},\dots,\lambda^d_{n_d})t}$ and normalized eigenfunctions $\varphi_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{d}\varphi^i_{n_i}(x_i)$, $x_i \in E_i$, $\mathbf{x} = (x_1, \dots, x_d)^{\top} \in E$, and possesses a continuous in $\mathbf{x}, \mathbf{y}$ density with respect to $m(d\mathbf{x})$ that is given by the bi-linear expansion*

$$p^{\phi}(t; \mathbf{x}, \mathbf{y}) = \sum_{\mathbf{n}\in\mathbb{N}^d} e^{-\phi(\lambda^1_{n_1},\dots,\lambda^d_{n_d})t}\varphi_{\mathbf{n}}(\mathbf{x})\varphi_{\mathbf{n}}(\mathbf{y}) \tag{2.20}$$

*uniformly convergent in $\mathbf{x}$, $\mathbf{y}$ on compacts in $E \times E$ for all $t > 0$. For each $f \in L^2(E, m)$ and $t > 0$ the function $\mathcal{P}^{\phi}_t f(\mathbf{x})$ has the eigenfunction expansion*

$$\mathcal{P}^{\phi}_t f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}^d} e^{-\phi(\lambda^1_{n_1},\dots,\lambda^d_{n_d})t} f_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{x}), \quad f_{\mathbf{n}} = (f, \varphi_{\mathbf{n}})_{L^2(E,m)} = \int_E f(\mathbf{x})\varphi_{\mathbf{n}}(\mathbf{x})m(d\mathbf{x}) \tag{2.21}$$

*uniformly convergent in $\mathbf{x}$ on compacts in $E$.*

Without the bound on the eigenfunctions of $X^i$ and the condition (2.19) on the Laplace exponent of the subordinator, the eigenfunction expansions (2.20) and (2.21) generally converge only in $L^2(E, m)$, and not uniformly. The bound on eigenfunctions is satisfied for many diffusions important in finance applications, such as OU, CIR, CEV, JDCEV. The condition (2.19) also turns out to be mild and is satisfied in many applications in finance. For example, it is satisfied for multidimensional subordinators constructed via linear factor models (see Remark 2.7) from tempered stable subordinators (2.16) with $\alpha \in (0, 1)$ when eigenvalues grow linearly in the eigenvalue number, as is the case for OU, CIR, CEV, and JDCEV diffusions.

## 3. EXTENSIONS WITH TIME-INHOMOGENEITY AND STOCHASTIC VOLATILITY

### 3.1. Additive Subordinators and Time Inhomogeneity

We now relax the assumption that the subordinator has stationary increments. Namely, each coordinate of a $d$-dimensional additive subordinator is now a 1D additive subordinator, that is, a nonnegative càdlàg process with independent increments, continuous in probability, and starting at the origin (see, e.g., Sato 1999, definition 1.6). Then the additive subordinator $\mathcal{T}$ has the Laplace transform given by the time-dependent Lévy-Khintchine formula:

$$\mathbb{E}\big[e^{-\langle\lambda,\,\mathcal{T}_t\rangle}\big] = \exp\Big(-\int_0^t \psi(\lambda, u)du\Big), \quad \lambda \in \mathbb{R}^d_{++},$$

with the Laplace exponent

$$\psi(\lambda, t) = \langle\lambda,\ \gamma(t)\rangle + \int_{\mathbb{R}^d_+\setminus\{0\}} \big(1 - e^{-\langle\lambda,\,\mathbf{s}\rangle}\big)\nu_t(d\mathbf{s})$$

with drift $\gamma(t) \in \mathbb{R}_+^d$ and time-dependent Lévy measure $\nu_t(d\mathbf{s})$ on $\mathbb{R}_+^d \backslash \{0\}$ that integrates $(\|\mathbf{s}\| \wedge 1)$ for each $t > 0$.

Examples of 1D additive subordinators can be generated by making the parameters of Lévy subordinators time dependent. For example, take the three-parameter family of Lévy measures (2.15) and make (all or some of) the parameters time dependent. The Laplace exponent $\psi(\lambda, t)$ has the same functional form of (2.16), where for all $t \geq 0$, $C(t) > 0$, $\alpha(t) < 1$, and $\gamma(t), \eta(t) \geq 0$, are now functions of time. In practice, when calibrating to the term structure of implied volatilities in option prices, one can take piece-wise constant or linear specifications with the nodes corresponding to different maturities in the data. Excellent fit across multiple maturities can be achieved by fitting the parameters on each subinterval. However, such specifications can be overparameterized.

Increasing Sato processes provide more parsimonious examples of additive subordinators. Let $\rho > 0$ denote a *self-similarity index*. $\mathcal{T}$ is a $\rho$-*Sato process* if it is additive and $\rho$-self-similar (i.e., $(\mathcal{T}_{ct})_{t\geq 0} \overset{(law)}{=} (c^\rho \mathcal{T}_t)_{t\geq 0}$ for all $c > 0$). From theorem 16.1 in Sato (1999), a random variable $Z$ is *self-decomposable* if and only if for every $\rho > 0$ there exists a $\rho$-Sato process $(\mathcal{T}_t)_{t\geq 0}$ such that $\mathcal{T}_1 \overset{(law)}{=} Z$. In particular, if the self-decomposable law of $Z$ has support on $\mathbb{R}_+$, then the Sato process has increasing paths. From corollary 15.11 in Sato (1999), the Laplace transform of $Z$ satisfies $\mathbb{E}[e^{-\lambda Z}] = e^{-\phi(\lambda)}$, where $\phi(\lambda) := \gamma\lambda + \int_0^\infty (1 - e^{-\lambda s})\frac{h(s)}{s} ds$ for all $\lambda > 0$ and $\gamma \geq 0$. The function $h(s)$ is positive and decreasing on $(0, \infty)$. Clearly, the function $\phi(\lambda)$ corresponds to the Lévy exponent of a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure of the special form $\nu(ds) = \frac{h(s)}{s} ds$ (the function $h(s)$ is such that $\int_{(0,\infty)} (1 \wedge s)\nu(ds) < \infty$). Given such a positive self-decomposable law, the $\rho$-Sato process $\mathcal{T}$ is defined via its characteristic exponent $\psi(\lambda, t)$. From theorem 1 in Carr et al. (2003), the $\rho$-Sato process $\mathcal{T}$ with increasing paths is an additive subordinator with drift $\gamma(t) = \gamma\rho t^{\rho-1}$ and Lévy measure $\nu(t, ds) = -\rho h'(st^{-\rho})t^{-\rho-1} ds$ such that $\mathbb{E}[e^{-\lambda \mathcal{T}_t}] = e^{-\int_0^t \psi(\lambda,u)du} = e^{-\phi(\lambda t^\rho)}$ for all $\rho, \lambda > 0$. Consequently, one can easily construct Sato-type additive subordinators from Lévy subordinators with self-decomposable distributions, such as tempered stable subordinators. In particular, Laplace exponents for the IG-Sato and Gamma-Sato additive subordinators can be readily obtained from the Laplace exponents $\phi(\lambda)$ for the IG and Gamma subordinators, see, equation (2.16),

$$\int_0^t \psi(\lambda, u)du = \begin{cases} \gamma\lambda t^\rho + 2C\sqrt{\pi}\left(\sqrt{\lambda t^\rho + \eta} - \sqrt{\eta}\right) & \textit{(IG-Sato)} \\ \gamma\lambda t^\rho + C\ln(1 + \lambda t^\rho/\eta) & \textit{(Gamma-Sato)} \end{cases}.$$

Such additive Sato subordinator specifications are parsimonious, as they involve only a single parameter, $\rho > 0$, in addition to the parameters in the corresponding Lévy subordinator. Nevertheless, they have proven to perform well in single-name equity (Carr et al. 2002, 2003), commodity (Li and Mendoza-Arriaga 2013a), and credit (Kokholm and Nicolato 2010) models.

Multivariate additive subordinators can be constructed by taking linear combinations of 1D additive subordinators as in equation (2.14). The results of Section 2 on multivariate subordination of Markov processes can be extended to additive subordinators. The result is a time-inhomogeneous Markov process in the product space. For time-inhomogeneous Markov processes the concept of a *Markov evolution* replaces the concept of a Markov semigroup. The Markov evolution $\{\mathcal{P}_{s,t}^\psi, 0 \leq s \leq t\}$ associated with a time-inhomogeneous Markov process on the state space $E$ is a family of transition operators, $\mathcal{P}_{s,t}^\psi f(x) := \int_E f(y)P(s, x; t, dy)$. If each of the $d$ independent time-homogeneous Markov processes $X^i$ has a semigroup $\mathcal{P}^i$ that is symmetric and trace-class, as defined in

Section 2.3, and $\mathcal{T}$ is a $d$-dimensional additive subordinator with the Laplace exponent $\psi$, then the time-inhomogeneous subordinate Markov process $X_t^{\psi,i} := X^i(\mathcal{T}^i(t))$ has a Markov evolution:

$$\mathcal{P}_{s,t}^{\psi} f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}^d} e^{-\int_s^t \psi(\lambda_{n_1}^1,\ldots,\lambda_{n_d}^d,u)du} f_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{x}),$$

with the eigenfunction expansion converging uniformly on compacts in $\mathbf{x}$ under a sufficient condition that the Laplace exponent satisfies $\sum_{\mathbf{n}\in\mathbb{N}^d} e^{-\int_s^t \psi(\lambda_{n_1}^1,\ldots,\lambda_{n_d}^d,u)du} < \infty$. This expression is a time-inhomogeneous counterpart of equation (2.21). Moreover, if each $X^i$ is a Feller jump-diffusion process, then $\mathcal{P}_{s,t}^{\psi}$ is a Feller evolution, and $X^{\psi}$ is a time-inhomogeneous Feller jump-diffusion. Feller evolutions or *propagators* are time-inhomogeneous counterparts of Feller semigroups associated to time-homogeneous Feller processes (see, e.g., Gulisashvili and van Casteren 2006; van Casteren 2011). Univariate time-inhomogeneous subordination of Feller semigroups has been recently considered by Mijatovic and Pistorius (2010).

Relaxing time-homogeneity is important for applications with strong time dependence. In particular, Li and Mendoza-Arriaga (2013a) construct a time-inhomogeneous commodity model via additive subordination of an OU diffusion based on a Sato subordinator. By adding a single additional parameter, they are able to calibrate to the implied volatility surface across multiple maturities in commodity options markets. Section 4.1 sketches a multivariate commodity model based on additive multivariate subordination.

### 3.2. Absolutely Continuous Time Changes and Stochastic Volatility

This section shows how absolutely continuous time changes used by Carr et al. (2002, 2003) and Mendoza-Arriaga et al. (2010) to introduce stochastic volatility into univariate Lévy process and Markov process models, respectively, can be extended to the multivariate setting of this paper. Let $V$ be an $\mathbb{R}_+^d$-valued conservative (i.e., no killing) Feller process. Define a multivariate time change by:

$$\mathcal{T}_t^i = \int_0^t V_u^i du.$$

The two key extensions of the results in Mendoza-Arriaga et al. (2010) are as follows. First, by extending the arguments in section 4.2 of Mendoza-Arriaga et al. (2010), if $\mathcal{G}^V$ is the infinitesimal generator of the Feller process $V$ (by Bony, Courrège, and Priouret 1968, acting as an integro-differential operator on twice continuously differential functions on $\mathbb{R}_+^d$ with compact support) and $\mathcal{G}^i$ are infinitesimal generators of each of the Markov processes $X^i$ (assumed to be Feller processes taking values in $\mathbb{R}^{n_i}$ in this section), then the process $(Y_t^i, V_t^i)_{i=1}^d$ with $Y_t^i := X^i(\mathcal{T}^i(t))$ is a Feller process in $\mathbb{R}^n \times \mathbb{R}_+^d$, $n = n_1 + \ldots + n_d$, with the infinitesimal generator

$$\mathcal{G}^{(Y,V)} f(\mathbf{x}, v) = \left(\sum_{i=1}^d v_i \mathcal{G}^i + \mathcal{G}^V\right) f(\mathbf{x}, v) \tag{3.1}$$

acting on functions $f \in C_c^2(\mathbb{R}^n \times \mathbb{R}_+^d)$. That is, the time changed process *together* with the multivariate activity rate process $V$ form a time-homogeneous Markov (Feller) process in the product state space. Each coordinate $V^i$ plays a role of stochastic volatility (stochastic activity) for the coordinate $Y^i = X^i(\mathcal{T}^i)$. Note that it scales the generator $\mathcal{G}^i$, so that the stochastic volatility is introduced into the diffusion, jump, and killing components of the corresponding process (see equation (2.7) for $\mathcal{G}^i$). Dependence among $V^i$ translates into the dependence among stochastic volatilities of $Y^i$.

The second key result is that, if the Laplace transform

$$\mathbb{E}_v[e^{-\langle\lambda,\mathcal{T}_t\rangle}] = \mathbb{E}_v[e^{-\int_0^t(\sum_{i=1}^d \lambda_i V_u^i)du}] =: \mathcal{L}(v, t, \lambda)$$

is available in closed form, then, under the trace-class assumption for the semigroups of $X^i$, the eigenfunction expansion

$$\mathbb{E}_{\mathbf{x},v}[f(Y)] = \sum_{\mathbf{n}\in\mathbb{N}^d} \mathcal{L}(v, t, \lambda) f_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x})$$

converges uniformly on compacts in $\mathbf{x}$ under the sufficient condition that $\sum_{\mathbf{n}\in\mathbb{N}^d} \mathcal{L}(v, t, \lambda) < \infty$ for each $t > 0$ and $v \in \mathbb{R}_+^d$. This expression is a counterpart of equation (2.21) with stochastic volatility. Here the Laplace transform explicitly depends on the initial value of the vector $V_0^i = v^i$ of stochastic volatilities.

We observe that the Laplace transform $\mathcal{L}(v, t, \lambda)$ can be interpreted as the price of a unit-face value zero-coupon bond with maturity $t$ in a model with the short rate $r_t := \sum_{i=1}^d \lambda_i V_t^i$. For example, any $d$-dimensional affine diffusion or jump-diffusion interest rate term structure model with nonnegative factors (e.g., Duffie, Filipovic, and Schachermayer 2003) generates a multivariate time change with the known Laplace transform in the affine form

$$\mathcal{L}(v, t, \lambda) = e^{-A(t,\lambda)-\sum_{i=1}^d B_i(t,\lambda)v^i},$$

with the functions $A$, $B$ of time and the Laplace transform parameters $\lambda$ found as solutions to the Riccati ODE can be used to time change a collection of $d$ independent Markov processes $X^i$. We also note that affine jump-diffusion processes are Feller (see Duffie et al. 2003).

### 3.3. Composite Time Changes

In this section, we sketch how time changes with Lévy subordinators of Section 2 can be composed with absolutely continuous time changes to build processes with state-dependent jumps and stochastic volatility. Let $\mathcal{S}_t^i$ be a $d$-dimensional Lévy subordinator and $\mathcal{A}_t = \int_0^t V_u du$ a univariate absolutely continuous time change, where $V$ is, say, a 1D nonnegative affine process (a CIR diffusion or a CIR diffusion with jumps, see Filipovic 2001) with the known Laplace transform, $\mathcal{L}(v, t, \lambda) = \mathbb{E}_v[e^{-\lambda\int_0^t V_u du}]$. Time changing the subordinator $\mathcal{S}$ with $\mathcal{A}$ yields a multivariate jump process $\mathcal{T}_t^i = \mathcal{S}^i(\mathcal{A}(t))$ with the arrival rates of jumps in all of its coordinates modulated with the common stochastic stochastic volatility process $V$. By conditioning

$$\mathbb{E}_v[e^{-\langle\lambda,\mathcal{S}_{\mathcal{A}_t}\rangle}] = \mathbb{E}_v[e^{-\phi(\lambda)\int_0^t V_u du}] = \mathcal{L}(v, t, \phi(\lambda)).$$

Time changing each of Feller processes $X^i$ with $\mathcal{T}^i$ yields a process $Z_t^i = X^i(\mathcal{T}^i(t)) = X^i(\mathcal{S}^i(\mathcal{A}(t))) = X_{\mathcal{A}_t}^{i,\phi}$ such that the process $(Z, V)$ is a Feller process in $\mathbb{R}^n \times \mathbb{R}_+$ with the generator

$$\mathcal{G}^{(Z,V)} f(\mathbf{x}, v) = \left(v\mathcal{G}^{\phi} + \mathcal{G}^V\right) f(\mathbf{x}, v),$$

where $\mathcal{G}^\phi$ is the generator of the process $X^\phi$ given by equation (2.10) and $\mathcal{G}^V$ is the generator of the process $V$. In this specification there is a single stochastic volatility process $V$ for all of the coordinates of the jump-diffusion process $X^\phi$. An increase (decrease) in the volatility process $V$ results in the simultaneous increase (decrease) in the conditional correlations among coordinates $Z^i$ (conditional on the level of stochastic volatility), as the volatility scales the Lévy measure (and the killing rates) of $Z^i$, as can be seen from the above expression for the generator (substitute equation (3.1) for $\mathcal{G}^\phi$ and observe that $Z$ has the Lévy measure $v\Pi^\phi(x, dy)$ and the killing rate $vk^\phi(x)$). This time change introduces a *procyclical* effect in conditional correlations, as the correlations increase (decrease) when the stochastic volatility factor $V$ increases (decreases).

## 4. EXAMPLES OF FINANCIAL APPLICATIONS

### 4.1. Highly Correlated Commodities and Spread Options

Li and Linetsky (2014) study a univariate commodity model where, under the risk-neutral probability measure chosen by the market, the spot price of a commodity follows the process

$$S_t = F(0, t)e^{X_t^\phi - G(t)},$$

where $\{F(0, t), t \geq 0\}$ is the futures curve observed at time zero, $X^\phi$ is a subordinate OU process (an OU diffusion time changed with a Lévy subordinator), and $G(t)$ is a deterministic function of time chosen so that $\mathbb{E}[S_t] = F(0, t)$ (i.e., $G(t) = \ln \mathbb{E}[e^{X_t^\phi}]$). This model can be thought of as an extension of the classical OU-diffusion–based commodity model (e.g., Schwartz 1997) to include jumps. Since jumps are introduced via subordination of the OU diffusion, this model exhibits mean-reverting jumps. Li and Linetsky (2014) derive the process for the future curve and futures options prices under this specification and show that this model calibrates well to a wide variety of implied volatility patterns across strikes in commodity markets, including metals, energies, and agricultural commodities; Li and Linetsky (2013) give a computational algorithm for American futures options in this model. In order to capture patterns across multiple option expirations, as well as across strikes, Li and Mendoza-Arriaga (2013a) further extend this model by replacing the Lévy subordinator in Li and Linetsky (2014) with an additive Sato subordinator, thus taking $X^\phi$ to be an OU diffusion time-changed with an additive Sato subordinator, such as Sato-IG. The resulting model, while still as analytically tractable as the original model, is able to synthesize the volatility surface for commodity options across both strikes and expiration dates.

In addition to single-commodity derivatives, spread options are an important class of contracts in commodities markets. A crack spread is the spread between the price of refined petroleum products and crude oil (the process of extraction of refined petroleum products from crude oil is known as cracking). Options on the unleaded gasoline/crude oil spread and the heating oil/crude oil spread are traded on the Chicago Mercantile Exchange and are used for risk management by refineries. Other important spreads are the crush spread and the spark spread. The crush spread refers to the difference between the price of soybeans and soybean meal and oil produced by crushing soybeans. The spark spread refers to the difference between the price of electricity sold by a generator and the price of the fuel used to generate it (such as natural gas).

Li and Mendoza-Arriaga (2013b) apply the procedure of multivariate subordination developed in the present paper to construct a model for highly correlated commodities and spread options. In this section, we sketch their model as an illustration and refer to Li and Mendoza-Arriaga (2013b) for details. Let $X_t^1$ and $X_t^2$ be two independent OU diffusions, $(\mathcal{T}_t^1, \mathcal{T}_t^2)$ a 2D additive subordinator with the time-dependent Laplace exponent $\phi(\lambda, t)$, and $(X_t^{1,\phi}, X_t^{2,\phi}) = (X^1(\mathcal{T}_t^1), X^2(\mathcal{T}_t^2))$ the subordinate process. Let $S_t^i$ be the prices of two related commodities. Li and Mendoza-Arriaga (2013b) model them as:

$$S_t^i = F^i(0, t) \exp\Big(\sum_{j=1}^{2} a_{i,j} X_t^{j,\phi} - G^i(t)\Big),$$

where $F^i(0, t)$ are the respective futures curves of the two commodities, $a_{i,j}$ are the factor loadings, and $G^i(t)$ are deterministic functions of time such that $\mathbb{E}[S_t^i] = F^i(0, t)$, that is, $G^i(t) = \ln \mathbb{E}[\exp(\sum_{j=1}^2 a_{i,j} X_t^{j,\phi})]$. If we take $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = 1$, then dependence among the prices of two commodities arises due to dependence in jumps of $X^\phi$, while the diffusion components of prices remain independent. If we take $a_{12} \neq 0$ and/or $a_{21} \neq 0$, diffusion components of the commodity prices are also dependent, as discussed in Remark 2.8. This flexibility allows to generate high correlation levels. Li and Mendoza-Arriaga (2013b) obtain the eigenfunction expansion for the pricing of spread options in this model. This specification is flexible enough to allow simultaneous and consistent calibration to the implied volatility surfaces of each commodity, as well as to the cross-section of spread options with different strikes and expiration dates. It can be extended to $n$ asset prices $S^i$ driven by $d$ factors $X^\phi$.

REMARK 4.1 (On Measure Changes). Since the main focus of this paper is on derivatives pricing, we assume that a risk-neutral probability measure in the incomplete market setting is chosen by the market, and work under this probability measure. In practice, the risk-neutral model parameters are obtained by calibrating to the prices of marketed derivative securities. We note that if we assume that under the risk-neutral measure chosen by the market the model has a particular structure, such as a subordinate OU process, in general the structure under the physical measure can be much more general. To simplify model development, it is natural to make a structural assumption and assume that the model has the *same structural form* under both physical and risk-neutral measures. This restricts the class of equivalent measure transformations that transform from the physical measure to risk-neutral measures to those transformations that preserve the structure. One can then study the class of equivalent measure transformations that satisfy the *structure condition*. Li and Linetsky (2014) characterize the class of equivalent measure transformations that transform a univariate subordinate OU process into another univariate subordinate OU process and show how the parameters of a subordinate OU process can change under an equivalent measure transformation. Similar analysis can be done for multivariate subordinate processes. Such mathematical analysis characterizes the entire class of equivalent measure transformations that preserve the model structure. To be able to address such interesting questions as how the correlation changes under the change from the physical to a risk-neutral measure, integrated empirical multivariate time series studies of the underlying assets and derivatives are needed to be able to simultaneously estimate parameters under both the physical and the risk-neutral measure and, hence, estimate risk premia. These topics are outside of the scope of the present paper.

## 4.2. Unified Multiname Credit-Equity Modeling

*4.2.1. General Model Architecture.* In this section, we develop an application to unified multi-name credit-equity modeling. We assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{Q}$ chosen by the market on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ as given. All stochastic processes and random variables in what follows are defined on this probability space, and all expectations are with respect to $\mathbb{Q}$ unless stated otherwise. We model the joint risk-neutral dynamics of stock prices $S_t^i$ of $n$ firms under the EMM as an $n$-dimensional stochastic process defined by:

$$S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{T}_t^i}, \quad i = 1, \dots, n. \tag{4.1}$$

We now describe the ingredients in this model architecture.

(i) *Independent 1D Diffusions $X^i$.* $X^i$ are $n$ time-homogeneous diffusion processes on $(0,\infty)$ starting from positive values $X_0^i = S_0^i > 0$ (initial stock prices at time zero) with the infinitesimal generators $\mathcal{G}_i$ acting on $C_c^2((0,\infty))$ by:

$$\mathcal{G}_i f(x) = \frac{1}{2}\sigma_i^2(x)x^2 f''(x) + (\mu_i + k_i(x))x f'(x) - k_i(x) f(x), \tag{4.2}$$

where $\sigma_i(x)$ is the state-dependent instantaneous volatility, $k_i(x)$ is the state-dependent killing rate, and $\mu_i + k_i(x)$ is the drift rate, where $\mu_i \in \mathbb{R}$ and the killing rate is added in the drift to compensate for the killing (interpreted as jump-to-default) to ensure that the discounted gain process (stock price changes and reinvested dividends) is a martingale. To simplify development, we assume that $\sigma_i(x)$ and $k_i(x)$ are continuous and differentiable on the open interval $(0,\infty)$, $\sigma_i(x) > 0$ on $(0,\infty)$, and $k_i(x) \ge 0$ on $(0,\infty)$. We also assume that infinity is a natural boundary for each diffusion (keeping $\sigma_i(x)$ and $k_i(x)$ bounded as $x \to \infty$ suffices). In general, the process may reach zero, depending on the behavior of $\sigma_i(x)$ and $k_i(x)$ as $x \to 0$. If zero is an accessible boundary for $X^i$, we specify it as a killing boundary and send $X^i$ to the cemetery state $\Delta_i$, where it remains for all subsequent times. We denote the lifetime of $X^i$ by $\zeta_i$. Under these assumptions, each $X^i$ is a Feller process on $(0,\infty)$ with killing at the rate $k_i$ inside the state space, and possibly at the boundary at zero if it is accessible. The diffusion $X^i$ with killing rate $k_i$ can be explicitly constructed by first considering a diffusion $Y^i$ with $k_i(x) \equiv 0$, and then defining $X^i$ as its subprocess with lifetime $\zeta_i := \inf\{t \in [0, H^i] : \int_0^t k_i(X_u^i)du \ge \mathcal{E}_i\}$, where $\mathcal{E}_i$ is an independent exponential random variable with unit mean and $H^i$ is the first hitting time of zero for the process $Y_i$ (by convention, $\inf\{\emptyset\} = H^i$). Under our assumptions, diffusions $X^i$ are Feller processes with their transition kernels defining (sub-Markovian) Feller semigroups on $C_0((0,\infty))$, as well as symmetric semigroups in $L^2((0,\infty), m)$ with the speed measure $m(dx) = m(x)dx$ with the speed density $m(x) = \frac{2}{\sigma_i^2(x)x^2}\exp(\int \frac{2(\mu_i + k_i(y))}{\sigma_i^2(y)y}dy)$.

(ii) *Multivariate Time Change $\mathcal{T}$.* $\mathcal{T}$ is an $n$-dimensional subordinator. In applications the linear factor model described in Remark 2.7 is particularly convenient.

(iii) *Default Times $\tau_i$.* The positive random variable $\tau_i$ models the time of default of the $i$th firm on its debt. We assume that in default strict priority rules are followed, so that while the corporate debt holders may receive some recovery, the stock becomes worthless (stock price is equal to zero in default, and stock holders receive no recovery). The time of default of the $i$th firm is defined by applying the time change $\mathcal{T}^i$ as discussed in Section 2, $\tau_i := \inf\{t \ge 0 : \mathcal{T}_t^i \ge \zeta_i\}$ (see equation (2.6)).

At the time of default of the $i$th firm, its stock price jumps to zero and stays at zero for all subsequent times (the indicator in equation (4.1)).

(iv) *Martingale Conditions.* This model architecture generalizes to $n$ firms the single-name model architecture of Mendoza-Arriaga et al. (2010). Marginally, each stock price process $S^i$ is modeled by a jump-to-default extended diffusion time-changed with a 1D subordinator $\mathcal{T}^i$. By theorem 4.2 of Mendoza-Arriaga et al. (2010), each single-name stock price process $S^i$ with the dividends reinvested and discounted at the risk-free interest rate is a nonnegative martingale under the EMM $\mathbb{Q}$ with respect to its natural filtration if and only if the constant $\mu_i$ in the drift of $X^i$ in (4.2) satisfies the following condition:

$$\int_{[1,\infty)} e^{\mu_i s} \nu_i(ds) < \infty,$$

where $\nu_i$ is the Lévy measure of the 1D subordinator $\mathcal{T}^i$ ($\nu_i(A) = \nu(B_1 \times \ldots \times A \times \ldots \times B_1)$ with $A$ in the $i$th place, for any Borel set $A \subset \mathbb{R}_+$, and for all $B_1, \ldots, B_n \subset \mathbb{R}_+$, bounded away from zero), and the constant $\rho_i$ in (4.1) is set to:

$$\rho_i = r - q_i + \phi_i(-\mu_i), \tag{4.3}$$

where $\phi_i(u)$ is the Laplace exponent of $\mathcal{T}^i$, $\phi_i(u) = \phi(0, \ldots, 0, u, 0, \ldots, 0)$ ($u$ is in the $i$th place), $r \geq 0$ is the risk-free interest rate, and $q_i \geq 0$ is the dividend yield of the $i$th stock. Under these conditions on $\mu_i$ and $\rho_i$, each stock price process in (4.1) is a nonnegative semimartingale under $\mathbb{Q}$, such that the process $e^{-(r-q_i)t} S^i_t$ is a nonnegative martingale under $\mathbb{Q}$.

Consider the pricing of contingent claims written on $n$ defaultable stocks. In particular, the price of a European-style derivative expiring at time $t > 0$ with the payoff function $f(S^1_t, \ldots, S^n_t)$ is given by $e^{-rt}\mathbb{E}[f(S^1_t, \ldots, S^n_t)]$. Because each of the $n$ firms may default by time $t$ (and its stock may become worthless) in this model, for each stock either $S^i_t > 0$ (survival to time $t$, i.e., $\tau_i > t$) or $S^i_t = 0$ (default by time $t$, i.e., $\tau_i \leq t$). We then have the following representation as a corollary of Theorem 2.4.

THEOREM 4.2. *(Representation of the Pricing Operator by the Subordinated Semigroup Operators) Let $\Upsilon = \{1, 2, \ldots, n\}$ denote the set of all $n$ firms. The expectation of the payoff can be decomposed as follows:*

$$\mathbb{E}[f(S^1_t, \ldots, S^n_t)] = \sum_{\Xi \subseteq \Upsilon} \mathbb{E}[\mathbf{1}_{\{\tau_\Xi > t\}} f_\Xi(S^{i_1}_t, \ldots, S^{i_k}_t)], \tag{4.4}$$

*where the sum is over all $2^n$ subsets $\Xi = \{i_1, \ldots, i_k\} \subseteq \Upsilon$ of the set of $n$ firms, $\tau_\Xi = \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}$ is the first time any of the firms in the subset $\Xi$ defaults (by convention $\tau_\emptyset = \infty$), and the $2^n$ functions $f_\Xi(S^{i_1}_t, \ldots, S^{i_k}_t)$ are constructed from the payoff function as follows ($|\Xi|$ denotes the cardinality of the set $\Xi$):*

$$f_\Xi(S^{i_1}_t, \ldots, S^{i_k}_t) = \sum_{\Theta \subseteq \Xi} (-1)^{|\Xi|-|\Theta|} f(\mathbf{1}_\Theta(1)S^1_t, \ldots, \mathbf{1}_\Theta(n)S^n_t), \tag{4.5}$$

*where $\mathbf{1}_\Theta(i) = 1(0)$ if $i \in \Theta$ ($i \in \Theta^c = \Upsilon \backslash \Theta$), by convention $f_\emptyset = f(0, \ldots, 0)$, and the sum is over all $2^k$ subsets of $\Xi$, $k = |\Xi|$.*

The representation (4.4)–(4.5) is suitable to apply the semigroup theory developed in Section 2. Each expectation on the right-hand side of equation (4.4) can be expressed in terms of the marginal semigroup operator $\mathcal{P}_t^{\Xi,\phi}$, so that for each fixed $t > 0$ we have:

$$\mathbb{E}[\mathbf{1}_{\{\tau_\Xi > t\}} f_\Xi(S_t^{i_1}, \dots, S_t^{i_k})] = (\mathcal{P}_t^{\Xi,\phi} \hat{f}_\Xi)(S_0^{i_1}, \dots, S_0^{i_k}), \tag{4.6}$$

where

$$\hat{f}_\Xi(x_{i_1}, \dots, x_{i_k}) := f_\Xi(e^{\rho_{i_1} t} x_{i_1}, \dots, e^{\rho_{i_k} t} x_{i_k}). \tag{4.7}$$

Here the Markov processes $\{X^i, i \in \Xi\}$ are $k$ 1D nonnegative diffusions with lifetimes $\zeta_i$, $\mathcal{T}^\Xi$ is the $k$-dimensional subordinator $(\mathcal{T}^{i_1}, \dots, \mathcal{T}^{i_k})^\top$, and the killing time of the $k$-dimensional subordinate process $X^{\Xi,\phi}$ is $\tau_\Xi = \tau_{i_1} \wedge \dots \wedge \tau_{i_k}$, where $\tau_i$ are defined by (2.6). The derivatives pricing problem is thus reduced to computing the action of the marginal semigroup operators (4.6) on the functions constructed from the payoff function by (4.7). In particular, if all $X^i$ have purely discrete spectra and all $f_\Xi \in L^2((0,\infty)^k, m_\Xi)$ with $m_\Xi(d\mathbf{x}) = m_{i_1}(dx_{i_1}) \dots m_{i_k}(dx_{i_k})$, where $m_i(dx)$ is the speed measure of the diffusion $X^i$, the expectations (4.6) have the eigenfunction expansions.

We stress that in our unified credit-equity approach, where we model defaultable stocks as the fundamental state variables, multiname credit derivatives are seen as special cases of multiname equity derivatives. In particular, a contingent claim that pays one dollar at expiration $t$ if all the firms in a given set $\Xi = \{i_1, \dots, i_k\}$ survive can be interpreted as a multiname equity derivative with the payoff $\mathbf{1}_{\{\tau_\Xi > t\}} = \mathbf{1}_{\{S_t^{i_1} > 0\}} \dots \mathbf{1}_{\{S_t^{i_k} > 0\}}$. This generalizes to the multiname setting the view taken in the unified single-name credit-equity modeling literature (Linetsky 2006; Carr and Linetsky 2006; Mendoza-Arriaga et al. 2010; Mendoza-Arriaga and Linetsky 2011; Linetsky and Mendoza-Arriaga 2011) that considers credit derivatives as special cases of equity derivatives written on defaultable stocks that can fall to zero in the event of bankruptcy.

*4.2.2. The Single-Name JDCEV Diffusion Process.* The CEV diffusion of Cox (1975) is a non-negative diffusion process with the instantaneous volatility $\sigma(x) = ax^\beta$, where $\beta < 0$ is the volatility elasticity parameter and $a > 0$ is the volatility scale parameter (see Schroder 1989; Davydov and Linetsky 2001, 2003; Linetsky and Mendoza 2010), and $k(x) \equiv 0$ in equation (4.2). The negative specification of the volatility elasticity parameter is consistent with the *leverage effect* (volatility increases when the stock price falls) and results in the implied volatility skew in stock options in this model. For $\beta < 0$, the CEV process hits zero with positive probability. In particular, when $\beta \in [-1/2, 0)$, the origin is the exit boundary for the CEV diffusion. When $\beta < -1/2$, the origin is a regular boundary for the CEV diffusion, and the killing boundary condition is imposed at the first hitting time of zero. The limiting case with $\beta = 0$ corresponds to the constant volatility assumption in the Black-Scholes-Merton model. Thus, the CEV model with $\beta < 0$ has the positive probability of bankruptcy built into the model due to the possibility of the stock price modeled by the CEV diffusion to hit zero. However, this probability is unrealistically small for reasonable parameter values for $a$ and $\beta$ that arise in calibrating the CEV model to implied volatility skews observed in stock options.

Carr and Linetsky (2006) generalized the CEV diffusion by introducing the possibility of a jump-to-default. To be consistent with the empirical evidence linking credit spreads with equity volatility, they introduced a killing rate (default intensity) into the CEV process that is specified to be an affine function of the CEV local volatility:

$$k(x) = b + c\,\sigma^2(x) = b + c\,a^2 x^{2\beta} \tag{4.8}$$

with $b \geq 0$ and $c \geq 0$. For $c > 0$, the default almost surely happens via a jump-to-default from a positive value. Carr and Linetsky (2006) give extensive references to the empirical literature documenting positive relationships between both historical equity volatility and implied volatility of equity options and corporate bond and CDS credit spreads. This literature continues to grow. We mention in particular Bandreddi, Das, and Fan (2007), Zhang, Zhou, and Zhu (2009), Carr and Wu (2010), Cao, Yu, and Zhong (2011), among others.

Carr and Linetsky (2006) solve the jump-to-default extended CEV (JDCEV) model by relating it to Bessel processes. In particular, they obtain explicit expressions for the (defective) continuous transition probability density $p(t, x, y)$ on $(0, \infty)$ and the probability of default by time $t$, $P_t(x, \{\Delta\}) = 1 - P_t(x, (0, \infty))$ (the killing probability). Mendoza-Arriaga et al. (2010) later obtained the eigenfunction expansions for the JDCEV semigroup. The next theorem summarizes the results about the JDCEV diffusion process relevant for the development of our multiname credit-equity model.

THEOREM 4.3. *(JDCEV Diffusion)*

*(i) The symmetric transition probability density of the JDCEV diffusion with respect to its speed measure m is given by:*

$$\begin{aligned} p(t, x, y) \;=\; & \frac{|\mu + b|(xy)^{\frac{1}{2}-c}(e^{\omega t})^{\nu/2}}{1 - e^{-\omega t}} \exp\left\{ -\varepsilon A \frac{\left(x^{-2\beta} + y^{-2\beta}\right)}{(1 - e^{-\varepsilon\omega t})} - \lambda_1 t \right\} \\ & I_\nu\left( \frac{A(xy)^{-\beta}}{\sinh\left((\omega t)/2\right)} \right), \end{aligned} \tag{4.9}$$

$$\begin{aligned} \nu :\;= \frac{1+2c}{2|\beta|}, \quad & \omega := 2|\beta(\mu + b)|, \quad A := \frac{|\mu + b|}{a^2|\beta|}, \\ \lambda_1 := & \begin{cases} 2(\mu + b)(|\beta| + c) + b, & \mu + b > 0 \\ |\mu|, & \mu + b < 0 \end{cases}, \end{aligned} \tag{4.10}$$

*where $I_\nu(z)$ is the modified Bessel function of order $\nu$. The speed density is given by:*

$$m(x) = \frac{2}{a^2} x^{2c-2-2\beta} e^{\varepsilon A x^{-2\beta}}, \quad \varepsilon := \operatorname{sign}(\mu + b). \tag{4.11}$$

*(ii) The default (killing) probability is $P_t(x, \{\Delta\}) = 1 - P_t(x, (0, \infty))$ with the survival probability given by:*

$$P_t(x, (0, \infty)) = \frac{\Gamma(c/|\beta| + 1)}{\Gamma(\nu + 1)} (\tau(t))^{\frac{1}{2|\beta|}} e^{-\tau(t) - bt}\, {}_1F_1\left( \begin{matrix} \frac{c}{|\beta|} + 1 \\ \nu + 1 \end{matrix} ; \tau(t) \right), \tag{4.12}$$

*where ${}_1F_1$ is the Kummer confluent hypergeometric function, $\Gamma(x)$ is the gamma function, and*

$$\tau(t) := \varepsilon A x^{-2\beta}/(1 - e^{-\varepsilon\omega t}).$$

(iii) *When $\mu + b \neq 0$, the JDCEV semigroup is a symmetric trace-class semigroup in $L^2((0,\infty), m)$ with the eigenvalues and continuous eigenfunctions of the negative of its infinitesimal generator $\mathcal{G}_i$ defined in $L^2((0,\infty), m)$ (as a self-adjoint extension of the operator* (4.2) *with $\sigma(x) = ax^\beta$ and $k(x)$ given by* (4.8) *with the Dirichlet (vanishing) boundary condition at* 0*):*

$$\lambda_n = \omega(n-1) + \lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\, x e^{-\frac{1}{2}(1+\varepsilon)Ax^{-2\beta}} L_{n-1}^{\nu}(Ax^{-2\beta}),$$

$$n = 1, 2, \ldots, \tag{4.13}$$

*where $L_n^\nu(x)$ are the generalized Laguerre polynomials. The principal eigenvalue $\lambda_1$ is given in* (4.10) *and $\varepsilon$ is defined in* (4.11). *On each compact interval $K \subset (0,\infty)$ there exists a constant $C_K$ independent of $n$ such that $|\varphi_n(x)| \leq C_K n^{-1/4}$ for all $n \geq 1$.*

(iv) *The JDCEV semigroup satisfies the estimates* (2.8) *and* (2.9).

Parts (i) and (ii) are proved in Carr and Linetsky (2006) based on the reduction of the JDCEV diffusion to Bessel processes. Part (iii) is proved in the Appendix. The estimates in part (iv) are verified by explicitly calculating the expressions on the right-hand side of (2.8) and (2.9) using equations (4.9)–(4.12) for the density and killing probability and considering the small time asymptotics of the resulting expressions. The explicit calculations are omitted to save space. One can verify directly that $\varphi_n(x)$ and $-\lambda_n$ are eigenfunctions and eigenvalues of the operator $\mathcal{G}_i$, that is, $\mathcal{G}\varphi_n(x) = -\lambda_n \varphi_n(x)$ with the Dirichlet boundary condition, $\varphi_n(0) = 0$. One can also verify directly that $\varphi_n(x)$ form an orthonormal basis in $L^2((0,\infty), m)$, that is, $(\varphi_l, \varphi_n) = \delta_{l,n}$ (which can be directly verified by integration using the known integrals of Laguerre polynomials). Different signs in the exponential in the speed density (4.11) in the cases $\mu + b > 0$ and $\mu + b < 0$ lead to the different eigenvalues and eigenfunctions. When $\mu + b = 0$, the spectrum is purely absolutely continuous. The spectral representation of the transition density in this case with continuous spectrum is given in Mendoza-Arriaga et al. (2010). To save space, in the rest of the paper we restrict our attention to the case with $\mu + b < 0$. Similar results can be obtained for the case with $\mu + b > 0$. The following lemma will be useful in the sequel for calculating eigenfunction expansions of credit and equity derivatives.

LEMMA 4.4.

(i) *For $p > 1/2 + \beta - c$ and $\mu + b < 0$, the power function $f_p(x) = x^p$ is in $L^2((0,\infty), m)$ and has the expansion in the eigenfunctions $\varphi_n(x)$:*

$$x^p = \sum_{n=1}^{\infty} c_n^p \varphi_n(x), \quad c_n^p = \frac{A^{\frac{\nu}{2} - \frac{p+2c}{2|\beta|}} \left(\frac{1-p}{2|\beta|}\right)_{n-1}}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}} \Gamma\left(\frac{p+2c}{2|\beta|} + 1\right),$$

$$n = 1, 2, \ldots, \tag{4.14}$$

*where $(z)_n = z(z-1)\ldots(z-n-1)$ is the Pochhammer symbol, and $A$ and $\nu$ are defined in* (4.10). *The series converges uniformly in $x$ on compacts.*

*(ii) While for $2(\beta - c) < p \leq 1/2 + \beta - c$ the power function $f_p(x) = x^p$ is not in $L^2((0, \infty), m)$, for all $t > 0$ the function $\mathcal{P}_t f_p(x) = \int_{(0,\infty)} p(t, x, y) y^p m(y) dy$ is in $L^2((0, \infty), m)$ and has the eigenfunction expansion*

$$\mathcal{P}_t f_p(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n^p \varphi_n(x)$$

*with the same coefficients $c_n^p$. Moreover, the convergence is uniform in $x$ on compacts for each $t > 0$.*

*4.2.3. The Single-Name Subordinate JDCEV Credit-Equity Model.* Before proceeding to the multiname subordinate JDCEV model, we present some results for the single-name model that will be useful for the development of the multi-name model. Due to Lemma 4.4 we can explicitly compute the $p$th moment of the stock price process in the subordinate JDCEV model.

PROPOSITION 4.5. *(Stock Price Moments) Let $S_t$ be the defaultable stock price process as defined in* (4.1) *with $d = 1$, where the corresponding diffusion $X$ is a JDCEV process with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$, and $\mu + b < 0$. Then, for $p > 2(\beta - c)$, the $p$th (risk neutral) moment of the stock price is given by:*

$$\mathbb{E}\big[S_t^p\big] = e^{p\rho t} \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t} c_n^p \varphi_n(x),$$

*where the JDCEV eigenvalues $\lambda_n$ and eigenfunctions $\varphi_n$ are given in Theorem 4.3 and the expansion coefficients are given in Lemma 4.4, and $x = S_0$, the initial stock price. If the Laplace exponent $\phi(s)$ of the subordinator $\mathcal{T}_t$ satisfies the condition* (2.19) *with $d = 1$ with $\lambda_n$ the eigenvalues of the JDCEV diffusion, then the convergence is uniform on compacts in $x$ for each $t > 0$. Otherwise, the convergence is in $L^2((0, \infty), m)$.*

The values $p = 0, 1, 2$ are of particular interest to us.

COROLLARY 4.6.

*(i) For $p = 0$ we obtain the survival probability up to time $t$:*

$$\mathbb{Q}(\tau > t) = \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t} c_n^0 \varphi_n(x).$$

*(ii) For $p = 1$, we obtain the forward stock price:*

$$\mathbb{E}[S_t] = e^{(r-q)t} S_0. \tag{4.15}$$

*(iii) For $p = 2$, we obtain the second (risk-neutral) moment of the stock price:*

$$\mathbb{E}[S_t^2] = e^{2\rho t} \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t} c_n^2 \varphi_n(x).$$

*4.2.4. The Multiname Subordinate JDCEV Credit-Equity Model.* Suppose the stocks of $n$ firms follow the $n$-dimensional process (4.1) with each $X^i$, $i \in \{1, 2, \ldots, n\} = \Upsilon$, a JDCEV diffusion with $\beta_i < 0$, $a_i > 0$, $c_i \geq 0$, $b_i \geq 0$, and $\mu_i + b_i < 0$, $\mathcal{T}$ an

$n$-dimensional subordinator, and default times $\tau_i$ defined by (2.6) (see also Section 4.2.1) with the JDCEV killing rates of the form (4.8).

Our first result is the eigenfunction expansion for the joint survival probability for any subset of firms.

THEOREM 4.7. *(Survival Probability for a Subgroup of Firms) The joint survival probability for a subset* $\Xi = \{i_1, \ldots, i_k\}$, $k \leq n$, *of firms is given by the eigenfunction expansion:*

$$\mathbb{Q}(\tau_\Xi > t) = \sum_{\mathbf{n}\in\mathbb{N}^k} e^{-\phi_\Xi(\lambda^\Xi_{\mathbf{n}})t} c^{0,\Xi}_{\mathbf{n}} \varphi^\Xi_{\mathbf{n}}(\mathbf{x}^\Xi), \tag{4.16}$$

*where* $\tau_\Xi = \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}$ *is the first default time among firms with indexes in* $\Xi$, $\mathbf{x}^\Xi = (x_{i_1}, \ldots, x_{i_k})^\top$, $x_i \in \mathbb{R}_+$, *the eigenfunctions are*

$$\varphi^\Xi_{\mathbf{n}}(\mathbf{x}^\Xi) = \prod_{i\in\Xi} \varphi^i_{n_i}(x_i),$$

*where* $\varphi^i_n(x)$ *and* $\lambda^i_n$ *are JDCEV eigenfunctions and eigenvalues given by equation* (4.13) *for the JDCEV diffusion* $X^i$ *(with* $\varepsilon = -1$ *when* $\mu_i + b_i < 0$*), the expansions coefficients are*

$$c^{0,\Xi}_{\mathbf{n}} = \prod_{j\in\Xi} c^{0,j}_{n_j}, \tag{4.17}$$

*where the coefficients* $c^{0,j}_n$ *are given by* (4.14) *with* $p = 0$,

$$\phi_\Xi(\mathbf{u}^\Xi) := \phi_\Xi(u_{i_1}, u_{i_2}, \ldots, u_{i_k}) = \phi(\mathbf{1}_\Xi(1)u_1, \mathbf{1}_\Xi(2)u_2, \ldots, \mathbf{1}_\Xi(n)u_n),$$

*where* $\phi_\Xi$ *is the Laplace exponent of the k-dimensional subordinator* $(\mathcal{T}_i, i \in \Xi)$ *and* $\mathbf{1}_\Xi(i) = 1\,(0)$ *if* $i \in \Xi$ $(i \in \Xi^c,\ \Xi^c = \Upsilon \setminus \Xi)$, *and* $\lambda^\Xi_{\mathbf{n}} = (\lambda^i_n, i \in \Xi)$. *If the Laplace exponent of the d-dimensional subordinator satisfies the condition* (2.19), *then the convergence is uniform on compacts in* $\boldsymbol{x}$. *Otherwise the convergence is in* $L^2(E, m)$.

Next we present the expressions for pairwise risk neutral stock return correlations and default indicator correlations for a fixed time horizon $t > 0$. To simplify notation we write these results for the two-firm case ($n = 2$).

THEOREM 4.8. *(Risk-Neutral Stock Return and Default Indicator Correlations). Let* $S^i$, $i = 1, 2$, *be the defaultable stock price processes* (4.1) *with* $n = 2$.

*(i) The correlation of stock returns,* $R^i_t := (S^i_t/S^i_0) - 1$, *is given by:*

$$\mathrm{Corr}(R^1_t, R^2_t) = \frac{e^{(2r-q_1-q_2)t}\left(e^{-(\phi(-\mu_1,-\mu_2)-\phi(-\mu_1,0)-\phi(0,-\mu_2))t} - 1\right)}{\sqrt{\mathrm{Var}(R^1_t)\mathrm{Var}(R^2_t)}},$$

*where* $\mathrm{Var}(R^i_t) = \mathrm{Var}(S^i_t)/(S^i_0)^2$ *with* $\mathrm{Var}(S^i_t) = \mathbb{E}[(S^i_t)^2] - (\mathbb{E}[S^i_t])^2$ *with the first and second moments calculated from Proposition 4.5 and Corollary 4.6.*

*(ii) The correlation of default indicators is given by:*

$$\mathrm{Corr}\left(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}}\right) = \frac{\sum_{\mathbf{n}\in\mathbb{N}^2}\left(e^{-\phi(\lambda^1_{n_1},\lambda^2_{n_2})t} - e^{-\left(\phi(\lambda^1_{n_1},0)+\phi(0,\lambda^2_{n_2})\right)t}\right) c_{\mathbf{n}}\varphi_{\mathbf{n}}(\mathbf{x})}{\prod_{i=1}^2 \sqrt{\mathbb{Q}(\tau_i > t)(1 - \mathbb{Q}(\tau_i > t))}}$$

*with* $c_{\mathbf{n}} = c^{0,1}_{n_1} c^{0,2}_{n_2}$, *where* $c^{0,i}_n$ *are given by equation* (4.14) *with* $p = 0$.

The stock return and default indicator correlations are zero if and only if $\phi(u_1, u_2) = \phi(u_1, 0) + \phi(0, u_2)$, that is, if and only if the coordinates $\mathcal{T}^1$ and $\mathcal{T}^2$ of the two-dimensional subordinator are independent. In general, $\phi(u_1, u_2) \le \phi(u_1, 0) + \phi(0, u_2)$, so that $e^{-\phi(\lambda^1_{n_1}, \lambda^2_{n_2})t} - e^{-(\phi(\lambda^1_{n_1}, 0) + \phi(0, \lambda^2_{n_2}))t} \ge 0$. We also observe that the stock return covariance $\mathrm{Cov}(R^1_t; R^2_t)$ is equal to $e^{(2r-q_1-q_2)t}(e^{-(\phi(-\mu_1,-\mu_2)-\phi_1(-\mu_1)-\phi_2(-\mu_2))t} - 1)$ and is independent of the initial stock prices and is determined solely by the dependence among the coordinates of the two-dimensional subordinator, while the correlation depends on the initial stock prices through the return variances $\mathrm{Var}(R^i_t)$ in the denominator. Since under the JDCEV model both volatility and default probability increase as the stock price falls, the stock return and default correlations are decreasing functions of the stock prices. This is consistent with some empirical evidence discussed in Section 5 (see also Section 4.2.2. for additional references).

Let $\{\tau_1, \dots, \tau_n\}$ be the default times and let $\tau_{(1)} \le \tau_{(2)} \le \cdots \le \tau_{(n)}$ be their order statistics. Then $\tau_{(N)}$ is the time of the $N$th default, $N \le n$. Note that the $N$th default time can be defined as:

$$\tau_{(N)} = \inf\left\{t \ge 0 : N_t = \sum_{i=1}^{n} \mathbf{1}_{\{\tau_i \le t\}} \ge N\right\}, \tag{4.18}$$

where the inequality in $N_t \ge N$ is to account for the possibility of more than one default occurring simultaneously. In addition, define the indicator process for the $N$th-to-default time $\tau_{(N)}$:

$$D^{(N)}_t := 1_{\{t \ge \tau_{(N)}\}} = \big(N_t - (N-1)\big)^+ - \big(N_t - N\big)^+.$$

Observe that the expectation of $D^{(N)}_t$ is the probability of observing $N$ or more defaults by time $t$, that is, $\mathbb{E}[D^{(N)}_u] = \mathbb{P}[N_t \ge N]$. This probability can be represented by equation (4.4) with the function $f(S^1_t, \dots, S^n_t) = \big(\sum_{i=1}^n \mathbf{1}_{\{S^i_t = 0\}} - (N-1)\big)^+ - \big(\sum_{i=1}^n \mathbf{1}_{\{S^i_t=0\}} - N\big)^+$. Hence, we obtain the following result.

THEOREM 4.9. *(Distribution of the Number of Defaults) Let $S^k$, $k \in \{1, 2, \dots, n\} = \Upsilon$, be the defaultable stock price processes as given in* (4.1). *Let the default counting process $N_t$ be defined as in* (4.18). *Then, the (risk neutral) probability of observing at least N defaults by time t is given by*

$$\mathbb{P}\big(N_t \ge N\big) = 1 - \sum_{\Xi \subseteq \Upsilon} (-1)^{(|\Xi|-1)-(n-N)} \binom{|\Xi|-1}{n-N} \mathbf{1}_{\{|\Xi| \ge n+1-N\}} \sum_{\mathbf{n} \in \mathbb{N}^{|\Xi|}} e^{-\phi_\Xi(\lambda^\Xi_{\mathbf{n}})t} c^{0,\Xi}_{\mathbf{n}} \varphi^\Xi_{\mathbf{n}}(\mathbf{x}),$$

*where expansion coefficients $c^{0,\Xi}_{\mathbf{n}}$ are given in equation* (4.17).

The complementary cumulative distribution function (CDF) of the number of defaults given in Theorem 4.9 can be used to price *Nth-to-default swaps* also known as *basket default swaps*. An $N$th-to-default swap is a portfolio credit derivative that provides protection against the $N$th default in a portfolio of $n$ debt instruments, see Chen and Glasserman (2008) and Herbertsson and Rootzén (2008) for details on this credit derivative. The protection buyer makes periodic premium payments to the protection seller until the $N$th default event occurs in the reference portfolio or until contract maturity $T > 0$, whichever comes first. The stream of premium payments is called the *premium leg* of the swap. If the $N$th default occurs before maturity $T$, the premium payments cease and the protection seller makes a payment to the protection buyer. This is called the *protection leg* of the swap. The payment is determined by the identity of the $N$th asset to default. If the $i$th asset is the $N$th to default, the payment is equal to the loss-given-default

(LGD) on the $i$th asset $LGD_i = F_i(1 - R_i)$, where $F_i$ and $R_i$ are the face value and the recovery rate on the $i$th asset, respectively. For simplicity, assume that $F_i = 1$ and $R_i = R$, thus $LGD_i = L$, is the same constant for all $i = 1, \dots, n$. Then the $N$th-to-default swap's protection payoff is $LGD\, 1_{\{\tau_{(N)} \le T\}}$ and is paid at the time $\tau_{(N)}$ of the $N$th default. Then the present value of the protection leg is given by:

$$L\mathbb{E}[e^{-r\tau_{(N)}} 1_{\{\tau_{(N)}\le T\}}] = L\mathbb{E}[\int_0^T e^{-ru} dD_u^{(N)}] = L\left(e^{-rT}\mathbb{E}[D_T^{(N)}] + r\int_0^T e^{-ru}\mathbb{E}[D_u^{(N)}]du\right),$$

where the last equality is due to integration by parts, and we assumed that the risk-free discount rate $r$ is constant.

The buyer makes regular premium payments every $\delta$ years until the $N$th default occurs or the contract expires at $T = M\delta$, whichever event occurs first. Typically the payments are made quarterly, that is, $\delta = 0.25$. When $n = 1$ the $N$th-to-default swap is reduced to a *(single-name) credit default swap (CDS)*. When $N = 1$ and $n > 1$, the contract is the *first-to-default swap*. For $N = n > 1$ the contract is the *last-to-default swap*. To value the $N$th-to-default swap, observe that $(1 - D_t^{(N)})$ is equal to 1 if the $N$th default has not yet occurred by time $t$ and zero otherwise. Therefore the present value of the premium leg at inception is given by

$$\mathcal{S}^{(N)}\sum_{i=1}^{M} e^{-r\,\delta i}\mathbb{E}\Big[\int_{\delta(i-1)}^{\delta i}\Big(1 - D_t^{(N)}\Big)\,du\Big] = \mathcal{S}^{(N)}\sum_{i=1}^{M} e^{-r\,\delta i}\Big(\delta - \int_{\delta(i-1)}^{\delta i}\mathbb{E}\Big[D_u^{(N)}\Big]du\Big).$$

The *fair Nth-to-default swap rate* $\mathcal{S}^{(N)}$ is the break even rate that makes the present value of the swap equal to zero at the contract's inception.

PROPOSITION 4.10. *(Nth-to-Default Swap Rate) The fair Nth-to-default swap rate $\mathcal{S}^{(N)}$ of a basket containing $n \ge N$ firms is given by*

$$\mathcal{S}^{(N)} = \frac{PV(Protection\ Leg)}{PV(Premium\ Leg)(1)},$$

*where the present value of the protection and premium legs are given by the following expressions:*

*(i) Present value of the protection leg:*

$$L\Big[1 - \sum_{\Xi\subseteq\Upsilon}(-1)^{(|\Xi|-1)-(n-N)}\binom{|\Xi|-1}{n-N}\mathbf{1}_{\{|\Xi|\ge n+1-N\}}$$
$$\times\sum_{\mathbf{n}\in\mathbb{N}^{|\Xi|}}\Big(\frac{r + \phi_\Xi(\lambda_{\mathbf{n}}^{\Xi})e^{-(r+\phi_\Xi(\lambda_{\mathbf{n}}^{\Xi}))T}}{r + \phi_\Xi(\lambda_{\mathbf{n}}^{\Xi})}\Big)c_{\mathbf{n}}^{0,\Xi}\varphi_{\mathbf{n}}^{\Xi}(\mathbf{x})\Big].$$

*(ii) Present value of the premium leg with the unit swap rate:*

$$\sum_{\Xi\subseteq\Upsilon}(-1)^{(|\Xi|-1)-(n-N)}\binom{|\Xi|-1}{n-N}\mathbf{1}_{\{|\Xi|\ge n+1-N\}}$$
$$\times\sum_{\mathbf{n}\in\mathbb{N}^{|\Xi|}}\frac{\big(e^{\phi_\Xi(\lambda_{\mathbf{n}}^{\Xi})\delta} - 1\big)\big(1 - e^{-(r+\phi_\Xi(\lambda_{\mathbf{n}}^{\Xi}))T}\big)}{\phi_\Xi(\lambda_{\mathbf{n}}^{\Xi})\big(e^{(r+\phi_\Xi(\lambda_{\mathbf{n}}^{\Xi}))\delta} - 1\big)}\,c_{\mathbf{n}}^{0,\Xi}\varphi_{\mathbf{n}}^{\Xi}(\mathbf{x}).$$

The proof is based on the result in Theorem 4.9 and straightforward algebra and is omitted to save space.

Next we consider a *basket put option* on the portfolio of $n$ stocks with the payoff $f(S_t^1, \ldots, S_t^n) = (K - \sum_{i=1}^n w_i S_t^i)^+$ at expiration time $t$. In our multiname credit-equity model, the interesting aspect of basket options is that it is possible that some of the names in the basket go bankrupt with their stocks dropping to zero, while the other stocks in the basket continue on. According to Theorem 4.2, we can decompose the put payoff as follows:

$$f(S_t^1, \ldots, S_t^n) = \sum_{\Xi \subseteq \Upsilon} \mathbf{1}_{\{\tau_\Xi > t\}} f_\Xi(S_t^{i_1}, \ldots, S_t^{i_k}),$$

$$f_\Xi(S_t^{i_1}, \ldots, S_t^{i_k}) = \sum_{\Theta \subseteq \Xi} (-1)^{|\Xi| - |\Theta|} \Big(K - \sum_{i=1}^n \mathbf{1}_\Theta(i) w_i S_t^i\Big)^+.$$

Clearly, $\mathbf{1}_{\{\tau_\Xi > t\}} = \mathbf{1}_{\{\tau_\Theta > t\}} \mathbf{1}_{\{\tau_{\Xi \setminus \Theta} > t\}}$, since $\tau_\Xi = \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}$ for all $\Xi \subseteq \Upsilon$. Therefore, we write,

$$f(S_t^1, \ldots, S_t^n) = \sum_{\Xi \subseteq \Upsilon} \sum_{\Theta \subseteq \Xi} (-1)^{|\Xi| - |\Theta|} \Big(K - \sum_{i=1}^n \mathbf{1}_\Theta(i) w_i S_t^i\Big)^+ \mathbf{1}_{\{\tau_\Theta > t\}} \mathbf{1}_{\{\tau_{\Xi \setminus \Theta} > t\}}.$$

Since $\Xi = \emptyset \subset \Upsilon$ and $\Theta = \emptyset \subseteq \Xi$ and since $\tau_\emptyset = \infty$, we observe that the basket put option has embedded a default claim that pays $K$ dollars in the event in which all firms default (*last-to-default* claim paid at maturity $t$):

$$K \sum_{\Xi \subseteq \Upsilon} (-1)^{|\Xi|} \mathbf{1}_{\{\tau_\Xi > t\}} = K\, \mathbf{1}_{\{\tau_1 \vee \cdots \vee \tau_n \leq t\}}.$$

Therefore, using the definition (4.7) of $\hat{f}_\Xi(\mathbf{x})$, the price at time zero of the basket put option is determined by

$$BPut(S_0^1, \ldots, S_0^n) = e^{-rt} \sum_{\Xi \subseteq \Upsilon} \mathcal{P}_t^{\Xi, \phi} \hat{f}_\Xi(\mathbf{x}^\Xi)$$

$$= e^{-rt} \sum_{\Xi \subseteq \Upsilon} \sum_{\Theta \subseteq \Xi} (-1)^{|\Xi| - |\Theta|} \mathbb{E}\Big[\Big(K - \sum_{i=1}^n \mathbf{1}_\Theta(i) w_i e^{\rho_i t} X_t^i\Big)^+ \mathbf{1}_{\{\tau_\Theta > t\}} \mathbf{1}_{\{\tau_{\Xi \setminus \Theta} > t\}}\Big]$$

$$= e^{-rt} \sum_{\Xi \subseteq \Upsilon} \sum_{\Theta \subseteq \Xi} (-1)^{|\Xi| - |\Theta|} \mathcal{P}_t^{\Xi, \phi} \hat{f}_\Theta(\mathbf{x}^\Theta) \tag{4.19}$$

with $\tilde{f}_\Theta(\mathbf{x}^\Theta) = \big(K - \sum_{i \in \Theta} w_i e^{\rho_i t} x_i\big)^+$. In particular, $\hat{f}_\emptyset(\mathbf{x}^\emptyset) = K$, thus $\mathcal{P}_t^{\Xi, \phi} \hat{f}_\emptyset(\mathbf{x}^\emptyset) = K \mathcal{P}_t^{\Xi, \phi} 1$ and $\mathcal{P}_t^{\emptyset, \phi} \hat{f}_\emptyset(\mathbf{x}^\emptyset) = K$. The following theorem gives the eigenfunction expansion for the value function of the basket put option.

THEOREM 4.11. *(Basket Option) Let $S^k$, $k \in \{1, 2, \ldots, n\} = \Upsilon$, be the defaultable stock price processes in the multi-name subordinate JDCEV model. Then the value* *function* $BPut(S_0^1,\ldots,S_0^n)$ *of the basket put option with the payoff* $f(S_t^1,\ldots,S_t^n)=(K-\sum_{i=1}^n w_i S_t^i)^+$ *has the following eigenfunction expansion:*

$$e^{-rt}K\sum_{\Xi\subseteq\Upsilon}\sum_{\Theta\subseteq\Xi}(-1)^{|\Xi|-|\Theta|}\sum_{\mathbf{n}\in\mathbb{N}^{|\Theta|}}\sum_{\mathbf{m}\in\mathbb{N}^{|\Xi|-|\Theta|}}e^{-\phi_\Xi(\lambda_{\mathbf{n}}^{\Theta},\lambda_{\mathbf{m}}^{\Xi\backslash\Theta})t}p_{\mathbf{n}}^{\Theta}(K)c_{\mathbf{m}}^{0,\Xi\backslash\Theta}\varphi_{\mathbf{n}}^{\Theta}(\mathbf{x}^{\Theta})\varphi_{\mathbf{m}}^{\Xi\backslash\Theta}(\mathbf{x}^{\Xi\backslash\Theta}).$$

*Here we use the following conventions* $\sum_{\mathbf{n}\in\mathbb{N}^{|\emptyset|}}\sum_{\mathbf{m}\in\mathbb{N}^{|\Xi|-|\emptyset|}}=\sum_{\mathbf{m}\in\mathbb{N}^{|\Xi|}}$, $\sum_{\mathbf{n}\in\mathbb{N}^{|\Xi|}}\sum_{\mathbf{m}\in\mathbb{N}^{|\emptyset|}}=\sum_{\mathbf{n}\in\mathbb{N}^{|\Xi|}}$, *and, since* $\mathcal{P}_t^{\emptyset,\phi}\hat{f}_\emptyset(\mathbf{x}^\emptyset)=K$ *by convention, then* $\sum_{\mathbf{n}\in\mathbb{N}^{|\emptyset|}}\sum_{\mathbf{m}\in\mathbb{N}^{|\emptyset|}}=1$. *The notation* $\phi_\Xi(\lambda_{\mathbf{n}}^{\Theta},\lambda_{\mathbf{m}}^{\Xi\backslash\Theta})$ *for the Laplace exponent* $\phi_\Xi(\lambda_{\mathbf{k}}^{\Xi})$ *signifies that the set* $\Xi$ *is divided into two nonoverlapping subsets* $\Theta$ *and* $\Xi\backslash\Theta$. *The expansion coefficients* $c_{\mathbf{m}}^{0,\Xi\backslash\Theta}$ *are given in equation* (4.17). *The expansion coefficients* $p_{\mathbf{n}}^{\Theta}(K)$ *are given by (here* $\mathbb{N}_0=\{0,1,2,\ldots\}$*):*

$$\begin{aligned}p_{\mathbf{n}}^{\Theta}(K)&=\prod_{j\in\Theta}\sqrt{\frac{\Gamma(\nu_j+n_j)}{\Gamma(n_j)|\mu_j+b_j|}}\frac{2|\beta_j|A_j^{\frac{\nu_j}{2}+1}}{\Gamma(1+\nu_j)}\Big(\frac{e^{-\rho_j t}}{w_j}\Big)^{2c_j-2\beta_j}\\&\quad\times\sum_{\mathbf{p}\in\mathbb{N}_0^{|\Theta|}}\frac{K^{2\sum_{j\in\Theta}(c_j-\beta_j(1+p_j))}}{\Gamma\left(2+2\sum_{j\in\Theta}(c_j-\beta_j(1+p_j))\right)}\\&\quad\prod_{j\in\Theta}\frac{(-1)^{p_j}(\nu_j+n_j)_{p_j}\Gamma\left(2c_j-2\beta_j(1+p_j)\right)}{(1+\nu_j)_{p_j}\,p_j!}A_j^{p_j}\Big(\frac{e^{-\rho_j t}}{w_j}\Big)^{-2\beta_j p_j}.\end{aligned}\tag{4.20}$$

In particular, for a basket put option on two stocks, the payoff $f(S_t^1,S_t^2)=(K-w_1S_t^1-w_2S_t^2))^+$ can be decomposed as,

$$\begin{aligned}f(S_t^1,S_t^2)&=(K-w_1S_t^1-w_2S_t^2)^+=K+\mathbf{1}_{\{\tau_1>t\}}\left((K-w_1S_t^1)^+-K\right)+\mathbf{1}_{\{\tau_2>t\}}\left((K-w_2S_t^2)^+-K\right)\\&\quad+\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}\left((K-w_1S_t^1-w_2S_t^2)^++K-(K-w_1S_t^1)^+-(K-w_2S_t^2)^+\right).\end{aligned}\tag{4.21}$$

We observe that the basket put includes the embedded multiname credit derivative with the notional amount equal to the strike price $K$ and paid at maturity $t$ if both firms default by $t$:

$$K(\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}+1-\mathbf{1}_{\{\tau_1>t\}}-\mathbf{1}_{\{\tau_2>t\}})=K\mathbf{1}_{\{\tau_1\vee\tau_2\le t\}}.$$

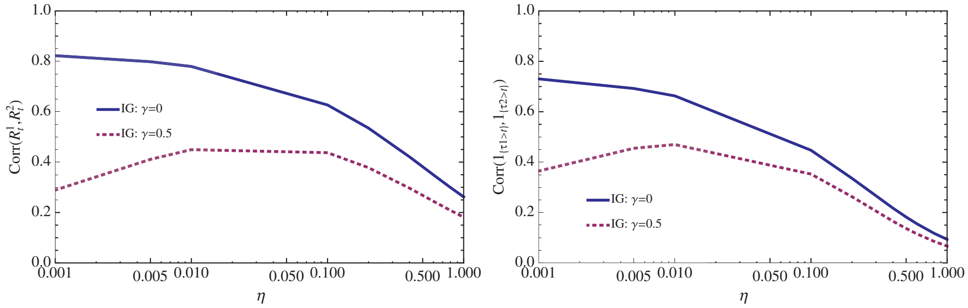
The price of this credit derivative is given by $e^{-rt}K(1+\mathbb{Q}(\tau_{\{1,2\}}>t)-\mathbb{Q}(\tau_1>t)-\mathbb{Q}(\tau_2>t))$, where the survival probabilities are given in Theorem 4.7.

To price the basket put on two firms (4.21), in addition to the embedded credit derivative, we further need to evaluate the prices of five component contingent claims: $\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K-w_1S_t^1+w_2S_t^2)^+$—the basket put that delivers the payoff if and only if both firms survive to maturity, $\mathbf{1}_{\{\tau_{\{1,2\}}>t\}}(K-w_kS_t^k)^+$, $k=1,2$—two single-name puts that deliver the payoffs if and only if both firms survive to maturity $t$, $\mathbf{1}_{\{\tau_k>t\}}(K-w_kS_t^k)^+$, $k=1,2$— two single-name puts that deliver the payoffs if and only if the firm whose stock the put payoff is made on survives to maturity. The price of each of the component claims can be explicitly represented by the eigenfunction expansions of Theorem 4.11.

We have presented the results for the basket put. The basket call is priced by the model-free put–call parity for basket options. For two firms it reads $BCall(t,S^1,S^2,K)-BPut(t,S^1,S^2,K)=e^{-q_1t}w_1S^1+e^{-q_2t}w_2S^2-e^{-rt}K$.

TABLE 5.1
JDCEV Parameter Values

| $X_0 = x$ | $a$ | $b$ | $c$ | $q$ | $\beta$ | $\mu$ | $r$ |
|---|---|---|---|---|---|---|---|
| 50 | 10 | 0.01 | 0.5 | 0 | $-1$ | $-0.3$ | 0.03 |




(a) Correlation of stock returns, $\mathrm{Corr}(R_t^1, R_t^2)$.

(b) Default correlation, $\mathrm{Corr}(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$.

FIGURE 5.1. One year correlation levels induced by different specifications of a single common subordinator $\mathcal{S}_t$. The decay parameter $\eta$ varies according to $\eta \in \{0.001, 0.005, 0.01, 0.1, 0.2, 0.4, 0.5, 0.6, 0.8, 1\}$. The scale factor $C$ is chosen such that for each specification, $\mathbb{E}[\mathcal{S}_1] = 1.5$.

## 5. NUMERICAL ILLUSTRATIONS

In this example, we consider a two-name defaultable stock model (4.1) with the two diffusions $X^i$ taken to be JDCEV with the same set of parameters given in Table 5.1. The volatility scale parameter $a$ in the local volatility function $\sigma(x) = ax^\beta$ is selected so that $\sigma(50) = 0.2$ (so that the volatility is 20% when $S = 50$).

For simplicity we consider a single-factor model for the 2D subordinator of the form $\mathcal{T}_t^i = \mathcal{S}_t$, $i = 1, 2$, where $\mathcal{S}_t$ is a 1D Inverse Gaussian (IG) subordinator. In other words, in this case both JDCEV diffusions are time changed with the common IG subordinator, $S_t^i = \mathbf{1}_{\{t<\tau_i\}} e^{\rho_i t} X^i_{\mathcal{S}_t}$. We consider IG subordinators without drift ($\gamma = 0$) and with drift ($\gamma = 0.5$). The IG Lévy measure is given by equation (2.15) with $\alpha = 1/2$. In our two examples we vary the tail decay parameter $\eta$, while also correspondingly varying the scale parameter $C$ so that the expected value of the IG subordinator at $t = 1$ remains fixed at $\mathbb{E}[\mathcal{S}_1] = 1.5$. We recall that the expected value of the IG subordinator with Lévy measure (2.15) with $\alpha \neq 0$ and drift $\gamma \geq 0$ is given by $\mathbb{E}[\mathcal{S}_t] = t\left(\gamma - \alpha C \Gamma(-\alpha)\eta^{\alpha-1}\right)$. We also recall that the scale parameter $C$ alters the arrival rates of jumps of all sizes simultaneously, while the parameter $\eta$ controls the decay rate of large jumps (see Cont and Tankov 2004, p. 115).

Figure 5.1 plots the stock return and default indicator correlations of Theorem 4.8 for the fixed initial stock prices $S_0^1 = S_0^2 = 50$ as we vary $\eta$ (while also varying $C$ to maintain $\mathbb{E}[\mathcal{S}_1] = 1.5$).[2] Recall that when the subordinator has zero drift ($\gamma = 0$), the subordinate

[2] All numerical examples in this section were produced in the Mathematica software package that includes all the special functions appearing in the eigenfunction expansions as built-in functions.

TABLE 5.2
IG Parameter Values

| | $\gamma$ | $\eta$ | $C$ |
|---|---|---|---|
| $\mathcal{S}_t^1$ | 0 | 1 | 0.7 |
| $\mathcal{S}_t^2$ | 0 | 1 | 0.7 |
| $\mathcal{S}_t^3$ | 0 | 0.001 | 0.025 |

process is a pure-jump process. When $\gamma > 0$, the subordinate process has both continuous and jump components. We observe that in the pure-jump case with the IG subordinator without drift, the correlations increase with decrease in $\eta$ due to the relative importance of large jumps in the subordinator in inducing correlation among the coordinates of the subordinate pure-jump process. When the subordinator has drift $\gamma > 0$ and, thus, the subordinate process has both continuous and jump components, the correlations are seen to be nonmonotone functions of $\eta$. The stock return and default correlations increase only up to a certain point as $\eta$ decreases, due to the increased importance of large jumps. Further decreasing $\eta$ beyond that point leads to decreased correlations. The reason for this behavior is as follows. Since we keep $\mathbb{E}[\mathcal{S}_1]$ fixed in this exercise by correspondingly varying $C$ when we vary $\eta$, the independent diffusion components become relatively more important as $C$ decreases to compensate for the decrease in $\eta$. That explains why the model with the IG subordinator with drift $\gamma = 0.5$ has generally lower correlation levels than the corresponding pure-jump model with $\gamma = 0$. However, we note that even in the jump-diffusion case with independent diffusion components, stock return and default indicator correlations can achieve substantial values.

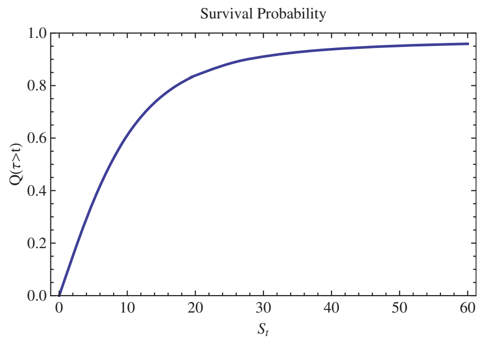
Next we illustrate dependence of correlations on the initial stock prices. In this example we consider the two-name defaultable stock model (4.1) with the same set of JDCEV parameters given in Table 5.1 and specify the 2D subordinator $\mathcal{T}$ as a linear combination of three independent subordinators $\mathcal{S}_t^i$, $i = 1, 2, 3$, as follows:

$$\mathcal{T}_t^k = 0.5\,\mathcal{S}_t^k + \mathcal{S}_t^3, \quad k = 1, 2. \tag{5.1}$$

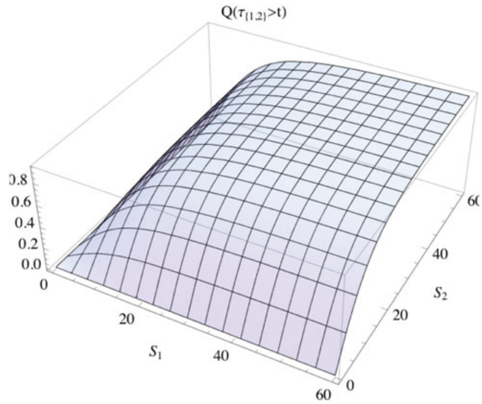
In this specification $\mathcal{S}^1$ and $\mathcal{S}^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and $\mathcal{S}_t^3$ is the systematic component common to both stocks. The three subordinators are taken to be pure-jump IG without drift and with parameters values given in Table 5.2

The parameters are selected so that the scale parameter $C$ of the idiosyncratic components is much larger than the scale parameter of the systematic component, but the Lévy measure tail decay parameter $\eta$ is also much large for the idiosyncratic components. Thus, for large $\epsilon$ the tail integral of the Lévy measure $\nu([\epsilon, \infty))$ is larger for the systematic component, while $\nu([\epsilon_1, \epsilon_2])$ is larger for the idiosyncratic components for moderate values of $0 < \epsilon_1 < \epsilon_2$. Intuitively, the idiosyncratic components primarily model the trading noise related to each stock, while the systematic component primarily models the arrival of less frequent but larger jumps (significant market events) that affect the market as a whole.

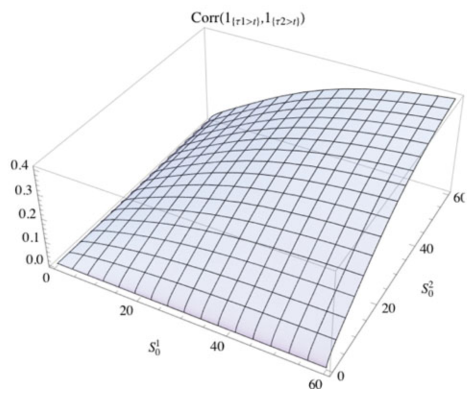
Since the 2D subordinator $\mathcal{T}$ is a pure-jump process, the subordinate process does not have any diffusion component. Thus, each stock is modeled by a pure-jump process,



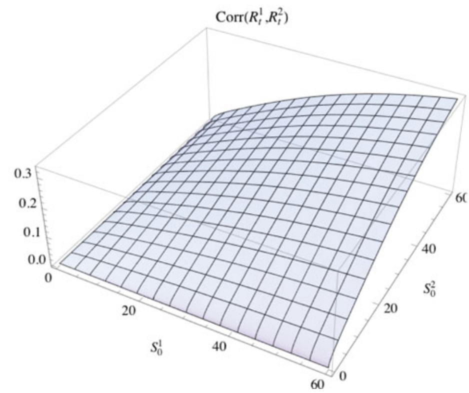

(a) Single-name survival probability, $\mathbb{Q}(\tau > t)$.




(b) Joint survival probability, $\mathbb{Q}(\tau_{\{1,2\}} > t)$.




(c) Default correlation, $\mathrm{Corr}(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$.




(d) Correlation of stock returns, $\mathrm{Corr}(R_t^1, R_t^2)$.

FIGURE 5.2. (a) Single-name survival probability $\mathbb{Q}(\tau > t)$, (b) joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$, (c) default correlation $\mathrm{Corr}(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}})$, and (d) correlation of stock returns $\mathrm{Corr}(R_t^1, R_t^2)$; for $t = 1$ year as functions of stock prices $S_0^1$ and $S_0^2$.

including the possibility of a jump to zero (jump to default). As the stock price falls, Figure 5.2(a) shows that the firm's survival probability decreases. As illustrated in Figure 5.2(b), as the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease (see Figure 5.2 c). When the stock price is relatively high, the survival probability is relatively high, and the default can only be triggered by a large catastrophic jump. The systematic component $\mathcal{S}^3$ primarily governs large jumps and, thus, primarily governs default arrival when the stock prices are high. In contrast, when the stock price is relatively low, a smaller jump can trigger default. The idiosyncratic components $\mathcal{S}^1$and $\mathcal{S}^2$ primarily govern small jumps. Thus, in this model specification, default from high stock price levels is primarily governed by the systematic factor, while default from low stock price levels is primarily governed by idiosyncratic factors. This results in the dependence of correlations on the stock price level, with default correlations increasing functions of stock prices (Figure 5.2 c and d).

This model behavior is in agreement with a number of empirical studies documenting that default correlations are higher for higher credit quality firms than for lower credit quality firms. In our single-factor model, the stock price is the state variable driving the default intensity and, thus, is *the* measure of credit quality of the firm.

According to the extensive empirical literature, lower credit quality firms often default on their own due to their idiosyncratic factors and generally have lower default correlations, while higher credit quality firms tend to exhibit higher default correlations, since it would generally take a significant systematic market shock to trigger default of a high quality firm. Lopez (2004) finds that the average asset correlation is a decreasing function of probability of default and an increasing function of the asset size (measured by the book value of assets). Xiao (2003) observes that credit spreads of higher rated bonds are more highly correlated than credit spreads of lower rated bonds. While analyzing the US credit crises of 2008, Koopman, Lucas, and Schwaab (2012) find that, on average across industries and time, 66% of total default risk is idiosyncratic, while the remaining 34% is systematic. However, for investment grade firms the percentage attributable to systematic factors is 60%. That supports the view that default events of lower credit quality firms tend to be more idiosyncratic, while those of higher credit quality firms tend to be more systematic. In fact, this is one of the assumptions of the Basel II Capital Accord. Recent analysis of Lee, Lin, and Yang (2011) corroborates the assumption that asset correlations are positively related to the firm size, but negatively related to the firm's default probability. In addition, these authors find that asset correlations are also industry specific, and that they tend to rise during economic downturns, but decline during economic upturns. The average magnitude of the rise is larger than that of the decline. That is, asset correlations are asymmetric and have a procyclical impact on the real economy. While the multiname subordinate JDCEV model in this section possesses the feature that higher credit quality firms (proxied by higher stock prices in this model) have higher correlations than lower credit quality firms (proxied by lower stock prices), this model does not exhibit the procyclical feature. It is a limitation of this model. One way to overcome this could be to introduce an additional common stochastic volatility factor following a mean-reverting diffusion process, such as via the absolutely continuous time change given by the integral of the CIR process, as described in Section 3.3. This stochastic volatility factor would drive the volatility of the diffusion and jump components of stock price processes, default intensities, and correlations.

Next we consider the pricing of two types of multi-name credit and equity derivatives: basket put options (options on the portfolio (basket) of stocks) and $N$th-to-default swaps. Table 5.3 and Figure 5.3 present the price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 50$ as a function of the initial stock prices $S_0^1$ and $S_0^2$. The parameters for the JDCEV process $X$ in credit-equity model are as in Table 5.1. The 2D subordinator is constructed as in (5.1) with the parameters of Table 5.2. When both firms are in default, $(S_0^1, S_0^2) = (0, 0)$, the price of the basket put is equal to the discounted strike $K$ (discounted because the strike is paid at maturity in the European put contract). When one of the two firms is in default, the basket put reduces to the single-name European-style put on the surviving stock with the strike $K$.

Next we present basket put options on the portfolios of 10 and 20 firms. To price a basket option on a portfolio of this size using the eigenfunction expansion is computationally infeasible due to the need to evaluate eigenfunction expansions for $2^n - 1$ marginal semigroups, each of the semigroups requiring evaluation of a multidimensional eigenfunction expansion. The only feasible alternative for high-dimensional problems is Monte Carlo simulation. Fortunately, our modeling framework is well suited for *exact simulation* due to the independence of the multivariate time change (the $n$-dimensional subordinator)

TABLE 5.3
Two-Name Basket Put Prices for the Range of Initial Stock Prices $S_0^1$ and $S_0^2$ from Zero to \$50 for One Year Time to Maturity and $K = 50$

| $S_0^1 \setminus S_0^2$ | 0.00 | 5.00 | 10.00 | 15.00 | 20.00 | 25.00 | 30.00 | 35.00 | 40.00 | 45.00 | 50.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 48.52 | 43.52 | 38.52 | 33.53 | 28.53 | 23.55 | 18.63 | 13.94 | 9.84 | 6.79 | 4.83 |
| 5.00 | 43.52 | 38.49 | 33.55 | 28.60 | 23.76 | 19.13 | 14.75 | 10.89 | 7.77 | 5.42 | 4.02 |
| 10.00 | 38.52 | 33.55 | 28.73 | 23.89 | 19.27 | 15.08 | 11.32 | 8.30 | 6.03 | 4.28 | 3.34 |
| 15.00 | 33.53 | 28.60 | 23.88 | 19.25 | 15.04 | 11.45 | 8.46 | 6.24 | 4.67 | 3.42 | 2.80 |
| 20.00 | 28.53 | 23.76 | 19.27 | 15.04 | 11.42 | 8.56 | 6.33 | 4.77 | 3.70 | 2.81 | 2.40 |
| 25.00 | 23.55 | 19.13 | 15.08 | 11.45 | 8.56 | 6.43 | 4.85 | 3.76 | 3.03 | 2.38 | 2.10 |
| 30.00 | 18.63 | 14.75 | 11.32 | 8.46 | 6.33 | 4.85 | 3.76 | 3.02 | 2.52 | 2.05 | 1.85 |
| 35.00 | 13.94 | 10.89 | 8.30 | 6.24 | 4.77 | 3.76 | 3.02 | 2.50 | 2.14 | 1.80 | 1.64 |
| 40.00 | 9.84 | 7.77 | 6.04 | 4.69 | 3.70 | 3.03 | 2.52 | 2.14 | 1.87 | 1.61 | 1.48 |
| 45.00 | 6.79 | 5.42 | 4.28 | 3.42 | 2.81 | 2.38 | 2.05 | 1.80 | 1.61 | 1.44 | 1.34 |
| 50.00 | 4.83 | 4.02 | 3.34 | 2.80 | 2.40 | 2.10 | 1.85 | 1.64 | 1.48 | 1.34 | 1.26 |

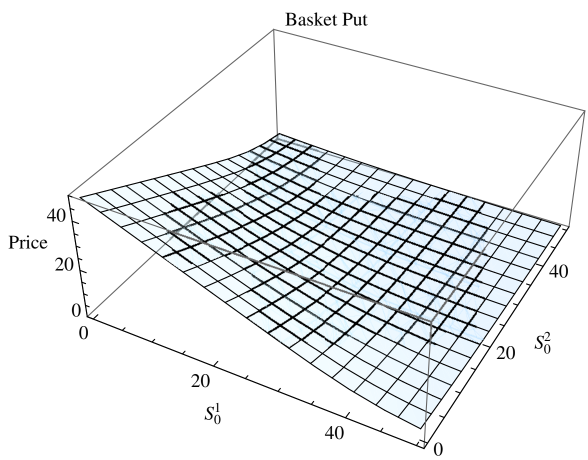



FIGURE 5.3. Two-name basket put prices for the range of initial stock prices $S_0^1$ and $S_0^2$ from zero to \$50 for one year time to maturity and $K = 50$.

and the $n$ JDCEV diffusions, as well as the availability of the analytical expression for the CDF of the JDCEV process. We thus separately sample the subordinator at the expiration time of the option $t > 0$ (our linear factor model for $n$-dimensional subordinators is particularly convenient for simulation, as it only requires simulating independent 1D subordinators $\mathcal{S}^i$) and the $n$ JDCEV diffusions $(X^1_{\mathcal{T}_t^1}, \ldots, X^n_{\mathcal{T}_t^n})$ at times $(\mathcal{T}_t^1, \ldots, \mathcal{T}_t^n)$ generated in the first step of sampling the subordinator. We then evaluate the basket put payoff on the vector of stock prices at option expiration $t$ obtained by equation (4.1). We stress that we do not have any discretization error stemming from discretizing the sample path, such as in the Euler scheme for solving numerically stochastic differential equations. Instead, we directly sample from the known JDCEV distribution at the fixed

TABLE 5.4
Basket Put Option Prices Using $10^5$ Samples

| $n$ | $BPut(S_0^1, \ldots, S_0^d)$ | Std. error |
|---|---|---|
| 2 | 6.402 | 0.0368 |
| 10 | 4.341 | 0.0236 |
| 20 | 3.574 | 0.0206 |

*Note* :
[a]When $n = 2$, the price corresponds to the basket put option of Table 5.3 with $S_0^1 = S_0^2 = 25$. The exact value for this option obtained by the eigenfunction expansion of Theorem 4.11 is $BPut(25, 25) = 6.4346$

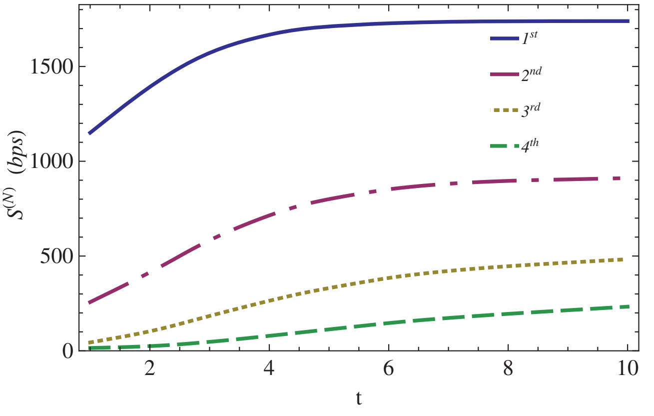



FIGURE 5.4. Fair $N$th-to-default swap rate $\mathcal{S}^{(N)}$ (bps) as a function of maturity time $t$ (years).

times given by the independently simulated subordinator $(\mathcal{T}_t^1, \ldots, \mathcal{T}_t^n)$. Thus, this simulation algorithm belongs to the class of *exact simulation* methods with no discretization error for the process.

For our numerical illustration with $n = 10$ and 20 firms, we randomly generate the set of parameters $\theta = (S_0^i, a_i, b_i, q_i, \beta_i, \mu_i, c_i)$, $i = 1, \ldots, n$, from the hypercube generated by $(\underline{\theta}, \overline{\theta})$, with $\underline{\theta} = (25, 5, 0.01, 0, -1.5, -0.4, 0.5)$ and $\overline{\theta} = (75, 15, 0.05, 0.02, -0.5, -0.2, 1.5)$. The set of portfolio weights $w_i$, $i = 1, \ldots, n$, were chosen such that the put option written on the basket portfolio is at-the-money. That is, by choosing $w_i = K/(n\, S_0^i)$ we obtain $\mathcal{B}_0 = \sum_{k=1}^n w_i S_0^i = K$, where $K = 50$ is the strike price. We assume that the interest rate is $r = 0.03$. We construct the $n$-dimensional subordinator using the linear factor model $\mathcal{T}_t^i = A_{i,1}\mathcal{S}_t^1 + A_{i,2}\mathcal{S}_t^2 + A_{i,3}\mathcal{S}_t^3$, $i = 1, \ldots, n$, with three IG subordinators of Table 5.2. The elements of the matrix $A$ are randomly chosen and are normalized so that $A_{i,1} + A_{i,2} + A_{i,3} = 1$, $i = 1, \ldots, n$. We generate $10^5$ samples. The results are given in Table 5.4. The basket options with 10 and 20 stocks were calculated with different sets of parameters, so the comparison of prices is not meaningful. The simulation with $10^5$ samples results in option price estimate with the standard errors of about two pennies, which constitutes practically reasonable accuracy.

TABLE 5.5
Parameter Set in the Calculation of the Fair $N$th-to-Default Swap Rate

(a) JDCEV parameters with $\beta_i = -1$ and $q_i = 0$, $i = 1, \ldots, 4$.

| $X_t^i$ | $X_0 = x$ | $a$ | $b$ | $c$ | $\mu$ |
|---|---|---|---|---|---|
| 1 | 100 | 20 | 0.02 | 0.5 | −0.4 |
| 2 | 95 | 25 | 0.01 | 1 | −0.3 |
| 3 | 90 | 20 | 0.02 | 0.5 | −0.4 |
| 4 | 85 | 25 | 0.01 | 1 | −0.3 |

(b) Factor loadings in the linear factor model, $\mathcal{T}_t^i = \sum_{a=1}^5 A_{i,a}\mathcal{S}_t^a$, $i = 1, \ldots, 4$.

| $A_{i,a}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.25 | 0.5 | 0 | 0 | 0.25 |
| 2 | 0.5 | 0.25 | 0 | 0 | 0.25 |
| 3 | 0 | 0 | 0.25 | 0.5 | 0.25 |
| 4 | 0 | 0 | 0.5 | 0.25 | 0.25 |

(c) Independent subordinators' parameter values.

| | $\gamma$ | $Y$ | $\eta$ | $C$ |
|---|---|---|---|---|
| $\mathcal{S}_t^1$ | 0 | 0.5 | 1 | 0.5 |
| $\mathcal{S}_t^2$ | 0 | 0.5 | 1 | 1 |
| $\mathcal{S}_t^3$ | 0 | 0.5 | 0.01 | 0.05 |
| $\mathcal{S}_t^4$ | 0 | 0.5 | 0.01 | 0.1 |
| $\mathcal{S}_t^5$ | 0.5 | −1 | 0.01 | 0.00005 |

To benchmark our simulation algorithm, we also re-price our previous example of a basket option with $n = 2$ firms. The parameters of the JDCEV processes $X_t^i$, $i = 1, 2$, are given as before in Table 5.2, and we set $S_0^1 = S_0^2 = 25$. We observe that our Monte Carlo estimate of the price given in the table is within one standard error of the exact price obtained by evaluating our analytical solution for basket options given in Theorem 4.11 that yields $BPut(25, 25) = 6.4346$ (see also Table 5.3). From the standpoint of numerical analysis, the usefulness of analytical solutions (that can usually only be obtained in special cases) is in providing highly accurate benchmarks for numerical methods that may have more general domain of applicability.

Our final example is the determination of the fair $N$th-to-default swap rate $\mathcal{S}^{(N)}$ for a four-name defaultable stock model (4.1) using analytical eigenfunction expansions obtained in Section 4.2. Each JDCEV process, $X_t^i$, $i = 1, \ldots, 4$, is specified by the set of parameters of Table 5.5(a). The 4D subordinator is constructed by the linear factor model (2.14) with five independent subordinators of Table 5.5(c) and the set of factor loadings of Table 5.5(b). We assume that the premium payments are made quarterly, that is, $\delta = 0.25$, and that the $LGD$ is 35% for all firms. Table 5.6 and Figure 5.4 show the fair $N$th-to-default swap rates for $N = 1, \ldots, 4$, as a function of maturity time $t$. While in this example with four names we have used the eigenfunction expansions, to value multiname credit derivatives with large number of names in the portfolio, Monte Carlo simulation

TABLE 5.6
Fair $N$th-to-Default Swap Rate $S^{(N)}$ (bps) as a Function of Maturity Time $t$ (Years)

| $N$th \ $t$ | 1 | 2 | 3 | 4 | 5 | 7 | 10 |
|---|---|---|---|---|---|---|---|
| 1st | 1150 | 1392 | 1572 | 1668 | 1711 | 1736 | 1739 |
| 2nd | 257 | 414 | 583 | 714 | 800 | 881 | 911 |
| 3rd | 44 | 103 | 184 | 264 | 331 | 421 | 483 |
| 4th | 15 | 25 | 47 | 79 | 113 | 173 | 233 |

has to be used. The analytical results for lower dimensional problems serve as useful benchmarks for simulation algorithms. We plan to further explore efficient simulation methods for high-dimensional credit-equity problems in a future article.

## 6. CONCLUSION

This paper proposed a modeling framework based on multivariate time changes of Markov processes. Starting with $d$ independent Markov processes with killing (e.g., diffusions) and time changing each of them with a coordinate of a $d$-dimensional Lévy subordinator, we obtain a Markov process with jumps and killing in the product space. Dependence among jumps is governed by the Lévy measure of the $d$-dimensional subordinator. This construction can be extended to introduce time inhomogeneity via additive subordinators and stochastic volatility via absolutely continuous time changes. When the semigroups of the underlying Markov processes have analytically tractable eigenfunctions expansions, the resulting time changed process turns out to be analytically tractable as well, facilitating derivatives pricing. Fortunately, many processes important in mathematical finance, including OU, CIR, and (JD)CEV, possess eigenfunction expansions, yielding analytically tractable building blocks for this framework. As applications, we sketch a model for correlated commodities that is further developed in Li and Mendoza-Arriaga (2013b) and develop in detail a multiname unified credit-equity model that describes joint dynamics of the stock prices of multiple firms, as well as their default times, as a multidimensional Markov process with jumps constructed by multivariate subordination of jump-to-default extended CEV (JDCEV) diffusions. Each of the stock prices experiences state-dependent jumps with the leverage effect (arrival rates of large jumps increase as the stock price falls), including the possibility of a jump to zero (jump to default). Some of the jumps are idiosyncratic to each firm, while some are either common to all firms (systematic), or common to a subgroup of firms. Pricing of multiasset equity and credit derivatives is accomplished via eigenfunction expansions. Further applications include more general multiname default intensity models with jumps and multifactor interest rate models with jumps and are left for future research.

## APPENDIX: PROOFS

*Proof of Theorem 2.3.* The proof of the strong Markov property follows by direct extension of the proof of theorem 15 on pp. 67–68 in Bouleau (1984) for univariate subordination of a single Markov process with a 1D subordinator to multivariate subordination of a set of $d$ independent Markov processes with a $d$-dimensional subordinator.

The representation (2.5) for the transition semigroup follows from Theorem 2.2 characterizing the subordinate semigroup. □

*Proof of Theorem 2.4.* Observing the identity

$$1=\sum_{\Theta\subseteq\Upsilon}\Big(\prod_{i\in\Theta}\mathbf{1}_{\{\tau_i>t\}}\prod_{j\in\Theta^c}(1-\mathbf{1}_{\{\tau_j>t\}})\Big),$$

we can write

$$\tilde{\mathcal{P}}_t^{\phi}f(x_1,\dots,x_d)=\mathbb{E}_{x_1,\dots,x_d}[f(\tilde{X}_t^{1,\phi},\dots,\tilde{X}_t^{d,\phi})]$$

$$=\sum_{\Theta\subseteq\Upsilon}\mathbb{E}_{x_1,\dots,x_d}\Big[\prod_{i\in\Theta}\mathbf{1}_{\{\tau_i>t\}}\prod_{j\in\Theta^c}(1-\mathbf{1}_{\{\tau_j>t\}})f(\mathbf{1}_{\Theta}(1)\tilde{X}_t^{1,\phi}+\mathbf{1}_{\Theta^c}(1)\Delta_1,\dots,\mathbf{1}_{\Theta}(d)\tilde{X}_t^{d,\phi}+\mathbf{1}_{\Theta^c}(d)\Delta_d)\Big]$$

$$=\sum_{\Theta\subseteq\Upsilon}\sum_{\Phi\subseteq\Theta^c}(-1)^{|\Phi|}\mathbb{E}_{x_1,\dots,x_d}\Big[\prod_{i\in\Theta\cup\Phi}\mathbf{1}_{\{\tau_i>t\}}f(\mathbf{1}_{\Theta}(1)\tilde{X}_t^{1,\phi}+\mathbf{1}_{\Theta^c}(1)\Delta_1,\dots,\mathbf{1}_{\Theta}(d)\tilde{X}_t^{d,\phi}+\mathbf{1}_{\Theta^c}(d)\Delta_d)\Big]$$

$$=\sum_{\Xi\subseteq\Upsilon}\sum_{\Theta\subseteq\Xi}(-1)^{|\Xi|-|\Theta|}\mathbb{E}_{x_1,\dots,x_d}\Big[\mathbf{1}_{\{\tau_\Xi>t\}}f(\mathbf{1}_{\Theta}(1)\tilde{X}_t^{1,\phi}+\mathbf{1}_{\Theta^c}(1)\Delta_1,\dots,\mathbf{1}_{\Theta}(d)\tilde{X}_t^{d,\phi}+\mathbf{1}_{\Theta^c}(d)\Delta_d)\Big]$$

$$=\sum_{\Xi\subseteq\Upsilon}\mathbb{E}_{x_{i_1},\dots,x_{i_k}}[\mathbf{1}_{\{\tau_\Xi>t\}}f_\Xi(X_t^{i_1,\phi},\dots,X_t^{i_k,\phi})]=\sum_{\Xi\subseteq\Upsilon}\mathcal{P}_t^{\Xi,\phi}f_\Xi(x_{i_1},\dots,x_{i_k}).$$ □

*Proof of Theorem 2.5.* Recall that a strongly continuous positive contraction semigroup on $C_0(E)$ is a Feller semigroup (cf. Kolokoltsov 2004, p. 115). By Theorem 2.2, if $X^i$ are Feller processes with Feller semigroups $\mathcal{P}^i$, the subordinate semigroup $\mathcal{P}_t^{\phi}$ of $\mathcal{P}^i$ with respect to the $d$-dimensional subordinator $\mathcal{T}$ is a strongly continuous one-parameter semigroup on $C_0(E)$. Similar to the proof of corollary 4.3.4 in Jacob (2001) for the subordination of one-parameter Feller semigroup, to show that $(\mathcal{P}_t^{\phi})$ is Feller, it suffices to show that it is a semigroup of positive contractions on $C_0(E)$. First observe that since each $\mathcal{P}^i$ is a Feller semigroup, then $\mathcal{P}_{\mathbf{t}}:=\Pi_{i=1}^d\mathcal{P}_{t_i}^i$ is a multiparameter Feller semigroup; see Proposition 2.1 in this paper and theorem 2.2.1 in Khoshnevisan (2002, p. 403). Let $u(\mathbf{x})\in C_0(E)$ and $u(\mathbf{x})\geq 0$ for all $\mathbf{x}\in E$. Then $\mathcal{P}_{\mathbf{t}}u(\mathbf{x})\geq 0$ for all $\mathbf{t}$ since $\mathcal{P}^i$ are all positive semigroups. Furthermore, $\mathcal{P}_{\mathbf{t}}1\leq 1$ since for each $\mathcal{P}_t^i 1\leq 1$. Thus, $(\mathcal{P}_{\mathbf{t}})$ is a multiparameter semigroup of positive contractions. Since $\pi_t(d\mathbf{s})$ is a positive transition kernel with $\pi_t(\mathbb{R}_+^d)=1$, integrating $\mathcal{P}_{\mathbf{s}}u(\mathbf{x})\geq 0$ against $\pi_t(d\mathbf{s})$ yields a nonnegative function. Therefore, $(\mathcal{P}_t^{\phi})$ is a strongly continuous one-parameter semigroup of positive contractions on $C_0(E)$. Therefore, it is a Feller semigroup. A Markov process with the Feller transition semigroup is a Feller process (theorem 2.7 in Ethier and Kurtz 1986, p. 169) and, hence, $X^{\phi}$ is a Feller process. □

*Proof of Theorem 2.6.* If $C_c^{\infty}(\mathbb{R}^{n_i})$ are in the domains of $\mathcal{G}_i$ for each $i$, then $C_c^{\infty}(\mathbb{R}^n)$ with $n=n_1+\ldots+n_d$ is in the domain of $\mathcal{G}^{\phi}$. This follows from the statement about the domains $\bigcap_{i=1}^d D(\mathcal{G}_i)\subset D(\mathcal{G}^{\phi})$ in Theorem 2.2. The explicit form of the generator $\mathcal{G}^{\phi}$ (2.10)–(2.13) is obtained as follows. Assume the estimates (2.8) and (2.9) are satisfied for each $P_t^i(x,dy)$. Then the integral in (2.4) can be rewritten as follows:

$$\int_{\mathbb{R}_+^d\setminus\{0\}}\Big(\int_{\mathbb{R}^n}f(\mathbf{y})\prod_{i=1}^d P_{s_i}^i\big(x_i,dy_i\big)-f(\mathbf{x})\Big)\nu(d\mathbf{s})$$

$$= \int_{\mathbb{R}_+^d \setminus \{0\}} \Big( \int_{\mathbb{R}^n} \Big( f(\mathbf{y}) - f(\mathbf{x}) - \mathbf{1}_{\{\|\mathbf{y}-\mathbf{x}\| \le 1\}} \sum_{i=1}^{d} \sum_{a_i=1}^{n_i} (y_{i,a_i} - x_{i,a_i}) \frac{\partial f}{\partial x_{i,a_i}}(\mathbf{x})$$

$$+ f(\mathbf{x}) + \mathbf{1}_{\{\|\mathbf{y}-\mathbf{x}\| \le 1\}} \sum_{i=1}^{d} \sum_{a_i=1}^{n_i} (y_{i,a_i} - x_{i,a_i}) \frac{\partial f}{\partial x_{i,a_i}}(\mathbf{x}) \Big) \prod_{i=1}^{d} P_{s_i}^i \big(x_i, dy_i\big) - f(\mathbf{x}) \Big) \nu(d\mathbf{s})$$

$$= \int_{\mathbb{R}^n} \Big( f(\mathbf{y}) - f(\mathbf{x}) - \mathbf{1}_{\{\|\mathbf{y}-\mathbf{x}\| \le 1\}} \sum_{i=1}^{d} \sum_{a_i=1}^{n_i} (y_{i,a_i} - x_{i,a_i}) \frac{\partial f}{\partial x_{i,a_i}}(\mathbf{x}) \Big) \int_{\mathbb{R}_+^d \setminus \{0\}} \prod_{i=1}^{d} P_{s_i}^i \Big(x_i, dy_i\Big) \nu(d\mathbf{s})$$

$$+ \int_{\mathbb{R}_+^d \setminus \{0\}} \int_{\mathbb{R}^n} \Big( (\mathbf{1}_{\{\|\mathbf{y}-\mathbf{x}\| \le 1\}} \sum_{i=1}^{d} \sum_{a_i=1}^{n_i} (y_{i,a_i} - x_{i,a_i}) \frac{\partial f}{\partial x_{i,a_i}}(\mathbf{x}) \Big) \prod_{i=1}^{d} P_{s_i}^i \Big(x_i, dy_i\Big) \nu(d\mathbf{s})$$

$$- f(\mathbf{x}) \int_{\mathbb{R}_+^d \setminus \{0\}} \Big( 1 - \prod_{i=1}^{d} P_{s_i}^i \Big(x_i, \mathbb{R}^{n_i}\Big) \Big) \nu(d\mathbf{s}). \tag{A.1}$$

In the first equality we added and subtracted terms in the integrand. In the second equality we broke up the resulting integral into three integrals and exchanged the order of integrations in the first of the three integrals. These operations are justified and the three resulting integrals are well defined due to the estimates (2.8) and (2.9) for each $i = 1, \dots, d$ and the integrability of the Lévy measure of the subordinator $\int_{\mathbb{R}_+^d \setminus \{0\}} (1 \wedge |\mathbf{s}|) \nu(d\mathbf{s}) < \infty$. Specifically, the estimate (2.8) and the first of the three estimates in (2.9) for each $i$ ensure that the first integral on the right-hand side of the last equality in (A.1) is well defined for each $f \in C_c^\infty(\mathbb{R}^n)$, as they ensure that the measure $\Pi^\phi(x, dy)$ (2.13) is a Lévy measure for each $x$; the proof is similar to Sato (1999, p. 200-1 proof that (30.8) is the Lévy measure of the subordinate Lévy process). The second estimate in (2.9) for each $i$ ensures that the second integral in the last equality in (A.1) is well defined for each $f \in C_c^\infty(\mathbb{R}^n)$, as it ensures that the drift (2.11) is well defined; the proof is similar to Sato (1999, proof that the drift (30.9) of the subordinate Lévy process is well defined). Finally, the third estimate in (2.9) for each $i$ ensures that the third integral in (A.1) is well defined for each $f \in C_c^\infty(\mathbb{R}^n)$, as it ensures that the killing rate (2.12) is well defined due to the integrand tending to zero at the rate $|\mathbf{s}|$ as $|\mathbf{s}| \to 0$. Finally, equation (2.10) follows by substituting our result for the integral and the expressions (2.7) for the generators $\mathcal{G}_i$ into equation (2.4). □

*Proof of Theorem 2.9.* We first observe that since $(\varphi_{n_i}^i)_{n_i \ge 1}$ are complete orthonormal bases in $L^2(E_i, m_i)$, $(\varphi_{\mathbf{n}}(\mathbf{x}))_{\mathbf{n} \in N^d}$ is a complete orthonormal basis in $L^2(E, m)$ (cf. Prugovecki 1981, section II.6.5). Moreover, for each $t > 0$ we have $\mathcal{P}_t^i \varphi_{\mathbf{n}}(\mathbf{x}) = e^{-t\lambda_{n_i}^i} \varphi_{\mathbf{n}}(\mathbf{x})$ (recall that we use the same notation for the operator on $L^2(E_i, m_i)$ and its extension to $L^2(E, m)$) and $\mathcal{P}_{\mathbf{s}} \varphi_{\mathbf{n}}(\mathbf{x}) = e^{-\sum_{i=1}^d s_i \lambda_{n_i}^i} \varphi_{\mathbf{n}}(\mathbf{x})$, where $\mathcal{P}_{\mathbf{s}} = \mathcal{P}_{s_1}^1 \dots \mathcal{P}_{s_d}^d$ (recall that $\mathcal{P}_{s_i}^i$ are bounded operators and commute with each other). From equation (2.3), for any $f \in L^2(E, m)$ and $t > 0$ we have

$$\mathcal{P}_t^\phi f(\mathbf{x}) = \int_{R_+^d} \big(\mathcal{P}_{s_1}^1 \dots \mathcal{P}_{s_d}^d f\big)(\mathbf{x}) \pi_t(d\mathbf{s}) = \int_{R_+^d} \Bigg( \sum_{\mathbf{n} \in \mathbb{N}^d} e^{-\sum_{i=1}^d s_i \lambda_{n_i}^i} f_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x}) \Bigg) \pi_t(d\mathbf{s})$$

$$= \sum_{\mathbf{n} \in \mathbb{N}^d} \Bigg( \int_{R_+^d} e^{-\sum_{i=1}^d s_i \lambda_{n_i}^i} \pi_t(d\mathbf{s}) \Bigg) f_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^d} e^{-\phi(\lambda_{n_1}^1, \dots, \lambda_{n_d}^d)t} f_{\mathbf{n}} \varphi_{\mathbf{n}}(\mathbf{x}),$$

where we used the eigenfunction expansions for each $\mathcal{P}_t^i$, the condition (2.19) for the Laplace exponent of the subordinator, Cauchy–Schwartz $|f_{\mathbf{n}}| \le \|f\|_{L^2(E,m)}$, and the bound on eigenfunctions $|\varphi_n^i(x)| \le C_K^i$ independent on $n$ for each compact set $K \subset \mathbb{R}^{n_i}$ to justify the interchange of the summation and integration, and the Lévy–Khintchine formula (2.1) for the subordinator. The expansion is uniformly convergent in $\mathbf{x}$ on compacts in $E$ for all $t > 0$. The bilinear expansion (2.20) is shown similarly. It is then immediate that $\mathcal{P}_t^\phi$ is a symmetric trace-class operator on $L^2(E, m)$ for each $t > 0$. □

*Proof of Theorem 4.3(iii).* Applying the the Hille–Hardy formula for Laguerre polynomials (Erdelyi (1953), p.189; valid for all $|t| < 1$, $\nu > -1$, $a, b > 0$)

$$\sum_{n=0}^{\infty} \frac{t^n n!}{\Gamma(n+\nu+1)} L_n^\nu(a)\, L_n^\nu(b) = \frac{(abt)^{-\nu/2}}{1-t} \exp\left\{-\frac{(a+b)\,t}{1-t}\right\} I_\nu\left(\frac{2\sqrt{abt}}{1-t}\right)$$

to the transition density (4.9) yields the spectral representation (2.18) with $\lambda_n$ and $\varphi_n(x)$ given by (4.13). □

*Proof of Lemma 4.4.* When $\mu + b < 0$, $\int_0^\infty x^{2p} m(x)dx < \infty$ for $p > 1/2 + \beta - c$. Moreover, one observes that for all $a > 0$ the integral $\int_0^\infty \mathbf{1}_{[a,\infty)}(x) x^{2p} m(x)dx$ is finite for all $p$ (i.e., $\mathbf{1}_{[a,\infty)}(x)x^p \in L^2(E, m)C$). Thus, by letting $a \to 0$ the expansion (4.14) remains valid for each $t > 0$ (strictly positive) as long as the coefficients $c_n^p(a) = (\mathbf{1}_{[a,\infty)}(x)x^p, \varphi_n)$ are finite, which relaxes the restriction on $p$ to $p > 2(\beta - c)$. The restriction $t > 0$ is required since the expansion fails to converge at $t = 0$. The expansion coefficients are given by (for $\mu + b < 0$):

$$c_n^p = \int_0^\infty y^p \varphi_n(y) m(y)dy = \frac{2}{a^2} A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}} \int_0^\infty y^{p+2c-2\beta-1} e^{-Ay^{-2\beta}}\, L_{n-1}^{(\nu)}(Ay^{-2\beta})dy.$$

The integral is calculated by doing the change of variable $x = y^{-2\beta}$ to reduce it to the integral (Prudnikov, Brychkov, and Marichev 1983, p. 463; valid for $\mathrm{Re}(c)$, $\mathrm{Re}(\alpha) > 0$):

$$\int_0^\infty x^{\alpha-1} e^{-cx} L_n^\lambda(cx)dx = \frac{(1-\alpha+\lambda)_n}{n!c^\alpha} \Gamma(\alpha).$$

One observes that the restrictions of this integral imply that this expansion coefficients are valid for all $p > 2(\beta - c)$ and not only for $p > 1/2 + \beta - c$. □

*Proof of Proposition 4.5.* First write $\mathbb{E}\big[(S_t)^p\big] = e^{\rho p t}\, \mathbb{E}_x\big[(X_t)^p \mathbf{1}_{\{\tau>t\}}\big] = e^{\rho p t}\, \mathcal{P}_t^\phi f_p(x)$ for $f_p(x) = x^p$ and $x = S_0$. The result then follows from the spectral representation of the JDCEV semigroup and the eigenfunction expansion of $x^p$ given in Lemma 4.4. □

*Proof of Corollary 4.6.* The results follow from Proposition 4.5 for the corresponding $p$. In particular, to obtain (ii), notice that $(0)_{n-1} = \delta_{n,1}$ (Kronecker's delta). Thus, only the first $c_1^1$ is not zero and all $c_n^1 = 0$ with $n > 1$. The sum thus collapses to $e^{\rho t - \phi(\lambda_1)t} c_1^1 \varphi_1(x) = e^{(\rho-\phi(\lambda_1))t} x$ with $\lambda_1 = |\mu|$ by (4.10). Recalling that $\rho = r - q + \phi(-\mu)$ by (4.3) and that in this case $\mu < 0$, we obtain $\rho - \phi(\lambda_1) = r - q$. This yields (4.15). □

*Proof of Theorem 4.7.* Observe that $\mathbb{Q}(\tau_\Xi > t) = \mathbb{E}^{\mathbf{x}}\left[\mathbf{1}_{\{\tau_\Xi > t\}}\right] = (\mathcal{P}_t^{\Xi,\phi} 1)(\mathbf{x})$, where $\mathcal{P}^{\Xi,\phi}$ is the semigroup of $X^{\Xi,\phi}$ as defined in Section 2. The result follows from equation (2.21) and Lemma 4.4. When the Laplace exponent of the subordinator satisfies the condition (2.19), the convergence is uniform in $\mathbf{x}$ on compacts for each $t > 0$. Otherwise, the convergence is in $L^2(E, m)$. □

*Proof of Theorem 4.8.* To prove part (i), we need to calculate $\mathbb{E}[S_t^1 S_t^2]$ in

$$\mathrm{Corr}(R_t^1, R_t^2) = \frac{\mathbb{E}[S_t^1 S_t^2] - \mathbb{E}[S_t^1]\mathbb{E}[S_t^2]}{\sqrt{\mathrm{Var}(S_t^1)\mathrm{Var}(S_t^2)}}.$$

Observe that $\mathbb{E}[S_t^1 S_t^2] = e^{(\rho_1+\rho_2)t}(\mathcal{P}^{\Xi,\phi} x_1 x_2)$ with $\Xi = \{1, 2\}$. Then, from equation (2.21) and Lemma 4.4, we obtain,

$$\mathbb{E}[S_t^1 S_t^2] = e^{(\rho_1+\rho_2)t} \sum_{\mathbf{n}\in\mathbb{N}^2} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{\mathbf{n}}^{1,\Xi} \varphi_{\mathbf{n}}^{\Xi}(\mathbf{x}) = e^{(2r-(q_1+q_2)+\phi(-\mu_1,0)+\phi(0,-\mu_2))t} e^{-\phi(-\mu_1,-\mu_2)t} x_1 x_2$$

The coefficients $c_{\mathbf{n}}^{1,\Xi}$ of the first equality are defined by $c_{\mathbf{n}}^{1,\Xi} = c_{n_1}^{1,1} c_{n_2}^{1,2}$, where $c_n^{1,i}$ are given by equation (4.14) with $p = 1$. The last equality is obtained by using the identity $(0)_{n-1} = \delta_{n,1}$ in the definition of $c_n^{1,i}$ and by substituting $\rho_i$ with equation (4.3). To prove part $(ii)$, substitute the result for the joint survival probability from Theorem 4.7 in the expression for the default indicator correlation

$$\mathrm{Corr}\left(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}}\right) = \frac{\mathbb{Q}\left(\tau_{\{1,2\}} > t\right) - \mathbb{Q}(\tau_1 > t)\,\mathbb{Q}(\tau_2 > t)}{\prod_{i=1}^2 \sqrt{\mathbb{Q}(\tau_i > t)(1 - \mathbb{Q}(\tau_i > t))}}. \qquad \square$$

$\square$

*Proof of Theorem 4.9.* Take the function $f(S_t^1, \ldots, S_t^n) = \left(\sum_{i=1}^n (1 - \mathbf{1}_{\{S_t^i>0\}}) - (N - 1)\right)^+ - \left(\sum_{i=1}^n (1 - \mathbf{1}_{\{S_t^i>0\}}) - N\right)^+$ and the equality of events $\{\tau_i > t\} = \{S_t^i > 0\}$, then using the definition (4.5) of the function $f_\Xi(S_t^{i_1}, \ldots, S_t^{i_k})$ we find that,

$$\begin{aligned} f_\Xi(S_t^{i_1}, \ldots, S_t^{i_k}) &= \sum_{\Theta\subseteq\Xi} (-1)^{|\Xi|-|\Theta|} \Big\{ \Big(n + 1 - N - \sum_{i=1}^n \mathbf{1}_\Theta(i)\mathbf{1}_{\{\tau_i>t\}}\Big)^+ \\ &\quad - \Big(n - N - \sum_{i=1}^n \mathbf{1}_\Theta(i)\mathbf{1}_{\{\tau_i>t\}}\Big)^+ \Big\}. \end{aligned}$$

We write the probability of observing at least $N$ defaults as

$$\mathbb{P}\big[N_t \geq N\big] = \sum_{\Xi\subseteq\Upsilon} \mathbb{E}\Big[\mathbf{1}_{\{\tau_\Xi>t\}} f_\Xi(S_t^{i_1}, \ldots, S_t^{i_k})\Big]$$

$$= \sum_{\Xi\subseteq\Upsilon} \mathbb{E}\Big[\mathbf{1}_{\{\tau_\Xi>t\}} \sum_{\Theta\subseteq\Xi} (-1)^{|\Xi|-|\Theta|} \Big\{ \Big(n + 1 - N - \sum_{i=1}^n \mathbf{1}_\Theta(i)\Big)^+ - \Big(n - N - \sum_{i=1}^n \mathbf{1}_\Theta(i)\Big)^+ \Big\}\Big]$$

$$= \sum_{\Xi\subseteq\Upsilon} \mathbb{E}\Big[\mathbf{1}_{\{\tau_\Xi>t\}} \sum_{\Theta\subseteq\Xi} (-1)^{|\Xi|-|\Theta|} \Big\{ \big(n + 1 - N - |\Theta|\big)^+ - \big(n - N - |\Theta|\big)^+ \Big\}\Big],$$

where the second equality is obtained by noticing that $\mathbf{1}_{\{\tau_\Xi>t\}}\mathbf{1}_\Theta(i)\mathbf{1}_{\{\tau_i>t\}} = \mathbf{1}_{\{\tau_\Xi>t\}}\mathbf{1}_\Theta(i)$. Now since the term in curly brackets is 1 for all $|\Theta| \leq n - N$ and zero otherwise,

$$\mathbb{P}\big[N_t \geq N\big] = \sum_{\Xi\subseteq\Upsilon} \mathbb{E}\Big[\mathbf{1}_{\{\tau_\Xi>t\}} \sum_{\Theta\subseteq\Xi} (-1)^{|\Xi|-|\Theta|} \mathbf{1}_{\{|\Theta|\leq n-N\}}\Big]$$

$$= \sum_{\Xi\subseteq\Upsilon} \mathbb{E}\Big[\mathbf{1}_{\{\tau_\Xi>t\}} \sum_{\Theta\subseteq\Xi} (-1)^{|\Xi|-|\Theta|} (1 - \mathbf{1}_{\{|\Theta|>n-N\}})\Big]$$

$$= \sum_{\Xi\subseteq\Upsilon} \mathbb{E}\Big[\mathbf{1}_{\{\tau_\Xi>t\}}\Big(\sum_{j=0}^{|\Xi|}(-1)^{|\Xi|-j}\binom{|\Xi|}{j} - \sum_{j=n+1-N}^{|\Xi|}(-1)^{|\Xi|-j}\binom{|\Xi|}{j}\Big)\Big].$$

Thus, we use the identities $\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} = \mathbf{1}_{\{n=0\}}$, $\sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j} = (-1)^{n-k}\binom{n-1}{k-1}$ and the fact that the binomial coefficient is zero for all $\forall k > n$ $(n \geq 1)$, and taking the expectation we obtain (recall that $\tau_\emptyset = \infty$),

$$\mathbb{P}\big[N_t \geq N\big] = 1 - \sum_{\Xi\subseteq\Upsilon}(-1)^{(|\Xi|-1)-(n-N)}\binom{|\Xi|-1}{n-N}\mathbf{1}_{\{|\Xi|\geq n+1-N\}}\mathbb{Q}(\tau_\Xi > t).$$

Finally, we use the spectral representation of $\mathbb{Q}(\tau_\Xi > t)$ as in equation (4.16) to conclude the proof. □

*Proof of Theorem 4.11.* Due to the representation (4.19) for the basket option, it is sufficient to obtain the eigenfunction expansion for the semigroup $\mathcal{P}_t^{\Xi,\phi}\hat{f}_\Theta(\mathbf{x}^\Theta)$ with $\hat{f}_\Theta(\mathbf{x}^\Theta) = \big(K - \sum_{j\in\Theta} w_j e^{\rho_j t} x_j\big)^+$. The eigenfunction expansion of the density $p^{\Xi,\phi}(t,\mathbf{x}^\Xi,\mathbf{y}^\Xi)$ corresponding to the the semigroup $\mathcal{P}_t^{\Xi,\phi}\hat{f}_\Theta(\mathbf{x}^\Theta)$ can be written as

$$p^{\Xi,\phi}(t,\mathbf{x}^\Xi,\mathbf{y}^\Xi) = \sum_{\mathbf{n}\in\mathbb{N}^{|\Theta|}}\sum_{\mathbf{m}\in\mathbb{N}^{|\Xi|-|\Theta|}} e^{-\phi_\Xi(\lambda_\mathbf{n}^\Theta,\lambda_\mathbf{m}^{\Xi\backslash\Theta})t}\varphi_\mathbf{n}^\Theta(\mathbf{x}^\Theta)\varphi_\mathbf{n}^\Theta(\mathbf{y}^\Theta)\varphi_\mathbf{m}^{\Xi\backslash\Theta}(\mathbf{x}^{\Xi\backslash\Theta})\varphi_\mathbf{m}^{\Xi\backslash\Theta}(\mathbf{y}^{\Xi\backslash\Theta}).$$

Therefore, the spectral representation of the semigroup $\mathcal{P}_t^{\Xi,\phi}\hat{f}_\Theta(\mathbf{x}^\Theta)$ is given by

$$\mathcal{P}_t^{\Xi,\phi}\hat{f}_\Theta(\mathbf{x}^\Theta) = \sum_{\mathbf{n}\in\mathbb{N}^{|\Theta|}}\sum_{\mathbf{m}\in\mathbb{N}^{|\Omega|}} e^{-\phi_\Xi(\lambda_\mathbf{n}^\Theta,\lambda_\mathbf{m}^{\Xi\backslash\Theta})t} q_\mathbf{n}^\Theta(K) c_\mathbf{m}^{\Xi\backslash\Theta}\varphi_\mathbf{n}^\Theta(\mathbf{x}^\Theta)\varphi_\mathbf{m}^{\Xi\backslash\Theta}(\mathbf{x}^{\Xi\backslash\Theta})$$

with $q_\mathbf{n}^\Theta(K) = (\hat{f}_\Theta, \varphi_\mathbf{n}^\Theta)_m$ and $c_\mathbf{m}^{0,\Xi\backslash\Theta} = (1, \varphi_\mathbf{m}^{\Xi\backslash\Theta})_m$. The coefficients $p_\mathbf{n}^\Theta(K)$ are obtained by dividing $q_\mathbf{n}^\Theta(K)$ by $K$ (i.e., $p_\mathbf{n}^\Theta(K) = q_\mathbf{n}^\Theta(K)/K$). The coefficients $q_\mathbf{n}^\Theta(K)$ are obtained by induction as follows. First, consider the case of one firm, that is, $|\Theta| = 1$ (i.e., $\Theta = \{i\}$). In this case, $\tilde{f}_\Theta(\mathbf{x}^\Theta) = \big(K - w_i e^{\rho_i t}x_i\big)^+ = w_i e^{\rho_i t}\big(k - x_i\big)^+$, with $k = e^{-\rho_i t}K/w_i$. Thus $(q_\mathbf{n}^\Theta(K) = q_{n_i}^1(K))$,

$$\begin{aligned}
q_{n_i}^1(K) &= w_i e^{\rho_i t}\int_0^k (k-y)\,\varphi_{n_i}(y) m_i(y)dy \\
&= 2|\beta_i|A_i^{\frac{\nu_i}{2}+1}\sqrt{\frac{(n_i-1)!}{|\mu_i+b_i|\Gamma(\nu_i+n_i)}}w_i e^{\rho_i t}\int_0^k (k-y)\,y^{2c_i-2\beta_i-1}e^{-A_i y^{-2\beta_i}}L_{n_i-1}^{\nu_i}(A_i y^{-2\beta_i})dy \\
&= A_i^{\frac{\nu_i}{2}+1}\sqrt{\frac{(n_i-1)!}{|\mu_i+b_i|\Gamma(\nu_i+n_i)}}w_i e^{\rho_i t}\int_0^{k^{2|\beta_i|}}\Big(k - x^{\frac{1}{2|\beta_i|}}\Big)x^{\frac{c_i}{|\beta_i|}}e^{-A_i x}L_{n_i-1}^{\nu_i}(A_i x)dx.
\end{aligned}$$

In the second equality we used the expressions (4.13) and (4.11) for the eigenfunctions and speed density, respectively. The integral in the last equality is calculated using the integral for the Laguerre polynomials (Prudnikov et al. 1983, pp. 463; valid for $a, r, \alpha, \beta > 0$),

$$\text{(A2)}\quad \int_0^a x^{\alpha-1}(a^r - x^r)^{\beta-1}e^{-cx}L_n^\lambda(cx)dx = \frac{a^{\alpha+r\beta-r}\Gamma(\beta)}{n!r}\sum_{p=0}^{\infty}\frac{(1+\lambda+p)_n\Gamma\left(\frac{\alpha+p}{r}\right)(-ca)^p}{\Gamma\left(\beta+\frac{\alpha+p}{r}\right)p!}.$$

Therefore, using elementary identities of the pochhammer symbol, such as $(a+p)_n = \frac{\Gamma(a+p+n)}{\Gamma(a+p)} = \frac{(a+n)_p\Gamma(a+n)}{(a)_p\Gamma(a)}$, we obtain,

$$q^1_{n_i}(K) = \sqrt{\frac{\Gamma(\nu_i+n_i)}{\Gamma(n_i)|\mu_i+b_i|}}\frac{2|\beta_i|A_i^{\frac{\nu_i}{2}+1}}{\Gamma(1+\nu_i)}\Big(\frac{e^{-\rho_i t}}{w_i}\Big)^{2c_i-2\beta_i}$$

$$\times\sum_{p_i=0}^{\infty}\frac{(-1)^{p_i}(\nu_i+n_i)_{p_i}\Gamma\left(2c_i-2\beta_i(1+p_i)\right)}{\Gamma\left(2c_i-2\beta_i(1+p_i)+2\right)(1+\nu_i)_{p_i}\,p_i!}\Big(A_i\Big(\frac{e^{-\rho_i t}}{w_i}\Big)^{-2\beta_i}\Big)^{p_i}K^{1+2c_i-2\beta_i(1+p_i)}.$$

Similarly, let us consider the case of two firms, that is, $|\Theta|=2$ (i.e., $\Theta=\{i,j\}$). In this case, $\tilde{f}_\Theta(\mathbf{x}^\Theta) = \left(K - w_i e^{\rho_i t}x_i - w_j e^{\rho_j t}x_j\right)^+ = \left(k-x_i\right)^+\mathbf{1}_{\{k\geq 0\}}$, with $k=(K-w_je^{\rho_j t}x_j)$. Thus ($q^\Theta_{\mathbf{n}}(K) = q^2_{n_i,n_j}(K)$),

$$q^2_{n_i,n_j}(K) = \int_0^\infty\Big[\int_0^\infty\left(K-w_ie^{\rho_i t}y_1 - w_je^{\rho_j t}y_2\right)^+\varphi_{n_i}(y_1)m_i(y_1)dy_1\Big]\varphi_{n_j}(y_2)m_j(y_2)dy_2$$

$$= \int_0^\infty \mathbf{1}_{\{k\geq 0\}}\Big[w_ie^{\rho_i t}\int_0^k\left(k-y_1\right)\varphi_{n_i}(y_1)m_i(y_1)dy_1\Big]\varphi_{n_j}(y_2)m_j(y_2)dy_2.$$

Clearly, the term in brackets corresponds to $q^1_{n_i}(K')$ with $K' = w_je^{\rho_j t}(k'-y_2)$, where $k' = Ke^{-\rho_j t}/w_j$. Hence we obtain,

$$q^2_{n_i,n_j}(K) = \int_0^{k'} p^1_{n_i}(K')\varphi_{n_j}(y_2)m_j(y_2)dy_2$$

$$= \sqrt{\frac{\Gamma(\nu_i+n_i)}{\Gamma(n_i)|\mu_i+b_i|}}\frac{2|\beta_i|A_i^{\frac{\nu_i}{2}+1}}{\Gamma(1+\nu_i)}\Big(\frac{e^{-\rho_i t}}{w_i}\Big)^{2c_i-2\beta_i}\sum_{p_i=0}^{\infty}\frac{(-1)^{p_i}(\nu_i+n_i)_{p_i}\Gamma\left(2c_i-2\beta_i(1+p_i)\right)}{\Gamma\left(2c_i-2\beta_i(1+p_i)+2\right)(1+\nu_i)_{p_i}\,p_i!}$$

$$\times\Big(A_i\Big(\frac{e^{-\rho_i t}}{w_i}\Big)^{-2\beta_i}\Big)^{p_i}\left(w_je^{\rho_j t}\right)^{1+2c_i-2\beta_i(p_i+1)}A_j^{\frac{\nu_j}{2}+1}\sqrt{\frac{(n_j-1)!}{|\mu_j+b_j|\Gamma(\nu_j+n_j)}}$$

$$\times\int_0^{k'^{2|\beta_j|}}(k'-x^{\frac{1}{2|\beta_j|}})^{1+2c_i-2\beta_i(1+p_i)}x^{\frac{c_j}{|\beta_j|}}e^{-A_jx}L^{\nu_j}_{n_j-1}(A_jx)dx.$$

Using the integral (A.2) we arrive at:

$$q^2_{n_i,n_j}(K) = \prod_{\ell\in\{i,j\}}\sqrt{\frac{\Gamma(\nu_\ell+n_\ell)}{\Gamma(n_\ell)|\mu_\ell+b_\ell|}}\frac{2|\beta_\ell|A_\ell^{\frac{\nu_\ell}{2}+1}}{\Gamma(1+\nu_\ell)}\Big(\frac{e^{-\rho_\ell t}}{w_\ell}\Big)^{2c_\ell-2\beta_\ell}$$

$$\times\sum_{p_i=0}^{\infty}\sum_{p_j=0}^{\infty}\frac{K^{1+2\sum_{\ell\in\{i,j\}}(c_\ell-\beta_\ell(1+p_\ell))}}{\Gamma\Big(2+2\sum_{\ell\in\{i,j\}}(c_\ell-\beta_\ell(1+p_\ell))\Big)}$$

$$\times \prod_{\ell \in \{i,j\}} \frac{(-1)^{p_\ell}\,(\nu_\ell + n_\ell)_{p_\ell} \Gamma\left(2c_\ell - 2\beta_\ell(1+p_\ell)\right)}{(1+\nu_\ell)_{p_\ell}\, p_\ell!} \Big( A_\ell \Big( \frac{e^{-\rho_\ell t}}{w_\ell} \Big)^{-2\beta_\ell} \Big)^{p_\ell}.$$

Clearly, $q_{\mathbf{n}}^{\Theta}(K) = Kp_{\mathbf{n}}^{\Theta}(K)$ of equation (4.20) holds for $|\Theta| = 1$ and 2. Now, let us assume that $q_{\mathbf{n}}^{\Theta}(K)$ holds for $N$ firms, that is, $|\Theta| = N$ (i.e., $\Theta = \{j_1, j_2, \dots, j_N\}$). Let $\tilde{\Theta} = \Theta \cup \{j_{N+1}\}$, that is, $|\tilde{\Theta}| = N+1$. In this case, $\tilde{f}_{\tilde{\Theta}}(\mathbf{x}^{\tilde{\Theta}}) = \big(K - \sum_{j_k \in \tilde{\Theta}} w_{j_k} e^{\rho_{j_k} t} x_{j_k}\big)^+ = \big(k - \sum_{j_k \in \Theta} w_{j_k} e^{\rho_{j_k} t} x_{j_k}\big)^+ \mathbf{1}_{\{k \geq 0\}}$, with $k = (K - w_{j_{N+1}} e^{\rho_{j_{N+1}} t} x_{j_{N+1}})$. In addition, let $\mathbf{n} = (n_{j_1}, \dots, n_{j_N})$ and $\mathbf{n}' = (n_{j_1}, \dots, n_{j_{N+1}})$. Then,

$$\begin{aligned} q_{\mathbf{n}'}^{\tilde{\Theta}}(K) &= \int_0^\infty \Big[ \int_{(0,\infty)^{|\Theta|}} \big(K - \sum_{j_k \in \tilde{\Theta}} w_{j_k} e^{\rho_{j_k} t} x_{j_k}\big)^+ \varphi_{\mathbf{n}}^{\Theta}(\mathbf{x}^{\Theta}) m_{\mathbf{n}}(\mathbf{x}^{\Theta}) d\mathbf{x}^{\Theta} \Big] \\ &\quad \times \varphi_{n_{j_{N+1}}}(x_{N+1}) m_{j_{N+1}}(x_{N+1}) dx_{N+1} \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty \mathbf{1}_{\{k \geq 0\}} \Big[ \int_{(0,\infty)^{|\Theta|}} \big(k - \sum_{j_k \in \Theta} w_{j_k} e^{\rho_{j_k} t} x_{j_k}\big)^+ \varphi_{\mathbf{n}}^{\Theta}(\mathbf{x}^{\Theta}) m_{\mathbf{n}}(\mathbf{x}^{\Theta}) d\mathbf{x}^{\Theta} \Big] \\ &\times \varphi_{n_{j_{N+1}}}(x_{N+1}) m_{j_{N+1}}(x_{N+1}) dx_{N+1}. \end{aligned}$$

Therefore, the term in brackets corresponds to $q_{\mathbf{n}}^{\Theta}(K')$ with $K' = w_{j_{N+1}} e^{\rho_{j_{N+1}} t}(k' - x_{N+1})$, where $k' = Ke^{-\rho_{j_{N+1}} t}/w_{j_{N+1}}$. Thus, rearranging terms we obtain,

$$q_{\mathbf{n}'}^{\tilde{\Theta}}(K) = \int_0^{k'} q_{\mathbf{n}}^{\Theta}(K') \varphi_{n_{j_{N+1}}}(x_{N+1}) m_{j_{N+1}}(x_{N+1}) dx_{N+1}$$

$$= \prod_{j_k \in \Theta} \sqrt{\frac{\Gamma(\nu_{j_k} + n_{j_k})}{\Gamma(n_{j_k})|\mu_{j_k} + b_{j_k}|}} \frac{2|\beta_{j_k}| A_{j_k}^{\frac{\nu_{j_k}}{2}+1}}{\Gamma(1+\nu_{j_k})} \Big( \frac{e^{-\rho_{j_k} t}}{w_{j_k}} \Big)^{2c_{j_k} - 2\beta_{j_k}}$$

$$\times \sum_{\mathbf{p} \in \mathbb{N}_0^{|\Theta|}} \frac{\prod_{j_k \in \Theta} \frac{(-1)^{p_{j_k}} (\nu_{j_k} + n_{j_k})_{p_{j_k}} \Gamma\left(2c_{j_k} - 2\beta_{j_k}(1+p_{j_k})\right)}{(1+\nu_{j_k})_{p_{j_k}}\, p_{j_k}!} \Big( A_{j_k} \Big( \frac{e^{-\rho_{j_k} t}}{w_{j_k}} \Big)^{-2\beta_{j_k}} \Big)^{p_{j_k}}}{\Gamma\Big(2 + 2\sum_{j_k \in \Theta}(c_{j_k} - \beta_{j_k}(1+p_{j_k}))\Big)}$$

$$\times \big(w_{j_{N+1}} e^{\rho_{j_{N+1}} t}\big)^{1+2\sum_{j_k \in \Theta}(c_{j_k} - \beta_{j_k}(1+p_{j_k}))} A_{j_{N+1}}^{\frac{\nu_{j_{N+1}}}{2}+1} \sqrt{\frac{(n_{j_{N+1}} - 1)!}{|\mu_{j_{N+1}} + b_{j_{N+1}}| \Gamma(\nu_{j_{N+1}} + n_{j_{N+1}})}}$$

$$\times \int_0^{k'^{2|\beta_{j_{N+1}}|}} (k' - x^{\frac{1}{2|\beta_{j_{N+1}}|}})^{1+2\sum_{j_k \in \Theta}(c_{j_k} - \beta_{j_k}(1+p_{j_k}))} x^{\frac{c_{j_{N+1}}}{|\beta_{j_{N+1}}|}} e^{-A_{j_{N+1}} x} L_{n_{j_{N+1}}-1}^{\nu_{j_{N+1}}}(A_{j_{N+1}} x) dx.$$

Finally, using the integral (A.2) and rearranging terms we arrive at:

$$q_{\mathbf{n}'}^{\tilde{\Theta}}(K) = \prod_{j_k \in \tilde{\Theta}} \sqrt{\frac{\Gamma(\nu_{j_k} + n_{j_k})}{\Gamma(n_{j_k})|\mu_{j_k} + b_{j_k}|}} \frac{2|\beta_{j_k}| A_{j_k}^{\frac{\nu_{j_k}}{2}+1}}{\Gamma(1+\nu_{j_k})} \left(\frac{e^{-\rho_{j_k} t}}{w_{j_k}}\right)^{2c_{j_k} - 2\beta_{j_k}}$$

$$\times \sum_{\mathbf{p} \in \mathbb{N}_0^{|\tilde{\Theta}|}} \frac{K^{1+2\sum_{j_k \in \tilde{\Theta}}(c_{j_k} - \beta_{j_k}(1+p_{j_k}))}}{\Gamma\left(2 + 2\sum_{j_k \in \tilde{\Theta}}(c_{j_k} - \beta_{j_k}(1+p_{j_k}))\right)}$$

$$\times \prod_{j_k \in \tilde{\Theta}} \frac{(-1)^{p_{j_k}} (\nu_{j_k} + n_{j_k})_{p_{j_k}} \Gamma\left(2c_{j_k} - 2\beta_{j_k}(1+p_{j_k})\right)}{(1+\nu_{j_k})_{p_{j_k}}\, p_{j_k}!} \left(A_{j_k} \left(\frac{e^{-\rho_{j_k} t}}{w_{j_k}}\right)^{-2\beta_{j_k}}\right)^{p_{j_k}},$$

as expected. □

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