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STOCHASTIC PROPERTIES OF THE LINEAR MULTIFRACTIONAL STABLE MOTION

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Abstract

We study a family of locally self-similar stochastic processes $Y = \{Y(t)\}_{t\in\mathbb{R}}$ with α-stable distributions, called linear multifractional stable motions. They have infinite variance and may possess skewed distributions. The linear multifractional stable motion processes include, in particular, the classical linear fractional stable motion processes, which have stationary increments and are self-similar with self-similarity parameter H. The linear multifractional stable motion process Y is obtained by replacing the self-similarity parameter H in the integral representation of the linear fractional stable motion process by a deterministic function $H(t)$. Whereas the linear fractional stable motion is always continuous in probability, this is not in general the case for Y. We obtain necessary and sufficient conditions for the continuity in probability of the process Y. We also examine the effect of the regularity of the function $H(t)$ on the local structure of the process. We show that under certain Hölder regularity conditions on the function $H(t)$, the process Y is locally equivalent to a linear fractional stable motion process, in the sense of finite-dimensional distributions. We study Y by using a related α-stable random field and its partial derivatives.

Keywords: Linear fractional stable motion; self-similarity; multifractional Brownian motion; local self-similarity; tangent process; heavy tail

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1. Introduction and summary of results

A commonly observed phenomenon in fast telecommunication networks is that the traffic process (e.g. the number of received bytes), viewed as a function of time, may exhibit self-similarity. See, for example, Leland *et al.* (1994), Paxson and Floyd (1995), and the collection of papers edited by Park and Willinger (2000) for further references. Many of the currently existing probabilistic models of Internet traffic over a single link incorporate either a self-similar or an approximately self-similar stochastic process $X = \{X(t),\ t \in \mathbb{R}\}$ to represent the fluctuations of the arriving traffic. A random process X is said to be self-similar with self-similarity parameter $H > 0$ (H-self-similar, in short) if, for all $c > 0$,

$$\{X(ct),\ t \in \mathbb{R}\} \stackrel{\text{FDD}}{=} \{c^H X(t),\ t \in \mathbb{R}\},$$

where '$\stackrel{\text{FDD}}{=}$' means equality in finite-dimensional distributions.

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In practice, however, the estimated scaling exponent H of the traffic traces is typically nonconstant in time. The changes of the parameter H may be related to changes in the operating regimes of the network and may have important performance implications. To be able to study and evaluate such effects, new traffic models are called for. These models should incorporate both self-similarity over short time scales *and* dependence of the scaling exponent H on time.

Another ubiquitous feature of Internet traffic, for example, is its *burstiness*, that is, the presence of rare but extremely busy periods of activity. Heavy-tailed, infinite-variance stochastic processes are natural candidates for modelling such behaviour.

In this paper, we extend an approach applied to Gaussian fractional Brownian motion by Peltier and Lévy-Véhel (1995) and Benassi *et al.* (1997) (see also Ayache and Lévy-Véhel (1999), Cohen (1999) and Ayache and Taqqu (2003) for related work). We focus on a model of an infinite-variance stochastic process $Y = \{Y(t)\}_{t\in\mathbb{R}}$, defined in (1.2) below, which exhibits local self-similarity. This means that the process $Y(t),\ t \in \mathbb{R}$, is approximately $H(t_0)$-self-similar when t is close to t_0, where $H(t) > 0$ is a continuous function of $t \in \mathbb{R}$. We will refer to the parameter $H(t)$ as the *local self-similarity* (or *scaling*) *exponent* of the process Y at time $t \in \mathbb{R}$. Our goal here is to study the stochastic properties of Y. We focus, in Stoev and Taqqu (2004), on the almost sure path properties of the process Y. We will then be able to address the problem of estimating the local scaling exponent function $H(t),\ t \in \mathbb{R}$.

Let $\alpha \in (0, 2)$ and let $H(t),\ t \in \mathbb{R}$, be a deterministic function such that

$$0 < H(t) < 1 \quad \text{for all } t \in \mathbb{R}. \tag{1.1}$$

We want to study the process $Y = \{Y(t)\}_{t\in\mathbb{R}}$, defined by the stochastic integral

$$\begin{aligned} Y(t) = \int_{\mathbb{R}} \{ & a^+((t+s)_+^{H(t)-1/\alpha} - (s)_+^{H(t)-1/\alpha}) \\ & + a^-((t+s)_-^{H(t)-1/\alpha} - (s)_-^{H(t)-1/\alpha})\} M_{\alpha,\beta}(\mathrm{d}s), \end{aligned} \tag{1.2}$$

where $a^+, a^- \in \mathbb{R}$, $|a^+| + |a^-| > 0$, and, for generic $x \in \mathbb{R}$,

$$(x)_+^a := \begin{cases} x^a & \text{when } x > 0, \\ 0 & \text{when } x \le 0, \end{cases} \qquad \text{and} \qquad x_-^a := (-x)_+^a. \tag{1.3}$$

$M_{\alpha,\beta}(\mathrm{d}s)$ is a (strictly) α-stable, independently scattered random measure with control measure $\mathrm{d}s$ and skewness intensity $\beta(s) \in [-1, 1],\ s \in \mathbb{R}$. This means, in particular, that if $A_1, \dots, A_n \subset \mathbb{R}$ are disjoint Borel sets, then the random variables (RVs) $M_{\alpha,\beta}(A_1), \dots, M_{\alpha,\beta}(A_n)$ are independent. Moreover, for any Borel set $A \subset \mathbb{R}$, the characteristic function of the RV $M_{\alpha,\beta}(A)$ is given by

$$\mathrm{E}\,\mathrm{e}^{\mathrm{i}\theta M_{\alpha,\beta}(A)} = \begin{cases} \exp\{-|A||\theta|^\alpha(1 - \mathrm{i}\beta_A \operatorname{sign}(\theta)\tan\frac{1}{2}\pi\alpha)\} & \text{if } \alpha \ne 1, \\ \exp\{-|A||\theta|\} & \text{if } \alpha = 1, \end{cases}$$

where $|A| := \int_A \mathrm{d}s$ and $\beta_A := \int_A \beta(s)\,\mathrm{d}s/|A|$. (For simplicity, we take the distribution of the α-stable random measure $M_{\alpha,\beta}(\mathrm{d}s)$ to be symmetric if $\alpha = 1$, that is, $\beta(\cdot) \equiv 0$, and, hence, in this case, $\mathrm{E}\,\mathrm{e}^{\mathrm{i}\theta M_{\alpha,\beta}(A)} = \mathrm{e}^{-|A||\theta|}$.)

The stochastic integral in (1.2) involves an infinite-variance white noise $M_{\alpha,\beta}(\mathrm{d}s)$. Because of (1.1) the integrals $\int_{\mathbb{R}} |(t+s)_\pm^{H(t)-1/\alpha} - (s)_\pm^{H(t)-1/\alpha}|^\alpha\,\mathrm{d}s$ are finite for all $t \in \mathbb{R}$, and hence the stochastic integral (1.2) is well defined (for more details see, for example, Chapter 3 in

Samorodnitsky and Taqqu (1994)). We call the process Y a *linear multifractional stable motion* (LMSM) with *local scaling exponent* $H(t),\ t \in \mathbb{R}$.

When the skewness intensity of the measure $M_{\alpha,\beta}(\mathrm{d}s)$ is constant, i.e. $\beta(s) \equiv \beta \in [-1, 1]$, $s \in \mathbb{R}$, and when the local scaling parameter is constant, i.e. $H(t) \equiv H, H \in (0, 1)$, then the process Y coincides with the *linear fractional stable motion* (LFSM) process $X_{H,\alpha} = \{X_{H,\alpha}(t),\ t \in \mathbb{R}\}$, where

$$X_{H,\alpha}(t) = \int_{\mathbb{R}} \{a^+((t+s)_+^{H-1/\alpha} - (s)_+^{H-1/\alpha}) \\ + a^-((t+s)_-^{H-1/\alpha} - (s)_-^{H-1/\alpha})\} M_{\alpha,\beta}(\mathrm{d}s). \tag{1.4}$$

The LFSM process $X_{H,\alpha}$ has stationary increments and is self-similar with self-similarity parameter H (see, for example, Proposition 7.3.6 in Samorodnitsky and Taqqu (1994)). In particular, if $H(t) \equiv 1/\alpha$, and if $a^+ = 1$ and $a^- = 0$, for example, the kernel function in (1.2) becomes

$$\mathbf{1}_{(-t,\infty)}(s) - \mathbf{1}_{(0,\infty)}(s) = \begin{cases} \mathbf{1}_{(-t,0]}(s) & \text{for } t > 0, \\ -\mathbf{1}_{(0,-t]}(s) & \text{for } t < 0, \end{cases}$$

and the process Y becomes the α-stable Lévy motion, a process with independent increments. Here $\mathbf{1}_A(\cdot)$ denotes the indicator function of the set $A \subset \mathbb{R}$.

In contrast to the Gaussian case $\alpha = 2$, the finite-dimensional distributions of the LFSM process $X_{H,\alpha}$ depend on a^+ and a^- (see, for example, p. 347, Theorem 7.4.5 in Samorodnitsky and Taqqu (1994)). We show in Theorem 5.2, below, that this is also the case for the LMSM process $Y(t)$. This is why we consider all possible values of a^+ and a^- instead of merely setting $a^+ = 1$ and $a^- = 0$, for example. When $a^+ = a^-$, the process

$$Y(t) = a^+ \int_{\mathbb{R}} (|t+s|^{H(t)-1/\alpha} - |s|^{H(t)-1/\alpha}) M_{\alpha,\beta}(\mathrm{d}s)$$

is called *well balanced.*

To study the LMSM process $Y(t),\ t \in \mathbb{R}$, it is convenient to introduce the α-stable field

$$X = \{X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\},$$

defined so that

$$Y(t) = X(t, H(t)), \qquad t \in \mathbb{R}.$$

In Section 2, we study the properties of the field X and, more generally, those of the α-stable derivative fields $\partial_v^n X,\ n \in \mathbb{N}$. The fields $\partial_v^n X$ are defined as $\partial_v^n X(u, v)/\partial v^n$ using convergence in probability (see Lemma 2.1, below). We establish, in Theorem 2.1, basic properties of these fields. We show that the fields $X(u, v)$ and $\partial_v^n X(u, v),\ n \in \mathbb{N}$, are continuous in probability in $(u, v) \in \mathbb{R} \times (0, 1)$, and that one can express the field $X(u, v)$ through a Taylor-type expansion involving the derivative fields $\partial_v^k X(u, v_0),\ k = 0, 1, \ldots, N-1$; $v, v_0 \in (0, 1)$; $N \in \mathbb{N}$. These results will be used in what follows and may be of independent interest.

In Section 3, we investigate the continuity in probability of the LMSM process $Y(t),\ t \in \mathbb{R}$, and show that the continuity of the function $H(t)$ is a necessary and sufficient condition for this. In Section 4, we discuss the choice of the range $(0, 1)$ for the function $H(t)$ and, in particular, what happens if one lets $H(t) \in (0, 1) \cup \{1/\alpha\}$ when $0 < \alpha \leq 1$.

In Section 5, we focus on the local asymptotic behaviour of $Y(t),\ t \in \mathbb{R}$, namely the limit $Z(t),\ t \in \mathbb{R}$, in the sense of the finite-dimensional distributions of

$$\frac{1}{d_\lambda}(Y(\lambda t + t_0) - Y(t_0)), \qquad t \in \mathbb{R},$$

as $\lambda \downarrow 0$, where d_λ is a suitable normalization factor. One can view the limiting process as a *tangent process* of Y at $t = t_0$. The limiting process $Z(t),\ t \in \mathbb{R}$, provides a magnified version of the local behaviour of $Y(t)$ at $t = t_0$. However, in contrast to Falconer (2003), we are not dealing with path-wise convergence in a Skorokhod space, but with the convergence of the finite-dimensional distributions (see also Falconer (2002)); we will say simply that $\{Y(t)\}_{t\in\mathbb{R}}$ is *locally equivalent* to $\{Z(t)\}_{t\in\mathbb{R}}$ at $t = t_0$.

To get some intuition about the nature of $Z(t)$, suppose that the function $H(t),\ t \in \mathbb{R}$, is continuous and nonconstant and, for simplicity, assume also that the α-stable measure $M_{\alpha,\beta}(\mathrm{d}s)$ has constant skewness intensity $\beta(s) \equiv \beta,\ s \in \mathbb{R}$. In view of (1.2) and (1.4), we expect the process $\{Y(t)\}_{t\in\mathbb{R}}$ to be close in probability at $t = t_0$ to the LFSM process $\{X_{H(t_0),\alpha}(t)\}_{t\in\mathbb{R}}$, defined in (1.4), with self-similarity parameter $H(t_0)$. And, therefore, by the self-similarity and stationarity of the increments of the LFSM process, we expect the process $\{Y(t)\}_{t\in\mathbb{R}}$ to be locally equivalent at $t = t_0$ to the LFSM process $\{X_{H(t_0),\alpha}(t)\}_{t\in\mathbb{R}}$. We show below (Theorem 5.1) that this is indeed the case when the function $H(t)$ has sufficient Hölder regularity at t_0 and when the skewness intensity function $\beta(s),\ s \in \mathbb{R}$, is continuous.

When $\alpha = 2$, the measure $M_{\alpha,\beta}(\mathrm{d}s),\ s \in \mathbb{R}$, becomes Gaussian and the LFSM process $X_{H,\alpha}$ becomes the fractional Brownian motion process. In this case the LFSM process Y, defined in (1.2), coincides (up to a multiplicative deterministic function) with the *multifractional Brownian motion* (MBM) process. The MBM process was introduced independently by Peltier and Lévy-Véhel (1995) and Benassi *et al.* (1997). In these papers, local self-similarity and path-regularity properties of the MBM were established under certain Hölder regularity conditions on the local scaling exponent function H. Since these two seminal works appeared, many authors have studied other finite-variance, locally self-similar models that extend the MBM process (see, for example, Ayache and Lévy-Véhel (2000) and Ayache and Taqqu (2003)). The recent works of Benassi *et al.* (2002), (2004) also establish interesting results on new classes of non-Gaussian locally self-similar Lévy processes and fields. Several techniques for the estimation of the local self-similarity exponent have also been proposed (see, for example, Benassi *et al.* (1998), (2000), Bardet and Bertrand (2003), and Cohen and Istas (2004)).

In this paper, we suppose that $\alpha \in (0, 2)$ and hence that the LMSM process Y has infinite variance. Most statements are also valid in the Gaussian case ($\alpha = 2$).

To summarize, in Section 2, we introduce the α-stable field

$$X = \{X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\},$$

and study its properties and those of the derivative fields

$$\partial_v^n X = \{\partial_v^n X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\}.$$

The results are used in Section 3 to obtain necessary and sufficient conditions for the continuity in probability of Y. We suppose throughout that $0 < H(t) < 1$, except in Section 4 where we study the case when $H(t) \in (0, 1) \cup \{1/\alpha\}$ and $0 < \alpha \leq 1$. In Section 5, we examine the effect of the regularity of the function H on the local structure of the process Y. Section 6 contains auxiliary results.

2. The derivative fields $\partial_v^n X$

The results in this section are used in the proofs of Sections 3 and 5. They may be also of independent interest.

Consider the following α-stable field $X = \{X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\}$:

$$X(u, v) := \int_{\mathbb{R}} f(u, v, s) M_{\alpha,\beta}(\mathrm{d}s) \tag{2.1}$$

with

$$f(u, v, s) = a^+ f_+(u, v, s) + a^- f_-(u, v, s),$$

where

$$f_\pm(u, v, s) := (u + s)_\pm^{v-1/\alpha} - (s)_\pm^{v-1/\alpha},$$

and where $M_{\alpha,\beta}(\mathrm{d}s)$ is the same α-stable random measure as in (1.2), assumed symmetric when $\alpha = 1$. The α-stable integral in (2.1) is well defined, since the kernel function $f(u, v, s)$ belongs to $L^\alpha(\mathrm{d}s) := \{g(s) : \int_{\mathbb{R}} |g(s)|^\alpha \,\mathrm{d}s < \infty\}$ for all $u \in \mathbb{R}$ and $v \in (0, 1)$. Observe that, because of (1.2), we have, for all $t \in \mathbb{R}$,

$$Y(t) = X(t, H(t)) \quad \text{a.s.} \tag{2.2}$$

In view of the last relation, the properties of the field X will provide tools for understanding the LMSM process Y.

The following *derivative fields*:

$$\partial_v^n X = \{\partial_v^n X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\}, \qquad n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N},$$

play an important role. They are defined by

$$\partial_v^n X(u, v) := \int_{\mathbb{R}} \partial_v^n f(u, v, s) M_{\alpha,\beta}(\mathrm{d}s), \tag{2.3}$$

where

$$\begin{aligned} \partial_v^n f(u, v, s) &= \frac{\partial^n}{\partial v^n} f(u, v, s) = a^+ \frac{\partial^n}{\partial v^n} f_+(u, v, s) + a^- \frac{\partial^n}{\partial v^n} f_-(u, v, s) \\ &= a^+ (\ln^n(u + s)_+ (u + s)_+^{v-1/\alpha} - \ln^n(s)_+ (s)_+^{v-1/\alpha}) \\ &\quad + a^- (\ln^n(u + s)_- (u + s)_-^{v-1/\alpha} - \ln^n(s)_- (s)_-^{v-1/\alpha}). \end{aligned} \tag{2.4}$$

Observe that the function $f(u, v, s)$ is infinitely differentiable with respect to $v \in (0, 1)$ for all $u, s \in \mathbb{R}$. This is because of (1.3). Note, for example, that when $s > -|u|$ and $s > 0$, $f(u, v, s) = a^+((u + s)^{v-1/\alpha} - s^{v-1/\alpha})$. In Theorem 2.1, below, we show that for all $u \in \mathbb{R},\ v \in (0, 1)$ and $n \in \mathbb{N}_0$, $\partial_v^n f(u, v, s) \in L^\alpha(\mathrm{d}s)$ and therefore that the α-stable integrals in (2.3) are well defined.

We denote by $\|\xi\|_\alpha$ the *scale coefficient* in the characteristic function $\varphi_\xi(\theta) = \mathrm{E}\exp\{\mathrm{i}\xi\theta\}$, $\theta \in \mathbb{R}$, of the α-stable ($0 < \alpha \le 2$) random variable ξ, as defined by

$$\varphi_\xi(\theta) = \begin{cases} \exp\{-\|\xi\|_\alpha^\alpha |\theta|^\alpha (1 - \mathrm{i}\beta_\xi \operatorname{sign}(\theta) \tan \frac{1}{2}\pi\alpha)\} & \text{if } \alpha \ne 1, \\ \exp\{-\|\xi\|_1 |\theta|\} & \text{if } \alpha = 1, \end{cases}$$

where $\beta_\xi \in [-1, 1]$ denotes the skewness coefficient of ξ.

A collection of RVs are said to be *jointly* α-stable if all their finite linear combinations are α-stable. The functional $\|\cdot\|_\alpha^{1\wedge\alpha}$, where $1\wedge\alpha$ denotes $\min\{1,\alpha\}$, metrizes the convergence in probability in the linear spaces of jointly α-stable random variables that we consider. (Recall that when $\alpha = 1$, we consider only symmetric 1-stable RVs – see also Proposition 3.5.1 in Samorodnitsky and Taqqu (1994), for example.) That is, if ξ and ξ_n, $n \in \mathbb{N}$, are jointly α-stable RVs, then, as $n \to \infty$, $\xi_n \stackrel{\mathrm{P}}{\to} \xi$ if and only if $\|\xi_n - \xi\|_\alpha \to 0$. Furthermore, if ξ and η are jointly α-stable, then the following triangle inequality holds:

$$\|\xi + \eta\|_\alpha^{1\wedge\alpha} \le \|\xi\|_\alpha^{1\wedge\alpha} + \|\eta\|_\alpha^{1\wedge\alpha}.$$

When $1 \le \alpha \le 2$, $\|\cdot\|_\alpha^{1\wedge\alpha} = \|\cdot\|_\alpha$ becomes a norm.

For all $g(s) \in L^\alpha(\mathrm{d}s)$, the stochastic integral $\xi = \int_{\mathbb{R}} g(s) M_{\alpha,\beta}(\mathrm{d}s)$ defines an α-stable RV with scale coefficient

$$\|\xi\|_\alpha = \left(\int_{\mathbb{R}} |g(s)|^\alpha \,\mathrm{d}s\right)^{1/\alpha} = \|g(s)\|_{L^\alpha(\mathrm{d}s)}.$$

In fact, the α-stable stochastic integral defines a linear isometry between the complete metric space of jointly α-stable RVs with the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$ and the complete metric space of functions $L^\alpha(\mathrm{d}s)$ with the metric $\|\cdot\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}$. Observe that the scale coefficient $\|\xi\|_\alpha$ of the RV ξ does *not* depend on the skewness intensity $\beta(s)$, $s \in \mathbb{R}$, of the α-stable measure $M_{\alpha,\beta}(\mathrm{d}s)$. For more detail, see, for example, Samorodnitsky and Taqqu (1994).

In the sequel, we shall add a subscript $\beta = \beta(\cdot)$ to X and Y to indicate the skewness intensity of the underlying α-stable measure. For example, $\partial_v^n X_\beta(u,v)$ with $u \in \mathbb{R}$, $v \in (0,1)$, $n \in \mathbb{N}_0$, denotes the α-stable field defined by (2.3), and $Y_\beta(t)$, $t \in \mathbb{R}$, denotes the LMSM process defined in (1.2). We omit the subscript β or $\beta(\cdot)$ when it is not necessary to indicate explicitly the skewness intensity of the underlying α-stable measure.

The next theorem states some basic properties of the fields $\partial_v^n X$, $n \in \mathbb{N}_0$.

Theorem 2.1. *Let* $\alpha \in (0,2]$. *Then,*

(a) *for all* $v \in (0,1)$, $u \in \mathbb{R}$ *and* $n \in \mathbb{N}_0$,

$$\partial_v^n f(u,v,s) \in L^\alpha(\mathrm{d}s);$$

(b) *for all* $n \in \mathbb{N}_0$, $\partial_v^n X_\beta(0,v) = 0$ *almost surely, and for all* $h > 0$,

$$\begin{aligned}&\{(\partial_v^n X_{\beta(\cdot)}(u+h,v) - \partial_v^n X_{\beta(\cdot)}(h,v)), (n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\}\\ &\quad\stackrel{\mathrm{FDD}}{=} \{\partial_v^n X_{\beta(\cdot - h)}(u,v), (n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\};\end{aligned} \tag{2.5}$$

(c) *for all* $c > 0$,

$$\begin{aligned}&\{\partial_v^n X_{\beta(\cdot)}(cu,v), (n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\}\\ &\quad\stackrel{\mathrm{FDD}}{=} \left\{c^v \sum_{k=0}^n \binom{n}{k} \ln^{n-k}(c)\partial_v^k X_{\beta(c\,\cdot)}(u,v), (n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\right\}.\end{aligned} \tag{2.6}$$

Proof. We first prove statement (a). Observe that the term $(\ln(cu+s))^n = (\ln(c) + \ln(u + s/c))^n$ with $c > 0$, $cu + s > 0$, schematically similar to certain terms in (2.4), can be expanded using the binomial formula. We obtain that, for all $s \in \mathbb{R}$ and $c > 0$, $\partial_v^n f(cu,v,s)$ is equal to

$$c^{v-1/\alpha} \sum_{k=0}^n \binom{n}{k} \ln^{n-k}(c) \sum_{\kappa \in \{+,-\}} a^\kappa (\ln^k(u+s/c)_\kappa (u+s/c)_\kappa^{v-1/\alpha} - \ln^k(s/c)_\kappa (s/c)_\kappa^{v-1/\alpha}).$$

Hence, for all $s \in \mathbb{R}$ and $c > 0$,

$$\partial_v^n f(cu, v, s) = c^{v-1/\alpha} \sum_{k=0}^{n} \binom{n}{k} \ln^{n-k}(c) \partial_v^k f(u, v, s/c). \tag{2.7}$$

In particular, for $u > 0$, we have

$$\partial_v^n f(u, v, s) = u^{v-1/\alpha} \sum_{k=0}^{n} \binom{n}{k} \ln^{n-k}(u) \partial_v^k f(1, v, s/u). \tag{2.8}$$

Observe that, by Proposition 6.1, below, applied with $v_0 \equiv v$, we have $\partial_v^k f(1, v, s/u) \in L^\alpha(\mathrm{d}s)$, $k = 0, 1, \dots, n$, and thus (2.8) implies that $\partial_v^n f(u, v, s) \in L^\alpha(\mathrm{d}s)$. Now since, by (2.4), for all $u, s \in \mathbb{R}$ and $v \in (0, 1)$,

$$\partial_v^n f(u, v, s) = -\partial_v^n f(-u, v, s + u), \tag{2.9}$$

we obtain that $\partial_v^n f(u, v, s) \in L^\alpha(\mathrm{d}s)$ for all $u \in \mathbb{R},\ v \in (0, 1)$.

Statement (b) follows directly from the stochastic integral representation (2.3). Indeed, by (2.4), we have that $\partial_v^n X(0, v) = 0$ almost surely (a.s.). Now fix $n \in \mathbb{N}_0$ and $h > 0$, and let $u_{j,k} \in \mathbb{R},\ v_{j,k} \in (0, 1),\ j = 1, \dots, m,\ k = 0, 1, \dots, n$, and $m \in \mathbb{N}$. Then, by (2.3), for all $\theta_{j,k} \in \mathbb{R},\ j = 1, \dots, m,\ k = 0, 1, \dots, n$, we have

$$\begin{aligned}
&\mathrm{E} \exp\Big\{ \mathrm{i} \sum_{j,k} \theta_{j,k} (\partial_v^k X_{\beta(\cdot)}(u_{j,k} + h, v_{j,k}) - \partial_v^k X_{\beta(\cdot)}(h, v_{j,k})) \Big\} \\
&\quad = \exp\Big\{ - \int_{\mathbb{R}} \Big| \sum_{j,k} \sum_{\kappa \in \{+,-\}} A_{j,k,\kappa}(s + h) \Big|^\alpha (1 - \mathrm{i} \tan(\pi\alpha/2) \beta(s) \operatorname{sign}(A_{j,k,\kappa}(s + h)))\, \mathrm{d}s \Big\} \\
&\quad = \exp\Big\{ - \int_{\mathbb{R}} \Big| \sum_{j,k} \sum_{\kappa \in \{+,-\}} A_{j,k,\kappa}(s) \Big|^\alpha (1 - \mathrm{i} \tan(\pi\alpha/2) \beta(s - h) \operatorname{sign}(A_{j,k,\kappa}(s)))\, \mathrm{d}s \Big\} \\
&\quad = \mathrm{E} \exp\Big\{ \mathrm{i} \sum_{j,k} \theta_{j,k} \partial_v^k X_{\beta(\cdot - h)}(u_{j,k}, v_{j,k}) \Big\},
\end{aligned} \tag{2.10}$$

where $\sum_{j,k} := \sum_{j=1}^m \sum_{k=0}^n$, and where

$$A_{j,k,\kappa}(s) := a^\kappa (\ln^k(u_{j,k} + s)_\kappa (u_{j,k} + s)_\kappa^{v_{j,k} - 1/\alpha} - \ln^k(s)_\kappa (s)_\kappa^{v_{j,k} - 1/\alpha}).$$

Since $\theta_{j,k} \in \mathbb{R}$ are arbitrary, the last relation implies (2.5).

The scaling property (c) follows from (2.7) and the stochastic integral representation in (2.3) by using a similar Wold-device-type argument, as in (2.10), above. Observe that

$$\{M_{\alpha,\beta(\cdot)}(\mathrm{d}[c^{-1}s])\}_{s \in \mathbb{R}} \overset{\text{FDD}}{=} \{c^{-1/\alpha} M_{\alpha,\beta(c\,\cdot)}(\mathrm{d}s)\}_{s \in \mathbb{R}}.$$

Therefore, we have a factor c^v in (2.6), rather than $c^{v-1/\alpha}$ as in (2.7), and the skewness intensity function corresponding to the right-hand side of (2.6) is $\beta(cs),\ s \in \mathbb{R}$, instead of $\beta(s),\ s \in \mathbb{R}$. This completes the proof of the theorem.

Remark 2.1. Observe that the α-stable fields $\partial_v^n X(u, v)$ appearing in (2.5) involve different skewness intensity functions. The skewness intensity function involved in the right-hand side of (2.5), i.e. $\beta(s-h),\ s \in \mathbb{R}$, is a *shifted* version of the skewness intensity on the left-hand side, namely $\beta(s),\ s \in \mathbb{R}$.

Similarly, the skewness intensity on the right-hand side of (2.6), $\beta(cs),\ s \in \mathbb{R}$, is a *rescaled* version of the skewness intensity $\beta(s),\ s \in \mathbb{R}$, appearing on the left-hand side.

When $\beta(s) \equiv \beta = \text{const.}$, $\beta \in [-1, 1]$, then part (b) of Theorem 2.1 implies that the processes $\{\partial_v^n X(u, v),\ u \in \mathbb{R}\},\ v \in (0, 1),\ n \in \mathbb{R}$, have jointly stationary increments. Also the statement in part (c) of Theorem 2.1 can be interpreted as a scaling property of the fields $\partial_v^n X(u, v)$. In particular, it follows that the process $X(u, v),\ u \in \mathbb{R}$, has stationary increments and is self-similar with self-similarity parameter $v \in (0, 1)$. In fact, by (2.1), we have that $X_{H,\alpha}(t) \equiv X(t, H),\ t \in \mathbb{R}$, where $X_{H,\alpha}(t)$ is the LFSM process defined in (1.4).

Consider the α-stable $(0 < \alpha < 2)$ fields $\partial_v^n X_{\beta_1}(u, v)$ and $\partial_v^n X_{\beta_2}(u, v)$, with $u \in \mathbb{R},\ v \in (0, 1)$ and $n \in \mathbb{N}_0$, defined as in (2.3), with possibly different skewness intensities $\beta_1(s)$ and $\beta_2(s),\ s \in \mathbb{R}$, respectively. Then, for all $u_{j,k}, \theta_{j,k} \in \mathbb{R}$ and $v_{j,k} \in (0, 1)$, with $j = 1, \dots, m$, $k = 0, 1, \dots, n$, $m \in \mathbb{N}$, and $n \in \mathbb{N}_0$, we have that

$$\Big\| \sum_{j,k} \theta_{j,k} \partial_v^k X_{\beta_1(\cdot)}(u_{j,k}, v_{j,k}) \Big\|_\alpha = \Big\| \sum_{j,k} \theta_{j,k} \partial_v^k X_{\beta_2(\cdot)}(u_{j,k}, v_{j,k}) \Big\|_\alpha, \tag{2.11}$$

where $\sum_{j,k} = \sum_{j=1}^m \sum_{k=0}^n$. This follows from the fact that the RVs in (2.11) are obtained by taking α-stable integrals of the same kernel functions, with respect to the α-stable measures $M_{\alpha,\beta_1}(\mathrm{d}s)$ and $M_{\alpha,\beta_2}(\mathrm{d}s)$, respectively. However, despite (2.11), the fields $\partial_v^n X_{\beta_1}(u, v)$ and $\partial_v^n X_{\beta_2}(u, v)$ will have, in general, different finite-dimensional distributions, unless the skewness intensities β_1 and β_2 coincide.

Lemma 2.1 below shows that the fields $\partial_v^n X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)$ are, in fact, the partial derivatives of the field $X(u, v)$, in the sense of convergence in probability.

Lemma 2.1. *For all $u \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $v \in (0, 1)$, we have that*

$$\left\| \frac{\partial_v^n X(u, v+h) - \partial_v^n X(u, v)}{h} - \partial_v^{n+1} X(u, v) \right\|_\alpha \to 0 \quad \text{as } h \to 0, \tag{2.12}$$

where $\partial_v^n X(u, v)$ are defined in (2.3).

Proof. By using (2.3) and the Taylor formula for the kernel functions $\partial_v^n f(u, v, s)$, we find that the left-hand side of (2.12) is equal to

$$\left\| \left(\frac{\partial_v^n f(u, v+h, s) - \partial_v^n f(u, v, s)}{h} \right) - \partial_v^{n+1} f(u, v, s) \right\|_{L^\alpha(\mathrm{d}s)} = |h| \|\partial_v^{n+2} f(u, v_\theta, s)\|_{L^\alpha(\mathrm{d}s)}, \tag{2.13}$$

where $v_\theta = v + \theta h$, $\theta = \theta(u, v, h, s) \in [0, 1]$.

We will now show that the right-hand side of (2.13) vanishes as $h \to 0$, which will complete the proof of the lemma.

The case $u = 0$ is trivial. In view of (2.9), without loss of generality we will assume that $u > 0$. Therefore, by (2.8), using the triangle inequality for the metric $\|\cdot\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}$, we get

$$\|\partial_v^{n+2} f(u, v_\theta, s)\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha} \le (u^{v-1/\alpha} + u^{v+h-1/\alpha})^{1\wedge\alpha} \sum_{k=0}^{n+2} \binom{n+2}{k}^{1\wedge\alpha} |\ln(u)|^{(n+2-k)(1\wedge\alpha)} \times \left(\int_{\mathbb{R}} \sup_{\theta\in[0,1]} |\partial_v^k f(1, v_\theta, s/u)|^\alpha \,\mathrm{d}s \right)^{1\wedge 1/\alpha}.$$

In the last inequality we used the fact that for all $u > 0$, $v \in (0, 1)$, and $\theta \in [0, 1]$, we have $u^{v+h\theta} \le u^v + u^{v+h}$. Now, by applying the inequality (6.1) (Proposition 6.1, below) to the integrals on the right-hand side of the last expression, we obtain

$$\|\partial_v^{n+2} f(u, v_\theta, s)\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha} = \mathcal{O}\left(\frac{1}{R(v, h)^{(1\wedge\alpha)(n+2)+1\wedge 1/\alpha}} \right) \quad \text{as } h \to 0,$$

where $R(v, h) := \min\{v, v + h, 1 - v, 1 - v - h\} \in (0, 1)$. Since $v \in (0, 1)$, $R(v, h) \to v \wedge (1 - v) > 0$ as $h \to 0$ and, hence, for sufficiently small $|h| > 0$, the term $R(v, h)$ will be bounded away from 0. Therefore, the right-hand side of (2.13) is of order $\mathcal{O}(h)$ as $h \to 0$.

Theorem 2.1 and Lemma 2.1 imply the following result.

Corollary 2.1. *For all $\alpha \in (0, 2]$ and $n \in \mathbb{N}_0$, the field*

$$\partial_v^n X = \{\partial_v^n X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\}$$

is continuous in the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$ with respect to u and v.

Proof. Let $n \in \mathbb{N}_0$. We first show that $\partial_v^n X(u, v)$ is continuous in $v \in (0, 1)$. By multiplying (2.12) by $|h|$ and using the triangle inequality for the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$, we obtain that, for all $u \in \mathbb{R}$ and $v \in (0, 1)$,

$$\Big| \|\partial_v^n X(u, v + h) - \partial_v^n X(u, v)\|_\alpha^{1\wedge\alpha} - |h|^{1\wedge\alpha} \|\partial_v^{n+1} X(u, v)\|_\alpha^{1\wedge\alpha} \Big| = o(|h|^{1\wedge\alpha}) \quad \text{as } h \to 0. \tag{2.14}$$

Since $v \in (0, 1)$, we have that $\|\partial_v^{n+1} X(u, v)\|_\alpha^{1\wedge\alpha} < \infty$, and therefore (2.14) implies that $\|\partial_v^n X(u, v+h) - \partial_v^n X(u, v)\|_\alpha^{1\wedge\alpha} \to 0$ as $h \to 0$, that is, the continuity of the field $\partial_v^n X(u, v)$ with respect to $v \in (0, 1)$.

Now let $u, u_0 \in \mathbb{R}$ and $v, v_0 \in (0, 1)$ be arbitrary. By using the triangle inequality for the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$, we get that

$$\begin{aligned} &\|\partial_v^n X(u, v) - \partial_v^n X(u_0, v_0)\|_\alpha^{1\wedge\alpha} \\ &\quad \le \|\partial_v^n X(u, v) - \partial_v^n X(u_0, v)\|_\alpha^{1\wedge\alpha} + \|\partial_v^n X(u_0, v) - \partial_v^n X(u_0, v_0)\|_\alpha^{1\wedge\alpha}. \end{aligned} \tag{2.15}$$

We have already shown that the second term on the right-hand side of (2.15) converges to 0 as $v \to v_0$. We will now show that, as $u \to u_0$, the first term on the right-hand side of (2.15) also converges to 0, uniformly in $v \in [v_0 - \varepsilon, v_0 + \varepsilon] \subset (0, 1)$ for some $\varepsilon > 0$. This will complete the proof of the corollary.

By using parts (b) and (c) of Theorem 2.1, and (2.11), we obtain

$$
\begin{aligned}
&\|\partial_v^n X_{\beta(\cdot)}(u,v) - \partial_v^n X_{\beta(\cdot)}(u_0,v)\|_\alpha^{1\wedge\alpha} \\
&\quad = \|\partial_v^n X_{\beta(\cdot - u_0)}(|u-u_0|, v)\|_\alpha^{1\wedge\alpha} \\
&\quad = \|\partial_v^n X_{\beta(\cdot)}(|u-u_0|, v)\|_\alpha^{1\wedge\alpha} \\
&\quad = \left\| |u-u_0|^v \sum_{k=0}^{n} \binom{n}{k} \ln^{n-k}|u-u_0| \partial_v^k X_{\beta(|u-u_0|\cdot)}(1,v) \right\|_\alpha^{1\wedge\alpha} \\
&\quad \le |u-u_0|^{(1\wedge\alpha)v} \sum_{k=0}^{n} \binom{n}{k}^{1\wedge\alpha} |\ln(|u-u_0|)|^{(1\wedge\alpha)(n-k)} \|\partial_v^k X(1,v)\|_\alpha^{1\wedge\alpha}. \qquad (2.16)
\end{aligned}
$$

The last relation follows by the triangle inequality for the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$, and the fact that the scale coefficient $\|\cdot\|_\alpha$ does not depend on the skewness intensity β (see (2.11)).

Now let $\varepsilon > 0$ be such that $[v_0-\varepsilon, v_0+\varepsilon] \subset (0,1)$. We have shown that the functions $v \mapsto \|\partial_v^k X(1,v)\|_\alpha^{1\wedge\alpha}$, $v \in (0,1)$, are continuous, and hence that $\sup_{v\in[v_0-\varepsilon,v_0+\varepsilon]} \|\partial_v^k X(1,v)\|_\alpha^{1\wedge\alpha} < \infty$ for all $0 \le k \le n$. This implies that the right-hand side of (2.16) converges to 0, as $u \to u_0$, uniformly in $v \in [v_0-\varepsilon, v_0+\varepsilon]$, since $v_0 - \varepsilon > 0$ and $(\ln|x|)^k = o(|x|^{-(v_0-\varepsilon)})$, $x \to 0$, for all $0 \le k \le n$.

The following results provide Taylor-series-type representations of the field X in terms of its partial derivatives $\partial_v^n X$, $n \in \mathbb{N}$.

Theorem 2.2. *Let* $\alpha \in (0,2]$, $u > 0$, *and* $v_0 \in (0,1)$. *For all* $N \in \mathbb{N}$ *and* $v \in (0,1)$ *such that*

$$
|v - v_0| < R_0 = \min\{v_0, 1 - v_0\},
$$

we have

$$
X(u,v) = \sum_{n=0}^{N-1} \frac{(v-v_0)^n}{n!} \partial_v^n X(u,v_0) + \xi_N(u,v,v_0). \qquad (2.17)
$$

The remainder term $\xi_N(u,v,v_0)$ *in (2.17) is an* α*-stable variable, such that*

$$
\|\xi_N(u,v,v_0)\|_\alpha^{1\wedge\alpha} \le K_\alpha(u,v,v_0)(N+1)^{(7/2)(1\wedge 1/\alpha)} \left(\frac{|v-v_0|}{R_0}\right)^{(1\wedge\alpha)N}, \qquad (2.18)
$$

where $K_\alpha(u,v,v_0) \in \mathbb{R}$ *is a continuous function whose arguments satisfy the above hypotheses. (See (2.29), below, for the exact expression for* K_α*.)*

Proof. By the power series expansion of the exponential function, for all $v, v_0 \in (0,1)$ and $x \in \mathbb{R}$, we have

$$
(x)_\kappa^{v-1/\alpha} = \mathrm{e}^{\ln(x)_\kappa (v-v_0)} (x)_\kappa^{v_0-1/\alpha} = \sum_{n=0}^{\infty} \frac{(v-v_0)^n}{n!} \ln^n(x)_\kappa (x)_\kappa^{v_0-1/\alpha}, \qquad \kappa \in \{+,-\}. \qquad (2.19)
$$

The last series is absolutely convergent for all $x > 0$ or $x < 0$ when $\kappa = +$ or $\kappa = -$, respectively, and by our convention the identity in (2.19) holds trivially for all $x \le 0$ or $x \ge 0$ when $\kappa = +$ or $\kappa = -$, respectively. Thus, by (2.4) and (2.19), we have that, for all

$v, v_0 \in (0,1)$ and $u, s \in \mathbb{R}$,

$$f(u,v,s) = \sum_{n=0}^{\infty} \frac{(v-v_0)^n}{n!} \partial_v^n f(u,v_0,s) = \sum_{n=0}^{N-1} \frac{(v-v_0)^n}{n!} \partial_v^n f(u,v_0,s) + g_{N,v_0}(u,v,s), \tag{2.20}$$

where

$$g_{N,v_0}(u,v,s) := \sum_{n=N}^{\infty} \frac{(v-v_0)^n}{n!} \partial_v^n f(u,v_0,s).$$

By Theorem 2.1, the functions $f(u,v,s)$ and $\partial_v^n f(u,v_0,s)$ belong to $L^\alpha(\mathrm{d}s)$ and therefore so too does $g_{N,v_0}(u,v,s)$. Thus, in view of (2.1), by integrating the expression in (2.20) with respect to the α-stable measure $M_{\alpha,\beta}(\mathrm{d}s)$, we establish (2.17), where

$$\xi_N(u,v,v_0) := X(u,v) - \sum_{n=0}^{N-1} \frac{(v-v_0)^n}{n!} \partial_v^n X(u,v_0) = \int_{\mathbb{R}} g_{N,v_0}(u,v,s) M_{\alpha,\beta}(\mathrm{d}s).$$

We shall now show (2.18), by estimating the quantity

$$A_N := \|\xi_N(u,v,v_0)\|_\alpha^{1\wedge\alpha} = \|g_{N,v_0}(u,v,s)\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}.$$

Relations (2.20) and (2.8) imply that

$$\begin{aligned} A_N &= \left\| \sum_{n=N}^{\infty} \frac{(v-v_0)^n}{n!} \partial_v^n f(u,v_0,s) \right\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha} \\ &= \left\| \sum_{n=N}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(v-v_0)^n}{n!} u^{v_0-1/\alpha} \ln(u)^{n-k} \partial_v^k f(1,v_0,s/u) \right\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}. \end{aligned}$$

By applying the triangle inequality for the metric $\|\cdot\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}$, and by interchanging the order of the summations in the last expression, we obtain

$$\begin{aligned} A_N \le \sum_{k=0}^{\infty} \Bigg\{ & u^{(1\wedge\alpha)v_0} \frac{|v-v_0|^{(1\wedge\alpha)k}}{(k!)^{1\wedge\alpha}} \|\partial_v^k f(1,v_0,s)\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha} \\ & \times \sum_{n=k\vee N}^{\infty} \left(\frac{|(v-v_0)\ln(u)|^{n-k}}{(n-k)!} \right)^{1\wedge\alpha} \Bigg\}, \end{aligned} \tag{2.21}$$

where in the last expression we used the fact that

$$u^{-1/\alpha} \|\partial_v^k f(1,v_0,s/u)\|_{L^\alpha(\mathrm{d}s)} = \|\partial_v^k f(1,v_0,s)\|_{L^\alpha(\mathrm{d}s)}.$$

By using the inequality

$$\sum_{r=n_0}^{\infty} \frac{a^r}{r!} = \sum_{r=0}^{\infty} \frac{a^{n_0+r}}{(n_0+r)!} \le \frac{a^{n_0}}{n_0!} \sum_{r=0}^{\infty} \frac{a^r}{r!} \quad \text{for } a>0,\ n_0 \in \mathbb{N},$$

we can express the inner summation in (2.21) as

$$\sum_{r=(N-k)\vee 0}^{\infty} \left(\frac{|(v-v_0)\ln(u)|^r}{r!} \right)^{1\wedge\alpha} \le \frac{|(v-v_0)\ln(u)|^{(1\wedge\alpha)(N-k)\vee 0}}{[((N-k)\vee 0)!]^{1\wedge\alpha}} S_\alpha((v-v_0)\ln(u)), \tag{2.22}$$

where

$$S_\alpha(x) := \sum_{r=0}^{\infty} \left(\frac{|x|^r}{r!} \right)^{1\wedge\alpha}, \qquad x \in \mathbb{R}. \tag{2.23}$$

Observe that, by the ratio test, for example, the last series converges uniformly in x over all compact intervals $[a, b]$, $a, b \in \mathbb{R}$, and therefore the function $S_\alpha(x)$ is continuous for all $x \in \mathbb{R}$.

Substituting for the inner summation in (2.21) with the right-hand side of (2.22), we obtain that

$$\begin{aligned} A_N &\le u^{(1\wedge\alpha)v_0} S_\alpha((v - v_0)\ln(u)) \\ &\quad \times \sum_{k=0}^{\infty} \frac{|v - v_0|^{(1\wedge\alpha)(N\vee k)}}{[((N-k)\vee 0)!]^{1\wedge\alpha}} \frac{1}{(k!)^{1\wedge\alpha}} \|\partial_v^k f(1, v_0, s)\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}. \end{aligned}$$

Using Proposition 6.1 (with $v = v_0$ therein) to bound $\|\partial_v^k f(1, v_0, s)\|_{L^\alpha(\mathrm{d}s)}^{1\wedge\alpha}$, we get

$$\begin{aligned} A_N &\le 2^{1\wedge 1/\alpha}(|a^-|^\alpha + |a^+|^\alpha)^{1\wedge 1/\alpha} \left(\frac{C}{\alpha^2}\right)^{(1\wedge 1/\alpha)} \frac{u^{(1\wedge\alpha)v_0}}{R_0^{(1\wedge 1/\alpha)}} S_\alpha((v - v_0)\ln(u)) \\ &\quad \times \sum_{k=0}^{\infty} \frac{|v - v_0|^{(1\wedge\alpha)(N\vee k)}}{[((N-k)\vee 0)!]^{1\wedge\alpha}} \frac{(k+1)^{(7/2)(1\wedge 1/\alpha)}}{R_0^{(1\wedge\alpha)k}}, \end{aligned} \tag{2.24}$$

where $R_0 := \min\{v_0, 1 - v_0\}$ and C is a constant. Let $C_\alpha(u, v, v_0)$ denote the coefficient appearing in front of the sum on the right-hand side of (2.24), namely,

$$C_\alpha(u, v, v_0) := 2^{1\wedge 1/\alpha}(|a^-|^\alpha + |a^+|^\alpha)^{1\wedge 1/\alpha} \left(\frac{C}{\alpha^2}\right)^{(1\wedge 1/\alpha)} \frac{u^{(1\wedge\alpha)v_0}}{R_0^{(1\wedge 1/\alpha)}} S_\alpha((v - v_0)\ln(u)). \tag{2.25}$$

By splitting the sum in (2.24) into two parts, we get

$$\begin{aligned} A_N &\le C_\alpha(u, v, v_0) \Bigg\{ \sum_{k=0}^{N} \left(\frac{|v - v_0|}{R_0}\right)^{(1\wedge\alpha)N} (k+1)^{(7/2)(1\wedge 1/\alpha)} \left(\frac{R_0^{N-k}}{(N-k)!}\right)^{1\wedge\alpha} \\ &\qquad + \sum_{k=1}^{\infty} \left(\frac{|v - v_0|}{R_0}\right)^{(1\wedge\alpha)(N+k)} (N + k + 1)^{(7/2)(1\wedge 1/\alpha)} \Bigg\} \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\le C_\alpha(u, v, v_0) \left(\frac{|v - v_0|}{R_0}\right)^{(1\wedge\alpha)N} (N+1)^{(7/2)(1\wedge 1/\alpha)} \\ &\quad \times \left\{ \sum_{k=0}^{N} \left(\frac{R_0^{N-k}}{(N-k)!}\right)^{1\wedge\alpha} + \sum_{k=1}^{\infty} \left(\frac{|v - v_0|}{R_0}\right)^{(1\wedge\alpha)k} \left(\frac{N+k+1}{N+1}\right)^{(7/2)(1\wedge 1/\alpha)} \right\}, \end{aligned} \tag{2.27}$$

where the last expression is obtained by factoring out the term $(|v - v_0|/R_0)^{(1\wedge\alpha)N}$ from the sums in (2.26) and then using the inequality $(k+1)^{(7/2)(1\wedge 1/\alpha)} \le (N+1)^{(7/2)(1\wedge 1/\alpha)}$, valid for all $k = 0, 1, \dots, N$.

Since $(N + k + 1)/(N + 1) \le k + 1$ for all $N, k \ge 1$, we obtain that the second sum in (2.27) is bounded above by

$$\sum_{k=1}^{\infty} \left(\frac{|v - v_0|}{R_0}\right)^{(1\wedge\alpha)k} (k+1)^{(7/2)(1\wedge 1/\alpha)} \le S\left(\frac{|v - v_0|^{1\wedge\alpha}}{R_0^{1\wedge\alpha}}\right), \tag{2.28}$$

where $S(x) := \sum_{k=1}^{\infty}(k+1)^{7/2}x^k$, $x \in \mathbb{R}$. Observe that the radius of convergence of the power series $S(x)$ is 1. Therefore, the right-hand side of (2.28) is finite, and in fact is a continuous function of v and v_0, for all $|v - v_0| < R_0$.

Now, by using (2.28) and the fact that the first sum in (2.27) is bounded above by $S_\alpha(R_0) < \infty$ (see (2.23)), we obtain that (2.18) holds, with

$$K_\alpha(u, v, v_0) = C_\alpha(u, v, v_0)(S_\alpha(R_0) + S(|v - v_0|^{1\wedge\alpha}/R_0^{1\wedge\alpha})). \tag{2.29}$$

In view of the definition of $C_\alpha(u, v, v_0)$ in (2.25) and the definitions of the functions $S_\alpha(\cdot)$ and $S(\cdot)$, above, it follows that the function $K_\alpha(u, v, v_0)$ is continuous for all $u > 0$, v, and v_0 such that $v \in (v_0 - R_0, v_0 + R_0) \subset (0, 1)$. This completes the proof of the theorem.

Relations (2.17) and (2.18) show that the local behaviour of the field $X(u, v)$ around v_0 is determined by the behaviour of the fields $\partial_v^n X(u, v_0)$, $n = 0, 1, \ldots, N - 1$, because the remainder $\xi_N(u, v, v_0)$ is negligible as $v \to v_0$. By setting $N = 1$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.2. *Let $K \subset \mathbb{R}$ be a compact set. As $v \to v_0$,*

$$X(u, v) - X(u, v_0) = \mathcal{O}_{\mathrm{P}}(v - v_0),$$

uniformly for $u \in K$, where $\mathcal{O}_{\mathrm{P}}$ denotes boundedness in probability.

Corollary 2.3. *Let $u \in \mathbb{R}$ and $v, v_0 \in (0, 1)$. If $|v - v_0| < R_0$, then we have*

$$X(u, v) = \sum_{n=0}^{\infty} \frac{(v - v_0)^n}{n!} \partial_v^n X(u, v_0), \tag{2.30}$$

where the series in (2.30) converges uniformly in (u, v) in the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$, over all compact sets $I \subset \mathbb{R} \times (v_0 - R_0, v_0 + R_0)$.

Proof. It suffices to prove that, as $N \to \infty$,

$$A_N(u, v) := \left\| X(u, v) - \sum_{n=0}^{N-1} \frac{(v - v_0)^n}{n!} \partial_v^n X(u, v_0) \right\|_\alpha^{1\wedge\alpha} \to 0, \tag{2.31}$$

uniformly in $(u, v) \in I$, for all $I = [a, b] \times [v_0 - R, v_0 + R]$, where $a < b$, $a, b \in \mathbb{R}$, and $0 < R < R_0$.

We first consider the case $I = [a, b] \times [v_0 - R, v_0 + R]$, where $a > 0$. Relation (2.18) in Theorem 2.2 implies the convergence in (2.31), for all $(u, v) \in I$, because $|v - v_0|/R_0 \leq R/R_0 < 1$ and therefore

$$A_N(u, v) \leq K_\alpha(u, v, v_0)(N + 1)^{(7/2)(1\wedge 1/\alpha)}(R/R_0)^{(1\wedge\alpha)N} \to 0 \tag{2.32}$$

as $N \to \infty$. The last convergence is uniform in $(u, v) \in I$, because the function $K_\alpha(u, v, v_0)$ is continuous in I and it does not depend on N. The fact that $0 < a \leq u$ is important, since Theorem 2.2 involves the assumption that $u > 0$.

Now let $I = [a, b] \times [v_0 - R, v_0 + R]$, where $a < b$, $a, b \in \mathbb{R}$ are arbitrary, and $0 < R < R_0$. Also, let $h > 0$ be such that $a + h > 0$. By using Theorem 2.1(b), we obtain

$$\begin{aligned}
&\left\| X_\beta(u+h, v) - X_\beta(h, v) - \sum_{n=0}^{N-1} \frac{(v - v_0)^n}{n!} (\partial_v^n X_\beta(u+h, v_0) - \partial_v^n X_\beta(h, v_0)) \right\|_\alpha^{1\wedge\alpha} \\
&\quad = \left\| X_{\beta(\cdot - h)}(u, v) - \sum_{n=0}^{N-1} \frac{(v - v_0)^n}{n!} \partial_v^n X_{\beta(\cdot - h)}(u, v_0) \right\|_\alpha^{1\wedge\alpha} = A_N(u, v), \qquad (2.33)
\end{aligned}$$

because the scale coefficient $\| \cdot \|_\alpha$ does not depend on β (see (2.11)). Now, by applying the triangle inequality for the metric $\| \cdot \|_\alpha^{1\wedge\alpha}$ to the left-hand side of (2.33), we get

$$A_N(u, v) \le A_N(u + h, v) + A_N(h, v). \qquad (2.34)$$

Since $u + h \ge a + h > 0$ for all $(u, v) \in I$, and since $h > 0$, (2.32) implies that the right-hand side of (2.34) converges to 0, as $N \to \infty$, uniformly in $(u, v) \in I$. This completes the proof of the corollary.

3. Stochastic continuity of LMSM

In this section, we give necessary and sufficient conditions for the continuity in probability (i.e. the stochastic continuity) of the process Y at a point $t_0 \in \mathbb{R}$. If a stable process has *stationary increments* or is *stationary*, then mere measurability implies continuity in probability. This is proved in Surgailis *et al.* (1998) in the case of a stationary increment process, and in Pipiras and Taqqu (2004) in the case of a stationary process. The LMSM process Y, however, is typically neither stationary nor has stationary increments. In fact, the following theorem shows that the process Y will *not* be continuous in probability if H is not continuous. Recall that we assume that $0 < H(t) < 1$.

Theorem 3.1. *Let $\alpha \in (0, 2]$ and let Y be an LMSM process defined in (1.2). The process $Y(t)$, $t \in \mathbb{R}$, is continuous in probability at $t_0 \in \mathbb{R}$, $t_0 \neq 0$, if and only if the local scaling exponent function $0 < H(t) < 1$, $t \in \mathbb{R}$, is continuous at t_0.*

Proof. The 'if' part of the statement follows, in view of (2.2), from Corollary 2.1.

Assume now that the process $Y(t)$ is continuous in probability at $t_0 \neq 0$. Let t_n, $n \in \mathbb{N}$, be an arbitrary real sequence such that $t_n \to t_0$ as $n \to \infty$. Since the function $H(t)$, $t \in \mathbb{R}$, is bounded, the Bolzano–Weierstrass theorem implies that

$$H(t_{n_k}) \to H^*, \qquad 0 \le H^* \le 1, \qquad (3.1)$$

as $k \to \infty$, for some subsequence n_k, $k \in \mathbb{N}$. We will now show that $H^* = H(t_0)$, which will complete the proof of the theorem.

(i) *The case* $\alpha \neq 1$. We will show that $0 < H^* < 1$. Indeed, suppose that $H^* = 0$ or $H^* = 1$. In view of (2.2), Theorem 2.1(c), (2.9), and (2.11) imply that

$$\begin{aligned}
\|Y_\beta(t_{n_k})\|_\alpha &= |t_{n_k}|^{H(t_{n_k})} \|X_{\beta(|t_{n_k}|\cdot)}(\operatorname{sign}(t_{n_k}), H(t_{n_k}))\|_\alpha \\
&= |t_{n_k}|^{H(t_{n_k})} \|X_\beta(1, H(t_{n_k}))\|_\alpha. \qquad (3.2)
\end{aligned}$$

Since $\alpha \neq 1$ and $H^* \neq 1/\alpha$, the bound in (6.17) of Lemma 6.3, below, applied with $v := H(t_{n_k})$, explodes as $k \to \infty$. Therefore $\|X_\beta(1, H(t_{n_k}))\|_\alpha = \|X(1, H(t_{n_k}))\|_\alpha \to \infty$ and, since

$|t_{n_k}|^{H(t_{n_k})} \to |t_0|^{H^*} > 0$ as $k \to \infty$, the right-hand side of (3.2) converges to infinity as $k \to \infty$. This, however, is impossible, because the continuity in probability of the process Y implies that $\|Y(t_n)\|_\alpha \to \|Y(t_0)\|_\alpha$ as $n \to \infty$, where $\|Y(t_0)\|_\alpha < \infty$.

Thus, we have that $0 < H^* < 1$. Now, by using the continuity in probability of the process Y and the fact that, for all $t \in \mathbb{R}$, $Y(t) = X(t, H(t))$ almost surely, we obtain that

$$Y(t_{n_k}) - Y(t_0) = X(t_{n_k}, H(t_{n_k})) - X(t_0, H(t_0)) \xrightarrow{\mathrm{P}} 0 \quad \text{as } k \to \infty. \tag{3.3}$$

On the other hand, by the continuity in probability of the field X (Corollary 2.1), we have that

$$X(t_{n_k}, H(t_{n_k})) - X(t_0, H^*) \xrightarrow{\mathrm{P}} 0 \quad \text{as } k \to \infty, \tag{3.4}$$

where $X(t_0, H^*)$ is well defined, since $0 < H^* < 1$. Relations (3.3) and (3.4) imply that $Y(t_0) \equiv X(t_0, H(t_0)) = X(t_0, H^*)$ and, therefore, $\int_{\mathbb{R}} |f(t_0, H(t_0), s) - f(t_0, H^*, s)|^\alpha \, \mathrm{d}s = 0$. Hence $f(t_0, H(t_0), s) = f(t_0, H^*, s)$ almost everywhere (a.e.) $\mathrm{d}s$. The last fact implies that $H(t_0) = H^*$, since $t_0 \neq 0$ (see Relation (2.1)).

(ii) *The case* $\alpha = 1$. If $0 < H^* < 1$, then we can use the same argument as in case (i) to show that $H(t_0) = H^*$. As in the case $\alpha \neq 1$, we cannot have $H^* = 0$, since the bounds in Lemma 6.3 explode at $v = 0$. We will now prove that $H^* \neq 1$, which will complete the proof of the theorem.

Without loss of generality, we assume that $a^+ \neq 0$ (recall that $|a^+| + |a^-| > 0$). Let $\mathbf{1}_{\{s \geq |t_0|+1\}}$ denote the indicator function of the set $\{s \geq |t_0| + 1\}$ and let

$$\begin{aligned} f_0(s) &= f(t_0, H(t_0), s)\mathbf{1}_{\{s \geq |t_0|+1\}} \\ &= a^+((t_0 + s)^{H(t_0)-1} - s^{H(t_0)-1}), \qquad s \geq |t_0| + 1. \end{aligned}$$

Since $t_{n_k} \to t_0 \neq 0$ as $k \to \infty$, for sufficiently large k we have that

$$\begin{aligned} f_k(s) &= f(t_{n_k}, H(t_{n_k}), s)\mathbf{1}_{\{s \geq |t_0|+1\}} \\ &= a^+((t_{n_k} + s)^{H(t_{n_k})-1} - s^{H(t_{n_k})-1}), \qquad s \geq |t_0| + 1. \end{aligned}$$

Since $\alpha = 1$, we have $\|f_k(s) - f_0(k)\|_{L^1(\mathrm{d}s)} \leq \|Y(t_{n_k}) - Y(t_0)\|_\alpha$, and (3.3) implies that

$$\|f_k(s) - f_0(s)\|_{L^1(\mathrm{d}s)} \to 0 \quad \text{as } n \to \infty.$$

Hence, in particular, the functions $f_k(s)$ converge in measure to the function $f_0(s)$ as $k \to \infty$ and, therefore, there exists a subsequence $k_m \in \mathbb{N}$, $m = 1, 2, \ldots$, such that, as $m \to \infty$,

$$f_{k_m}(s) \to f_0(s) \quad \text{a.e. } \mathrm{d}s. \tag{3.5}$$

Suppose now that $H^* = 1$ and let $\tau_k := t_{n_k}$, $k \in \mathbb{N}$. Relation (3.1) implies that $H(\tau_{k_m}) \to H^* = 1$ as $m \to \infty$. Since $\tau_{k_m} \to t_0 > 0$ as $m \to \infty$, for all $s \geq |t_0| + 1$ we have that

$$f_{k_m}(s) = a^+((\tau_{k_m} + s)^{H(\tau_{k_m})-1} - s^{H(\tau_{k_m})-1}) \to a^+(1 - 1) = 0 \quad \text{as } m \to \infty.$$

Thus, (3.5) implies that $f_0(s) = 0$ a.e. $\mathrm{d}s$. However, in view of (2.1), this is impossible, since $t_0 \neq 0$, $a^+ \neq 0$, and $0 < H(t_0) < 1$. This shows that $H^* \neq 1$, and completes the proof of the theorem.

Theorem 3.1 gave necessary and sufficient conditions for the continuity of Y at a point $t_0 \neq 0$. The conditions for the continuity of Y at the origin $t_0 = 0$ are quite different.

Theorem 3.2. *Let $\alpha \in (0, 2]$ and let Y be an LMSM process defined in (1.2). The process $Y(t)$, $t \in \mathbb{R}$, is continuous in probability at $t = 0$ if and only if,*

(a) *for $\alpha \neq 1$,*

$$|t|^{H(t)}\left(\frac{1}{H(t)^{1/\alpha}} + \frac{1}{|1 - H(t)|^{1/\alpha}}\right) \to 0 \quad as\ t \to 0, \tag{3.6}$$

(b) *for $\alpha = 1$,*

$$|t|^{H(t)}\left(\frac{1}{H(t)^{1/\alpha}}\right) \to 0 \quad as\ t \to 0. \tag{3.7}$$

Proof. Since $Y(0) = 0$ almost surely, the process Y is stochastically continuous at $t = 0$ if and only if

$$\|Y(t)\|_\alpha = |t|^{H(t)}\|X(1, H(t))\|_\alpha \to 0 \quad \text{as } t \to 0 \tag{3.8}$$

(see (3.2)). We will now show that the conditions in (3.6) and (3.7) are equivalent to (3.8).

We first prove part (a) of the theorem. Let $\alpha \neq 1$ and assume that (3.6) holds. Then, by using (6.16) with $v := H(t)$, and the fact that $|H(t) - 1/\alpha| < (1/\alpha) \vee (1 - 1/\alpha)$, we get that

$$|t|^{H(t)}\|X(1, H(t))\|_\alpha \le \text{const.}\, |t|^{H(t)}\left(\frac{1}{H(t)^{1/\alpha}} + \frac{1}{(1 - H(t))^{1/\alpha}}\right) \to 0$$

as $t \to 0$. This, by (3.8), implies the continuity in probability of the process Y at $t = 0$.

We will now show that the convergence in (3.6) follows from (3.8), which will complete the proof of part (a). Using the bound in (6.17) of Lemma 6.3, and (3.8), we get that

$$|t|^{H(t)}\frac{|H(t) - 1/\alpha|}{(1 - H(t))^{1/\alpha}} \to 0 \quad \text{and} \quad |t|^{H(t)}\frac{\mathbf{1}_{(0,1/\alpha)}(H(t))}{2^{(\alpha+1)/(\alpha|H(t)\alpha-1|)}}\frac{1}{H(t)^{1/\alpha}} \to 0 \tag{3.9}$$

as $t \to 0$. Assume that (3.6) does not hold. Then, for some sequence t_n, such that $t_n \to 0$ as $n \to \infty$, either

$$\frac{|t_n|^{H(t_n)}}{H(t_n)^{1/\alpha}} \to c \quad \text{as } n \to \infty \tag{3.10}$$

or

$$\frac{|t_n|^{H(t_n)}}{(1 - H(t_n))^{1/\alpha}} \to c \quad \text{as } n \to \infty, \tag{3.11}$$

where $c > 0$ or $c = \infty$.

If (3.10) holds, then (3.9) implies (in particular) that *either* $\mathbf{1}_{(0,1/\alpha)}(H(t_n)) \to 0$ *or* $H(t_n) \to 1/\alpha$ as $n \to \infty$, for some sequence $t_n \to 0$, $n \to \infty$. Thus, for sufficiently large n, $H(t_n) \ge \frac{1}{2}\alpha$ and, hence, since $t_n \to 0$, $n \to \infty$,

$$\frac{|t_n|^{H(t_n)}}{H(t_n)^{1/\alpha}} \le (2\alpha)^{1/\alpha}|t_n|^{(1/2)\alpha} \to 0 \quad \text{as } n \to \infty.$$

This contradicts (3.10).

Suppose now that (3.11) holds. Then, by (3.9), we have that $H(t_n) - 1/\alpha \to 0$ as $n \to \infty$ and, since $0 < H(t_n) < 1$, $\alpha \in (0, 2]$, and $\alpha \neq 1$, this implies that $1 - H(t_n) \to 1 - 1/\alpha > 0$ as $n \to \infty$. Hence $|t_n|^{H(t_n)}/(1 - H(t_n))^{1/\alpha} \to 0$ as $n \to \infty$, which contradicts (3.11). We have thus shown that neither (3.10) nor (3.11) is possible, which completes the proof of part (a) of the theorem.

Now we focus on part (b), where $\alpha = 1$. The inequality (6.16) implies that

$$|t|^{H(t)} \|X(1, H(t))\|_1 \le |t|^{H(t)} (|a^-| + |a^+|) \left(1 + \frac{10}{H(t)}\right) \le \text{const.}\, |t|^{H(t)}/H(t),$$

where the last inequality follows from the fact that $1 \le 1/H(t)$. Therefore (3.7) implies (3.8) and, hence, the continuity in probability of the process $Y(t)$ at $t = 0$.

Suppose now that (3.8) holds. By (6.17), we have that

$$|t|^{H(t)} \frac{\mathbf{1}_{(0,1)}(H(t))}{2^{2/|H(t)-1|}} \frac{1}{H(t)} \to 0 \quad \text{as } t \to 0.$$

As in the proof of part (a) above, we can show that the last relation implies (3.7). This completes the proof of the theorem.

Remark 3.1. Theorem 3.1 states that the continuity of the function H is a necessary and sufficient condition for the continuity in probability of the process Y. This is true at points $t_0 \neq 0$. If $t_0 = 0$, the necessary and sufficient condition on H, given by (3.6) and (3.7) in Theorem 3.2, is weaker. It is sufficient, for example, that $H(t)$ be continuous at $t = 0$ since this implies (3.6) and (3.7), given that $0 < H(t) < 1$. The continuity of $H(t)$ at $t = 0$ is, however, not necessary. Using Theorem 3.2, the following, less restrictive condition is also sufficient:

$$0 < \liminf_{t\to 0} H(t) \le \limsup_{t \to 0} H(t) < 1, \tag{3.12}$$

namely, that $H(t)$ be bounded away from 0 and 1 as $t \to 0$. Condition (3.12) is not necessary either. Indeed, if $\alpha = 1$, for example, let $H(0) := \frac{1}{2}$ and $H(t) := (-1/\ln(t))^{\gamma}$, $0 < t < \mathrm{e}^{-1}$ with $0 < \gamma < 1$. Then $\liminf_{t\to 0} H(t) = 0$ and Y is nevertheless continuous in probability at 0, since condition (3.7) holds.

Remark 3.2. Consider the function $H(t) := (-1/\ln(t))^{\gamma}, 0 < t < \mathrm{e}^{-1}$, with $H(0) := \frac{1}{2}$, where now $\gamma \ge 1$. The LMSM process $Y(t)$ with local scaling exponent function $H(t)$, $t \in [0, \mathrm{e}^{-1})$, is *not* continuous in probability at $t = 0$, because $|t|^{H(t)}/H(t)^{1/\alpha} \not\to 0$, as $t \downarrow 0$, and hence the conditions (3.6) and (3.7) of Theorem 3.2 do not hold.

4. The case $H(t) \in (0, 1) \cup \{1/\alpha\}$ when $0 < \alpha \le 1$

In the definition of the LMSM process $Y(t)$ (see (1.2)) we have supposed that $0 < H(t) < 1$, but $H(t)$ can also take the value $1/\alpha$, which can be greater than or equal to 1 when $0 < \alpha \le 1$. Indeed, by (1.3), when $H(t) \equiv 1/\alpha$, the kernel function of the integral in (1.2) becomes

$$\begin{aligned} f(t, 1/\alpha, s) &:= a^+(\mathbf{1}_{(-t,\infty)}(s) - \mathbf{1}_{(0,\infty)}(s)) - a^-(\mathbf{1}_{(-\infty,-t)}(s) - \mathbf{1}_{(-\infty,0)}(s)) \\ &= \begin{cases} a^+\,\mathbf{1}_{(-t,0]}(s) - a^-\,\mathbf{1}_{[-t,0)}(s), & t > 0, \\ -a^+\,\mathbf{1}_{(0,-t]}(s) + a^-\,\mathbf{1}_{[0,-t)}(s), & t \le 0. \end{cases} \end{aligned}$$

In this case, for all $\alpha \in (0, 2]$, the integral in (1.2) is well defined and is equal to

$$X(t, 1/\alpha) = Y(t) = (a^+ - a^-) \int_{-t}^{0} M_{\alpha,\beta}(\mathrm{d}s), \qquad t \in \mathbb{R}, \tag{4.1}$$

where $\int_a^b = -\int_b^a$, $a, b \in \mathbb{R}$, by convention. If the skewness intensity of $M_{\alpha,\beta}(\mathrm{d}s)$ is constant then the process $Y(t),\ t \in \mathbb{R}$, in (4.1) is $(1/\alpha)$-self-similar, and it has *stationary* and *independent increments*. $Y(t)$ is an α-stable Lévy motion if $a^+ \neq a^-$. Observe that for an LMSM process $Y(t),\ t \in \mathbb{R}$, we have the following lemma.

Lemma 4.1. *If $t \neq 0$ then $Y(t) = 0$ a.s. if and only if $a^+ = a^-$ and $H(t) = 1/\alpha$.*

Proof. By (4.1), $Y(t) = 0$ if $a^+ = a^-$ and $H(t) = 1/\alpha$. Suppose now that $Y(t) = 0, t \neq 0$. Then, (1.2) implies that

$$f(t, H(t), s) = \sum_{\kappa \in \{+,-\}} a^\kappa ((t+s)_\kappa^{H(t)-1/\alpha} - (s)_\kappa^{H(t)-1/\alpha}) = 0,$$

for almost all $s \in \mathbb{R}$. If $a^+ \neq 0$ then $f(t, H(t), s) = a^+((t+s)^{H(t)-1/\alpha} - s^{H(t)-1/\alpha}) = 0$ for all $s \geq |t|$. Since $t \neq 0$, this implies that $H(t) = 1/\alpha$. If $a^+ = 0$ and $a^- \neq 0$, one similarly gets $H(t) = 1/\alpha$, by considering that $f(t, H(t), s) = 0$ for $s \leq -|t|$. Finally, since $H(t) = 1/\alpha$, (4.1) holds and implies that $a^+ = a^-$.

When $1 < \alpha \leq 2$, setting $H(t) = 1/\alpha$ yields nothing new, since $1/\alpha \in [\frac{1}{2}, 1)$, which falls in the range $(0, 1)$ of $H(t)$. This case is covered in Section 3. When $0 < \alpha \leq 1$, however, the process $Y(t)$ is defined for functions $H(t),\ t \in \mathbb{R}$, taking values in the range $(0, 1) \cup \{1/\alpha\}$ and, correspondingly, one can define the field $X(u, v)$ for all $u \in \mathbb{R}$ and $v \in (0, 1) \cup \{1/\alpha\}$. The goal of this section is to describe what happens in this situation.

The following two results extend Theorems 3.1 and 3.2 of Section 3, respectively.

Theorem 4.1. *Let $0 < \alpha < 1$ and let $Y(t)$ be an LMSM process with local scaling exponent function $H(t) \in (0, 1) \cup \{1/\alpha\},\ t \in \mathbb{R}$. Then $Y(t)$ is continuous in probability at $t = t_0$, where $t_0 \neq 0$, if and only if the function $H(t)$ is continuous at $t = t_0$.*

Proof. Suppose that $H(t)$ is continuous at $t = t_0$. Since $1/\alpha \notin [0, 1]$, we have that *either* $H(t) \in (0, 1)$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ *or* $H(t) \equiv 1/\alpha$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, for some $\varepsilon > 0$. In the first case, the continuity in probability of $Y(t)$ at $t = t_0$ follows from Theorem 3.1 (see also Corollary 2.1). In the second case, $Y(t),\ t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, coincides with the process $X(t, 1/\alpha) = X_\beta(t, 1/\alpha),\ t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, which, by (4.1), is continuous in probability.

Conversely, suppose that $Y(t) \xrightarrow{\mathrm{P}} Y(t_0),\ t \in t_0$, and consider an arbitrary sequence $t_n \to t_0$, $n \to \infty$.

(i) If $H(t_n) \in (0, 1)$ for all sufficiently large $n \in \mathbb{N}$, then by using the same argument as in the proof of Theorem 3.2, we can show that necessarily $H(t_n) \to H(t_0)$, where $H(t_0) \in (0, 1)$.

(ii) Assume now that $H(t_n) \equiv 1/\alpha$ for all sufficiently large $n \in \mathbb{N}$; then (4.1) implies that $Y(t_n) = X(t_n, 1/\alpha) \xrightarrow{\mathrm{P}} X(t_0, 1/\alpha)$. Hence $Y(t_0) = X(t_0, H(t_0)) = X(t_0, 1/\alpha)$ and therefore $\int_{\mathbb{R}} |f(t_0, H(t_0), s) - f(t_0, 1/\alpha, s)|^\alpha \,\mathrm{d}s = 0$, by (2.1). Since $t_0 \neq 0$, this implies that $H(t_0) = 1/\alpha$.

The case when $H(t_n) \in (0, 1)$ for infinitely many n and $H(t_n) = 1/\alpha$ for infinitely many (other) n cannot occur. Indeed, by taking appropriate subsequences and applying the arguments from (i) and (ii), we conclude that $H(t_0) \in (0, 1)$ and $H(t_0) = 1/\alpha$, which is impossible since $1 < 1/\alpha$. Thus, $H(t_n) \to H(t_0)$ as $n \to \infty$, which concludes the proof of the theorem.

The result of Theorem 3.2, on the continuity of the LMSM process at the origin, remains valid when $0 < \alpha \leq 1$ and the function $H(t)$ is allowed to take values in the extended range $(0, 1) \cup \{1/\alpha\}$. Indeed, we can prove the following theorem.

Theorem 4.2. *Let $0 < \alpha \le 1$ and let $Y(t)$ be an LMSM process with local scaling exponent function $H(t) \in (0,1) \cup \{1/\alpha\}$, $t \in \mathbb{R}$. Then $Y(t)$ is continuous in probability at $t = 0$ if and only if condition (3.6) or (3.7) holds (for the appropriate value of α).*

Proof. Let $t_n \to 0$, $n \to \infty$, be an arbitrary sequence. We will show that, as $n \to \infty$,

$$Y(t_n) \xrightarrow{\mathrm{P}} Y(0) \Longleftrightarrow \begin{cases} |t_n|^{H(t_n)}(1/H(t_n)^{1/\alpha} + 1/|1 - H(t_n)|^{1/\alpha}) \to 0 & \text{if } 0 < \alpha < 1, \\ |t_n|^{H(t_n)}(1/H(t_n)^{1/\alpha}) \to 0 & \text{if } \alpha = 1. \end{cases} \tag{4.2}$$

(i) If $H(t_n) \in (0,1)$ for sufficiently large n, then one can show (4.2) by using the same argument as in the proof of Theorem 3.2.

(ii) If $H(t_n) = 1/\alpha$ for sufficiently large n, then the convergences on the right-hand side of (4.2) hold trivially and, by (4.1), one also has that $Y(t_n) = X(t_n, 1/\alpha) \xrightarrow{\mathrm{P}} 0 = Y(0)$ as $n \to \infty$.

(iii) Suppose now that $\{n_k\}_{k\in\mathbb{N}} \cup \{m_k\}_{k\in\mathbb{N}} = \mathbb{N}$, where $n_k \to \infty$ and $m_k \to \infty$ as $k \to \infty$, and where $H(t_{n_k}) \in (0,1)$ and $H(t_{m_k}) = 1/\alpha$. Then the equivalence (4.2) follows by applying the arguments in (i) and (ii) above to the sequences $\{t_{n_k}\}_k$ and $\{t_{m_k}\}_k$, respectively.

We have thus shown (4.2) for an arbitrary sequence $t_n \to 0, n \to \infty$. This implies the statement of the theorem.

Theorem 4.1 covers the case $0 < \alpha < 1$, where the range $(0,1) \cup \{1/\alpha\}$ is disconnected, and it implies that Y cannot be continuous if the values of $H(t)$ jump from $(0,1)$ to $1/\alpha$ or vice versa. When $\alpha = 1$, however, the range $(0,1) \cup \{1/\alpha\}$ becomes the interval $(0,1]$, which is connected. It is interesting to see whether or not, in this case, the process $Y(t)$ can be continuous in probability at $t = t_0$, $t_0 \neq 0$, if $H(t) \to H(t_0) = 1$ as $t \to t_0$. In fact, Corollary 4.1, below, shows that this is not possible. This follows from the next result.

Theorem 4.3. *Let $\alpha = 1$ and let $u_n \in \mathbb{R}$ and $v_n \in (0,1)$, $n \in \mathbb{N}$, be arbitrary sequences such that*

$$u_n \to u \neq 0 \quad \textit{and} \quad v_n \to 1 \quad \textit{as } n \to \infty.$$

Then the sequence of random variables $\xi_n := X(u_n, v_n)$ does not converge in probability as $n \to \infty$.

Proof. Suppose that $\xi_n \xrightarrow{\mathrm{P}} \xi$ as $n \to \infty$, for some random variable ξ. Since ξ_n, $n \in \mathbb{N}$, have the integral representations

$$\xi_n = \int_{\mathbb{R}} f(u_n, v_n, s) M_{\alpha,\beta}(\mathrm{d}s),$$

with $\beta(s) \equiv 0$, $s \in \mathbb{R}$ (see (2.1)), it follows that

$$\xi = \int_{\mathbb{R}} g(s) M_{\alpha,\beta}(\mathrm{d}s)$$

for some $g(s) \in L^\alpha(\mathrm{d}s)$ and, in fact,

$$\|\xi_n - \xi\|_\alpha^\alpha = \|\xi_n - \xi\|_1 = \int_{\mathbb{R}} |f(u_n, v_n, s) - g(s)|\,\mathrm{d}s \to 0 \tag{4.3}$$

as $n \to \infty$. Relation (4.3) implies that the functions $|f(u_n, v_n, s) - g(s)|$ converge in measure to 0 as $n \to \infty$. Therefore, there exists a subsequence $n_k \to \infty$ such that $|f(u_{n_k}, v_{n_k}, s) - g(s)| \to 0$ a.e. ds as $k \to \infty$.

However, observe that, for all $s \in \mathbb{R}$,

$$\begin{aligned} f(u_{n_k}, v_{n_k}, s) &= a^+((u_{n_k} + s)_+^{v_{n_k}-1/\alpha} - s_+^{v_{n_k}-1/\alpha}) + a^-((u_{n_k} + s)_-^{v_{n_k}-1/\alpha} - s_-^{v_{n_k}-1/\alpha}) \\ &\to a^+((u+s)_+^0 - s_+^0) + a^-((u+s)_-^0 - s_-^0) \end{aligned}$$

as $k \to \infty$, because $v_{n_k} - 1/\alpha = v_{n_k} - 1 \to 0$. Hence $g(s) = a^+((u+s)_+^0 - s_+^0) + a^-((u+s)_-^0 - s_-^0)$ a.e. ds and, thus,

$$\xi = (a^+ - a^-) \int_{-u}^{0} M_{\alpha,\beta}(\mathrm{d}s). \tag{4.4}$$

We will now show that this is impossible. Without loss of generality, we assume that $a^+ \neq 0$ (recall that $|a^+| + |a^-| > 0$). Since $u_n \to u$, $n \to \infty$, for sufficiently large n, $|u_n| + 1 \geq |u|$ and, therefore, by (4.4), we have

$$\|\xi_n - \xi\|_1 \geq \int_{|u_n|+1}^{\infty} |f(u_n, v_n, s) - g(s)|\,\mathrm{d}s = |a^+| \int_{|u_n|+1}^{\infty} |(u_n + s)^{v_n - 1} - s^{v_n - 1}|\,\mathrm{d}s, \tag{4.5}$$

because $g(s) = a^+((u+s)_+^0 - s_+^0) \equiv 0$, for all $s > |u|$. By applying the mean value theorem to the integrand in (4.5), and using the fact that $v_n - 2 < 0$, we find that

$$\begin{aligned} \|\xi_n - \xi\|_1 &\geq |a^+||u_n| \int_{|u_n|+1}^{\infty} |v_n - 1|(s + |u_n|)^{v_n - 2}\,\mathrm{d}s \\ &= |a^+||u_n|(2|u_n| + 1)^{v_n - 1} \to |a^+||u| > 0 \end{aligned}$$

as $n \to \infty$. This contradicts (4.3) and shows that the convergence $\xi_n \xrightarrow{\mathrm{P}} \xi, n \to \infty$, is impossible.

Theorem 4.3 implies the following result.

Corollary 4.1. *Suppose that* $\alpha = 1$. *Then,*

(i) *for all* $u \in \mathbb{R}, u \neq 0$, *the process* $\{X(u, v),\ v \in (0, 1]\}$ *is discontinuous in probability at* $v = 1$;

(ii) *the LMSM process* $Y(t)$ *with continuous local scaling exponent function* $H(t) \in (0, 1]$ *is discontinuous in probability if* $H(t) \not\equiv 1$ *and* $H(t_0) = 1$ *for some* $t_0 \neq 0$.

5. The local asymptotic behaviour of the LMSM process

In this section, we study the local behaviour of $Y(t)$ in terms of its finite-dimensional distributions. We show that the process $Y(t),\ t \in \mathbb{R}$, defined in (1.2), is locally equivalent, in the sense of finite-dimensional distributions, to an LFSM process when the scaling exponent function $H(t),\ t \in \mathbb{R}$, has sufficient pointwise Hölder regularity and when the skewness intensity function $\beta(s)$ of the measure $M_{\alpha,\beta}(\mathrm{d}s),\ s \in \mathbb{R}$, is continuous. If the scaling exponent function H is not *sufficiently* regular, then the LMSM process Y can be *locally asymptotically degenerate*.

Theorem 5.1. *Let $\alpha \in (0, 2]$ and let Y be an LMSM process defined in (1.2), where the function $H(t)$, $t \in \mathbb{R}$, is continuous at $t_0 \in \mathbb{R}$ and satisfies (1.1).*

(a) *Assume that the skewness intensity function $\beta(s)$, $s \in \mathbb{R}$, of the measure $M_{\alpha,\beta}(\mathrm{d}s)$ is continuous. If $t_0 \neq 0$ and*

$$H(t) - H(t_0) = o(|t - t_0|^{H(t_0)}) \quad \text{as } t \to t_0, \tag{5.1}$$

then, for all $t \in \mathbb{R}$, as $\lambda \to 0$, $\lambda > 0$, we have that

$$\left\{ \frac{1}{\lambda^{H(t_0)}} (Y_\beta(\lambda t + t_0) - Y_\beta(t_0)), t \in \mathbb{R} \right\} \xrightarrow{\text{FDD}} \{X_{H(t_0),\alpha,\beta(-t_0)}(t), t \in \mathbb{R}\}, \tag{5.2}$$

where

$$X_{H(t_0),\alpha,\beta(-t_0)}(t) = \int_{\mathbb{R}} f(t, H(t_0), s) M_{\alpha,\beta(-t_0)}(\mathrm{d}s)$$

is an LFSM process, defined as in (1.4), and '$\xrightarrow{\text{FDD}}$' denotes convergence in finite-dimensional distributions. Its α-stable measure $M_{\alpha,\beta(-t_0)}(\mathrm{d}s)$ has a constant skewness intensity equal to $\beta(-t_0) \in [-1, 1]$.

(b) *If $t_0 \neq 0$ and, for all $t > t_0$,*

$$H(t) - H(t_0) = (t - t_0)^\rho L(t - t_0) \quad \text{with } 0 < \rho < H(t_0),$$

then, for all $t \in \mathbb{R}$, $t > 0$, we have that

$$\frac{1}{\lambda^\rho L(\lambda)} (Y_\beta(\lambda t + t_0) - Y_\beta(t_0)) \xrightarrow{\text{P}} t^\rho \xi_\beta \quad \text{as } \lambda \to 0, \lambda > 0, \tag{5.3}$$

where $\xi_\beta = \partial_v X_\beta(t_0, H(t_0))$ is a nondegenerate α-stable random variable, defined in (2.3). Here, $L(s)$ is a slowly varying function as $s \to 0$, $s > 0$, and is defined on the interval $(0, \infty)$.

(c) *Assume that the skewness intensity function $\beta(s)$, $s \in \mathbb{R}$, of the measure $M_{\alpha,\beta}(\mathrm{d}s)$ is continuous. If $t_0 = 0$ and $H(t) - H(0) = o(1/\ln|t|)$ as $t \to 0 = t_0$, then the convergence in (5.2) also holds.*

We shall first comment on the above result and then present its proof.

Remark 5.1. From a physical perspective, focusing on

$$\frac{1}{d(\lambda)} (Y(\lambda t + t_0) - Y(t_0)),$$

as $\lambda \downarrow 0$, means zooming in on the process Y at the point t_0. The normalization factor $d(\lambda)$, which tends to 0 as $\lambda \downarrow 0$ (it equals $\lambda^{H(t_0)}$ in the cases (a) and (c), and equals $\lambda^\rho L(\lambda)$ in case (b)), indicates the amount of spatial rescaling that is involved.

Remark 5.2. Part (b) of Theorem 5.1 involves a function $H(t)$ that is less *regular* than in part (a). In fact, the conditions on $H(t)$ in parts (a) and (b) are mutually exclusive. Indeed, if the condition of part (b) holds, then

$$\frac{|H(t) - H(t_0)|}{(t - t_0)^{H(t_0)}} = (t - t_0)^{\rho - H(t_0)} |L(t - t_0)|, \qquad t > t_0,$$

which converges to infinity as $t \to t_0$, $t > t_0$, since $0 < \rho < H(t_0)$.

Remark 5.3. Observe that the assumption in part (a) of Theorem 5.1 is satisfied if the function $H(t),\ t \in \mathbb{R}$, is differentiable at $t_0 \neq 0$. For example, the process $Y(t),\ t \in \mathbb{R}$, is *locally equivalent* to an LFSM process, in the sense of (5.2), if the local scaling exponent function $H(t),\ t \in \mathbb{R}$, is continuously differentiable.

Part (b) of the theorem shows that the process Y is *locally degenerate*, in the sense of (5.3), for a large class of *regularly varying* functions $H(t),\ t \in \mathbb{R}$, which have pointwise Hölder regularity exponents lower than $H(t)$. The convergence in probability in (5.3) also implies convergence in the sense of finite-dimensional distributions.

Remark 5.4. Part (c) of Theorem 5.1 shows, in particular, that at time 0 the process Y will be locally equivalent to an LFSM, provided both that the function $H(t)$ is Hölder continuous at $t = 0$, that is, $|H(t) - H(0)| = o(|t|^\delta),\ t \to 0$, for some $\delta > 0$, and that the measure $M_{\alpha,\beta}(\mathrm{d}s)$ has continuous skewness intensity $\beta(s),\ s \in \mathbb{R}$. In this case, the limiting process for $Y(\lambda t)/\lambda^{H(t_0)}$ is the LFSM $X_{H(t_0),\alpha}(t)$ and we do not observe a degenerate limit as in (5.3) of part (b), even when the function $H(t)$ satisfies the condition of part (b).

Remark 5.5. Weak convergence in a function space is addressed in Stoev and Taqqu (2004).

Proof of Theorem 5.1. We first prove part (c). It is easy to see that

$$\{\lambda^{-H(t_0)} X_{H(t_0),\alpha,\beta(\cdot)}(\lambda t)\}_{t\in\mathbb{R}} \xrightarrow{\text{FDD}} \{X_{H(t_0),\alpha,\beta(-t_0)}(t)\}_{t\in\mathbb{R}}$$

as $\lambda \downarrow 0$. Therefore, it suffices to show that $\lambda^{-H(t_0)}(Y_{\beta(\cdot)}(\lambda t) - X_{H(t_0),\alpha,\beta(\cdot)}(\lambda t)) \to 0$ in probability as $\lambda \downarrow 0$, for any fixed t. Assume that

$$H(t) - H(t_0) = o(1/\ln|t|)$$

as $t \to 0$. Then, by using the integral representations in (1.2) and (1.4), we obtain that

$$\begin{aligned} Y_\beta(\lambda t) - X_{H(t_0),\alpha,\beta}(\lambda t) &\overset{\text{FDD}}{=} \lambda^{H(\lambda t)} \int_{\mathbb{R}} f(t, H(\lambda t), s) M_{\alpha,\beta(\lambda\,\cdot)}(\mathrm{d}s) \\ &\quad - \lambda^{H(t_0)} \int_{\mathbb{R}} f(t, H(0), s) M_{\alpha,\beta(\lambda\,\cdot)}(\mathrm{d}s) \\ &= \lambda^{H(\lambda t)} X_{\beta(\lambda\,\cdot)}(t, H(\lambda t)) + \lambda^{H(t_0)} X_{\beta(\lambda\,\cdot)}(t, H(t_0)), \end{aligned}$$

where f and X_β are defined in (2.1). Thus, we have that

$$\begin{aligned} \frac{1}{\lambda^{H(t_0)}}(Y_{\beta(\cdot)}(\lambda t) - X_{H(t_0),\alpha,\beta(\cdot)}(\lambda t)) &\overset{\text{FDD}}{=} (\mathrm{e}^{\ln(\lambda)(H(\lambda t) - H(t_0))} - 1) X_{\beta(\lambda\,\cdot)}(t, H(\lambda t)) \\ &\quad + (X_{\beta(\lambda\,\cdot)}(t, H(\lambda t)) - X_{\beta(\lambda\,\cdot)}(t, H(t_0))). \end{aligned} \tag{5.4}$$

Now, the continuity (in probability) of the field $X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)$ (Corollary 2.1), and the continuity of the function $H(t)$ at $t = 0$, imply that the second term on the right-hand side of (5.4) vanishes as $\lambda \downarrow 0$. The first term also converges in probability to 0 as $\lambda \downarrow 0$, since, by assumption, $\ln(\lambda)(H(\lambda t) - H(t_0)) = o(1)$. Hence, the left-hand side of (5.4) converges to 0, which implies (5.2).

Now we focus on part (a). By (1.2), (1.4), and (2.1), we have that $Y_\beta(t) = X_\beta(t, H(t))$, $X_{H(t_0),\alpha,\beta}(t) = X_\beta(t, H(t_0))$, and $Y_\beta(t_0) = X_{H(t_0),\alpha,\beta}(t_0)$. Therefore,

$$\begin{aligned} \frac{1}{\lambda^{H(t_0)}}(Y_\beta(\lambda t + t_0) - Y_\beta(t_0)) &= \frac{1}{\lambda^{H(t_0)}}(X_{H(t_0),\alpha,\beta}(\lambda t + t_0) - X_{H(t_0),\alpha,\beta}(t_0)) \\ &\quad + \frac{1}{\lambda^{H(t_0)}}(X_\beta(\lambda t + t_0, H(\lambda t + t_0)) - X_\beta(\lambda t + t_0, H(t_0))). \end{aligned} \tag{5.5}$$

By Corollary 2.2, the second difference on the right-hand side of (5.5) is

$$\mathcal{O}_{\mathrm{P}}\left(\frac{H(\lambda t+t_0)-H(t_0)}{\lambda^{H(t_0)}}\right)\xrightarrow{\mathrm{P}}0$$

as $\lambda \downarrow 0$, because $H(\lambda t+t_0)-H(t_0)=o(|\lambda|^{H(t_0)})$ as $\lambda\to 0$. By a change of variables, and the continuity of $\beta(\cdot)$, the first difference on the right-hand side of (5.5) converges in finite-dimensional distributions to the LFSM process $\{X_{H(t_0),\alpha,\beta(-t_0)}(t)\}_{t\in\mathbb{R}}$. This implies (5.2) and completes the proof of part (a).

Assume now that the conditions of part (b) hold. By applying Theorem 2.2 with $N=2$, $v=H(\lambda t+t_0)$, $v_0=H(t_0)$, and $u=\lambda t+t_0$, we have, for all $t\in\mathbb{R}$, that the left-hand side of (5.3) equals

$$\frac{1}{\lambda^{\rho}L(\lambda)}(X_{H(t_0),\alpha,\beta}(\lambda t+t_0)-X_{H(t_0),\alpha,\beta}(t_0))+\frac{(H(\lambda t+t_0)-H(t_0))}{\lambda^{\rho}L(\lambda)}\partial_v X_{\beta}(\lambda t+t_0,H(t_0))$$
$$+\frac{1}{\lambda^{\rho}L(\lambda)}\mathcal{O}_{\mathrm{P}}((H(\lambda t+t_0)-H(t_0))^2)\quad \text{as } \lambda\to 0, \lambda>0. \tag{5.6}$$

By using parts (b) and (c) of Theorem 2.1 we get that

$$\begin{aligned}\|X_{H(t_0),\alpha,\beta}(\lambda t+t_0)-X_{H(t_0),\alpha,\beta}(t_0)\|_{\alpha}&=\|X_{\beta(\cdot\,-t_0)}(\lambda t,H(t_0))\|_{\alpha}\\&=\lambda^{H(t_0)}\|X_{\beta(\lambda\,\cdot\,-t_0)}(t,H(t_0))\|_{\alpha}=\lambda^{H(t_0)}\|X(t,H(t_0))\|_{\alpha}.\end{aligned}$$

Therefore the first term in (5.6) vanishes as $\lambda\downarrow 0$, because it is of order $\mathcal{O}_{\mathrm{P}}(\lambda^{H(t_0)-\rho}/L(\lambda))$, $\lambda\downarrow 0$, where $\rho<H(t_0)$. Also, the third term in (5.6) converges in probability to 0 since, by assumption,

$$(H(\lambda t+t_0)-H(t_0))^2=(\lambda t)^{2\rho}L^2(\lambda t)=o(\lambda^{\rho}L(\lambda))$$

as $\lambda\downarrow 0$. Using the assumptions again, we get that the second term in (5.6) equals

$$\frac{(\lambda t)^{\rho}L(\lambda t)}{\lambda^{\rho}L(\lambda)}\partial_v X_{\beta}(\lambda t+t_0,H(t_0))\xrightarrow{\mathrm{P}}t^{\rho}\partial_v X_{\beta}(t_0,H(t_0))\quad\text{as } \lambda\to 0, \lambda>0,$$

because the process $\{\partial_v X_{\beta}(u,v),\ u\in\mathbb{R},\ v\in(0,1)\}$ is continuous in probability (Corollary 2.1) and since $L(\lambda t)/L(\lambda)\to 1$. This completes the proof of the theorem.

The following theorem shows that the finite-dimensional distributions of the LMSM process Y_{β} depend, in general, on the function $H(\cdot)$ and on the parameters a^+ and a^-.

Theorem 5.2. *Let $\alpha\in(0,2)$ and let $Y^i_{\beta}(t),\ t\in\mathbb{R},\ i=1,2$, be two LMSM processes, defined by (1.2) with parameters $a^+:=a_i^+$, $a^-:=a_i^-$ and local scaling functions $H(t):=H_i(t),\ t\in\mathbb{R}$. Suppose that the functions $H_i(t)\in(0,1),\ i=1,2$, are continuous and satisfy condition (5.1) of Theorem 5.1(a) for all $t_0\in\mathbb{R}$, $t_0\neq 0$. Assume also that the skewness intensity function $\beta(s)$ of the underlying α-stable measure $M_{\alpha,\beta}(\mathrm{d}s),\ s\in\mathbb{R}$, is continuous and that*

$$Y=\{Y^1_{\beta}(t)\}_{t\in\mathbb{R}}\overset{\mathrm{FDD}}{=}\{Y^2_{\beta}(t)\}_{t\in\mathbb{R}}.$$

(a) *If $H_1(t)\not\equiv 1/\alpha,\ t\in\mathbb{R}$, then*

$$a_1^+=\varepsilon a_2^+,\quad a_1^-=\varepsilon a_2^-,\qquad \text{and}\qquad H_1(t)\equiv H_2(t),\quad t\in\mathbb{R},$$

where $\varepsilon\in\{+1,-1\}$.

(b) *If* $H_1(t) \equiv 1/\alpha,\ t \in \mathbb{R}$, *then*

$$a_1^+ - a_1^- = \varepsilon(a_2^+ - a_2^-) \quad \textit{and} \quad H_1(t) \equiv H_2(t) = 1/\alpha, \quad t \in \mathbb{R}, \tag{5.7}$$

where $\varepsilon \in \{+1, -1\}$.

Proof. We first prove that

$$H_1(t) = H_2(t), \qquad t \in \mathbb{R}. \tag{5.8}$$

Consider the processes

$$D^i_{t_0,\beta}(\lambda) = \{D^i_{t_0,\beta}(\lambda, t)\}_{t\in\mathbb{R}} := \{Y^i_\beta(\lambda t + t_0) - Y^i_\beta(t_0)\}_{t\in\mathbb{R}}, \qquad i = 1, 2,$$

where $t_0 \in \mathbb{R}, t_0 \neq 0$ and where $\lambda > 0$. Since the functions $H_i(\cdot),\ i = 1, 2$, satisfy the assumption of Theorem 5.1(a), as $\lambda \downarrow 0$,

$$\lambda^{-H_i(t_0)} D^i_{t_0,\beta}(\lambda, \cdot) \xrightarrow{\text{FDD}} X^i_{H_i(t_0),\alpha,\beta(-t_0)}(\cdot), \qquad i = 1, 2. \tag{5.9}$$

Here $X^i_{H_i(t_0),\alpha,\beta(-t_0)}(t),\ i = 1, 2$, is the LFSM process, defined by (1.4), where $M_{\alpha,\beta}(\mathrm{d}s)$ is replaced by $M_{\alpha,\beta(-t_0)}(\mathrm{d}s)$ and where $H := H_i(t_0)$ and $a^+ := a_i^+, a^- := a_i^-, i = 1, 2$. Recall that the LFSM process $X_{H,\alpha}$ is identically equal to 0 if and only if $a^+ = a^-$ and $H = 1/\alpha$ (Lemma 4.1).

Suppose first that both limit processes in (5.9) are nonzero. By assumption, we have that

$$D^1_{t_0,\beta}(\lambda, \cdot) \stackrel{\text{FDD}}{=} D^2_{t_0,\beta}(\lambda, \cdot). \tag{5.10}$$

Thus, by comparing the rates of convergence $\lambda^{-H_1(t_0)}$ and $\lambda^{-H_2(t_0)}$ in (5.9), we conclude that $H_1(t_0) = H_2(t_0)$, because the right-hand side of (5.9) is nontrivial for each i.

Now assume, for example, that $X^1_{H_1(t_0),\alpha} \equiv 0$. Then, we have that $a_1^+ = a_1^-$ and $H_1(t_0) = 1/\alpha$ and, therefore, that $Y_1(t_0) = 0$ a.s. Since $Y_1(t_0) \stackrel{\text{FDD}}{=} Y_2(t_0)$, this implies that $Y_2(t_0) = 0$ a.s. and, hence, that $H_2(t_0) = 1/\alpha = H_1(t_0)$.

We have shown that $H_1(t_0) = H_2(t_0)$ for all $t_0 \in \mathbb{R}, t_0 \neq 0$, and thus (5.8) holds by continuity.

We will now prove part (a). Let $H(t) = H_1(t) = H_2(t),\ t \in \mathbb{R}$. By (5.9) and (5.10),

$$X^1_{H(t_0),\alpha,\beta(-t_0)}(\cdot) \stackrel{\text{FDD}}{=} X^2_{H(t_0),\alpha,\beta(-t_0)}(\cdot). \tag{5.11}$$

Since $H(t) \not\equiv 1/\alpha$, there exists $t_0 \neq 0$ such that $H(t_0) \neq 1/\alpha$. Then the limit processes $X^i_{H(t_0),\alpha,\beta(-t_0)}(\cdot),\ i = 1, 2$, are not equal to the α-stable Lévy motion and, therefore, by Theorem 7.4.5 in Samorodnitsky and Taqqu (1994), (5.11) implies that $a_1^+ = \varepsilon a_2^+$ and $a_1^- = \varepsilon a_2^-$, for some $\varepsilon \in \{+1, -1\}$, which completes the proof of part (a).

We now focus on part (b). Since $H_1(t) = H_2(t) = 1/\alpha$, by (4.1), we get that, for all $t \in \mathbb{R}$,

$$Y^i_\beta(t) = (a_i^+ - a_i^-) \int_{-t}^0 M_{\alpha,\beta}(\mathrm{d}s), \qquad i = 1, 2. \tag{5.12}$$

Since $\{Y^1_\beta(t)\}_{t\in\mathbb{R}} \stackrel{\text{FDD}}{=} \{Y^2_\beta(t)\}_{t\in\mathbb{R}}$, (5.12) implies that $\|Y^1_\beta(t)\|_\alpha = \|Y^2_\beta(t)\|_\alpha = |a_i^+ - a_i^-||t|^{1/\alpha}$, $i = 1, 2$, for all $t \in \mathbb{R}$. This yields (5.7) and completes the proof of the theorem.

6. Auxiliary results

The following results were used in the proof of Lemma 2.1 and Theorem 2.2.

Proposition 6.1. *Let $\partial_v^n f(u, v, s)$ be defined as in (2.4). Then, for all $\alpha \in (0, 2]$, $v, v_0 \in (0, 1)$, and $n \in \mathbb{N}_0$, we have*

$$I \equiv \int_{\mathbb{R}} \sup_{\theta\in[0,1]} |\partial_v^n f(1, v_0 + (v - v_0)\theta, s)|^\alpha \, \mathrm{d}s \\ \leq 2(|a^+|^\alpha + |a^-|^\alpha)\frac{C}{\alpha^2}(n+1)^{7/2}\frac{(n!)^\alpha}{R(v, v_0)^{\alpha n+1}}, \tag{6.1}$$

where $R(v, v_0) = \min\{R(v), R(v_0)\}$, $R(v) := v \wedge (1 - v)$ and where C is an absolute constant that does not depend on α, v, v_0 and n.

Proof. By using the fact that $\partial_v^n f(1, v, s) = a^+\partial_v^n f_+(1, v, s) + a^-\partial_v^n f_-(1, v, s)$ for $v \in (0, 1)$, $s \in \mathbb{R}$, and the inequality

$$|x + y|^\alpha \leq 2|x|^\alpha + 2|y|^\alpha, \tag{6.2}$$

valid for all $x, y \in \mathbb{R}$ and $\alpha \in (0, 2]$, we obtain that

$$I \leq 2|a^+|^\alpha \int_{\mathbb{R}} \sup_{\theta\in[0,1]} |\partial_v^n f_+(1, v_\theta, s)|^\alpha \, \mathrm{d}s + 2|a^-|^\alpha \int_{\mathbb{R}} \sup_{\theta\in[0,1]} |\partial_v^n f_-(1, v_\theta, s)|^\alpha \, \mathrm{d}s \\ =: 2|a^+|^\alpha J^+ + 2|a^-|^\alpha J^-, \tag{6.3}$$

where $v_\theta = v_0 + (v - v_0)\theta$. Observe that, since $\partial_v^n f_+(1, v, s) \equiv -\partial_v^n f_-(1, v, -s - 1)$ for $v \in (0, 1)$, $s \in \mathbb{R}$, the two integrals on the right-hand side of the last expression coincide. Therefore, it suffices to show that

$$J = J^+ \equiv J^- \leq \frac{C}{\alpha^2}(n+1)^{7/2}\frac{(n!)^\alpha}{R(v, v_0)^{\alpha n+1}}.$$

We will examine the integral J^+ in (6.3) separately over the regions $-1 \leq s \leq 0$, $0 \leq s \leq 1$, and $1 \leq s$. From (2.3) and (2.4), we find that

$$J = \int_0^1 \sup_{\theta\in[0,1]} |\ln^n(s)s^{v_\theta-1/\alpha}|^\alpha \, \mathrm{d}s \\ + \int_0^1 \sup_{\theta\in[0,1]} |\ln^n(s+1)(s+1)^{v_\theta-1/\alpha} - \ln^n(s)s^{v_\theta-1/\alpha}|^\alpha \, \mathrm{d}s \\ + \int_1^\infty \sup_{\theta\in[0,1]} |\varphi_n(s+1, v_\theta) - \varphi_n(s, v_\theta)|^\alpha \, \mathrm{d}s =: J_{1,n} + J_{2,n} + J_{3,n}, \tag{6.4}$$

where $\varphi_n(s, v) := \ln^n(s)s^{v-1/\alpha}$, $s \geq 1$. Now, by applying the inequality (6.2) to the term $J_{2,n}$ in (6.4), we obtain

$$J \leq 3J_{1,n} + 2\int_1^2 \sup_{\theta\in[0,1]} |\ln^n(s)s^{v_\theta-1/\alpha}|^\alpha \, \mathrm{d}s + J_{3,n} =: 3J_{1,n} + 2J_{4,n} + J_{3,n}. \tag{6.5}$$

In the rest of the proof we focus on bounding the integrals $J_{1,n}$, $J_{4,n}$ and $J_{3,n}$. By using the change of variables $u = -\ln(s),\ s \in (0, 1)$, we get

$$\begin{aligned} J_{1,n} &= \int_0^\infty \sup_{\theta\in[0,1]} \{u^{n\alpha} \mathrm{e}^{-(\alpha v_\theta)u}\}\,\mathrm{d}u \le \int_0^\infty u^{n\alpha}\mathrm{e}^{-\alpha(v\wedge v_0)u}\,\mathrm{d}u \\ &= \frac{\Gamma(n\alpha+1)}{(\alpha(v\wedge v_0))^{n\alpha+1}} \le \frac{\Gamma(n\alpha+1)}{(\alpha R)^{n\alpha+1}}, \end{aligned} \tag{6.6}$$

since $v \wedge v_0 := \min\{v, v_0\} \le v_\theta$ and $\alpha R \le \alpha(v \wedge v_0)$.

Consider now the integral $J_{4,n}$. For all $\alpha \in (0, 2]$, $\theta \in [0, 1]$, and $s \in (1, 2)$, we have that $(\ln^n(s)s^{v_\theta - 1/\alpha})^\alpha < s^{\alpha-1} \le s$, because $0 < \ln(s) < 1$ and $0 < v_\theta < 1$. Thus, from (6.5), we find that

$$J_{4,n} \le \int_1^2 s\,\mathrm{d}s = \tfrac{3}{2}. \tag{6.7}$$

We now focus on the integral $J_{3,n}$. The mean value theorem implies that $\varphi_n(s+1, v_\theta) - \varphi_n(s, v_\theta) = \varphi_n'(s+\eta, v_\theta)$ for some $\eta = \eta(v_\theta) \in [0, 1]$, where

$$\varphi_n'(s+\eta, v_\theta) = (s+\eta)^{v_\theta - 1/\alpha - 1}(n\ln^{n-1}(s+\eta) + (v_\theta - 1/\alpha)\ln^n(s+\eta)), \tag{6.8}$$

and where, by convention, the term $n\ln^{n-1}(s+\eta)$ vanishes when $n = 0$. By using (6.8) and (6.2), we obtain that

$$\begin{aligned} J_{3,n} \le 2\biggl\{ & n^\alpha \int_1^\infty s^{\alpha(v\vee v_0 - 1/\alpha - 1)} \ln^{(n-1)\alpha}(s+1)\,\mathrm{d}s \\ & + (|v - 1/\alpha| \vee |v_0 - 1/\alpha|)^\alpha \int_1^\infty s^{\alpha(v\vee v_0 - 1/\alpha - 1)} \ln^{n\alpha}(s+1)\,\mathrm{d}s \biggr\}, \end{aligned} \tag{6.9}$$

because $\ln(s+\eta) \le \ln(s+1)$, $(s+\eta)^{-\delta} \le s^{-\delta}$, and $s^{v_\theta - 1/\alpha - 1} \le s^{v\vee v_0 - 1/\alpha - 1}$ for all $s \ge 1$, $\eta \in [0, 1]$, and $\delta > 0$. Applying the inequality (6.11), below, to the integrals in (6.9), we get that

$$\begin{aligned} J_{3,n} &\le 2\mathrm{e}^{\alpha(1+1/\alpha - v\vee v_0)} \biggl\{ \frac{n^\alpha \Gamma((n-1)\alpha+1)}{(\alpha(1 - v\vee v_0))^{(n-1)\alpha+1}} + \frac{(|v - 1/\alpha| \vee |v_0 - 1/\alpha|)^\alpha \Gamma(n\alpha+1)}{(\alpha(1 - v\vee v_0))^{n\alpha+1}} \biggr\} \\ &\le \frac{4\mathrm{e}^3}{(\alpha R)^{n\alpha+1}}(n^\alpha\Gamma((n-1)\alpha+1) + \Gamma(n\alpha+1)) \le C(n^\alpha+1)\frac{\Gamma(n\alpha+1)}{(\alpha R)^{n\alpha+1}}, \end{aligned} \tag{6.10}$$

where C is an absolute constant and v, v_0, and R are defined above. The second inequality in (6.10) follows from the bounds

$$\begin{gathered} \alpha(1 + 1/\alpha - v\vee v_0) < \alpha(1+1/\alpha) \le 3, \\ (|v - 1/\alpha| \vee |v_0 - 1/\alpha|)^\alpha \le (1/\alpha \vee |1 - 1/\alpha|)^\alpha \le \sup_{0<\alpha\le 2} \alpha^{-\alpha} = \mathrm{e}^{1/\mathrm{e}} < 2, \end{gathered}$$

and

$$\alpha R = \alpha\min\{(1 - v\vee v_0), v\wedge v_0\} \le \alpha/2 \le 1.$$

We will now justify the last inequality in (6.10). If $n = 0$, then, in view of (6.8), the term $n^\alpha\Gamma((n-1)\alpha+1)$ is not present and thus the inequality is trivial. If $(n-1)\alpha \ge 1$, then the inequality follows from the fact that the function $\Gamma(x+1)$ is monotonically increasing for all

$x \geq 1$. Suppose now that $0 \leq (n-1)\alpha \leq 1$. Since the function $\Gamma(x+1)$, $x \geq 0$, is positive and continuous, we see that

$$\Gamma((n-1)\alpha+1) \leq \sup_{x\in[0,1]} \Gamma(x+1)$$
$$\leq \left(\frac{\sup_{x\in[0,1]}\Gamma(x+1)}{\inf_{x\in[0,3]}\Gamma(x+1)}\right)\Gamma(n\alpha+1) = C\Gamma(n\alpha+1),$$

where C is an absolute positive constant. This implies the last inequality in (6.10).

We can now combine the bounds in (6.6), (6.7), and (6.10). Observe that, since $\alpha R < 1$, the right-hand side of (6.6) is bounded away from 0 and, hence,

$$J \leq 3\frac{\Gamma(n\alpha+1)}{(\alpha R)^{n\alpha+1}} + 3 + C(n^{\alpha}+1)\frac{\Gamma(n\alpha+1)}{(\alpha R)^{n\alpha+1}} \leq C'(n^{\alpha}+1)\frac{\Gamma(n\alpha+1)}{(\alpha R)^{n\alpha+1}},$$

for some $C' > 0$. The inequality in (6.1) then follows, by Lemma 6.2. This completes the proof of the proposition.

The following inequality was used in the proof of Proposition 6.1 above.

Lemma 6.1. *For all $x, y \in \mathbb{R}$, $x > 1$, $y \geq 0$, we have that*

$$\int_1^{\infty} s^{-x}\ln^y(s+1)\,\mathrm{d}s \leq \mathrm{e}^x\frac{\Gamma(y+1)}{(x-1)^{y+1}}. \tag{6.11}$$

Proof. By making the change of variable $u = \ln(s+1)$, the left-hand side of (6.11) equals

$$\int_2^{\infty}(\mathrm{e}^u-1)^{-x}\mathrm{e}^u u^y\,\mathrm{d}u \leq \int_{\ln(2)}^{\infty}\mathrm{e}^{-(u-1)x}\mathrm{e}^u u^y\,\mathrm{d}u \leq \mathrm{e}^x\int_0^{\infty}\mathrm{e}^{-(x-1)u}u^y\,\mathrm{d}u, \tag{6.12}$$

where the first inequality in (6.12) follows from the fact that, for all $u \geq \ln(2)$, $\mathrm{e}^u - 1 \geq \mathrm{e}^{u-1}$. This completes the proof of the lemma, since the right-hand sides of (6.11) and (6.12) are equal.

The following result, which follows from Stirling's formula, was used in the proof of Proposition 6.1.

Lemma 6.2. *For all $\alpha \in (0,2]$ and $n \in \mathbb{N}_0$, we have that*

$$\Gamma(n\alpha+1) \leq \frac{c}{\alpha}(n+1)^{3/2}(n!)^{\alpha}\alpha^{n\alpha}, \tag{6.13}$$

where c is a positive constant that does not depend on n and α.

Proof. If $n = 0$ then the statement of the lemma is trivial, so let $n \geq 1$ and suppose, firstly, that $n\alpha \leq 1$. Since $\Gamma(n\alpha+1)$ is bounded it is sufficient to show that the right-hand side of (6.13) is bounded away from 0. This is true because $(n+1)^{3/2}(n!)^{\alpha} \geq 1$, and the inequalities $0 < \alpha \leq 1$ and $1 \leq n \leq 1/\alpha$ imply that $\alpha^{n\alpha}/\alpha \geq \alpha^{(1/\alpha)\alpha-1} = 1$.

Secondly, suppose that $n\alpha > 1$. By using $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we find that $\Gamma(n\alpha+1) \leq C\lfloor n\alpha+1\rfloor!$, where $C = \sup_{0\leq x\leq 1}\Gamma(x+1) < \infty$ and where $\lfloor x\rfloor$ denotes the integer-part function. We let the constant C change from line to line in what follows. By Stirling's formula $n! \sim C\mathrm{e}^{-n}n^{n+1/2}$, $n \to \infty$, we have

$$\Gamma(n\alpha+1) \leq C\mathrm{e}^{-\lfloor n\alpha+1\rfloor}\lfloor n\alpha+1\rfloor^{\lfloor n\alpha+1\rfloor+1/2} \leq C\mathrm{e}^{-n\alpha}(n\alpha+1)^{n\alpha+3/2}, \tag{6.14}$$

where the last inequality holds because the function $g(x) := x^x$ is nondecreasing for all $x \geq 1$. We now express the right-hand side of (6.14) in terms of $n!$ by reapplying Stirling's formula. This yields

$$\begin{aligned}\Gamma(n\alpha+1) &\leq C(n!)^\alpha \left(\frac{n\alpha+1}{n\alpha}\right)^{n\alpha} \alpha^{n\alpha} \frac{(n\alpha+1)^{3/2}}{n^{\alpha/2}} \\ &\leq C(n+1)^{3/2}(n!)^\alpha \alpha^{n\alpha}\left(\frac{n\alpha+1}{n\alpha}\right)^{n\alpha}, \end{aligned} \tag{6.15}$$

for some new constant C. Since $\sup_{x\geq 1}((1+x)/x)^x < \infty$, (6.15) implies the inequality (6.13) and the proof of the lemma is complete.

The next result was used in the proofs of Theorems 3.1 and 3.2, above. It provides upper and lower bounds for $\|X(1,v)\|_\alpha^\alpha$.

Lemma 6.3. *Let* $\alpha \in (0,2]$. *For all* $v \in (0,1)$*, the following two inequalities hold:*

$$\|X(1,v)\|_\alpha^\alpha \leq \frac{(|a^-|^\alpha + |a^+|^\alpha)}{\alpha}\left(\frac{|v-1/\alpha|^\alpha}{1-v} + \frac{10}{v}\right) \tag{6.16}$$

and

$$\frac{(|a^-|^\alpha + |a^+|^\alpha)}{2\alpha}\left(\frac{|v-1/\alpha|^\alpha}{1-v} + \frac{\mathbf{1}_{(0,1/\alpha)}(v)}{2^{(1+\alpha)/|\alpha v-1|}}\frac{1}{v}\right) \leq \|X(1,v)\|_\alpha^\alpha, \tag{6.17}$$

where, recall, $\mathbf{1}_{(0,1/\alpha)}(\cdot)$ *denotes the indicator function of the interval* $(0,1/\alpha)$.

Proof. By (2.1), we have that

$$\begin{aligned}\|X(1,v)\|_\alpha^\alpha &= |a^-|^\alpha\left\{\int_{-2}^{-1} + \int_{-\infty}^{-2}\right\}|(1+s)_-^{v-1/\alpha} - s_-^{v-1/\alpha}|^\alpha\, \mathrm{d}s \\ &\quad + \int_{-1}^{0} |a^+(1+s)_+^{v-1/\alpha} - a^- s_-^{v-1/\alpha}|^\alpha\, \mathrm{d}s \\ &\quad + |a^+|^\alpha\left\{\int_0^1 + \int_1^\infty\right\}|(1+s)^{v-1/\alpha} - s^{v-1/\alpha}|^\alpha\, \mathrm{d}s \\ &=: |a^-|^\alpha\{A_1 + A_2\} + B + |a^+|^\alpha\{C_1 + C_2\}, \end{aligned} \tag{6.18}$$

where A_1, A_2, B, C_1, and C_2 denote the first, second, third, fourth, and fifth integrals in (6.18), respectively. Observe that by making the change of variable $\tilde{s} := -(1+s)$, one obtains $A_1 \equiv C_1$ and $A_2 \equiv C_2$. Hence, from (6.18), we have that

$$\|X(1,v)\|_\alpha^\alpha = (|a^-|^\alpha + |a^+|^\alpha)(C_1 + C_2) + B. \tag{6.19}$$

We can now prove the inequality in (6.16). From (6.2), we obtain that

$$B \leq 2|a^+|^\alpha \int_{-1}^0 (1+s)_+^{\alpha v-1}\, \mathrm{d}s + 2|a^-|^\alpha \int_{-1}^0 s_-^{\alpha v-1}\, \mathrm{d}s = 2\frac{(|a^+|^\alpha + |a^-|^\alpha)}{\alpha v} \tag{6.20}$$

and that

$$C_1 \leq 2\int_0^1 (1+s)^{\alpha v-1}\, \mathrm{d}s + 2\int_0^1 s^{\alpha v-1}\, \mathrm{d}s = 2\frac{2^{\alpha v}}{\alpha v} \leq \frac{8}{\alpha v}. \tag{6.21}$$

Now, by applying the mean value theorem for the function $g(x) := x^{v-1/\alpha},\ x > 0$, over the interval $[s, s+1]$, we get that, for all $s > 0$,

$$|v - 1/\alpha|(1+s)^{v-1/\alpha-1} \le |(1+s)^{v-1/\alpha} - s^{v-1/\alpha}| \le |v - 1/\alpha| s^{v-1/\alpha-1}, \tag{6.22}$$

because $v - 1/\alpha - 1 < 0$. By using the upper bound in (6.22), we obtain that

$$C_2 \le |v - 1/\alpha|^\alpha \int_1^\infty s^{\alpha(v-1)-1}\,\mathrm{d}s = \frac{|v-1/\alpha|^\alpha}{(1-v)\alpha}. \tag{6.23}$$

The inequality (6.16) follows from (6.19), (6.20), (6.21), and (6.23).

We will now prove (6.17). Firstly, by using the lower bound in (6.22), we obtain that

$$C_1 + C_2 \ge |v - 1/\alpha|^\alpha \int_0^\infty (1+s)^{\alpha(v-1/\alpha-1)}\,\mathrm{d}s = \frac{|v-1/\alpha|^\alpha}{(1-v)\alpha}. \tag{6.24}$$

Secondly, we obtain a lower bound for the term B by considering two cases for the coefficients a^- and a^+.

(i) *The case $a^-a^+ \le 0$.* Since $(1+s)_+^{v-1/\alpha}$ and $s_-^{v-1/\alpha}$ are positive for all $s \in (-1, 0)$, and since the coefficients a^+ and $-a^-$ are either 0 or have the same sign, we obtain that

$$\begin{aligned} B &\ge \max\left\{|a^+|^\alpha \int_{-1}^0 (1+s)_+^{\alpha v-1}\,\mathrm{d}s,\ |a^-|^\alpha \int_{-1}^0 s_-^{\alpha v-1}\,\mathrm{d}s\right\} \\ &= \frac{\max\{|a^+|^\alpha, |a^-|^\alpha\}}{\alpha v} \ge \frac{(|a^+|^\alpha + |a^-|^\alpha)}{2\alpha v}. \end{aligned} \tag{6.25}$$

(ii) *The case $a^-a^+ > 0$.* Without loss of generality, assume that $a^+ > a^- > 0$ and consider the two subcases in which $v - 1/\alpha < 0$ and $v - 1/\alpha > 0$. Suppose first that $v - 1/\alpha < 0$, which is the delicate subcase, since $\int_{-1}^0 ((1+s)_+^{v-1/\alpha})^\alpha\,\mathrm{d}s \to \infty$ as $v \to 0$. Since $a^+ > a^- > 0$, we have, for all $s \in (-1, -\frac{1}{2})$,

$$\begin{aligned} a^+(1+s)_+^{v-1/\alpha} - a^- s_-^{v-1/\alpha} &\ge a^+((1+s)_+^{v-1/\alpha} - s_-^{v-1/\alpha}) \\ &\ge a^+((1+s)_+^{v-1/\alpha} - (\tfrac{1}{2})^{v-1/\alpha}). \end{aligned}$$

Let $c = c(\alpha, v) := (\frac{1}{2})^{v-1/\alpha} = 2^{|v-1/\alpha|}$. The last inequality implies that

$$B \ge |a^+|^\alpha \int_{-1}^{-1/2} ((1+s)_+^{v-1/\alpha} - c)^\alpha\,\mathrm{d}s = \frac{|a^+|^\alpha}{|v-1/\alpha|} \int_0^\infty x^\alpha (x+c)^{1/(v-1/\alpha)-1}\,\mathrm{d}x,$$

where, in the last integral, we made the change of variable $x := (1+s)^{v-1/\alpha} - c$. By using the last relation, and the fact that $x^\alpha/(x+c)^\alpha \ge 1/(1+c)^\alpha$ for all $c > 0$ and $x > 1$, we obtain that

$$\begin{aligned} B &\ge \frac{|a^+|^\alpha}{(1+c)^\alpha} \int_1^\infty \frac{(x+c)^{\alpha+1/(v-1/\alpha)-1}}{|v-1/\alpha|}\,\mathrm{d}x \\ &= -\frac{|a^+|^\alpha}{(1+c)^\alpha} \frac{(1+c)^{\alpha+1/(v-1/\alpha)}}{(\alpha + 1/(v-1/\alpha))|v-1/\alpha|} \\ &= |a^+|^\alpha \frac{(1+c)^{1/(v-1/\alpha)}}{\alpha v}, \end{aligned} \tag{6.26}$$

since $\alpha + 1/(v - 1/\alpha) = \alpha v/(v - 1/\alpha)$, where $v - 1/\alpha < 0$. Since $0 < v < 1/\alpha$, we have that $c = 2^{|v-1/\alpha|} \le 2^{1/\alpha}$ and, hence, $(1 + c)^{1/(v-1/\alpha)} = (1 + c)^{-1/|v-1/\alpha|} \ge (1 + 2^{1/\alpha})^{-1/|v-1/\alpha|}$. Therefore, (6.26) implies that

$$\begin{aligned} B &\ge |a^+|^\alpha \frac{(1 + 2^{1/\alpha})^{-1/|v-1/\alpha|}}{\alpha v} \\ &\ge (|a^+|^\alpha + |a^-|^\alpha) \frac{2^{-(\alpha+1)/|\alpha v - 1|}}{2\alpha v}, \end{aligned} \tag{6.27}$$

since $(1 + 2^{1/\alpha}) \le 2^{1+1/\alpha} = 2^{(1+\alpha)/\alpha}$.

Now take $v - 1/\alpha \ge 0$. In this case, we trivially bound the term B below by 0 and, therefore, by combining the bounds in (6.25) and (6.27), we obtain that

$$B \ge (|a^+|^\alpha + |a^-|^\alpha) \frac{2^{-(\alpha+1)/|\alpha v - 1|}}{2\alpha v} \mathbf{1}_{(0,1/\alpha)}(v).$$

This, in view of (6.24), implies (6.17) and completes the proof of the lemma.

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