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FIRE SALES FORENSICS: MEASURING ENDOGENOUS RISK

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We propose a tractable framework for quantifying the impact of loss-triggered fire sales on portfolio risk, in a multi-asset setting. We derive analytical expressions for the impact of fire sales on the realized volatility and correlations of asset returns in a fire sales scenario and show that our results provide a quantitative explanation for the spikes in volatility and correlations observed during such deleveraging episodes. These results are then used to develop an econometric framework for the forensic analysis of fire sales episodes, using observations of market prices. We give conditions for the identifiability of model parameters from time series of asset prices, propose a statistical test for the presence of fire sales, and an estimator for the magnitude of fire sales in each asset class. Pathwise consistency and large sample properties of the estimator are studied in the high-frequency asymptotic regime. We illustrate our methodology by applying it to the forensic analysis of two recent deleveraging episodes: the Quant Crash of August 2007 and the Great Deleveraging following the default of Lehman Brothers in Fall 2008.

KEY WORDS: fire sales, financial contagion, feedback effects, price impact, liquidity, diffusion approximation, diffusion models, correlations, endogenous risk.

1. INTRODUCTION

Fire sales or, more generally, the sudden deleveraging of large financial portfolios, have been recognized as a destabilizing factor in recent (and less recent) financial crises, contributing to unexpected spikes in volatility and correlations of asset returns and resulting in spirals of losses for investors (Carlson 2006; Brunnermeier 2008; Khandani and Lo 2011). In particular, unexpected increases in correlations across asset classes have frequently occurred during market downturns (Bailey, Kapetanios, and Pesaran 2012; Cont and Wagalath 2013), leading to a loss of diversification benefits for investors, precisely when such benefits were desirable.

For instance, during the first week of August 2007, when a large fund deleveraged its positions in long–short market neutral equity strategies, other long–short market neutral equity funds experienced huge losses, while in the meantime, index funds were left unaffected (Khandani and Lo 2011). On a larger scale, the Great Deleveraging of

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[Correction added on 15 September 2015, after first online publication: The affiliation of Lakshithe Wagalath has been changed from “Université Pierre & Marie Curie” to “IESEG School of Management - LEM (CNRS)”.]

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financial institutions' portfolios subsequent to the default of Lehman Brothers in Fall 2008 led to an unprecedented peak in correlations across asset returns (Fratzscher 2012).

The importance of fire sales as a factor of market instability is recognized in the economic literature. Shleifer and Vishny (1992, 2011) characterize an asset fire sale by a financial institution as a forced sale in which potential high valuation buyers are affected by the same shocks as this financial institution, resulting in a sale of the asset at a discounted price to nonspecialist buyers. They underline the fact that in the presence of fire sales, losses by financial institutions with overlapping holdings become self-reinforcing, leading to downward spirals for asset prices and, ultimately, to systemic risk. Bouchaud and Cont (1998) study how feedback effects from distressed selling may generate price instability. Pedersen (2009) describes qualitatively the effects of investors running for the exit and the spirals of losses and spillover effects they generate. Shin (2010) and Ozdenoren and Yuan (2008) propose equilibrium models which show how feedback effects generated by investors reacting to a price move contribute to the amplification of volatility and market instability. Boyer, Kumagai, and Yuan (2006) emphasize the role of institutional investors in price-mediated contagion, suggesting that crisis spread through the asset holdings of international investors rather than through changes in fundamentals. Brunnermeier (2008) points to the role of fire sales as a channel of contagion through which losses in mortgage-backed-securities during the recent financial crisis led to huge losses in equity markets.

The empirical link between fire sales and increase in correlation across asset returns has been documented in several recent studies. Coval and Stafford (2007) give empirical evidence for fire sales by open-end mutual funds by studying the transactions caused by capital flows. They show that funds in distress experience outflows of capital by investors which result in fire sales in existing positions, creating a price pressure in the securities held in common by distressed funds. Jotikasthira, Lundblad, and Ramadorai (2012) lead an empirical investigation on the effects of fund flows from developed countries to emerging markets. They show that such investment flows generate distressed trading by fund managers, affecting asset prices and correlations between emerging markets and creating a new channel through which shocks are transmitted from developed markets to emerging markets. Anton and Polk (2008) find empirically that common active mutual fund ownership predicts cross-sectional variation in realized covariance of returns.

Although the empirical examples cited above are related to the liquidation of large (multi-asset) *portfolios*, most theoretical studies have focused on fire sales in a single asset market and thus were not able to investigate the effect of fire sales on asset return correlations and the resulting limits to diversification. On the other hand, there is considerable empirical evidence that correlations across asset returns increase during periods of market stress (Forbes and Rigobon 2002; Fratzscher 2012).

Kyle and Xiong (2001) propose an equilibrium model, which takes into account the supply and demand of noise traders, long-term investors, and convergence traders, in a market with two risky assets and find that convergence traders react to a price shock in one asset by deleveraging their positions in both markets, leading to contagion effects. Greenwood and Thesmar (2011) propose a simple framework for modeling price dynamics which takes into account the ownership structure of financial assets, considered as given exogenously. Cont and Wagalath (2013) model the systematic supply and demand generated by investors exiting a large distressed fund and quantify its impact on asset returns.

We propose here a tractable framework in a multi-asset setting for modeling and estimating the impact of fire sales in multiple funds on the volatility and correlations of asset returns. We explore the mathematical properties of the model in the continuous-time limit and derive analytical results linking the realized covariance of asset returns to the parameters describing the volume of fire sales. In particular, we show that, starting from homoscedastic inputs, feedback effects from fire sales lead to endogenous heteroscedasticity in the covariance structure of asset returns.

Our results may also be viewed as providing an economic mechanism for some recently proposed models of heteroscedasticity in the covariance structure of returns (Engle 2002; Da Fonseca, Grasselli, and Tebaldi 2008; Gouriéroux, Jasiak, and Sufana 2009; Stelzer 2010), linking variations in cross-sectional correlations structure to market stress events as in Bailey et al. (2012).

The analytically tractable nature of these results enables to explore in detail the problem of *estimating* the magnitude of fire sales from time series of market prices. We explore the corresponding identification problem, propose a method for estimating the magnitude of fire sales in each asset class, and study the consistency and large sample properties of the proposed estimator. These results provide a quantitative framework for the "forensics analysis" of the impact of fire sales and distressed selling, which we illustrate with two empirical examples: the August 2007 hedge fund losses and the Great Deleveraging of bank portfolios following the default of Lehman Brothers in September 2008.

Our framework links sudden shifts in the realized covariance structure of asset returns with the liquidation of large portfolios, in a framework versatile enough to be used in the forensic analysis of empirical data. This provides a toolbox for risk managers and regulators in view of investigating unusual market events and their impact on the risk of portfolios in a systematic way, moving a step in the direction proposed by Fielding, Lo, and Yang (2011), who underlined the importance of systematically investigating all "systemic risk" events in financial markets, as done by the National Transportation Safety Board for major civil transportation accidents.

Outline

This paper is organized as follows. Section 2 presents a simple framework for modeling the impact of fire sales in various funds on asset returns. Section 3 resolves the question of the identification and estimation of the model parameters, characterizing the fire sales. Section 4 displays the results of our estimation procedure on liquidations occurring after the collapse of Lehman Brothers whereas Section 5 is focused on uncovering the liquidations in US equity markets during the "Quant Crash" of August 2007.

2. FIRE SALES AND ENDOGENOUS RISK

2.1. A Multi-Asset Model of Fire Sales

Consider a financial market where assets labeled $i = 1..n$ are traded at dates $(t_k, k \geq 1)$. The value of asset i at date t_k is denoted S^i_k. Time is expressed in years and we set $t_k = \frac{k}{N}$ where $N = 250$ in the examples below.

We consider J institutional investors trading in these assets: fund j initially holds α^j_i units of asset i. The value of this (benchmark) portfolio at date t_k is denoted

$$V^j_k = \sum_{i=1}^{n} \alpha^j_i S^i_k. \tag{2.1}$$

The impact of exogenous economic factors ("fundamentals") on prices is modeled through an IID sequence $(\xi_k)_{k\geq 1}$ of $\mathbb{R}^n$-valued centered random variables such that, in the absence of fire sales, the return of asset i during period $[t_k, t_{k+1}]$ is given by

$$\exp\left(\frac{1}{N}\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\frac{1}{N}}\xi^i_{k+1}\right) - 1.$$

Here m_i represents the expected return of asset i in the absence of fire sales and the "fundamental" covariance matrix Σ, defined by

$$\Sigma_{i,j} = \mathrm{cov}(\xi^i_k, \xi^j_k)$$

represents the covariance structure of returns in the absence of large systematic trades by institutional investors. The scaling factor $\sqrt{N}$ means that the quantities $\Sigma_{i,j}$ are annualized.

Typically, over short time horizons of a few days, institutional investors do not alter their portfolio allocations but the occurrence of large losses may induce them to sell off part of its assets (Coval and Stafford 2007; Jotikasthira et al. 2012; Shleifer and Vishny 2011). Such *distressed selling* may be triggered endogenously by capital requirements set by regulators or target leverage ratios set by fund managers (Danielsson, Shin, and Zigrand 2004; Greenwood and Thesmar 2011). Both mechanisms may lead financial institutions to deleverage their portfolios when faced with trading losses, as shown in the following example.

Consider a fund committed to a maximal leverage ratio of 12. Initially this fund possesses \$10 million of equity and borrows \$90 million to build a portfolio of assets worth \$100 million. The initial leverage of this fund is hence equal to $\frac{Assets}{Assets-Debt} = \frac{100}{100-90} = 10 < 12$.

A decline of d (expressed in percent) in the value of the assets held by the fund modifies the fund's leverage to a value of $\frac{100\times(1-d)}{100\times(1-d)-90}$. As a consequence, a decline in asset value of more than 1.8% leads to a spike in the fund's leverage ratio above the maximum leverage ratio of 12. In order to respect the constraint on its leverage ratio, the fund can either raise equity (which can be costly, especially at a time when its portfolio value is decreasing) or, most likely, engage in fire sales. As illustrated in Figure 2.1, a drop in asset value of 5% leads a liquidation of 35% of initial assets in this example. We also note that leverage ratios for large US and European banks in 2008 were much higher than in this example. On the contrary, as long as the drop in asset value is lower than 1.8%, the leverage of the fund remains below 12 and there is no distressed selling.

Note that this mechanism is asymmetric with respect to losses/gains: large losses trigger fire sales, but large gains do not necessarily result in massive buying. Once the capital requirement constraints or leverage constraints are not binding, they may cease to influence the fund managers' actions in a decisive manner.

Fire sales may also result from:

- Investors redeeming (or expanding) their positions depending on the performance of the funds, causing inflows and outflows of capital. This mechanism is described by Coval and Stafford (2007), who show empirically that funds in distress experience outflows of capital by investors and explain that, as the ability of borrowing is reduced for distressed funds and regulation and self-imposed constraints prevent them from short-selling other securities, such outflows of capital result in fire sales in existing positions.

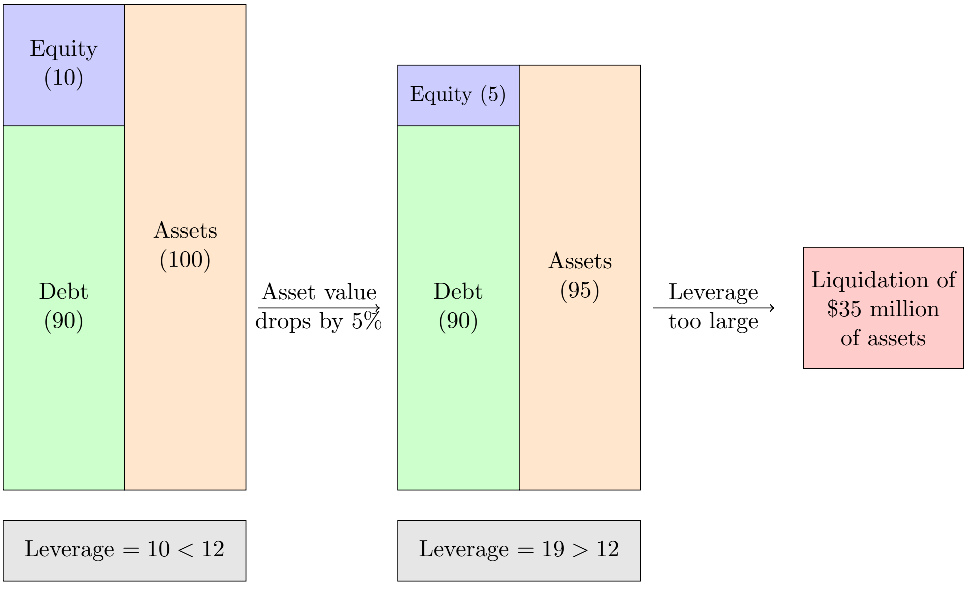

FIGURE 2.1. When portfolios are leveraged, even a moderate loss may trigger substantial deleveraging in presence of leverage constraints or capital ratio constraints. In this example, a 5% loss in asset values in a portfolio with initial leverage 10 and leverage cap of 12 leads to liquidation of 35% of initial assets.

- Rule-based strategies—such as portfolio insurance—which result in selling when a fund underperforms (Gennotte and Leland 1990),
- Sale of assets held as collateral by creditors of distressed funds (Shleifer and Vishny 2011).

The impact of fire sales may also be exacerbated by short-selling and predatory trading. Brunnermeier and Pedersen (2005) show that, in the presence of fire sales in a distressed fund, the mean-variance optimal strategy for other investors is to short-sell the assets held by the distressed fund and buy them back after the period of distress. A common feature of these mechanisms is that they are triggered by a large initial loss.

Rather than modeling a specific example of these triggering mechanisms in detail, we focus here on their outcome, in particular their impact on price dynamics. This impact may be modeled in a parsimonious manner by introducing a *deleveraging schedule*, represented by a function f_j which measures the systematic supply/demand generated by the fund j as a function of the fund's return: when, due to market shocks, the value of the portfolio j moves over $[t_k, t_{k+1}]$ from V_k^j to

$$\sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}\left(m_l - \frac{\Sigma_{l,l}}{2}\right) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)$$

and a portion

$$f_j\left(\frac{V_k^j}{V_0^j}\right) - f_j\left(\frac{1}{V_0^j}\sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}\left(m_l - \frac{\Sigma_{l,l}}{2}\right) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)\right) \tag{2.2}$$

of fund j is liquidated between t_k and t_{k+1}, proportionally in each asset held by the fund.

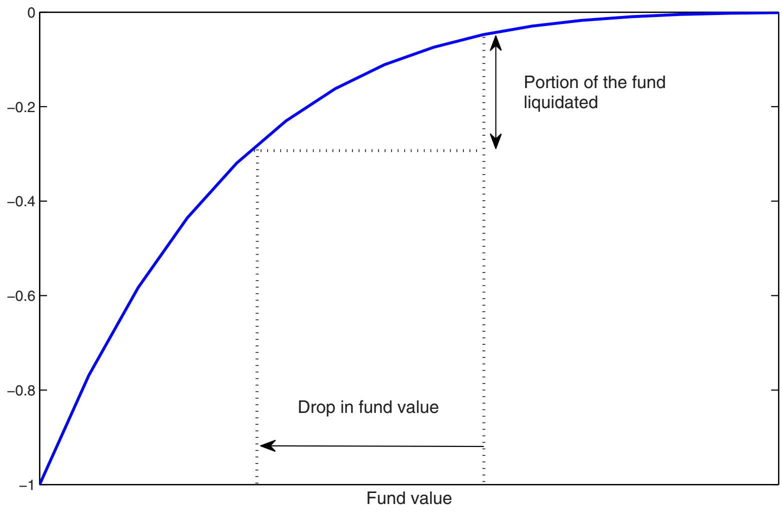

FIGURE 2.2. Example of a deleveraging schedule f_j.

As shown in the previous example and by Jotikasthira et al. (2012), negative returns for a fund lead to outflows of capital from this fund: this implies that f_j is an increasing function. Fire sales occur when a fund underperforms significantly and its value goes below a threshold and it ends when the fund is entirely liquidated: as a consequence, we choose f_j to be constant for small and large values of its argument (i.e., constant outside an interval $[\beta_j^{\text{liq}}, \beta_j]$) with $\beta_j < 1$. Figure 2.2 displays an example of such a deleveraging schedule f_j. As long as fund j's value remains above $\beta_j V_0^j$, the portion liquidated, given in (2.2), is equal to zero, as f_j is constant on $[\beta_j, +\infty[$: there are no fire sales. A drop in fund value below that threshold generates liquidation of a portion, given in (2.2), of the fund j.

When the trades are sizable with respect to the average trading volume, the supply/demand generated by this deleveraging strategy may have a nonnegligible impact on asset prices. We introduce, for each asset i, a *price impact* function $\phi_i(.)$, which captures this effect: the impact of buying v shares (where $v < 0$ represents a sale) on the return of asset i is $\phi_i(v)$. We assume that $\phi_i : \mathbb{R} \mapsto \mathbb{R}$ is increasing and $\phi_i(0) = 0$.

The impact of fire sales on the return of asset i is then equal to

$$\phi_i\left[\sum_{j=1}^{J} \alpha_i^j \left(f_j\left(\frac{1}{V_0^j}\sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}\left(m_l - \frac{\Sigma_{l,l}}{2}\right) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)\right) - f_j\left(\frac{V_k^j}{V_0^j}\right)\right)\right].$$

The price dynamics can be summed up as follows:

$$S_{k+1}^i = S_k^i \exp\left(\frac{1}{N}\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\frac{1}{N}}\xi_{k+1}^i\right) \times \tag{2.3}$$
$$\left(1 + \phi_i\left[\sum_{j=1}^{J} \alpha_i^j \left(f_j\left(\frac{1}{V_0^j}\sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)\right) - f_j\left(\frac{V_k^j}{V_0^j}\right)\right)\right]\right),$$

where V_k^j is the benchmark portfolio value of fund j at date t_k, defined in (2.1).

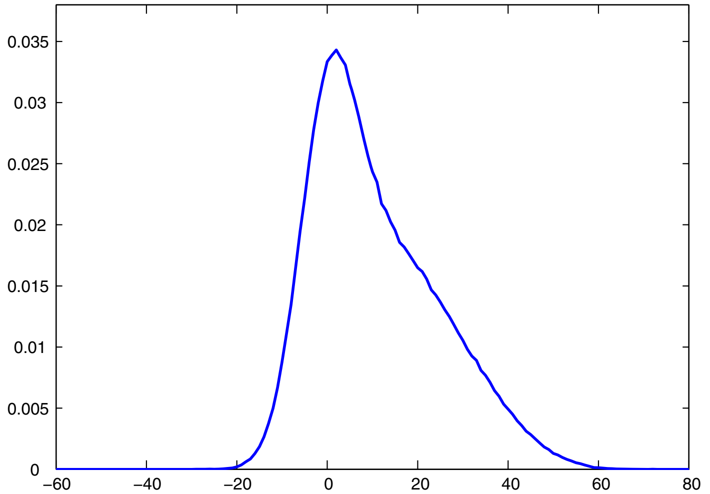

FIGURE 2.3. Distribution of realized correlation between two securities in the presence of fire sales (case of zero fundamental correlation).

At each period, the return of asset i can be decomposed into a fundamental component, which is independent from the past, and an endogenous component due to the impact of fire sales. Note that when there are no fire sales, this endogenous term is equal to zero and the return of asset i is equal to its fundamental return.

ASSUMPTION 2.1. $S_0 \in (\mathbb{R}_+^*)^n$ *and* $\min\limits_{1 \le i \le n} \ \phi_i(-2\sum_{j=1}^n |\alpha_i^j| \times \|f_j\|_\infty) > -1$.

PROPOSITION 2.2. *Under Assumption 2.1, there exists a unique discrete-time Markov process* $(S(t_k), k \ge 1)$ *with values in* $\left(\mathbb{R}_+^*\right)^n$*, satisfying* (2.1)–(2.3).

Proof. Equations (2.1) and (2.3) show that S_{k+1} depends only on its value at t_k and on the (IID) sequence ξ_{k+1}. The price vector S is thus a discrete-time Markov process. Furthermore, when $\min\limits_{1 \le i \le n} \ \phi_i(-2\sum_{j=1}^n |\alpha_i^j| \times \|f_j\|_\infty) > -1$, the endogenous price impact due to fire sales, is strictly larger than -1, which ensures that the Markov process stays in $\left(\mathbb{R}_+^*\right)^n$. □

This multiperiod model exhibits interesting properties as shown in Cont and Wagalath (2013); even with homoscedastic fundamentals, the presence of loss-induced distressed selling induces an endogenous heteroscedasticity and asymmetry in the covariance structure of returns, which in turn leads to realized correlations conditional on large losses being different from correlations in "fundamental" driving factors.

Figure 2.3 illustrates the magnitude of such endogenous correlations. Using a million independent sample paths simulated from a model with zero "fundamental" correlations (using parameters given in Cont and Wagalath 2013), we have computed the average realized correlation across all pairs of assets: Figure 2.3 displays the distribution across sample paths of this average realized correlation. We observe that even in the case where the exogenous shocks driving the asset values are independent (i.e., the "fundamental" covariance matrix Σ is diagonal), the presence of fire sales may lead to significant realized correlations in some scenarios, thereby increasing the volatility experienced by investors holding the fund during episodes of fire sales. This phenomenon may substantially decrease the benefits of diversification. Moreover, the breadth of the distribution across scenarios illustrates the fact that this realized correlation is scenario-dependent, even though driving noise terms are homoscedastic. We now investigate the mechanism underlying this phenomenon in more detail, using the continuous-time limit of the multiperiod model above.

2.2. Diffusion Limit

The multiperiod model described above is rather cumbersome to study directly; in the sequel we focus on its continuous-time limit, which is analytically tractable and related to commonly used diffusion models for price dynamics. This will enable us to compute realized covariances between asset returns in the presence of feedback effects from fire sales.

Denote $x.y = \sum_{i=1}^{n} x_i y_i$ the scalar product between two vectors x and y and M^t the transpose of a matrix $M \in \mathcal{M}_n(\mathbb{R})$. $\mathcal{S}_n(\mathbb{R})$ (respectively, $\mathcal{S}_n^+(\mathbb{R})$) denotes the set of real-valued symmetric matrices (respectively, real-valued symmetric positive semidefinite matrices). For $(a, b) \in \mathbb{R}^2$, we denote $a \wedge b = \min(a, b)$.

In order to study the continuous-time limit of the multiperiod model described in the previous section, we make the following assumption:

ASSUMPTION 2.3. *For* $i = 1..n,\ j = 1..J,$

$$\phi_i \in \mathcal{C}^3(\mathbb{R})\,, \quad f_j \in \mathcal{C}_0^3(\mathbb{R}) \ \text{ and } \ \alpha_i^j \geq 0$$

$$\exists \eta > 0, \mathbb{E}(\|\exp(\eta\xi)\|) < \infty \ \text{ and } \ \mathbb{E}(\|\xi\|^{\eta+4}) < \infty,$$

where $\mathcal{C}_0^p(\mathbb{R})$ *denotes the set of real-valued, p-times continuously differentiable maps whose first derivative has compact support.*

Note that if $f_j \in \mathcal{C}_0^p(\mathbb{R})$, all its derivatives of order $1 \leq l \leq p$ have compact support. In particular f_j is constant for large values and very small values of its argument. This assumption has a natural interpretation in our context: fire sales occur when funds underperform, i.e., when the value of the fund relative to a benchmark falls below a threshold, and cease when the fund defaults.

THEOREM 2.4. *Under Assumptions 2.1 and 2.3, the process* $(S_{\lfloor Nt \rfloor})_{t\geq 0}$ *converges weakly on the Skorokhod space* $D([0, \infty[, \mathbb{R}^n)$*, as* $N \to \infty$*, to a diffusion process* $(P_t)_{t\geq 0}$ *solution of the stochastic differential equation*

$$\frac{\mathrm{d}P_t^i}{P_t^i} = \mu_i(P_t)\mathrm{d}t + (\sigma(P_t)\mathrm{d}W_t)_i \qquad 1 \leq i \leq n, \tag{2.4}$$

where μ *(respectively,* σ*) is an* $\mathbb{R}^n$*-valued (respectively, matrix-valued) mapping defined by*

$$\sigma_{i,k}(P_t) = A_{i,k} + \phi_i'(0) \sum_{j=1}^{J} \alpha_i^j f_j' \left(\frac{V_t^j}{V_0^j} \right) \frac{\left(A\pi_t^j \right)_k}{V_0^j} \tag{2.5}$$

$$\mu_i(P_t) = m_i + \frac{\phi_i'(0)}{2} \sum_{j=1}^{J} \frac{\alpha_i^j}{(V_0^j)^2} f_j'' \left(\frac{V_t^j}{V_0^j} \right) \pi_t^j.\Sigma\pi_t^j \tag{2.6}$$

$$+\sum_{j=1}^{J}\phi_i'(0)\frac{\alpha_i^j}{V_0^j}f_j'\left(\frac{V_t^j}{V_0^j}\right)\left(\pi_t^j.\overline{m}+(\Sigma\pi_t^j)_i\right)+\frac{\phi_i''(0)}{2}\sum_{j,r=1}^{J}\frac{\alpha_i^j\alpha_i^r}{V_0^jV_0^r}f_j'\left(\frac{V_t^j}{V_0^j}\right)f_r'\left(\frac{V_t^r}{V_0^r}\right)\pi_t^j.\Sigma\pi_t^r,$$

where W_t is an n-dimensional Brownian motion, $\pi_t^j = \begin{pmatrix}\alpha_1^j P_t^1\\ \vdots \\ \alpha_n^j P_t^n\end{pmatrix}$ is the (dollar) allocation of fund j, $V_t^j = \sum_{k=1}^n \alpha_k^j P_t^k$ is the benchmark value of fund j, $\overline{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$, and A is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

The proof of this theorem is given in Appendix 5.

REMARK 2.5. The limit price process that we exhibit in Theorem 2.4 depends on the price impact functions ϕ_i only through their first and second derivatives $\phi_i'(0)$ and $\phi_i''(0)$ at 0. In particular, the expression of σ in (2.5) shows that realized volatilities and realized correlations of asset returns depend only on the slope $\phi_i'(0)$ of the price impact function. As a consequence, under our assumptions, a linear price impact function would lead to the same realized covariance structure for asset returns in the continuous-time limit.

REMARK 2.6. The drift coefficient of the diffusion process P involves the second derivative f'' of the deleveraging schedule f_j. Under the (natural) assumption that deleveraging intensifies as the fund value decreases, $f'' < 0$ in the range $]\beta_j^{\text{liq}}, \beta_j[$ where fire sales occur. This leads to the well-known effect that "fire sales push down prices," which translates in our setting into a negative drift term.

In the remainder of this paper, which is dedicated to the study of the impact of fire sales on the covariance structure of asset returns, we hence use the assumption of linear price impact: $D_i = \frac{1}{\phi_i'(0)}$ then corresponds to the market depth for asset i and is interpreted as the number of shares an investor has to buy in order to increase the price of asset i by 1%.

COROLLARY 2.7 (Case of linear price impact). *When $\phi_i(x) = \frac{x}{D_i}$, the drift and volatility of the stochastic differential equation* (2.4) *verified by the continuous-time price process are:*

$$\sigma_{i,k}(P_t) = A_{i,k} + \frac{1}{D_i}\sum_{j=1}^{J}\alpha_i^j f_j'\left(\frac{V_t^j}{V_0^j}\right)\frac{\left(A\pi_t^j\right)_k}{V_0^j}, \tag{2.7}$$

$$\begin{aligned}\mu_i(P_t) = m_i + \frac{1}{D_i}\sum_{j=1}^{J}\Bigg(&\frac{\alpha_i^j}{2(V_0^j)^2}f_j''\left(\frac{V_t^j}{V_0^j}\right)\pi_t^j.\Sigma\pi_t^j\\ &+\frac{\alpha_i^j}{V_0^j}f_j'\left(\frac{V_t^j}{V_0^j}\right)\left(\pi_t^j.\overline{m}+(\Sigma\pi_t^j)_i\right)\Bigg),\end{aligned} \tag{2.8}$$

where W_t, π_t^j, V_t^j, $\overline{m}$, and A are defined in Theorem 2.4.

When market depths are infinite, the price dynamics follows a multivariate exponential Brownian motion. In the presence of fire sales by distressed sellers, the fundamental dynamics of the assets is modified.

2.3. Realized Covariance in the Presence of Fire Sales

The *realized covariance* matrix (Andersen et al. 2003; Barndorff-Nielsen and Shephard 2004) over the period $[t_1, t_2]$ computed on a time grid with step $\Delta t = \frac{1}{N}$ is defined as

$$\widehat{C}^{(N)}_{[t_1,t_2]} = \frac{1}{t_2 - t_1}\left([X, X]^{(N)}_{t_2} - [X, X]^{(N)}_{t_1}\right), \tag{2.9}$$

where X is the log price process defined by $X^i_t = \ln P^i_t$, $[X, X]^{(N)}_t = \left([X^i, X^k]^{(N)}_t\right)_{1\le i,k\le n}$ and

$$[X^i, X^k]^{(N)}_t = \sum_{1\le l\le \lfloor tN\rfloor}\left(X^i_{l/N} - X^i_{(l-1)/N}\right)\left(X^k_{l/N} - X^k_{(l-1)/N}\right). \tag{2.10}$$

As N goes to infinity, the process $([X, X]^{(N)}_t)_{t\ge 0}$ converges in probability on the Skorokhod space $D([0,\infty[, \mathbb{R}^n)$ to an increasing, $\mathcal{S}^+_n(\mathbb{R})$-valued process $([X, X]_t)_{t\ge 0}$, the quadratic covariation process of X (Jacod and Protter 2012, theorem 3.3.1). We define the $\mathcal{S}^+_n(\mathbb{R})$-valued process $c = (c_t)_{t\ge 0}$, which corresponds intuitively to the "instantaneous covariance" of returns, as the derivative of the quadratic covariation process. The realized covariance matrix of returns between t_1 and t_2 is denoted $C_{[t_1,t_2]}$.

$$[X, X]_t = \int_0^t c_s \mathrm{d}s \qquad C_{[t_1,t_2]} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} c_t\, \mathrm{d}t. \tag{2.11}$$

Theorem 2.7 enables to compute the realized covariance matrix for the n assets.

PROPOSITION 2.8. *The instantaneous covariance matrix of returns, c_t, defined in* (2.11)*, is given by*

$$c_t = \Sigma + \sum_{j=1}^{J}\left[\frac{1}{V^j_0} f^{'}_j\left(\frac{V^j_t}{V^j_0}\right)\left(\Lambda_j(\pi^j_t)^t\Sigma + \Sigma\pi^j_t\Lambda^t_j\right)\right]$$
$$+ \sum_{j,k=1}^{J}\frac{\pi^j_t.\Sigma\pi^k_t}{V^j_0 V^k_0} f^{'}_j\left(\frac{V^j_t}{V^j_0}\right) f^{'}_k\left(\frac{V^k_t}{V^k_0}\right)\Lambda_j\Lambda^t_k,$$

where

$\pi^j_t = \begin{pmatrix}\alpha^j_1 P^1_t\\ \vdots\\ \alpha^j_n P^n_t\end{pmatrix}$ *denotes the (dollar) holdings of fund j and* $\Lambda_j = \begin{pmatrix}\frac{\alpha^j_1}{D_1}\\ \vdots\\ \frac{\alpha^j_n}{D_n}\end{pmatrix}$ *represents the positions of fund j in each market as a fraction of the respective market depth.*

Fire sales impact realized covariances between assets. In the presence of fire sales, realized covariance is the sum of the fundamental covariance matrix Σ and an excess realized covariance which is liquidity-dependent and path-dependent. The magnitude of this endogenous impact is measured by the vectors Λ_j, which represent the positions of each fund as a fraction of asset market depths. The volume generated by fire sales in fund j on each asset i is equal to $\alpha^j_i \times f^{\prime}_j$ and its impact on the return of asset i is equal to $\frac{\alpha^j_i}{D_i} \times f^{\prime}_j$. This impact can be significant even if the asset is very liquid, when the positions liquidated are large enough compared to the asset's market depth. Thus, even starting with homoscedastic inputs, fire sales naturally lead to endogenous patterns of heteroscedasticity in the covariance structure of asset returns—in particular spikes or plateaux of high correlation during liquidation periods—similar to those observed in empirical data.

More precisely, we observe that the excess realized covariance terms due to fire sales contain a term of order one in $\|\Lambda\|$ plus higher order terms:

$$c_t = \Sigma + \sum_{j=1}^{J} \left[\frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + O(\|\Lambda\|^2), \tag{2.12}$$

where

$$\Lambda = (\Lambda_1, ..., \Lambda_J) \in \mathcal{M}_{n \times J}(\mathbb{R}), \tag{2.13}$$

where Λ_j is defined in Proposition 2.8 and $\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2}$ is bounded as $\|\Lambda\| \to 0$. This result is due to the fact that under Assumption 2.3, the second-order terms $\frac{\pi_t^j . \Sigma \pi_t^k}{V_0^j V_0^k} f_j'(\frac{V_t^j}{V_0^j}) f_k'(\frac{V_t^k}{V_0^k})$ in the expression of c_t in Proposition 2.8 are bounded because for all $1 \leq j \leq n$, f_j' has a compact support.

In addition, if we denote γ_j the average rate of liquidation (for example, $\gamma_j = \frac{f_j(\beta_j) - f_j(\beta_j^{\text{liq}})}{\beta_j - \beta_j^{\text{liq}}}$), we can approximate the terms of order one in $\|\Lambda\|$ in (2.12) as follows:

$$\begin{aligned} &\sum_{j=1}^{J} \left[\frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] \\ &= \sum_{j=1}^{J} \left[\frac{\gamma_j}{V_0^j} \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + O(\|f''\|), \end{aligned}$$

where $\|f''\| = \sum_{j=1}^{J} \|f_j''\|_\infty$.

As a consequence, Proposition 2.8 may be interpreted as follows: if there are no fire sales between 0 and T, the realized covariance of returns between 0 and T is given by

$$C_{[0,T]} = \frac{1}{T} \int_0^T c_t \, \mathrm{d}t = \Sigma,$$

while the realized covariance between T and $T + \tau_{\text{liq}}$ (where liquidations could have occurred) contains an endogenous component, whose leading terms will be

$$C_{[T, T+\tau_{\text{liq}}]} = \frac{1}{\tau_{\text{liq}}} \int_T^{T+\tau_{\text{liq}}} c_t \, \mathrm{d}t = \Sigma + L M_0 \Pi \Sigma + \Sigma \Pi M_0 L + O(\|\Lambda\|^2, \|f''\|), \tag{2.14}$$

where the remainder is composed of higher order corrections in $\|\Lambda\|^2$ and $\|f''\|$, and

$$M_0 = \sum_{j=1}^{J} \frac{\gamma_j}{V_0^j} \times \alpha^j (\alpha^j)^t, \tag{2.15}$$

where $\alpha^j = \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_n^j \end{pmatrix}$ is the vector of positions of fund j and L and Π are diagonal matrices with ith diagonal term equal, respectively, to $\frac{1}{D_i}$ and $\frac{1}{\tau_{\text{liq}}} \int_T^{T+\tau_{\text{liq}}} P_t^i \, \mathrm{d}t$. In practice, as shown by simulation studies in Cont and Wagalath (2013), this first-order approximation is precise enough and we will focus on this approximation in the numerical examples.

In the absence of fire sales between 0 and T, the realized covariances between asset returns during this period are equal to their fundamental value. Between T and $T+\tau_{\mathrm{liq}}$, fire sales can affect the realized covariance between asset returns. The excess realized covariance is characterized by a matrix M_0, defined in (2.15), which reflects the magnitude of the fire sales. Note that we do not assume that all the funds are liquidating between T and $T+\tau_{\mathrm{liq}}$. A fund j which is not subject to fire sales during this period of time has a rate of liquidation γ_j equal to zero.

In (2.15), $\alpha^j(\alpha^j)^t$ is an $n\times n$ symmetric matrix representing an orthogonal projection on fund j's positions and hence M_0 is a sum of projectors. The symmetric matrix M_0 captures the direction and intensity of liquidations in the J funds.

2.4. Spillover Effects: Price-Mediated Contagion

Consider now the situation where a reference fund with positions $(\alpha_1, \ldots, \alpha_n)$ is subject to fire sales. As argued above, this leads to endogenous volatility and correlations in asset prices, which then modifies the volatility experienced by any other fund holding the same assets.

Proposition 2.8 enables to compute the magnitude of this *volatility spillover* effect (Cont and Wagalath 2013). The following result shows that the realized variance of a (small) fund with positions $(\mu_t^i, i=1..n)$ is the sum of the realized variance in the absence of fire sales and an endogenous term which represents the impact of fire sales in the reference fund.

COROLLARY 2.9 (Spillover effects). *In the presence of fire sales in a reference fund with positions $(\alpha_1, \ldots, \alpha_n)$, the realized variance for a small fund with positions $(\mu_t^i)_{1\le i\le n}$ between t_1 and t_2 is equal to $\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\gamma_s\,\mathrm{d}s$ where*

$$(2.16)\quad \gamma_s M_s^2 = \pi_s^{\mu}.\Sigma\pi_s^{\mu} + \frac{2f'\left(\frac{V_s}{V_0}\right)}{V_0}(\pi_s^{\mu}.\Sigma\pi_s^{\alpha})(\Lambda.\pi_s^{\mu}) + \frac{f'\left(\frac{V_s}{V_0}\right)^2}{V_0^2}(\pi_s^{\alpha}.\Sigma\pi_s^{\alpha})(\Lambda.\pi_s^{\mu})^2,$$

where $\pi_s^{\alpha} = \begin{pmatrix}\alpha_1 P_s^1\\ \vdots\\ \alpha_n P_s^n\end{pmatrix}$ *and* $\pi_s^{\mu} = \begin{pmatrix}\mu_t^1 P_s^1\\ \vdots\\ \mu_t^n P_s^n\end{pmatrix}$ *denote the (dollar) holdings of the reference fund and the small fund, respectively,* $M_s = \sum_{i=1}^{n}\mu_s^i P_s^i$ *is the small fund's value, and* $\Lambda = (\frac{\alpha_1}{D_1}, \ldots, \frac{\alpha_n}{D_n})^t$ *represents the positions of the reference fund in each market as a fraction of the respective market depth.*

The second and third terms in (2.16), which represent the price-mediated contagion of endogenous risk from the distressed fund to other funds holding the same assets, are maximal for funds whose positions are colinear to those of the distressed fund. Both of these terms depend on the *overlap* between portfolios μ and α, defined as

$$(2.17)\qquad \Lambda.\pi_t^{\mu} = \sum_{i=1}^{n}\frac{\alpha_i}{D_i}\mu_t^i P_t^i.$$

As can be observed from this expression, the portfolio overlap is a market depth-weighted scalar product between the two portfolio vectors.

3. FORENSIC ANALYSIS OF FIRE SALES

Theorem 2.4 describes the convergence of the multiperiod model to its diffusion limit under the assumption that the funds liquidate long positions. However, the continuous-time model given in Theorem 2.4 makes sense in a more general setting where we relax the constraint on the sign of α_i^j, i.e., when long–short portfolios are liquidated: in this case, the coefficients of the stochastic differential equation are still locally Lipschitz, so by Ikeda and Watanabe (1981, theorem 3.1, ch. 4) the equation still has a unique strong solution on some interval $[0, \tau[$, where τ is a stopping time (possibly infinite).

In the sequel, we consider the continuous-time model given in Theorem 2.4 in this more general setting which enables for the liquidation of long–short portfolios. Note that the expressions for covariances and spillover effects are not modified.

3.1. Inverse Problem and Identifiability

Equation (2.14) describes the leading term in the impact of fire sales on the realized covariance matrix of returns. Conversely, given that realized covariances can be estimated from observation of prices series, one can use this relation to recover information about the volume of liquidation during a fire sales episode.

We now consider the inverse problem of explaining "abnormal" patterns in realized covariance and volatility in the presence of fire sales and estimating the parameters of the liquidated portfolio from observations of prices. Mathematically, this boils down to answering the following question: for a given time period $[T, T+\tau_{\text{liq}}]$ where liquidations could have occurred, is it possible, given Σ, $C_{[T,T+\tau_{\text{liq}}]}$, L and Π, to find M such that

$$C_{[T,T+\tau_{\text{liq}}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML. \tag{3.1}$$

The following proposition gives conditions under which this inverse problem is well-posed, i.e., the parameter M is identifiable:

PROPOSITION 3.1 (Identifiability). *Let L and Π be diagonal matrices with*

$$L_{ii} = \frac{1}{D_i} \qquad \Pi_{ii} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \, \mathrm{d}t.$$

Assume there exists an invertible matrix Ω and $\phi_1,\dots,\phi_n$ such that

$$\Omega^{-1}\Pi\Sigma L^{-1}\Omega = \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{pmatrix}$$

and for all $1 \le p, q \le n$

$$\phi_p + \phi_q \neq 0,$$

then there exists a unique symmetric $n \times n$ matrix M verifying (3.1)*, which is given by*

$$M = \Phi\left(\Sigma, C_{[T, T+\tau_{liq}]}\right), \tag{3.2}$$

where $\Phi(\Sigma, C)$ is an $n \times n$ matrix defined by

$$\left[\Omega^t \Phi(\Sigma, C)\Omega\right]_{p,q} = \frac{1}{\phi_p + \phi_q} \times \left[\Omega^t L^{-1}(C - \Sigma)L^{-1}\Omega\right]_{p,q}. \tag{3.3}$$

In this case, the unique solution M of (3.1) *verifies*

$$M = M_0 + O(\|\Lambda\|^2, \|f''\|), \tag{3.4}$$

where M_0 is defined in (2.15).

The proof of this proposition is given in Appendix 5. Thanks to (3.4), we deduce the following corollary:

COROLLARY 3.2. *The knowledge of M enables to estimate, up to an error term of order one in $\|\Lambda\|$ and zero in $\|f''\|$, the volume of fire sales in asset class i between T and $T + \tau_{liq}$:*

$$\sum_{j=1}^{J} \frac{\alpha_i^j P_T^i}{V_T^j} \times \gamma_j \times \left(\frac{V_T^j - V_{T+\tau_{liq}}^j}{V_0^j}\right) \times V_T^j$$
$$= (0, ..., 0, P_T^i, 0, ..., 0) M(P_T - P_{T+\tau_{liq}}) + O(\|\Lambda\|^2, \|f''\|).$$

Note that the knowledge of M does not enable in general to reconstitute the detail of fire sales in each fund. Indeed, the decomposition of M given in (2.15) is not always unique. Nevertheless, when different funds engage in similar patterns of fire sales, the common component of these patterns may be recovered from the principal eigenvector of M. In the empirical examples, we find that M has one large eigenvalue, meaning that liquidations were concentrated in one direction.

3.2. Uncovering Fire Sales: Consistency and Large Sample Properties

In the remainder of the paper, we make the following assumption, which guarantees that the identification problem is well-posed in the sense of Proposition 3.1:

ASSUMPTION 3.3. *$\Pi\Sigma L^{-1}$ is diagonalizable with distinct eigenvalues $\phi_1, ..., \phi_n$ such that for all $1 \leq p, q \leq n$:*

$$\phi_p + \phi_q \neq 0.$$

As a consequence, (3.2), (3.3), and (3.4) hold. We require that the eigenvalues of $\Pi\Sigma L^{-1}$ are distinct so that the set of matrices Σ verifying Assumption 3.3 is an open subset of $\mathcal{S}_n(\mathbb{R})$ which enables for the study of the differentiability of Φ defined in (3.3).

Proposition 3.1 states that if we know $L = diag(\frac{1}{D_i})$, $\Pi = diag(\frac{1}{\tau_{\text{liq}}} \int_T^{T+\tau_{\text{liq}}} P_t^i \, \mathrm{d}t)$, the fundamental covariance matrix, Σ, and the realized covariance matrix between T and $T + \tau_{\text{liq}}$, $C_{[T, T+\tau_{\text{liq}}]}$, we can reconstitute M and hence the aggregate characteristics of the liquidation between T and $T + \tau_{\text{liq}}$, according to Corollary 3.2.

The market depth parameters (L) may be estimated using intraday data, following the methods outlined in Obizhaeva (2011) and Cont, Kukanov, and Stoikov (2014). This is further discussed in Section 4. Π may be computed from time series of prices.

Σ and $C_{[T,T+\tau_{\text{liq}}]}$ are estimated using the realized covariance matrices computed on a time grid with step $\frac{1}{N}$, defined in (2.9). In order to estimate Σ, we have to identify a period of time with no fire sales. Denote

$$\tau = \inf\{t \geq 0 \mid \exists\, 1 \leq j \leq J,\ V_t^j < \beta_j V_0^j\} \wedge T. \tag{3.5}$$

τ is the first time, prior to T, when fire sales occur. In our model, fire sales begin when the value of a fund j drops below a certain threshold $\beta_j V_0^j$, with $\beta_j < 1$. Given Corollary 2.7, asset prices and hence fund values are continuous, which implies that τ is a stopping time, bounded by T. Furthermore, as $\beta_j < 1$ for all $1 \leq j \leq J$, τ is strictly positive almost surely: $\mathbb{P}(\tau = 0) = 0$. As a consequence, we estimate the fundamental covariance matrix Σ using the sample realized covariance matrix on $[0, \tau]$, denoted $\widehat{\Sigma}^{(N)}$. In addition, a natural estimator for $C_{[T,T+\tau_{\text{liq}}]}$ is the sample realized covariance matrix between T and $T + \tau_{\text{liq}}$, denoted $\widehat{C}^{(N)}$. By Jacod and Protter (2012, theorem 3.3.1), we find that the estimators of Σ and $C_{[T,T+\tau_{\text{liq}}]}$ are consistent:

$$\widehat{\Sigma}^{(N)} = \frac{1}{\tau}[X, X]_\tau^{(N)} \xrightarrow[N\to\infty]{\mathbb{P}} \Sigma, \tag{3.6}$$

$$\widehat{C}^{(N)} = \frac{1}{\tau_{\text{liq}}}\left([X, X]_{T+\tau_{\text{liq}}}^{(N)} - [X, X]_T^{(N)}\right) \xrightarrow[N\to\infty]{\mathbb{P}} C_{[T,T+\tau_{\text{liq}}]}, \tag{3.7}$$

where the process $[X, X]^{(N)}$ is defined in (2.10) and τ is defined in (3.5). We can hence define an estimator $\widehat{M}^{(N)}$ of M by

$$\widehat{M}^{(N)} = \Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}), \tag{3.8}$$

where Φ is defined in (3.3).

PROPOSITION 3.4 (Consistency). *$\widehat{M}^{(N)}$ defined in* (3.8) *is a consistent estimator of M:*

$$\widehat{M}^{(N)} = \Phi\left(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}\right) \xrightarrow[N\to\infty]{\mathbb{P}} M.$$

The proof of this proposition is given in Appendix 5. Proposition 3.4 shows that $\widehat{M}^{(N)}$ defined in (3.8) is a consistent estimator of M, which contains the information on liquidations between T and $T + \tau_{\text{liq}}$. The following proposition gives us the rate of this estimator $\widehat{M}^{(N)}$ and its asymptotic distribution.

PROPOSITION 3.5 (Asymptotic distribution of estimator).

$$\sqrt{N}\left(\widehat{M}^{(N)} - M\right) \underset{N\to\infty}{\Rightarrow} \nabla\Phi\left(\Sigma, C_{[T,T+\tau_{liq}]}\right) \cdot \begin{pmatrix} \frac{1}{\tau}\overline{Z}_\tau \\ \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) \end{pmatrix}, \tag{3.9}$$

where τ is defined in (3.5)*, $\nabla\Phi$ is the gradient of Φ, defined in* (3.3)*, and*

$$\overline{Z}_t^{ij} = \frac{1}{\sqrt{2}} \sum_{1\leq k,l\leq n} \int_0^t \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right) \mathrm{d}\tilde{W}_s^{kl}, \tag{3.10}$$

where $\tilde{W}$ is an n^2-dimensional Brownian motion independent from W and $\tilde{V}$ is an $\mathcal{M}_{n^2\times n^2}(\mathbb{R})$-valued process verifying

$$(\tilde{V}_t \tilde{V}_t^t)^{ij,kl} = [\sigma\sigma^t(P_t)]_{i,k}[\sigma\sigma^t(P_t)]_{j,l}, \tag{3.11}$$

where σ is defined in (2.7).

The proof of this proposition is given in Appendix 5. The Brownian motion $\tilde{W}$ describes the estimation errors in (3.8): the fact that it is asymptotically independent from the randomness W driving the path of the price process enables to compute the asymptotic distribution of the estimator, conditioned on a given price path and derive confidence intervals, as explained below.

3.3. Testing for the Presence of Fire Sales

Proposition 3.5 enables to test whether $M \neq 0$, i.e., if significant fire sales occurred between T and $T + \tau_{\text{liq}}$. Consider the null hypothesis

$$M = 0 \qquad (H_0).$$

Under hypothesis (H_0), there are no fire sales between T and $T + \tau_{\text{liq}}$. The central limit theorem given in Proposition 3.5 can be simplified as follows:

PROPOSITION 3.6. *Under the null hypothesis* (H_0)*, the estimator* $\widehat{M}^{(N)}$ *verifies the following central limit theorem:*

$$\sqrt{N}\widehat{M}^{(N)} \underset{N\to\infty}{\Rightarrow} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{\tau}\overline{Z}_\tau\right),$$

where $\overline{Z}$ is an n^2-dimensional Brownian motion with covariance

$$cov(\overline{Z}^{i,j}, \overline{Z}^{k,l}) = \Sigma_{i,k}\Sigma_{j,l} + \Sigma_{i,l}\Sigma_{j,k}$$

and Φ and τ are defined in (3.3) *and* (3.5)*, respectively.*

The proof of this proposition is given in Appendix 5. τ is given in (3.5) and can be simulated thanks to Corollary 2.7. This result enables to test whether the variability in the realized covariance of asset returns during $[T, T + \tau_{\text{liq}}]$ may be explained by the superposition of homoscedastic fundamental covariance structure and feedback effects from fire sales. To do this, we estimate the matrix M and test the nullity of the liquidation volumes derived in Corollary 3.2. In practice, it may be possible, for economic reasons, to identify a period $[0, T]$ with no fire sales and hence test the presence of fire sales during $[T, T + \tau_{\text{liq}}]$.

COROLLARY 3.7. *Under the null hypothesis* (H_0) *and if there are no fire sales between* 0 *and T,*

$$\sqrt{N}\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}})\right) \underset{N\to\infty}{\Rightarrow} \mathcal{N}\left(0, \left(\frac{1}{T} + \frac{1}{\tau_{liq}}\right) \sum_{i,j,k,l=1}^{n} m_{ij} m_{kl} \left(\Sigma_{ik}\Sigma_{jl} + \Sigma_{jk}\Sigma_{il}\right)\right)$$

with $m_{ij} = \sum_{p,q=1}^{n} \frac{[\Omega^{-1}P_T]_p[\Omega^{-1}(P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q} \Omega_{ip}\Omega_{jq} D_i D_j$*, where Ω and $(\phi_i)_{1\le i\le n}$ are defined in Proposition 3.1, P_t is the vector of prices at date t and $(D_i)_{1\le i\le n}$ are the asset market depths.*

The proof of this corollary is given in Appendix 5. Corollary 3.7 gives the asymptotic law of $\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}})\right)$, the estimated volume of liquidations, under the null hypothesis (H_0) and if there are no fire sales during $[0, T]$. We can then define a level l such that

$$\mathbb{P}\left(\left|P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}})\right| > l\right) \leq 5\%.$$

If we find that $\left|P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}})\right| > l$ and *if we know that there were no fire sales during* $[0, T]$, then the null hypothesis of no fire sales between T and $T + \tau_{\text{liq}}$ may be rejected at 95% confidence level.

3.4. Numerical Experiments

To assess the accuracy of these estimators in samples of realistic size, we first apply this test to a simulated data set from our model. We consider the case of one fund investing in $n = 20$ assets, with fundamental volatility 30% and zero fundamental correlation. Furthermore, we assume that all assets have the same market depth D and that the fund is initially equally weighted across these assets: $\frac{\alpha_i P_0^i}{V_0} = \frac{1}{n}$. The size of the fund can be captured by the vector Λ, defined in Proposition 2.8, which represents the size of the fund's position in each asset as a fraction of the asset's market depth. In our simulations, we choose this ratio equal to 20%.

We examine the results of our estimation method in the two following cases:

- the fund is not subject to fire sales;
- the fund is subject to fire sales: when the fund value drops below $\beta_0 = 95\%$ of its initial value, the manager deleverages the fund portfolio.

We consider a market where trading is possible every day ($\frac{1}{N} = \frac{1}{250}$). We calculate $\widehat{\Sigma}^{(N)}$ and $\widehat{C}^{(N)}$ and we apply our estimation procedure and calculate in each case (no liquidation and liquidation cases) an estimate for the volume of liquidations. Using 3.7, we can determine, at confidence level 95%, for example, whether there has been a liquidation or not.

Under the assumption (H_0) that $M = 0$ and using Lemma 3.7, we find that

$$\mathbb{P}\left(\left|P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}})\right| > 3.2 \times 10^3\right) \leq 5\%.$$

We find that

- when there are no fire sales, $P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}}) = 203 < 3.2 \times 10^3$ and we cannot reject assumption (H_0);
- when fire sales occur, $P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}}) = 7 \times 10^3 > 3.2 \times 10^3$ and we reject (H_0) at a 95% confidence level.

Using our estimation procedure in the case where there were liquidations, we find that the estimates for the proportions liquidated $\frac{\alpha_i P_0^i}{V_0}$ are all positive and ranging from 2% to 10%, around the true value which is $\frac{1}{20} = 5\%$.

4. THE GREAT DELEVERAGING OF FALL 2008

Lehman Brothers was the fourth largest investment bank in the United States. During the year 2008, it experienced severe losses, caused mainly by the subprime mortgage crisis, and on September 15, 2008, it filed for chapter 11 bankruptcy protection, citing bank

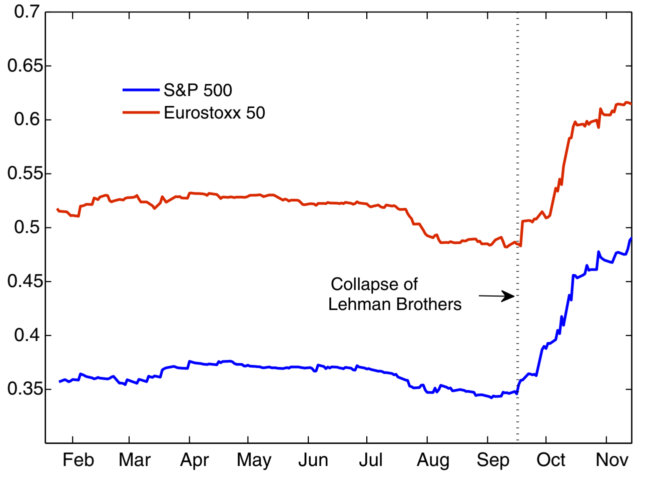

FIGURE 4.1. One-year exponentially weighted moving average estimator of average pairwise correlations of daily returns in S&P 500 and EuroStoxx 50 index.

debt of $613 billion, $155 billion in bond debt, and assets worth $639 billion, becoming the largest bankruptcy filing in the US history.

The failure of Lehman Brothers generated liquidations and deleveraging in all asset classes all over the world. The collapse of this huge institution was such a shock to financial markets—major equity indices all lost around 10% on that day—that it triggered stop loss and deleveraging strategies among a remarkable number of financial institutions worldwide. Risk measures of portfolios, for example, the value-at-risk, increased sharply, obliging financial institutions to hold more cash, which they got by deleveraging their portfolios, rather than by issuing debt which would have been very costly at such distressed times.

This massive deleveraging has been documented in several empirical studies. Fratzscher (2012) studies the effect of key events, such as the collapse of Lehman Brothers, on capital flows. He uses a data set on portfolio capital flows and performance at the fund level from EPFR, containing daily, weekly, and monthly flows for more than 16,000 equity funds and 8,000 bond funds, domiciled in 50 countries. He aggregates the net capital flows (i.e., net of valuation changes) for each country and finds that they are negative for all the countries of the study. This means that fund managers of such funds deleveraged their positions after the collapse of Lehman Brothers, sometimes in dramatic proportions: in some cases, the outflows can represent up to 30% of the assets under management by the funds.

Our method enables to estimate the net effect of liquidations during this period. We report below the results of the estimation method described in Section 3 SPDRs and components of the Eurostoxx 50 index. Figure 4.1 shows that the increase of average correlation in these two equity baskets lasted for around 3 months after September 15, 2008. As a consequence, we examine liquidations that occurred between September 15, 2008 and December 31, 2008.

We calculate the realized covariance matrices, respectively, between $T = 02/01/2008$ and $T = 09/15/2008$ and between $T = 09/15/2008$ and $T + \tau_{\text{liq}} = 12/31/2008$ and apply the estimation procedure described in Section 3.

TABLE 4.1
Estimated Market Depth for SPDRs

Sector SPDR	Estimated market depth $\times 10^8$ shares
Financials	34.8
Consumer discretionary	4.4
Consumer staples	6.2
Energy	8.8
Health care	6.4
Industrials	8.1
Materials	7.0
Technology	7.9
Utilities	7.1

We model price impact as linear, following Obizhaeva (2011) and Cont et al. (2014). To calibrate the market depth parameters D_i, we follow the approach proposed in Obizhaeva (2011): denoting by σ_i the average daily volatility of asset i and ADV_i the average daily trading volume, it was shown in Obizhaeva (2011) for a large panel of US stocks that the ratio $\frac{1}{D}\frac{ADV}{\sigma_r}$ does not vary significantly from one asset to another and

$$\frac{1}{D}\frac{ADV}{\sigma_r} \approx 0.3 \tag{4.1}$$

on average, across a wide range of stocks. Obizhaeva (2011) shows empirically that the difference in price impact of buy-originated trades and sell-originated trades is not statistically significant. We use average daily volumes and average daily volatility to estimate the market depth of each asset, using (4.1). Alternatively one could use intraday data, following the methodology proposed in Cont et al. (2014).

4.1. Sector Exchange-Traded Funds

We first study fire sales among sector SPDRs, which are sector subindices of the S&P 500. There exist nine sector SPDRs: Financials (XLF), Consumer Discretionary (XLY), Consumer Staples (XLP), Energy (XLE), Health Care (XLV), Industrials (XLI), Materials (XLB), Technology (XLK), and Utilities (XLU) and our goal is to determine how economic actors investing in those SPDRs liquidated their portfolios following the collapse of Lehman Brothers.

In order to compute our estimation procedure, we need to know the market depth of each SPDR, which we can estimate as described in the previous section. Market depths are given in Table 4.1. We find that financials have the highest market depth and that other SPDRs have similar market depths.

We can then apply the estimation method described in Section 3 and find the magnitude of fire sales in each SPDR between September 15, 2008 and December 31, 2008.

Our method yields an estimate of 86 billion dollars for fire sales afffecting SPDRs between September 15, 2008 and December 31, 2008. Using Corollary 3.7, we can reject the hypothesis of no liquidation at a 95% confidence level for this period. The liquidation

TABLE 4.2
Daily Liquidations, Percentage of Market Depth Liquidated, and Proportions of Fire Sales for SPDR between September 15, 2008 and December 31, 2008

Sector SPDR	Daily liquidation $\times 10^6$\$	Percent of market depth (%)	Weight (%)
Financials	320	35	28
Consumer discretionary	55	31	5
Consumer staples	38	16	3.5
Energy	300	40	26
Health care	63	23	5.5
Industrials	90	25	8
Materials	110	32	9.5
Technology	65	30	5.5
Utilities	100	30	9

volume that we find is equivalent to a daily liquidation volume of 1.2 billion dollars per day. In comparison, the average volume on SPDRs before Lehman Brother's collapse was 5.1 billion dollars per day. This shows how massive the liquidations were after this market shock.

Corollary 3.2 enables us to determine the aggregate composition of liquidations between September 15, 2008 and December 31, 2008. The daily liquidated volumes and the proportions of each SPDR are given in Table 4.2. This shows that the aggregate portfolio liquidated after Lehman Brother's collapse was a long portfolio. This is consistent with the observation that many financial institutions liquidated equity holdings in order to meet capital requirements during this period, due to the increase of the risk associated with Lehman Brother's collapse. The highest volume of liquidations is associated with financial stocks, followed by the energy sector. These two sectors represent 60% of the liquidations and more that 50 billion dollars liquidated before December 31, 2008.

As discussed in Section 3.1, the principal eigenvector of M reflects the common patterns of fire sales. Table 4.3 gives the proportions of fire sales associated to the principal eigenvector of M. We see that this portfolio is essentially made of financials, which have a weight of 78%. The large weight of XLF, the financial sector index, may be explained in terms of the loss of confidence in banks in the aftermath of the Lehman's collapse.

4.2. Eurostoxx 50

We now conduct our analysis on stocks which belong to the Eurostoxx 50 in order to determine the average composition of portfolios diversified among the components of the Eurostoxx 50 that were liquidated after Lehman Brother's filing for bankruptcy. The Eurostoxx 50 is an equity index regrouping the 50 largest capitalizations of the Euro zone. It is the most actively traded index in Europe and is used as a benchmark to measure the financial health of the euro zone.

We use the same methodology as in the previous section (choice of dates, estimation of Σ, and market depths). Note that we restricted our study to 45 stocks of the index,

TABLE 4.3
Proportions of Fire Sales between September 15, 2008 and December 31, 2008 Associated to the Principal Eigenvector of M

Sector SPDR	Weight (%)
Financials	78
Consumer discretionary	0
Consumer staples	2.5
Energy	4
Health care	0
Industrials	0
Materials	2.5
Technology	10
Utilities	3

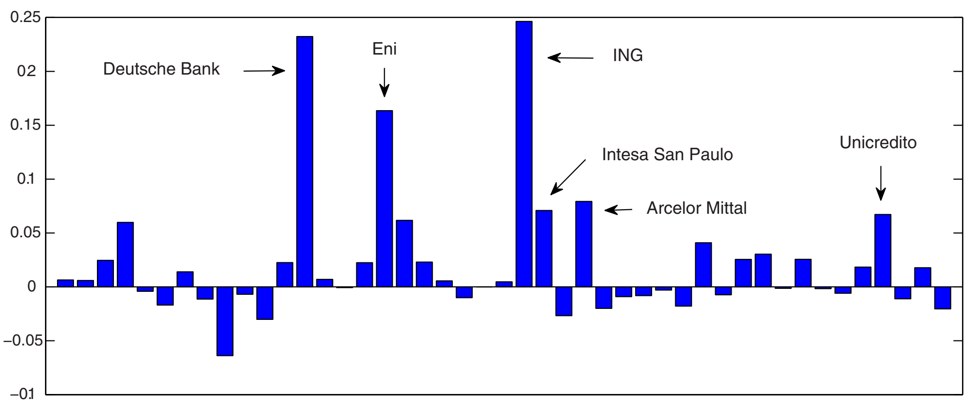

FIGURE 4.2. Fire sales in Eurostoxx 50 stocks in Fall 2008: each bar represents the weight of one stock in the aggregate liquidated portfolio.

for which we had clean data. The five stocks left correspond to the lowest capitalizations among the index components, with very low liquidity.

We find that 350 billion euros were liquidated on stocks belonging to the Eurostoxx 50 between September 15, 2008 and December 31, 2008. Our statistical test described in Corollary 3.7 enables us to reject the null hypothesis of no liquidation at a 99% confidence level. Our estimate for the liquidated volume is equivalent to a daily liquidation of 5 billion euros, which is equal to one-third of the average daily volume of the index components before September 15, 2008.

Figure 4.2, where each bar represents the weight of a stock in the aggregate liquidated portfolio, shows that most of the liquidations following Lehman Brother's collapse involved liquidation of long positions in stocks.

Figure 4.2 shows that fire sales are more intense for some stocks than others. Table 4.4 gives the detail of those stocks. As suggested by the previous section, we see that the fire

TABLE 4.4
Most Liquidated Stocks in the Eurostoxx 50 during the 3 Months Following September 15, 2008

Stock	Amount liquidated $\times 10^6$	Weight (%)
ING	1,100	25
Deutsche Bank	1.000	23
Eni	750	16
Arcelor Mittal	350	8
Intesa San Paolo	320	7
Unicredito	300	6.5

sales in the Eurostoxx 50 index were concentrated in the financial and energy sectors. ING and Deutsche Bank account for almost half of the volume liquidated on the whole index.

5. THE HEDGE FUND LOSSES OF AUGUST 2007

From August 6 to August 9, 2007, long–short market neutral equity funds experienced large losses: many funds lost around 10% per day and experienced a rebound of around 15% on August 10, 2007. During this week, as documented by Khandani and Lo (2011), market neutral equity funds whose returns previously had a low historical volatility exhibited negative returns exceeding 20 standard deviations, while no major move was observed in US equity market indices, which was puzzling for many market observers: it seemed as if prices moved in a coordinated manner as to maximize the loss of this particular category of hedge funds.

Khandani and Lo (2011) suggested that this event was due to a large market neutral fund deleveraging its positions. They simulate a contrarian long–short equity market neutral strategy implemented on all stocks in the CRSP Database and were able to reconstitute qualitatively the empirically observed profile of returns of quantitative hedge funds : low volatility before August 6, huge losses during 3 days, and a rebound on August 10. We reconstituted empirically the returns for Khandani and Lo's market-neutral strategy on the S&P 500 for the first three quarters of 2007. Figure 5.1 shows that this strategy underperforms significantly during the second week of August 2007, while no major move occurred in the S&P 500. Such empirical results tend to confirm the hypothesis of the unwind of a large portfolio which generated, through price impact, large losses across similar portfolios, as predicted by our model.

Using historical data on returns of 487 stocks from the S&P 500 index, we have reconstituted the composition of the fund that deleveraged its positions during the second week of August 2007 using the estimation procedure described in Section 3 for the periods $[0, T] = [08/03/2006, 08/03/2007]$ and $[T, T + \tau_{\text{liq}}] = [08/06/2007, 08/09/2007]$.

Figure 5.2 displays the composition of the aggregate portfolio liquidated on the S&P 500 during this period and found by our estimation method. The first and striking difference with the case of the deleveraging after Lehman Brother's collapse is that, during this quant event, the liquidated portfolio was a long–short portfolio. We clearly

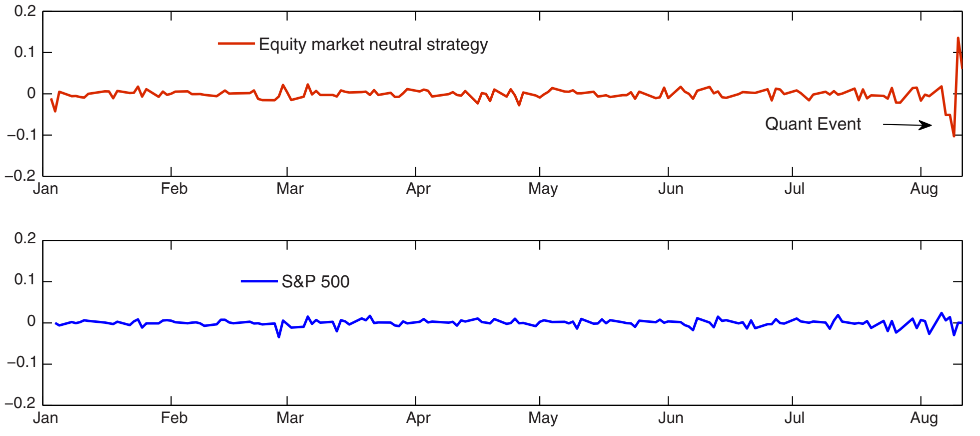

FIGURE 5.1. Returns of a market neutral equity portfolio in 2007, compared with S&P 500 returns.

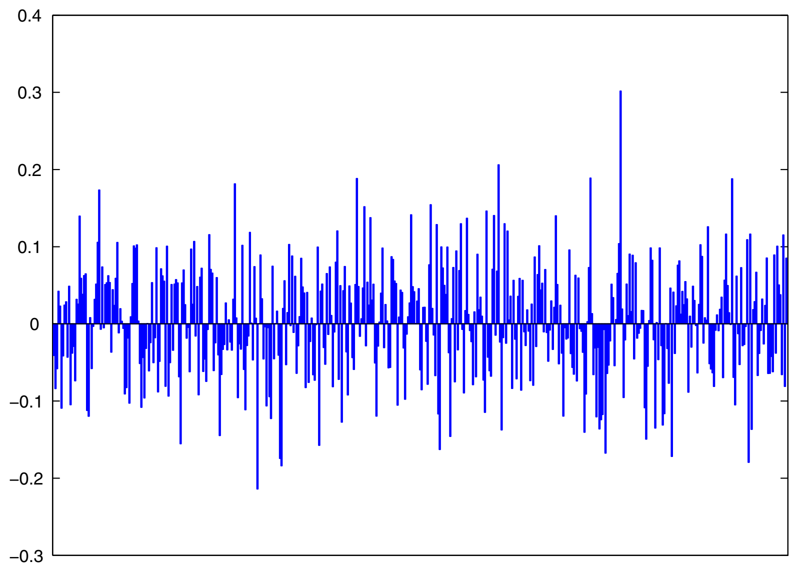

FIGURE 5.2. Equity positions liquidated during the second week of August 2007.

see in Figure 5.2 that for some stocks the liquidated position is significantly negative, meaning that a short position is being exited. More precisely, 250 stocks have positive weights in the liquidated portfolio, whereas 237 have negative weights. Furthermore, we find that the liquidated portfolio was highly leveraged: for each dollar of capital, 15 dollars are invested in long positions and 14 dollars are invested in short positions.

Interestingly, we find that the estimated portfolio is market neutral: the overlap, in the sense of (2.17), with the index is very close to zero. Using the notations of Section 2.4, we find

$$\frac{\hat{\Lambda}.\pi_t^{\hat{\mu}}}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = \frac{\sum_{i=1}^{n} \frac{\alpha_i}{D_i} \mu_t^i P_t^i}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = 0.0958,$$

which corresponds to an angle of 0.47π—i.e., very close to orthogonality—between the vector $\hat{\Lambda}$, of estimated liquidations, and $\pi_t^{\hat{\mu}}$, the composition of the S&P 500 index. This provides a quantitative explanation for the fact that, although a large number of equity funds took massive losses during this episode, index funds were not affected by this event. Note that, unlike other explanations proposed at the time, our explanation does not involve any assumption on the dry up of liquidity during this period: indeed, our estimation assumes constant equity market depth.

APPENDIX

A.1. Proof of Theorem 2.4. We work under Assumptions 2.1 and 2.3. We denote $Z_{k+1} = \frac{1}{N}\overline{m} + \sqrt{\frac{1}{N}}\xi_{k+1} \in \mathbb{R}^n$ where $\overline{m_i} = m_i - \frac{\Sigma_{i,i}}{2}$. We can write the price dynamics (2.3) as follows:

$$S_{k+1}^i = S_k^i \exp\left(Z_{k+1}^i\right)\left[1 + \phi_i\left(\sum_{j=1}^{J} \alpha_i^j \left(f_j\left(\sum_{l=1}^{n} \frac{\alpha_l^j S_k^l}{V_0^j} \exp\left(Z_{k+1}^l\right)\right) - f_j\left(\sum_{l=1}^{n} \frac{\alpha_l^j S_k^l}{V_0^j}\right)\right)\right)\right].$$

As a consequence, we have $S_{k+1} = \theta(S_k, Z_{k+1})$ where $\theta : \left(\mathbb{R}_+^*\right)^n \times \mathbb{R}^n \mapsto \left(\mathbb{R}_+^*\right)^n$ is $\mathcal{C}^3(\mathbb{R})$ as f_j and ϕ_i are $\mathcal{C}^3(\mathbb{R})$ for all $1 \le j \le J$ and $1 \le i \le n$.

Define now a (respectively, b) an $\mathcal{M}_n(\mathbb{R})$-valued (respectively, $\mathbb{R}^n$-valued) mapping such that

$$a_{i,j}(S) = \sum_{l=1}^{n} \frac{\partial \theta_i}{\partial z_l}(S, 0) \times A_{l,j}, \tag{A.1}$$

$$b_i(S) = \sum_{j=1}^{n} \frac{\partial \theta_i}{\partial z_j}(S, 0)\overline{m_j} + \frac{1}{2}\sum_{j,l=1}^{n} \frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(S, 0)\Sigma_{j,l}. \tag{A.2}$$

In order to show Theorem 2.4, we first show the following lemma:

LEMMA A.1. *Under Assumptions 2.1 and 2.3, for all $\epsilon > 0$ and $r > 0$:*

$$\lim_{N\to\infty} \sup_{\|S\|\le r} N \times \mathbb{P}\left(\|S_{k+1} - S_k\| \ge \epsilon | S_k = S\right) = 0, \tag{A.3}$$

$$\lim_{N\to\infty} \sup_{\|S\|\le r} \|N \times \mathbb{E}\left(S_{k+1} - S_k | S_k = S\right) - b(S)\| = 0, \tag{A.4}$$

$$\lim_{N\to\infty} \sup_{\|S\|\le r} \left\|N \times \mathbb{E}\left((S_{k+1} - S_k)(S_{k+1} - S_k)^t | S_k = S\right) - aa^t(S)\right\| = 0, \tag{A.5}$$

where a and b are defined, respectively, in (A.1) *and* (A.2).

Proof. Fix $\epsilon > 0$ and $r > 0$. As θ is $\mathcal{C}^1$, for $\|S\| \leq r$, there exists $C > 0$ such that for all $Z \in \mathbb{R}^n$

$$\|\theta(S, Z) - \theta(S, 0)\| \leq C\|Z\|.$$

As $S_{k+1} = \theta(S_k, Z_{k+1})$ and $S_k = \theta(S_k, 0)$, we find that

$$\mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S, \|S\| \leq r) \leq \mathbb{P}(C\|Z_{k+1}\| \geq \epsilon)$$

$$\leq \mathbb{P}\left(C\|\frac{\overline{m}}{N} + \sqrt{\frac{1}{N}}\xi_{k+1}\| \geq \epsilon\right) \leq \mathbb{P}\left(\|\xi_{k+1}\| \geq \frac{\epsilon - \frac{\|\overline{m}\|C}{N}}{C\sqrt{\frac{1}{N}}}\right)$$

$$\leq \mathbb{E}\left[\left(\|\xi_{k+1}\|\frac{C\sqrt{\frac{1}{N}}}{\epsilon - \frac{\|\overline{m}\|C(r)}{N}}\right)^{2+\eta}\right] \leq \frac{1}{N^{1+\frac{\eta}{2}}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{C}{\epsilon - \frac{\|\overline{m}\|C(r)}{N}}\right)^{2+\eta}$$

which implies (A.3).

As θ is $\mathcal{C}^2$, we can write the Taylor expansion of θ_i in 0, for $1 \leq i \leq n$:

$$\theta_i(S, Z) - \theta_i(S, 0) = \frac{\partial \theta_i}{\partial z}(S, 0)Z + \frac{1}{2}Z.\frac{\partial^2 \theta_i}{\partial z \partial z'}(S, 0)Z + Z.R_i(S, Z)Z,$$

where R_i converges uniformly to 0 when Z goes to 0, when $\|Z\| \leq \epsilon$ and $\|S\| \leq r$. We have

$$\mathbb{E}\left(\frac{\partial \theta_i}{\partial z}(S, 0)Z_{k+1}\right) = \frac{1}{N}\sum_{j=1}^{n}\frac{\partial \theta_i}{\partial z_j}(S, 0)\overline{m}_j$$

and

$$\mathbb{E}\left(Z_{k+1}.\frac{\partial^2 \theta_i}{\partial z \partial z'}(S, 0)Z_{k+1}\right) = \frac{1}{N}\sum_{j,l=1}^{n}\frac{\partial^2 \theta_i}{\partial z_l z_j}\Sigma_{j,l} + o\left(\frac{1}{N}\right).$$

Recalling that $S^i_{k+1} - S^i_k = \theta(S_k, Z_{k+1}) - \theta(S_k, 0)$, we find that

$$\lim_{N\to\infty} \sup_{\|S\|\leq r} \|N\mathbb{E}[(S_{k+1} - S_k)|S_k = S, \|Z_{k+1}\| \leq \epsilon] - b(S)\| = 0. \tag{A.6}$$

We remark that

$$\left\|N\mathbb{E}\left((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\|\leq\epsilon}|S_k = S\right) - b(S)\right\|$$

$$\leq \|(N\mathbb{E}((S_{k+1} - S_k)|S_k = S, \|Z_{k+1}\| \leq \epsilon) - b(S))\|\,\mathbb{P}(\|Z_{k+1}\| \leq \epsilon) + \|b(S)\|\mathbb{P}(\|Z_{k+1}\| \geq \epsilon).$$

As we saw that $\mathbb{P}(\|Z_{k+1}\| \geq \epsilon) \leq \frac{1}{N^{1+\frac{\eta}{2}}}\mathbb{E}[(\|\xi_{k+1}\|)^{2+\eta}] \times (\frac{1}{\epsilon - \frac{\|\overline{m}\|}{N}})^{2+\eta}$ and given (A.6) and the fact that b is continuous, we find that

$$\lim_{N\to\infty} \sup_{\|S\|\leq r} \left\|N\mathbb{E}\left((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\|\leq\epsilon}|S_k = S\right) - b(S)\right\| = 0. \tag{A.7}$$

Similarly, we show that

$$\lim_{N\to\infty} \sup_{\|S\|\leq r} \left\|N\mathbb{E}\left((S_{k+1} - S_k)(S_{k+1} - S_k)^t\mathbf{1}_{\|Z_{k+1}\|\leq\epsilon}|S_k = S\right) - aa^t(S)\right\| = 0. \tag{A.8}$$

Given (2.3), we have the following inequality:

$$S^i_{k+1} \leq S^i_k \exp\left(\frac{\overline{m}_i}{N} + \sqrt{\frac{1}{N}}\xi^i_{k+1}\right)\left(1+\phi_i\left(2\sum_{j=1}^{J}\frac{\alpha_i^j}{D_i}\|f_j\|_\infty\right)\right),$$

which implies that, conditional on $S_k = S$ and for $p > 0$ such that $p\sqrt{\frac{1}{N}} < \eta$, $S_{k+1} \in L^p$. Using this result for $p = 2$, we find that for $\sqrt{\frac{1}{N}} < \frac{\eta}{2}$, $S_{k+1} \in L^2$ and we can use Cauchy Schwarz inequality:

$$\begin{aligned}
&\left|\mathbb{E}\left((S^i_{k+1} - S^i_k)\mathbf{1}_{\|Z_{k+1}\|\geq\epsilon}|S_k = S\right)\right| \\
&\leq \sqrt{\mathbb{E}\left((S^i_{k+1} - S^i_k)^2|S_k = S\right)\mathbb{P}\left(\|Z_{k+1}\| \geq \epsilon\right)} \\
&\leq \frac{1}{N^{1+\frac{\eta}{4}}}\sqrt{\mathbb{E}\left((S^i_{k+1} - S^i_k)^2|S_k = S\right)}\sqrt{\mathbb{E}\left(\frac{\|\xi_{k+1}\|}{\epsilon - \frac{\|\overline{m}\|}{N}}\right)^{4+\eta}}.
\end{aligned}$$

As $\mathbb{E}(\|\xi_{k+1}\|^{4+\eta}) < \infty$, $S_{k+1} \in L^2$ and S_{k+1} stays L^2 bounded conditional on $S_k = S$ and $\|S\| \leq r$. As a consequence, we obtain

$$\lim_{N\to\infty}\sup_{\|S\|\leq r}\left\|N\mathbb{E}\left((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\|\geq\epsilon}|S_k = S\right)\right\| = 0. \tag{A.9}$$

Using the same property with $p = 4$, we show that

$$\lim_{N\to\infty}\sup_{\|S\|\leq r}\left\|N\mathbb{E}\left((S_{k+1} - S_k)(S_{k+1} - S_k)^t\mathbf{1}_{\|Z_{k+1}\|\geq\epsilon}|S_k = S\right)\right\| = 0. \tag{A.10}$$

Equations (A.7) and (A.9) give (A.4). Similarly, (A.8) and (A.10) give (A.5). $\square$

The following lemma gives the expressions of a and b by direct computation of (A.1) and (A.2).

LEMMA A.2. *Equations* (A.1) *and* (A.2)*, respectively, can be written as*

$$a_{i,k}(S) = S^i\left[A_{i,k} + \phi_i'(0)\sum_{j=1}^{J}\frac{\alpha_i^j}{V_0^j}f'\left(\frac{V_j(S)}{V_0^j}\right)(A^t\pi_j(S))_k\right], \tag{A.11}$$

$$\begin{aligned}
b_i(S) = {} & S^i m_i + S^i\frac{\phi_i'(0)}{2}\sum_{j=1}^{J}\frac{\alpha_i^j}{(V_0^j)^2}f_j''\left(\frac{V_j(S)}{V_0^j}\right)\pi_j(S).\Sigma\pi_j(S) \\
& + S^i\phi_i'(0)\sum_{j=1}^{J}\frac{\alpha_i^j}{V_0^j}f_j'\left(\frac{V_j(S)}{V_0^j}\right)\left(\pi_j(S).\overline{m} + (\Sigma\pi_j(S))_i\right) \\
& + S^i\frac{\phi_i''(0)}{2}\sum_{j,r=1}^{J}\frac{\alpha_i^j\alpha_i^r}{V_0^jV_0^r}f_j'\left(\frac{V_j(S)}{V_0^j}\right)f_r'\left(\frac{V_r(S)}{V_0^r}\right)\pi_j(S).\Sigma\pi_r(S),
\end{aligned} \tag{A.12}$$

where $\pi_j(S) = \begin{pmatrix} \alpha_1^j S^1 \\ \vdots \\ \alpha_n^j S^n \end{pmatrix}$ *and* $V_j(S) = \sum_{l=1}^n \alpha_l^j S^l$.

Because f_j is $\mathcal{C}^3$ for $1 \leq j \leq J$, a and b are $\mathcal{C}^2$ and $\mathcal{C}^1$, respectively. Furthermore, because f_j', and hence f_j'' and $f_j^{(3)}$, have a compact support, there exists $R > 0$ such that, for all $1 \leq j \leq J$, when $\|S\| \geq R$, $f_j'(\frac{V_j(S)}{V_0^j}) = f_j''(\frac{V_j(S)}{V_0^j}) = f_j^{(3)}(\frac{V_j(S)}{V_0^j}) = 0$. As a consequence, there exists $K > 0$ such that for all $S \in \left(\mathbb{R}_+^*\right)^n$:

$$\|a(S)\| + \|b(S)\| \leq K\|S\|. \tag{A.13}$$

Furthermore, as the first derivatives of a and b are bounded, a and b are Lipschitz.

Define the differential operator $G : C_0^\infty \left(\mathbb{R}_+^*\right)^n \mapsto C_0^1 \left(\mathbb{R}_+^*\right)^n$ by

$$Gh(x) = \frac{1}{2} \sum_{1 \leq i,j \leq n} (aa^t)_{i,j}(x) \partial_i \partial_j h + \sum_{1 \leq i \leq n} b_i(x) \partial_i h.$$

As a and b verify (A.13), one can apply Ethier and Kurtz (1986, theorem 2.6, ch. 8) to conclude that the martingale problem associated to the operator G is well-posed for any initial condition $S_0 \in \left(\mathbb{R}_+^*\right)^n$. In fact, as a and b are Lipschitz continuous, the solution of this martingale problem is given by the unique strong solution of the stochastic differential equation:

$$\mathrm{d}P_t = b(P_t)\mathrm{d}t + a(P_t)\mathrm{d}W_t \quad \textit{with} \quad P_0 = S_0.$$

As we have shown in Lemma A.1, by Ethier and Kurtz (1986, theorem 4.2, ch. 7), when $N \to \infty$, $\left(S_{\lfloor Nt \rfloor}\right)_{t \geq 0}$ converges in distribution to the solution of the martingale problem associated to the operator G, which concludes the proof of Theorem 2.4.

A.2. Proofs of Propositions 3.1 and 3.4. Let us invert (3.1) under the assumptions of Proposition 3.1. Denote

$$\Omega^{(i)} = \begin{pmatrix} \Omega_{1,i} \\ \vdots \\ \Omega_{n,i} \end{pmatrix}$$

the ith column of the matrix Ω. By definition, we know that $\Pi \Sigma L^{-1} \Omega^{(p)} = \phi_p \Omega^{(p)}$ which is equivalent to $\left(\Omega^{(p)}\right)^t L^{-1} \Sigma \Pi = \phi_p \left(\Omega^{(p)}\right)^t$. As (3.1) is equivalent to $M \Pi \Sigma L^{-1} + L^{-1} \Sigma \Pi M = L^{-1}(C_{[T,T+\tau_{\mathrm{liq}}]} - \Sigma) L^{-1}$ and multiplying this equality on the left by $\left(\Omega^{(p)}\right)^t$ and on the right by $\Omega^{(q)}$, we find that

$$(\phi_p + \phi_q)[\Omega^t M \Omega]_{p,q} = [\Omega^t L^{-1} \left(C_{[T,T+\tau_{\mathrm{liq}}]} - \Sigma\right) L^{-1} \Omega]_{p,q},$$

which gives the matrix $\Omega^t M \Omega$ as a function of Σ and $C_{[T,T+\tau_{\mathrm{liq}}]}$. As Ω is invertible, this characterizes the matrix M, as a function, denoted Φ of Σ and $C_{[T,T+\tau_{\mathrm{liq}}]}$, proving (3.2) and (3.3).

Furthermore, notice that $M_0 = \Phi\left(\Sigma, C_{[T,T+\tau_{\mathrm{liq}}]} + O(\|\Lambda\|^2, \|f''\|)\right)$. Given the expression for Φ in (3.3), (3.4) follows directly. This concludes the proof of Proposition 3.1.

LEMMA A.3. *The mapping* Φ *defined in* (3.3) *is* $\mathcal{C}^\infty$ *in a neighborhood of* (Σ, C).

Proof. The following map

$$F : \mathcal{S}_n^3(\mathbb{R}) \mapsto \mathcal{S}_{\mathrm{n}}(\mathbb{R}), (\mathrm{S, C, N}) \mapsto \mathrm{LN\Pi S + S\Pi NL + S - C} \tag{A.14}$$

is infinitely differentiable, its gradient with respect to N given by

$$\frac{\partial F}{\partial N}(S, C, N).H_3 = LH_3\Pi S + S\Pi H_3 L.$$

As Σ verifies Assumption 3.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible for all C. As $\Phi(\Sigma, C)$ is defined as the only matrix verifying $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, the implicit function theorem states that Φ is C^∞ in a neighborhood of (Σ, C). □

As convergence in probability implies that a subsequence converges almost surely, we assume from now on that the estimators defined in (3.6) and (3.7) converge almost surely. Since the set of matrices Σ verifying Assumption 3.3, is an open set, for N large enough, $\widehat{\Sigma}^{(N)}$ also verifies Assumption 3.3. We can thus define $\widehat{M}^{(N)}$ as in (3.8).

Lemma A.3 implies in particular that Φ is continuous and hence that $\Phi\left(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}\right)$ converges almost surely, and hence in probability, to $\Phi\left(\Sigma, C_{[T,T+\tau_{\mathrm{liq}}]}\right)$. As a consequence, $\Phi\left(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}\right)$ is a consistent estimator of $\Phi\left(\Sigma, C_{[T,T+\tau_{\mathrm{liq}}]}\right)$, meaning that $\widehat{M}^{(N)}$ is a consistent estimator of M. This shows Proposition 3.4.

A.3. Proof of Proposition 3.5. Using Theorem 2.7 and Ito's formula, we deduce that the log price $X_t^i = \ln(P_t^i)$ verifies the following stochastic differential equation:

$$\mathrm{d}X_t^i = \left(\mu_i\left(e^{X_t}\right) - \frac{1}{2}\left(\sigma\left(e^{X_t}\right)\sigma\left(e^{X_t}\right)^t\right)_{i,i}\right)\mathrm{d}t + \left(\sigma\left(e^{X_t}\right)\mathrm{d}W_t\right)_i,$$

where σ, μ, and W are defined in Theorem 2.7 and e^{X_t} is an n-dimensional column vector with ith term equal to $\exp X_t^i$. As a consequence, X is an Ito process which verifies, for $t \geq 0$,

$$\int_0^t \left(\sum_{1\leq i\leq n}\left(\mu_i(P_t) - \frac{1}{2}\left(\sigma(P_t)\sigma_t(P_t)^t\right)_{i,i}\right)^2 + \|\sigma\sigma^t(P_t)\|^2\right)\mathrm{d}s < \infty.$$

We are thus in the setting of Jacod and Protter (2012, theorem 5.4.2, ch. 5) which describes the asymptotic distribution of the quadratic covariation of an Ito process with well-behaved coefficients. We need to extend $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ to a larger probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$. There exists $\tilde{W}$ an n^2-dimensional Brownian motion, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ and independent from W, such that

$$\sqrt{N}\left([X, X]^{(N)} - [X, X]\right) \underset{N\to\infty}{\overset{s.l.}{\longrightarrow}} \overline{Z},$$

where the $n \times n$ dimensional process $\overline{Z}$ is defined in (3.10) and s.l. means stable convergence in law (Jacod and Protter 2012, see section 2.2.1). The auxiliary Brownian motion $\tilde{W}$ represents the estimation error. Furthermore, Jacod and Protter (2012, equation 2.2.5) shows that

$$\left(\sqrt{N}\left([X, X]^{(N)} - [X, X]\right), \tau\right) \underset{N\to\infty}{\Rightarrow} \left(\overline{Z}, \tau\right).$$

This implies that the estimators $\left(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}\right)$ defined in (3.6) and (3.7) verify the following central limit theorem:

$$\sqrt{N}\left[\begin{pmatrix}\widehat{\Sigma}^{(N)}\\ \widehat{C}^{(N)}\end{pmatrix} - \begin{pmatrix}\Sigma\\ C_{[T,T+\tau_{\text{liq}}]}\end{pmatrix}\right] \underset{N\to\infty}{\Rightarrow} \begin{pmatrix}\frac{1}{\tau}\overline{Z}_\tau\\ \frac{1}{\tau_{\text{liq}}}(\overline{Z}_{T+\tau_{\text{liq}}} - \overline{Z}_T)\end{pmatrix}. \tag{A.15}$$

Since $\Phi \in \mathcal{C}^1$, one can then apply the "delta method" to $\left(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}\right)$ to derive the result in Proposition 3.5.

A.4. Proofs of Proposition 3.6 and Corollary 3.7. Under the null hypothesis (H_0), $\frac{1}{\tau_{\text{liq}}}\int_T^{T+\tau_{\text{liq}}} c_t \, \mathrm{d}t = \Sigma$ and hence

$$\Phi\left(\Sigma, \frac{1}{\tau_{\text{liq}}}\int_T^{T+\tau_{\text{liq}}} c_t \, \mathrm{d}t\right) = \Phi(\Sigma, \Sigma) = 0.$$

To compute the gradient $\nabla\Phi(\Sigma, \Sigma)$, recall that $\Phi(\Sigma, C)$ is implicitly defined as the unique solution of $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, where F is defined in (A.14). F is affine in each component and thus $\mathcal{C}^\infty$; its derivatives $\frac{\partial F}{\partial S}(S, C, N)$, $\frac{\partial F}{\partial C}(S, C, N)$, and $\frac{\partial F}{\partial N}(S, C, N)$ are linear mappings from $\mathcal{S}_n(\mathbb{R})$ to $\mathcal{S}_n(\mathbb{R})$ given by

$$\frac{\partial F}{\partial S}(S, C, N).H_1 = LN\Pi H_1 + H_1\Pi NL + H_1$$

$$\frac{\partial F}{\partial C}(S, C, N).H_2 = -H_2$$

$$\frac{\partial F}{\partial N}(S, C, N).H_3 = LH_3\Pi S + S\Pi H_3 L.$$

We therefore have

$$\nabla F(S, C, N).(H_1, H_2, H_3) = LN\Pi H_1 + H_1\Pi NL + H_1 - H_2 + LH_3\Pi S + S\Pi H_3 L.$$

In the proof of Lemma A.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible. As a consequence, we can apply the implicit function theorem in order to compute the gradient of Φ. As $F(\Sigma, C, \Phi(\Sigma, C)) = 0$ and $\Phi(\Sigma, \Sigma) = 0$, we find that $\frac{\partial F}{\partial S}(\Sigma, \Sigma, 0).H_1 = H_1$, $\frac{\partial F}{\partial C}(\Sigma, \Sigma, 0).H_2 = -H_2$ and $\frac{\partial F}{\partial N}(\Sigma, \Sigma, 0).H_3 = LH_3\Pi\Sigma + \Sigma\Pi H_3 L$ and hence the derivative of Φ on (Σ, Σ) is given by

$$\nabla\Phi(\Sigma, \Sigma).(H_1, H_2) = \left(\frac{\partial F}{\partial N}(\Sigma, \Sigma, 0)\right)^{-1}(H_2 - H_1),$$

which is equivalent to

$$\nabla\Phi(\Sigma, \Sigma).(H_1, H_2) = \Phi(\Sigma, \Sigma + H_2 - H_1).$$

Using Proposition 3.5, we find that

$$\sqrt{N}\widehat{M}^{(N)} \overset{\mathcal{L}}{\Rightarrow} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{\text{liq}}}(\overline{Z}_{T+\tau_{\text{liq}}} - \overline{Z}_T) - \frac{1}{\tau}\overline{Z}_\tau\right),$$

which concludes the proof of Proposition 3.6.

If there are no fire sales between 0 and T, then $\tau = T$ almost surely. In addition, under (H_0), we have $\sigma\sigma^t = \Sigma$ and the expression for the process $\tilde{V}_t$ defined in (3.11) is simplified as

$$(\tilde{V}_t \tilde{V}_t^t)^{ij,kl} = \Sigma_{i,k}\Sigma_{j,l}, \tag{A.16}$$

which implies that the process $\overline{Z}$ defined in (3.10) is a Brownian motion.

Furthermore, given Proposition 3.6, under (H_0), $\sqrt{N}\left(P_T^t \widehat{M}^{(N)}\left(P_T - P_{T+\tau_{\text{liq}}}\right)\right)$ converges in law when N goes to infinity to the random variable

$$P_T^t \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{\text{liq}}}\left(\overline{Z}_{T+\tau_{\text{liq}}} - \overline{Z}_T\right) - \frac{1}{T}\overline{Z}_T\right)\left(P_T - P_{T+\tau_{\text{liq}}}\right).$$

Given the expression for Φ given in (3.3), we find the expression for

$$P_T^t \Phi(\Sigma, C)\left(P_T - P_{T+\tau_{\text{liq}}}\right)$$
$$= \sum_{1\le p,q\le n} \left(\Omega^{-1}P_T\right)_p \frac{\left[\Omega^t L^{-1}(C-\Sigma)L^{-1}\Omega\right]_{p,q}}{\phi_p + \phi_q}\left(\Omega^{-1}\left(P_T - P_{T+\tau_{\text{liq}}}\right)\right)_q.$$

Given the fact that $L^{-1} = diag(D_i)$, we have $(\Omega^t L^{-1})_{p,i} = \Omega_{i,p} D_i$ and $(L^{-1}\Omega)_{j,q} = \Omega_{j,q} D_j$. As a consequence, denoting

$$m_{i,j} = \sum_{1\le p,q\le n} \frac{\left[\Omega^{-1}P_T\right]_p \left[\Omega^{-1} \le (P_T - P_{T+\tau_{\text{liq}}})\right]_q}{\phi_p + \phi_q}\Omega_{ip}\Omega_{jq} D_i D_j$$

we can write $P_T^t \Phi(\Sigma, C)\left(P_T - P_{T+\tau_{\text{liq}}}\right)$ as $\sum_{1\le i,j\le n} m_{ij}(C_{i,j} - \Sigma_{i,j})$. Hence, the limit of $\sqrt{N}\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{\text{liq}}})\right)$ is equal to

$$\sum_{1\le i,j\le n} m_{ij}\left(\frac{1}{\tau_{\text{liq}}}\left(\overline{Z}_{T+\tau_{\text{liq}}} - \overline{Z}_T\right) - \frac{1}{T}\overline{Z}_T\right)_{i,j}.$$

Under the assumptions of Corollary 3.7, $\overline{Z}$ is a Brownian motion on $[0, T+\tau_{\text{liq}}]$ (see (A.16)), so the limit process is a mean-zero Gaussian process. To compute its variance, we first compute the variance of $\sum_{1\le i,j\le n} m_{ij}\overline{Z}_t^{i,j}$ which, given the expression of $\overline{Z}$ in (3.10), can be written as

$$\sum_{1\le k,l\le n}\int_0^t \frac{1}{\sqrt{2}}\sum_{1\le i,j\le n} m_{i,j}\left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right) \mathrm{d}\tilde{W}_s^{kl}.$$

Using the Ito isometry formula, its variance is thus equal to

$$\sum_{1\le k,l\le n}\int_0^t \left(\sum_{1\le i,j\le n}\frac{1}{\sqrt{2}} m_{i,j}\left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right)\right)^2 \mathrm{d}s$$
$$= \frac{t}{2}\sum_{1\le k,l\le n}\left(\sum_{1\le i,j,p,q\le n} m_{i,j}m_{p,q}\left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right)\left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl}\right)\right)$$

$$= \frac{t}{2} \sum_{1 \leq i,j,p,q \leq n} m_{i,j} m_{p,q} \left(\sum_{1 \leq k,l \leq n} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right)$$

$$= t \sum_{1 \leq i,j,p,q \leq n} m_{i,j} m_{p,q} \left(\Sigma_{i,p} \Sigma_{j,q} + \Sigma_{i,q} \Sigma_{j,p} \right)$$

using the fact that $\sum_{1 \leq k,l \leq n} \tilde{V}_s^{ij,kl} \tilde{V}_s^{pq,kl} = \Sigma_{i,p} \Sigma_{j,q}$ as $\tilde{V}$ verifies (A.16). Given the fact that $\overline{Z}_{T+\tau_{\text{liq}}} - \overline{Z}_T$ and $\overline{Z}_T$ are independent, we find that the variance of the limit $\sum_{1 \leq i,j \leq n} m_{ij} \left(\frac{1}{\tau_{\text{liq}}} (\overline{Z}_{T+\tau_{\text{liq}}} - \overline{Z}_T) - \frac{1}{T} \overline{Z}_T \right)_{i,j}$ is equal to

$$\left(\frac{1}{T} + \frac{1}{\tau_{\text{liq}}} \right) \sum_{1 \leq i,j,k,l \leq n} m_{ij} m_{kl} \left(\Sigma_{ik} \Sigma_{jl} + \Sigma_{jk} \Sigma_{il} \right)$$

which concludes the proof of Corollary 3.7.

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