J Theor Probab (2009) 22: 375–401
DOI 10.1007/s10959-008-0147-9

Multifractional, Multistable, and Other Processes with Prescribed Local Form

K.J. Falconer · J. Lévy Véhel

Received: 4 September 2007 / Revised: 4 January 2008 / Published online: 16 February 2008

Abstract We present a general method for constructing stochastic processes with prescribed local form, encompassing examples such as variable amplitude multifractional Brownian and multifractional α-stable processes. We apply the method to Poisson sums to construct multistable processes, that is, processes that are locally $\alpha(t)$-stable but where the stability index $\alpha(t)$ varies with t. In particular we construct multifractional multistable processes, where both the local self-similarity and stability indices vary.

Keywords Stochastic process · Localisable · Multifractional · Multistable · Stable process

1 Introduction

Stochastic processes where the local Hölder regularity varies with a parameter t are important both in theory and in practical applications. The best known example is multifractional Brownian motion (mBm) [1–3, 10, 13], where the Hurst index h of fractional Brownian motion is replaced by a functional parameter $h(t)$, permitting the Hölder exponent to vary in a prescribed manner. This allows local regularity and long range dependence to be decoupled to give sample paths that are both highly irregular and highly correlated, a useful feature for terrain or TCP traffic modeling.

K.J. Falconer (✉)
Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland, UK
e-mail: kjf@st-and.ac.uk

J. Lévy Véhel
Projet Fractales, INRIA Rocquencourt, 78153 Le Chesnay Cedex, France
e-mail: Jacques.Levy-Vehel@inria.fr

For modelling financial or medical data the presence of jumps is often important. Stable non-Gaussian processes give good models for data containing discontinuities, with the stability index α controlling the distribution of jumps. Recently, multifractional stable processes, generalising mBm, were introduced to provide jump processes with varying local regularity [17, 18]. However, a further step is needed for situations where both local regularity and jump intensity vary with time, for example to model financial data or epileptic episodes in EEG, where for some periods there may be only small jumps and at other instants very large ones. Our general method may be applied to certain Poisson sums to construct processes where both h and α vary in a prescribed way: thus there are two parameters which might correspond to distinct aspects of financial risk, to different sources of irregularity leading to the onset of epilepsy, or to textured images where both Hölder regularity and the distribution of discontinuities vary.

It is natural to construct processes $Y = \{Y(t) : t \in \mathbb{R}\}$ that are localisable with a prescribed scaling limit or local form Y'_u at each u, in the sense of (2.1) below. Our general construction will allow known localisable processes $X(\cdot, v) = \{X(t, v) : t \in \mathbb{R}\}$ to be pieced together over a range of v to yield a localisable 'diagonal' process $Y = \{X(t, t) : t \in \mathbb{R}\}$ with local form depending on t. The basic setting is akin to that adopted in [2, 17]. Thus we seek a random field $\{X(t, v) : (t, v) \in \mathbb{R}^2\}$ such that for each u the local form $X'_u(\cdot, u)$ of $X(\cdot, u)$ at u is the desired local form Y'_u of Y at u. Clearly the interplay of $X(\cdot, v)$ for v in a neighbourhood of u will be crucial to the local behaviour of Y near u and we derive general criteria that guarantee the transference of localisability from the $X(\cdot, v)$ to Y.

We illustrate the general method with *multifractional* processes such as mBm and multifractional α-stable motions. We then develop *multistable* processes, with varying stability index $\alpha(t)$, using sums over Poisson processes. In particular we construct *multifractional multistable* processes, where both the local self-similarity index and the stability index vary.

2 Localisable and Strongly Localisable Processes

Let $C(T)$ be the space of continuous functions on $T \subset \mathbb{R}$ endowed with the uniform metric d and let $D(T)$ be the "càdlàg" functions on T, that is functions which are continuous on the right and have left limits at all $t \in T$, endowed with the Skorohod metric d_S, see for example [4, 14]. When we write $X = Y$ it will be clear from the context whether this refers to equality in law (i.e. in finite dimensional distributions) or in distribution.

For convenience we define localisability at u for random processes with domain $\mathbb{R}$, but the definitions will also apply in the obvious way where the domain is a real interval with u as an interior point. (A technicality here is that our functions or processes may have a domain U that is a proper interval of $\mathbb{R}$. This presents no difficulty, since we will generally work with a sequence of enlargements of a process about some u interior to U, and the domain of definition will eventually include each bounded interval $[a, b]$.)

We say that $Y = \{Y(t) : t \in \mathbb{R}\}$ is *h-localisable* at u with *local form* the random process $Y'_u = \{Y'_u(t) : t \in \mathbb{R}\}$ if

$$\frac{Y(u+rt) - Y(u)}{r^h} \to Y'_u(t) \tag{2.1}$$

as $r \searrow 0$, where convergence is of finite dimensional distributions. If Y and Y'_u have versions in $C(\mathbb{R})$ or $D(\mathbb{R})$ and convergence in (2.1) is in distribution, we say that Y is *h-strongly localisable* at u with *strong local form* Y'_u.

It is clear from (2.1) that the h-local form Y'_u at u, if it exists, must itself be h-self-similar, that is $Y'_u(rt) = r^h Y'_u(t)$ in law for $r > 0$. However, much more is true: under quite general conditions Y'_u must be h-self-similar with stationary increments (h-sssi) at almost all u at which it is strongly localisable, that is $r^{-h}(Y'_u(u+rt) - Y'_u(u)) = Y'_u(t)$ for all u and $r > 0$, see [8, 9]. Thus if we wish to construct processes with given local forms, the local forms should themselves be sssi, and the next proposition shows that such processes are themselves localisable.

Proposition 2.1 *Let* $\{Y(t) : t \in \mathbb{R}\}$ *be a process that is h-self-similar (that is* $Y(rt) = r^h Y(t)$ *in law for all* $r > 0$*) with stationary increments (that is* $Y(t+u)$ *and* $Y(t)$ *are equal in law for* $u \in \mathbb{R}$*). Then* Y *is h-localisable at all* $u \in \mathbb{R}$ *with* $Y'_u = Y$*. If in addition* Y *is in* $C(\mathbb{R})$ *or* $D(\mathbb{R})$ *then* Y *is strongly h-localisable at all* $u \in \mathbb{R}$.

Proof If Y is h-sssi, then

$$\frac{Y(u+rt) - Y(u)}{r^h} = \frac{Y(rt) - Y(0)}{r^h} = \frac{Y(rt)}{r^h} = Y(t)$$

(where the equalities are in law) for all $r \neq 0$, so Y is localisable at u.

If Y is in $C(\mathbb{R})$ or $D(\mathbb{R})$ then $(Y(u+rt) - Y(u))/r^h$ and $Y(t)$ have identical probability distributions, since probability distributions on $C(\mathbb{R})$ and $D(\mathbb{R})$ are completely determined by their finite dimensional distributions, see [4]. Thus Y is strongly localisable. □

There are several important processes which are sssi so are strongly localisable by Proposition 2.1. For $0 < h < 1$, index-h fractional Brownian motion (fBm) on $\mathbb{R}$ may be defined as a stochastic integral with respect to Wiener measure W:

$$B_h(t) = c(h)^{-1} \int_{-\infty}^{\infty} \big((t-x)_+^{h-1/2} - (-x)_+^{h-1/2}\big) W(dx), \tag{2.2}$$

where $(a)_+ = \max\{0, a\}$ and $c(h)$ is a normalising constant that ensures that $\mathrm{var} B_h(1) = 1$. (We make the convention that expressions involving the difference of two positive parts represent an indicator function when the exponent is 0, so for example, if $h = 1/2$ then $(t-x)_+^{h-1/2} - (-x)_+^{h-1/2}$ is just $\mathbf{1}_{[0,t)}(x)$.) It is well-known [6, 7, 12, 16] that index-h fBm is h-sssi with a version in $C(\mathbb{R})$, so is strongly localisable at all $u \in \mathbb{R}$ with $(B_h)'_u = B_h$.

Many α-stable processes are sssi, see [16, Corollary 7.3.4], and so are strongly h-localisable by Proposition 2.1. A particular instance is linear stable fractional motion:

$$L_{\alpha,h}(t) = \int_{-\infty}^{\infty} \big[a\big((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}\big) \\ + b\big((t-x)_-^{h-1/\alpha} - (-x)_-^{h-1/\alpha}\big)\big] M(dx), \tag{2.3}$$

where $0 < \alpha < 2$ and M is an α-stable random measure with constant skewness β and control measure Lebesgue measure, $0 < h < 1$ and a and b are constants, see [16, Sect. 7.4 and Chap. 10]. The process is h-sssi and so is h-localisable at all $u \in \mathbb{R}$ with $(L_{\alpha,h})'_u = L_{\alpha,h}$. Provided that $h > 1/\alpha$ it has a version in $C(\mathbb{R})$, so is strongly localisable. However, if $h < 1/\alpha$ then almost surely Y is unbounded on every interval and so is not a process of $D(\mathbb{R})$, though it is nevertheless localisable.

An α-stable Lévy motion, $0 < \alpha < 2$ is a process in $D(\mathbb{R})$ with stationary independent increments which have a strictly α-stable distribution. It may be represented as $L_\alpha(t) = M([0,t])$ where M is an α-stable random measure on $\mathbb{R}$ with constant skewness intensity, see [16, Sect. 7.5]. Then L_α is $1/\alpha$-sssi, and so is strongly $1/\alpha$-localisable.

Localisability behaves well under reasonably smooth changes of coordinates and the following proposition allows us to vary the 'local amplitude' of localisable processes.

Proposition 2.2 *Let U be an interval with u an interior point. Suppose that $\{Y(t) : t \in U\}$ is h-localisable (resp. strongly h-localisable) at u. Let $a : U \to \mathbb{R}$ satisfy an η-Hölder condition on U where $\eta > h$. Then $aY = \{a(t)Y(t) : t \in U\}$ is h-localisable (resp. strongly h-localisable) at u with $(aY)'_u = a(u)Y'_u$.*

Proof We have

$$\frac{a(u+rt)Y(u+rt) - a(u)Y(u)}{r^h} \\ = a(u+rt)\frac{Y(u+rt) - Y(u)}{r^h} + Y(u)\frac{a(u+rt) - a(u)}{r^h}.$$

The result now follows on letting $r \to 0$ with the appropriate form of convergence. □

Our main aim is to construct localisable and strongly localisable processes with prescribed local form Y'_u that varies with u. We do this by 'joining together' localisable processes $\{X(t,v) : t \in U\}$ over a range of v, and we obtain conditions that ensure $Y = \{X(t,t) : t \in U\}$ 'looks locally' like $\{X(t,u) : t \in U\}$ when t is close to u.

Let U be an interval with u an interior point. Let $\{X(t,v) : (t,v) \in U \times U\}$ be a random field and let Y be the diagonal process $Y = \{X(t,t) : t \in U\}$. We want Y and $X(\cdot,u)$ to have the same local forms at u, that is $Y'_u(\cdot) = X'_u(\cdot,u)$ where $X'_u(\cdot,u)$ is the local form of $X(\cdot,u)$ at u. Thus we require

$$\frac{X(u+rt, u+rt) - X(u,u)}{r^h} \to X'_u(t,u) \tag{2.4}$$

in finite dimensional distributions or in distribution as $r \searrow 0$. The following two theorems give sufficient conditions for this to occur.

Theorem 2.3 *Let U be an interval with u an interior point. Suppose that for some $0 < h < \eta$ the process $\{X(t,u) : t \in U\}$ is h-localisable at $u \in U$ with local form $X'_u(\cdot, u)$ and*

$$\mathsf{P}(|X(v,v) - X(v,u)| \geq |v-u|^{\eta}) \to 0 \tag{2.5}$$

as $v \to u$. Then $Y = \{X(t,t) : t \in U\}$ is h-localisable at u with $Y'_u(\cdot) = X'_u(\cdot, u)$.

Proof For $r \neq 0$

$$\begin{aligned}\frac{Y(u+rt) - Y(u)}{r^h} &= \frac{X(u+rt, u+rt) - X(u,u)}{r^h} \\ &= \frac{X(u+rt, u+rt) - X(u+rt, u)}{r^h} \\ &\quad + \frac{X(u+rt, u) - X(u,u)}{r^h}.\end{aligned} \tag{2.6}$$

Fix $t \in \mathbb{R}$ and $c > 0$. Let r_0 be sufficiently small to ensure that if $0 < r < r_0$ then both $u \pm rt \in U$ and $cr^h \geq (r|t|)^{\eta}$. Then for $0 < r < r_0$

$$\begin{aligned}&\mathsf{P}\left(\frac{|X(u+rt, u+rt) - X(u+rt, u)|}{r^h} \geq c\right) \\ &\quad \leq \mathsf{P}(|X(u+rt, u+rt) - X(u+rt, u)| \geq |(u+rt) - u|^{\eta}) \to 0\end{aligned}$$

as $r \searrow 0$, by (2.5). Thus for all $t \in \mathbb{R}$,

$$\frac{X(u+rt, u+rt) - X(u+rt, u)}{r^h} \to 0$$

in probability and so in finite dimensional distributions. Moreover,

$$\frac{X(u+rt, u) - X(u,u)}{r^h} \xrightarrow{\text{fdd}} X'_u(t,u),$$

since $X(\cdot, u)$ is localisable at u. We conclude from (2.6) that Y is localisable at u with local form $Y'_u(\cdot) = X'_u(\cdot, u)$. □

We next obtain an analogue of Theorem 2.3 in the strongly localisable case, that is a criterion for convergence in distribution in (2.1).

Theorem 2.4 *Let $F(\mathbb{R})$ be either $C(\mathbb{R})$ endowed with the uniform metric d or $D(\mathbb{R})$ with the Skorohod metric d_S. Let U be an interval with u an interior point. Suppose that for some $h > 0$ the process $\{X(t,u) : t \in U\}$ of $F(U)$ is strongly h-localisable at u, with local form $X'_u(\cdot, u)$ a random function of $F(\mathbb{R})$. Suppose that for all $c > 0$*

$$\mathsf{P}\left(\sup_{0<|v-u|<\epsilon} \frac{|X(v,v) - X(v,u)|}{|v-u|^h} > c\right) \to 0 \tag{2.7}$$

as $\epsilon \to 0$. *If the process* $Y = \{X(t,t) : t \in U\}$ *is in* $F(U)$ *then* Y *is strongly* h*-localisable at* u *with* $Y'_u(\cdot) = X'_u(\cdot, u)$.

In particular, this conclusion holds if for some $\eta > h$ *we have*

$$\sup_{v \in U, v \neq u} \frac{|X(v,v) - X(v,u)|}{|v-u|^\eta} < \infty \tag{2.8}$$

almost surely.

Proof First consider $(C(\mathbb{R}), d)$. For each positive τ and r sufficiently small,

$$\begin{aligned} &\mathsf{P}\left(\sup_{0<|t|\le\tau} \frac{|X(u+rt,u+rt) - X(u+rt,u)|}{r^h} > c \right) \\ &\quad \le \mathsf{P}\left(\sup_{0<|v-u|\le r\tau} \frac{|X(v,v) - X(v,u)|}{|v-u|^h} > \frac{c}{\tau^h} \right) \to 0 \end{aligned} \tag{2.9}$$

as $r \to 0$. Thus,

$$\frac{X(u+rt,u+rt) - X(u+rt,u)}{r^h} \xrightarrow{\text{p}} 0 \tag{2.10}$$

in $(C(\mathbb{R}), d)$ since convergence in probability on every bounded interval implies convergence in probability on $(C(\mathbb{R}), d)$. Then

$$\begin{aligned} \frac{Y(u+rt) - Y(u)}{r^h} &= \frac{X(u+rt,u+rt) - X(u+rt,u)}{r^h} + \frac{X(u+rt,u) - X(u,u)}{r^h} \\ &\xrightarrow{\text{d}} X'_u(t,u), \end{aligned} \tag{2.11}$$

as $r \to 0$, since X is localisable at u. Here we use a standard property [4, Theorem 4.1], that for random elements Z_r, Z, W_r of some metric space (M, ρ), if $Z_r \xrightarrow{\text{d}} Z$ and $\rho(Z_r, W_r) \xrightarrow{\text{p}} 0$, then $W_r \xrightarrow{\text{d}} Z$.

Turning to $(D(\mathbb{R}), d_S)$, if $X(t,u) \in D(\mathbb{R})$ and (2.7) holds, the same argument using (2.9) implies convergence in probability in (2.10) and convergence in distribution in (2.11) with respect to the metric d_S on $D(\mathbb{R})$.

Finally, (2.7) is an immediate consequence of (2.8) if $h < \eta$. □

To utilise Theorem 2.4 we need to verify (2.8), that is to show that $Z(v) = (X(v,v) - X(v,u))/|v-u|^\eta$ is bounded as v ranges across an interval. Kolmogorov's continuity theorem [15, Theorem 25.2] is extremely useful for this.

To illustrate our method we give a simple construction of multifractional Brownian motion, that is a strongly localisable process with local form fractional Brownian motion of prescribed local index and amplitude, see [1–3, 13] for other constructions. As in [13] we model our definition on (2.2) but allow h to vary.

Example 2.5 (Multifractional Brownian motion) Let $u \in \mathbb{R}$ and let U be a closed interval with u an interior point. Suppose that $h : U \to (0,1)$ and $a : U \to \mathbb{R}^+$ both satisfy an η-Hölder condition where $h(u) < \eta \le 1$. Define

$$Y(t) = a(t) \int_{-\infty}^{\infty} \big((t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2}\big) W(dx) \quad (t \in U). \tag{2.12}$$

Then Y is strongly $h(u)$-localisable at u with $Y'_u = a(u)c(h(u))B_{h(u)}$ where B_h is index-h fBm and where $c(h)$ is the normalisation constant in (2.2).

Sketch of proof By Proposition 2.2 it is enough to consider the case where $a(v) \equiv 1$. We define

$$X(t,v) = \int_{-\infty}^{\infty} \big((t-x)_+^{h(v)-1/2} - (-x)_+^{h(v)-1/2}\big) W(dx) \quad (t, v \in U). \tag{2.13}$$

By estimating the integral of the square of the integrand of (2.13) and using a mean value estimate, Kolmogorov's theorem gives (2.8) (with η replaced by $\eta - \epsilon$). But $X(\cdot, u) = B_{h(u)}(\cdot)$ which is sssi so is strongly $h(u)$-localisable at u by Proposition 2.1. Theorem 2.4 implies Y is strongly $h(u)$-localisable at u with $Y'_u(\cdot) = X'_u(\cdot, u) = (B_{h(u)})'_u(\cdot) = B_{h(u)}(\cdot)$. $\square$

3 Multifractional Stable Processes

Multifractional Brownian motion generalizes fractional Brownian motion by allowing the parameter h to vary with time. By working with stochastic integrals with respect to an α-stable measure instead of Wiener measure we construct multifractional stable processes with the local scaling exponent depending on t. In this section we only sketch proofs since more general results (though with slightly stronger hypotheses) are obtained in Sect. 4 and related constructions may be found in [17, 18].

Recall that a process $\{X(t) : t \in T\}$, where T is generally a subinterval of $\mathbb{R}$, is called *α-stable* $(0 < \alpha \le 2)$ if all its finite-dimensional distributions are α-stable, see the encyclopaedic work on stable processes [16].

Many stable processes admit a stochastic integral representation. Write $S_\alpha(\sigma, \beta, \mu)$ for the α-stable distribution with scale parameter σ, skewness β and shift-parameter μ; we will assume throughout that $\mu = 0$. Let $(E, \mathcal{E}, m)$ be a sigma-finite measure space (which will be Lebesgue measure in our examples). Taking m as the control measure and $\beta : E \to [-1, 1]$ a measurable function, this defines an α-stable random measure M on E such that for $A \in \mathcal{E}$ we have that $M(A) \sim S_\alpha(m(A)^{1/\alpha}, \int_A \beta(x) m(dx)/m(A), 0)$. If $\beta = 0$ then the process is *symmetric* α-stable or SαS.

Let

$$\mathcal{F}_\alpha \equiv \mathcal{F}_\alpha(E, \mathcal{E}, m) = \{f : f \text{ is measurable and } \|f\|_\alpha < \infty\},$$

where $\|\ \|_\alpha$ is the quasinorm (or norm if $1 < \alpha \le 2$) given by

$$\|f\|_\alpha = \begin{cases} \left(\int_E |f(x)|^\alpha m(dx)\right)^{1/\alpha} & (\alpha \ne 1), \\ \int_E |f(x)| m(dx) + \int_E |f(x)\beta(x) \ln |f(x)|| m(dx) & (\alpha = 1). \end{cases} \tag{3.1}$$

The stochastic integral of $f \in \mathcal{F}_\alpha(E, \mathcal{E}, m)$ with respect to M exists [16, Chap. 3] with

$$I(f) = \int_E f(x) M(dx) \sim S_\alpha(\sigma_f, \beta_f, 0), \tag{3.2}$$

where

$$\sigma_f = \|f\|_\alpha, \qquad \beta_f = \frac{\int f(x)^{\langle\alpha\rangle} \beta(x) m(dx)}{\|f\|_\alpha^\alpha},$$

writing $a^{\langle b\rangle} \equiv \operatorname{sign}(a)|a|^b$, see [16, Sect. 3.4]. In particular,

$$\mathsf{E}|I(f)|^p = \begin{cases} c(\alpha, \beta, p)\|f\|_\alpha^p & (0 < p < \alpha), \\ \infty & (p \geq \alpha), \end{cases} \tag{3.3}$$

where $c(\alpha, \beta, p) < \infty$, see [16, Property 1.2.17].

When $0 < \alpha < 1$ there is a non-negative stable subordinator measure M' associated with M so that $M'(A) \sim S_\alpha(m(A)^{1/\alpha}, 1, 0)$. In particular, for $f \in \mathcal{F}_\alpha$,

$$|I(f)| \leq \int_E |f(x)| M'(dx). \tag{3.4}$$

In this section we will be concerned with processes that may be expressed as stochastic integrals $X(t) = \int_E f(t, x) M(dx)$ $(t \in T)$, where $f(t, \cdot)$ is a jointly measurable family of functions in $\mathcal{F}_\alpha(E, \mathcal{E}, m)$. (Note that if $\operatorname{esssup}_{a \leq t \leq b} f(t, x) = \infty$ for all $x \in A$ for some $A \subset E$ with $m(A) > 0$ then $X(t)$ will be unbounded a.s. on the interval $[a, b]$, see [16, Sect. 10].) We consider the localisability at u of processes defined in terms of random fields $X(t, v) = \int_E f(t, v, x) M(dx) (t, v \in U)$ where $f(t, v, .) \in \mathcal{F}_\alpha$ for all $t, v \in U$ for some interval U. We assume throughout that $f(t, v, x)$ is measurable on $U \times U \times E$.

We first give appropriate conditions for Y to have a continuous or bounded version, which is needed for strong localisability to be meaningful. These conditions and their proofs are variants of those in [16, Chaps. 10, 12].

Proposition 3.1 *Let U be a closed interval. Let X be a random field defined by*

$$X(t, v) = \int_E f(t, v, x) M(dx) \quad (t, v \in U) \tag{3.5}$$

where $f(t, v, \cdot) \in \mathcal{F}_\alpha$ are jointly measurable and M is an α-stable random measure with control measure m and measurable skewness.

(a) *Let $0 < \alpha < 1$. If*

$$\left\| \sup_{t, v \in U} |f(t, v, x)| \right\|_\alpha < \infty, \tag{3.6}$$

then the random field (3.5) has a bounded version.

If in addition $\{f(t, v, x) : x \in E\}$ is an equiuniformly continuous family for $t, v \in U$, then (3.5) has a continuous version.

(b) *Let* $1 < \alpha < 2$ *and* $1/\alpha < \eta \le 1$. *If*

$$\left\| f(t, v, \cdot) - f(t', v', \cdot) \right\|_\alpha \le k\big(|v - v'|^\eta + |t - t'|^\eta\big) \quad (t, t', v, v' \in U), \tag{3.7}$$

then $Y = \{X(t,t) : t \in U\}$ *has a continuous version for* $t \in U$, *satisfying an a.s.* β*-Hölder condition for all* $0 < \beta < (\eta\alpha - 1)/\alpha$.

Sketch of proof (a) Since $0 < \alpha < 1$ there exists a stable subordinator measure M' associated with M. Using (3.4) and (3.6) to estimate the integral (3.5) leads to a uniform bound for $|X(t, v)|$ by an a.s. finite random variable.

Similarly, given the equicontinuity condition on f, a subordinator estimate gives a.s. small uniform bounds for $|X(t, v) - X(t', v')|$ when (t', v') is close to (t, v) and an application of the Borel-Cantelli lemma gives continuity, compare [16, Chaps. 10, 12].

(b) We estimate $\mathsf{E}|X(t, v) - X(t', v')|^p$ for $0 < p < \alpha$ using (3.3) and (3.7). Kolmogorov's theorem then gives that X has a Hölder continuous version. □

The following calculus lemma, whose proof is left to the reader, is useful in deriving many of the estimates that we will require.

Lemma 3.2 *Let* U *be an interval and let* $f : U \to \mathbb{R}$ *be continuously differentiable with* f' *satisfying an* η*-Hölder condition*

$$|f'(v) - f'(w)| \le k|v - w|^\eta \quad (v, w \in U) \tag{3.8}$$

for some $0 < \eta \le 1$. *Let* $v, w, u \in U$ *with* $v \ne u$, $w \ne u$. *Then*

$$\left| \frac{f(v) - f(u)}{v - u} - \frac{f(w) - f(u)}{w - u} \right| \le 2^\eta k|v - w|^\eta. \tag{3.9}$$

The following theorem gives conditions that allow the transfer of localisability properties from $X(\cdot, v)$ to $Y = \{X(t,t) : t \in U\}$ in the α-stable case.

Theorem 3.3 *Let* U *be a closed interval with* u *an interior point. Let* X *be a random field defined by*

$$X(t, v) = \int f(t, v, x) M(dx) \quad (t, v \in U) \tag{3.10}$$

where $f(t, v, \cdot) \in \mathcal{F}_\alpha$ *are jointly measurable and* M *is an* α*-stable random measure with control measure* m *and measurable skewness.*

(a) *Suppose that* $0 < \alpha \le 2$ *and the process* $X(\cdot, u)$ *is* h*-localisable at* u *with* $h > 0$. *Suppose that for some* $\eta > h$

$$\left\| f(t, v, \cdot) - f(t, u, \cdot) \right\|_\alpha \le k_1 |v - u|^\eta \quad (t, v \in U). \tag{3.11}$$

Then $Y = \{X(t,t) : t \in U\}$ *is* h*-localisable at* u *with local form* $Y'_u(\cdot) = X'_u(\cdot, u)$.

(b) *Suppose that* $0 < \alpha < 1$ *and that* $X(\cdot, u)$ *is strongly* h*-localisable in* $C(\mathbb{R})$ (*resp.* $D(\mathbb{R})$) *at* u. *Suppose that for some* $\eta > h$

$$\left| f(t, v, x) - f(t, u, x) \right| \le k_1(x)|v - u|^{\eta} \quad (t, v \in U, x \in E), \tag{3.12}$$

where $k_1(\cdot) \in \mathcal{F}_\alpha$. *If* $Y = \{X(t,t) : t \in U\}$ *has a version in* $C(U)$ (*resp.* $D(U)$) *then* Y *is strongly* h*-localisable at* u *in* $C(\mathbb{R})$ (*resp.* $D(\mathbb{R})$) *with* $Y'_u(\cdot) = X'_u(\cdot, u)$.

(c) *Suppose that* $1 < \alpha \le 2$, *that* $\eta > 1/\alpha$ *and that* $X(\cdot, u)$ *is strongly* h*-localisable in* $C(\mathbb{R})$ *or* $D(\mathbb{R})$ *at* u. *Suppose that for all* $t, v \in U$ *the partial derivative* $f_v(t, v, \cdot) \in \mathcal{F}_\alpha$ *with*

$$\left| f_v(t, v, x) - f_v(t, v', x) \right| \le k_1(t, x)|v - v'|^{\eta} \quad (t, v, v' \in U, x \in E), \tag{3.13}$$

where $\sup_{t \in U} \|k_1(t, \cdot)\|_\alpha < \infty$, *and that*

$$\sup_{v \in U} \left| f_v(t, v, x) - f_v(t', v, x) \right| \le k_2(t, t', x) \quad (t, t' \in U, x \in E), \tag{3.14}$$

where $\|k_2(t, t', \cdot)\|_\alpha \le c|t - t'|^{\eta}$. *Then* $Y = \{X(t,t) : t \in U\}$ *is strongly* h*-localisable at* u *in* $C(\mathbb{R})$ *with* $Y'_u(\cdot) = X'_u(\cdot, u)$.

Sketch of proof (a) A routine estimate shows that, for $0 < p < \alpha$,

$$\mathsf{E}\left| X(t, v) - X(t, u) \right|^p \le c|v - u|^{\eta p},$$

so the conclusion follows from Theorem 2.3.

(b) We use that M has a stable subordinator to show that

$$\left| X(t, v) - X(t, u) \right| \le |v - u|^{\eta} Z,$$

where Z is an a.s. finite random variable, and apply Theorem 2.4.

(c) For $t, v \in U$, $v \ne u$ we define

$$Z(t, v) = \frac{X(t, v) - X(t, u)}{v - u} = \int g(t, v, x) M(dx),$$

where

$$g(t, v, x) = (f(t, v, x) - f(t, u, x))/(v - u).$$

With estimates using Lemma 3.2 and the mean value theorem, applying Proposition 3.1(b) gives that $Z(v, v)$ has a version that is a.s. continuous and bounded for $v \in U$. Thus (2.8) holds and strong localisability follows from Theorem 2.4. □

We illustrate Theorem 3.3 by constructing processes whose local forms are linear stable fractional motions $L_{\alpha, h(t)}$, see (2.3). Overlapping results with a different emphasis are given in [17, 18]. The following process is termed a *linear stable multifractional motion*:

$$\begin{aligned} Y(t) = \int_{-\infty}^{\infty} \Big[& a\big((t - x)_+^{h(t) - 1/\alpha} - (-x)_+^{h(t) - 1/\alpha}\big) \\ & + b\big((t - x)_-^{h(t) - 1/\alpha} - (-x)_-^{h(t) - 1/\alpha}\big)\Big] M(dx) \quad (t \in \mathbb{R}), \end{aligned} \tag{3.15}$$

where M is an α-stable random measure ($0<\alpha<2$) with constant skewness intensity β and control measure Lebesgue measure, with $h(t)\in(0,1)$ for all $t\in\mathbb{R}$, and $a,b\in\mathbb{R}$.

To investigate localisability, we introduce the random field

$$\begin{aligned} X(t,v)=\int_{-\infty}^{\infty}\Big[\,& a\big((t-x)_+^{h(v)-1/\alpha}-(-x)_+^{h(v)-1/\alpha}\big) \\ &+b\big((t-x)_-^{h(v)-1/\alpha}-(-x)_-^{h(v)-1/\alpha}\big)\Big]M(dx)\quad (t,v\in\mathbb{R}). \end{aligned} \tag{3.16}$$

Then $X(t,v)$ is well-defined since for each (t,v) the α-th power of the integrand is Lebesgue integrable. For each fixed v the process $X(\cdot,v)$ is just a linear stable fractional motion (2.3) so is $h(v)$-localisable, with $X'_u(\cdot,v)=L_{\alpha,h(v)}(\cdot)$ for all $u\in\mathbb{R}$. Provided that $h(v)>1/\alpha$ it is in $C(\mathbb{R})$ and is strongly localisable.

Theorem 3.4 (Linear multifractional stable motion) *Let U be a closed interval with u an interior point. Let $0<\alpha<2$ and $h:U\to(0,1)$. Define $\{Y(t):t\in U\}$ by (3.15).*

(a) *Assume that h satisfies a η-Hölder condition at u*

$$|h(v)-h(u)|\le k|v-u|^\eta \quad (v\in U)$$

where $h(u)<\eta\le 1$. Then Y is $h(u)$-localisable at u with local form $Y'_u=L_{\alpha,h(u)}$.

(b) *If $1<\alpha<2$ and h is differentiable with $1/\alpha<h(u)<1$ and*

$$|h'(v)-h'(v')|\le k|v-v'|^\eta \quad (v,v'\in U) \tag{3.17}$$

where $1/\alpha<\eta\le 1$, then Y is strongly $h(u)$-localisable at u with local form $Y'_u=L_{\alpha,h(u)}$.

Sketch of proof We indicate the proof for well-balanced linear multifractional stable motion, that is with $a=b=1$ in (3.15); the general case is similar. Thus we take

$$f(t,v,x)=|t-x|^{h(v)-1/\alpha}-|x|^{h(v)-1/\alpha}$$

in Theorem 3.3 with $X(t,v)=\int f(t,v,x)M(dx)$ and $Y(t)=\int f(t,t,x)M(dx)$.

(a) Mean value estimates show that, if U is a sufficiently small interval, $|f(t,v,x)-f(t,u,x)|\le k_1(t,x)|v-u|^\eta$, where $\int k_1(t,x)^\alpha dx$ is uniformly bounded for $t\in U$, so as $X(\cdot,u)$ is $h(u)$-localisable at u Theorem 3.3(a) gives the conclusion.

(b) This is similar to (a) except that mean value theorem estimates on the derivatives $|f_v(t,v,x)-f_v(t,v',x)|$ are required to verify both (3.13) and (3.14) in Theorem 3.3(c). □

4 Sums over Poisson Processes

To set up 'multistable processes' it is convenient to express the random field $X(t,v)$ as a sum over a suitable Poisson point process and in this section we develop the required properties of Poisson sums.

Let $(E,\mathcal{E},m)$ be a σ-finite measure space. We work throughout with a Poisson point process Π on $E\times\mathbb{R}$, with mean measure $m\times\mathcal{L}$ where $\mathcal{L}$ is Lebesgue measure. Thus Π is a random countable subset of $E\times\mathbb{R}$ such that, writing $N(A)$ for the number of points in a measurable $A\subset E\times\mathbb{R}$, the random variable $N(A)$ has a Poisson distribution of mean $(m\times\mathcal{L})(A)$ with $N(A_1),\dots,N(A_n)$ independent for disjoint $A_1,\dots,A_n\subset\mathbb{R}^2$, see [11].

For $0<a\le b<2$ we define spaces of measurable functions in terms of quasi-norms by

$$\mathcal{F}_{a,b}\equiv\mathcal{F}_{a,b}(E,\mathcal{E},m)=\{f : f \text{ is } m\text{-measurable with } \|f\|_{a,b}<\infty\}$$

where

$$\|f\|_{a,b}=\left(\int_E |f(x)|^a m(dx)\right)^{1/a}+\left(\int_E |f(x)|^b m(dx)\right)^{1/b}. \tag{4.1}$$

(Of course $\|\ \|_{a,b}$ is a norm if $1\le a\le b$.) Note that if $a\le a'\le b'\le b$ then $\mathcal{F}_{a,b}\subset\mathcal{F}_{a',b'}$ and $\|f\|_{a',b'}\le c\|f\|_{a,b}$ where c depends on a,a',b',b. Moreover, $\mathcal{F}_{a,a}=\mathcal{F}_a$.

We work with kernel functions $g:E\times\mathbb{R}\to\mathbb{R}$ which are $m\times\mathcal{L}$-measurable such that

$$|g(x,y)|\le h(x)\big(|y|^{-1/a}+|y|^{-1/b}\big)\quad\text{for some } h\in\mathcal{F}_{a,b}, \tag{4.2}$$

where $0<a\le b<2$. Note that expressions such as (4.2) have two parts since we need different orders of estimate of $g(x,y)$ at small and large values of y.

Lemma 4.1 *Let g satisfy* (4.2). *Then there is a constant c depending only on a and b such that*

$$\begin{aligned}&\iint \sin^2\Big(\frac{1}{2}\theta g(x,y)\Big)m(dx)dy\\&\quad\le c\left(\theta^a\int |h(x)|^a m(dx)+\theta^b\int |h(x)|^b m(dx)\right)\quad(\theta\ge 0).\end{aligned} \tag{4.3}$$

Proof We have

$$\begin{aligned}\iint \sin^2\Big(\frac{1}{2}\theta g(x,y)\Big)m(dx)dy &\le \iint \min\Big\{\frac{1}{4}\theta^2|g(x,y)|^2,1\Big\}m(dx)dy\\&\le c_1\iint \min\big\{\theta^2|h(x)|^2|y|^{-2/a},1\big\}m(dx)dy\\&\quad+c_1\iint \min\big\{\theta^2|h(x)|^2|y|^{-2/b},1\big\}m(dx)dy,\end{aligned} \tag{4.4}$$

where c_1 is a constant, using (4.2) and making a simple estimate. But

$$\begin{aligned}
&\iint \min\{\theta^2|h(x)|^2|y|^{-2/a}, 1\} m(dx)dy \\
&\quad \le \int \left[\int_{|y|\le|\theta h(x)|^a} dy + \theta^2|h(x)|^2 \int_{|y|>|\theta h(x)|^a} |y|^{-2/a} dy \right] m(dx) \\
&\quad \le c_2\theta^a \int |h(x)|^a m(dx)
\end{aligned}$$

where c_2 depends only on a, so along with a similar estimate with b replacing a, (4.4) gives (4.3). □

The next proposition gives criteria for the convergence of Poisson sums. We write (X, Y) for a random point of $E \times \mathbb{R}$ of the Poisson process Π.

Proposition 4.2 *Let g satisfy* (4.2).

(a) *If $0 < a \le b < 1$ then the series*

$$\Sigma \equiv \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X}, \mathsf{Y}) \tag{4.5}$$

converges absolutely almost surely.

(b) *Suppose that $0 < a \le b < 2$ and that g is symmetric in the sense that*

$$g(x, -y) = -g(x, y), \quad (x, y) \in E \times \mathbb{R}. \tag{4.6}$$

Let E_n be an increasing sequence of m-measurable subsets of E with $m(E_n) < \infty$ for all n and $\bigcup_{n=1}^{\infty} E_n = E$ and write R_n for the rectangle $\{(x, y) : x \in E_n, |y| \le n\} \subset E \times \mathbb{R}$. Then we may define

$$\Sigma \equiv \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X}, \mathsf{Y}) = \lim_{n\to\infty} \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} g(\mathsf{X}, \mathsf{Y}), \tag{4.7}$$

where the series converges almost surely.

(c) *Provided the symmetry condition* (4.6) *holds, the characteristic function of Σ, taking either definition* (4.5) *or definition* (4.7), *is given by*

$$\mathsf{E}(e^{i\theta\Sigma}) = \exp\left(-2\iint \sin^2\left(\frac{1}{2}\theta g(x, y)\right) m(dx)dy\right) \quad (\theta \in \mathbb{R}). \tag{4.8}$$

Proof If $0 < a \le b < 1$, (4.2) easily implies that $\int \min\{|g(x, y)|, 1\} m(dx)dy < \infty$. By Campbell's theorem [11, Sect. 3.2] the random sum (4.5) is absolutely convergent almost surely with characteristic function

$$\mathsf{E}(e^{i\theta\Sigma}) = \exp\left(\iint \left(e^{i\theta g(x,y)} - 1\right) m(dx)dy\right) \quad (\theta \in \mathbb{R}).$$

If the symmetry condition (4.6) holds, this reduces to (4.8).

In case (b) write $\Sigma_n = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} g(\mathsf{X},\mathsf{Y}) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X},\mathsf{Y})\mathbf{1}_{R_n}(\mathsf{X},\mathsf{Y})$, where $\mathbf{1}_{R_n}$ is the indicator function of R_n. Then by (4.2) $\int \min\{|g(x,y)\mathbf{1}_{R_n}(x,y)|, 1\}\, m(dx)dy < \infty$, so using Campbell's theorem just as before

$$\begin{aligned}\mathsf{E}(e^{i\theta\Sigma_n}) &= \exp\left(-2\iint \sin^2\left(\frac{1}{2}\theta g(x,y)\right)\mathbf{1}_{R_n}(x,y)m(dx)dy\right)\\ &\to \exp\left(-2\iint \sin^2\left(\frac{1}{2}\theta g(x,y)\right)m(dx)dy\right),\end{aligned}$$

as $n\to\infty$ for all θ, by monotone convergence. By (4.3) there is a number $c_1 > 0$ such that

$$1 \geq \exp\left(-2\iint \sin^2\left(\frac{1}{2}\theta g(x,y)\right)m(dx)dy\right) \geq \exp\left(-2c_1|\theta|^a\right) \geq 1-2c_1|\theta|^a$$

for $|\theta|\leq 1$, using that $1-e^{-x}\leq x$ if $x\geq 0$. Thus $\lim_{n\to\infty}\mathsf{E}(e^{i\theta\Sigma_n})$ exists for all θ and is continuous at $\theta=0$, so by Lévy's continuity theorem [5, Sect. 10.6], Σ_n converges in distribution to a random variable Σ with characteristic function (4.8).

We may write

$$\lim_{n\to\infty}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} g(\mathsf{X},\mathsf{Y}) = \sum_{n=1}^{\infty}\ \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap(R_n\setminus R_{n-1})} g(\mathsf{X},\mathsf{Y})$$

(taking $R_0=\emptyset$), which is an infinite sum of independent random variables that converges in distribution, so by a theorem of Lévy [5, Chap. 12] it also converges almost surely. □

Proposition 4.3 *Let* $\Sigma = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X},\mathsf{Y})$ *be as in* (4.5) *or* (4.7) *where* g *satisfies* (4.2) *and* $g(x,-y) = -g(x,y)$. *Then for* $0<p<a$,

$$\mathsf{E}|\Sigma|^p \leq c\|h\|_{a,b}^p, \tag{4.9}$$

where c *depends only on* a, b *and* p.

Proof A simple calculation using characteristic functions (see [4, p. 47]) gives

$$\begin{aligned}\mathsf{P}\left\{|\Sigma|\geq\lambda\right\} &\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\left(1-\mathsf{E}(\exp(i\theta\Sigma))\right)d\theta\\ &= \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\left(1-\exp\left(-2\iint\sin^2\left(\frac{1}{2}\theta g(x,y)\right)m(dx)dy\right)\right)d\theta\\ &\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\left(1-\exp\left(-2c\left(\theta^a\int|h(x)|^a m(dx)\right.\right.\right.\\ &\qquad \left.\left.\left. +\theta^b\int|h(x)|^b m(dx)\right)\right)\right)d\theta\end{aligned}$$

$$\begin{aligned}
&\le c_0\lambda \int_{-2/\lambda}^{2/\lambda} \left(\theta^a \int |h(x)|^a m(dx) + \theta^b \int |h(x)|^b m(dx)\right) d\theta \\
&\le c_1 \lambda^{-a} \int |h(x)|^a m(dx) + c_1 \lambda^{-b} \int |h(x)|^b m(dx) \\
&\equiv \lambda^{-a} h_a + \lambda^{-b} h_b,
\end{aligned}$$

say, where c_0 and c_1 depend only on a and b, using (4.8) and (4.3). Then

$$\begin{aligned}
\mathsf{E}|\Sigma|^p &= p\int_0^\infty \lambda^{p-1} \mathsf{P}(|\Sigma| \ge \lambda) d\lambda \\
&\le p\int_0^\infty \lambda^{p-1} \min\{1, \lambda^{-a} h_a\} d\lambda + p\int_0^\infty \lambda^{p-1} \min\{1, \lambda^{-b} h_b\} d\lambda \\
&\le p\int_0^{h_a^{1/a}} \lambda^{p-1} d\lambda + p h_a \int_{h_a^{1/a}}^\infty \lambda^{p-a-1} d\lambda \\
&\quad + p\int_0^{h_b^{1/b}} \lambda^{p-1} d\lambda + p h_b \int_{h_b^{1/b}}^\infty \lambda^{p-b-1} d\lambda \\
&\le c_2 (h_a^{p/a} + h_b^{p/b}) \le c\|h\|_{a,b}^p
\end{aligned}$$

where c_2, c depend on a, b and p. □

We will sometimes need the following variant of Proposition 4.3.

Corollary 4.4 *Let $0 < p < a < a_1 < b_1 < b < 2$ and let $f_1, f_2 \in \mathcal{F}_{a,b}$. Let*

$$\Sigma \equiv \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \left(f_1(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha_1\rangle} - f_2(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha_2\rangle}\right),$$

where $a_1 \le \alpha_1, \alpha_2 \le b_1$. Then

$$\mathsf{E}|\Sigma|^p \le c\|f_1 - f_2\|_{a,b}^p + c\|f_2\|_{a,b}^p |\alpha_1 - \alpha_2|^p$$

where c depends only on a, a_1, b, b_1 and p.

Proof Splitting the sum and using the mean value theorem

$$\begin{aligned}
\Sigma &= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (f_1(\mathsf{X}) - f_2(\mathsf{X}))\mathsf{Y}^{\langle -1/\alpha_1\rangle} + f_2(\mathsf{X}) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \left(\mathsf{Y}^{\langle -1/\alpha_1\rangle} - \mathsf{Y}^{\langle -1/\alpha_2\rangle}\right) \\
&= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (f_1(\mathsf{X}) - f_2(\mathsf{X}))\mathsf{Y}^{\langle -1/\alpha_1\rangle} + f_2(\mathsf{X}) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} (\alpha_1 - \alpha_2)\mathsf{Y}^{\langle -1/\alpha\rangle} \alpha^{-2} \log|\mathsf{Y}|
\end{aligned}$$

where $\alpha \in [\alpha_1, \alpha_2]$, and the corollary follows from Proposition 4.3. □

Note that the introduction of a_1 and b_1 in Corollary 4.4 is necessitated by the 'log' term to ensure uniformity of the constant c.

5 Multistable Processes

We use the localisability results to construct multistable processes, seeking an analogue of Theorem 3.3 but with the local form Y'_u an $\alpha(u)$-stable process where $\alpha(u)$ depends on u. The development of this section mirrors that of Sect. 3, but depends heavily on the properties of Poisson sums derived in Sect. 4.

We first define a random field analogous to (3.10), but where the stable random measure M is not allied to a particular value of α. It would be possible, but technically complicated, to set up a random measure that resembles an $\alpha(u)$-stable measure close to u, so we favour an approach using a representation by sums over Poisson processes. In particular this permits $X(\cdot, v)$ to be specified using the same underlying Poisson process for different v.

As before $(E, \mathcal{E}, m)$ is a σ-finite measure space, and Π is a Poisson process on $E \times \mathbb{R}$ with mean measure $m \times \mathcal{L}$. If α is constant and M is a symmetric α-stable random measure on E with control measure m and skewness 0, the stochastic integral (3.2) may be expressed as a Poisson process sum

$$I(f) = \int f(x)M(dx) = c(\alpha) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha \rangle} \quad (0 < \alpha < 2), \tag{5.1}$$

with the sum taken in the sense of (4.5) or (4.7), where $c(\alpha) = (2\alpha^{-1}\Gamma(1-\alpha)\cos(\frac{1}{2}\pi\alpha))^{-1/\alpha}$, see [16, Sect. 3.12]. (As before $a^{\langle b \rangle} = \text{sign}(a)|a|^b$.)

Particularly relevant in (5.1) is that the stability index α occurs only as an exponent of Y, since the underlying Poisson process does not depend on α, so by varying this exponent we can vary the stability index. Thus the random field

$$X(t, v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, v, \mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v) \rangle} \tag{5.2}$$

gives rise to a *multistable* process with varying α, of the form

$$Y(t) \equiv X(t, t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, t, \mathsf{X})\mathsf{Y}^{\langle -1/\alpha(t) \rangle}. \tag{5.3}$$

We first consider continuity and boundedness of the processes.

Proposition 5.1 *Let U be a closed interval. Let X be the random field defined by*

$$X(t, v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, v, \mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v) \rangle} \quad (t, v \in U) \tag{5.4}$$

where $f(t, v, \cdot) \in \mathcal{F}_{a,b}$ are jointly measurable and $\alpha : U \to (a, b)$ is continuous.

(a) *Suppose $0 < a < \alpha(v) < b < 1$ for $v \in U$. If*

$$\sup_{t,v\in U} |f(t, v, x)| \le k(x), \tag{5.5}$$

where $k \in \mathcal{F}_{a,b}$, then $\{X(t, v) : t, v \in U\}$ has a bounded version.

If in addition $\{f(t,v,x) : x \in E\}$ *is an equiuniformly continuous family for* t, $v \in U$, *then* X *has a continuous version.*

(b) *Suppose that* $1 < a < \alpha(v) < b < 2$ *for* $v \in U$ *and* $1/a < \eta \le 1$. *Suppose that*

$$\big|\alpha(v) - \alpha(v')\big| \le k_1 |v - v'|^{\eta} \quad (v, v' \in U), \tag{5.6}$$

that

$$\sup_{t,v \in U} \|f(t,v,\cdot)\|_{a,b} < \infty, \tag{5.7}$$

and

$$\big\|f(t,v,\cdot) - f(t',v',\cdot)\big\|_{a,b} \le k_2\big(|v - v'|^{\eta} + |t - t'|^{\eta}\big) \quad (t,t',v,v' \in U). \tag{5.8}$$

Then $Y = \{X(t,t) : t \in U\}$ *has a continuous version satisfying an a.s.* β*-Hölder condition for all* $0 < \beta < (\eta a - 1)/a$.

Proof (a) From (5.4), for all $t, v \in U$,

$$\begin{aligned}|X(t,v)| &\le \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t,v,\mathsf{X})||\mathsf{Y}|^{-1/\alpha(v)}\\ &\le \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \sup_{t,v\in U} |f(t,v,\mathsf{X})|(|\mathsf{Y}|^{-1/a} + |\mathsf{Y}|^{-1/b}) \equiv Z\end{aligned}$$

where Z is an a.s. finite random variable, by Proposition 4.2(a). Thus $\{X(t,v) : t, v \in U\}$ is a.s. bounded.

Assuming also the equicontinuity condition, given $\epsilon > 0$ we may choose $r \ge 1$ such that $\|k(x)\mathbf{1}_{\{|x|>r\}}(x)\|_{a,b} < \epsilon$, where $\mathbf{1}$ is the indicator function. By equiuniform continuity we may find $\delta > 0$ such that for all $x \in \mathbb{R}$ and $|(t,v) - (t',v')| < \delta$ we have $|f(t,v,x) - f(t',v',x)| < r^{-1/a}\epsilon$, and $|\alpha(v) - \alpha(v')| < \epsilon$. Then if $|(t,v) - (t',v')| < \delta$, making several estimates in the obvious way,

$$\begin{aligned}&|X(t,v) - X(t',v')|\\ &\quad\le \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t,v,\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle} - f(t',v',\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v')\rangle}|\\ &\quad\le \sum_{|\mathsf{X}|\le r} |f(t,v,\mathsf{X}) - f(t',v',\mathsf{X})||\mathsf{Y}|^{-1/\alpha(v)} + 2\sum_{|\mathsf{X}|>r} \sup_{t,v\in U} |f(t,v,\mathsf{X})||\mathsf{Y}|^{-1/\alpha(v)}\\ &\qquad + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t',v',\mathsf{X})||\mathsf{Y}^{\langle -1/\alpha(v)\rangle} - \mathsf{Y}^{\langle -1/\alpha(v')\rangle}|\\ &\quad\le \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} r^{-1/a}\epsilon\mathbf{1}_{\{|x|\le r\}}(\mathsf{X})|\mathsf{Y}|^{-1/\alpha(v)} + 2\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} k(\mathsf{X})\mathbf{1}_{\{|x|>r\}}(\mathsf{X})|\mathsf{Y}|^{-1/\alpha(v)}\\ &\qquad + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t',v',\mathsf{X})|\frac{1}{\alpha^2}|\mathsf{Y}|^{-1/\alpha}|\log|\mathsf{Y}||\,|\alpha(v) - \alpha(v')|\end{aligned}$$

$$\leq \Bigg(\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} r^{-1/a}\epsilon \mathbf{1}_{\{|x|\leq r\}}(\mathsf{X}) + 2 \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} k(\mathsf{X})\mathbf{1}_{\{|x|>r\}}(\mathsf{X})$$

$$+ \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} c_1|k(\mathsf{X})|\frac{1}{a^2}\epsilon\Bigg)(|\mathsf{Y}|^{-1/a} + |\mathsf{Y}|^{-1/b}) \equiv Z_\epsilon$$

where Z_ϵ is a random variable, and we have used the mean value theorem in the third term of the sum with $\alpha \in [\alpha(v), \alpha(v')]$.

Fix $0 < p < \alpha$. By (4.9) there is a constant c independent of ϵ such that

$$\mathsf{E}|Z_\epsilon|^p \leq c\epsilon^p.$$

Thus choosing $\epsilon(n)$ $(n = 1, 2, \dots)$ such that $\mathsf{E}|Z_{\epsilon(n)}|^p \leq 2^{-n}$, there are corresponding δ_n such that

$$\sup_{|(t,v)-(t',v')|<\delta_n} |X(t, v) - X(t', v')| \leq Z_{\epsilon(n)}.$$

Since $\sum_{n=1}^{\infty} \mathsf{E}|Z_{\epsilon(n)}|^p < \infty$, the Borel-Cantelli lemma gives that $Z_{\epsilon(n)} \to 0$ almost surely, so $\sup_{|(t,v)-(t',v')|<\delta_n} |X(t, v) - X(t', v')| \to 0$ a.s. as $n \to \infty$, giving continuity of $X(t, v)$ a.s.

(b) We estimate

$$X(t, v) - X(t', v') = (X(t, v) - X(t, v')) + (X(t, v') - X(t', v'))$$
$$(t, t', v, v' \in U) \tag{5.9}$$

by considering its two parts in turn. Firstly

$$X(t, v) - X(t, v') = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \big(f(t, v, \mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle} - f(t, v', \mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v')\rangle} \big).$$

Thus Corollary 4.4 gives, for $0 < p < a$,

$$\begin{aligned} \mathsf{E}|X(t, v) - X(t, v')|^p &\leq c_1 \| f(t, v, \cdot) - f(t, v', \cdot)\|_{a,b}^p \\ &\quad + c_1 \| f(t, v', \cdot)\|_{a,b}^p |\alpha(v) - \alpha(v')|^p \\ &\leq c_2 |v - v'|^{\eta p} \end{aligned} \tag{5.10}$$

by (5.8), (5.7) and (5.6).

For the second term of (5.9)

$$X(t, v) - X(t', v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \big(f(t, v, \mathsf{X}) - f(t', v, \mathsf{X})\big)\mathsf{Y}^{\langle -1/\alpha(v)\rangle}.$$

Then

$$|(f(t, v, x) - f(t', v, x))y^{\langle -1/\alpha(v)\rangle}| \leq |f(t, v, x) - f(t', v, x)|\big(|y|^{-1/a} + |y|^{-1/b}\big)$$

so, for $0 < p < a$, Proposition 4.3 and (5.8) give

$$\mathsf{E}|X(t,v) - X(t',v)|^p \le c_3 \| f(t,v,\cdot) - f(t',v,\cdot)\|_{a,b}^p \le c_4|t-t'|^{\eta p}.$$

Combining with (5.10) we estimate (5.9) to get

$$\mathsf{E}|X(t,v) - X(t',v')|^p \le c_5(|v-v'|^{\eta p} + |t-t'|^{\eta p}) \quad (t,t',v,v' \in U),$$

so specialising,

$$\mathsf{E}|Y(t) - Y(t')|^p = \mathsf{E}|X(t,t) - X(t',t')|^p \le 2c_5|t-t'|^{\eta p} \quad (t,t \in U)$$

for $t, t' \in U$.

Since $\eta > 1/a$ we may choose $0 < p < a$ such that $\eta p > 1$. Kolmogorov's theorem gives that $\{Y(t) : t \in U\}$ has a continuous version that is a.s. β-Hölder for all $0 < \beta < (\eta p - 1)/p$ for all $p < a$. □

Here is the main result on the localisability of processes with varying stability index.

Theorem 5.2 *Let U be a closed interval with u an interior point and let $0 < a < b < 2$. Let X be the random field defined by*

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,v,\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle} \quad (t,v \in U) \tag{5.11}$$

where $f(t,v,\cdot) \in \mathcal{F}_{a,b}$ are jointly measurable and $\alpha : U \to (a,b)$.

(a) *Suppose $X(\cdot,u)$ is h-localisable at u for $h > 0$. Suppose that $\sup_{t\in U} \|f(t,u,\cdot)\|_{a,b} < \infty$, and that for some $\eta > h$*

$$|\alpha(v) - \alpha(u)| \le k_1|v-u|^\eta \quad (v \in U), \tag{5.12}$$

and

$$\|f(t,v,\cdot) - f(t,u,\cdot)\|_{a,b} \le k_2|v-u|^\eta \quad (t,v \in U). \tag{5.13}$$

Then $Y = \{X(t,t) : t \in U\}$ is h-localisable at u with local form $Y_u'(\cdot) = X_u'(\cdot,u)$.

(b) *Suppose that $0 < \alpha(u) < 1$ and that $X(\cdot,u)$ is strongly h-localisable in $C(\mathbb{R})$ (resp. $D(\mathbb{R})$) at u. Suppose that for some $\eta > h$*

$$|\alpha(v) - \alpha(u)| \le k_1|v-u|^\eta \quad (v \in U), \tag{5.14}$$

and

$$|f(t,u,x)| \le k_2(x) \quad (t \in U, x \in E), \tag{5.15}$$

and

$$|f(t,v,x) - f(t,u,x)| \le k_3(x)|v-u|^\eta \quad (t,v \in U, x \in E), \tag{5.16}$$

where $k_2(\cdot), k_3(\cdot) \in \mathcal{F}_{a,b}$. *If* $Y = \{X(t,t) : t \in U\}$ *has a version in* $C(U)$ (*resp.* $D(U)$) *then* Y *is strongly* h*-localisable in* $C(\mathbb{R})$ (*resp.* $D(\mathbb{R})$) *at* u *with* $Y'_u(\cdot) = X'_u(\cdot, u)$.

(c) *Suppose that* $1 < \alpha(u) < 2$ *and that* $X(\cdot, u)$ *is strongly* h*-localisable in* $C(\mathbb{R})$ (*resp.* $D(\mathbb{R})$) *at* u. *Let* η *satisfy* $1/\alpha(u) < \eta \le 1$. *Suppose that* α *is continuously differentiable on* U *with*

$$|\alpha'(v) - \alpha'(v')| \le k_1|v - v'|^\eta \quad (v, v' \in U). \tag{5.17}$$

Suppose that the partial derivatives $f_v(t, v, \cdot) \in \mathcal{F}_{a,b}$ *for all* $t, v \in U$, *and the following estimates hold*:

$$\sup_{t\in U} \|f(t,u,\cdot)\|_{a,b} < \infty, \qquad \sup_{t,v\in U} \|f_v(t,v,\cdot)\|_{a,b} < \infty, \tag{5.18}$$

$$\|f(t,v,\cdot) - f(t',v,\cdot)\|_{a,b} \le k_2|t - t'|^\eta \quad (t, t', v \in U), \tag{5.19}$$

$$\big|f_v(t,v,x) - f_v(t,v',x)\big| \le k_3(t,x)|v - v'|^\eta \quad (t, v, v' \in U, x \in E), \tag{5.20}$$

and

$$\big|f_v(t,v,x) - f_v(t',v,x)\big| \le k_4(t,t',x) \quad (t, t', v \in U, x \in E), \tag{5.21}$$

where $\sup_{t\in U} \|k_3(t,\cdot)\|_{a,b} < \infty$, *and* $\|k_4(t,t',\cdot)\|_{a,b} \le k|t - t'|^\eta$ *for all* $t, t' \in U$. *Then* $Y = \{X(t,t) : t \in U\}$ *is strongly* h*-localisable in* $C(\mathbb{R})$ *at* u *with* $Y'_u(\cdot) = X'_u(\cdot, u)$.

Proof (a) We have

$$X(t,v) - X(t,u) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \big(f(t,v,\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle} - f(t,u,\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(u)\rangle}\big). \tag{5.22}$$

With $0 < p < a$, Corollary 4.4 gives that there are constants c_1, c_2 such that

$$\begin{aligned} \mathsf{E}|X(t,v) - X(t,u)|^p &\le c_1\|f(t,v,\cdot) - f(t,u,\cdot)\|_{a,b}^p \\ &\quad + c_1|\alpha(v) - \alpha(u)|^p\|f(t,u,\cdot)\|_{a,b}^p \\ &\le c_2|v - u|^{\eta p}, \end{aligned}$$

for all $t, v \in U$, by (5.12) and (5.13). Part (a) now follows from Theorem 2.3.

(b) We may assume that $a < \alpha(v) < b < 1$ for $v \in U$, if necessary using the continuity of α to replace U by a subinterval to decrease the value of b. Splitting and estimating (5.22), using the mean value theorem as in the proof of Corollary 4.4, we get, with $\alpha' \in [\alpha(v), \alpha(u)]$ (where α' depends on v),

$$
\begin{aligned}
|X(t,v)-X(t,u)| \le &\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t,v,\mathsf{X})-f(t,u,\mathsf{X})|\,|\mathsf{Y}|^{-1/\alpha(v)} \\
&+\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |f(t,u,\mathsf{X})||\alpha(v)-\alpha(u)||\mathsf{Y}|^{-1/\alpha'}\alpha'^{-2}|\log|\mathsf{Y}|| \\
&\le |v-u|^{\eta}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |k_3(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b}) \\
&\quad +c_1|v-u|^{\eta}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |k_2(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b})
\end{aligned}
$$

for all $t, v \in U$, using (5.14)–(5.16). By Proposition 4.2(a) $|k_2(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b})$ and $|k_3(\mathsf{X})|(|\mathsf{Y}|^{-1/a}+|\mathsf{Y}|^{-1/b})$ are a.s. finite random variables, so (2.8) holds and Theorem 2.4 implies that Y is strongly localisable at u.

(c) The conditions of Proposition 5.1(b) are easily checked, so Y has a continuous version. We may assume that $1<a<\alpha(v)<b$ for $v\in U$ and that $1/a<\eta$, using continuity of α to replace U by a subinterval and to increase the value of a if necessary.

Define

$$
Z(t,v)=\frac{X(t,v)-X(t,u)}{v-u}\quad (t,v\in U, v\neq u);
$$

again we use Kolmogorov's theorem to show that $\{Z(v,v): v\in U\}$ is almost surely bounded to get (2.8). We write

$$
Z(t,v)=Z_1(t,v)+Z_2(t,v)\quad (t,v\in U) \tag{5.23}
$$

where

$$
Z_1(t,v)=\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(t,v,\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle}\quad \text{with } g(t,v,x)=\frac{f(t,v,x)-f(t,u,x)}{v-u} \tag{5.24}
$$

and

$$
Z_2(t,v)=\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,u,\mathsf{X})\frac{\mathsf{Y}^{\langle -1/\alpha(v)\rangle}-\mathsf{Y}^{\langle -1/\alpha(u)\rangle}}{v-u}. \tag{5.25}
$$

For $p<a$ we estimate $\mathsf{E}|Z(t,v)-Z(t',v')|^p$ by breaking it into four parts.

(i) Applying Lemma 3.2 to (5.20) gives $|g(t,v,x)-g(t,v',x)|\le 2^{\eta}k_3(t,x)\times|v-v'|^{\eta}$. Thus Corollary 4.4 on (5.24) and then (5.18) with the mean value theorem gives

$$
\begin{aligned}
&\mathsf{E}|Z_1(t,v)-Z_1(t,v')|^p \\
&\quad\le c_1\|g(t,v,\cdot)-g(t,v',\cdot)\|_{a,b}^p+c_1\|g(t,v',\cdot)\|_{a,b}^p|\alpha(v)-\alpha(v')|^p \\
&\quad\le c_2\|k_3(t,\cdot)\|_{a,b}^p|v-v'|^{\eta p}+c_2\sup_{t,v\in U}\|f_v(t,v,\cdot)\|_{a,b}^p|v-v'|^{\eta p} \\
&\quad\le c_3|v-v'|^{\eta p}.
\end{aligned} \tag{5.26}
$$

(ii) Using the mean value theorem and (5.21)

$$\begin{aligned}
&|g(t,v,x)-g(t',v,x)| \\
&\quad = \frac{1}{|v-u|}\big|(f(t,v,x)-f(t',v,x))-(f(t,u,x)-f(t',u,x))\big| \\
&\quad = \big|f_v(t,v_1,x)-f_v(t',v_1,x)\big| \le k_4(t,t',x)
\end{aligned}$$

where $v_1 \in (v,u)$ depends on t, t', v and x. Proposition 4.3 with (5.24) now gives

$$\mathsf{E}|Z_1(t,v)-Z_1(t',v)|^p \le c_4\|k_4(t,t',\cdot)\|_{a,b}^p \le c_5|t-t'|^{\eta p}. \tag{5.27}$$

(iii) Turning to (5.25), a simple estimate using (5.17) gives that

$$\left|\left[\frac{d}{dv}y^{\langle -1/\alpha(v)\rangle}\right]_{v'}^{v}\right| \le c_6|v-v'|^\eta\big(|y|^{-1/a}+|y|^{-1/b}\big).$$

By Lemma 3.2

$$\begin{aligned}
&\left|\frac{y^{\langle -1/\alpha(v)\rangle}-y^{\langle -1/\alpha(u)\rangle}}{v-u}-\frac{y^{\langle -1/\alpha(v')\rangle}-y^{\langle -1/\alpha(u)\rangle}}{v'-u}\right| \\
&\quad \le 2^\eta c_6|v-v'|^\eta\big(|y|^{-1/a}+|y|^{-1/b}\big).
\end{aligned}$$

Thus Proposition 4.3 applied to (5.25) gives

$$\mathsf{E}|Z_2(t,v)-Z_2(t,v')|^p \le c_7\|f(t,u,\cdot)\|_{a,b}^p|v-v'|^{\eta p} \le c_8|v-v'|^{\eta p}. \tag{5.28}$$

(iv) Finally, a further mean value estimate gives

$$\begin{aligned}
&\left|(f(t,v,x)-f(t',v,x))\frac{y^{\langle -1/\alpha(v)\rangle}-y^{\langle -1/\alpha(u)\rangle}}{v-u}\right| \\
&\quad \le c_9|(f(t,v,x)-f(t',v,x))|\big(|y|^{1/a}+|y|^{1/b}\big).
\end{aligned}$$

By Proposition 4.3 and (5.19)

$$\mathsf{E}|Z_2(t,v)-Z_2(t',v)|^p \le c_{10}\|f(t,v,\cdot)-f(t',v,\cdot)\|_{a,b}^p \le c_{11}|t-t'|^{\eta p}. \tag{5.29}$$

Taking (5.23) with (5.26), (5.27), (5.28) and (5.29), we conclude that for some c_{12} independent of $t, t', v, v' \in U$,

$$\mathsf{E}|Z(t,v)-Z(t',v')|^p \le c_{12}(|v-v'|^{\eta p}+|t-t'|^{\eta p})$$

if $0<p<a$. Specialising,

$$\mathsf{E}|Z(v,v)-Z(v',v')|^p \le 2c_{12}|v-v'|^{\eta p} \quad (v,v' \in U).$$

Since $\eta > 1/a$ we may choose $0<p<a$ such that $\eta p > 1$. Applying Kolmogorov's theorem to $\{Z(v,v) : v \in U\}$ we conclude that $Z(v,v) = \frac{X(v,v)-X(v,u)}{v-u}$

has a version that is a.s. bounded, so (2.8) holds and strong localisability follows from Theorem 2.4. □

We now show how Theorem 5.2 may be used to construct some specific multistable processes. In these examples we take $(E, \mathcal{E}, m)$ to be Lebesgue measure on $\mathbb{R}$ so that Π is the Poisson process on $\mathbb{R}^2$ with mean measure $\mathcal{L}^2$. We first construct a multistable analogue of the linear multifractional motion of Theorem 3.4.

Theorem 5.3 (Linear multistable multifractional motion) *Let $a : \mathbb{R} \to \mathbb{R}^+$ be C^1 and $\alpha : \mathbb{R} \to (0, 2)$ and $h : \mathbb{R} \to (0, 1)$ be C^2. Define*

$$Y(t) = a(t)c(\alpha(t)) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathsf{Y}^{\langle -1/\alpha(t)\rangle}\big(|t-\mathsf{X}|^{h(t)-1/\alpha(t)} - |\mathsf{X}|^{h(t)-1/\alpha(t)}\big) \quad (t \in \mathbb{R}). \tag{5.30}$$

(a) *The process Y is $h(u)$-localisable at all $u \in \mathbb{R}$, with $Y_u' = a(u)L_{\alpha(u),h(u)}$, where $L_{\alpha,h}$ is linear stable motion.*
(b) *If u is such that $h(u) > 1/\alpha(u)$ then Y is strongly $h(u)$-localisable in $C(\mathbb{R})$ at u, with $Y_u' = a(u)L_{\alpha(u),h(u)}$.*

Proof By Proposition 2.2 the term $a(t)c(\alpha(t))$ in (5.30) does not affect localisability, so it is enough to prove this with $a(t)c(\alpha(t)) = 1$. Define a random field by

$$\begin{aligned} X(t, v) &= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathsf{Y}^{\langle -1/\alpha(v)\rangle}\big(|t-\mathsf{X}|^{h(v)-1/\alpha(v)} - |\mathsf{X}|^{h(v)-1/\alpha(v)}\big) \\ &= \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t, v, \mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle} \quad (t, v \in \mathbb{R}) \end{aligned}$$

where

$$f(t, v, x) = \big(|t-x|^{h(v)-1/\alpha(v)} - |x|^{h(v)-1/\alpha(v)}\big).$$

Then

$$\begin{aligned} f_v(t, v, x) &= \big(|t-x|^{h(v)-1/\alpha(v)} \log|t-x| - |x|^{h(v)-1/\alpha(v)} \log|x|\big) \\ &\quad \times \big(h'(v) + \alpha'(v)/\alpha(v)^2\big). \end{aligned}$$

Given $u \in \mathbb{R}$ we may use continuity of h and α to choose U to be a small enough closed interval with u an interior point, and numbers a, b, h_-, h_+, such that $0 < a < \alpha(v) < b < 2$ and $0 < h_- < h(v) < h_+ < 1$ for all $v \in U$, and such that $\frac{1}{a} - \frac{1}{b} < h_- < h_+ < 1 - (\frac{1}{a} - \frac{1}{b})$. Routine mean value theorem estimates give

$$|f(t, v, x)|, |f_v(t, v, x)| \le k_1(t, x) \quad (t, v \in U, x \in \mathbb{R}) \tag{5.31}$$

and

$$\begin{aligned} &|f(t, v, x) - f(t, v', x)|, |f_v(t, v, x) - f_v(t, v', x)| \\ &\quad \le k_1(t, x)|v - v'| \quad (t, v, v' \in U, x \in \mathbb{R}) \end{aligned} \tag{5.32}$$

where

$$k_1(t,x)=\begin{cases} c_1\max\{1,|t-x|^{h_--1/a}+|x|^{h_--1/a}\} & (|x|\le 1+2\max_{t\in U}|t|),\\ c_2|x|^{h_+-1/b-1} & (|x|>1+2\max_{t\in U}|t|)\end{cases} \tag{5.33}$$

for constants c_1 and c_2. By virtue of the conditions on a, b, h_-, h_+ it follows that $\sup_{t\in U}\|k_1(t,\cdot)\|_{a,b}<\infty$. Since $X(\cdot,u)=c(\alpha(u))^{-1}L_{\alpha(u),h(u)}(\cdot)$, Theorem 5.2(a) gives $h(u)$-localisability of Y with $Y'_u(\cdot)=X'_u(\cdot,u)=c(\alpha(u))^{-1}(L_{\alpha(u),h(u)})'_u(\cdot)=c(\alpha(u))^{-1}L_{\alpha(u),h(u)}(\cdot)$.

For part (b), we choose U and the numbers a, b, h_-, h_+ to satisfy the conditions stipulated in the proof of (a) but also to satisfy $h_->1/a+(1/a-1/b)>0$, so in particular $h(v)-1/\alpha(v')>0$ for all $v, v'\in U$. Further routine estimates give

$$|f(t,v,x)-f(t',v,x)|, |f_v(t,v,x)-f_v(t',v,x)|\le k_2(t,t',x)$$

for $t, v\in U$, $x\in\mathbb{R}$, where

$$k_2(t,t',x)=\begin{cases} c_3|t-t'|^{h_--1/a} & (|x-\frac{1}{2}(t-t')|\le|t-t'|),\\ c_4|x-\frac{1}{2}(t-t')|^{h_+-1/b-1}|t-t'| & (|x-\frac{1}{2}(t-t')|>|t-t'|)\end{cases} \tag{5.34}$$

for constants c_3, c_4. Then $\|k_2(t,t',\cdot)\|_{a,b}\le c_5|t-t'|^{1/a}$. The conditions of Theorem 5.2(c) are satisfied with $\eta=1/a>1/\alpha(u)$, so strong localisability follows. □

Note that the differentiability conditions in Theorem 5.3 could be weakened slightly to Hölder conditions for which Theorem 5.2 would still be applicable.

An α-stable Lévy motion, $0<\alpha<2$, is a process of $D(\mathbb{R})$ with stationary independent increments which have a strictly α-stable distribution. Taking M as a symmetric α-stable random measure on $\mathbb{R}$, the α-stable Lévy motion may be represented as

$$L_\alpha(t)=M[0,t]=\int \mathbf{1}_{[0,t]}(x)M(dx)=c(\alpha)\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha\rangle}\quad (t\in\mathbb{R}), \tag{5.35}$$

where Π is the Poisson process on $\mathbb{R}^2$ with $\mathcal{L}^2$ as mean measure, $\mathbf{1}_{[0,t]}$ is the indicator function and $c(\alpha)=(2\alpha^{-1}\Gamma(1-\alpha)\cos(\frac{1}{2}\pi\alpha))^{-1/\alpha}$. Then L_α is $1/\alpha$-sssi and is strongly $1/\alpha$-localisable in $D(\mathbb{R})$.

Theorem 5.4 (Multistable Lévy motion) *Let $\alpha:\mathbb{R}\to(0,2)$ and $a:\mathbb{R}\to\mathbb{R}^+$ be continuously differentiable, and define*

$$Y(t)=a(t)c(\alpha(t))\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(t)\rangle}\quad (t\in\mathbb{R}). \tag{5.36}$$

(a) *If $1<\alpha(u)<2$ then Y is $1/\alpha(u)$-localisable at u, with $Y'_u=a(u)L_{\alpha(u)}$.*

(b) *If* $0<\alpha(u)<1$ *and* $\alpha'(u)\neq 0$ *then* Y *is* 1*-localisable at* u *with* $\{Y_u'(t):t\in\mathbb{R}\}=\{tW:t\in\mathbb{R}\}$, *where* W *is the random variable*

$$W=a(u)c(\alpha(u))\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\mathbf{1}_{[0,u]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(u)\rangle}\times\left(\frac{\alpha'(u)}{\alpha(u)^2}|\log|\mathsf{Y}||+\frac{d}{du}(a(u)c(\alpha(u)))\right).$$

Proof (a) By Proposition 2.2 the term $a(t)c(\alpha(t))$ does not affect localisability. Define

$$X(t,v)=\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(v)\rangle}\quad(t,v\in\mathbb{R}).$$

Taking $f(t,v,x)=\mathbf{1}_{[0,t]}(x)$ the conditions of Theorem 5.2(a) are satisfied with $h=1/\alpha(u)<1\equiv\eta$, so the result follows from the localisability of L_α.

(b) In the case where $a(t)c(\alpha(t))=1$

$$\frac{Y(u+rt)-Y(u)}{r}=\frac{1}{r}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\mathbf{1}_{[0,u]}(\mathsf{X})(\mathsf{Y}^{\langle -1/\alpha(u+rt)\rangle}-\mathsf{Y}^{\langle -1/\alpha(u)\rangle})+\frac{1}{r}\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\mathbf{1}_{[u,u+rt]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(u+rt)\rangle}.$$

Letting $r\to 0$ the second term vanishes if $1/\alpha(u)>1$ and the first term converges to W in finite dimensional distributions. The general case is similar. □

Theorem 5.4(b) illustrates a general phenomenon that occurs when the process $\{X(t,u):t\in\mathbb{R}\}$ is $h(u)$-localisable at u where $h(u)>1$. The process $Y(t)\equiv X(t,t)$ will typically be 1-localisable at u, with the dominant component of $Y_u'(t)$ derived from $(X(u+rt,u+rt)-X(u+rt,u))/r$ in (2.6) rather than from $X_u'(t,u)$.

As explained in [16, Sect. 7.6], there are two ways to extend the linear fractional stable motion to the case $H=1/\alpha$. Apart from the Lévy motion considered above, one may define the following process, called log-fractional stable motion:

$$\Lambda_\alpha(t)=\int_{-\infty}^{\infty}\big(\log(|t-x|)-\log(|x|)\big)M(dx)\quad(t\in\mathbb{R})\tag{5.37}$$

where M is an α-stable random measure. This process is well-defined only for $\alpha\in(1,2]$ (the integrand is not in $\mathcal{F}_\alpha$ for $\alpha\leq 1$). It is $1/\alpha$-self-similar with stationary increments but, unlike the Lévy motion, its increments are not independent. Also log-fractional stable motion does not have a version in $D(\mathbb{R})$, so we cannot speak of strong localisability.

Theorem 5.5 (Log-fractional multistable motion) *Let* $\alpha:\mathbb{R}\to(1,2)$ *and* a *be continuously differentiable, and define*

$$Y(t)=a(t)\sum_{(\mathsf{X},\mathsf{Y})\in\Pi}\big(\log|t-\mathsf{X}|-\log|\mathsf{X}|\big)\mathsf{Y}^{\langle -1/\alpha(t)\rangle}\quad(t\in\mathbb{R}).\tag{5.38}$$

Then Y is $1/\alpha(u)$-localisable at all $u \in \mathbb{R}$, with $Y'_u = a(u)\Lambda_{\alpha(u)}$.

Proof The proof is similar to that of Theorem 5.4, by considering the field

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \big(\log|t-\mathsf{X}| - \log|\mathsf{X}|\big)\mathsf{Y}^{\langle -1/\alpha(v)\rangle} \quad (t, v \in \mathbb{R}), \tag{5.39}$$

with Theorem 5.2(a) applied to $f(t,v,x) = \log|t-x| - \log|x|$. □

6 Further Work

There are a great many possible variants and extensions of this work. Localisable processes of many other forms may be constructed. For example multistable processes with skewness and the class of stationary localisable processes deserve investigation. There may be advantages in seeking other representations of multistable processes such as by sums involving arrival times of a Poisson process or as stochastic integrals with respect to suitably constructed random measures. Our conditions for localisability could certainly be weakened and further techniques for establishing localisability and in particular strong localisability developed. It would be interesting to study properties such as long range dependence of multistable processes. Effective techniques for simulation and inference on parameters for these processes are also needed. We will be addressing some of these matters in a sequel to this paper.

References

1. Ayache, A.: The generalized multifractional field: a nice tool for the study of the generalized multifractional Brownian motion. J. Fourier Anal. Appl. **8**, 581–601 (2002)
2. Ayache, A., Lévy Véhel, J.: The generalized multifractional Brownian motion. Stat. Inference Stoch. Process. **3**, 7–18 (2000)
3. Benassi, A., Jaffard, S., Roux, D.: Gaussian processes and pseudodifferential elliptic operators. Rev. Mat. Iberoam. **13**, 19–89 (1997)
4. Billingsley, P.: Convergence of Probability Measures. Wiley, New York (1968)
5. Eisen, M.: Introduction to Mathematical Probability Theory. Prentice Hall, Englewood Cliffs (1969)
6. Embrechts, P., Maejima, M.: Selfsimilar Processes. Princeton University Press, Princeton (2002)
7. Falconer, K.J.: Fractal Geometry—Mathematical Foundations and Applications, 2nd edn. Wiley, New York (2003)
8. Falconer, K.J.: Tangent fields and the local structure of random fields. J. Theor. Probab. **15**, 731–750 (2002)
9. Falconer, K.J.: The local structure of random processes. J. Lond. Math. Soc. (2) **67**, 657–672 (2003)
10. Herbin, E.: From N-parameter fractional Brownian motions to N-parameter multifractional Brownian motion. Rocky Mt. J. Math. **36**, 1249–1284 (2006)
11. Kingman, J.F.C.: Poisson Processes. Oxford University Press, London (1993)
12. Mandelbrot, B.B., Van Ness, J.: Fractional Brownian motion, fractional noises and applications. SIAM Rev. **10**, 422–437 (1968)
13. Peltier, R.F., Lévy Véhel, J.: Multifractional Brownian motion: definition and preliminary results. Rapport de recherche de l'INRIA, No. 2645 (1995). Available at: http://www-rocq1.inria.fr/fractales/index.php?page=publications
14. Pollard, D.: Convergence of Stochastic Processes. Springer, Berlin (1984)
15. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes and Martingales, 2nd edn., vol. 1. Cambridge University Press, Cambridge (2000)

16. Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes. Chapman and Hall, London (1994)
17. Stoev, S., Taqqu, M.S.: Stochastic properties of the linear multifractional stable motion. Adv. Appl. Probab. **36**, 1085–1115 (2004)
18. Stoev, S., Taqqu, M.S.: Path properties of the linear multifractional stable motion. Fractals, **13**, 157–178 (2005)