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OPTION PRICING THEORY AND ITS APPLICATION

Presiding JOHN C. COX†

The Pricing of Commodity-Linked Bonds

EDUARDO S. SCHWARTZ*

I. Introduction

DURING 1980 SUNSHINE MINING CO., operator of the largest silver mine in the United States, made two $25 million bond issues backed by silver. Each $1000 bond is linked to 50 ounces of silver, pays a coupon rate of 8½% and has a maturity of 15 years. At maturity the company promises to pay the bondholders either the $1000 face value or the market value of the 50 ounces of silver, whichever is greater.[1] At the time of the first issue in April 1980, silver was trading at $16 an ounce so that the value of 50 ounces was $800.

In August of 1979, an agency of the Mexican Government issued bonds in local currency backed by oil. Each 1,000 pesos bond was linked to 1.95354 barrels of oil, had a coupon rate of 12.65823% and a maturity of three years. At maturity they would be redeemed at face value plus the amount by which the market value of the reference oil bundle exceeded the face value plus all coupons received during the life of the bond, if this amount were positive. This was the third successful issue of 'petrobonds' by the Mexican agency.

These are two recent examples of a corporation or government seeking funds in financial markets and being willing to share the potential price appreciation of the underlying commodity with the purchaser of the bond, in exchange for a lower coupon rate, more favorable bond indentures or the acceptance of a weaker currency by foreign investors. In both cases, the underlying commodity was produced by the issuing firm or country. In early references to the potential use of commodity-linked bonds by less developed countries, Lessard [1977a, 1977b, 1979] had suggested that producers could transfer a substantial proportion of

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[1] Despite the fact that the price of silver had plunged from a record high of about $52 an ounce in January 1980 to about $13 in January 1981, both issues were trading at the later date at premiums of as much as 68% over the value of the underlying silver.

commodity price risks to the financial markets through commodity backed securities.

This paper develops a model for pricing commodity-linked bonds, using the option pricing framework as pioneered by Black and Scholes [1973] and extended by Merton [1973] and Cox and Ross [1976]. The key assumptions of the model are that the underlying commodities, the commodity-linked bonds and the equities of the firms issuing the bonds are continuously traded in frictionless markets. A general model is developed in Section II which considers commodity price risk, default risk and interest rate risk, and takes the form of a second-order partial differential equation in four variables which governs the value of the commodity-linked bond at any point in time. The following sections discuss three special cases in which closed-form solutions can be obtained. Section III considers default free bonds and assumes a constant interest rate: the solution to this problem is a direct application of the Black-Scholes [1973] equation for pricing call options. In Section IV the assumption of no-default risk is relaxed: commodity price risk and default risk are taken jointly into account and an integral solution is derived. Section V returns to the no-default risk situation, but interest rate risk is now introduced: Merton's [1973] solution for the pricing of options when interest rates are stochastic is used for this case. Finally, Section VI summarizes the results and offers some concluding remarks.

II. The General Model

It is assumed that all assets are traded in perfect markets (no taxes, no transaction costs, perfect divisibility, etc.), that continuous trading is allowed and that commodity price, firm value and the interest rate follow continuous time diffusion processes.[2] In particular the analysis assumes that there are no costs of carrying to the commodity (evaporation, obsolescence, insurance, etc.) and that the commodity is held for speculative purposes like a stock. Let P be the value the reference commodity bundle, V the value of the firm issuing the bonds and r the instantaneously riskless rate of interest, and assume that they follow continuous paths described by the following stochastic differential equations:

$$\frac{dP}{P} = \alpha_p\, dt + \sigma_p\, dz_p \tag{1}$$

$$\frac{dV}{V} = \left(\alpha_v - \frac{D(V, t)}{V}\right) dt + \sigma_v\, dz_v \tag{2}$$

$$dr = \alpha_r(r)\, dt + \sigma_r(r)\, dz_r \tag{3}$$

where D, the rate of total payouts to all the securityholders of the firm (dividends, interest, etc.), is a deterministic function of the value of the firm and time. $(\alpha_v - D/V)$ is therefore the expected rate of appreciation in the value of the firm. It is assumes that σ_p and σ_v are constants. dz_p, dz_v and dz_r are Gauss-Wiener processes

[2] For a detailed discussion of the assumptions underlying the option pricing model, see Black-Scholes [1973], Merton [1973] and Smith [1976]; and for details on Itô processes and Itô's Lemma, see Merton [1971, 1973a] and McKean [1969].

with:

$$dz_p \cdot dz_v = \rho \cdot dt, \qquad dz_p \cdot dz_r = \rho_{pr} \cdot dt$$
$$dz_v \cdot dz_r = \rho_{vr} \cdot dt$$

It is assumed that the value of any default free discount bond, G, depends only on r and time to maturity τ: $G(r, \tau)$. Then application of Itô's Lemma gives

$$\frac{dG}{G} = \alpha_G \, dt + \sigma_G \, dz_r \tag{4}$$

where

$$\alpha_G = (\alpha_r G_r + \tfrac{1}{2}\sigma_r^2 G_{rr} - G_\tau)/G$$
$$\sigma_G = \sigma_r G_r / G$$

and G_r, G_{rr}, G_τ are partial derivatives.

It can be shown[3] that for discount bonds of all maturities:

$$\frac{\alpha_G - r}{\sigma_G} = \lambda(r) \tag{5}$$

where λ is the risk per unit of risk ('market price of interest rate risk'). In general λ will be a function of r and t but independent of the maturity of the bond.

Under the above assumptions, the total value of the commodity linked bond can be expressed as: $B \equiv B(P, V, r, \tau)$. Applying Itô's Lemma again gives:

$$\frac{dB}{B} = \left(\alpha_B - \frac{C}{B}\right) dt + \eta_p \, dz_p + \eta_v \, dz_v + \eta_r \, dz_r, \tag{6}$$

where

$$\begin{aligned}\alpha_B = [&\alpha_p P B_p + (\alpha_v V - D) B_v + \alpha_r B_r + \tfrac{1}{2}\sigma_p^2 P^2 B_{pp} + \tfrac{1}{2}\sigma_v^2 V^2 B_{vv} \\ &+ \tfrac{1}{2}\sigma_r^2 B_{rr} + \sigma_{pv} P V B_{pv} + \sigma_{pr} P B_{pr} + \sigma_{vr} V B_{vr} - B_\tau + C]/B,\end{aligned} \tag{7}$$

and

$$\eta_p = P\sigma_p B_p / B, \qquad \eta_v = V\sigma_v B_v / B, \qquad \eta_r = \sigma_r B_r / B, \tag{7'}$$

and C is the rate of total payouts from the firm to the bonds, and

$$\sigma_{pv} = \rho \, \sigma_v \sigma_p, \qquad \sigma_{pr} = \rho_{pr} \sigma_p \sigma_r, \qquad \sigma_{vr} = \rho_{vr} \sigma_v \sigma_r.$$

Consider forming a portfolio by investing amounts:

X_1 in the underlying commodity,
X_2 in the firm,
X_3 in a riskless discount bond, and
X_4 in the commodity-linked bond.

The instantaneous total return on this portfolio, dY, will then be

$$dY = X_1 \frac{dP}{P} + X_2 \frac{dV + D\,dt}{V} + X_3 \frac{dG}{G} + X_4 \frac{dB + C\,dt}{B} \tag{8}$$

[3] See, for example, Brennan and Schwartz [1980].

Substitution of (1), (2), (4) and (6) into (8) gives:

$$dY = (X_1\alpha_p + X_2\alpha_v + X_3\alpha_G + X_4\alpha_B)dt + (X_1\sigma_p + X_4\eta_p)dz_p + (X_2\sigma_v + X_4\eta_v)dz_v + (X_3\sigma_G + X_4\eta_r)dz_r \tag{9}$$

By choosing X_1, X_2, X_3 and X_4 such that

$$X_1\sigma_p + X_4\eta_p = 0, \qquad X_2\sigma_v + X_4\eta_v = 0, \qquad X_3\sigma_G + X_4\eta_r = 0, \tag{10}$$

the portfolio return in (9) becomes riskless. To avoid the possibility of arbitrage profits, the return on this portfolio must equal the risk-free rate of interest, so that:

$$X_1(\alpha_p - r) + X_2(\alpha_v - r) + X_3(\alpha_G - r) + X_4(\alpha_B - r) = 0. \tag{11}$$

Substitution of (10) and (7′) into (11) gives

$$-\frac{PB_p}{B}(\alpha_p - r) - \frac{VB_v}{B}(\alpha_v - r) - \frac{\sigma_r B_r}{B}\frac{\alpha_G - r}{\sigma_G} + (\alpha_B - r) = 0.$$

Substitution from (5) and (7) and simplification finally yields the partial differential equation governing the value of the commodity-linked bond at every point in time.

$$\tfrac{1}{2}\sigma_p^2P^2B_{pp} + \tfrac{1}{2}\sigma_v^2V^2B_{vv} + \tfrac{1}{2}\sigma_r^2B_{rr} + \sigma_{pv}PVB_{pv} + \sigma_{pr}PB_{pr} + \sigma_{vr}VB_{vr} + rPB_p + (rV - D)B_v + (\alpha_r - \lambda\sigma_r)B_r - B_\tau - rB + C = 0. \tag{12}$$

Note that the value of the bonds will be independent of the expected return on the commodity and on the firm; it will only depend on the current values of the reference commodity bundle and the firm.

It should be pointed out that the same equation (12) could be derived in the general equilibrium framework of Cox, Ingersoll and Ross [1978] with appropriate assumptions about preferences and technologies.

The promised payment on the bonds at maturity is equivalent to the face value of the bonds (F) for sure plus an option to buy the reference commodity bundle at a specified exercise price (E). The promised payment can only be made if the value of the firm at maturity is greater than that amount. If it is assumed that in case of default the bondholders take over the firm, the boundary condition at maturity can be expressed as:

$$B(P, V, r, 0) = \min[V, F + \max(0, P - E)] \tag{13}$$

This condition is shown schematically in Figure 1.[4]

Solution to equation (12) subject to boundary condition (13) is very difficult, even by numerical methods. In the following sections therefore three simplified versions of the general model are considered for which closed formed solutions can be obtained.

[4] The exercise price of the option was set equal to the face value of the bonds in the Sunshine bonds; for the Mexican petro-bonds, however, it was set equal to the face value of the bonds plus total coupon payments.

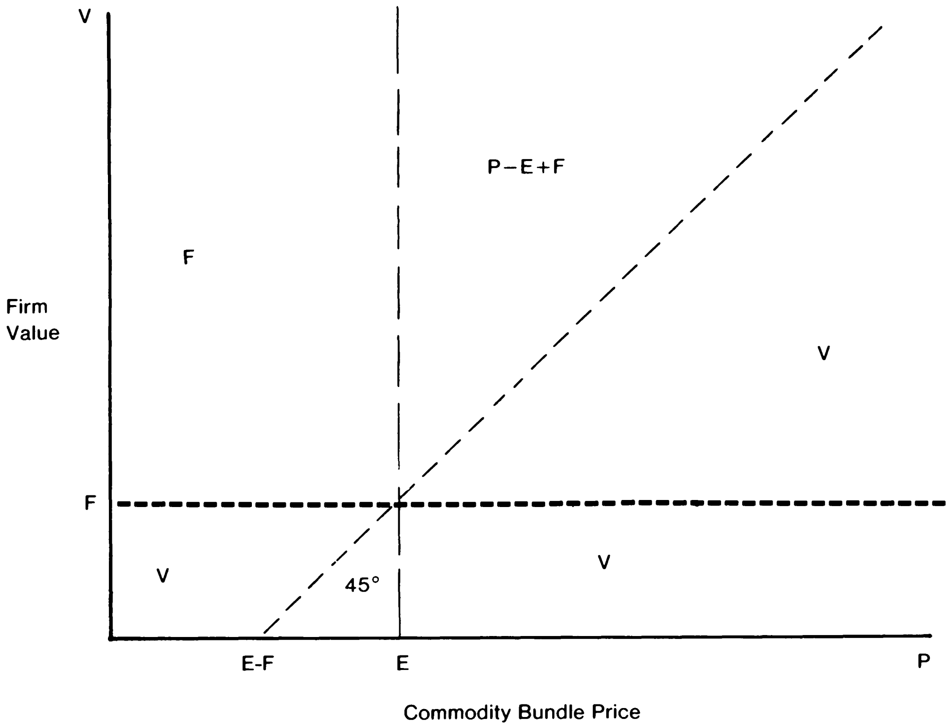

Figure 1.

III. Uncertain Commodity Price

This section discusses the case in which there is no default risk, and the interest rate is constant. Under these conditions equation (12) and boundary condition (13) can be simplified to:

$$\tfrac{1}{2}\sigma_p^2 P^2 B_{pp} + rPB_p - B_\tau - rB + C = 0, \tag{14}$$

and

$$B(P, 0) = F + \max(0, P - E). \tag{15}$$

The solution to (14) subject to (15) is,

$$B(P, \tau) = \frac{C}{r}(1 - e^{-r\tau}) + Fe^{-r\tau} + W(P, \tau) \tag{16}$$

where $W(P, \tau)$ is the Black-Scholes [1973] solution to the value of a call option with exercise price E:

$$W(P, \tau) = PN(h) - Ee^{-r\tau}N(h - \sigma_p\sqrt{\tau}), \tag{17}$$

where

$$h = \frac{\ln P/E + r\tau}{\sigma_p\sqrt{\tau}} + \frac{1}{2}\sigma_p\sqrt{\tau},$$

and $N(\cdot)$ is the cumulative normal density function.

Expression (16) simply says that the value of the commodity bond is equal to the discounted value of future coupon payments and face value of the bond plus a call option to buy the reference commodity bundle at the agreed exercise price. From (16) it is also easy to solve for the coupon rate (C/F) that the issuer must offer, in equilibrium, to sell the bonds at face value at time of issue $(\tau = T)$, given the reference commodity bundle price at that time and the exercise price set for the option:

$$\frac{C}{F} = r - \frac{r}{[1 - e^{-rT}]} \cdot \frac{W(P, T)}{F} \tag{18}$$

Note that the issuer has three parameters under his control to influence the price of the bond at the time of issue: the coupon rate, the exercise price of the option, and the amount of the commodity to be included in the reference bundle. Even if he decides to set the exercise price of the option equal to the face value of the bonds, he can still play with increases in the coupon rate against decreases in the amount included in the reference bundle.

IV. Default Risk

This section extends the analysis of the previous section to include default risk, in addition to commodity price risk. The assumption of a constant interest rate is retained, and to further simplify the problem, it is assumed that there are no payouts from the firm to shareholders or bondholders before the maturity of the bonds, and that the capital structure of the firm consists solely of equity and the single issue of commodity-linked discount bonds that is being valued.

Under the above assumptions, equation (12) simplifies to

$$\tfrac{1}{2}\sigma_p^2 P^2 B_{pp} + \sigma_p \sigma_v \rho P V B_{pv} + \tfrac{1}{2}\sigma_v^2 V^2 B_{vv} + rPB_p + rVB_v - B_\tau - rB = 0 \tag{19}$$

subject to the same boundary condition (13).

Let S be the total value of the equity of the firm, then

$$V = B + S \tag{20}$$

To solve this problem, it is conceptually and computationally easier to value the equity of the firm and then use (20) to compute the value of the bonds. It can easily be verified that the same partial differential equation (19) holds for the equity of the firm, and that the corresponding boundary condition at maturity can be expressed as

$$S(P, V, 0) = \begin{cases} \max(0, V - F) & \text{for } P \leq E \\ \max(0, V - P + E - F) & \text{for } P > E \end{cases} \tag{21}$$

Given that the partial differential equation governing the value of the equity (or the bonds) is independent of preferences, following Cox and Ross [1976] it is possible to proceed as if the market were composed of risk-neutral investors.[5] Then, the equilibrium rate of return on all assets would be equal to the risk free

[5] See Smith (1976) for an extension of this idea and Cox, Ingersoll and Ross (1978) for a proof that this argument is also valid when the contingent claim depends on more than one stochastic variable.

rate (i.e., $\alpha_p = \alpha_v = r$), and the equity would be priced so that its current price is the discounted terminal price at the maturity of the bonds:

$$S(P, V, \tau) = e^{-r\tau}E[S(P, V, 0)] \tag{22}$$

The expected value of the equity at the maturity of the bonds can be computed from the boundary condition (21) and the adjusted joint probability distribution of commodity price and firm value. From (1) and (2), the assumptions of this section and risk neutrality, this distribution is known to be joint-lognormal with means $(r - \frac{1}{2}\sigma_p^2)\tau$ and $(r - \frac{1}{2}\sigma_v^2)\tau$, variances $\sigma_p^2\tau$ and $\sigma_v^2\tau$, respectively, and covariance $\sigma_{pv}\tau = \rho\sigma_p\sigma_v\tau$.

From (21) and (22) the solution to the partial differential equation for the equity of the firm can be expressed as

$$S(P, V, \tau) = e^{-r\tau}\int_{P^*=0}^{E}\int_{V^*=F}^{\infty}(V^* - F)L'(P^*, V^*)\,dV^*\,dP^*$$

$$+ e^{-r\tau}\int_{P^*=E}^{\infty}\int_{V^*=P^*-E+F}^{\infty}(V^* - P^* + E - F)L'(P^*, V^*)\,dV^*\,dP^* \tag{23}$$

where P^* and V^* represent the values of the commodity and the firm at maturity, P and V represent these same values at current time, and $L'(P^*, V^*)$ is the joint log-normal density function.

Let

$$x = \frac{\ln(V^*/V) - (r - \frac{1}{2}\sigma_v^2)\tau}{\sigma_v\sqrt{\tau}}$$

$$y = \frac{\ln(P^*/P) - (r - \frac{1}{2}\sigma_p^2)\tau}{\sigma_p\sqrt{\tau}} \tag{24}$$

Then

$$g(x, y, \rho)dx\,dy = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}\left(\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}\right)\right\}dx\,dy$$

$$= L'(P^*, V^*)dV^*\,dP^* \tag{25}$$

Substitution of (24) and (25) into (23) gives:

$$S(P, V, \tau) = V\int_{y=-\infty}^{y'}\int_{x=x'}^{\infty}\exp\{\sigma_v\sqrt{\tau}x - \tfrac{1}{2}\sigma_v^2\tau\}g(x, y, \rho)\,dx\,dy$$

$$- e^{-r\tau}F\int_{y=-\infty}^{y'}\int_{x=x'}^{\infty}g(x, y, \rho)\,dx\,dy$$

$$+ V \int_{y=y'}^{\infty} \int_{x=x^*(y)}^{\infty} \exp\{\sigma_v \sqrt{\tau} x - \tfrac{1}{2}\sigma_v^2 \tau\} g(x, y, \rho)\, dx\, dy$$

$$- P \int_{y=y'}^{\infty} \int_{x=x^*(y)}^{\infty} \exp\{\sigma_p \sqrt{\tau} y - \tfrac{1}{2}\sigma_p^2 \tau\} g(x, y, \rho)\, dx\, dy$$

$$+ e^{-r\tau}(E - F) \int_{y=y'}^{\infty} \int_{x=x^*(y)}^{\infty} g(x, y, \rho)\, dx\, dy \qquad (26)$$

where

$$x' = \frac{\ln(F/V) - (r - \tfrac{1}{2}\sigma_v^2)\tau}{\sigma_v \sqrt{\tau}}$$

$$y' = \frac{\ln(E/P) - (r - \tfrac{1}{2}\sigma_p^2)\tau}{\sigma_p \sqrt{\tau}}$$

$$x^*(y) = \frac{\ln\left\{\dfrac{P}{V} \exp\left[\sigma_p \sqrt{\tau} y + \left(r - \dfrac{1}{2}\sigma_p^2\right)\tau\right] + (F - E)\right\} - \left(r - \dfrac{1}{2}\sigma_v^2\right)\tau}{\sigma_v \sqrt{\tau}}$$

A second transformation of the type

$$u(x, y) = \beta_1 x + \beta_2 y + \beta_3$$

$$w(y) = \gamma_1 y + \gamma_2$$

with different β's and γ's for each one of the integrals in (26), can be used to write (26) as

$$S(P, V, \tau) = V \cdot M_a(u_1, w_1) - e^{-r\tau} F \cdot M_a(u_2, w_2)$$
$$+ V \cdot M_b(u_3, w_3) - P \cdot M_b(u_4, w_4) + e^{-r\tau}(E - F) \cdot M_b(u_5, w_5) \qquad (27)$$

where

$$M_a(u, k) = \frac{1}{2\sqrt{\pi}} \int_{w=-\infty}^{k} \exp(-w^2) erfc[u(w)]\, dw$$

$$M_b(u, k) = \frac{1}{2\sqrt{\pi}} \int_{w=k}^{\infty} \exp(-w^2) erfc[u(w)]\, dw$$

$$w_1 = w_3 = \frac{\ln(E/P) - (r - \tfrac{1}{2}\sigma_p^2 + \rho\sigma_v\sigma_p)\tau}{\sigma_p \sqrt{2\tau}}$$

$$w_2 = w_5 = \frac{\ln(E/P) - (r - \frac{1}{2}\sigma_p^2)\tau}{\sigma_p\sqrt{2\tau}}$$

$$w_4 = \frac{\ln(E/P) - (r + \frac{1}{2}\sigma_p^2)\tau}{\sigma_p\sqrt{2\tau}}$$

$$u_1(w) = \frac{\ln(F/V) - (r + \frac{1}{2}\sigma_v^2)\tau - \rho\sigma_v\sqrt{2\tau}w}{\sigma_v\sqrt{2\tau(1-\rho^2)}}$$

$$u_2(w) = \frac{\ln(F/V) - (r - \frac{1}{2}\sigma_v^2)\tau - \rho\sigma_v\sqrt{2\tau}w}{\sigma_v\sqrt{2\tau(1-\rho^2)}}$$

$$u_3(w) = \frac{1}{\sigma_v\sqrt{2\tau(1-\rho^2)}}\left[\ln\left\{\frac{P}{V}\exp[\sigma_p\sqrt{2\tau}\,w + (r + \rho\sigma_p\sigma_v - \frac{1}{2}\sigma_p^2)\tau\right]\right.$$
$$\left.+ (F-E)\right\} - \left(r + \frac{1}{2}\sigma_v^2\right)\tau - \rho\sigma_v\sqrt{2\tau}w\Bigg]$$

$$u_4(w) = \frac{1}{\sigma_v\sqrt{2\tau(1-\rho^2)}}\left[\ln\left\{\frac{P}{V}\exp\left[\sigma_p\sqrt{2\tau}w + \left(r + \frac{1}{2}\rho_p^2\right)\tau\right] + (F-E)\right\}\right.$$
$$\left. - \left(r + \rho\sigma_v\sigma_p - \frac{1}{2}\sigma_v^2\right) - \rho\sigma_v\sqrt{2\tau}w\right]$$

$$u_5(w) = \frac{1}{\sigma\sqrt{2\tau(1-\rho^2)}}\left[\ln\left\{\frac{P}{V}\exp\left[\sigma_p\sqrt{2\tau}w + \left(r - \frac{1}{2}\sigma_p^2\right)\tau\right] + (F-E)\right\}\right.$$
$$\left. - \left(r + \frac{1}{2}\sigma_v^2\right)\tau - \rho\sigma_v\sqrt{2\tau}w\right]$$

and $erfc(\cdot)$ is the complement of the error function.

Finally, from (20) and (27) the value of the commodity-linked bonds can be expressed as

$$\begin{aligned} B(P, V, \tau) &= V[1 - M_a(u_1, w_1) - M_b(u_3, w_3)] \\ &\quad + e^{-r\tau}F[M_a(u_2, w_2) + M_b(u_5, w_5)] \\ &\quad + P\cdot M_b(u_4, w_4) - e^{-r\tau}E\cdot M_b(u_5, w_5). \end{aligned} \tag{28}$$

It can be easily verified that when there is no default risk, i.e., $V \to \infty$: $u_i \to -\infty$ and $erfc(u_i) \to 2$. This implies that

$$\begin{aligned} M_a(u_1, w_1) + M_b(u_3, w_3) &= 1 \\ M_a(u_2, w_2) + M_b(u_5, w_5) &= 1 \\ M_b(u_4, w_4) &= N(h) \\ M_b(u_5, w_5) &= N(h - \sigma_p\sqrt{\tau}) \end{aligned}$$

and the solution in (28) collapses to that in (16) and (17) for the no-default risk case.

V. Interest Rate Risk

This section returns to the no-default risk situation of Section III, but considers stochastic interest rates. Under these assumptions, equation (12) can be simplified to

$$\tfrac{1}{2}\sigma_p^2 P^2 B_{pp} + \sigma_{pr} P B_{pr} + \tfrac{1}{2}\sigma_r^2 B_{rr} + rPB_p + (\alpha_r - \lambda\sigma_r)B_r - B_\tau - rB + C = 0 \quad (29)$$

subject to boundary condition at maturity:

$$B(P, r, 0) = F + \max(0, P - E). \quad (30)$$

This equation is very similar to the equation derived by Brennan and Schwartz [1980, eq. (13), pp. 912] for the valuation of convertible bonds when interest rates are stochastic. To be able to solve the question it is necessary to make specific assumptions about the functional forms of the drift (α_r) and standard deviation (σ_r) of the interest rate process, and of the market price of interest rate risk (λ).[6] Their numerical algorithm with slight modifications could then be used to obtain a numerical solution to equation (29) subject to boundary condition (30).

In the spirit of trying to find closed form solutions, however, this paper will follow a different approach suggested by Merton [1973]. Thus, instead of assuming process (3) for the instantaneous risk-free rate, it is assumed that the price of a default free discount bond with time to maturity τ, $Q(\tau)$, satisfies:

$$\frac{dQ}{Q} = \mu(\tau)\,dt + \delta(\tau)\,dq(t;\tau) \quad (31)$$

where μ is the instantaneous expected return on the bond, δ^2 is the instantaneous variance and dq is a Gauss-Wiener process for maturity τ, with $dz_p \cdot dq = \rho_{PQ}\,dt$. Then if the commodity-linked bond is of the discount type, its value can be expressed as

$$B(P, Q, \tau) = F \cdot Q + W(P, Q, \tau) \quad (32)$$

where the value of the option can be obtained from Merton [1973, equation (40), pp. 167]

$$W(P, Q, \tau) = [P\,erfc(h_1) - EQ\,erfc(h_2)]/2 \quad (33)$$

where

$$h_1 = [\ln(P/EQ) + \tfrac{1}{2}I]/\sqrt{2I}$$

$$h_2 = [\ln(P/EQ) - \tfrac{1}{2}I]/\sqrt{2I}$$

$$I = \int_0^\tau [\sigma_p^2 + \delta(s)^2 - 2\rho_{pQ}\sigma_p\delta(s)]\,ds$$

[6] For example, Brennan and Schwartz assume a mean reverting process with standard deviation proportional to the level of the interest rate, and a constant market price of risk.

VI. A Numerical Example

Figure 2 shows the coupon rates that must be offered when there is no risk of default in order to sell the bonds at face value at the time of issue for different commodity bundle prices. Equation (18) and the relevant parameter values given in Table 1 were used in this computation. Naturally, the higher the standard deviation, the higher is the value of the option and the lower the required coupon rate. Note that when the value of the reference bundle goes to zero, the bond becomes riskless and all curves converge to 12%. Note also that when the value of the reference bundle equals the face value of the bonds ($P/F = 1.0$), the equilibrium coupon rate is negative. The reason for this is that under the above assumptions, equation (16) represents the well known put-call parity relationship, and the first term in the RHS is the negative of the value of the corresponding put option.

Figures 3, 4 and 5 are derived from the model with commodity and default risks but no interest rate risk which is described in Section IV. Boundary condition (13) and Figure 1 indicate that default at maturity depends not only on the value of the firm, but also on the value of the commodity bundle. A higher standard deviation on the return on the commodity (σ_p) has two opposing effects on bond values: first, it is well known that the value of an option increases with the standard deviation of its underlying security; but second, the probability of

Table 1

Parameters for Numerical Example

$\sigma_p = 0.8$, 0.4 and 0.2	$r = 0.12$
$\sigma_v = 0.3$	$\rho_{PQ} = 0$
$\rho = 0.7$	$\delta(\tau)^2 = 0.003\ \tau$

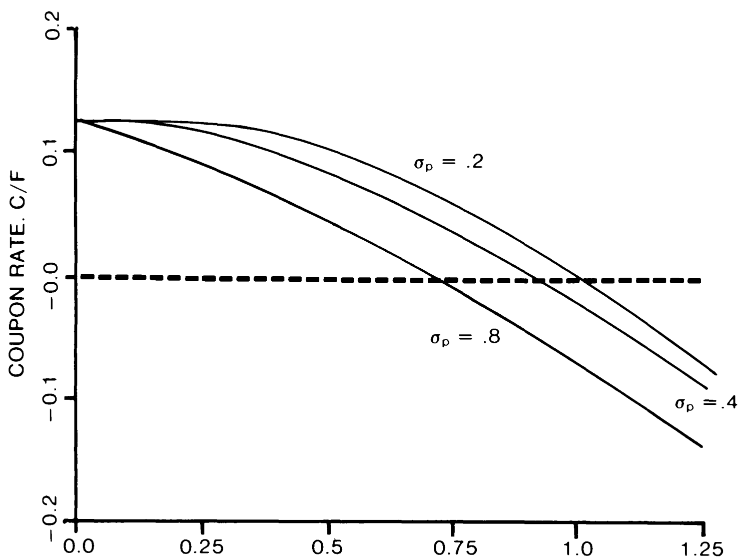

Figure 2.

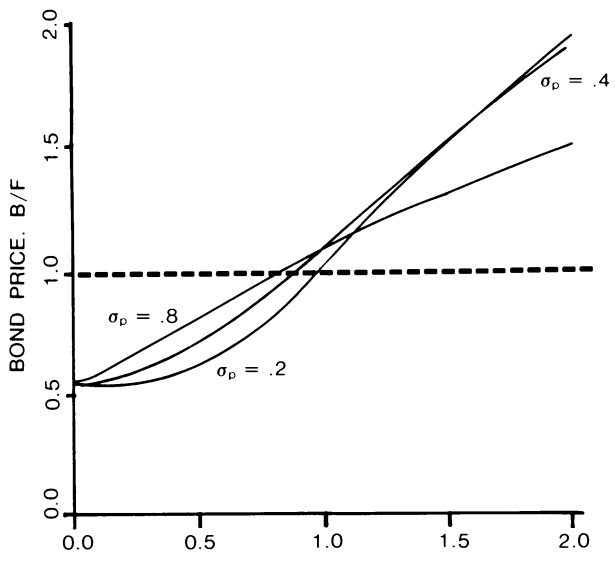

Figure 3.

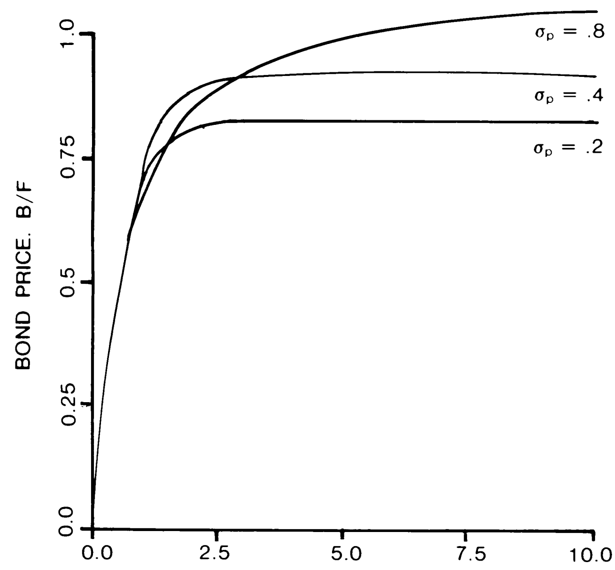

Figure 4.

default also increases with σ_p, and this tends to lower bond values. The first effect is shown to dominate the second for low commodity bundle prices in Figure 3, for high firm values in Figure 4 and for shorter maturity dates in Figure 5. These figures and the values reported in Table 2 indicate that default risk has a significant impact on bond values and that most of this risk comes not from the firm being unable to pay the face value of the commodity bonds, as is the case for

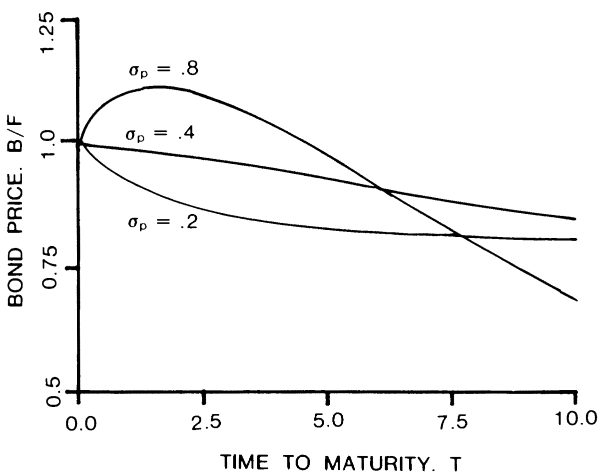

Figure 5.

regular corporate bonds, but from the firm being unable to pay the value of the option for high commodity prices, even under substantial increases in the value of the firm. For this reason, a higher correlation between the return on the commodity and the return on the firm increases bond values. Table 2 also illustrates that as the risk of default decreases the value of the bonds approaches the solution for the no default constant interest rate case shown in the penultimate column.

For the parameter values considered in the numerical example, the effect on bond values of introducing stochastic interest rates in the manner of Merton (1973) is equivalent to the effect of increasing the variance of the rate of return on the commodity by less than one tenth, and therefore the commodity-linked bond values obtained are only marginally higher than the ones obtained when the interest rate is assumed constant. This can be observed by comparing the last two columns of Table 2. This analysis indicates that when pricing commodity-linked bonds, it is quite safe to use the constant interest rate model, as long as the relevant interest rate used is the one to the maturity of the bond.

VII. Concluding Remarks

This paper has dealt with the problem of valuing commodity-backed bonds, a new security whose importance seems to be increasing rapidly.[7] The model developed here may be useful as an aid for companies and governments issuing these bonds to set the terms of the issue. The trade-offs between the various characteristics of an issue: the coupon rate, the reference commodity bundle, the exercise price of the option and time to maturity can only be established within a general model for the valuation of the bonds. Secondly, the model may be used to help investors in their search for "undervalued" or "overvalued" commodity-

[7] The *Wall Street Journal* in an article entitled "Wary That Greenbacks Won't Outlast Inflation, the Timid Turn to Commodity-Backed Bonds" published on February 2, 1981, indicates that "dozens of companies are now said to be studying the idea".

Table 2

Commodity-Linked Bond Values for Different Commodity Bundle Prices, Firm Values and Correlations: $E = F = 100$, $r = .12$, $\tau = 5.0$, $\sigma_p = .4$ and $\sigma_v = .3$

		$V = 200$			$V = 400$			$V = 1000$			No Default Risk	
		$\rho = 0.0$	.35	.70	$\rho = 0.0$	.35	.70	$\rho = .00$	.35	.70	Constant Interest Rate	Stochastic Interest Rate
Commodity Bundle Value	$P = 100$	85.45	93.34	102.54	99.00	104.66	108.70	107.15	108.92	109.40	109.41	109.85
	$P = 80$	77.34	83.20	89.26	86.57	90.02	92.35	91.45	92.41	92.60	92.60	93.06
	$P = 50$	65.01	67.67	69.62	68.89	70.14	70.58	70.39	70.61	70.64	70.64	71.03

linked bonds. In principle, if market values differ from model prices, the same hedging arguments used to derive the equations can be employed to take positions in the market and make arbitrage profits.[8]

Some of the assumptions used to derive the model are questionable. The model assumes, for example, that the underlying commodity is perfectly tradeable: this might be a good approximation in the case of gold or silver, but perhaps not so good in the case of oil. The model neglects taxes completely: yet one reason why investors might find these bonds attractive is to transform regular income into capital gains. Also, like most of the option pricing literature, the model assumes constant variances. The importance of these restrictive assumptions is an empirical question, and as more bonds appear in the market, the data will become available to carry out the appropriate tests.

More complex bond characteristics, such as call features, sinking funds and convertibility into the reference commodity bundle before maturity could be introduced at the cost of having to use numerical procedures to solve the appropriate partial differential operations. This is also the case if the model is to be applied to more complicated capital structures than the ones considered here.

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[8] Such as in Black and Scholes [1972] and many other later studies.