VIX OPTION PRICING

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Substantial progress has been made in developing more realistic option pricing models for S&P 500 index (SPX) options. Empirically, however, it is not known whether and by how much each generalization of SPX price dynamics improves VIX option pricing. This article fills this gap by first deriving a VIX option model that reconciles the most general price processes of the SPX in the literature. The relative empirical performance of several models of distinct interest is examined. Our results show that state-dependent price jumps and volatility jumps are important for pricing VIX options. © 2009 Wiley Periodicals, Inc. Jrl Fut Mark 29:523–543, 2009

INTRODUCTION

The 1987 crash brought volatility products to the attention of academics and practitioners and the Chicago Board Options Exchange (CBOE) successively launched Volatility Index (VIX) futures on March 26, 2004 and VIX options on

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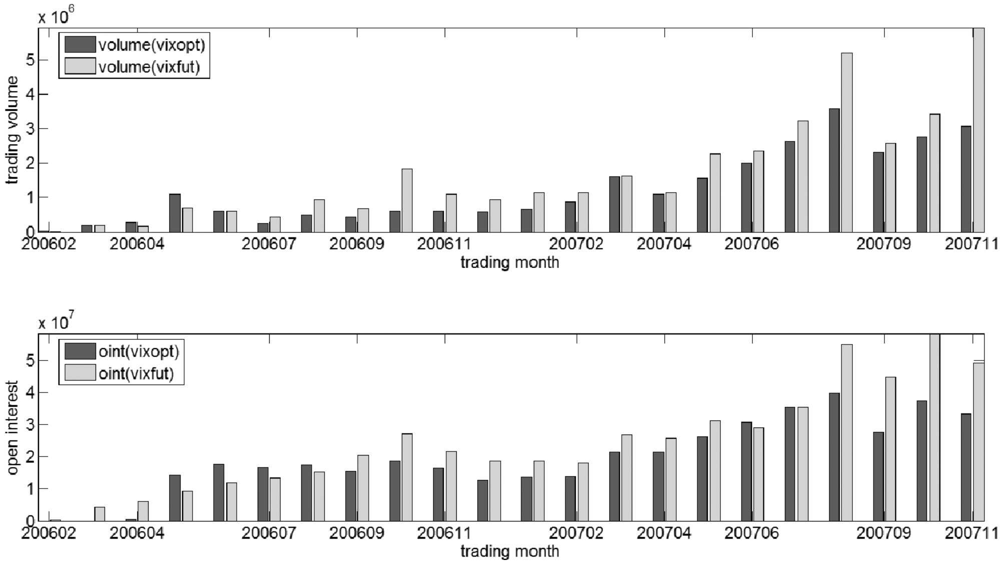

FIGURE 1
The trading volume (volume) and open interest (oint) of VIX options and VIX futures across trading months, February 2006–November 2007.

February 24, 2006. These were the first of an entire family of volatility products to be traded on exchanges. As shown in Figure 1, the trading volume and open interest of VIX options and VIX futures grew significantly over our sample period of February 2006 to November 2007, reflecting their economic importance.

In contrast to the implied volatility extracted from an option pricing model, the VIX uses a model-free formula to extract expected volatility directly from the prices of a weighted strip of S&P 500 index (SPX) options over a wide range of strike prices, which incorporates information from the volatility skew. Current prices of both VIX futures and VIX options consequently reflect the market's expectation of the VIX level at expiration. The current VIX futures price, rather than the spot VIX itself, is the underlying price of VIX options. Although the construction of VIX is model-free, the fair value of VIX options involves the variance of VIX futures prices from the current time to their expiry. Hence, this study examines the impact of alternate variance models on VIX option pricing.

The dynamics of the new and evolving volatility market have been explored in the literature. Brenner and Galai (1989, 1993) first suggested options written on a volatility index. Various models to price volatility options written on instantaneous volatility have also been developed (see Detemple & Osakwe, 2000; Grünbichler & Longstaff, 1996; Whaley, 1993). These models differ in their specification of the assumed stochastic process, and the assumptions made about the volatility risk premium. There have been extensive studies to extend one-factor volatility models. Examples of multifactor pricing models and related tests include, among others, the unified theory of local volatility (Dupire, 1996), stochastic implied volatility (Derman & Kani, 1998), and variance curve models (Balland, 2006; Bergomi, 2005; Buehler, 2006; Dupire, 1993).

The literature on the price behavior of VIX spot and futures markets is growing fast (see, for example, Carr & Wu, 2006; Dotsis, Psychoyios, & Skiadopoulos, 2007; Dupire, 2006; Lin, 2007; Zhang & Zhu, 2006; Zhu & Zhang, 2007). Nevertheless, research on the valuation of VIX options is not concluded. Rather than directly modeling volatility dynamics, this study reconciles the growing literature on SPX price processes by investigating how much each generalization of the SPX price dynamics improves VIX option pricing. The affine stochastic-volatility model with simultaneous jumps both in the asset price and variance process is one of the most general specifications of the SPX price in the literature (see, Alizadeh, Brandt, & Diebold, 2002; Andersen, Benzoni, & Lund, 2002; Duffie, Pan, & Singleton, 2000; Eraker, 2004; Eraker, Johannes, & Polson, 2003). Hence, this study evaluates VIX options that allow for stochastic volatility, price jumps, and volatility jumps in SPX returns.

Our sample covers the period February 24, 2006 to November 30, 2007. The resultant parameter estimates of alternative models are used to investigate each model's internal consistency and out-of-sample pricing. These two yardsticks judge the alternative models from different perspectives. The reason for the consistency test adopted by Bates (1996) and Bakshi, Cao, and Chen (1997) is that if an option model is correctly specified, its structural parameters implied by option prices will be consistent with those implicit in the observed time-series data. Out-of-sample pricing errors give a direct measure of model misspecification. Although a more complex model will generally lead to a better in-sample fit, it will not necessarily perform better in out-of-sample pricing as any overfitting may be penalized. Our results show that complex jump specifications add explanatory power in fitting options. However, incorporating price jumps mainly improves short-dated out-of-sample pricing, whereas allowing for volatility jumps fares better in fitting medium- and long-dated VIX options.

Our model for the VIX option is presented in the next section. Model parameters are then estimated using historical VIX futures and VIX options, and VIX options pricing formulas are tested against the market data. A conclusion is presented in the last section. Proof of pricing equations and formulas are provided in Appendix A.

THEORY

The VIX Model

The forward price of the SPX, denoted $F_t(T)$, under a risk-neutral measure, is modeled by a jump-diffusion process with stochastic instantaneous variance ν_t, denoted "ERAKER" (Eraker et al., 2003; Eraker, 2004)

$$\frac{dF_t(T)}{F_t(T)} = \sqrt{\nu_t}\, d\omega_{S,t} + J_t dN_t - \lambda_t \kappa dt \tag{1}$$

$$d\nu_t = \kappa_\nu(\theta_\nu - \nu_t)dt + \sigma_\nu \sqrt{\nu_t}\, d\omega_{\nu,t} + z_\nu dN_t \tag{2}$$

where $J_t = \exp(z_S) - 1$ is the percentage price jump size with mean κ. Satisfying the no-arbitrage condition, $\kappa = \exp(\mu_j + \sigma_j^2/2)/(1 - \rho_j\mu_\nu) - 1$. ω_S,t and ω_ν,t are correlated Brownian motions with $\rho dt = \mathrm{corr}(d\omega_{S,t}, d\omega_{\nu,t})$. They are independent of the compound Poisson processes $z_S \mathrm{dN_t}$ and $z_\nu dN_t$. The instantaneous variance ν follows a mean-reverting square-root process with exponentially distributed jump size z_ν that is correlated with price jump size z_S through $z_S = \mu_j + \rho_j z_\nu$. Formally, jumps in volatility are assumed to have an exponential distribution, i.e., $z_\nu \sim \exp(\mu_\nu)$, whereas jumps in asset log-prices are normally distributed conditional on the realization of z_ν, i.e., $z_S|z_\nu \sim N(\mu_j + \rho_j z_\nu, \sigma_j^2)$. Thus, z_S has mean $E(z_S) = \mu_j + \rho_j\mu_\nu$, variance $\mathrm{var}(z_S) = \sigma_j^2 + \rho_j^2\mu_v^2$, and is correlated with z_ν through $\rho_j\mu_\nu/\sqrt{\sigma_j^2 + \rho_j^2\mu_\nu^2}$. The underlying return and its volatility share the same jump arrival uncertainty followed by a Poisson process N_t with state-dependent intensity $\lambda_t = \lambda_0 + \lambda_1\nu_t$. Further, $\kappa_\nu - \lambda_1\mu_\nu$, $(\kappa_\nu\theta_\nu + \lambda_0\mu_\nu)/(\kappa_\nu - \lambda_1\mu_\nu)$, and $\sigma_\nu^2\nu_t + 2\lambda_t\mu_\nu^2$ are the speed of adjustment, the long-run mean, and the variation of instantaneous variance ν.

Because the square of VIX (denoted VIX squared) is defined to be the variance swap rate, we are able to evaluate it by computing the conditional expectation under the risk-neutral measure Q

$$\mathrm{VIX}_t^2 \equiv \frac{1}{\tau} E_t^Q\left[\int_t^{t+\tau} (\nu_u du + J_u^2 dN_u)\right] \tag{3}$$

where τ is 30 calendar days by definition. From Equations (1)–(3) we have the following result.

Proposition 1: The VIX squared is a linear function of the instantaneous variance

$$\mathrm{VIX}_t^2 \equiv \frac{\zeta_1}{\tau} a_\tau \nu_t + \frac{\zeta_1}{\tau} b_\tau + \zeta_2 \tag{4}$$

where $\zeta_1 = 1 + 2\lambda_1[\kappa - (\mu_j + \rho_j\mu_\nu)]$, $\zeta_2 = 2\lambda_0[\kappa - (\mu_j + \rho_j\mu_\nu)]$, $\tau = 30/365$,

$$a_\tau = \frac{1 - e^{-(\kappa_\nu - \lambda_1\mu_\nu)\tau}}{\kappa_\nu - \lambda_1\mu_\nu} \text{ and } b_\tau = \left(\frac{\kappa_\nu\theta_\nu + \lambda_0\mu_\nu}{k_\nu - \lambda_1\mu_\nu}\right)\left(\tau - \frac{1 - e^{-(\kappa_\nu - \lambda_1\mu_\nu)\tau}}{\kappa_\nu - \lambda_1\mu_\nu}\right).$$

VIX Options

Consider a European call option written on VIX with strike price K and expiry T (or time-to-maturity τ_C). Its time-t price $C(t, \tau_C)$ must solve the following partial differential equation (PDE)

$$\begin{aligned}
&\frac{1}{2}\nu\frac{\partial^2 C}{\partial L^2} + \left[r - \lambda_0\kappa - \left(\lambda_1\kappa + \frac{1}{2}\right)\nu\right]\frac{\partial C}{\partial L} + \rho\sigma_\nu\nu\frac{\partial^2 C}{\partial L\partial\nu} \\
&\quad + \frac{1}{2}\sigma_\nu^2\nu\frac{\partial^2 C}{\partial\nu^2} + \kappa_\nu(\theta_\nu - \nu)\frac{\partial C}{\partial\nu} - \frac{\partial C}{\partial\tau_C} - rC \\
&\quad + E_t^Q\{[\lambda_0 + \lambda_1(\nu + z_\nu)]C(t, \tau_C; L + z_S, \nu + z_\nu) \\
&\quad - (\lambda_0 + \lambda_1\nu)C(t, \tau_C; L, \nu)] = 0 \qquad (5)
\end{aligned}$$

subject to $C(t + \tau_C, 0) = \max(\text{VIX}_T - K, 0)$ with $\text{VIX}_T = \sqrt{\zeta_1 a_\tau \nu_T/\tau + \zeta_1 b_\tau/\tau + \zeta_2}$. $L = \ln S$ and r is the risk-free rate of interest. The following proposition gives the solution to the PDE.

Proposition 2: The price of a VIX call option with strike K and maturity T is given by the following formula:

$$C(t, \tau_C) = \exp(-r\tau_C)[F_t^{VIX}(T)\Pi_1 - K\,\Pi_2] \qquad (6)$$

where the risk-adjusted probabilities, Π_1 and Π_2, are recovered from inverting the respective characteristic functions of the log VIX squared:

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-i\phi\ln K^2} f_2(t, \tau_C; i\phi + 1/2)}{i\phi f_2(t, \tau_C; 1/2)}\right]d\phi \qquad (7)$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-i\phi\ln K^2} f_2(t, \tau_C; i\phi)}{i\phi}\right]d\phi. \qquad (8)$$

Re[] denotes the real part of a complex number. The characteristic function $f_2(t, \tau_C; i\phi)$ of $\ln \text{VIX}_T^2$ is given in Appendix A.

The price of a VIX put can be determined from put–call parity. The fair value of the VIX futures can be obtained by setting $i\phi = 1/2$ in $f_2(t, \tau_C; i\phi)$, i.e., $F_t^{VIX}(T) = f_2(t, \tau_c; 1/2)$. The option pricing model in Equation (6) has several distinctive features. First, it applies to economies with the index price process containing stochastic volatility and correlated jumps in returns and volatility. Second, it reconciles most existing equity dynamics as special cases. Each nested model is obtained by the choice of specific parameters. We obtain (i) the stochastic volatility model of Heston (1993), denoted HESTON, by setting the jump frequency to zero, i.e., $\lambda_0 = \lambda_1 = 0$; (ii) the price jump diffusion with stochastic volatility model of Bates (1996), denoted BATES, by setting the volatility jump size to zero, i.e., $\mu_v = 0$; and (iii) the stochastic volatility and correlated simultaneous jumps in price and volatility model of Duffie et al. (2000), denoted DPS, by imposing a state-independent jump frequency, i.e., $\lambda_1 = 0$.

DATA

The sample period spans from February 24, 2006, through November 30, 2007. Spot VIX, daily midpoints of the last bid and last ask quotations for VIX options, and daily settlement prices for VIX futures are obtained from the CBOE. The data on daily U.S. Treasury-bill bid and ask discounts with maturities up to one year are obtained from Datastream. By convention, the average of the bid and ask U.S. Treasury-bill discounts is used and converted to an annualized continuously compounded interest rate. As Treasury bills mature on Thursdays whereas options and futures on VIX settle on Wednesdays, that is 30 days before the third Friday of the calendar month immediately following the month in which the contract expires, we utilize the two Treasury-bill rates straddling an option's expiration date to obtain the interest rate corresponding to the option's maturity. This is done for each contract and each day in the sample.

Several exclusion filters are applied to construct the option price data. First, as options with less than six days to expiration may induce liquidity-related biases, they are excluded from the sample. Second, to mitigate the impact of price discreteness on option valuation, prices lower than \$3/8 are excluded. Finally, the VIX call option prices not satisfying the arbitrage restrictions

$$\max\{0, \exp(-r\tau_C)[F_t^{VIX}(T) - K)]\} \leq C(t, \tau_C) \leq \exp(-r\tau_C)F_t^{VIX}(T) \tag{9}$$

and the VIX put option prices violating the boundaries

$$\max\{0, \exp(-r\tau_C)[K - F_t^{VIX}(T)]\} \leq P(t, \tau_C) \leq \exp(-r\tau_C)K \tag{10}$$

are excluded from the sample. Based on these criteria, 19,589 observations (approximately 29.43% of the original sample) are eliminated. A total of 46,969 records of joint futures and options prices on VIX are used for parameter estimation. Of these, 22,109 are calls and 24,860 are puts. Table I presents characteristics of the data sample across maturity and moneyness, where moneyness is defined as the VIX futures price divided by the option strike price, i.e., $F_t^{VIX}(T)/K$. Average VIX option prices range from \$0.8214 for deep out-of-the-money (DOTM) short-term (SR) calls to \$9.0448 for deep in-the-money

TABLE I
Sample Characteristics of VIX Options and VIX Futures

	Days to Expiration					
	Calls			Puts		
Moneyness	*<60*	*60–180*	*>180*	*<60*	*60–180*	*>180*
DOTM						
Option price	\$0.8214	\$1.0851	\$1.3180	\$0.8501	\$1.0239	\$1.2186
Futures price	16.4892	16.9383	16.3520	19.0767	18.2766	17.0761
Observations	2357	4476	4543	771	2138	2413
OTM						
Option price	\$1.2940	\$2.0751	\$2.4410	\$1.1448	\$1.8476	\$2.1701
Futures price	15.5783	16.9965	16.5446	15.6918	17.1513	16.0475
Observations	262	327	267	240	285	268
AIM1						
Option price	\$1.4981	\$2.1193	\$2.5023	\$1.2824	\$2.0287	\$2.4883
Futures price	16.0649	16.5600	16.0585	15.2760	16.4529	16.3282
Observations	253	286	309	252	250	252
ATM2						
Option price	\$1.5188	\$2.2664	\$2.7604	\$1.7217	\$2.3559	\$2.7353
Futures price	15.1965	16.4529	16.3282	16.0300	16.5600	16.0585
Observations	257	250	252	255	286	309
ITM						
Option price	\$1.7835	\$2.5875	\$2.8483	\$1.9336	\$2.8567	\$3.1758
Futures price	15.4616	17.1513	16.0491	15.4194	16.9965	16.5448
Observations	256	285	269	273	327	266
DITM						
Option price	\$4.2986	\$4.0056	\$4.0630	\$8.1798	\$9.0448	\$8.5087
Futures price	17.9294	17.8422	16.9714	15.3433	16.4938	16.2413
Observations	2196	2833	2431	5257	6010	5008

Note. The reported figures are, respectively, the average VIX option price, the average VIX futures price, and the total number of observations of VIX options for each moneyness–maturity category. The sample period extends from February 24, 2006, through November 30, 2007 for a total of 22,109 calls and 24,860 puts. Daily information from the midpoints of last quotations for each option contract and daily settlement futures prices are used to obtain the summary statistics. $F_t^{VIX}(T)$ indicates the time-t VIX futures price with expiry T and K is the option exercise price. *Moneyness* is defined as $F_t^{VIX}(T)/K$. DOTM (DITM), OTM (ITM), ATM1 (ATM2), ATM2 (ATM1), ITM (OTM), and DITM (DOTM) for calls (puts) are defined by *Moneyness* <0.94, 0.94–0.97, 0.97–1.00, 1.00–1.03, 1.03–1.06, and >1.06, respectively.

(DITM) medium-term (MR) puts. The average VIX futures prices range from \$15.1965 corresponding to the SR at-the-money (ATM) calls to \$19.0767 for the SR DOTM puts.

MODEL ESTIMATION

The vector of structural parameters Φ is backed out by minimizing the sum of the squared pricing errors between option model and market prices. The minimization is given by

$$\min_{\Phi} \sum_{t=1}^{N_T} \sum_{n=1}^{N_t} [C_n - C_n^*(\Phi)]^2 \tag{11}$$

where N_T is the number of trading days in the estimation sample, N_t is the number of options on day t, and C_n and C_n^* are the observed and model option prices, respectively. The model is estimated separately each month and thus Φ is assumed to be constant over a month. The assumption that the structural parameters are constant over a month is justified by an appeal to parameter stability (Bates, 1996; Eraker, 2004; Zhang & Zhu, 2006). Table II reports the monthly average of each estimated parameter series and the monthly averaged in-sample mean squared errors for the HESTON, BATES, DPS, and ERAKER models, respectively.

Several observations are in order. First, the estimated structural parameters for the SPX price process generally differ across models. The estimate of the *variance adjustment speed* is highest for the ERAKER model, partly attributable to the existence of volatility jumps. The estimates of the *long-run variance mean* for the HESTON and BATES models are 0.0594 and 0.0498, which are relatively low, whereas those under the DPS and ERAKER models are

TABLE II

Implied Parameters Estimation

Model Fit	*HESTON*	*BATES*	*DPS*	*ERAKER*
In-sample pricing error	8.9036	7.511	3.4225	2.9443
Adjustment speed of ν	5.2028	7.4837	8.0532	8.5118
Long-run mean of ν	0.0594	0.0498	0.2848	0.1475
Total variation of ν	0.0095	0.0052	6.4580	1.6621
Correlation between diffusion Brownian motions	−0.8248	−0.6723	−0.5112	−0.3101
Mean jump intensity		1.3289	1.2113	1.2242
Mean of price jump-size innovations		−0.2673	−0.5849	−0.8097
Variance of price jump-size innovations		0.1047	0.1364	0.0883
Correlation between jumps-size components			−0.7328	−0.8799
Implied volatility (%)				
Variation attributed to stochastic volatility	0.5265	0.5246	0.2172	0.1779
Variation attributed to volatility jumps			0.3225	0.3522

Note. The values of *in-sample pricing error* and estimated structural parameters reported here are their averages over 20 nonoverlapping estimation months from February 24, 2006, to October 16, 2007, with a total of 20,077 calls and 22,885 puts. The reported *in-sample pricing error* is the average of the sum of the squared pricing errors between the market price and the model price for each option in the sample. $\kappa_\nu - \lambda_1\mu_\nu$, $(\kappa_\nu\theta_\nu + \lambda_0\mu_\nu)/(\kappa_\nu - \lambda_1\mu_\nu)$ and $\sigma_\nu^2\nu_t + 2\lambda_t\mu_\nu^2$ are the speed of adjustment, long-run mean, and variation of instantaneous variance ν. The correlation between diffusion Brownian motions for the asset price and volatility is ρ. $\lambda_t = \lambda_0 + \lambda_1\nu_t$ is the mean jump intensity and $\nu_t \equiv (\tau/\zeta_1 a_\tau)(\mathrm{VIX}_t^2 - \zeta_2 - \zeta_1 b_\tau/\tau)$ where $\zeta_1 = 1 + 2\lambda_1[\kappa - (\mu_j + \rho_j\mu_\nu)]$, $\zeta_2 = 2\lambda_0[\kappa - (\mu_j + \rho_j\mu_\nu)]$, $\tau = 30/365$, $a_\tau = [1 - e^{-(\kappa_\nu - \lambda_1\mu_\nu)\tau}]/(\kappa_\nu - \lambda_1\mu_\nu)$, and $b_\tau = \left(\frac{\kappa_\nu\theta_\nu + \lambda_0\mu_\nu}{\kappa_\nu - \lambda_1\mu_\nu}\right)\left(\tau - \frac{1 - e^{-(\kappa_\nu - \lambda_1\mu_\nu)\tau}}{\kappa_\nu - \lambda_1\mu_\nu}\right)$. For the HESTON and BATES models, $\lambda_t = \lambda_J$. z_S is the jump in asset log-prices with $\mathrm{E}(z_S) = \mu_j + \rho_j\mu_\nu$, $\mathrm{var}(z_S) = \sigma_j^2 + \rho_j^2\mu_\nu^2$, and is correlated with the jumps in volatility, z_ν, according to $\mathrm{corr}(z_S, z_\nu) = \rho_j\mu_\nu/\sqrt{\sigma_j^2 + \rho_j^2\mu_\nu^2}$. The estimated implied variances of the VIX futures price changes are $\nu_{t,F_t^{VIX}(T)} + \nu_{J,F_t^{VIX}(T)}$ where $\nu_{t,F_t^{VIX}(T)}$ is the variation attributed to stochastic volatility and $\nu_{J,F_t^{VIX}(T)}$ is the variance attributed to volatility jumps.

much higher, suggesting that the jump components explain a significant portion of the unconditional return variance. The HESTON model has the strongest negative correlation between Brownian increments in volatility and index returns. More specifically, negative estimates of ρ are -0.8248, -0.6723, -0.5112, and -0.3101 for the HESTON, BATES, DPS, and ERAKER models, respectively.

Second, the volatility jump size (μ_v) under the DPS model of 1.6324 is greater than its ERAKER model counterpart of 0.8236, which is expected because the extra parameters (related to the state-dependent volatility jump process) make the ERAKER model fit the data better than the DPS model. The negative conditional correlation between price jumps and volatility jumps ρ_j (-0.1658 for DPS and -0.3175 for ERAKER) indicates the asymmetry of volatility jumps across the index price level. The negative unconditional correlation between price jumps and volatility jumps $\rho_j\mu_v/\sqrt{\sigma_j^2 + \rho_j^2\mu_v^2}$ (-0.7328 for DPS and -0.8799 for ERAKER) further demonstrates that the leverage effect still holds when the SPX price process contains volatility jumps. Comparing the correlation of diffusion motions with those of jump-size innovations, the leverage effect between asset prices and volatility is mainly attributed to the jump component. In particular, state-dependence in jump frequency enhances the jump portion of the leverage effect.

Third, the BATES, DPS, and ERAKER models attribute part of the negative skewness and excess kurtosis to the possibility of jumps. The jumps occur extremely rarely: The λ_t estimates in Table II indicate that one can expect about four to six jumps in a stretch of 1,000 trading days. The unconditional jump frequency is only marginally higher under the state-dependent ERAKER model. Whenever spot volatility is high, say an annualized variance of 3%, the estimate of $\lambda_1 = 0.7658$ is indicative of an instantaneous jump probability of about 0.023, or a 2% increase over the constant arrival intensity specifications. The possibility of asset price jumps occurs with an average jump size, $\mu_j + \rho_j\mu_v$, of -0.2673, -0.5849, and -0.8097 (with jump size uncertainty, $\sigma_j^2 + \rho_j^2\mu_v^2$, estimated at 0.1047, 0.1364, and 0.0883) for the BATES, DPS, and ERAKER models, respectively. Note that a larger negative price jump size mean with smaller variance for the ERAKER model gives a reduced possibility for the asset to jump up. A possible explanation for this observation is that the state-dependence of jump intensity enhances the probability of a jump occurring once a large price jump-down takes place. Hence, the ERAKER model can help to explain volatility clustering in financial markets.

Together, these estimates show that, to the extent the pricing structure of options prices can be explained by each model, the ERAKER or DPS models' demands on the $\nu(t)$ process are the most stringent as they require both the highest variation and the greatest covariance (in magnitude) with underlying returns. The estimated implied variances of the VIX futures price changes are, however, generally very close among alternate models. This is consistent with the observation that total variance ($\sigma_v^2 v_t + 2\lambda_t \mu_v^2$) for the DPS model is significantly reduced by incorporating a *state-dependent* jump frequency, i.e., the ERAKER model.

The parameter estimates in Table II are interesting in light of estimates obtained in prior studies. Bakshi et al. (1997) estimate the jump frequency for the BATES model as a 0.59 annualized jump probability, and the jump-size parameter μ_j and σ_j are -5.37 and -7%, respectively. Their estimate of κ_v and σ_v are 2.03 and 0.38. Pan (2002) and Bates (2000) assume that the jump frequency depends on the spot volatility, and hence the jump size and jump frequency parameters are comparable to those reported in Eraker's (2004) model and here. The average jump intensity point estimates in Pan (2002) are in the range [0.07, -0.3%] across different model specifications, whereas Eraker (2004) indicates two to three jumps in a stretch of 1,000 trading days. Interestingly, Bates (2000) obtains quite different results, with an average jump intensity of 0.005.[1] Bates (2000) also reports a jump size mean ranging from -5.4 to -9.5% and standard deviations of about 10 to 11%. Hence, Bates' estimates imply more frequent and more severe crashes than the parameter estimates reported in Bakshi et al. (1997) and Eraker (2004). Our estimates are in the same ballpark as those reported in Bates (2000). The practical implication of the difference is that Bates (2000) and our estimates generate more skewness and kurtosis in the conditional index returns distributions.

Finally, the fact that allowing jumps to occur enhances the HESTON model's fit is illustrated by each model's sum of the squared pricing errors between the market price and the model price in an average month (MSE). The MSE is 8.90 for the HESTON model, 7.51 for the BATES model, 3.42 for the DPS model, and 2.94 for the ERAKER model. The in-sample mean squared errors are consistently smaller for more complicated models.

TESTING THE VIX OPTION PRICE FORMULA

Internal Consistency of Implicit Parameters

Another way to gauge model misspecification is to follow the approach taken by Bates (1996, 2000), Bakshi et al. (1997), Pan (2002), and Eraker (2004) and examine whether each model's implied parameters are consistent with those

[1]Pan (2002) and Bates (2000) model the jump frequencies as $\lambda = \lambda_1 v_t$ with spot volatility v_t, whereas Eraker (2004) assumes the jump frequency $\lambda = \lambda_0 + \lambda_1 v_t$. The average jump intensities reported here are obtained from the articles of Pan, Bates, and Eraker as the multiple of the long-run volatility mean, θ_v, and the proportionality parameter, λ_1, plus λ_0.

implicit in the time series of the VIX futures prices. That is, is the forward view offered by volatility implied by the market prices of VIX options similar in magnitude to that from their time series counterparts? The closer the implied parameters, the closer the implied time-series path to its observed counterpart and hence the less misspecified the model. The justification for using VIX futures prices to test internal consistency is that VIX options are priced based on VIX futures rather than on the spot VIX. Eraker (2004) reports that estimates of σ_v from the simulated values of the historical volatilities match almost exactly the option implied volatility of volatility. He points out that the match is partly attributed to the Markov Chain Monte Carlo (MCMC) method providing much more erratically behaving volatility paths than other methods based on Kalman filtering, and quasi-maximum likelihood methods.

This study examines the consistency of the VIX option-implied volatility of the forward VIX with the sample volatility of daily VIX futures price changes. Two distinct approaches are adopted for estimating the volatility function $\sigma(t, T)$. Implicit volatility estimation uses the parameters recovered from the market prices of VIX options (generated from Table II), whereas historic volatility estimation uses time-series observations of VIX futures.

For the overall sample period of February 24, 2006, to November 30, 2007, the annualized daily VIX futures price changes have a mean of 0.03%, a volatility of 32.44%, a skewness of -1.90, and a kurtosis of 83.60. The historical volatility is indeed lower than its option-implied counterpart, 72.56% for HESTON, 72.43% for BATES, 73.47% for DPS and 72.81% for ERAKER. This departure between the average implied and the time-series estimate is remarkably similar across alternate models. Hence, each of the four models is equally misspecified. Figure 2 contains time series graphs of the VIX futures prices and the estimated forward VIX from the market prices of VIX options over the entire time period studied. There are 445 observations of daily means of VIX futures prices across maturities in this time period. In this graph, the forward VIX curves are piecewise constant. The forward VIX processes recovered from the market prices of both VIX futures and VIX options are remarkably similar in time-series patterns. However, the implied value for each model is about two times its time-series estimate. The volatility process of the forward VIX implicit in option prices is therefore much too volatile, relative to each volatility time series implicit in futures prices. According to these yardsticks, each of the four models is significantly misspecified. On a relative scale, however, this departure between the average implied and the time-series estimate is the weakest for the BATES, DPS, and ERAKER models, and the strongest for the HESTON model.

In summary, the four models rely on implausible levels of volatility variation of forward VIX to rationalize the observed option prices. This finding is similar

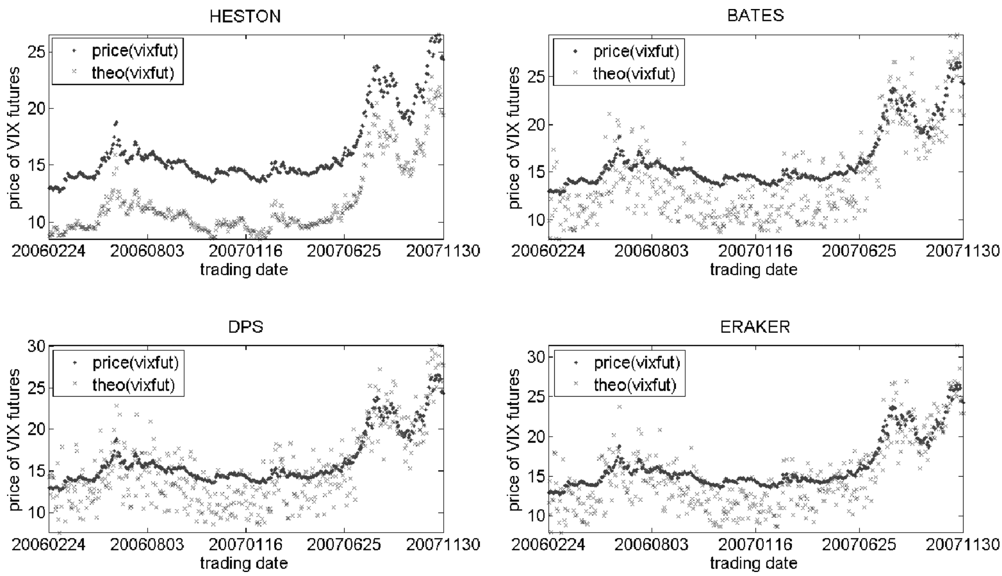

FIGURE 2
Forward VIX Evolutions over February 24, 2006–November 30, 2007. The dotted line is the average VIX futures prices and the line with cross symbols is the forward VIX recovered from VIX option prices.

to those of Bates (1996, 2000) using currency and SPX futures options and Bakshi et al. (1997) and Pan (2002) using SPX options. Although the HESTON, BATES, DPS, and ERAKER models are clearly misspecified (though BATES, DPS, and ERAKER to a lesser degree than HESTON), how do they perform in forecasting VIX options? We answer this question in the next section.

Out-of-Sample Pricing

This section provides a comparison of out-of-sample pricing. Out-of-sample pricing is carried out with the *previous month's* structural parameters and the *current day's* VIX and VIX futures prices to calculate the *current day's* VIX option model price. The mean absolute pricing error (MAE) is employed to assess the out-of-sample pricing performance of the VIX option pricing models on the HESTON, BATES, DPS, and ERAKER specifications.

Table III reports MAE values for several categories according to time to expiration and moneyness. Of the maturity combinations reported in Table III, MAE is lowest for the BATES (ERAKER) model for the SR (MR and long-term, denoted LR) options contracts. Thus, improvements are generated for short-dated VIX options under the BATES model and for medium- and long-dated VIX options under the ERAKER model.

Bakshi et al. (1997) point out that the price jump and the volatility diffusion features can in principle improve the pricing of, respectively, SR and relatively

TABLE III
Out-of-Sample Pricing Errors

		Mean Absolute Pricing Error		
		Days to Expiration		
Moneyness	*Model*	*<60*	*60–180*	*>180*
DOTM	HESTON	\$2.54	\$2.50	\$2.77
	BATES	0.86	1.13	1.04
	DPS	0.92	1.78	0.54
	ERAKER	1.03	0.60	0.42
OTM	HESTON	2.54	2.79	2.83
	BATES	0.85	1.18	1.22
	DPS	1.12	1.03	0.93
	ERAKER	1.09	0.52	0.44
ATM1	HESTON	1.71	2.15	2.12
	BATES	0.57	1.83	1.26
	DPS	1.27	1.78	1.23
	ERAKER	0.86	0.84	0.96
ATM2	HESTON	1.23	1.19	2.36
	BATES	0.94	1.22	1.16
	DPS	1.20	1.08	2.12
	ERAKER	1.01	0.87	1.03
ITM	HESTON	0.93	1.65	1.71
	BATES	0.35	1.22	1.55
	DPS	0.54	1.03	1.08
	ERAKER	0.43	0.99	0.95
DITM	HESTON	1.72	2.37	2.25
	BATES	0.23	1.24	1.27
	DPS	0.58	1.16	2.05
	ERAKER	0.42	0.70	0.87

Note. For a given model, we compute the price of each option using the *previous month*'s structural parameters and the *current day*'s VIX and VIX futures prices. The reported mean absolute pricing error is the sample average of the absolute difference between the market price and the model price for each option in a given moneyness–maturity category. The out-of-sample period is March 22, 2006–November 30, 2007, with a total of 21,939 calls and 24,660 puts. *Moneyness* is defined as the VIX futures price divided by the option strike price, i.e., $F_t^{VIX}(T)/K$. DOTM (DITM), OTM (ITM), ATM1 (ATM2), ATM2 (ATM1), ITM (OTM), and DITM (DOTM) for calls (puts) are defined by *Moneyness* <0.94, 0.94–0.97, 0.97–1.00, 1.00–1.03, 1.03–1.06, and >1.06, respectively.

LR options. Therefore, the BATES model enhances the flexibility of permissible return distributions and thus provides a better pricing fit for SR and LR options than the HESTON model (unless the *covariance* and *volatility variation* of the HESTON model are unreasonably high). Table III confirms this argument. The BATES model provides a better out-of-sample pricing fit for VIX options across maturities than the HESTON model.

Further, Eraker et al. (2003) point out that jumps in returns can generate large movements such as the crash of 1987, but the impact of a jump is transient: A jump in returns today has no impact on the future distribution of returns. On the other hand, diffusive volatility is highly persistent, but its dynamics are driven by a Brownian motion. For this reason, diffusive stochastic volatility can only increase gradually via a sequence of small normally distributed increments. Jumps in volatility fill the gap between jumps in returns and diffusive volatility by providing a rapidly moving but persistent factor that drives the conditional volatility of returns.

It is important to note that the presence of jumps in volatility does not eliminate the need for jumps in returns. With both types of jumps, jumps in returns occur less often ($\lambda_J^{BATES} = 1.33 > \lambda_J^{DPS} = 1.21$), but they still play an important role, as they generate the large, though infrequently observed, crash-like movements. This indicates that jumps in volatility and returns play a greater role than diffusive stochastic volatility in generating the index dynamics.

In summary, like jumps in returns and unlike diffusive stochastic volatility, jumps in volatility are a rapidly moving factor driving returns. Like diffusive stochastic volatility and unlike jumps in returns whose impact on returns is transient, a jump in volatility persists. Thus, jumps in volatility provide a rapidly moving but persistent factor driving volatility. Therefore, each factor (diffusive volatility, price jump, and volatility jump) generates very different behavior. In particular, the persistent feature of volatility jumps enhances the valuation of long-dated derivatives. This is because as maturity increases, the fat-tails and asymmetries in the conditional distribution are driven to a larger extent by diffusive volatility and the volatility jump through its persistence, rather than by price jumps. Similarly, as maturity decreases, the fat-tails and asymmetries in the conditional distribution are driven to a larger extent by price jumps, rather than the other two factors. Hence, according to the yardstick of out-of-sample pricing fits the BATES model performs best for short-dated VIX options and the ERAKER model outperforms for medium- and long-dated VIX options.

To further understand the structure of remaining pricing errors, we appeal to a regression analysis to study the association between the errors and factors that are either contract-specific or market condition-dependent. We first fix an option pricing model, and let $\varepsilon_n(t)$ denote the nth option's percentage pricing error on day t. Then, we run the regression below for the entire sample:

$$\varepsilon_n(t) = \beta_0 + \beta_1 \mathrm{mon}_n(t) + \beta_2 \tau_{C,n}(t) + \beta_3 \mathrm{BA}_n(t) + \beta_4 \mathrm{VJ}_n(t) + u_n(t) \tag{12}$$

where $\mathrm{mon}_n(t)$ indicates the option *moneyness*, defined as $F_t^{VIX}(T)/K_n$ for calls and $K_n/F_t^{VIX}(T)$ for puts, $\tau_{C,n}(\mathrm{t})$ the remaining time to expiration, and $\mathrm{BA}_n(t)$ the percentage bid–ask spread at date t of the option (i.e., (Ask − Bid)/ 0.5 (Ask + Bid)], all of which are contract-specific variables. The variable, $\mathrm{VJ}_n(t)$, is the date-t *skewness premium* introduced by Bates (1991) as a proxy for fears of a jump in volatility, and is included in the regression to see whether volatility jumps cause systematic pricing biases. As volatility shocks tend to follow negative rather than positive shocks to the value of the underlying asset in the stock market, this study uses the *skewness premium* introduced by Bates (1991) as a proxy for fears of a jump in volatility:

$$\text{skew}_t(T^{SPX}) \equiv \frac{C_t^{SPX}(F_t(T^{SPX}), T^{SPX}, X_C^{SPX})}{P_t^{SPX}(F_t(T^{SPX}), T^{SPX}, X_P^{SPX})} - 1 \tag{13}$$

where $\text{skew}_t(T^{SPX})$ is the *skewness premium* at time t for SPX options maturating at T^{SPX}, i.e., two days after the comparable maturity T of the VIX option. C_t^{SPX} and P_t^{SPX} are the prices of call and put options on the SPX as a function of the time-t SPX forward price, $F_t(T^{SPX})$, and the strike prices X_C^{SPX} and X_P^{SPX}. The strike prices of both options are defined to be $x\%$ of out-of-the-money ($x > 0$) and spaced geometrically around the SPX forward price in the following way:

$$X_P^{SPX} = \frac{F_t(T^{SPX})}{1 + x} < F_t(T^{SPX}) < F_t(T^{SPX})(1 + x) = X_C^{SPX}. \tag{14}$$

As described in Bates (1991), the *skewness premium* can be used as a diagnostic for the symmetry or skewness in the risk-neutral distribution implicit in option prices. Theoretically, negative skewness in the risk-neutral distribution reflects either the existence of crash fears or that volatility is expected to rise if the market falls. As unexpected increases in volatility are much more often associated with negative shocks to the underlying in stock markets than with positive shocks (Andersen, Bollerslev, Diebold, & Ebens, 2001; Figlewski & Wang, 2000; French, Schwert, & Stambaugh, 1987; Low, 2004), this study assumes this is a good proxy for the markets' expectations of a positive jump in volatility. The assumption of a strong negative relationship between rates of SPX returns and volatility in the stock market is confirmed by the negative estimates of ρ and $\text{cov}(d \ln S, dv)$ in Table II. This study uses SPX options prices that are 4% out-of-the-money to calculate the *skewness premium*. As options exist only for specific strike prices, this study interpolates the relevant option prices by fitting a constrained cubic spline through observed option price/forward price ratios as a function of observed strike price/forward price ratios (see Appendix A in Bates, 2000, for detailed information on the calculation of these constrained cubic splines). Similar to Bates (1991, 1997, 2000), this study requires that prices exist for at least four call strikes and four put strikes.

In some sense, the contract-specific variables help detect the existence of cross-sectional pricing biases, whereas VJ(t) serves to indicate whether the pricing errors over time are related to dynamically changing market conditions. Table IV reports the regression results based on the entire sample period, where the standard error for each coefficient estimate is adjusted according to

TABLE IV
Regression Analysis of Out-of-Sample Pricing Errors

Coefficient	*HESTON*	*BATES*	*DPS*	*ERAKER*
Constant	−1.73**	1.52*	1.32*	1.29*
	(−2.18)	(1.66)	(1.65)	(1.89)
mon	−2.25*	−2.01*	−1.24*	−1.53*
	(−1.65)	(−1.89)	(−2.04)	(−1.77)
τ_C	0.15*	0.17*	−0.21*	−0.22**
	(1.93)	(1.66)	(−2.98)	(−2.95)
BA	1.75**	1.42**	2.53**	1.09**
	(5.81)	(4.73)	(5.12)	(5.35)
VJ	2.31**	1.15**	−0.55**	−0.34**
	(2.66)	(2.82)	(−2.73)	(−2.68)
Adj. R^2	0.14	0.06	0.06	0.02

Note. The regression results are based on the equation:

$$\varepsilon_n(t) = \beta_0 + \beta_1 \,\text{mon}_n(t) + \beta_2\, \tau_{C,n}(t) + \beta_3\, \text{BA}_n(t) + \beta_5\, \text{VJ}_n(t) + u_n(t)$$

where $\varepsilon_n(t)$ is the percentage pricing error of the nth call on date t. $\text{mon}_n(t)$ indicates the option *moneyness*, defined as $F_t^{VIX}(T)/K_n$ for calls and $K_n/F_t^{VIX}(T)$ for puts. $\tau_{C,n}(t)$ represents the time-to-expiration of the option contract. The variable $\text{BA}_n(t)$ is the percentage bid–ask spread. $\text{VJ}_n(t)$ proxies the volatility jump computed from Bates' (1991) skewness premium using the ratio of deep out-of-the-money S&P 500 index puts and calls with comparable maturity to the VIX option. The standard errors, reported in parentheses, are White's (1980) heteroskedasticity consistent estimators. The percentage pricing errors are obtained using the parameters implied by all of the previous month's options. The sample period is March 22, 2006–November 30, 2007 giving a total of 46,599 observations. HESTON, BATES, DPS, and ERAKER, respectively, stand for the stochastic-volatility model, the stochastic-volatility model with random price jumps, the stochastic-volatility model with state-independent but correlated jumps in both S&P 500 index returns and their volatility, and the stochastic-volatility model with state-dependent and correlated jumps in both S&P 500 index returns and their volatility.

White's (1980) heteroskedasticity-consistent estimator and is given in the parentheses. Regardless of the model, each independent variable has statistically significant explanatory power for the remaining pricing errors. That is, the pricing errors from each model have some moneyness, maturity, volatility jump, and bid—ask spread-related biases. The magnitude and sign of each such bias, however, differ among the models. The pricing errors due to the two models with volatility jumps are always biased in the same direction. To look at some point estimates, the HESTON and the BATES percentage pricing errors are on average 1.75 and 1.42 points higher when the bid–ask spread BA(t) increases by one point, whereas the DPS and the ERAKER percentage errors are, respectively, 2.53 and 1.09 points higher in response. Other noticeable patterns include the following. The HESTON and BATES pricing errors are more sensitive to VJ(t) than are the DPS and the ERAKER pricing errors, which are weakly decreasing in the SPX's volatility jumps. This confirms that modeling volatility jumps is important. The deeper in-the-money are the options, the lower is the mispricing of the four models. In contrast to the DPS and ERAKER models, mispricing for the HESTON and BATES models is increasing with the options' time to maturity. Even though the pricing errors of all four models are, in most cases, significantly related to each independent variable, the collective explanatory power of these variables is quite high only for the HESTON model but not so for the others. The adjusted R^2 is 14% for the HESTON formula's pricing errors, 6% for BATES, 6% for DPS, and 2% for ERAKER.

CONCLUSION

This study derives closed-form pricing formulas for VIX options that reconcile the most general price processes of the SPX in the literature: stochastic volatility, price jumps, and volatility jumps. Using both VIX options and VIX futures, we examine several alternative models for the internal consistency of implied parameters from options and relevant parameters from time-series data. The out-of-sample pricing error of each model is used to examine the overfitting problem and the explanatory power of complex jump specifications.

Our empirical evidence indicates that regardless of performance yardstick, taking price jumps and volatility jumps into account is of first-order importance in improving upon the VIX option pricing formulas. In terms of internal consistency, the HESTON, BATES, DPS, and ERAKER models remain significantly misspecified. In particular, the four models rely on implausible levels of volatility variation of the forward VIX to rationalize the observed option prices. However, such structural misspecifications do not necessarily preclude these models from performing better otherwise. According to the out-of-sample pricing measures, adding the random price jump feature to the HESTON model can further improve its performance, especially in pricing SR VIX options; whereas modeling volatility jumps can enhance the fit of LR VIX options. For the BATES, DPS, and ERAKER models, the remaining pricing errors show the least contract-specific or market-conditions-related biases. Overall, the two performance yardsticks employed in this study can rank a given set of models differently as they capture and reveal distinct aspects of a pricing model. Our results support the claim that a model with stochastic volatility and state-dependent correlated jumps in SPX returns and volatility (i.e., the ERAKER model) is a better alternative to the others in terms of pricing VIX options.

APPENDIX A

Closed-Form Solution to the VIX Call Option

Following the work of Feller (1971) and Kendall and Stuart (1977), one can recover risk-adjusted probabilities, Π_1 and Π_2, from respective characteristic functions $f_1(t,\ \tau_C;\ i\phi)$ and $f_2(t,\ \tau_C;\ i\phi)$ of the log VIX squared as shown in Equations (7) and (8). Inserting the conjectured solution in Equation (6) into Equation (5) produces the PDEs for the risk-neutralized probabilities, Π_j for $j=1,\ 2$. The corresponding characteristic functions for Π_j also satisfy a similar PDE with the boundary condition $f_2(t+\tau_C,\ 0;\ i\phi) = \exp(i\phi \ln \mathrm{VIX}_T^2)$. The closed-form expressions of the characteristic functions are given by

$$f_1(t,\ \tau_C;\ i\phi) = \frac{f_2(t,\ \tau_C;\ i\phi + 1/2)}{f_2(t,\ \tau_C;\ 1/2)}$$

$$f_2(t,\ \tau_C;\ i\phi) = \exp[C_2(\tau_C) + J_2(\tau_C) + D_2(\tau_C)\ln \mathrm{VIX}_t^2]$$

where

$$C_2(\tau_C) = \frac{B}{A}\kappa_\nu\tau_C - \frac{\kappa_\nu}{A}\left\{B\tau_C - \ln\left\{\frac{A}{B} + \left[\left(i\phi + \frac{B}{A}\right)^{-1} - \frac{A}{B}\right]e^{B\tau_C}\right\} + \ln\left[\left(i\phi + \frac{B}{A}\right)^{-1}\right]\right\}$$

$$D_2(\tau_C) = \frac{-B}{A} + \left\{\frac{A}{B} + \left[\left(i\phi + \frac{B}{A}\right)^{-1} - \frac{A}{B}\right] \times e^{B\tau_C}\right\}^{-1}$$

$$J_2(\tau_C) = -\lambda_0\tau_C - \lambda_1\tau_C\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{b_t}{a_\tau} - \frac{\tau\zeta_2}{a_\tau\zeta_1}\right) + \left[\lambda_0 + \lambda_1\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau\zeta_2}{a_\tau\zeta_1}\right) + \lambda_1\mu_\nu\right] \times [\vartheta_3(\mathrm{VIX}_t^2,\ t) + \vartheta_4(\mathrm{VIX}_t^2,\ t)]$$

$$A = \frac{1}{2}\sigma_\nu^2\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)^2\left(\frac{1}{\ln \mathrm{VIX}_t^2}\right)$$

$$B = \left[\kappa_\nu\theta_\nu - \frac{1}{2}\sigma_\nu^2\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\left(\frac{\tau \mathrm{VIX}_t^2}{a_\tau\zeta_1} - \frac{b_\tau}{a_\tau} - \frac{\tau\zeta_2}{a_\tau\zeta_1}\right) + \kappa_\nu\left(\frac{b_\tau}{a_t} + \frac{\tau\zeta_2}{a_\tau\zeta_1}\right)\right]\left(\frac{a_\tau\zeta_1}{\tau \mathrm{VIX}_t^2}\right)\left(\frac{1}{\ln VIX_t^2}\right)$$

$$\vartheta_3(\mathrm{VIX}_t^2,\ t) = \frac{(i\phi)\mu_\nu}{(1+M)\mathrm{VIX}_t^4}\left(1 + \frac{\mu_\nu}{\mathrm{VIX}_t^2}\right)^{i\phi}\left[\left(\tau_C - \frac{(1-\alpha_{T-t})}{(\kappa_\nu - \lambda_1\mu_\nu)}\right)\mathrm{VIX}_t^2 - \tau_C a_\tau\beta_{T-t}\frac{\zeta_1}{\tau} - \left(\frac{\zeta_1}{\tau}b_\tau + \zeta_2\right)\left(\tau_C - \frac{(1-\alpha_{T-t})}{(\kappa_\nu - \lambda_1\mu_\nu)}\right)\right]$$

$$\vartheta_4(\mathrm{VIX}_t^2,\ t) = \left[\frac{(i\phi)\mu_\nu}{(1+M)\mathrm{VIX}_t^6} + \frac{(i\phi)^2\mu_\nu^2}{2(1+M)^2\mathrm{VIX}_t^8}\right]\left(1 + \frac{\mu_\nu}{\mathrm{VIX}_t^2}\right)^{i\phi}\vartheta_5(\mathrm{VIX}_t^2,\ t)$$

$$
\begin{aligned}
\vartheta_5(\mathrm{VIX}_t^2, t) = & \left[1 + \frac{1}{2}\left(\frac{1-\alpha_{T-t}^2}{\kappa_\nu - \lambda_1\mu_\nu}\right) - 2\left(\frac{1-\alpha_{T-t}}{\kappa_\nu - \lambda_1\mu_\nu}\right)\right]\mathrm{VIX}_t^4 \\
& + \left[4\left(\frac{\zeta_1}{\tau}b_\tau + \zeta_2\right)\left(\frac{1-\alpha_{T-t}}{\kappa_\nu - \lambda_1\mu_\nu}\right) - \left(\zeta_2 + \frac{\zeta_1}{\tau}b_\tau\right)\left(\frac{1-\alpha_{T-t}^2}{\kappa_\nu - \lambda_1\mu_\nu}\right)\right. \\
& + \frac{\zeta_1 a_\tau}{\tau}\left(\frac{\kappa_\nu\theta_\nu + \lambda_0\mu_\nu}{\kappa_\nu - \lambda_1\mu_\nu}\right)\left(\frac{(1-\alpha_{T-t})(3-\alpha_{T-t})}{(\kappa_\nu - \lambda_1\mu_\nu)} - 2\tau_C\right) \\
& \left. + \frac{a_\tau\zeta_1}{2\tau}\left(\frac{\sigma_\nu^2 + 2\lambda_1\mu_\nu^2}{\kappa_\nu - \lambda_1\mu_\nu}\right)\frac{(1-\alpha_{T-t})^2}{(\kappa_\nu - \lambda_1\mu_\nu)} - 2\left(\zeta_2 + \frac{\zeta_1}{t}b_\tau\right)\right]\mathrm{VIX}_t^2 \\
& + \left(\frac{(1-\alpha_{T-t})(\alpha_{T-t}-3)}{2(\kappa_\nu - \lambda_1\mu_\nu)}\right) \times \left[\zeta_2^2 + 2\frac{\zeta_1\zeta_2}{\tau}b_\tau + \left(\frac{b_\tau\zeta_1}{\tau}\right)^2\right] \\
& + \frac{1}{2}\left(\frac{b_\tau\zeta_1}{\tau}\right)^2\left(\frac{1-\alpha_{T-t}^2}{\kappa_\nu - \lambda_1\mu_\nu}\right) + \left[\frac{a_\tau\zeta_1}{\tau}\zeta_2 + \left(\frac{a_\tau\zeta_1}{\tau}\right)^2\frac{b_\tau}{a_\tau}\right] \\
& \times \left[\left(\frac{\kappa_\nu\theta_\nu + \lambda_0\mu_\nu}{\kappa_\nu - \lambda_1\mu_\nu}\right)\left(2\tau_C - \frac{(1-\alpha_{T-t}^2)}{(\kappa_\nu - \lambda_1\mu_\nu)}\right) - \frac{1}{2}\left(\frac{\sigma_\nu^2 + 2\lambda_1\mu_\nu^2}{\kappa_\nu - \lambda_1\mu_\nu}\right)\frac{(1-\alpha_{T-t})^2}{(\kappa_\nu - \lambda_1\mu_\nu)}\right] \\
& + \left(\frac{a_\tau\zeta_1}{\tau}\right)^2\left(\frac{\kappa_\nu\theta_\nu + \lambda_0\mu_\nu}{\kappa_\nu - \lambda_1\mu_\nu}\right)\left[\frac{(1-\alpha_{T-t})(\alpha_{T-t}-3)}{2(\kappa_\nu - \lambda_1\mu_\nu)} + \tau_C\right]\left[\left(\frac{\kappa_\nu\theta_\nu + \lambda_0\mu_\nu}{\kappa_\nu - \lambda_1\mu_\nu}\right)\right. \\
& \left. + \frac{1}{2}\left(\frac{\sigma_\nu^2 + 2\lambda_1\mu_\nu^2}{\kappa_\nu - \lambda_1\mu_\nu}\right)\right] + \left(\frac{a_\tau\zeta_1}{\tau}\right)^2\left(\frac{\lambda_0\mu_\nu^2}{\kappa_\nu - \lambda_1\mu_\nu}\right)\left[\tau_C - \frac{(1-\alpha_{T-t}^2)}{2(\kappa_\nu - \lambda_1\mu_\nu)}\right] \\
& + \zeta_2^2 + 2\frac{\zeta_1\zeta_2}{\tau}b_\tau.
\end{aligned}
$$

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