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Abstract One of the earliest system that was used to asset prices description is Black-Scholes model. It is based on geometric Brownian motion and was used as a tool for pricing various financial instruments. However, when it comes to data description, geometric Brownian motion is not capable to capture many properties of present financial markets. One can name here for instance periods of constant values. Therefore we propose an alternative approach based on subordinated tempered stable geometric Brownian motion which is a combination of the popular geometric Brownian motion and inverse tempered stable subordinator. In this paper we introduce the mentioned process and present its main properties. We propose also the estimation procedure and calibrate the analyzed system to real data.

Keywords Geometric Brownian motion · Tempered inverse subordinator · Calibration

1 Introduction

Geometric Brownian motion (GBM) was served for many years as a main tool in financial modeling, since the ground-breaking work of Fischer Black, Myron Scholes and Robert Merton [4, 20]. The geometric Brownian motion (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift. It is an important example of extension for classical Brownian motion; in particular, it is used in mathematical finance to model stock prices in the Black-Scholes (BS) model. The BS model is still the most popular model of financial market. Within the framework of BS model authors

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of [4, 20] determined consistent formulas for the fair prices of European options. Two of them, namely Merton and Scholes, were awarded the Nobel Prize for Economics in 1997 due to their outstanding ideas. Nowadays, after the recent findings, we observe that many economical processes are inconsistent with this classical model. GBM cannot capture such properties of present markets as: long range correlations, heavy-tailed and skewed marginal distributions as well as periods of constant values. In order to overcome such difficulties many modifications of standard GBM have been introduced, [7].

Data analysis of many real-life data observed in economics display characteristic periods when the prices stay on the same level, [9]. This phenomenon is especially visible for prices of assets with low liquidity. This constant time periods (called also trapping events) are also observed in variety of physical systems exhibiting subdiffusion. One can name here: in biophysical context [11, 25, 26], charge carrier transport in amorphous semiconductors [22], single molecule spectroscopy [1]. For a review and discussion see [15, 21]. Subdiffusion is in statistical physics a well established phenomenon. Its mathematical description is in terms of fractional Fokker-Planck equation (FFPE) based on the Continuous Time Random Walk (CTRW) scenario with heavy-tailed waiting times [2, 21]. This heavy-tailed waiting times correspond to periods of constant values in which a test particle gets immobilized. Equivalently, one can represent subdiffusion in terms of subordinated Langevin equation [16, 17]. Such representation reveals that subdiffusion is a combination of two independent mechanisms: the first one is based on standard diffusion represented by some Itô process $X(\tau)$, where the second one is the waiting time distribution represented by the so-called inverse α-stable subordinator $S_\alpha(t)$.

Instead of heavy-tailed distributions one can use more general class of distributions, namely nonnegative infinitely divisible (ID), equivalently ID inverse subordinators $S_\Psi(t)$ [18]. The ID distributions with special importance are one-sided Lévy stable, Pareto, Mittag-Leffler, gamma and tempered stable distributions. Among of all this distributions the tempered stable are the most appropriate in modeling of waiting times, especially in cases between sub and normal diffusion [5, 6]. Tempered stable distributions posses many interesting properties (i.e. finite moments of all orders) but they stay close to the purely α-stable distributions [24]. Popularity of tempered stable distributions resulted in many applications for instance in physics to description of anomalous diffusion [5, 6], biology [8], plasma physics [12] and finance [13, 14].

In this paper we consider tempered stable geometric Brownian motion that extends the classical GBM through the use of tempered stable subordinator. More general model with ID subordinator was recently applied to option pricing [19]. The subdiffusive GBM with tempered stable waiting-times demonstrates the subdiffusive behavior for short times and Gaussian for large times. Combination of classical GBM with inverse tempered stable subordinator makes this model not complicated from the theoretical point of view. From the practical point of view the parameters of the model can be easily estimated through the similar methods presented in [23].

The rest of the paper is organized as follows: in Sect. 2 we introduce the subordinated GBM driven by inverse tempered stable process. In Sect. 3 we present main properties of examined system while in Sect. 4—details of the estimation procedure for unknown parameters of subordinated GBM. We also check the effectiveness of this method using Monte Carlo simulations. In Sect. 5 we calibrate the examined process to real financial data. Last section contains conclusions.

2 GBM with Tempered Stable Waiting Times

2.1 Inverse Tempered Stable Subordinator

The inverse tempered stable subordinator is the process $\{S_{\alpha,\lambda}(t)\}$ defined as follows:

$$S_{\alpha,\lambda}(t) = \inf\{\tau > 0 : T_{\alpha,\lambda}(\tau) > t\}, \quad t \geq 0,$$

where $\{T_{\alpha,\lambda}(\tau)\}$ is tempered α-stable Lévy process defined via its Laplace transform:

$$E\left(e^{-uT_{\alpha,\lambda}(\tau)}\right) = e^{-\tau((u+\lambda)^{\alpha}-\lambda^{\alpha})}. \tag{1}$$

The parameter $\lambda > 0$ is called the tempering index (or tempering parameter) while $0 < \alpha < 1$—the stability index. There is a strong relation between the tempered stable Lévy process and strictly increasing α-stable process examined in [3, 27], namely the probability density function (PDF) of $T_{\alpha,\lambda}(\tau)$ can be formulated as:

$$p_{T_{\alpha,\lambda}(\tau)}(x) = e^{-\lambda x+\lambda^{\alpha}\tau} p_{S_{\alpha,\tilde{\sigma},1,0}(\tau)}(x), \tag{2}$$

where $\tilde{\sigma} = (\cos\frac{\pi\alpha}{2})^{1/\alpha}$ and $p_{S_{\alpha,\tilde{\sigma},\beta,\eta}(\tau)}(\cdot)$ is the PDF of the α-stable Lévy motion with the index of stability α, scale parameter $\tilde{\sigma}$, skewness β and shift η. One observes that for $\lambda = 0$ the tempered stable Lévy process defined by its Laplace transform in (1) is strictly increasing α-stable process. Because for the α-stable distribution the PDF for large values of arguments can be approximated by the power function $x^{-(\alpha+1)}$, therefore:

$$p_{T_{\alpha,\lambda}(\tau)}(x) \sim e^{-\lambda x} x^{-(\alpha+1)}, \quad x \to \infty. \tag{3}$$

The above relation provides to evident fact that the right tail of tempered stable Lévy process $\{T_{\alpha,\lambda}(\tau)\}$ behaves like $e^{-\lambda x} x^{-\alpha}$, see [27]. This result can be useful in the estimation problem of unknown parameters from tempered stable distribution (see Sect. 4).

The relation between tempered stable and inverse tempered stable processes is discussed in details in [27]. We only mention here the relation between PDFs of such systems, namely:

$$f_{S_{\alpha,\lambda}(t)}(x) = \frac{1}{\alpha x} t f_{T_{\alpha,\lambda}(x)}(t) + \lambda \frac{1}{\alpha x} \int_0^t u f_{T_{\alpha,\lambda}(x)}(u)du - \lambda^{\alpha} \int_0^t f_{T_{\alpha,\lambda}(x)}(u)du, \tag{4}$$

where $f_{S_{\alpha,\lambda}(t)}(\cdot)$ and $f_{T_{\alpha,\lambda}(\tau)}(\cdot)$ denote PDFs of inverse tempered stable $\{S_{\alpha,\lambda}(t)\}$ and tempered stable $\{T_{\alpha,\lambda}(\tau)\}$ Lévy processes, respectively.

2.2 Subordinated GBM Driven by Inverse Tempered Stable Subordinator

We consider GBM subordinated with tempered stable waiting times i.e. a process $\{Y(t)\}$ defined as follows [19]:

$$Y(t) = X\left(S_{\alpha,\lambda}(t)\right), \tag{5}$$

where $\{X(\tau)\}$ is GBM with the drift parameter $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$ given by the following stochastic differential equation:

$$dX(t) = X(t)\sigma dB(t) + X(t)\left(\mu + \frac{1}{2}\sigma^2\right)dt, \tag{6}$$

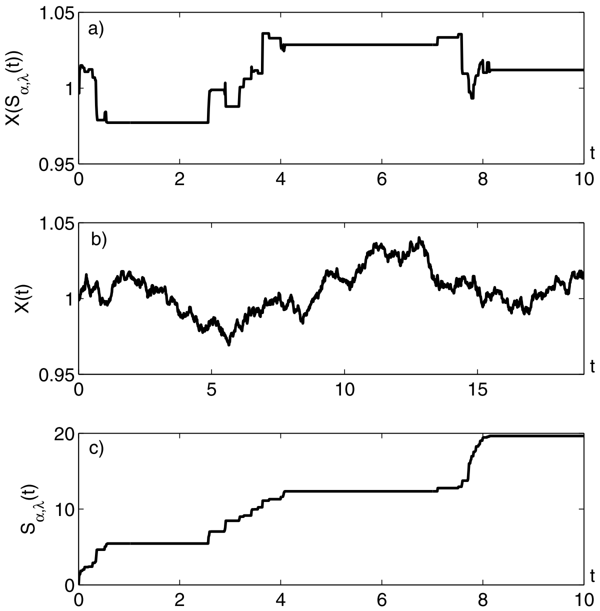

Fig. 1 An exemplary sample path of the subdiffusive process $X(S_{\alpha,\lambda}(t))$ on the interval [0, 10] (panel **a**), classical geometric Brownian motion $X(t)$ (panel **b**) and inverse tempered stable subordinator $S_{\alpha,\lambda}(t)$ (panel **c**). The parameters are: $\alpha = 0.3$, $\lambda = 0.55$, $\mu = 0$, $\sigma = 0.012$

where $\{B(t)\}$ is the standard Brownian motion. Moreover we assume the processes $\{X(t)\}$ and $\{S_{\alpha,\lambda}(t)\}$ are independent. The $\{S_{\alpha,\lambda}(t)\}$ is responsible for trapping events i.e. it slows down the overall motion thus in trajectory we observe periods of stagnation. Adapting inverse tempered stable subordinator as a new clock of the system we are able to model a real data with visible periods of constant values.

The simulation procedure for the process $\{Y(t)\}$ one can find in [5, 19]. In Fig. 1 we present an exemplary sample path of the $\{Y(t)\}$, $\{X(t)\}$ and the $\{S_{\alpha,\lambda}(t)\}$. The constant periods in the trajectory of $\{Y(t)\}$ correspond to waiting-times distributed according to tempered stable law.

3 Properties of Subordinated GBM with Tempered Stable Inverse Subordinator

In this section we analyze the basic characteristics like the first moment and variance for GBM with tempered stable waiting times that can be useful in applications of such process. Let us first note that geometric Brownian motion given by Eq. (6) have the following solution:

$$X(t) = X(0)\exp\bigl\{\mu t + \sigma B(t)\bigr\}.$$

In the further analysis we assume $X(0) = 1$ with probability 1. Thus it follows that $\{X(t)\}$ has lognormal distribution with parameters μt and $\sigma^2 t$, (LN(μt, $\sigma^2 t$)), i.e. for given $t > 0$, $X(t)$ has following PDF:

$$f_{X(t)}(x) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} e^{-(\log(x)-\mu t)^2/2\sigma^2 t}, \quad x > 0. \tag{7}$$

Moreover the distribution function (DF) can be expressed as

$$F_{X(t)}(x) = \Phi\left(\frac{\log(x) - \mu t}{\sigma\sqrt{t}}\right), \quad x > 0, \tag{8}$$

where $\Phi(\cdot)$ is a DF of standard normal distribution. The main characteristics, such as first and second central moments of $X(t)$, take following forms:

$$E\big(X(t)\big) = e^{\mu t + \frac{1}{2}\sigma^2 t}, \tag{9}$$

and

$$Var\big(X(t)\big) = e^{2\mu t + \sigma^2 t}\big(e^{\sigma^2 t} - 1\big). \tag{10}$$

Using Eq. (9) we can derive expression for the first moment of the subordinated geometric Brownian motion with tempered stable inverse subordinator, namely:

$$E\big(X\big(S_{\alpha,\lambda}(t)\big)\big) = E\big(E\big(X(\tau)|S_{\alpha,\lambda}(t) = \tau\big)\big) = E\big(e^{S_{\alpha,\lambda}(t)(\mu + \frac{1}{2}\sigma^2)}\big). \tag{11}$$

The last expression is simply Laplace transform of $S_{\alpha,\lambda}(t)$. Thus using numerical integration one can calculate the first moment of geometric Brownian motion with tempered stable waiting times via:

$$E\big(X\big(S_{\alpha,\lambda}(t)\big)\big) = \int_0^\infty e^{(\mu + \frac{1}{2}\sigma^2)\tau} f_{S_{\alpha,\lambda}(t)}(\tau)d\tau, \tag{12}$$

where $f_{S_{\alpha,\lambda}(t)}(\cdot)$ is given in (4).

For the variance we have:

$$\begin{aligned}
Var\big(X\big(S_{\alpha,\lambda}(t)\big)\big) &= E\big(Var\big(X(\tau)|S_{\alpha,\lambda}(t) = \tau\big)\big) + Var\big(E\big(X(\tau)|S_{\alpha,\lambda}(t) = \tau\big)\big) \\
&= E\big(e^{2\mu S_{\alpha,\lambda}(t) + \sigma^2 S_{\alpha,\lambda}(t)}\big(e^{\sigma^2 S_{\alpha,\lambda}(t)} - 1\big)\big) + Var\big(e^{\mu S_{\alpha,\lambda}(t) + \frac{1}{2}\sigma^2 S_{\alpha,\lambda}(t)}\big) \\
&= E\big(e^{2\mu S_{\alpha,\lambda}(t) + 2\sigma^2 S_{\alpha,\lambda}(t)}\big) - E\big(e^{2\mu S_{\alpha,\lambda}(t) + \sigma^2 S_{\alpha,\lambda}(t)}\big) \\
&\quad + E\big(e^{2\mu S_{\alpha,\lambda}(t) + \sigma^2 S_{\alpha,\lambda}(t)}\big) - E^2\big(e^{\mu S_{\alpha,\lambda}(t) + \frac{1}{2}\sigma^2 S_{\alpha,\lambda}(t)}\big) \\
&= E\big(e^{2\mu S_{\alpha,\lambda}(t) + 2\sigma^2 S_{\alpha,\lambda}(t)}\big) - E^2\big(e^{\mu S_{\alpha,\lambda}(t) + \frac{1}{2}\sigma^2 S_{\alpha,\lambda}(t)}\big) \\
&= \int_0^\infty e^{2\tau(\mu + \sigma^2)} f_{S_{\alpha,\lambda}(t)}(\tau)d\tau - \left(\int_0^\infty e^{\tau(\mu + \frac{1}{2}\sigma^2)} f_{S_{\alpha,\lambda}(t)}(\tau)d\tau\right)^2.
\end{aligned}$$

In Fig. 2 we present sample moments of subordinated geometric Brownian motion with tempered stable waiting times (with $\alpha = 0.3$, $\lambda = 0.55$, $\mu = 0$ and $\sigma = 0.012$) calculated via Monte Carlo methods (1000 trajectories).

4 Estimation Procedure

The estimation procedure for subordinated GBM driven by inverse tempered stable process proceeds as follows:

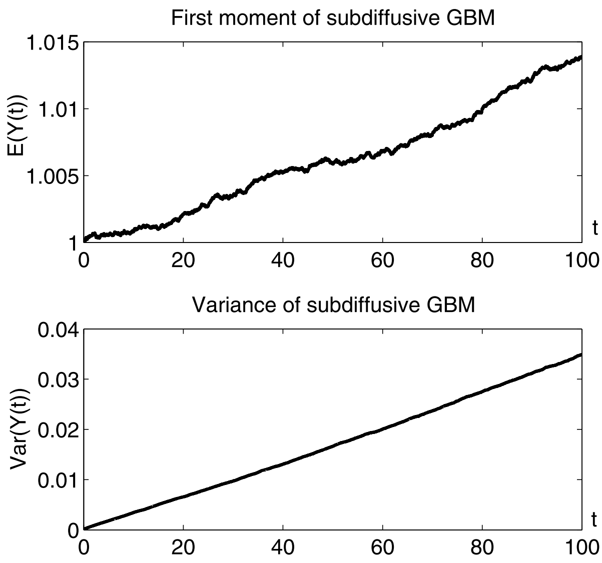

Fig. 2 Sample mean (*top panel*) and variance (*bottom panel*) of subdiffusive geometric Brownian motion with parameters $\alpha = 0.3$, $\lambda = 0.55$, $\mu = 0$ and $\sigma = 0.012$

– In the first step we decompose the vector $Y(t_1), Y(t_2), \ldots, Y(t_n)$ into two separate time series. The first one, represents lengths of constant time periods (we call it by L) that are typical for subdiffusive processes, while the second one arises after removing the constant time periods (this vector we call P). This approach was used in [9, 10] for the α-stable subordinator but also in [23] in tempered stable case.
– The vector L, that represents constant time periods constitute sample from tempered stable distribution, i.e. realizations of the process $\{T_{\alpha,\lambda}(t)\}$ defined in (1) by its Laplace transform. In order to estimate the parameters α and λ we propose to use the tail behavior of tempered stable distribution. Namely, to empirical right tail we fit the function $e^{-\lambda x}x^{-\alpha}$ using the least squares method. Similar approach was used in [9, 10] in case of α-stable subordinator. In this case to the right tail we fitted the power function of the form $x^{-\alpha}$. Another approach was used in [23] where authors estimated the parameters α and λ of the tempered stable subordinator using method of moments.
– After estimation of the α and λ parameters, we can estimate the μ and σ indexes. Since the vector P is related to the external process $\{X(t)\}$ then in this case the difference of logarithm of this series has Gaussian distribution with mean μ and variance σ^2. To estimate these parameters we use the maximum likelihood method.

In order to show the effectiveness of the presented procedure we use the Monte Carlo simulations. We simulate 1000 trajectories from subordinated GBM driven by tempered stable inverse subordinator on the interval $[0, 1]$ with parameters $\alpha = 0.3$, $\lambda = 0.55$, $\mu = 0$ and $\sigma = 0.012$, then we estimate the unknown parameters using the presented methodology. In Fig. 3 we present the boxplots of obtained estimators. As we observe, the calculated values are close to the theoretical ones.

5 Applications

In this section we examine real data set that describes Germany Interbank Rates (BBA LIBOR, 12 Month) expressed in EUR from the period 12.06.1996–11.05.2005. In Fig. 4 we

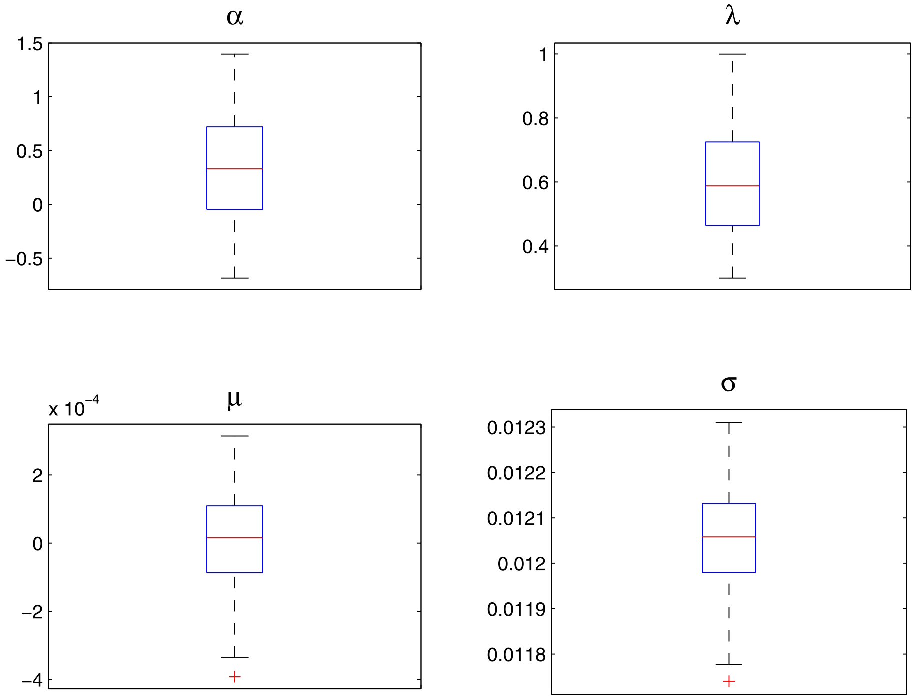

Fig. 3 The boxplots of estimators for unknown parameters of subordinated GBM driven by tempered stable inverse subordinator. The theoretical values are: $\alpha = 0.3$, $\lambda = 0.55$, $\mu = 0$ and $\sigma = 0.012$

present the analyzed time series. One observes characteristic trap-behavior which is typical for subdiffusive processes. This observed constant time periods indicate that the data cannot be modeled using another system adequate for subdiffusive processes like for example the persistent fractional Brownian motion.

In the further analysis, we take every observation from the underlying series with the accuracy 0.01. Next, we divide considered data into two vectors, one represents lengths of the observed periods of stagnation (vector L), while another describes data after removing the constant time periods (vector P). We stress at this point that constant time period of length n take place if there are $n+1$ the same values come one after another. From theoretical considerations we deduce that vector L constitute the lengths of constant periods of the inverse tempered subordinator. Number of traps detected in the series is 453 and it states that tempered subordinator is a significant component of the data, see Fig. 5. The rest of the series, namely vector P, has lognormal distribution.

To reject the hypothesis that the purely α-stable distribution better describes waiting-times behavior and sustain the hypothesis that tempered stable distribution is appropriate in this case, we have examined vector L. From this data set applying McCulloch, regression and moments method as well as method based on the behavior of the right tail, [9], we have estimated the α parameter under the assumption that lengths of the traps forms an i.i.d. series from strictly α-stable distribution. All estimators returned $\hat{\alpha} > 1$ which contradicts the assumption about stable distribution of the subordinator, thus in α-stable case we should obtain $0 < \alpha < 1$. As an alternative to purely stable distribution we propose tempered stable one. Estimation of the parameters, calculated by using the method based on the tail behavior,

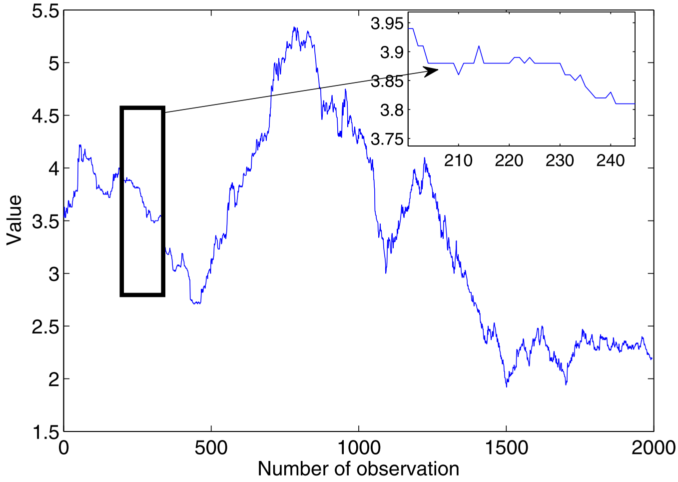

Fig. 4 The examined real data set of Germany, Interbank Rates expressed in EUR. The considered period is 12.06.1996–11.05.2005. The data demonstrate characteristic trap-behavior typical for the subdiffusive processes

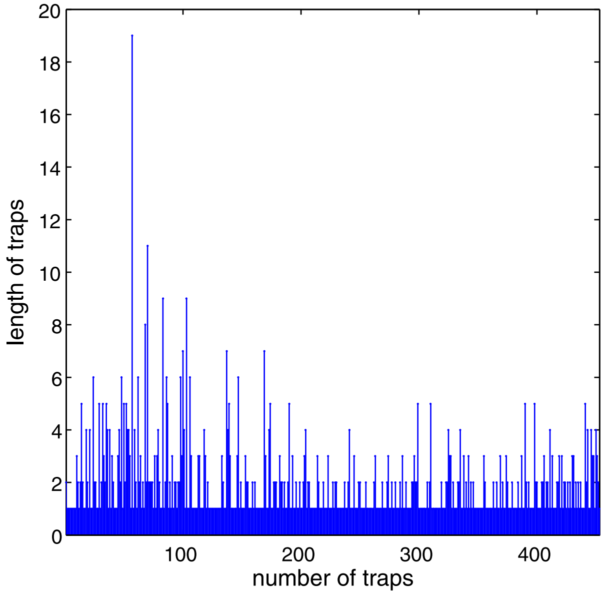

Fig. 5 The lengths of constant time periods (vector L)

returns:

$$\hat{\alpha} = 0.28, \qquad \hat{\lambda} = 0.52. \tag{13}$$

For the vector P after calculation the empirical mean and standard deviation from the differenced (with order 1) logarithm of the series we obtain:

$$\hat{\mu} = -0.004, \qquad \hat{\sigma} = 0.0128. \tag{14}$$

6 Conclusions

Subdiffusive geometric Brownian motion with tempered stable waiting-times is the most appropriate model for intermediate cases between sub and normal diffusion. We have shown the main characteristics of analyzed system and present in details the new estimation procedure. We have applied successfully the examined model to data describing Germany Interbank Rates. We believe that proposed model and presented methodology will serve as one of the useful tool in real data analysis.

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