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Abstract - In this paper, we obtain an upper bound of the modulus of continuity of linear multifractional multistable random motions. Such processes are generalizations of linear multifractional α-stable motions for which the stability index α is also allowed to vary in time. In the case of linear multifractional α-stable motions, we improve the recent result of [2]. The main idea is to consider some conditionnally sub-Gaussian LePage series representations to fit the framework of [5].

Key words and phrases : stable and multistable random fields, modulus of continuity.

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1 Introduction

Self-similar random fields are required to model persistent phenomena in internet traffic, hydrology, geophysics or financial markets, e.g. [1, 22]. The fractional Brownian motion ([15, 9]) provides the most famous self-similar model. Nevertheless, in image modeling, in finance or in biology for example, the phenomena under study are rarely Gaussian. Then, α-stable random processes have been proposed as an alternative to Gaussian modeling, since they allow to model data with heavy tails, such as in internet traffic [16]. The linear fractional stable motion, which has been proposed in [21, 14], is one of the numerous stable extensions of the fractional Brownian motion. Let us recall how this self-similar random motion can be defined through a stochastic integral representation. To this way, let us consider $H_1 \in (0,1)$, $\alpha_1 \in (0,2)$ and M_{α_1} a real-valued symmetric α_1-stable random measure with Lebesgue control measure (see [17] p.281 for details on such measures). Then, a linear fractional stable motion is defined by

$$X_{\alpha_1,H_1}(t) = \int_{\mathbb{R}} f_+(\alpha_1, H_1, t, \xi) M_{\alpha_1}(d\xi), \quad t \in \mathbb{R} \tag{1.1}$$

where f_+ is defined by

$$f_+(\alpha_1, H_1, t, \xi) = (t-\xi)_+^{H_1-1/\alpha_1} - (-\xi)_+^{H_1-1/\alpha_1} \tag{1.2}$$

with for $c \in \mathbb{R}$,

$$(x)_+^c = \begin{cases} x^c & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Since the self-similarity property is a global property which can be too restrictive for applications, a multifractional generalization $X_{\alpha_1,h}$ of this process has also been introduced by [18] to model internet traffic, by replacing H_1 by a real function h with values on $(0,1)$. Some necessary and sufficient conditions for the stochastic continuity of the linear multifractional stable motion $X_{\alpha_1,h}$ have been given in [18] and its Hölder sample path regularity has been studied in [19]. The Hölder sample path properties have also been improved in [2] by establishing upper and lower bounds for the modulus of continuity. In the following, we will improve the upper bound, using the results we established in [5]. Let us mention that in the case where $h \equiv H_1$ is constant, that is when $X_{\alpha_1,h}$ is a linear fractional stable motion, sample path regularity properties have previously been studied in [17, 20, 10].

Moreover, the framework of [5] allows to study $X_{\alpha_1,h}$ as well as some multistable generalizations for which the stability index α_1 is also allowed to vary with t. Multistable processes have been defined in [7] using sums over Poisson processes or in [6] using a Klass-Ferguson LePage series.

In this paper we consider a random field S_m defined using a Lepage series representation of the linear fractional α_1-stable motion and such that

$$S_m(\alpha(t), h(t), t), \quad t \in \mathbb{R}$$

is a linear multifractional multistable motion. This auxiliary random field S_m allows to study the variations due to the functions α, h and to the position t separately. Then, to study sample path regularity of linear multistable motions, our first step is to establish an upper bound for the modulus of continuity of the field S_m considering a conditionnally sub-Gaussian representation and applying [5]. The main property of sub-Gaussian random variables, which have been introduced by [8], is that their tail distributions decrease exponentially as the Gaussian ones. This property is one of the main tool used in [5] to study the sample path regularity property of conditionnally sub-Gaussian random series.

The paper is organized as follows. Section 2 introduces LePage series random fields under study. An upper bound of their modulus of continuity and a rate of convergence are stated in Section 3. Section 4 focuses on linear multifractional multistable motions. Some technical proofs are postponed to the appendix for reader convenience.

2 LePage series models

In order to define LePage series, let us introduce some notation.

Hypothesis 2.1 Let $(g_n)_{n\geq 1}$, $(\xi_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ be three independent sequences of random variables satisfying the following conditions.

1. $(g_n)_{n\geq 1}$ is a sequence of independent identically distributed (i.i.d.) real-valued symmetric sub-Gaussian random variables, that is such that there exists $s \in [0, +\infty)$ for which
$$\forall \lambda \in \mathbb{R},\ \mathbb{E}(\mathrm{e}^{\lambda g_n}) \leq \mathrm{e}^{\frac{s^2\lambda^2}{2}}. \tag{2.3}$$

2. $(\xi_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with common law
$$\mu(d\xi) = m(\xi)d\xi$$
equivalent to the Lebesgue measure (that is such that $m(\xi) > 0$ for almost every ξ).

3. T_n is the nth arrival time of a Poisson process with intensity 1.

Let us now introduce the random field $(S_m(\alpha, H, t))_{(\alpha,H,t)\in(0,2)\times(0,1)\times\mathbb{R}}$ we study in this paper.

Proposition 2.1 (LePage series representation) *Assume that Hypothesis 2.1 is fulfilled and let f_+ be defined by* (1.2)*. Then, for any $(\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$, the sequence*
$$S_{m,N}(\alpha, H, t) = \sum_{n=1}^{N} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad N \geq 1 \tag{2.4}$$
converges almost surely and its limit is denoted by
$$S_m(\alpha, H, t) := \sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n. \tag{2.5}$$

Proof. Let $(\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$. Then, since Hypothesis 2.1 holds, the variables
$$W_n := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad n \geq 1,$$
are i.i.d., symmetric and such that
$$\mathbb{E}(|W_1|^\alpha) = \mathbb{E}(|g_1|^\alpha) \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^\alpha \, d\xi < +\infty,$$
since g_1 and ξ_1 are independent (see e.g. [17]). Therefore, by Theorem 5.1 of [13], the sequence
$$\left(\sum_{n=1}^{N} T_n^{-1/\alpha} W_n \right)_{N\geq 1}$$

converges almost surely as $N \to +\infty$, that is $(S_{m,N}(\alpha, H, t))_{N \geq 1}$ converges almost surely. $\square$

Let us conclude this section by some remarks.

Remark 2.1 According to Proposition 5.1 of [5], the finite dimensional distributions of S_m do not depend on m as soon as Condition 2 of Hypothesis 2.1 holds. Moreover, when studying the sample path regularity of S_m, Proposition 5.1 of [5] allows us to change m by a more convenient function $\tilde{m}$ if necessary.

Remark 2.2 When $\alpha = \alpha_1 \in (0, 2)$ is fixed, $(S_m(\alpha_1, H, t))_{(H,t)\in(0,1)\times\mathbb{R}}$ is an α_1-stable symmetric random field, which can also be represented as an integral under an α_1-stable random measure M_{α_1} with Lebesgue control measure. More precisely, for every $\alpha_1 \in (0, 2)$,

$$(S_m(\alpha_1, H, t))_{(H,t)\in(0,1)\times\mathbb{R}} \overset{fdd}{=} d_{\alpha_1}(Y_{\alpha_1}(H, t))_{(H,t)\in(0,1)\times\mathbb{R}} \tag{2.6}$$

where $\overset{fdd}{=}$ means equality of finite distributions and

$$Y_{\alpha_1}(H, t) := \int_{\mathbb{R}} f_+(\alpha_1, H, t, \xi) M_{\alpha_1}(d\xi), \quad (H, t) \in (0, 1) \times \mathbb{R}, \tag{2.7}$$

for M_{α_1} a real-valued symmetric α_1-stable random measure with Lebesgue control measure and

$$d_{\alpha_1} := \mathbb{E}(|g_1|^{\alpha_1})^{1/\alpha_1} \left(\int_0^{+\infty} \frac{\sin x}{x^{\alpha_1}} dx \right)^{1/\alpha_1}. \tag{2.8}$$

One can check Equation (2.6) following the proof of Proposition 5.1 of [5] or Proposition 4.2 of [4], which is a consequence of Lemma 4.1 of [11].

3 Sample path properties

Several papers [20, 10, 18, 19, 2] have already investigated sample path properties of the linear fractional stable motion X_{α_1,H_1} defined by Equation (1.1) or of its multifractional generalization $X_{\alpha_1,h}$ defined on $\mathbb{R}$ by

$$X_{\alpha_1,h}(t) := Y_{\alpha_1}(h(t), t), \quad t \in \mathbb{R} \tag{3.9}$$

where $\alpha_1 \in (0, 2)$, Y_{α_1} is given by (2.7) and h is a function with values in $(0, 1)$. In the following, we improve the upper bound of the global modulus of continuity of $X_{\alpha_1,h}$ stated in [2]. Our first step is to establish an upper bound for the global modulus of continuity of the field S_m defined by (2.5) on a compact set K of $(0, 2) \times (0, 1) \times \mathbb{R}$. To obtain our upper bound, we use the results we established in [5] on conditionally sub-Gaussian random series.

Let us first recall (see [17] for example) that the α_1-stable random process $X_{\alpha_1,H_1} = (Y_{\alpha_1}(H_1,t))_{t\in\mathbb{R}}$ is unbounded almost surely on each compact set with non-empty interior when $H_1 < 1/\alpha_1$. A similar result holds for S_m as stated in the following proposition.

Proposition 3.1 *Assume that* $K = [\alpha_1,\alpha_2] \times [H_1,H_2] \times [a,b] \subset (0,2) \times (0,1) \times \mathbb{R}$ *with* $0 < \alpha_1 \leq \alpha_2 < 2$, $0 < H_1 \leq H_2 < 1$ *and* $a < b$.

1. *If* $H_1 < 1/\alpha_1$, *then the random field* S_m *is almost surely unbounded on* K.
2. *If* $H_1 = 1/\alpha_1$, *then* S_m *does not have almost surely continuous sample paths on the compact set* K.

Proof. By Equation (2.6)

$$(S_m(\alpha_1,H_1,t))_{t\in\mathbb{R}} \stackrel{fdd}{=} d_{\alpha_1}(X_{\alpha_1,H_1}(t))_{t\in\mathbb{R}}, \tag{3.10}$$

where d_{α_1} is defined by Equation (2.8) and X_{α_1,H_1} is the linear fractional stable motion given by (1.1).
Let us first assume that $H_1 < 1/\alpha_1$. Then, since $a < b$, by Corollary 10.2.4 of [17], $(S_m(\alpha_1,H_1,t))_{t\in\mathbb{R}}$ is unbounded almost surely on the compact set $[a,b]$. It follows that

$$\sup_{(\alpha,H,t)\in K} |S_m(\alpha,H,t)| = +\infty \text{ a.s.}$$

since $\sup_{(\alpha,H,t)\in K} |S_m(\alpha,H,t)| \geq \sup_{t\in[a,b]} |S_m(\alpha_1,H_1,t)|$.

Let us now assume that $H_1 = 1/\alpha_1$ (which implies that $\alpha_1 > 1$). Then,

$$X_{\alpha_1,H_1} = (M_{\alpha_1}([0,t))\mathbf{1}_{t>0} + M_{\alpha_1}((t,0])\mathbf{1}_{t<0})_{t\in\mathbb{R}}$$

is a Lévy α_1-stable motion and by Equation (3.10), so is the process $(S_m(\alpha_1,H_1,t))_{t\in\mathbb{R}}$. Since $\alpha_1 < 2$, the stable motion $(S_m(\alpha_1,1/\alpha_1,t))_{t\in\mathbb{R}}$ is not a Brownian motion and then does not have almost surely continuous sample paths (see Exercice 2.7 p.64 of [12] for instance). This concludes the proof. □

Therefore, it remains to study the sample paths on a compact set

$$K = [\alpha_1,\alpha_2] \times [H_1,H_2] \times [-A,A] \subset (0,2) \times (0,1) \times \mathbb{R}$$

such that $H_1 > 1/\alpha_1$, which implies that $\alpha_1 \in (1,2)$ and $H_1 > 1/2$.

The main result of this paper is the following theorem, which states an upper bound for the modulus of continuity of S_m on K, and for some m a rate of uniform convergence on K for the series $S_{m,N}$ defined by (2.4).

Theorem 3.1 *Assume that Hypothesis 2.1 is fulfilled. Let $S_{m,N}$ and S_m be defined by* (2.4) *and* (2.5) *and let us consider the compact set*

$$K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1, 2) \times (1/2, 1) \times \mathbb{R}$$

with $A > 0$ and $H_1 > 1/\alpha_1$.

1. *As $N \to +\infty$, the series $(S_{m,N})_{N \geq 1}$ converges uniformly on K to S_m and almost surely*

$$\sup_{\substack{x,x' \in K \\ x \neq x'}} \frac{|S_m(x) - S_m(x')|}{\tau(x - x')\sqrt{|\log(\tau(x - x'))| + 1}} < +\infty$$

   with $\tau(z) = |\alpha| + |H| + |t|^{H_1 - 1/\alpha_1}$ for $z = (\alpha, H, t) \in \mathbb{R}^3$.

2. *For $\eta > 0$, let us consider $m = m_\eta$ defined by*

$$m_\eta(\xi) = c_\eta |\xi|^{-1} (1 + |\log(|\xi|)|)^{-1-\eta}, \qquad (3.11)$$

   with $c_\eta > 0$ such that $\int_{\mathbb{R}} m_\eta(\xi) d\xi = 1$. Then, almost surely

$$\sup_{N \geq 1} N^{\varepsilon} \sup_{x \in K} \left| S_{m_\eta,N}(x) - S_{m_\eta}(x) \right| < +\infty$$

   for any $\varepsilon \in (0, 1/\alpha_2 - 1/2)$.

Proof. For all $x = (\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$ and all integer $n \geq 1$, we consider

$$V_{m,n}(x) := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha}, \qquad (3.12)$$

so that

$$S_{m,N}(x) = \sum_{n=1}^{N} T_n^{-1/\alpha} V_{m,n}(x) g_n \quad \text{and} \quad S_m(x) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_{m,n}(x) g_n.$$

Let us also remark that for all $x = (\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$,

$$\mathbb{E}(|V_{m,n}(x)|^\alpha) = \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^\alpha d\xi < +\infty.$$

Note that if in Equation (2.3) the sub-Gaussian parameter s of g_n is less than 1, Equation (2.3) also holds for $s = 1$. Moreover, if s is greater than 1 we may write $V_{m,n}(x) g_n = (s V_{m,n}(x)) g_n / s$ so that g_n/s is sub-Gaussian with parameter 1. Hence without loss of generality we may and will assume that $s = 1$. It follows that $(g_n)_{n \geq 1}$, $(T_n)_{n \geq 1}$ and $(V_{m,n})_{n \geq 1}$ are three independent sequences that satisfy Assumption 4 in [5] on $(0, 2) \times (0, 1) \times \mathbb{R}$. Then, by Theorem 4.2 of [5], the result follows once we prove $\mathbb{E}\left(|V_{m,1}(x_0)|^2\right) < +\infty$

for some $x_0 \in K$ and Equation (15) of [5] for $p = 1$, namely (in our setting) if there exists $r > 0$ such that

$$\mathbb{E}\left(\left[\sup_{\substack{x,x' \in K\\ 0<\|x-x'\|\leq r}} \frac{|V_{m,1}(x) - V_{m,1}(x')|}{\tau(x-x')}\right]^2\right) < +\infty. \tag{3.13}$$

The following proposition, whose proof is postponed to the appendix, allows to find some m satisfying such conditions.

Proposition 3.2 *There exists a finite deterministic constant $c_{3,1}(K) > 0$ such that a.s. for all $x, x' \in K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A]$,*

$$|V_{m,1}(x) - V_{m,1}(x')| \leq c_{3,1}(K)\tau(x-x')h_{m,K}(\xi_1),$$

with, for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned} h_{m,K}(\xi) &= \max\left(m(\xi)^{-1/\alpha_1}, m(\xi)^{-1/\alpha_2}\right)(1 + |\log m(\xi)|) \\ &\quad \times \left(\mathbf{1}_{|\xi|\leq \mathrm{e}} + |\xi|^{-1+H_2-1/\alpha_2}\log|\xi|\mathbf{1}_{|\xi|>\mathrm{e}}\right). \end{aligned} \tag{3.14}$$

Let us first consider $m = m_\eta$ given by (3.11) for some $\eta > 0$. In view of Proposition 3.2, since $V_{m_\eta,1}(\alpha, H, 0) = 0$ for all $(\alpha, H, 0) \in K$, up to use a finite covering of K, it is enough to prove that there exists $r > 0$ with

$$\mathbb{E}\left(h_{m_\eta,K}(\xi_1)^2\right) < +\infty, \tag{3.15}$$

for $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A]$ with $\alpha_2 - \alpha_1 \leq r$. One has

$$\begin{aligned} \mathbb{E}(h_{m_\eta,K}(\xi_1)^2) &= \int_{\mathbb{R}} h_{m_\eta,K}(\xi)^2 m_\eta(\xi)d\xi \\ &= \int_{|\xi|\leq \mathrm{e}} + \int_{|\xi|>\mathrm{e}} := I_1 + I_2. \end{aligned}$$

On the one hand,

$$\begin{aligned} I_1 &= \int_{|\xi|\leq \mathrm{e}} m_\eta(\xi)\max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2})(1 + |\log(m_\eta(\xi))|)^2 d\xi \\ &\leq c_{3,2}(\eta, K)\int_{|\xi|\leq \mathrm{e}} |\xi|^{-1+2/\alpha_2}\left(1 + |\log(|\xi|)|\right)^{(1+\eta)(2/\alpha_1-1)}\left(1 + |\log(m_\eta(\xi))|\right)^2 d\xi, \end{aligned}$$

with $c_{3,2}(\eta, K)$ a positive finite constant. It follows that $I_1 < +\infty$ since $\alpha_2 > 0$. On the other hand,

$$\begin{aligned} I_2 &= \int_{|\xi|>\mathrm{e}} m_\eta(\xi)\max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2})(1 + |\log(m_\eta(\xi))|)^2|\xi|^{2(H_2-1/\alpha_2)-2}\log(|\xi|)^2 d\xi \\ &\leq c_{3,3}(\eta, K)\int_{|\xi|>\mathrm{e}} |\xi|^{2(H_2+1/\alpha_1-1/\alpha_2)-3}\log(|\xi|)^{(1+\eta)(2/\alpha_1-1)+2}(1 + |\log(m_\eta(\xi))|)^2 d\xi, \end{aligned}$$

with $c_{3,3}(\eta, K)$ a positive finite constant. Since $\alpha_1 > 1$, note that $\alpha_2 - \alpha_1 < 1 - H_2$ implies that $H_2 + 1/\alpha_1 - 1/\alpha_2 < H_2 + \alpha_2 - \alpha_1 < 1$ and thus $I_2 < +\infty$. Therefore choosing $r \in (0, 1 - H_2)$, Equation (3.15) and then (3.13) hold for $m = m_\eta$. By Theorem 4.2 of [5], $(S_{m_\eta,N})_{N\geq 1}$ and S_{m_η} satisfy *1.* and *2.* of the theorem.
Since for almost every $\xi \in \mathbb{R}$ the map $(\alpha, H, t) \mapsto f_+(\alpha, H, t, \xi)$ is continuous on K, by Assertion *2.* of Proposition 5.1 of [5], S_m satisfies Assertion *1.* whatever m is.

□

Remark 3.1 Assertion *2.* in Theorem 3.1 holds for any m satisfying Equation (3.15) instead of m_η.

4 Linear multifractional multistable and stable motions

From now on let us consider $\alpha : \mathbb{R} \mapsto (0, 2)$ and $h : \mathbb{R} \mapsto (0, 1)$ two continuous functions. Under Hypothesis 2.1, by Proposition 2.1, we may consider the linear multifractional multistable motion defined on $\mathbb{R}$ by

$$\tilde{S}_m(t) := S_m(\alpha(t), h(t), t), \tag{4.16}$$

with S_m given by (2.5).

4.1 Regularity and rate of convergence

We may also define $\tilde{S}_{m,N}(t) := S_{m,N}(\alpha(t), h(t), t)$, for all $N \geq 1$. The following theorem is a direct consequence of Theorem 3.1.

Theorem 4.1 *Let us consider $\alpha : \mathbb{R} \mapsto (0, 2)$ and $h : \mathbb{R} \mapsto (0, 1)$ two continuous functions and two real numbers $a < b$. Then let us set*

$$\alpha_1 = \min_{t\in[a,b]} \alpha(t),\ \alpha_2 = \max_{t\in[a,b]} \alpha(t) \ \text{ and } \ H_1 = \min_{t\in[a,b]} h(t).$$

Assume that $H_1 > 1/\alpha_1$ and that α and h are $(H_1 - 1/\alpha_1)$-Hölder continuous functions on $[a, b]$.

1. *Then, as $N \to +\infty$, the series $\left(\tilde{S}_{m,N}\right)_{N\geq 1}$ converges uniformly on $[a, b]$ to $\tilde{S}_m$ and almost surely*

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|\tilde{S}_m(t) - \tilde{S}_m(t')\right|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log|t - t'|| + 1}} < +\infty.$$

2. *Moreover if $m = m_\eta$ is defined by (3.11) with $\eta > 0$, then, almost surely*

$$\sup_{N\geq 1} N^{\varepsilon} \sup_{t\in[a,b]} \left|\tilde{S}_{m_\eta,N}(t) - \tilde{S}_{m_\eta}(t')\right| < +\infty$$

for any $\varepsilon \in (0, 1/\alpha_2 - 1/2)$.

Note that one can use $\tilde{S}_{m_\eta,N}$ to simulate $\tilde{S}_{m_\eta}$. The error of approximation is then given by N^{ε}.

4.2 Stochastic integral and series representation

Assuming that α is a constant function equal to α_1, we have already seen that $\tilde{S}_m \overset{fdd}{=} d_{\alpha_1} X_{\alpha_1,h}$ where $X_{\alpha_1,h}$ is the linear multifractional α_1-stable motion defined by (3.9) and d_{α_1} is given by (2.8). Using the previous theorem we will prove the following one.

Theorem 4.2 *Let $\alpha_1 \in (0,2)$ and $h : \mathbb{R} \mapsto (0,1)$ be a continuous function. Let us also consider $X_{\alpha_1,h}$ the linear multifractional α_1-stable motion defined by (3.9) and two real numbers $a < b$. If $H_1 := \min_{t\in[a,b]} h(t) > 1/\alpha_1$ and if h is $(H_1 - 1/\alpha_1)$-Hölder continuous on $[a,b]$, then there exists a continuous modification $X^*_{\alpha_1,h}$ of $X_{\alpha_1,h}$ such that almost surely*

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t')\right|}{|t-t'|^{H_1-1/\alpha_1}\sqrt{|\log|t-t'||+1}} < +\infty.$$

Proof. Let $\alpha : \mathbb{R} \to (0,2)$ be the constant function equal to α_1 and let $\tilde{S}_m$ be defined by (4.16). Since $\tilde{S}_m \overset{fdd}{=} d_{\alpha_1} X_{\alpha_1,h}$ with $d_{\alpha_1} \neq 0$ defined by (2.8), by Theorem 4.1, we already know that a.s.

$$\sup_{\substack{t,t'\in[a,b]\cap\mathcal{D}\\ t\neq t'}} \frac{|X_{\alpha_1,h}(t) - X_{\alpha_1,h}(t')|}{|t-t'|^{H_1-1/\alpha_1}\sqrt{|\log|t-t'||+1}} < +\infty,$$

where $\mathcal{D}$ is the dense set of dyadic real numbers. Moreover, since h is continuous with values in $(0,1)$, the stochastic continuity of the linear multifractional α_1-stable motion $X_{\alpha_1,h}$ has been established in [19]. This implies that there exists a modification $X^*_{\alpha_1,h}$ of $X_{\alpha_1,h}$ such that

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t')\right|}{|t-t'|^{H_1-1/\alpha_1}\sqrt{|\log|t-t'||+1}} < +\infty,$$

see e.g. Section D.2 of [5] for the construction of $X^*_{\alpha_1,h}$. Then, the proof is complete. □

In [2], using a wavelet series expansion, under our assumptions of Proposition 3.9, the authors obtained a continuous modification $X^*_{\alpha_1,h}$ satisfying a.s. for all $\eta > 0$,

$$\sup_{\substack{t,t' \in [a,b] \\ t \neq t'}} \frac{\left| X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t') \right|}{|t-t'|^{H_1 - 1/\alpha_1} \left(|\log |t-t'|| + 1 \right)^{2/\alpha_1 + \eta}} < +\infty.$$

Since $1/2 < 2/\alpha_1$, our result is sharper. Moreover it is quasi-optimal since, for $\eta > 0$, one can find h such that a.s.

$$\sup_{\substack{t,t' \in [a,b] \\ t \neq t'}} \frac{\left| X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t') \right|}{|t-t'|^{H_1 - 1/\alpha_1} \left(|\log |t-t'|| + 1 \right)^{-\eta}} = +\infty,$$

by Theorem 6.1 of [2]. Let us also quote that following our method based on [5], one may obtain an upper bound for the global modulus of continuity of linear fractional stable sheets, which is sharper than the one given in [3].

A Proof of Proposition 3.2

Let us consider $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1,2) \times (1/2, 1) \times \mathbb{R}$ such that $1/\alpha_1 < H_1 \leq H_2 < 1$. Let us note that it is enough to prove Proposition 3.2 for A large enough. Then, in this proof, we assume, without loss of generality that $A > \mathrm{e}$ (so that $\log \xi > 1$ for $\xi > A$).

For all $x = (\alpha, H, t) \in K$, we set

$$\beta(x) = H - 1/\alpha \in (0,1)$$

and remark that $\beta(x) \in [\beta_1, \beta_2] \subset (0,1)$ with

$$\beta_1 = H_1 - 1/\alpha_1 \text{ and } \beta_2 = H_2 - 1/\alpha_2.$$

Moreover, for all $x = (\alpha, H, t) \in K$ and all $\xi \in \mathbb{R}$, let us note that

$$f_+(\alpha, H, t, \xi) = g(\beta(x), t, \xi)$$

with g defined on $(0,1) \times \mathbb{R} \times \mathbb{R}$ by

$$g(\beta, t, \xi) := (t - \xi)_+^\beta - (-\xi)_+^\beta.$$

Let us now consider $x = (\alpha, H, t) \in K$ and $x' = (\alpha', H', t') \in K$. Then, by (3.12),

$$V_{m,n}(x) - V_{m,n}(x') = \left(g(\beta(x), t, \xi_n) m(\xi_n)^{-1/\alpha} - g\big(\beta(x'), t', \xi_n\big) m(\xi_n)^{-1/\alpha'} \right).$$

Proposition 3.2 follows from the following lemma, which proof is given at the end of this section.

Lemma A.1 *Let $0 < \beta_1 \leq \beta_2 < 1$ and $A > \mathrm{e}$.*

1. *There exists a finite positive constant $c_1(A, \beta_1, \beta_2)$ such that for all $\beta, \beta' \in [\beta_1, \beta_2]$, all $t, t' \in [-A, A]$ and all $\xi \in \mathbb{R}$,*

$$\left|g(\beta, t, \xi) - g\big(\beta', t', \xi\big)\right| \leq c_1(A, \beta_1, \beta_2)\Big(\left|t - t'\right|^{\beta_1} + \left|\beta - \beta'\right|\Big)h_{A,1}(\xi, \beta_2)$$

with

$$h_{A,1}(\xi, c) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{c-1} \log|\xi|\, \mathbf{1}_{|\xi| > 2A}.$$

2. *Moreover, there exists a finite positive constant $c_2(A, \beta_1)$ such that for all $\beta \in [\beta_1, \beta_2]$ and $t \in [-A, A]$,*

$$|g(\beta, t, \xi)| \leq c_2(A, \beta_1) h_{A,2}(\xi, \beta_2)$$

with

$$h_{A,2}(\xi, c) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{c-1}\, \mathbf{1}_{|\xi| > 2A}.$$

Setting for almost every $\xi \in \mathbb{R}$

$$\left\{ \begin{array}{lcl} F_1(x, x', \xi) & := & |g(\beta(x), t, \xi) - g(\beta(x'), t', \xi)| m(\xi)^{-1/\alpha}, \\ F_2(x, x', \xi) & := & |g(\beta(x'), t', \xi)| \left| m(\xi)^{-1/\alpha} - m(\xi)^{-1/\alpha'} \right|, \end{array} \right.$$

we then have

$$\left|V_{m,1}(x) - V_{m,1}\big(x'\big)\right| \leq F_1\big(x, x', \xi_1\big) + F_2\big(x, x', \xi_1\big).$$

Before we apply Lemma A.1 to bound F_1 and F_2, let us remark that for all $\xi \in \mathbb{R}$,

$$h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2) \leq c_3(A, \beta_2)\Big(\mathbf{1}_{|\xi| \leq \mathrm{e}} + |\xi|^{\beta_2 - 1} \log|\xi| \mathbf{1}_{|\xi| > \mathrm{e}}\Big) \quad \text{(A.17)}$$

with $c_3(A, \beta_2)$ a finite positive constant, which does not depend on ξ. Then, combining this remark with Lemma A.1, for almost every $\xi \in \mathbb{R}$,

$$F_1\big(x, x', \xi\big) \leq c_1(A, \beta_1, \beta_2) c_3(A, \beta_2)\Big(\left|t - t'\right|^{\beta_1} + \left|\beta(x) - \beta(x')\right|\Big) h_{m,K}(\xi)$$

with $h_{m,K}$ defined by Equation (3.14). Since $\alpha_1 > 1$, by definition of the function β, it follows that for almost every $\xi \in \mathbb{R}$,

$$F_1\big(x, x', \xi\big) \leq c_1(A, \beta_1, \beta_2) c_3(A, \beta_2) \tau\big(x - x'\big) h_{m,K}(\xi),$$

with $\tau(x - x') = |t - t'|^{\beta_1} + |H - H'| + |\alpha - \alpha'|$.
Moreover, applying Assertion 2 of Lemma A.1, Equation (A.17) and the mean value theorem, for almost every $\xi \in \mathbb{R}$,

$$F_2\big(x, x', \xi\big) \leq c_2(A, \beta_1) c_3(A, \beta_2) \left|\alpha - \alpha'\right| h_{m,K}(\xi).$$

In view of the previous computations, we have: almost surely,

$$\left|V_{m,1}(x) - V_{m,1}\left(x'\right)\right| \leq c_{3,1}(K)\tau\left(x - x'\right)h_{m,K}(\xi_1)$$

with $c_{3,1}(K) := c_3(A, \beta_2)(c_1(A, \beta_1, \beta_2) + c_2(A, \beta_1))$. This concludes the proof of Proposition 3.2. □

We conclude this section by the proof of Lemma A.1.

Proof. [Proof of Lemma A.1] Let $0 < \beta_1 < \beta_2 < 1$ and $A > \mathrm{e}$. Let $\beta, \beta' \in [\beta_1, \beta_2] \subset (0, 1)$ and $t, t' \in [-A, A]$. Let us write for all $\xi \in \mathbb{R}$,

$$\left|g(\beta, t, \xi) - g\left(\beta', t', \xi\right)\right| \leq g_1\left(\beta', t, t', \xi\right) + g_2\left(\beta, \beta', t, \xi\right)$$

with

$$\left\{\begin{array}{lcl} g_1(\beta', t, t', \xi) & := & |g(\beta', t', \xi) - g(\beta', t, \xi)| \\ g_2(\beta, \beta', t, \xi) & := & |g(\beta', t, \xi) - g(\beta, t, \xi)|. \end{array}\right.$$

Step 1: Control of g_1. Let us note that if $t = t'$, $g_1(\beta', t, t', \xi) = 0$ for all $\xi \in \mathbb{R}$. Then, in this step, we assume now, without loss of generality that $t < t'$. This implies that

$$g_1\left(\beta', t, t', \xi\right) = \left\{\begin{array}{ll} 0 & \text{if } \xi \geq t' \\ (t' - \xi)^{\beta'} & \text{if } t \leq \xi < t' \\ \left|(t - \xi)^{\beta'} - (t' - \xi)^{\beta'}\right| & \text{if } \xi < t. \end{array}\right.$$

Let $\xi \in \mathbb{R}$ with $|\xi| > 2A$. If $\xi < 0$ it follows that $\xi < t < t'$. Since $\beta' > 0$, applying the mean value theorem,

$$g_1(\beta', t, t', \xi) \quad \leq \quad \beta' \left|t - t'\right| \left|c_{_{\xi,t,t'}} - \xi\right|^{\beta' - 1}$$

with $c_{_{\xi,t,t'}} \in (t, t') \subset [-A, A]$. Moreover, since $|\xi| > 2A$

$$\left|c_{_{\xi,t,t'}} - \xi\right| \geq |\xi| - \left|c_{_{\xi,t,t'}}\right| \geq |\xi| - A \geq |\xi|/2$$

and then

$$g_1\left(\beta', t, t', \xi\right) \leq 2^{1-\beta'} \left|t - t'\right| |\xi|^{\beta' - 1}$$

since $\beta' \in (0, 1)$. Therefore, for $|\xi| > 2A$,

$$g_1\left(\beta', t, t', \xi\right) \leq 4A\left|t - t'\right|^{\beta_1} |\xi|^{\beta_2 - 1} \tag{A.18}$$

since $|t - t'| \leq 2A$, $\beta' \in [\beta_1, \beta_2] \subset (0, 1)$ and $2A > 1$.

Now let $\xi \in \mathbb{R}$ with $|\xi| \leq 2A$. Since $0 < \beta' < 1$, we have

$$\left|a^{\beta'} - b^{\beta'}\right| \leq |a - b|^{\beta'}$$

for all $a, b \geq 0$. By definition of g, it follows that

$$g_1\big(\beta', t, t', \xi\big) \leq \Big|\big(t' - \xi\big)_+ - (t - \xi)_+\Big|^{\beta'} \leq \big|t' - t\big|^{\beta'} \leq 2A\big|t' - t\big|^{\beta_1}$$

since $-A \leq t < t' \leq A$, $0 < \beta_1 \leq \beta' < 1$ and $A > 1$. From this last inequality and Equation (A.18), we deduce that for all $\xi \in \mathbb{R}$,

$$g_1\big(\beta', t, t', \xi\big) \leq 4A\big|t - t'\big|^{\beta_1} h_{A,2}(\xi, \beta_2) \tag{A.19}$$

with $h_{A,2}(\xi, \beta_2) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{\beta_2 - 1}\mathbf{1}_{|\xi| > 2A}$.

Step 2: Control of g_2. Let us recall that for all $\xi \in \mathbb{R}$,

$$g_2\big(\beta, \beta', t, \xi\big) = \Big|(t - \xi)_+^{\beta'} - (t - \xi)_+^{\beta} + (-\xi)_+^{\beta} - (-\xi)_+^{\beta'}\Big|.$$

Then, applying the mean value theorem, for all $\xi \in \mathbb{R}$,

$$g_2\big(\beta, \beta', t, \xi\big) \leq \big|\beta - \beta'\big| \sup_{\beta_1 \leq c \leq \beta_2} \big|(t - \xi)_+^c \log(t - \xi)_+ - (-\xi)_+^c \log(-\xi)_+\big|$$

where for $c > 0$,

$$(x)_+^c \log(x)_+ = \begin{cases} x^c \log x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let us first consider $\xi \in [-2A, 2A]$. Then, $(-\xi)_+ \in [0, 2A]$ and $(t - \xi)_+ \in [0, 3A]$ since $t \in [-A, A]$. Therefore,

$$g_2\big(\beta, \beta', t, \xi\big) \leq \tilde{c}_1(A, \beta_1, \beta_2)\big|\beta - \beta'\big| \tag{A.20}$$

with

$$\tilde{c}_1(A, \beta_1, \beta_2) = 2 \max_{\beta_1 \leq c \leq \beta_2} \max_{0 < u \leq 3A} u^c|\log u| = 2\max\left(\frac{1}{\mathrm{e}\beta_1}, (3A)^{\beta_2}\log(3A)\right) < +\infty.$$

Let us now assume that $\xi < -2A$. Then, $\xi < t$ and

$$g_2\big(\beta, \beta', t, \xi\big) \leq \big|\beta - \beta'\big| \sup_{\beta_1 \leq c \leq \beta_2} |(t - \xi)^c \log(t - \xi) - (-\xi)^c \log(-\xi)|$$

with $t - \xi > 0$ and $-\xi > 0$. Let us remark that $-\xi \in (-\xi/2, -3\xi/2)$ since $-\xi > 0$ and that

$$-\xi/2 < -A - \xi \leq t - \xi \leq A - \xi < -3\xi/2$$

since $t \in [-A, A]$ and $\xi < -2A$. Then, for each $c \in [\beta_1, \beta_2] \subset (0, 1)$, by the mean value theorem,

$$|(t - \xi)^c \log(t - \xi) - (-\xi)^c \log(-\xi)| \leq |u_{t,\xi,c}|^{c-1}(c|\log u_{t,\xi,c}| + 1)$$

with $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$. Since $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$ and $-\xi/2 > A > \mathrm{e}$, we get

$$|(t-\xi)^c \log(t-\xi) - (-\xi)^c \log(-\xi)| \leq 4|\xi|^{\beta_2-1} \log|\xi|$$

for all $c \in [\beta_1, \beta_2] \subset (0,1)$. Hence, for $\xi < -2A$,

$$g_2\big(\beta, \beta', t, \xi\big) \leq 4\big|\beta - \beta'\big||\xi|^{\beta_2-1} \log|\xi|.$$

Note that this last inequality still holds for $\xi > 2A$ since in this case, $g_2(\beta, \beta', t, \xi) = 0$.

Then, we have proved that for all $\xi \in \mathbb{R}$,

$$g_2\big(\beta, \beta', t, \xi\big) \leq \tilde{c}_2(A, \beta_1, \beta_2)\big|\beta - \beta'\big| h_{A,1}(\xi, \beta_2) \tag{A.21}$$

with $\tilde{c}_2(A, \beta_1, \beta_2) = \max\left(\tilde{c}_1(A, \beta_1, \beta_2), 4\right)$ and

$$h_{A,1}(\xi, \beta_2) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{\beta_2-1} \log|\xi| \mathbf{1}_{|\xi|>2A}.$$

Step 3: Proof of Assertion 1. It follows from Equations (A.19) and (A.21) choosing $c_1(A, \beta_1, \beta_2) = \tilde{c}_2(A, \beta_1, \beta_2) + 4A \in (0, +\infty)$ and using the fact that $h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2)$ since $A > \mathrm{e}$.

Step 4: Proof of Assertion 2. Let us remark that

$$g\big(\beta', t', \xi\big) = g\big(\beta', t', \xi\big) - g\big(\beta', 0, \xi\big)$$

since $g(\beta', 0, \xi) = (-\xi)_+^{\beta'} - (-\xi)_+^{\beta'} = 0$. Hence, applying Equation (A.19) with $t = 0$ and $\beta' = \beta$,

$$\big|g\big(\beta', t', \xi\big)\big| \leq 4A\big|t'\big|^{\beta_1} h_{A,2}(\xi, \beta_2) \leq 4A^{\beta_1+1} h_{A,2}(\xi, \beta_2),$$

which concludes the proof. □

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