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A NEW LOOK AT SHORT-TERM IMPLIED VOLATILITY IN ASSET PRICE MODELS WITH JUMPS

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We analyze the behavior of the implied volatility smile for options close to expiry in the exponential Lévy class of asset price models with jumps. We introduce a new renormalization of the strike variable with the property that the implied volatility converges to a nonconstant limiting shape, which is a function of both the diffusion component of the process and the jump activity (Blumenthal–Getoor) index of the jump component. Our limiting implied volatility formula relates the jump activity of the underlying asset price process to the short-end of the implied volatility surface and sheds new light on the difference between finite and infinite variation jumps from the viewpoint of option prices: in the latter, the wings of the limiting smile are determined by the jump activity indices of the positive and negative jumps, whereas in the former, the wings have a constant model-independent slope. This result gives a theoretical justification for the preference of the infinite variation Lévy models over the finite variation ones in the calibration based on short-maturity option prices.

KEY WORDS: exponential Lévy models, Blumenthal–Getoor index, short-dated options, implied volatility

1. INTRODUCTION

In financial markets, the price of a vanilla call or put option on a risky asset with strike e^k and maturity t is often quoted in terms of the *implied volatility* $\widehat{\sigma}(t, k)$ (see (3.3) in Section 3 for the definition and Gatheral 2006 for more information on implied volatility). Similarly, given a risk-neutral pricing model, one can define a function $(t, k) \mapsto \widehat{\sigma}(t, k)$ via the prices of the vanilla options under that model. The implied volatility is a central object in option markets and it is, therefore, not surprising that understanding the properties and computing the function $(t, k) \mapsto \widehat{\sigma}(t, k)$ for widely used pricing models has been of considerable interest in the mathematical finance literature. Typically, for a given modeling framework, the implied volatility $\widehat{\sigma}(t, k)$ is not available in closed form. Hence, the study of the asymptotic behavior in a variety of asymptotic regimes [e.g., fixed t and $k \to \pm\infty$ (Lee 2004; Friz and Benaim 2009; Gulisashvili 2010); $t \to \infty$ with k constant (Tehranchi 2009) or proportional (Jacquier, Keller-Ressel, and Mijatović 2013)

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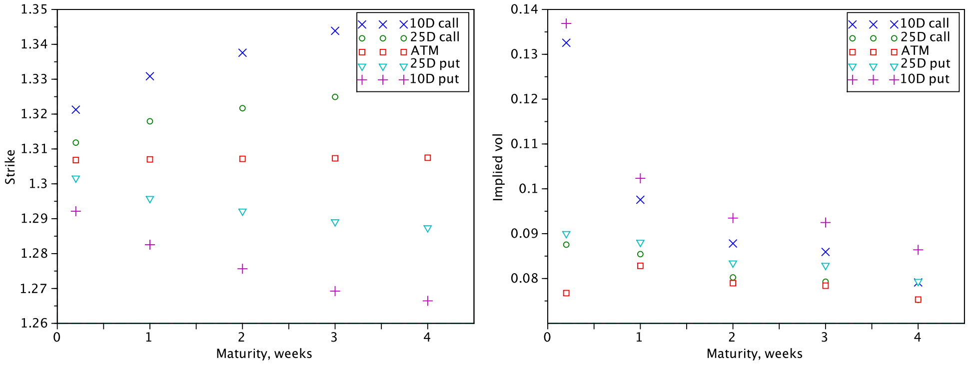

FIGURE 1.1. The liquid at-the-money, 10- and 25-delta strikes (left panel) and the corresponding implied volatilities (right panel) for the market-defined maturities t in the set {1 day, 1 week, 2 weeks, 3 weeks, 1 month} in the EURUSD option market (taken on the January 4, 2013) suggest the following: as maturity t becomes small, the relevant strikes k_t approach the at-the-money strike and the implied volatilities $\widehat{\sigma}(t, k_t)$ remain bounded. These features of liquid strikes and implied volatilities are persistent in time and across option markets.

to t; $t \to 0$ and k constant (Roper and Rutkowski 2009; Tankov 2010; Figueroa-López and Forde 2012) etc.] has attracted a lot of attention in the recent years.

In this paper, we assume that the returns of the risky asset $S = e^X$ are modeled by a Lévy process X and study the relationship between the jump activity of X and the implied volatility at short maturities in the model S. Most existing approaches analyze either the at-the-money (ATM) case, when the implied volatility is determined exclusively by the diffusion component and converges to zero in the pure-jump models (Tankov 2010; Muhle-Karbe and Nutz 2011; Houdré, Gong, and Figueroa-López 2011), or the fixed-strike out-of-the-money (OTM) case, when the implied volatility for short maturities explodes in the presence of jumps (Roper 2009; Tankov 2010; Figueroa-López and Forde 2012). However, in option markets, (a) although the implied volatility for liquid strikes grows with decreasing t, it remains within a range of reasonable values and appears not to explode, and (b) the liquid strikes become concentrated around the money as the maturity gets shorter. For instance, in foreign exchange (FX) option markets, which are among the most liquid derivatives markets in the world, options with fixed values of the Black–Scholes delta are quoted for each maturity (see Section 5.2 and Andersen and Lipton 2013 for the conventions in FX option markets and the natural delta parameterization of the smile).

The market data in Figure 1.1, therefore, suggests that, in order to understand the behavior of the volatility surface at short maturities, one should look for a *moving log-strike* $k_t \neq 0$, for $t > 0$, such that (i) the corresponding implied volatility has a nontrivial limit $\lim_{t\downarrow 0} \widehat{\sigma}(t, k_t)$ and (ii) the log-strike k_t converges to the ATM log-strike value as maturity t tends to zero (i.e., $\lim_{t\downarrow 0} k_t = 0$ if one assumes that $S_0 = 1$).

This paper defines a new universal and model-free parameterization of the log-strike given by

$$k_t = \theta\sqrt{t \log(1/t)} \quad \text{where} \quad \theta \in \mathbb{R}\backslash\{0\}.$$

For fixed θ, the corresponding strike value tends to the ATM strike as $t \downarrow 0$ but is OTM for each short maturity $t > 0$. We prove that under suitable assumptions the limiting implied volatility $\sigma_0(\theta) = \lim_{t\downarrow 0} \widehat{\sigma}(t, k_t)$ takes the following form as a function of θ:

$$(1.1) \quad \sigma_0(\theta) = \max\left\{ \frac{-\theta}{\sqrt{1-(\alpha_- - 1)^+}}, \sigma, \frac{\theta}{\sqrt{1-(\alpha_+ - 1)^+}} \right\} \quad \text{for any} \quad \theta \in \mathbb{R}\backslash\{0\}.$$

In this formula σ denotes the volatility of the Gaussian component of the underlying Lévy process X and α_+ (resp. α_-) denotes the jump activity (Blumenthal–Getoor) index of the positive (resp. negative) jumps of X. More precisely, if the jump measure of X is denoted by ν, α_+ and α_- are given by

$$\alpha_+ = \inf\{p \geq 0 : \int_{(0,1)} |x|^p \nu(dx) < \infty\} \quad \text{and} \quad \alpha_- = \inf\{p \geq 0 : \int_{(-1,0)} |x|^p \nu(dx) < \infty\}.$$

Unlike in the case of fixed strike, where short maturity smile explodes in the presence of jumps, our parameterization of the strike as a function of time yields a nonconstant formula for the limiting implied volatility, which depends on the balance between the size of the Gaussian volatility parameter and the activity of small jumps. It allows us to make the following observations about the relationship between the short-dated option prices and the characteristics of the underlying model:

(i) the formula for $\sigma_0(\theta)$ depends on the jump measure of the log-spot process X only if the jumps are of infinite variation; put differently, if the jumps of X are of finite variation, then the absolute value of the slope of the limiting smile for large $|\theta|$ is equal to one and, in particular, $\sigma_0(\theta)$ does not depend on the structure of jumps;
(ii) the limiting smile $\sigma_0(\theta)$ is "V-shaped" in the absence of the diffusion component (i.e., when $\sigma = 0$) and is "U-shaped" otherwise.

Remark (i) provides a theoretical basis for distinguishing between the models with jumps of finite and infinite variation in terms of the observed prices of vanilla options with short maturity. It is well known that, for any short maturity t, the market-implied smile $k \mapsto \widehat{\sigma}(t, k)$ exhibits pronounced skewness and/or curvature, due, in particular, to the risk of large moves over short time horizons perceived by the investors. Hence, jumps are typically introduced into the risk-neutral pricing models with the aim to capture this risk and modulate the ATM skew of the implied volatility $\widehat{\sigma}(t, k)$ at small t (see, e.g., equation (5.10) in Gatheral 2006). However, since this task can be accomplished by jumps of either finite or infinite variation, this requirement tells us little about the options implied jump activity of the underlying risk-neutral model. On the other hand, the formula for $\sigma_0(\theta)$ implies that, if we need to control the tails (in the parameter θ) of the implied volatility for short maturities, we must use jumps of infinite variation. This finding complements the analysis in Carr and Wu (2003) of the pathwise structure of the risk-neutral process implied by the option prices on the S&P 500 index.

The formula for the limiting implied volatility given in (1.1) should be compared to the recent results (for Lévy models) on the limiting behavior of the implied volatility at a fixed OTM strike (i.e., $k_t \equiv \text{const.} \neq 0$). In Roper (2009), Figueroa-López and Forde (2012), and Tankov (2010), it is shown that, in the presence of jumps, the implied volatility explodes at the rate $(t \log(1/t))^{-1/2}$ as maturity t tends to zero. Furthermore, this rate is independent of the jump structure and is insensitive to the presence of the diffusion component. Hence, in the fixed-strike OTM asymptotic regime little can be deduced about the relation between the jump structure and the diffusion component of asset returns as the maturity t decreases to zero, since this makes the implied volatility tend to infinity in a model-independent way. In the ATM case (i.e., $k_t \equiv 0$), the limit of the implied volatility as the maturity decreases is equal to the diffusion component of the Lévy triplet (Tankov 2010, proposition 5), making it zero for a pure-jump Lévy process. In the light of these results, the formula for $\sigma_0(\theta)$ in (1.1) provides new insight into the relation between the jump structure and the diffusion component implied by the short-maturity smile. It should be noted that the extension of the formula $\sigma_0(\theta)$ to a more general class of processes with jumps (e.g., jump diffusions, stochastic volatility processes with jumps, or even general semimartingales) is likely to hold under the appropriate assumptions. In particular, it is reasonable to expect that the model-independent parametrization of the strike k_t given above will lead to analogous limit results for the implied volatility as the maturity t tends to zero, as the "tangential" Lévy process at $t = 0$ to a general model (Muhle-Karbe and Nutz 2011) will control the limiting behavior of the smile.

In recent years, there has been a lot of interest in the literature on the statistics of stochastic process in the question of the estimation of the Blumenthal–Getoor index of models with jumps based on high-frequency data. For example, it is shown in Aït-Sahalia and Jacod (2009) that the jump activity (measured by the Blumenthal–Getoor index) estimated on high-frequency stock returns for two large US corporates is well beyond one, implying that the underlying model for stock returns should have jumps of infinite variation. Likewise, the formula in (1.1) suggests that jumps of infinite variation are needed in order to capture the correct tails (in θ) of the quoted short-dated option prices (cf. Section 2.1.1). In contrast to the high-frequency setting, a spectral estimation algorithm for the Blumenthal–Getoor index of a Lévy process based on low-frequency historical and options data was proposed in Belomestny (2010). Unlike formula (1.1), which would require option prices of arbitrarily short maturities for the estimation of the Blumenthal–Getoor index, the algorithm in Belomestny (2010) relates the distributional properties of the Lévy process to the index, thus enabling its estimation using options with a fixed maturity.

1.1. Structure of the Results

The formula in (1.1) follows from Corollary 3.3, which gives the expansion of the implied volatility $\widehat{\sigma}(t, k_t)$, where $k_t = \theta\sqrt{t\log(1/t)}$, up to order $o(1/\log(1/t))$. This expansion is a consequence of (A) Theorem 3.1, which itself gives an expansion of the implied volatility for a general log-strike k_t that tends to zero as $t \downarrow 0$, and (B) Theorem 2.1 and Proposition 2.3, which describe the asymptotic behavior of the option prices under Lévy processes with infinite and finite jump variations, respectively. Theorem 3.1 relates the asymptotic behavior of the vanilla option prices under a general semimartingale model to the asymptotic behavior of the implied volatility as the log-strike k_t tends to zero [it should be noted that the asymptotic regime (t, k_t) in Theorem 3.1 is not covered by the analysis in Gao and Lee (2011), see Remark (iv) after Theorem 3.1 for more details]. The asymptotic formula in Corollary 3.3 then follows by combining Theorem 3.1 with the asymptotic behavior of the vanilla option prices established in Theorem 2.1 (for the case of jumps of infinite variation) and Proposition 2.3 (for jumps of finite variation).

In a certain sense, Theorem 2.1 and Proposition 2.3 represent the main contributions of this paper. The asymptotic formulae for the call and put options, struck at e^{k_t} and

e^{-k_t}, respectively, have the same structure in both results: the leading order term is a sum of two contributions, one coming from the diffusion component of the process and the other from the jump measure. Which of the two summands dominates in the limit depends on the level of the parameter θ. This structure of the asymptotic formulae is also reflected in the expression for $\sigma_0(\theta)$, as it is clear from (1.1) that $\sigma_0(\theta) \equiv \sigma$ if θ is between $-\sigma\sqrt{1-(\alpha_- - 1)^+}$ and $\sigma\sqrt{1-(\alpha_+ - 1)^+}$, and $\sigma_0(\theta)$ only depends on the jump measure otherwise. However, the proofs of Theorem 2.1 and Proposition 2.3 differ greatly: the finite variation case follows from the Itô–Tanaka formula, which can, in this case, be applied directly to the hockey-stick payoff function, while the case of jumps with infinite variation requires a detailed analysis of the asymptotic behavior of the option prices.

1.2. Structure of the Paper

Section 2 defines the setting and states Theorem 2.1 and Proposition 2.3. In Section 3, we state the asymptotic formulae for the implied volatility and derive the limit in (1.1). Section 4 presents numerical results that demonstrate the convergence of option prices and implied volatilities given in the previous two sections, in the context of the CGMY (Carr, Geman, Madan, and Yor 2002) model and the CGMY model with an additional diffusion component. In Section 5, we present a qualitative comparison, based on the observed market quotes in FX, between our theoretical predicted shape (1.1) of the short-maturity smile and the actual market smiles. Also, our θ-parameterization of the strike variable is compared to the parameterization in terms of the option delta, commonly used in FX markets. Section 6 concludes the paper by proving Theorem 2.1, Proposition 2.3, Theorem 3.1, and Corollary 3.3 in that order. The Appendix contains a short technical lemma, which is applied in Section 6.

2. OPTION PRICE ASYMPTOTICS CLOSE TO THE MONEY

In this paper, we study the behavior of option prices close to maturity in an exponential Lévy model $S = e^X$, where X is a Lévy process with the characteristic triplet (σ^2, ν, γ). Throughout the paper we assume the following:

- S is a true martingale (i.e., the interest rate and the dividend yield are assumed to be zero);
- S is normalized to start at $S_0 = 1$ (i.e., as usual the Lévy process X starts at $X_0 = 0$);
- the tails of the Lévy measure ν admit exponential moments:

$$\int_{|z|>1} e^{|z|(1+\delta)}\nu(dz) < \infty \quad \text{for some } \delta > 0. \tag{2.1}$$

In particular, assumption (2.1) guarantees the finiteness of vanilla option prices for any maturity $t > 0$. Section 2.1 describes the asymptotic behavior of option prices for short maturities in the case the process X has jumps of infinite variation. Section 2.2 deals with the case where the pure-jump part of X has finite variation.

2.1. Lévy Processes with Jumps of Infinite Variation

Theorem 2.1 describes the asymptotic behavior of option prices in the case the tails of the Lévy measure of X around zero have asymptotic power-like behavior. This assumption does not exclude any exponential Lévy models that appear in the literature but yields sufficient analytical tractability to characterize a nontrivial limit as maturity tends to zero for the option prices around the ATM. Before stating the theorem, we recall standard notation used throughout the paper: functions $f(t)$ and $g(t)$, where $g(t) > 0$ for all small $t > 0$, satisfy

$$f(t) \sim g(t) \quad \text{as } t \downarrow 0 \textit{ if } \quad \lim_{t\downarrow 0} \frac{f(t)}{g(t)} = 1, \tag{2.2a}$$

$$f(t) = o(g(t)) \quad \text{as } t \downarrow 0 \textit{ if } \quad \lim_{t\downarrow 0} \frac{f(t)}{g(t)} = 0, \tag{2.2b}$$

$$f(t) = O(g(t)) \quad \text{as } t \downarrow 0 \textit{ if } \quad \frac{f(t)}{g(t)} \text{ is bounded for all small } t > 0. \tag{2.2c}$$

Furthermore, we denote $x^+ := \max\{x, 0\}$ for any $x \in \mathbb{R}$.

THEOREM 2.1. *Let X be a Lévy process as described at the beginning of the section and assume that the following holds*

$$\lim_{x\downarrow 0} x^{\alpha_+} \nu((x, \infty)) = c_+, \quad \lim_{x\downarrow 0} x^{\alpha_-} \nu((-\infty, -x)) = c_-, \tag{2.3}$$

for $\alpha_+, \alpha_- \in (1, 2)$ and $c_+, c_- \in [0, \infty)$. Let k_t be a deterministic function satisfying

$$k_t > 0 \quad \forall t > 0, \quad \lim_{t\downarrow 0} k_t = 0,$$

and

$$\textit{if } \sigma^2 = 0, \quad \lim_{t\downarrow 0} \tfrac{t^{1/\alpha}}{k_t} = 0 \quad \textit{for some } \alpha \in (\max(\alpha_-, \alpha_+), 2),$$
$$\textit{if } \sigma^2 > 0, \quad \lim_{t\downarrow 0} \tfrac{\sqrt{t}}{k_t} = 0.$$

Then, if $c_+ > 0$, we have

$$\mathbb{E}[(e^{X_t} - e^{k_t})^+] \sim \mathbb{E}[(e^{\sigma W_t - \frac{\sigma^2 t}{2}} - e^{k_t})^+] + \frac{t k_t^{1-\alpha_+} c_+}{\alpha_+ - 1} \quad \textit{as } t \downarrow 0, \tag{2.4}$$

and, if $c_- > 0$, it holds

$$\mathbb{E}[(e^{-k_t} - e^{X_t})^+] \sim \mathbb{E}[(e^{-k_t} - e^{\sigma W_t - \frac{\sigma^2 t}{2}})^+] + \frac{t k_t^{1-\alpha_-} c_-}{\alpha_- - 1} \quad \textit{as } t \downarrow 0. \tag{2.5}$$

REMARKS 2.2.

(i) Theorem 2.1 implies that the price of a call (resp. put) option struck at e^{k_t} (resp. e^{-k_t}) tends to zero at a rate strictly slower than t if the paths of the pure-jump part of X have infinite variation. In particular, combining the notation in (2.2) and (2.3),

we get that the following equalities hold as $t \downarrow 0$:

$$\mathbb{E}[(e^{X_t} - e^{k_t})^+] = \mathbb{E}[(e^{\sigma W_t - \frac{\sigma^2 t}{2}} - e^{k_t})^+] + \frac{t k_t^{1-\alpha_+} c_+}{\alpha_+ - 1} + o\big(t k_t^{1-\alpha_+}\big),$$

$$\mathbb{E}[(e^{-k_t} - e^{X_t})^+] = \mathbb{E}[(e^{-k_t} - e^{\sigma W_t - \frac{\sigma^2 t}{2}})^+] + \frac{t k_t^{1-\alpha_-} c_-}{\alpha_- - 1} + o\big(t k_t^{1-\alpha_-}\big).$$

(ii) The proof of Theorem 2.1 is given in Section 6.1.

2.1.1. Blumenthal–Getoor index and the short-dated option prices. Recall that for any Lévy process Y with a nontrivial Lévy measure ν_Y, the *Blumenthal–Getoor index*, introduced in Blumenthal and Getoor (1961), is defined as

$$\text{(2.6)} \qquad \mathrm{BG}(Y) := \inf\left\{ r \geq 0 \ : \ \int_{(-1,1)\setminus\{0\}} |x|^r \, \nu_Y(dx) < \infty \right\}.$$

The Blumenthal–Getoor index is a measure of the jump activity of the Lévy process Y, since the following holds: $r > \mathrm{BG}(Y)$, if and only if $\sum_{s\leq t} |\Delta Y_s|^r < \infty$ almost surely, where $\Delta Y_s := Y_s - Y_{s-}$ denotes the size of the jump of Y at time s. Furthermore, it is clear from (2.6) that $\mathrm{BG}(Y)$ lies in the interval $[0, 2]$.

In recent years, there has been renewed interest in the Blumenthal–Getoor index from the point of view of estimation of the jump structure of stochastic processes based on high-frequency financial data. For example, it was estimated in Aït-Sahalia and Jacod (2009) that the value of $\mathrm{BG}(Y)$ is around 1.7 (i.e., the stock price process has jumps of infinite variation) based on high-frequency transactions (taken at 5 and 15 time intervals) for *Intel* and *Microsoft* stocks throughout 2006. Since the pricing measure is equivalent to the real-world measure, the Blumenthal–Getoor index of the process under the pricing measure is in this case also close to 1.7 (by theorem 7.23(b) in Jacod and Shiryaev 2003, chapter III, which relates the semimartingale characteristics of the price process under the two measures).

Let X^+ and X^- be the pure-jump parts of the Lévy process X from Theorem 2.1. In other words, X^+ (resp. X^-) is a Lévy process with the characteristic triplet $(0, \nu^+, 0)$ (resp. $(0, \nu^-, 0)$), where $\nu^+(dx) := 1_{\{x>0\}}\nu(dx)$ (resp. $\nu^-(dx) := 1_{\{x<0\}}\nu(dx)$). Then assumption (2.5) implies

$$\mathrm{BG}(X^+) = \alpha_+ \quad \text{and} \quad \mathrm{BG}(X^-) = \alpha_-,$$

and relations (2.6) and (2.7) of Theorem 2.1 describe how the Blumenthal–Getoor indices of the positive and negative jumps of X influence the asymptotic behavior of option prices at short maturities. These results clearly depend on the asymptotic behavior of the log-strike k_t. In Section 3, we will prescribe a specific parametric form of k_t (see (3.4)) and give explicit formulae for the asymptotic expansion and the limit of the implied volatility as maturity tends to zero in terms of the Blumenthal–Getoor indices of X^+ and X^- (see Corollary 3.3 for details).

2.2. Lévy Processes with Jumps of Finite Variation

In this section, we study the option price asymptotics at short maturities in the case the process X has a (possibly trivial) Brownian component and a pure-jump part of finite variation.

PROPOSITION 2.3. *Let X be a Lévy process as described at the beginning of Section 2. Assume further that the jump part of X has finite variation, i.e.,*

$$\int_{\mathbb{R}\setminus\{0\}} |x|\nu(dx) < \infty.$$

Let k_t be a deterministic function satisfying

$$k_t > 0 \quad \forall t > 0, \quad \lim_{t\downarrow 0} k_t = 0,$$

and

$$\begin{aligned} &\text{if } \sigma^2 = 0, \quad \lim_{t\downarrow 0} \tfrac{t}{k_t} = 0, \\ &\text{if } \sigma^2 > 0, \quad \lim_{t\downarrow 0} \tfrac{\sqrt{t}}{k_t} = 0. \end{aligned}$$

Then, as $t \downarrow 0$, it holds:

$$\mathbb{E}[(e^{X_t} - e^{k_t})^+] = \mathbb{E}[(e^{\sigma W_t - \frac{\sigma^2 t}{2}} - e^{k_t})^+] + t\int_{(0,\infty)} (e^x - 1)\nu(dx) + o(t), \tag{2.7}$$

and

$$\mathbb{E}[(e^{-k_t} - e^{X_t})^+] = \mathbb{E}[(e^{-k_t} - e^{\sigma W_t - \frac{\sigma^2 t}{2}})^+] + t\int_{(-\infty,0)} (1 - e^x)\nu(dx) + o(t). \tag{2.8}$$

REMARKS 2.4.

(i) *Proposition 2.3 implies that, in the absence of a Brownian component, the call and put prices of options struck at e^{k_t} and e^{-k_t}, respectively, tend to zero at the rate equal to t if X has paths of finite variation (cf. Remark (i) after Theorem 2.1).*

(ii) *The Blumenthal–Getoor indices of the positive and negative jump processes X^+ and X^- of X, defined in Section 2.1.1, are both smaller or equal to one by the assumption in Proposition 2.3. Furthermore, unlike in the case of jumps of infinite variation, Proposition 2.3 implies that the asymptotic behavior of short-dated option prices (as maturity t tends to zero) does not depend, up to order $o(t)$, on the indices $BG(X^+)$ and $BG(X^-)$. Hence, the same will hold for the short-dated implied volatility (cf. Corollary 3.3).*

(iii) *It should be stressed that the proof of Proposition 2.3, given in Section 6.2, is fundamentally different from that of Theorem 2.1, as it relies on the pathwise version of the Itô–Tanaka formula for the processes of finite variation, which cannot be applied in the context of Theorem 2.1.*

3. ASYMPTOTIC BEHAVIOR OF IMPLIED VOLATILITY

The value $C^{\mathrm{BS}}(t,k,\sigma)$ of the European call option with strike e^k (for any $k \in \mathbb{R}$) and expiry t under a Black–Scholes model (with log-spot $X_t = \sigma W_t - t\sigma^2/2$ of constant volatility $\sigma > 0$) is given by the Black–Scholes formula

$$C^{\mathrm{BS}}(t,k,\sigma) = N(d_+) - e^k N(d_-), \quad \text{where} \quad d_\pm = -\frac{k}{\sigma\sqrt{t}} \pm \frac{\sigma\sqrt{t}}{2}, \tag{3.1}$$

and $N(\cdot)$ is the standard normal cumulative distribution function. The price of a put option with the same strike and maturity is given by $P^{\mathrm{BS}}(t,k,\sigma) = e^k N(-d_-) - N(-d_+)$. Let S be a positive martingale, with $S_0 = 1$, that models a risky security and denote by

$$C(t,k) := \mathbb{E}\big[\big(S_t - e^k\big)^+\big] \quad \text{and} \quad P(t,k) := \mathbb{E}\big[\big(e^k - S_t\big)^+\big], \tag{3.2}$$

the prices of call and put options on S struck at e^k with maturity t, respectively. The *implied volatility* in the model S for any log-strike $k \in \mathbb{R}$ and maturity $t > 0$ is the unique positive number $\widehat{\sigma}(t,k)$ that satisfies the following equation in σ:

$$C^{\mathrm{BS}}(t,k,\sigma) = C(t,k). \tag{3.3}$$

Implied volatility is well defined since the function $\sigma \mapsto C^{\mathrm{BS}}(t,k,\sigma)$ is strictly increasing on the positive half-line and the right-hand side of (3.3) lies in the image of the Black–Scholes formula by a simple no-arbitrage argument. Put–call parity, which holds since S is a true martingale, implies the identity $P^{\mathrm{BS}}(t,k,\widehat{\sigma}(t,k)) = P(t,k)$.

In order to study the limiting behavior of the implied volatility close to the ATM strike $1 = e^0$ for short maturities, we define the following parameterization of the log-strike k_t:

$$k_t := \theta \left(t \log \frac{1}{t}\right)^{1/2}, \quad \text{where} \quad \theta \in \mathbb{R}\backslash\{0\}. \tag{3.4}$$

We can now define the implied volatility $\sigma_t : \mathbb{R}\backslash\{0\} \to (0,\infty)$ as a function of θ in the asymptotic maturity-strike regime (t, k_t), given by (3.4), for a short maturity t:

$$\sigma_t(\theta) := \widehat{\sigma}(t, k_t). \tag{3.5}$$

The implied volatility $\sigma_t(\theta)$ is of interest in the context of processes with jumps, because its limit $\sigma_0(\theta)$, as $t \downarrow 0$, exists and is finite for each θ, depends on both the jump and the diffusion components of the process and can be computed explicitly in terms of the parameters. In order to find the asymptotic behavior of $\sigma_t(\theta)$, we first state Theorem 3.1, which relates the asymptotics of $\sigma_t(\theta)$ to the asymptotic behavior of the OTM option price

$$I_t(\theta) := C(t,k_t)1_{\{\theta>0\}} + P(t,k_t)1_{\{\theta<0\}}, \tag{3.6}$$

under the model S as maturity t tends to zero.

THEOREM 3.1. *Let S be a martingale model for a risky security with $S_0 = 1$ and k_t a log-strike given in (3.4) for a fixed $\theta \in \mathbb{R}\backslash\{0\}$. Let $\widehat{C}_t$ and $\widehat{P}_t$ be deterministic functions such that $C(t,k_t) \sim \widehat{C}_t$ and $P(t,k_t) \sim \widehat{P}_t$ as $t \downarrow 0$, where $C(t,k_t)$ and $P(t,k_t)$ are given in (3.2),* *and define* $\widehat{I}_t(\theta) := \widehat{C}_t 1_{\{\theta>0\}} + \widehat{P}_t 1_{\{\theta<0\}}$. *Assume further that the OTM option price* $I_t(\theta)$, *given in (3.6), satisfies:*

$$\frac{1}{2} < \liminf_{t\downarrow 0} \frac{\log I_t(\theta)}{\log t} \leq \limsup_{t\downarrow 0} \frac{\log I_t(\theta)}{\log t} < \infty. \tag{3.7}$$

Then the implied volatility $\sigma_t(\theta)$, *defined in (3.5), can be expressed by*

$$\sigma_t(\theta) = \frac{|\theta|}{\sqrt{2L_t(\theta)-1}} + \frac{|\theta| \log \frac{(2L_t(\theta)-1)^{\frac{3}{2}}\sqrt{2\pi}}{|\theta|}}{(2L_t(\theta)-1)^{\frac{3}{2}}} \frac{1}{\log\frac{1}{t}} + O\left(\frac{1}{\log^2 \frac{1}{t}}\right), \quad as \quad t \downarrow 0, \tag{3.8}$$

and

$$\sigma_t(\theta) = \frac{|\theta|}{\sqrt{2\widehat{L}_t(\theta)-1}} + \frac{|\theta| \log \frac{(2\widehat{L}_t(\theta)-1)^{\frac{3}{2}}\sqrt{2\pi}}{|\theta|}}{(2\widehat{L}_t(\theta)-1)^{\frac{3}{2}}} \frac{1}{\log\frac{1}{t}} + o\left(\frac{1}{\log \frac{1}{t}}\right), \quad as \quad t \downarrow 0, \tag{3.9}$$

where $L_t(\theta) := J_t(I_t(\theta))$ *and* $\widehat{L}_t(\theta) := J_t(\widehat{I}_t(\theta))$ *are defined by the formula*

$$J_t(x) := \frac{\log x}{\log t} - \frac{\log\log\frac{1}{t}}{\log\frac{1}{t}} \quad for \ any \quad x, t > 0.$$

Before proceeding with the application of Theorem 3.1 given in Corollary 3.3 below (see also Section 6.4), we make the following remarks in order to place its statement in context.

REMARKS 3.2.

(i) *In the Black–Scholes model with volatility* $\sigma > 0$, *the following well-known expansion of the call option price in the* (t, k_t) *maturity-strike regime (3.4) holds (e.g., a straightforward calculation using Gao and Lee 2011, equation (3.11) yields the expansion):*

$$C^{BS}(t, k_t, \sigma) = \frac{\sigma}{\sqrt{2\pi}} t^{\frac{1}{2}+\frac{\theta^2}{2\sigma^2}} \left\{ \frac{\sigma^2}{\theta^2}\frac{1}{\log\frac{1}{t}} - 3\frac{\sigma^4}{\theta^4}\frac{1}{\log^2\frac{1}{t}} + O\left(\frac{1}{\log^3\frac{1}{t}}\right)\right\} \quad as \quad t \downarrow 0. \tag{3.10}$$

In particular, we have $\log C^{BS}(t, k_t, \sigma) = (\frac{1}{2} + \frac{\theta^2}{2\sigma^2})\log t + o(\log(1/t))$ *as* $t \downarrow 0$ *and hence the assumption in (3.7) is satisfied in the Black–Scholes model.*

(ii) *Note that the log-strike* k_t *in (3.4) satisfies the assumptions of Theorem 2.1. For any Lévy process* X *as in Theorem 2.1, formula (2.6) and Remark (i) above imply*

$$\log C(t, k_t) = \min\left\{\frac{3-\alpha_+}{2}, \frac{1}{2} + \frac{\theta^2}{2\sigma^2}\right\} \log t + o(\log(1/t)) \quad as \quad t \downarrow 0. \tag{3.11}$$

Since the minimum of the constants in front of $\log t$ *is clearly larger than* $1/2$, *assumption (3.7) of Theorem 3.1 is satisfied. As we shall soon see, it is the balance (as a function of* θ*) between the two constants in (3.11) that determines the value of the limiting smile* $\sigma_0(\theta)$.

(iii) *Let a Lévy process X be as in Proposition 2.3 (i.e., with jumps of finite variation). Formulae (2.9) and (3.10) imply that the call option price* $C(t, k_t)$ *under the model S has the following asymptotic behavior*

$$\log C(t, k_t) = \min\left\{1, \frac{1}{2} + \frac{\theta^2}{2\sigma^2}\right\} \log t + o(\log(1/t)) \quad \text{as} \quad t \downarrow 0. \tag{3.12}$$

In particular, note that assumption (3.7) is satisfied and that, in the case of jumps with finite variation, the constant in front of $\log t$ *does not depend on the Lévy measure but solely on the diffusion component of the model.*

(iv) *In Gao and Lee (2011), the authors present a general result, which translates the asymptotic behavior of the option prices, in a generic maturity-strike regime, to the asymptotics of the corresponding implied volatilities. Unfortunately the results in Gao and Lee (2011) do not apply to the regime* (t, k_t), *for* k_t *in (3.4), since the essential standing assumption* $\max\{0, \log(1/k_t)\} = o(\log(1/C(t, k_t)))$ *in Gao and Lee (2011, equation (4.3)) is not satisfied in our setting by (3.11) and (3.12). We therefore have to establish Theorem 3.1, which is applicable in our context as remarked in (ii) and (iii) above.*

The main asymptotic formula of the paper is given in the following corollary.

COROLLARY 3.3. *Let X be a Lévy process with the jump measure* ν *and the Gaussian component* $\sigma^2 \geq 0$. *Pick* $\theta \in \mathbb{R} \setminus \{0\}$, *let* k_t *be the log-strike from (3.4) and let* $\sigma_t(\theta)$ *be the implied volatility defined in (3.5). Then the following statements hold.*

(a) *Let X be a Lévy process satisfying the assumptions of Theorem 2.1. Then the implied volatility* $\sigma_t(\theta)$ *takes the form*

$$\sigma_t(\theta) = \begin{cases} \frac{\pm\theta}{\sqrt{2-\alpha_\pm}}\left[1 + I_\pm(t, \theta)\right] + o\left(\frac{1}{\log\frac{1}{t}}\right), & \\ \quad \textit{if}\ \pm\theta \geq \sigma\sqrt{2-\alpha_\pm} \quad \textit{and}\ c_\pm > 0, & \\ \sigma + o\left(\frac{1}{\log\frac{1}{t}}\right), & \\ \quad \textit{if}\ 0 < \pm\theta < \sigma\sqrt{2-\alpha_\pm} \quad \textit{and}\ c_\pm > 0, \quad \textit{as}\ t \downarrow 0, & \end{cases} \tag{3.13}$$

where

$$I_\pm(t, \theta) := \frac{3-\alpha_\pm}{2(2-\alpha_\pm)} \frac{\log\log\frac{1}{t}}{\log\frac{1}{t}} + \frac{1}{(2-\alpha_\pm)} \log\left(\frac{(2-\alpha_\pm)^{\frac{3}{2}} c_\pm \sqrt{2\pi}}{|\theta|^{\alpha_\pm}(\alpha_\pm - 1)}\right) \frac{1}{\log\frac{1}{t}}, \quad \textit{for}\ t > 0, \tag{3.14}$$

and the sign $\pm$ *denotes either* $+$ *or* $-$ *throughout the formulae in (3.13) and (3.14). In particular, the limiting smile* $\sigma_0(\theta) := \lim_{t\downarrow 0} \sigma_t(\theta)$ *exists for any* $\theta \in \mathbb{R} \setminus \{0\}$ *and takes the form*

$$\sigma_0(\theta) = \max\left\{\frac{\pm\theta}{\sqrt{2-\alpha_\pm}}, \sigma\right\} \quad \textit{if}\ c_\pm > 0.$$

(b) Let a Lévy process X be as in Proposition 2.3 and let $\gamma_+, \gamma_- \geq 0$ be equal to the following integrals

$$\gamma_+ := \int_{(0,\infty)} (e^x - 1)\nu(dx), \quad \gamma_- := \int_{(-\infty,0)} (1 - e^x)\nu(dx).$$

Then the implied volatility $\sigma_t(\theta)$ for short maturity t is given by

$$\sigma_t(\theta) = \begin{cases} \pm\theta\,[1 + F_\pm(t,\theta)] + o\left(\frac{1}{\log \frac{1}{t}}\right), & \text{if } \pm\theta \geq \sigma \quad \text{and } \gamma_\pm > 0, \\ \sigma + o\left(\frac{1}{\log\frac{1}{t}}\right), & \text{if } 0 < \pm\theta < \sigma \quad \text{and } \gamma_\pm > 0, \end{cases} \quad \text{as } t \downarrow 0, \tag{3.15}$$

where

$$F_\pm(t,\theta) := \frac{\log\log\frac{1}{t}}{\log\frac{1}{t}} + \log\left(\frac{\gamma_\pm\sqrt{2\pi}}{|\theta|}\right)\frac{1}{\log\frac{1}{t}}, \quad \text{for } t > 0, \tag{3.16}$$

and $\pm$ denotes either $+$ or $-$ throughout the formulae in (3.15) and (3.16). The limit of the implied volatility smile as maturity tends to zero, $\sigma_0(\theta) := \lim_{t\downarrow 0}\sigma_t(\theta)$, exists for $\theta \in \mathbb{R} \setminus \{0\}$ and is equal to

$$\sigma_0(\theta) = \max\{\pm\theta, \sigma\} \quad \text{if } \gamma_\pm > 0.$$

REMARKS 3.4.

(i) Recall display (2.3) in Theorem 2.1 and note that the assumptions $c_+ > 0$ and $c_- > 0$ of Corollary 3.3 (a) mean that, as $x \downarrow 0$, the tails around zero of the Lévy measure ν of X behave as $\nu((x,\infty)) \sim c_+ x^{-\alpha_+}$ and $\nu((-\infty,-x)) \sim c_- x^{-\alpha_-}$. Note further that, once we have identified the precise rate of the tail behavior of ν at zero, the constants c_+ and c_- do not feature in the limiting formula $\sigma_0(\theta)$.

(ii) The assumption $\gamma_\pm > 0$ in Corollary 3.3 (b) ensures that the process X has positive jumps when $\theta > 0$ and negative jumps when $\theta < 0$ as we are only interested in the asymptotic behavior of the implied volatility in the presence of jumps.

In Section 6.3 we establish Theorem 3.1 and in Section 6.4 we derive Corollary 3.3 from Theorem 3.1.

4. NUMERICAL RESULTS

In this section, we present some numerical illustrations for the convergence results discussed in Section 3. We assume that the process X follows the widely used CGMY model (Carr, Geman, Madan, and Yor 2002) with Lévy density

$$\nu(x) = \frac{ce^{-\lambda_+ x}}{|x|^{1+\alpha}}1_{\{x>0\}} + \frac{ce^{-\lambda_-|x|}}{|x|^{1+\alpha}}1_{\{x<0\}}. \tag{4.1}$$

For this process, the price of a European call option with payoff $(S_0e^{X_t} - K)^+$ at time t can be computed as

$$\frac{K}{2\pi}\int_{\mathbb{R}}\left(\frac{K}{S_0}\right)^{iu-R}\frac{\phi_t(-u - iR)}{(R - iu)(R - 1 - iu)}du, \tag{4.2}$$

where ϕ_t is the characteristic function of X_t and $R > 1$ (see, e.g., Carr and Madan 1998 or Tankov 2010). We compute the integral in (4.2) with an adaptive integration algorithm.

4.1. Testing the Algorithm

To ensure that the prices returned by our algorithm are correct, we first compare them to the values computed in Wang, Wan and Forsyth (2007) with their approximate "fixed point" algorithm (discretization of the partial differential equation). The following table shows that the values we obtain are very similar to those computed in Wang et al. (2007) with small discrepancies probably due to the discretization error of Wang et al. (2007).

S	K	T	r	c	λ_+	λ_-	α	Value (Wang et al. 2007)	Our value
90	98	0.25	0.06	16.97	29.97	7.08	0.6442	16.212578	16.211904
90	98	0.25	0.06	0.42	191.2	4.37	1.0102	2.2307031	2.2306558
10	10	0.25	0.1	1	9.2	8.8	1.8	4.3714972	4.3898433

4.2. Convergence of the ATM Options

In this section, we fix the parameters of the process at

$$c = 1, \quad \lambda_+ = \lambda_- = 3, \quad \alpha = 1.5, \tag{4.3}$$

and $S_0 = 1$. First, we analyze the rate of convergence to zero of the ATM options. It follows from the results in Muhle-Karbe and Nutz (2011) that the ATM option price satisfies

$$\mathbb{E}[(e^{X_t} - 1)^+] \sim t^{1/\alpha}\mathbb{E}[(Z^*)^+],$$

where Z^* is a stable random variable with the Lévy density $\frac{c}{|x|^{1+\alpha}}$. Furthermore, it is known (see, e.g., Samorodnitsky and Taqqu 1994, property 1.2.17) that

$$\mathbb{E}[(Z^*)^+] = \frac{(2c)^{1/\alpha}}{\pi}\Gamma(1 - 1/\alpha)\left(\Gamma(-\alpha)\left|\cos\frac{\pi\alpha}{2}\right|\right)^{1/\alpha} =: C.$$

Figure 4.1 plots the dependence of the normalized option price $t^{-1/\alpha}\mathbb{E}[(e^{X_t} - 1)^+]$ and the normalized "Bachelier" price $t^{-1/\alpha}\mathbb{E}[(X_t)^+]$ on $\log t$, i.e., on time to maturity expressed on the log-scale. The horizontal line in Figure 4.1 corresponds to the value of the constant C. The desired convergence is clearly visible.

4.3. Convergence of Option Prices with Variable Strike

In this section, we investigate numerically the convergence of the OTM option prices given in Theorem 2.1. The parameter values for the underlying process are given in (4.3). Note that in the case of the CGMY model with Lévy density (4.1), the limits in (2.5) of

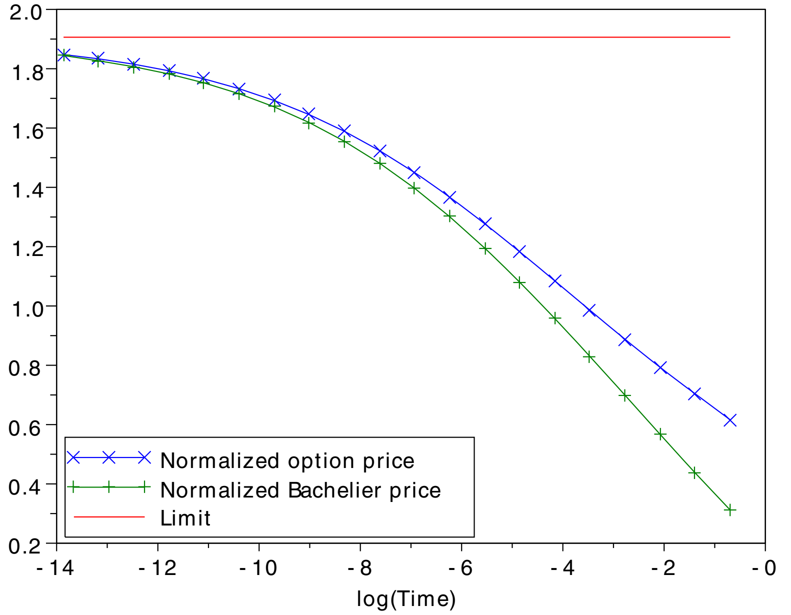

FIGURE 4.1. Convergence of the renormalized price for ATM options: the parameters of the CGMY process are given in (4.3).

Theorem 2.1 take the form

$$\lim_{x\downarrow 0} x^\alpha \nu((x,\infty)) = \lim_{x\downarrow 0} x^\alpha \nu((-\infty,-x)) = \frac{c}{\alpha}.$$

Figure 4.2 shows the dependence of the normalized option and "Bachelier" prices, respectively, given by

$$\frac{\mathbb{E}[(e^{X_t}-e^{k_t})^+]}{tk_t^{1-\alpha}} \quad \text{and} \quad \frac{\mathbb{E}[(X_t-k_t)^+]}{tk_t^{1-\alpha}},$$

on time to maturity in log-scale, where

$$k_t = t^{1/\alpha'} \quad \text{with} \quad \alpha' = 1.9.$$

The horizontal dotted line shows the limiting value $\frac{c}{\alpha(\alpha-1)} = \frac{4}{3}$ predicted by Theorem 2.1.

Similarly, Figure 4.3 plots the dependence on time to maturity (on the log-scale) of the normalized option price

$$\frac{\mathbb{E}[(e^{X_t}-e^{k_t})^+]}{tk_t^{1-\alpha}} \quad \text{for} \quad k_t = \theta\sqrt{t\log\frac{1}{t}} \quad \text{and} \quad \theta \in \{0.1, 0.2, 0.3\}.$$

As in Figure 4.2, the limiting horizontal dotted line is given by $\frac{c}{\alpha(\alpha-1)} = \frac{4}{3}$.

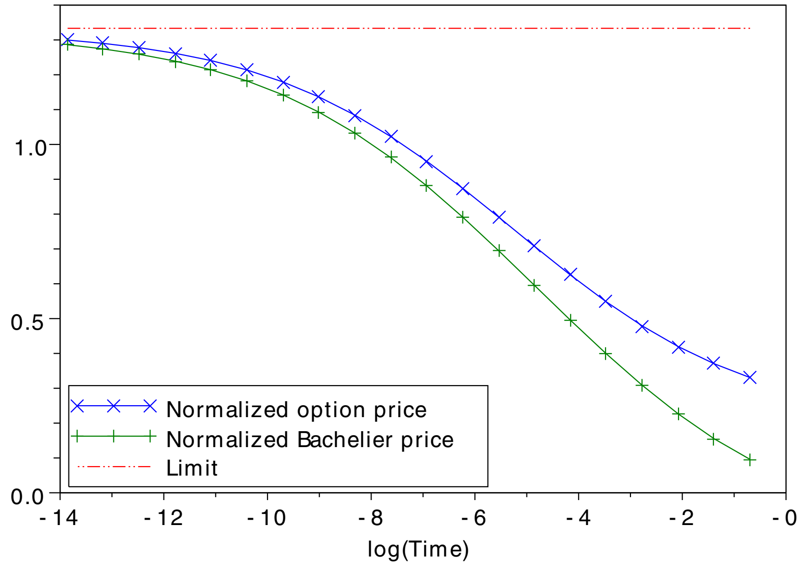

FIGURE 4.2. Convergence of the renormalized price for OTM options: $k_t = t^{\frac{1}{\alpha'}}$ with $\alpha' = 1.9$ and the parameters of the process X are given in (4.3).

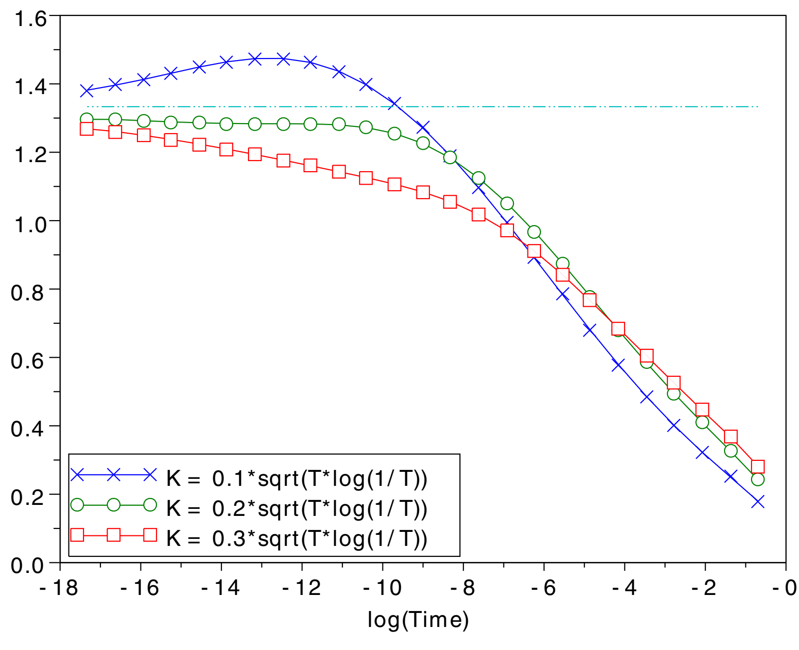

FIGURE 4.3. Convergence of the renormalized price for OTM options: $k_t = \theta\sqrt{t \log \frac{1}{t}}$ with $\theta \in \{0.1, 0.2, 0.3\}$ and the parameters of the process X are given in (4.3).

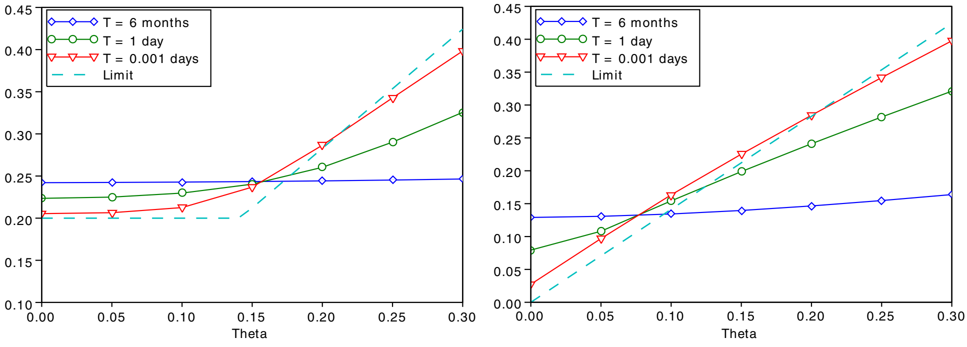

FIGURE 4.4. Convergence of the implied volatilities. Left: a diffusion component is present. Right: no diffusion component.

4.4. Convergence of the Implied Volatilities to the Limiting Smile

In this section, we illustrate the convergence of the implied volatility (expressed as a function of the renormalized strike θ) to the limit $\sigma_0(\theta)$ given in Corollary 3.3. In order to test the formula both with and without the diffusion component, we fix two models: the first model is a pure-jump CGMY Lévy process with the following parameter values

$$c = 0.01, \quad \lambda_+ = \lambda_- = 3, \quad \alpha = 1.5,$$

which corresponds to the unit annualized volatility of about 14%. The second model is the same CGMY process with an additional diffusion component of volatility $\sigma = 0.2$.

Recall that the limiting formula for positive θ is $\sigma_0(\theta) = \max\{\sigma, \frac{\theta}{\sqrt{2-\alpha}}\}$. Figure 4.4 plots the right wing of the implied volatility smile (as a function of θ) for different times to maturity when the diffusion component is present (left graph) and the diffusion component is absent (right graph), together with the limiting shape $\sigma_0(\theta)$. The convergence to the limit is visible in both graphs but slow, because the error terms in Corollary 3.3 are logarithmic in time. Nevertheless, the following observations can be made already at "not such small" times:

- The smile is remarkably stable in time, when it is expressed as function of the renormalized variable θ. In particular, the slope of the wings predicted by Corollary 3.3 is achieved rather quickly.
- The distinction between the U-shaped smile in the presence of a diffusion component and the V-shaped smile in the pure-jump case, is clearly visible.

4.5. Approximation of the Implied Volatility for Small Times to Maturity

In this section, we illustrate the approximation of the implied volatility at small times by the asymptotic formula (3.13). We take the same parameters of the CGMY process as in Section 4.4 and consider the case $\sigma = 0$ (when the diffusion component is present, in the region where the pure-jump component dominates, the asymptotic formula is the same, and in the diffusion-dominated region, there are no additional terms added to the constant limit). Figure 4.5 illustrates the quality of the approximation for $t = 1$ day and $t = 0.1$ days.

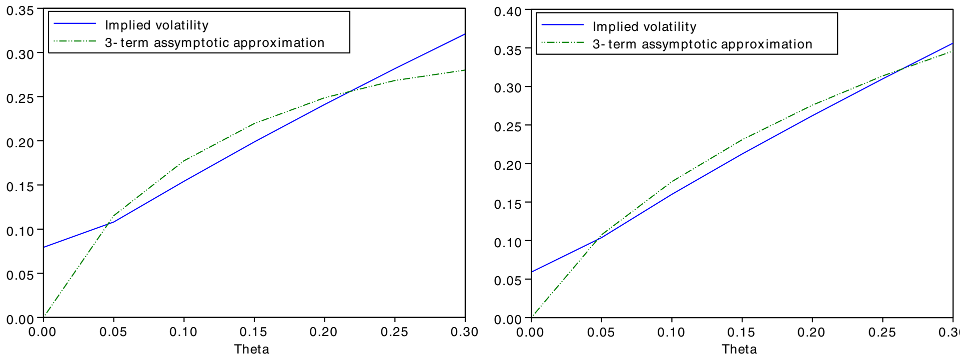

FIGURE 4.5. Approximation of the implied volatilities by the asymptotic formula (3.13). Left: $t = 1$ day. Right: $t = 0.1$ days.

5. A QUALITATIVE COMPARISON WITH MARKET SMILES

5.1. Aim

In this section, we aim to compare our theoretical insights into the shape of the short-maturity smile with the actual market smiles and to assess, using observed market quotes, the parameterizations of the implied volatility smiles in terms of the theta, delta, and strike. We base our qualitative analysis on the option prices for the two most liquid currency pairs: USDJPY and EURUSD. Figures 5.1 and 5.2 depict the implied volatilities corresponding to the five option prices for each currency pair, expressed in terms of the aforementioned parameterizations (midday quotes for two recent dates for each currency pair are used). The plotted implied volatilities give the market prices for the options with the following strikes: ATM, 25-delta call and put, 10-delta call and put, and maturities ranging from 1 day to 2 months. The options on the two currency pairs, which correspond to these strikes and maturities, are chosen as the basis of our analysis as they are the most liquid options in FX markets.

5.2. FX Option Quotes

In FX derivative markets, the option prices are typically quoted in terms of the implied volatility σ for a given amount of the Black–Scholes call (resp. put) delta $\Delta_C(K, \sigma)$ (resp. $\Delta_C(K, \sigma)$):

$$\Delta_C(K,\sigma) = N\left(\frac{\log(e^{(r_f - r_d)t} S_0/K)}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}\right) \quad (\text{resp. } \Delta_P(K,\sigma) = \Delta_C(K,\sigma) - 1),$$

where $N(\cdot)$ is the standard normal cumulative distribution function, S_0 is the current exchange rate, t is the maturity, and $(r_f - r_d)$ is the interest rate differential between the two currencies. Note that the ATM strike corresponds to a 50-delta call strike and the 25-delta (resp. 10-delta) put strike is equal to the 75-delta (resp. 90-delta) call strike.[1]

[1]The implied volatilities in the markets are quoted for the OTM and ATM options only. For representational convenience, in Figures 5.1 and 5.2, we plot the smiles in terms of the call delta only. Note also that the definition of the delta used for option quotes in FX markets is different from the actual delta of the option, which includes an additional $e^{-r_f t}$ factor.

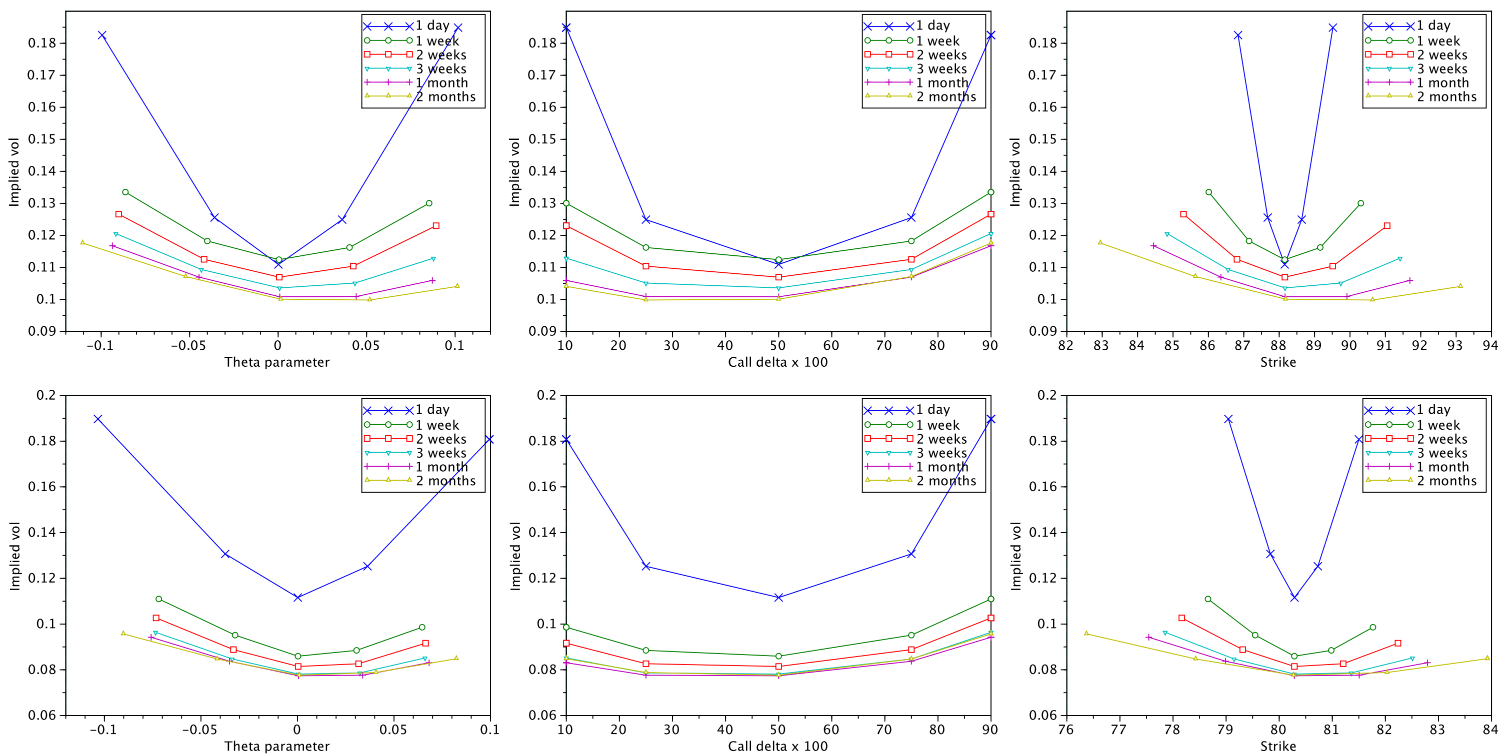

FIGURE 5.1. USDJPY option prices: January 4, 2013 (above) and November 5, 2012 (below).

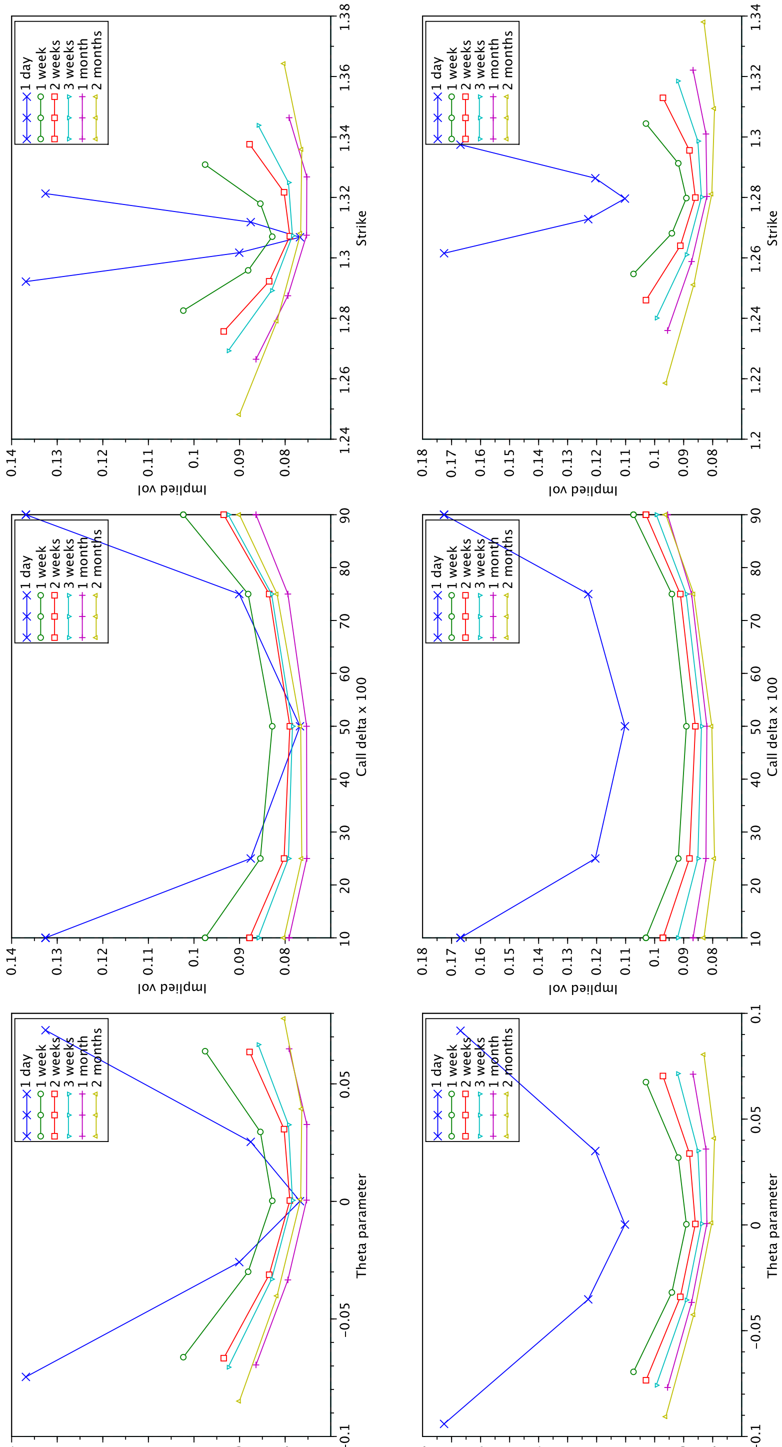

FIGURE 5.2. EURUSD option prices: Januray 4, 2013 (above) and November 5, 2012 (below).

The parameterization in terms of the delta is convenient for the traders as it expresses by its definition the amount of delta risk contained in the quoted option. However, in order to obtain the parameterization of the implied volatility smile in terms of the strike (the rightmost graphs in Figures 5.1 and 5.2), one has to solve the nonlinear equation in the strike K with the right-hand side given by the market quote for the implied volatility and the left-hand side equal to the formula for either $\Delta_C(K, \sigma)$ or $\Delta_P(K, \sigma)$. By contrast, the theta parameterization of the implied volatility smile (the leftmost graphs in Figures 5.1 and 5.2) is given by the simple formula in (3.4), which relates explicitly the strikes of the quoted options to the values of the parameter θ.

5.3. Discussion

In the context of this paper, it is natural to ask how formula (3.4) for the strike behaves across different maturities and whether the theta parameterization of the smile (1.1) relates to the market data. We now briefly discuss these questions.

5.3.1. Stability across maturities. It is clear from the graphs in Figures 5.1 and 5.2 that the parameterization of the implied volatility smile in terms of delta is more stable across different maturities than the parameterization based on strike. As described in Section 5.2, the parameterization of the smile in terms of theta possesses a much simpler relationship to the parameterization based on strike than the one based on delta. And yet, Figures 5.1 and 5.2 suggest that the stability of the smile across maturities is similar to that exhibited by the delta parameterization.

5.3.2. Implied volatility formula as a function of θ. It can be observed in Figures 5.1 and 5.2 that the 1-day market-implied volatilities have a qualitatively similar shape to that predicted by the limiting formula in (1.1). In particular, it appears that the slope of the wings of the simile in the leftmost graphs in Figures 5.1 and 5.2, computed from the two extreme points in the graph, is close to one as predicted by the limiting formula in (1.1) for the finite variation case. Furthermore, it appears that on November 5, 2012, the smiles for both currency pairs were converging to a flat smile (in θ) close to the ATM (i.e., $\theta = 0$), which, by formula (1.1), implies the presence of the diffusion component in the underlying model. Analogously, the market data on January 4, 2013 would appear to suggest that on that day no diffusion component was present.

It should be stressed, however, that the maturity t equal to 1 day is, in the context of the smile formula in (1.1), still far from the limit since the magnitude of the error is of order of $\frac{\log\log(1/t)}{\log(1/t)}$ (see Corollary 3.3). This fact makes it difficult to quantify, based on the market data, the observations on the structure of the underlying model made in the paragraph above. In particular, it is not feasible to estimate the Blumenthal–Getoor indices of the positive and negative jumps of the underlying process, based on the smile formula in (1.1), if only 1-day options data are available. We stress that the main aim of our study is not to develop quantitative estimation algorithms from short maturity options, but provide explicit insights into the qualitative behavior of the short-maturity smile in jump models.

6. PROOFS

6.1. Proof of Theorem 2.1

By Lemma A.1, to prove Theorem 2.1, it is sufficient to show that

$$\mathbb{E}[(X_t - k_t)^+] = \mathbb{E}[(\sigma W_t - k_t)^+] + \frac{t k_t^{1-\alpha_+} c_+}{\alpha_+ - 1} + o(t k_t^{1-\alpha_+} + \mathbb{E}[(\sigma W_t - k_t)^+]), \tag{6.1}$$

as $t \downarrow 0$ for the call case and

$$\mathbb{E}[(-k_t - X_t)^+] = \mathbb{E}[(-k_t - \sigma W_t)^+] + \frac{t k_t^{1-\alpha_-} c_-}{\alpha_- - 1} + o(t k_t^{1-\alpha_-} + \mathbb{E}[(-k_t - \sigma W_t)^+]), \tag{6.2}$$

as $t \downarrow 0$ for the put case. Note that (6.2) follows from (6.1) by a substitution $X \mapsto -X$. Therefore, from now on we concentrate on the proof of (6.1), assuming with no loss of generality that $c_+ > 0$.

Step 1. In this first step, we assume that $\nu((-\infty, 0)) = 0$ and would like to prove

$$\mathbb{E}[(X_t - k_t)^+] = \mathbb{E}[(\sigma W_t - k_t)^+] + \frac{t k_t^{1-\alpha_+} c_+}{\alpha_+ - 1} + o(t k_t^{1-\alpha_+}). \tag{6.3}$$

Fix $t > 0$, and $\varepsilon > 0$ with $\varepsilon < \frac{1}{32}$, let X^t be a Lévy process with no diffusion part, Lévy measure $\nu(dx) 1_{\{0 < x \le \varepsilon k_t\}}$ and the third component of the characteristic triplet

$$\gamma_t = \gamma - \int_{(\varepsilon k_t, 1]} z \nu(dz).$$

Let $(\xi_i^t)_{i \ge 1}$ be a sequence of i.i.d. random variables with the probability distribution

$$\frac{\nu(dz) 1_{\{z > \varepsilon k_t\}}}{\nu(\{z : z > \varepsilon k_t\})},$$

and N^t a standard Poisson process with intensity $\lambda_t := \nu(\{z : z > \varepsilon k_t\})$. Furthermore, we assume that X^t, N^t, and $(\xi_i^t)_{i \ge 1}$ are independent. Then the following equality in law holds

$$X_t \stackrel{d}{=} \sigma W_t + X_t^t + \sum_{i=1}^{N_t^t} \xi_i^t, \tag{6.4}$$

and it follows that

$$\mathbb{E}\left[(X_t - k_t)^+\right] = e^{-\lambda_t t} \mathbb{E}\left[\left(\sigma W_t + X_t^t - k_t\right)^+\right] \tag{6.5}$$

$$+\lambda_t t e^{-\lambda_t t} \mathbb{E}\left[\left(\sigma W_t + X_t^t + \xi_1^t - k_t\right)^+\right] \tag{6.6}$$

$$+e^{-\lambda_t t} \sum_{k \ge 2} \frac{(\lambda_t t)^k}{k!} \mathbb{E}\left[\left(\sigma W_t + X_t^t + \sum_{i=1}^k \xi_i^t - k_t\right)^+\right]. \tag{6.7}$$

As a preliminary computation, we deduce from the assumptions of the theorem that the following asymptotic behavior holds as $t \downarrow 0$ (recall definition (2.2a)):

$$\lambda_t \sim c_+(\varepsilon k_t)^{-\alpha_+}, \quad \Sigma_t := \int_{(0,k_t\varepsilon]} z^2\nu(dz) \sim \tfrac{2c_+}{2-\alpha_+}(\varepsilon k_t)^{2-\alpha_+}, \tag{6.8}$$

$$\gamma_t \sim \tfrac{c_+}{(\varepsilon k_t)^{\alpha_+-1}}, \quad \mathbb{E}[(X_t^t)^2] = t^2(\gamma_t)^2 + t\Sigma_t \sim t\Sigma_t, \quad \mathbb{E}[\xi_1^t] \sim \tfrac{\alpha_+}{\alpha_+-1}\varepsilon k_t, \tag{6.9}$$

$$\begin{aligned}\mathbb{E}[(X_t^t)^4] &= t\int_{(0,k_t\varepsilon]} z^4\nu(dz) + 4t^2\gamma_t\int_{(0,k_t\varepsilon]} z^3\nu(dz) + 3t^2\Sigma_t^2 + 6t^3\gamma_t^2\Sigma_t + t^4\gamma_t^4 \\ &\sim \tfrac{4c_+}{4-\alpha_+} t k_t^{4-\alpha_+}.\end{aligned} \tag{6.10}$$

To estimate the term in (6.5), we apply the argument inspired by lemma 2 in Rüschendorf and Woerner (2002). In the current notation this implies

$$\mathbb{P}[X_t^t > k_t] \le \exp\left\{-t\int_{\gamma_t}^{k_t/t} \tau(z)dz\right\}, \tag{6.11}$$

where $\tau : [\gamma_t, \infty) \to \mathbb{R}$ is the inverse function of $s : [0,\infty) \to \mathbb{R}$ defined by

$$s(x) = \gamma_t + \int_{(0,k_t\varepsilon]} z(e^{zx} - 1)\nu(dz).$$

By Taylor's theorem, this function satisfies

$$s(x) \le \gamma_t + xe^{k_t\varepsilon x}\Sigma_t \le \gamma_t + \frac{e^{2k_t\varepsilon x} - 1}{k_t\varepsilon}\Sigma_t.$$

This implies that

$$\tau(z) \ge \frac{1}{2k_t\varepsilon}\log\left\{1 + \frac{z-\gamma_t}{\Sigma_t}k_t\varepsilon\right\},$$

and, therefore, substituting this into (6.11),

$$\begin{aligned}\mathbb{P}[X_t^t > k_t] &\le \exp\left\{-\tfrac{t\Sigma_t}{2(k_t\varepsilon)^2}\int_0^{\frac{k_t\varepsilon}{\Sigma_t}(k_t/t-\gamma_t)} \log(1+s)ds\right\} \\ &\le \exp\left\{-\tfrac{k_t-\gamma_t t}{2k_t\varepsilon}\log\left(\tfrac{k_t\varepsilon}{e\Sigma_t}(k_t/t-\gamma_t)\right)\right\}.\end{aligned} \tag{6.12}$$

From the assumptions of the theorem and (6.8)–(6.10), there exists $t_1 > 0$ such that $t < t_0$ implies

$$\mathbb{P}[X_t^t > k_t] \le \exp\left\{-\frac{1}{4\varepsilon}\log\left(\frac{k_t^2\varepsilon}{2et\Sigma_t}\right)\right\} = \left(\frac{k_t^2\varepsilon}{2et\Sigma_t}\right)^{-\frac{1}{4\varepsilon}} \le C(tk_t^{-\alpha_+})^{\frac{1}{4\varepsilon}} \le C(tk_t^{-\alpha_+})^8,$$

for some constant $C < \infty$. By similar arguments it can be shown that

$$\mathbb{P}\left[X_t^t > \frac{k_t}{2}\right] \le C(tk_t^{-\alpha_+})^4. \tag{6.13}$$

Coming back to the estimation of (6.5), we first deal with the case $\sigma = 0$. In this case, the Cauchy–Schwartz inequality allows to conclude that

$$\mathbb{E}\left[\left(X_t^t - k_t\right)^+\right] \le \mathbb{E}\left[\left(X_t^t\right)^2\right]^{\frac{1}{2}} \mathbb{P}[X_t^t > k_t]^{\frac{1}{2}} = O(k_t(tk_t^{-\alpha_+})^2),$$

because the first factor remains bounded by (6.8)–(6.10).

Let us now focus on the case $\sigma > 0$. Let $f(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty (z-x)e^{-\frac{z^2}{2}}\,dz$. The expectation in (6.5) can be expressed as

$$\mathbb{E}[(\sigma W_t + X_t^t - k_t)^+] = \sigma\sqrt{t}\mathbb{E}\left[f\left(\frac{k_t - X_t^t}{\sigma\sqrt{t}}\right)\right].$$

By Taylor's formula, we then get

$$\begin{aligned}
\mathbb{E}[(\sigma W_t + X_t^t - k_t)^+] &= \sigma\sqrt{t} f\left(\tfrac{k_t}{\sigma\sqrt{t}}\right) - f'\left(\tfrac{k_t}{\sigma\sqrt{t}}\right)\mathbb{E}[X_t^t] \\
&\quad + \tfrac{1}{\sigma\sqrt{t}}\mathbb{E}\left[(X_t^t)^2 \int_0^1 (1-\theta) f''\left(\frac{k_t - \theta X_t^t}{\sigma\sqrt{t}}\right) d\theta\right] \\
&= \mathbb{E}[(\sigma W_t - k_t)^+] + \gamma_t t\mathbb{P}[\sigma W_t > k_t] + \tfrac{1}{\sigma\sqrt{2\pi t}}\mathbb{E}\left[(X_t^t)^2 \int_0^1 (1-\theta) e^{-\frac{1}{2}\left(\frac{k_t - \theta X_t^t}{\sigma\sqrt{t}}\right)^2} d\theta\right].
\end{aligned}$$

We now need to show that the second and the third terms do not contribute to the limit. Since by assumption $\frac{\sqrt{t}}{k_t} \to 0$, we have that $\mathbb{P}[\sigma W_t > k_t] \to 0$ as $t \to 0$, and, therefore, by (6.8)–(6.10),

$$\gamma_t t\mathbb{P}[\sigma W_t > k_t] = o(tk_t^{1-\alpha_+}).$$

The last term can be split into two terms, which are easy to estimate using (6.8)–(6.10):

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\mathbb{E}\left[(X_t^t)^2 1_{\{X_t^t \le \frac{k_t}{2}\}} \int_0^1 (1-\theta) e^{-\frac{1}{2}\left(\frac{k_t - \theta X_t^t}{\sigma\sqrt{t}}\right)^2} d\theta\right] \le \tfrac{1}{\sqrt{t}}\mathbb{E}[(X_t^t)^2] e^{-\frac{1}{8}\left(\frac{k_t}{\sigma\sqrt{t}}\right)^2} \\
&\quad = O(tk_t^{1-\alpha_+})\tfrac{k_t}{\sqrt{t}} e^{-\frac{1}{8}\left(\frac{k_t}{\sigma\sqrt{t}}\right)^2} = o(tk_t^{1-\alpha_+}),
\end{aligned}$$

because by assumption of the theorem, $\frac{k_t}{\sqrt{t}} \to \infty$. On the other hand,

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\mathbb{E}\left[(X_t^t)^2 1_{\{X_t^t > \frac{k_t}{2}\}} \int_0^1 (1-\theta) e^{-\frac{1}{2}\left(\frac{k_t - \theta X_t^t}{\sigma\sqrt{t}}\right)^2} d\theta\right] \le \tfrac{1}{\sqrt{t}}\mathbb{E}\left[(X_t^t)^2 1_{\{X_t^t > \frac{k_t}{2}\}}\right] \\
&\quad \le \tfrac{1}{\sqrt{t}}\mathbb{E}[(X_t^t)^4]^{\frac{1}{2}}\mathbb{P}[X_t^t > \tfrac{k_t}{2}]^{\frac{1}{2}} = O(k_t^{2-\frac{\alpha_+}{2}})O((tk_t^{-\alpha_+})^2) = o(tk_t^{1-\alpha_+}),
\end{aligned}$$

by (6.10) and (6.13). We have, therefore, shown that

$$\mathbb{E}[(\sigma W_t + X_t^t - k_t)^+] = \mathbb{E}[(\sigma W_t - k_t)^+] + o(tk_t^{1-\alpha_+}).$$

From (6.8), the assumption on k_t in Theorem 2.1 and the Lipschitz property of the function $x \mapsto x^+$, it follows that

$$e^{-\lambda_t t}\mathbb{E}[(\sigma W_t + X_t^t - k_t)^+] = \mathbb{E}[(\sigma W_t - k_t)^+] + o(tk_t^{1-\alpha_+})$$

as well.

For the term in (6.6), the Lipschitz property of the function $x \mapsto x^+$, (6.8)–(6.10) and the assumption of the theorem (i.e., the first assumption on k_t in Theorem 2.1 in the case $\sigma = 0$ and the second one otherwise) imply the following estimate:

$$\begin{aligned}\lambda_t t\left|\mathbb{E}\left[\left(\sigma W_t + X_t^t + \xi_1^t - k_t\right)^+\right] - \mathbb{E}\left[\left(\xi_1^t - k_t\right)^+\right]\right| &\le \lambda_t t\{\mathbb{E}[|X_t^t|] + \sigma\mathbb{E}[|W_t|]\} \\ \le \lambda_t t\{\mathbb{E}[|X_t^t|^2]^{1/2} + \sigma\sqrt{t}\} = O(\lambda_t t^{\frac{3}{2}}\Sigma_t^{\frac{1}{2}}) + \sigma\lambda_t t^{\frac{3}{2}} &= o(tk_t^{1-\alpha_+}) \quad \text{as } t \to 0.\end{aligned}$$

On the other hand, integration by parts implies

$$\lambda_t t\mathbb{E}\left[\left(\xi_1^t - k_t\right)^+\right] = t\int_{k_t}^{\infty}(z - k_t)\,\nu(dz) = t\int_{k_t}^{\infty}U(z)dz \sim \frac{tk_t^{1-\alpha_+}c_+}{\alpha_+ - 1} \quad \text{as } t \to 0,$$

where $U(z) := \nu((z, \infty))$, which yields the second term in (2.6).

To treat the summand in (6.7), observe that by (6.8)–(6.10), for $k \ge 2$,

$$\begin{aligned}\mathbb{E}\left[\left(\sigma W_t + X_t^t + \textstyle\sum_{i=1}^k \xi_i^t - k_t\right)^+\right] &\le \sigma\sqrt{t} + \mathbb{E}[|X_t^t|^2]^{1/2} + k\mathbb{E}[\xi_1^t] \\ &= \sigma\sqrt{t} + O(t^{1/2}k_t^{1-\alpha_+/2}) + kO(k_t) = kO(k_t).\end{aligned}$$

Therefore, the summand in (6.7) is of order $O(k_t\lambda_t^2 t^2) = O(k_t(tk_t^{-\alpha_+})^2)$ and hence $o(tk_t^{1-\alpha_+})$.

Step 2. We now treat the case when $\nu((-\infty, 0)) \neq 0$. Let X^- be a spectrally negative Lévy process with zero mean and zero diffusion part and Y be a spectrally positive Lévy process such that $X^- + Y \stackrel{d}{=} X$. Let $\beta \in (\max(\alpha_+, \alpha_-), \alpha)$ (where we take $\alpha = 2$ is $\sigma > 0$) and $\chi_t = t^{\frac{1}{\beta}}$. As before, we fix $\varepsilon > 0$ and let $\bar{X}^t$ be a Lévy process with no diffusion part, zero mean and Lévy measure $\nu(dx)1_{\{-\varepsilon\chi_t \le x < 0\}}$, let $\bar{\gamma}_t = \int_{(-\infty, -\varepsilon\chi_t)} z\nu(dz)$, let $(\bar{\xi}_i^t)_{i \ge 1}$ be a sequence of i.i.d. random variables with the probability distribution

$$\frac{\nu(dz)1_{\{z < \varepsilon\chi_t\}}}{\nu(\{z : z < \varepsilon\chi_t\})},$$

and finally $\bar{\lambda}_t = \nu(\{z : z < \varepsilon\chi_t\})$. Decomposing X^- similarly to (6.4) in terms of $\bar{X}^t$ and $(\bar{\xi}_i^t)_{i \ge 1}$, it is easy to show that the option price $\mathbb{E}[(X_t - k_t)^+]$ admits an upper bound

$$\begin{aligned}\mathbb{E}[(X_t - k_t)^+] &= \mathbb{E}[(X_t^- + Y_t - k_t)^+] \le \mathbb{E}[(\bar{X}_t^t + \bar{\gamma}_t t + Y_t - k_t)^+] \\ &\le \mathbb{E}[(Y_t + \chi_t - k_t)^+]\mathbb{P}[\bar{X}_t^t \le \chi_t] + \mathbb{E}[X_t^2]^{\frac{1}{2}}\mathbb{P}[\bar{X}_t^t > \chi_t]^{\frac{1}{2}},\end{aligned}$$

and a lower bound

$$
\begin{aligned}
\mathbb{E}[(X_t - k_t)^+] &= \mathbb{E}[(X_t^- + Y_t - k_t)^+] \geq e^{-\bar{\lambda}_t t}\mathbb{E}[(\bar{X}_t^t + \bar{\gamma}_t t + Y_t - k_t)^+] \\
&\geq e^{-\bar{\lambda}_t t}\mathbb{E}[(-\chi_t + \bar{\gamma}_t t + Y_t - k_t)^+] - e^{-\bar{\lambda}_t t}\mathbb{P}[\bar{X}_t^t < -\chi_t]\mathbb{E}[(-\chi_t + \bar{\gamma}_t t + Y_t - k_t)^+].
\end{aligned}
$$

Similarly to (6.8)–(6.10), we have

$$
\bar{\Sigma}_t := \int_{(-\varepsilon\chi_t, 0)} z^2 \nu(dz) \sim \frac{2}{2-\alpha_-}(\varepsilon\chi_t)^{2-\alpha_-},
$$

and with the same logic as in (6.12), we have that

$$
\mathbb{P}[\bar{X}_t^t > \chi_t] \leq \left(\frac{\chi_t^2 \varepsilon}{e\bar{\Sigma}_t t}\right)^{\frac{1}{2\varepsilon}} \sim \left(t^{\frac{\alpha_-}{\beta}-1}\right)^{\frac{1}{2\varepsilon}}, \quad t \to 0.
$$

It is now clear that one can choose $\varepsilon > 0$ so that $\sqrt{\mathbb{P}[\bar{X}_t^t > \chi_t]}$ is of order of $o(tk_t^{1-\alpha_+})$. Since $\mathbb{P}[\bar{X}_t^t < -\chi_t]$ admits the same estimate, and $t\bar{\lambda}_t \to 0$ as $t \to 0$, we get that for some deterministic functions m_t and M_t which converge to 1 as $t \to 0$,

$$
\begin{aligned}
\mathbb{E}[(X_t - k_t)^+] &\geq m_t\mathbb{E}[(Y_t - \chi_t + \bar{\gamma}_t t - k_t)^+], \\
\mathbb{E}[(X_t - k_t)^+] &\leq M_t\mathbb{E}[(Y_t + \chi_t - k_t)^+] + o(tk_t^{1-\alpha_+}).
\end{aligned}
$$

Since $\chi_t = o(k_t)$ and $\bar{\gamma}_t t = o(k_t)$, from (6.3), we then get

$$
\begin{aligned}
\mathbb{E}[(X_t - k_t)^+] &\geq m_t\mathbb{E}[(\sigma W_t - \chi_t + \bar{\gamma}_t t - k_t)^+] + m_t\frac{t(k_t + \chi_t - \bar{\gamma}_t t)^{1-\alpha_+}c_+}{\alpha_+ - 1} + o(tk_t^{1-\alpha_+}) \\
&= m_t\mathbb{E}[(\sigma W_t - \chi_t + \bar{\gamma}_t t - k_t)^+] + \frac{tk_t^{1-\alpha_+}c_+}{\alpha_+ - 1} + o(tk_t^{1-\alpha_+}), \\
\mathbb{E}[(X_t - k_t)^+] &\leq M_t\mathbb{E}[(\sigma W_t + \chi_t - k_t)^+] + M_t\frac{t(k_t - \chi_t)^{1-\alpha_+}c_+}{\alpha_+ - 1} + o(tk_t^{1-\alpha_+}) \\
&= M_t\mathbb{E}[(\sigma W_t + \chi_t - k_t)^+] + \frac{tk_t^{1-\alpha_+}c_+}{\alpha_+ - 1} + o(tk_t^{1-\alpha_+}).
\end{aligned}
$$

Finally, since we also have $\chi_t = o(\sqrt{t})$ and $\bar{\gamma}_t t = o(\sqrt{t})$, we get that $\mathbb{E}[(\sigma W_t - \chi_t + \bar{\gamma}_t t - k_t)^+] \sim \mathbb{E}[(\sigma W_t - k_t)^+]$ and $\mathbb{E}[(\sigma W_t + \chi_t - k_t)^+] \sim \mathbb{E}[(\sigma W_t - k_t)^+]$, which allows to complete the proof of Theorem 2.1. □

6.2. Proof of Proposition 2.3

We first concentrate on the proof of (2.10). Let (σ^2, ν, b) be the characteristic triplet of X with respect to the zero truncation function, meaning that

$$
X_t = bt + \sigma W_t + \sum_{s \leq t} \Delta X_s,
$$

where as usual for any $s > 0$ we define $\Delta X_s = X_s - X_{s-}$.

Assume first that $\sigma = 0$. The left derivative of the function

$$x \mapsto (e^{-k_t} - e^x)^+ \quad \text{is} \quad x \mapsto -e^x 1_{\{x \le -k_t\}},$$

and hence Itô–Tanaka formula (Protter 2004, chapter IV, theorem 70) applied to the process $(e^{-k_t} - e^X)^+$ yields

$$\begin{aligned}
(e^{-k_t} - e^{X_t})^+ &= -\int_{(0,t]} e^{X_{s-}} 1_{\{X_{s-} \le -k_t\}} dX_s \\
&\quad + \sum_{0<s\le t} \left[(e^{-k_t} - e^{X_s})^+ - (e^{-k_t} - e^{X_{s-}})^+ + e^{X_{s-}} 1_{\{X_{s-} \le -k_t\}} \Delta X_s \right] \\
&= -b \int_0^t e^{X_{s-}} 1_{\{X_{s-} \le -k_t\}} ds + \sum_{0<s\le t} \left[(e^{-k_t} - e^{X_{s-} + \Delta X_s})^+ - (e^{-k_t} - e^{X_{s-}})^+ \right],
\end{aligned}$$

for any $t \ge 0$, since, in this case, X has paths of finite variation. Since $(\Delta X_s)_{s \ge 0}$ is a Poisson point process with intensity measure $\nu(dy) \times ds$, and $X_{s-} \ne X_s$ for at most countably many time s in the interval $[0, t]$ almost surely, taking expectations on both sides of the pathwise representation above and applying the compensation formula for point processes, we obtain

$$\begin{aligned}
\text{(6.14)}\ \mathbb{E}[(e^{-k_t} - e^{X_t})^+] &= -b\mathbb{E}\left[\int_0^t e^{X_s} 1_{\{X_s \le -k_t\}} ds \right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}\setminus\{0\}} \left\{ (e^{-k_t} - e^{X_s + y})^+ - (e^{-k_t} - e^{X_s})^+ \right\} \nu(dy) ds \right].
\end{aligned}$$

From theorem 43.20 in Sato (1999), we have that $\frac{X_t}{t} \to b$ almost surely as $t \to 0$. Recall also that by assumption $k_t/t \to \infty$ as $t \downarrow 0$. Therefore, for any $\varepsilon > 0$, each path $X(\omega)$ satisfies the following inequalities

$$k_t > t(b+\varepsilon) > X_t(\omega) > t(b-\varepsilon) > -k_t \quad \text{for all small enough} \quad t > 0 \quad \text{and all}$$

$$\text{(6.15)} \quad s \le t.$$

Therefore, it holds $\frac{1}{t}\int_0^t e^{X_s} 1_{\{X_s \le -k_t\}} ds \to 0$ almost surely. Since on the other hand we have

$$\frac{1}{t}\int_0^t e^{X_s} 1_{\{X_s \le -k_t\}} ds \le \frac{1}{t}\int_0^t e^{-k_t} ds = e^{-k_t},$$

the dominated convergence theorem implies

$$\mathbb{E}\left[\int_0^t e^{X_s} 1_{\{X_s \le -k_t\}} ds \right] = o(t) \quad \text{as } t \to 0.$$

To deal with the second term in (6.14), we deduce from (6.15) that $X(\omega)$ also satisfies the following inequalities for any $y \in \mathbb{R} \setminus \{0\}$, all sufficiently small times $t > 0$ and all $s \le t$:

$$(e^{-k_t} - e^{X_s(\omega) + y})^+ - (e^{-k_t} - e^{X_s(\omega)})^+ \le (e^{-k_t} - e^{(b-\varepsilon)s + y})^+ - (e^{-k_t} - e^{(b-\varepsilon)s})^+,$$

and

$$(e^{-k_t} - e^{X_s(\omega)+y})^+ - (e^{-k_t} - e^{X_s(\omega)})^+ \geq (e^{-k_t} - e^{(b+\varepsilon)s+y})^+ - (e^{-k_t} - e^{(b+\varepsilon)s})^+.$$

The second terms in both sides of the above two inequalities is in fact always zero for sufficiently small t due to (6.15). Therefore, we get the following almost sure convergence:

$$\frac{1}{t}\int_0^t \int_{\mathbb{R}\setminus\{0\}} \left\{(e^{-k_t} - e^{X_s+y})^+ - (e^{-k_t} - e^{X_s})^+\right\} \nu(dy)ds \to \int_{\mathbb{R}\setminus\{0\}} (1-e^y)^+\nu(dy) \quad \text{as } t \downarrow 0.$$

Since the function $y \mapsto (e^{-k_t} - e^{X_s+y})^+$ is Lipschitz with a Lipschitz constant that does not depend on the path $X(\omega)$, the dominated convergence theorem and the representation in (6.14) yield

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}[(e^{-k_t} - e^{X_t})^+] = \int_{\mathbb{R}\setminus\{0\}} (1-e^y)^+\nu(dy).$$

Assume now that $\sigma > 0$. Define

$$f(t,x) := \mathbb{E}\left[\left(1 - e^{x+k_t+\sigma W_t - \frac{\sigma^2}{2}t}\right)^+\right] \quad \text{and} \quad Z_t := \left(b + \frac{\sigma^2}{2}\right)t + \sum_{0<s\leq t} \Delta X_s,$$

and note that

$$\mathbb{E}[(e^{-k_t} - e^{X_t})^+] = e^{-k_t}\mathbb{E}[f(t,Z_t)]. \tag{6.16}$$

The derivative of f with respect to x is given by

$$f'(t,x) := -\mathbb{E}\left[e^{x+k_t+\sigma W_t-\frac{\sigma^2}{2}t} 1_{\{x+k_t+\sigma W_t - \frac{\sigma^2}{2}t \leq 0\}}\right] \geq -1, \tag{6.17}$$

It can be computed explicitly as

$$f'(t,x) = -e^{x+k_t} N\left(-\frac{\frac{1}{2}\sigma^2 t + x + k_t}{\sigma\sqrt{t}}\right), \tag{6.18}$$

where $N(\cdot)$ denotes the standard normal cumulative distribution function. Note also for future use that

$$f''(t,x) = -e^{x+k_t} N\left(-\frac{\frac{1}{2}\sigma^2 t + x + k_t}{\sigma\sqrt{t}}\right) + \frac{1}{\sigma\sqrt{t}} n\left(\frac{-\frac{1}{2}\sigma^2 t + x + k_t}{\sigma\sqrt{t}}\right) \geq -1, \tag{6.19}$$

with $n(x) = N'(x)$.

Applying Itô's formula to the process $f(t,Z)$ as a function of Z with t fixed, yields

$$\begin{aligned} f(t,Z_t) &= f(t,0) + \int_{(0,t]} f'(t,Z_{s-})dZ_s + \sum_{0<s\leq t}\left[f(t,Z_s) - f(t,Z_{s-}) - f'(t,Z_{s-})\Delta Z_s\right] \\ &= f(t,0) + \left(b + \frac{\sigma^2}{2}\right)\int_0^t f'(t,Z_{s-})ds + \sum_{0<s\leq t}\left[f(t,Z_{s-}+\Delta X_s) - f(t,Z_{s-})\right], \end{aligned}$$

since $\Delta Z_s = \Delta X_s$ for all $s > 0$. By taking the expectation and applying (6.16) we find that

$$\mathbb{E}[(e^{-k_t} - e^{X_t})^+] = e^{-k_t}\mathbb{E}[f(t,0)] + e^{-k_t}\mathbb{E}\left[\int_0^t f'(t, Z_s)ds\right] + e^{-k_t}\mathbb{E}\left[\int_0^t \int_{\mathbb{R}\setminus\{0\}} \{f(t, Z_s + y) - f(t, Z_s)\}\, \nu(dy)ds\right]. \tag{6.20}$$

The first term on the right-hand side of (6.20) is equal to the first term on the right-hand side of (2.10). As in the case $\sigma = 0$, using the almost sure convergence $\frac{Z_t}{t} \to b + \frac{\sigma^2}{2}$, the explicit form (6.18) of $f'(t,x)$ and the assumption that $\frac{k_t}{\sqrt{t}} \to \infty$ as $t \downarrow 0$, we get that

$$\frac{1}{t}\int_0^t f'(t, Z_s)ds \to 0,$$

almost surely. Since $|f'(t, Z_s)| \le 1$ for all $t, s \ge 0$ by (6.17), the dominated convergence theorem yields

$$\mathbb{E}\left[\int_0^t f'(t, Z_s)ds\right] = o(t).$$

To treat the last term in (6.20), we use the fact that for any $\varepsilon > 0$, each path $Z(\omega)$ satisfies the inequalities

$$t(b + \sigma^2/2 - \varepsilon) \le Z_t(\omega) \le t(b + \sigma^2/2 + \varepsilon),$$

for all sufficiently small t. Therefore, since $f''(t,x) \ge -1$, the following inequalities hold

$$\begin{aligned} f'(t, t(b - \varepsilon + \sigma^2/2) + \theta y) - 2t\varepsilon &\le f'(t, Z_s(\omega) + \theta y) \\ &\le f'(t, t(b + \varepsilon + \sigma^2/2) + \theta y) + 2t\varepsilon, \end{aligned} \tag{6.21}$$

for any trajectory $s \mapsto Z_s(\omega)$, where $s \in [0, t]$, and all sufficiently small t. The random variable under the expectation in the last term on the right-hand side of (6.20) can be expressed as follows:

$$\begin{aligned} &\frac{1}{t}\int_0^t \int_{\mathbb{R}\setminus\{0\}} \{f(t, Z_s + y) - f(t, Z_s)\}\, \nu(dy)ds \\ &= \frac{1}{t}\int_0^t ds \int_0^1 d\theta \int_{\mathbb{R}\setminus\{0\}} y f'(t, Z_s + \theta y)\nu(dy). \end{aligned} \tag{6.22}$$

The pathwise bounds in (6.21) can be used to estimate (6.22) from above and below. For each path $Z(\omega)$, we have the following bound for $y \in (-\infty, 0)$ and all sufficiently small t:

$$\begin{aligned} &2t\varepsilon \int_{(-\infty,0)} y\nu(dy) + \int_0^1 d\theta \int_{(-\infty,0)} y f'(t, t(b + \varepsilon + \sigma^2/2) + \theta y)\nu(dy) \\ &\le \tfrac{1}{t}\int_0^t ds \int_0^1 d\theta \int_{(-\infty,0)} y f'(t, Z_s(\omega) + \theta y)\nu(dy) \end{aligned}$$

$$\leq -2t\varepsilon \int_{(-\infty,0)} y\nu(dy) + \int_0^1 d\theta \int_{(-\infty,0)} yf'(t, t(b-\varepsilon+\sigma^2/2)+\theta y)\nu(dy).$$

The explicit form (6.18) of $f'(t, x)$ implies that for all $y < 0$ and $\theta > 0$ we have

$$f'(t, t(b \pm \varepsilon + \sigma^2/2) + \theta y) \to -e^{\theta y} \quad \text{as } t \to 0.$$

Since $f'(t, x)$ is bounded, the dominated convergence theorem yields

$$\int_0^1 d\theta \int_{(-\infty,0)} yf'(t, t(b \pm \varepsilon + \sigma^2/2) + \theta y)\nu(dy) \to -\int_0^1 d\theta \int_{(-\infty,0)} ye^{\theta y}\nu(dy)$$
$$= \int_{(-\infty,0)} \nu(dy)(1-e^y),$$

as $t \downarrow 0$. Formula (6.18) for $f'(t, x)$ implies that for all $y \in (0, \infty)$ and $\theta > 0$ we have

$$f'(t, t(b \pm \varepsilon + \sigma^2/2) + \theta y) \to 0 \quad \text{as } t \to 0.$$

An analogous argument for $y \in (0, \infty)$ to the one above and the representation in (6.22) imply the almost sure convergence

$$\frac{1}{t}\int_0^t \int_{\mathbb{R}\setminus\{0\}} \{f(t, Z_s + y) - f(t, Z_s)\}\, \nu(dy)ds \to \int_{\mathbb{R}\setminus\{0\}} (1-e^y)^+\nu(dy) \quad \text{as } t \to 0.$$

Finally, since $f(t, x)$ is Lipschitz in x, with the Lipschitz constant independent of t, the dominated convergence theorem implies

$$\frac{1}{t}\mathbb{E}\left[\int_0^t \int_{\mathbb{R}\setminus\{0\}} \{f(t, Z_s + y) - f(t, Z_s)\}\, \nu(dy)ds\right] \to \int_{\mathbb{R}\setminus\{0\}} (1-e^y)^+\nu(dy).$$

This concludes the proof of (2.8). Note that in this proof, we did not use the condition in (2.1), but only the assumption $\int_{\mathbb{R}\setminus\{0\}} |x|\nu(dx) < \infty$.

We now concentrate on the proof of (2.1). Since the Lévy process X satisfies (2.1), we can define the share measure $\widetilde{\mathbb{P}}$, via $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{X_t}$, as in the proof of Theorem 3.1. Analogous to the equality in (6.24), we have

$$\mathbb{E}[(e^{X_t} - e^{k_t})^+] = e^{k_t}\widetilde{\mathbb{E}}[(e^{-k_t} - e^{-X_t})^+], \tag{6.23}$$

where $\widetilde{\mathbb{E}}$ denotes the expectation under the share measure $\widetilde{\mathbb{P}}$. Furthermore, by Sato (1999, theorem 33.1), under the measure $\widetilde{\mathbb{P}}$, the process X is again a Lévy process with a characteristic triplet $(\sigma^2, \widetilde{\nu}, \widetilde{\gamma})$, where $\widetilde{\nu}(dx) = e^x\nu(dx)$, and e^{-X} is a positive $\widetilde{\mathbb{P}}$-martingale started at one. The Lévy measure $\widetilde{\nu}$ clearly satisfies

$$\int_{\mathbb{R}\setminus\{0\}} |x|\widetilde{\nu}(dx) < \infty.$$

Therefore, we can apply (2.8) to the process $-X$ under the measure $\widetilde{\mathbb{P}}$. Hence, the identity in (6.23) yields:

$$\mathbb{E}[(e^{X_t} - e^{k_t})^+] = e^{k_t}\mathbb{E}[(e^{-k_t} - e^{\sigma W_t - \frac{\sigma^2 t}{2}})^+] + te^{k_t}\int_{(0,\infty)} (1-e^{-x})\widetilde{\nu}(dx) + o(t)$$

$$= \mathbb{E}[(e^{\sigma W_t - \frac{\sigma^2 t}{2}} - e^{k_t})^+] + t \int_{(0,\infty)} (e^x - 1)\nu(dx) + o(t),$$

where we used the Black–Scholes put–call symmetry given in (6.27), the fact $e^{k_t} = 1 + o(1)$ and the equality $\widetilde{\nu}(dx) = e^x \nu(dx)$. This establishes the formula in (2.9) and concludes the proof of Proposition 2.3. □

6.3. Proof of Theorem 3.1

We first assume that $\theta > 0$. Equality (3.10) implies the following

$$\frac{\log C^{\mathrm{BS}}(t, k_t, s)}{\log t} - \frac{\log\log\frac{1}{t}}{\log\frac{1}{t}} = \frac{1}{2} + \frac{\theta^2}{2s^2} - \frac{1}{\log\frac{1}{t}} \log\frac{s^3}{\theta^2\sqrt{2\pi}} + 3\frac{s^2}{\theta^2}\frac{1}{\log^2\frac{1}{t}} + O\left(\frac{1}{\log^3\frac{1}{t}}\right),$$

as $t \downarrow 0$ for any $s > 0$. Define

$$F(s, z) := -z\log C^{\mathrm{BS}}(e^{-1/z}, k_{e^{-1/z}}, s) + z\log z,$$

and note that $F(s, z)$ corresponds to the left-hand side of the above formula with the change of variable $z = \frac{1}{\log\frac{1}{t}}$. The expansion shows that $F(s, z)$ is regular as $z \to 0$ and the following equality holds

$$F(s, z) = \frac{1}{2} + \frac{\theta^2}{2s^2} - z\log\frac{s^3}{\theta^2\sqrt{2\pi}} + 3\frac{s^2}{\theta^2}z^2 + O\left(z^3\right).$$

The expansion for the inverse mapping can be deduced from this expression as follows. To keep the formulae simple, we give the expansion up to $O(z^2)$:

$$F(s, z) = a(s) + zb(s) + O(z^2), \quad \text{where} \quad a(s) = \frac{1}{2} + \frac{\theta^2}{2s^2}, \quad b(s) = \log\frac{s^3}{\theta^2\sqrt{2\pi}}.$$

Denote by $F^{-1}(y, z)$ the unique positive solution of the equation $F(s, z) = y$, where y equals $J_{e^{-1/z}}(x)$ (see the statement of Theorem 3.1 for the definition of $J_t(x)$) and x is any arbitrage-free call option price with maturity $e^{-1/z}$ and strike $k_{e^{-1/z}}$. The uniqueness of the quantity $F^{-1}(y, z)$ is equivalent to the fact that the implied volatility is a well-defined quantity.

An approximate expression for y is given by

$$y = a(F^{-1}(y, z)) + zb(F^{-1}(y, z)) + O(z^2),$$

and hence we find

$$a^{-1}(y) = a^{-1}(a(F^{-1}(y, z)) + zb(F^{-1}(y, z)) + O(z^2)).$$

Using the regularity of the coefficient a in the neighborhood of the point $F^{-1}(y, z) > 0$, we can expand the inverse a^{-1} around the point $a(F^{-1}(y, z))$ as follows:

$$a^{-1}(y) = F^{-1}(y, z) + (a^{-1})'(a(F^{-1}(y, z)))b(F^{-1}(y, z))z + O(z^2).$$

In view of this expression, and using once again the regularity of the coefficients a and b, we can replace $F^{-1}(y,z)$ with $a^{-1}(y)$ in the second term, obtaining

$$a^{-1}(y) = F^{-1}(y,z) + (a^{-1})'(y)b(a^{-1}(y))z + O(z^2).$$

Hence, the following asymptotic equalities hold true:

$$\begin{aligned} F^{-1}(y,z) &= a^{-1}(y) - \tfrac{b(a^{-1}(y))}{a'(a^{-1}(y))}z + O(z^2) \\ &= \frac{\theta}{\sqrt{2y-1}} + \frac{\theta \log \frac{(2y-1)^{\frac{3}{2}}\sqrt{2\pi}}{\theta}}{(2y-1)^{\frac{3}{2}}} z + O(z^2). \end{aligned}$$

Making the substitution $y = J_{e^{-1/z}}(x)$ in the above formula, we find an expansion for the implied volatility $\sigma_t(\theta)$ given in (3.8). Now, $C(t,k_t) \sim \widehat{C}_t$ implies that $\frac{\log C(t,k_t)}{\log t} - \frac{\log \widehat{C}_t}{\log t} = o(\log^{-1} t^{-1})$. Since all the coefficients in expansion (3.8) are regular, the additional term arising from this difference may be ignored in an expansion up to order $o(\log^{-1} t^{-1})$ and (3.9) follows.

The formulae in the theorem in the case $\theta < 0$ will be established by applying the result for the positive log-strike under the share measure. More precisely, let $\mathbb{P}$ denote the original risk-neutral measure under which the process S is a positive martingale starting at one. For each time t, we define the share measure $\widetilde{\mathbb{P}}$ on the σ-algebra $\mathcal{F}_t$ of events that can occur up to time t via its Radon–Nikodym derivative $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} := S_t$ and note that the following relationship holds for any log-strike $k \in \mathbb{R}$:

$$P(t,k) = \mathbb{E}\left[(e^k - S_t)^+\right] = e^k \widetilde{\mathbb{E}}\left[(S_t^{-1} - e^{-k})^+\right] = e^k \widetilde{C}(t,-k), \tag{6.24}$$

where $\widetilde{C}(t,-k) := \widetilde{\mathbb{E}}\left[(S_t^{-1} - e^{-k})^+\right]$ denotes the expectation under the share measure $\widetilde{\mathbb{P}}$ of a call payoff with strike e^{-k}, where the evolution of the risky asset is given by S^{-1}. Note that S^{-1} is a positive martingale starting at one under $\widetilde{\mathbb{P}}$ and hence $\widetilde{C}(t,-k)$ represents an arbitrage-free call option price. Furthermore, the put–call symmetry formula in the Black–Scholes model (see (6.27)) and the equality in (6.24) mean that the implied volatility $\widehat{\sigma}(t,k)$ defined by the put price $P(t,k)$ coincides with the implied volatility $\widehat{\widetilde{\sigma}}(t,-k)$ defined by the call price $\widetilde{C}(t,-k)$ (see beginning of Section 3 for the definition of $\widehat{\sigma}(t,k)$).

Note that, since $\theta < 0$, we now have $-k_t > 0$ and $\sigma_t(\theta) = \widetilde{\sigma}_t(-\theta)$, where $\widetilde{\sigma}_t(-\theta)$ denotes $\widehat{\widetilde{\sigma}}(t,-k_t)$. In order to apply the formula in (3.8) to $\widetilde{C}(t,-k_t)$, we have to ensure that assumption (3.7) is satisfied. Since (3.7) holds for $P(t,k_t)$ and $k_t = o(\log t)$, the equality in (6.24) implies (3.7) for $\widetilde{C}(t,-k_t)$. Therefore, formula (3.8) gives an asymptotic expansion of $\sigma_t(\theta) = \widetilde{\sigma}_t(-\theta)$ in terms of $\widetilde{L}_t(-\theta) := J_t(\widetilde{C}(t,-k_t))$. Since equality (6.24) implies

$$L_t(\theta) = \widetilde{L}_t(-\theta) - \theta\sqrt{t/\log(1/t)} = \widetilde{L}_t(-\theta) + O\left(\frac{1}{\log^2 \frac{1}{t}}\right) \quad \text{as } t \downarrow 0,$$

and the two leading order terms in (3.8) are regular in $\widetilde{L}_t(-\theta)$, the asymptotic expansion in (3.8) also holds when $\widetilde{L}_t(-\theta)$ is replaced by $L_t(\theta)$. The formula in (3.9) now follows by the same argument as in the case of the positive log-strike. This concludes the proof of the theorem. $\square$

6.4. Proof of Corollary 3.3

(a) Assume first that $\sigma\sqrt{2-\alpha_+} > \theta > 0$. Define $\widehat{C}_t := C^{\mathrm{BS}}(t, k_t, \sigma)$ and note that (3.10), the definition of k_t in (3.4) and (2.6) of Theorem 2.1 imply

$$\text{(6.25)}\quad C(t,k_t) \sim \widehat{C}_t, \quad \text{and hence} \quad \frac{\log C(t,k_t)}{\log t} = \frac{\log \widehat{C}_t}{\log t} + o\left(\frac{1}{\log \frac{1}{t}}\right), \quad \text{as } t \downarrow 0,$$

where $C(t,k_t)$ denotes the call option price with maturity t and strike e^{k_t} under the exponential Lévy model e^X. Assumption (3.7) of Theorem 3.1 is, therefore, satisfied by Remark (i) after Theorem 3.1. The formula for $\widehat{L}_t(\theta) = \log \widehat{C}_t / \log t - (\log\log \frac{1}{t}) \log \frac{1}{t}$ takes the form

$$\text{(6.26)}\qquad \widehat{L}_t(\theta) = \frac{1}{2} + \frac{\theta^2}{2\sigma^2} - \log\left(\frac{\sigma^3}{\theta^2\sqrt{2\pi}}\right)\frac{1}{\log\frac{1}{t}} + o\left(\frac{1}{\log\frac{1}{t}}\right), \quad \text{as } t \downarrow 0.$$

The formula in (3.9) of Theorem 3.1, together with (6.26) and the Taylor expansions in $\log(1/t)$ as $t \downarrow 0$

$$\frac{\theta}{\sqrt{2\widehat{L}_t(\theta)-1}} = \sigma\left[1 + \frac{\sigma^2}{\theta^2}\log\left(\frac{\sigma^3}{\theta^2\sqrt{2\pi}}\right)\frac{1}{\log\frac{1}{t}}\right] + o\left(\frac{1}{\log\frac{1}{t}}\right),$$

$$\frac{\theta \log \frac{(2\widehat{L}_t(\theta)-1)^{\frac{3}{2}}\sqrt{2\pi}}{\theta}}{\left(2\widehat{L}_t(\theta)-1\right)^{\frac{3}{2}}}\frac{1}{\log\frac{1}{t}} = \frac{\sigma^3 \log \frac{\theta^2\sqrt{2\pi}}{\sigma^3}}{\theta^2}\frac{1}{\log\frac{1}{t}} + o\left(\frac{1}{\log\frac{1}{t}}\right),$$

yield the formula in (3.13).

In the case $\sigma\sqrt{2-\alpha_+} \le \theta$, the relation (6.25) is satisfied by $\widehat{C}_t := \frac{t k_t^{1-\alpha_+} c_+}{\alpha_+ - 1}$. This follows directly from the definition of k_t in (3.4) and Theorem 2.1 (see formula (2.6)). An analogous argument as the one above shows that in this case the assumptions of Theorem 3.1 are also satisfied. By the definition of $\widehat{L}_t(\theta)$ in Theorem 3.1, we find

$$2\widehat{L}_t(\theta) - 1 = (2-\alpha_+)\left[1 - \frac{3-\alpha_+}{2-\alpha_+}\frac{\log\log\frac{1}{t}}{\log\frac{1}{t}} - \frac{2}{2-\alpha_+}\log\left(\frac{\theta^{1-\alpha_+}c_+}{\alpha_+-1}\right)\frac{1}{\log\frac{1}{t}}\right].$$

By Taylor's formula the following asymptotic relations hold as $t \downarrow 0$:

$$\frac{\theta}{\sqrt{2\widehat{L}_t(\theta)-1}} = \frac{\theta}{\sqrt{2-\alpha_+}}\left[1 + \frac{3-\alpha_+}{2(2-\alpha_+)}\frac{\log\log\frac{1}{t}}{\log\frac{1}{t}} + \frac{1}{2-\alpha_+}\log\left(\frac{\theta^{1-\alpha_+}c_+}{\alpha_+-1}\right)\frac{1}{\log\frac{1}{t}}\right] + o\left(\frac{1}{\log\frac{1}{t}}\right),$$

and

$$\frac{\theta \log \frac{(2\widehat{L}_t(\theta)-1)^{\frac{3}{2}}\sqrt{2\pi}}{\theta}}{\left(2\widehat{L}_t(\theta)-1\right)^{\frac{3}{2}}}\frac{1}{\log\frac{1}{t}} = \frac{\theta \log \frac{(2-\alpha_+)^{\frac{3}{2}}\sqrt{2\pi}}{\theta}}{(2-\alpha_+)^{\frac{3}{2}}}\frac{1}{\log\frac{1}{t}} + o\left(\frac{1}{\log\frac{1}{t}}\right).$$

Substituting these expressions into (3.9) establishes the formula in (3.13).

Assume now that $-\sigma\sqrt{2-\alpha_-} < \theta < 0$. Define $\widehat{P}_t := P^{\mathrm{BS}}(t, k_t, \sigma)$, where $P^{\mathrm{BS}}(t, k_t, \sigma)$ is the put option price in the Black–Scholes model, and recall the well-known put–call symmetry

$$P^{\mathrm{BS}}(t, k_t, \sigma) = e^{k_t} C^{\mathrm{BS}}(t, -k_t, \sigma), \tag{6.27}$$

which holds since the laws of minus the log-spot under the share measure (i.e., the pricing measure where the risky asset is a numeraire) and the log-spot under the risk-neutral measure (i.e., the measure where the riskless asset is the numeraire) coincide. Analogous to the case above, (3.10) with the put–call symmetry, the definition of k_t in (3.4) and (2.5) of Theorem 2.1 imply

$$P(t, k_t) \sim \widehat{P}_t, \quad \text{and hence} \quad \frac{\log P(t, k_t)}{\log t} = \frac{\log \widehat{P}_t}{\log t} + o\left(\frac{1}{\log \frac{1}{t}}\right), \quad \text{as } t \downarrow 0, \tag{6.28}$$

where $P(t, k_t)$ is the put option price under the exponential Lévy model e^X. Therefore, the assumptions of Theorem 3.1 are satisfied and $\widehat{L}_t(\theta)$ takes the form (6.26). Note that the right-hand side of (6.26) depends solely on the even powers of θ and hence the fact $\theta < 0$ does not influence the asymptotic behavior of $\widehat{L}_t(\theta)$. The proof of formula (3.13) now follows in the same way as in the call case above.

In the case $-\sigma\sqrt{2-\alpha_-} \geq \theta$, we define $\widehat{P}_t := \frac{t(-k_t)^{1-\alpha_+} c_+}{\alpha_+ - 1}$. Under this assumption, the relation (6.28) is satisfied by (2.7) of Theorem 2.1 and the rest of the proof follows along the same lines as in the case $\sigma\sqrt{2-\alpha_+} \leq \theta$. This proves formula (3.13).

(b) The proof of part (b) of the corollary is based on Proposition 2.3 and Theorem 3.1. The steps are analogous to the ones in the proof of part (a):

$$\text{if} \quad \sigma > \theta > 0, \quad \text{define } \widehat{C}_t := C^{\mathrm{BS}}(t, k_t, \sigma); \quad \text{if} \quad \sigma \leq \theta, \quad \text{define } \widehat{C}_t := t\gamma_+;$$
$$\text{if} \quad -\sigma < \theta < 0, \quad \text{define } \widehat{P}_t := P^{\mathrm{BS}}(t, k_t, \sigma); \text{ if} \quad -\sigma \geq \theta, \quad \text{define } \widehat{P}_t := t\gamma_-.$$

The details of the calculations are left to the reader. □

APPENDIX

LEMMA A.1. *Let X be a Lévy process satisfying (2.1) and k_t a deterministic function such that*

$$k_t > 0 \quad \forall t > 0 \quad \textit{and} \quad \lim k_t = 0 \quad \textit{as } t \downarrow 0.$$

Then for any $b \in \mathbb{R}$ we have

$$\mathbb{E}[(e^{X_t+bt} - e^{k_t})^+] = e^{k_t}\mathbb{E}[(X_t - k_t)^+] + O(t) \quad \textit{as } t \downarrow 0,$$
$$\mathbb{E}[(e^{-k_t} - e^{X_t+bt})^+] = e^{-k_t}\mathbb{E}[(-k_t - X_t)^+] + O(t) \quad \textit{as } t \downarrow 0.$$

Proof. Since $0 \leq (X_t + bt - k_t)^+ - (X_t - k_t)^+ \leq b^+ t = O(t)$, it is clearly sufficient to prove the formula for the call in the case $b = 0$. Let $f(x, k) = (e^x - e^k)^+ - e^k(x - k)^+$ and note the following: $f'_x(x, k) = (e^x - e^k)^+$ for all $x \in \mathbb{R}$ and $f''_x(x, k) = e^x 1_{\{x \geq k\}}$ for all $x \in \mathbb{R} \setminus \{k\}$. By Taylor's formula we have $f(x, k) = (x - k)^2 \int_0^1 (1 - \theta) f''_x((1 - \theta)k + \theta x) d\theta$

for any $x \neq k$, and, considering k_t fixed, we find

$$\mathbb{E}[f(X_t, k_t)] = \mathbb{E}\left[(X_t - k_t)^2 \int_0^1 (1-\theta) e^{k_t + \theta(X_t - k_t)} 1_{\{k_t + \theta(X_t - k_t) \geq k_t\}} d\theta\right] \leq C_0 \mathbb{E}[X_t^2 e^{X_t}],$$

for some constant $C_0 > 0$. Under the assumption of the lemma, the right-hand side can be computed as

$$\mathbb{E}[X_t^2 e^{X_t}] = \frac{\partial^2}{\partial u^2} \mathbb{E}[e^{u X_t}]\Big|_{u=1}.$$

A direct computation using the Lévy–Khintchine formula then shows that $\mathbb{E}[X_t^2 e^{X_t}] = O(t)$ as $t \downarrow 0$. The put case is treated in a similar manner. $\square$

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