

ON THE MARTINGALE PROPERTY IN STOCHASTIC VOLATILITY MODELS BASED ON TIME-HOMOGENEOUS DIFFUSIONS

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Lions and Musiela give sufficient conditions to verify when a stochastic exponential of a continuous local martingale is a martingale or a uniformly integrable martingale. Blei and Engelbert and Mijatović and Urusov give necessary and sufficient conditions in the case of perfect correlation ($\rho = 1$). For financial applications, such as checking the martingale property of the stock price process in correlated stochastic volatility models, we extend their work to the arbitrary correlation case ($-1 \leq \rho \leq 1$). We give a complete classification of the convergence properties of both perpetual and capped integral functionals of time-homogeneous diffusions and generalize results in Mijatović and Urusov with direct proofs avoiding the use of *separating times* (concept introduced by Cherny and Urusov and extensively used in the proofs of Mijatović and Urusov).

KEY WORDS: Martingale property, local martingale, stochastic volatility, Engelbert Schmidt zero-one law.

1. INTRODUCTION

There are several recent papers proposing sufficient conditions (Lions and Musiela 2007) or necessary and sufficient conditions (Delbaen and Shirakawa 2002; Blei and Engelbert 2009; Mijatović, Novak, and Urusov 2012; Mijatović and Urusov 2012c) to verify when the stochastic exponential of a continuous local martingale is a true martingale or a uniformly integrable (UI) martingale. A relevant application in finance is to check if the discounted stock price is a true martingale in a general stochastic volatility model with arbitrary correlation.

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This problem has been extensively studied and dates back to Girsanov (1960) who poses the problem of deciding whether a stochastic exponential is a true martingale or not. Gikhman and Skorohod (1972), Liptser and Shiryaev (1972), Novikov (1972), and Kazamaki (1977) provide sufficient conditions for the martingale property of a stochastic exponential. Novikov's criterion is easy to apply in practical situations, but it may not always be verified in models in mathematical finance. In the setting of Brownian motions, refer to Kramkov and Shiryaev (1998), Cherny and Shiryaev (2001), and Ruf (2013b) for improvements of the criteria of Novikov (1972) and Kazamaki (1977). For affine processes, similar questions are considered by Kallsen and Shiryaev (2002), Kallsen and Muhle-Karbe (2010), and Mayerhofer, Muhle-Karbe, and Smirnov (2011). Kotani (2006) and Hulley and Platen (2011) obtain necessary and sufficient conditions for a one-dimensional regular strong Markov continuous local martingale to be a true martingale. In the strand of stochastic exponentials based on time-homogeneous diffusions, Engelbert and Schmidt (1984) provide analytic conditions for the martingale property, and Stummer (1993) gives further analytic conditions when the diffusion coefficient is the identity. Delbaen and Shirakawa (2002) first provide deterministic criteria to check if a stochastic exponential is a true martingale under a slightly restrictive assumption requiring certain functions to be locally bounded on $(0, \infty)$. Mijatović and Urusov (2012c) removed the restriction of locally boundedness and extend their results utilizing a new tool, called *separating times*, introduced in Cherny and Urusov (2004). In the context of stochastic volatility models, Sin (1998), Andersen and Piterbarg (2007), and Lions and Musiela (2007) provide easily verifiable sufficient conditions. Blanchet and Ruf (2016) describe a method to decide on the martingale property of a nonnegative local martingale based on weak convergence arguments. Through the study of the classical solutions to the valuation partial differential equation associated with the stochastic volatility model, Bayraktar, Kardaras, and Xing (2012) establish a necessary and sufficient condition when the asset price is a martingale. In the context of stochastic differential equations (SDEs), Doss and Lenglart (1978) provide a detailed study of their asymptotics and other properties. Ruf (2013a) studies the martingale property of a nonnegative local martingale that is given as a nonanticipative functional of a solution to an SDE. A recent paper by Karatzas and Ruf (2016) provides the precise relationship between explosions of one-dimensional SDEs and the martingale properties of related stochastic exponentials. For an overview of stochastic exponentials and the related problem of martingale properties, refer to Rheinländer (2010) and the references therein.

This paper makes two contributions to the current literature. First, we provide a complete classification of the convergence or divergence properties of perpetual and capped integral functionals of time-homogeneous diffusions based on the local integrability of certain deterministic test functions. Theorem 3.6 provides necessary and sufficient conditions, similar to, but weaker than those in Salminen and Yor (2006), Khoshnevisan, Salminen, and Yor (2006). Mijatović and Urusov (2012a) derive a similar result. Theorem 3.6 permits two absorbing boundaries, while Engelbert and Tittel (2002) assume that there is exactly one absorbing boundary. Theorem 3.7 deals with the *capped* integral functional and, to the best of the authors' knowledge, is new. We also extend some results in Mijatović and Urusov (2012b, c) from the case $\rho = 1$ to the case $-1 \leq \rho \leq 1$ (see Propositions 4.1 and 4.2). Our proofs do not require the concept of *separating times* introduced by Cherny and Urusov (2004). As examples, we give necessary and sufficient conditions for the (UI) martingale property of the stock price in popular stochastic

volatility models (Hull and White 1987; Heston 1993; Schöbel and Zhu 1999; and 3/2 models).

Section 2 uses the probabilistic setting and technical tools of Ruf (2013b) and Carr, Fisher and Ruf (2014). Section 3 provides a complete classification of the convergence or divergence properties of perpetual and capped integral functionals of time-homogeneous diffusions. The main result of the paper is given in Section 4: we generalize some results in Mijatović and Urusov (2012b, c) to the arbitrary correlation case with new direct proofs. Section 5 studies in detail the martingale properties in four popular stochastic volatility models. Section 6 concludes.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE MARTINGALE PROPERTY

2.1. Probabilistic Setup

Throughout the paper, we fix a time horizon $T \in (0, \infty]$. As in Carr et al. (2014), we define a stochastic basis by $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. This basis is assumed rich enough to support the processes described below and satisfies the regularity conditions outlined in Appendix A. For any stopping time τ , we define $\mathcal{F}_\tau := \{A \in \mathcal{F}_T \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in [0, T]\}$ and $\mathcal{F}_{\tau-} := \sigma(\{A \cap \{\tau > t\} \in \mathcal{F}_t \mid A \in \mathcal{F}_t \text{ for some } t \in [0, T] \cup \mathcal{F}_0\})$. In general, nonnegative random variables are permitted to take values in the set $[0, \infty]$ and stopping times τ are permitted to take values in the set $[0, \infty] \cup \mathbb{T}$ for some transfinite time $\mathbb{T} > T$ as in appendix A of Carr et al. (2014). In special cases, we will restrict the range.

For an \mathcal{F}_t -adapted Brownian motion process W_t , assume that Y satisfies the SDE

$$(2.1) \quad dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = x_0,$$

where $\mu, \sigma : J \rightarrow \mathbb{R}$ are Borel functions, $x_0 \in J$, and that μ, σ satisfy the Engelbert–Schmidt condition

$$(2.2) \quad \forall x \in J, \quad \sigma(x) \neq 0, \quad \text{and} \quad \frac{1}{\sigma^2(\cdot)}, \quad \frac{\mu(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J).$$

Here, $L^1_{loc}(J)$ denotes the class of locally integrable functions, i.e., the functions $J \rightarrow \mathbb{R}$ that are integrable on compact subsets of the state space, $J = (\ell, r)$, $-\infty \leq \ell < r \leq \infty$, of the process $Y = (Y_t)_{t \in [0, T]}$. We set $\bar{J} = [\ell, r]$.

The Engelbert–Schmidt condition (2.2) guarantees that the SDE (2.1) has a unique in law weak solution that possibly exits its state space J (see theorem 5.15, page 341, Karatzas and Shreve 1991). Denote the possible exit time¹ of Y from its state space by ζ , i.e., $\zeta = \inf\{u > 0, Y_u \notin J\}$, P -a.s. which means that on $\{\zeta = \infty\}$ the trajectories of Y do not exit J , P -a.s., and on $\{\zeta < \infty\}$, $\lim_{t \rightarrow \zeta} Y_t = r$ or $\lim_{t \rightarrow \zeta} Y_t = \ell$, P -a.s. Observe that Y is defined such that it stays at its exit point, which means that ℓ and r are absorbing boundaries. The following terminology will be used: “ Y may exit the state space J at r ” means $P(\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = r) > 0$.

¹Refer to Karatzas and Ruf (2016) for a detailed study of the distribution of this exit time in a one-dimensional time-homogeneous diffusion setting.

We then introduce a standard Brownian motion $W^{(2)}$ independent of (Y, W) . Let $Z = (Z_t)_{t \in [0, T]}$ denote the (discounted) stock price with $Z_0 = 1$, and define

$$(2.3) \quad Z_t = \exp \left\{ \rho \int_0^{t \wedge \xi} b(Y_u) dW_u + \sqrt{1 - \rho^2} \int_0^{t \wedge \xi} b(Y_u) dW_u^{(2)} - \frac{1}{2} \int_0^{t \wedge \xi} b^2(Y_u) du \right\}, \quad t \in [0, \infty),$$

where $b : J \rightarrow \mathbb{R}$ is a Borel function, and the constant correlation satisfies $-1 \leq \rho \leq 1$.

Denote $W^{(1)} = \rho W + \sqrt{1 - \rho^2} W^{(2)}$, we have

$$(2.4) \quad Z_t = \exp \left\{ \int_0^{t \wedge \xi} b(Y_u) dW_u^{(1)} - \frac{1}{2} \int_0^{t \wedge \xi} b^2(Y_u) du \right\}, \quad t \in [0, \infty),$$

and it is easy to verify that Z and Y satisfy the following system of SDEs:

$$(2.5) \quad \begin{aligned} dZ_t &= Z_t b(Y_t) dW_t^{(1)}, & Z_0 &= 1, \\ dY_t &= \mu(Y_t) dt + \sigma(Y_t) dW_t, & Y_0 &= x_0. \end{aligned}$$

The Borel sigma algebra $\mathcal{B}(\mathbb{R})$ in \mathbb{R} is the smallest σ -algebra that contains the open intervals of \mathbb{R} . In what follows, $\lambda(\cdot)$ denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. We require that $\lambda(x \in (\ell, r) : b^2(x) > 0) > 0$, and assume the following local integrability condition:

$$(2.6) \quad \forall x \in J, \quad \sigma(x) \neq 0, \quad \text{and} \quad \frac{b^2(\cdot)}{\sigma^2(\cdot)} \in L_{loc}^1(J).$$

REMARK 2.1. In the literature (e.g., Andersen and Piterbarg 2007), there is a more general class of stochastic volatility models where the (discounted) stock price has a nonlinear diffusion coefficient in Z . For example, a general model is as follows:

$$\begin{aligned} dZ_t &= Z_t^\alpha b(Y_t) 1_{t \in [0, \xi)} dW_t^{(1)}, & Z_0 &= 1, \\ dY_t &= \mu(Y_t) 1_{t \in [0, \xi)} dt + \sigma(Y_t) 1_{t \in [0, \xi)} dW_t, & Y_0 &= x_0, \end{aligned}$$

where $W_t^{(1)}$ and W_t are standard \mathcal{F}_t -Brownian motions, with $E[dW_t^{(1)} dW_t] = \rho dt$. ρ is the constant correlation coefficient and $-1 \leq \rho \leq 1$. Here, $1 \leq \alpha \leq 2$. The difficulty of dealing with this model lies mainly in obtaining an explicit representation of Z in terms of functionals of only Y . Thus, in this paper, we only focus on model (2.5).

LEMMA 2.2 (Mijatović and Urusov 2012c). *Assume conditions (2.2) and (2.6), and $0 < t < \infty$. Then,*

$$\int_0^t b^2(Y_u) du < \infty \text{ P-a.s. on } \{t < \xi\}.$$

²Note that this is the same condition as in Mijatović and Urusov (2012b, c), and Cherny and Urusov (2006).

Fix an arbitrary constant $c \in J$ and introduce the scale function $s(\cdot)$ of the SDE (2.1) under P

$$(2.7) \quad s(x) := \int_c^x \exp \left\{ - \int_c^y \frac{2\mu}{\sigma^2}(u) du \right\} dy, \quad x \in \bar{J}.$$

The following result and its proof can be found in Cherny and Urusov (2006), here translated into our notation:

LEMMA 2.3 (lemma 5.7, page 149 of Cherny and Urusov 2006). *Assume conditions (2.2) and (2.6) for the SDE (2.1), and $s(\ell) = -\infty$, $s(r) = \infty$. Then $\int_0^\infty b^2(Y_u) du = \infty$, P -a.s.*

2.2. Properties of Nonnegative Continuous Local Martingales

In this section, we fix a time horizon $T \in (0, \infty]$, and work under the canonical probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, P)$. This space must be rich enough to support processes with distributions described below, and the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ must satisfy additional conditions outlined in the appendix. We begin by applying some results from Ruf (2013b) and Carr, et al. (2014), which deal with nonnegative continuous local martingales, to time-homogeneous diffusions as in (2.4). Ruf (2013b) does not specify the form of the continuous local martingale $(L_t)_{t \in [0, T]}$, which is, in our setting

$$(2.8) \quad L_t = \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)}.$$

To cast the setting of Ruf (2013b) into the current notation, the process in (2.4) under P can be rewritten as $Z_t = \mathcal{E}(L_t) = \exp(L_t - \langle L \rangle_t / 2)$ where L_t in (2.8) is a continuous local martingale under P .

LEMMA 2.4 (lemma 1, Ruf 2013b). *Assume conditions (2.2) and (2.6) for the SDE (2.1). Under P , consider the continuous local martingale $(L_t)_{t \in [0, T]}$ given in (2.8), and its quadratic variation $\langle L \rangle_t = \int_0^{t \wedge \zeta} b^2(Y_u) du$. For a predictable positive stopping time $0 < \tau \leq \infty$, define $Z_t = \mathcal{E}(L_t)$, $t \in [0, \tau)$. Then the random variable $Z_\tau := \lim_{t \uparrow \tau} Z_t$ exists, is nonnegative and satisfies*

$$\left\{ \int_0^{\tau \wedge \zeta} b^2(Y_u) du < \infty \right\} = \{Z_\tau > 0\}, \quad P\text{-a.s.}$$

As an application of Lemma 2.4, we have the following result:

COROLLARY 2.5. *Assume³ conditions (2.2) and (2.6) for the SDE (2.1). Under P , with the process Z defined in (2.4), for $t \in [0, T]$*

$$\{Z_t = 0\} = \left\{ \zeta \leq t, \int_0^\zeta b^2(Y_u) du = \infty \right\}, \quad P\text{-a.s.}$$

Proof. From Lemma 2.4,

$$\{Z_t = 0\} = \left\{ \int_0^{t \wedge \zeta} b^2(Y_u) du = \infty \right\}, \quad P\text{-a.s.}$$

³This is stated without proof after equation (7) on page 4, Mijatović and Urusov (2012c), and after equation (2.4) on page 228, Mijatović and Urusov (2012b). Here, we provide a proof.

From Lemma 2.2, $P(\int_0^{t \wedge \zeta} b^2(Y_u) du < \infty) = P(\int_0^t b^2(Y_u) du < \infty) = 1$ on the set $\{t < \zeta, t \in [0, T]\}$. Therefore

$$\{Z_t = 0\} = \left\{ \zeta \leq t, \int_0^{t \wedge \zeta} b^2(Y_u) du = \infty \right\}, \quad P\text{-a.s.} \quad \square$$

In the following, for notation convenience, denote $T_\infty := R$ and $T_0 := S$ as the first hitting times of ∞ and 0, respectively, by Z , where R and S are defined in Section 2.1. Both may take values in $[0, \infty] \cup \tilde{\mathcal{T}}$. The next result is theorem 2.1 of Carr et al. (2014) given in our notation.

PROPOSITION 2.6 (theorem 2.1 of Carr et al.⁴ 2014). *Consider the canonical probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, P)$, with the process Z defined in (2.4) (so that $Z_0 = 1$) and assume conditions (2.2) and (2.6). Then, there exists a unique probability measure \tilde{P} on $(\Omega, \mathcal{F}_{T_\infty-})$ such that, for any stopping time $0 < \nu < \infty$, g*

(1)

$$(2.9) \quad \tilde{P}(A \cap \{T_\infty > \nu \wedge T\}) = E^P[1_A Z_{\nu \wedge T}]$$

for all $A \in \mathcal{F}_{\nu \wedge T}$;

(2) for all nonnegative $\mathcal{F}_{\nu \wedge T}$ -measurable random variables U taking values in $[0, \infty]$,

$$(2.10) \quad E^{\tilde{P}}[U 1_{\{T_\infty > \nu \wedge T\}}] = E^P[U Z_{\nu \wedge T} 1_{\{T_0 > \nu \wedge T\}}],$$

and, with $\tilde{Z}_t = \frac{1}{Z_t} 1_{\{T_\infty > t\}}$ ⁵,

$$(2.11) \quad E^P[U 1_{\{T_0 > \nu \wedge T\}}] = E^{\tilde{P}}[U \tilde{Z}_{\nu \wedge T}];$$

(3) Z is an UI P martingale on $[0, T]$ if and only if

$$(2.12) \quad \tilde{P}(T_\infty > T) = 1.$$

Notice that from (2.9), for any stopping time $\nu < T$, $\tilde{P}(Z_\nu = 0) = 0$ so that the measure \tilde{P} assigns zero mass to paths Z_t that hit 0. The condition (2.12) is equivalent to

$$\tilde{P}(\sup_{t \in (0, T]} Z_t < \infty) = \tilde{P}(\inf_{t \in (0, T]} \tilde{Z}_t > 0) = 1 \text{ or } \tilde{P}(\sup_{t \in [0, T]} Z_t = \infty) = 0.$$

PROPOSITION 2.7. (1) Under P , for $t \in [0, T_0)$, define the continuous P -local martingale L_t as in (2.8). Then, under \tilde{P} , for $t \in [0, T_\infty)$, $\tilde{L}_t := L_t - \langle L \rangle_t = \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)} - \int_0^{t \wedge \zeta} b^2(Y_u) du$ is a continuous \tilde{P} -local martingale.

(2) Under \tilde{P} , for $t \in [0, T_\infty)$

$$\tilde{Z}_t = \mathcal{E}(-\tilde{L}_t) = \exp \left\{ - \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)} + \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u) du \right\}.$$

Proof. For statement (1), we need to show that $\tilde{L}_t = \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)} - \int_0^{t \wedge \zeta} b^2(Y_u) du$ is a \tilde{P} -local-martingale on $[0, T_\infty)$. Recall R_n is the first hitting time of Z_t to the level n , and put $\tau_n = R_n \wedge n$ for all $n \in \mathbb{N}$. We will show that $\tilde{L}_{t \wedge \tau_n} = \int_0^{t \wedge \zeta \wedge \tau_n} b(Y_u) dW_u^{(1)} - \int_0^{t \wedge \zeta \wedge \tau_n} b^2(Y_u) du$

⁴Theorem 2.1, page 6 of Carr et al. (2014) is a general result for nonnegative local martingales. See also Ruf (2013b) for a similar result for continuous nonnegative local martingales.

⁵By definition this is 0 whenever $t \geq T_\infty$ even if $Z_t = 0$.

$dW_u^{(1)} - \int_0^{t \wedge \zeta \wedge \tau_n} b^2(Y_u) du$ is a \tilde{P} -local martingale. This follows from the Girsanov theorem (ch. VIII, theorem 1.4 in Revuz and Yor 1999), the facts that $\tilde{P} \ll P$ on \mathcal{F}_{τ_n} and $\tilde{P}(\lim_{n \rightarrow \infty} \tau_n = T_\infty) = 1$.

For statement (2), under \tilde{P} , for $t < T_\infty$

$$\begin{aligned} \tilde{Z}_t &= \exp \left\{ - \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)} + \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u) du \right\} \\ &= \exp \left\{ - \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)} + \int_0^{t \wedge \zeta} b^2(Y_u) du - \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u) du \right\} \\ &= \mathcal{E}(-\tilde{L}_t^*). \end{aligned}$$

Now we seek to determine the SDE satisfied by Y under \tilde{P} . □

PROPOSITION 2.8. *Assume conditions (2.2) and (2.6) for the SDE (2.1). Under \tilde{P} , for $-1 \leq \rho \leq 1$, the diffusion Y satisfies the following SDE up to ζ :*

$$(2.13) \quad dY_t = (\mu(Y_t) + \rho b(Y_t)\sigma(Y_t))1_{t \in [0, \zeta]} dt + \sigma(Y_t)1_{t \in [0, \zeta]} d\tilde{W}_t, \quad Y_0 = x_0.$$

Proof. Consider the system of SDEs in (2.5), from the Cholesky decomposition, $dW_t^{(1)} = \rho dW_t + \sqrt{1 - \rho^2} dW_t^{(2)}$, where W and $W^{(2)}$ are standard independent Brownian motions under P . Define for $t \in [0, T]$

$$(2.14) \quad \tilde{W}_t := \begin{cases} W_t - \rho \int_0^t b(Y_u) du, & \text{if } t < \zeta, \\ W_\zeta - \rho \int_0^\zeta b(Y_u) du + \tilde{\beta}_{t-\zeta}, & \text{if } t \geq \zeta, \end{cases}$$

where $\tilde{\beta}$ is a standard \tilde{P} -Brownian motion independent of W with $\tilde{\beta}_0 = 0$.

Define $\xi_n = \zeta \wedge \tau_n$, where $\tau_n = R_n \wedge n$ and consider the process \tilde{W} up to ξ_n . As $\mathcal{F}_{\xi_n} \subset \mathcal{F}_{\tau_n}$, it follows from Proposition 2.6 that \tilde{P} restricted to \mathcal{F}_{ξ_n} is absolutely continuous with respect to P restricted to \mathcal{F}_{ξ_n} for $n \in N$. Then, from Girsanov theorem (ch. VIII, theorem 1.12, page 331 of Revuz and Yor 1999)

$$\begin{aligned} \tilde{W}_t &:= W_t - \left\langle W_t, \int_0^t b(Y_u) dW_u^{(1)} \right\rangle \\ &= W_t - \left\langle W_t, \rho \int_0^t b(Y_u) dW_u \right\rangle - \left\langle W_t, \sqrt{1 - \rho^2} \int_0^t b(Y_u) dW_u^{(2)} \right\rangle \\ &= W_t - \rho \int_0^t b(Y_u) du \end{aligned}$$

is a \tilde{P} -Brownian motion for $t \in [0, \xi_n)$ and $n \in N$. It is easy to see from the construction (2.14), that the finite dimensional distributions of \tilde{W} are those of a Brownian motion under \tilde{P} on $[0, \xi_n)$. Thus Y is governed by the following SDE under \tilde{P} for $t \in [0, \xi_n)$

$$\begin{aligned} dY_t &= \mu(Y_t) dt + \sigma(Y_t) (d\tilde{W}_t + \rho b(Y_t) dt) \\ (2.15) \quad &= (\mu(Y_t) + \rho b(Y_t)\sigma(Y_t)) dt + \sigma(Y_t) d\tilde{W}_t, \quad Y_0 = x_0. \end{aligned}$$

The result will follow from the next lemma which shows that $\xi_n = \zeta \wedge R_n \wedge n \rightarrow \zeta \wedge T_\infty = \zeta$, P -a.s. □

LEMMA 2.9. Assume conditions (2.2) and (2.6), then $\zeta \leq T_0 \wedge T_\infty$, P -a.s. and \tilde{P} -a.s.

Proof. We prove by contradiction that $P(T_0 \wedge T_\infty < \zeta) = 0$. Suppose that $T_\infty < \zeta$ with positive probability, so that for some t , $P(T_\infty < t < \zeta) > 0$. As $T_\infty < t$, $P(Z_t = \infty) > 0$. By Lemma 2.2,

$$(2.16) \quad P\left(\int_0^t b^2(Y_u)du < \infty, Z_t = \infty\right) > 0.$$

Note that $Z_t = \exp(L_t - \frac{1}{2}\langle L \rangle_t) = \infty$ if and only if $L_t = \infty$. By the Dambis–Dubins–Schwartz theorem (ch. V, theorem 1.6, Revuz and Yor 1999), for some Brownian motion B on an extended probability space, we can write $L_t - \frac{1}{2}\langle L \rangle_t = \langle L \rangle_t(\frac{B_{\langle L \rangle_t}}{\langle L \rangle_t} - \frac{1}{2})$ and from the continuity of the Brownian motion, $P(\langle L \rangle_t < \infty, L_t - \frac{1}{2}\langle L \rangle_t = \infty) = P(\langle L \rangle_t < \infty, B_{\langle L \rangle_t} = \infty) = 0$, so that

$$P\left(\int_0^t b^2(Y_u)du < \infty, Z_t = \infty\right) = P\left(\langle L \rangle_t < \infty, L_t - \frac{1}{2}\langle L \rangle_t = \infty\right) = 0$$

contradicting (2.16). Similarly, suppose that, for some t , $P(T_0 < t < \zeta) > 0$. Then $P(Z_t = 0) > 0$, and since $t < \zeta$, from Lemma 2.2,

$$P\left(\int_0^t b^2(Y_u)du < \infty, Z_t = 0\right) > 0$$

contradicting Lemma 2.4. We have thus shown that $P(T_\infty < \zeta) = P(T_0 < \zeta) = 0$. To demonstrate a similar statement under the probability measure \tilde{P} , note that \tilde{P} is a probability measure on $(\Omega, \mathcal{F}_{R_-})$ such that, for a stopping time R_n ,

$$\tilde{P}(\zeta > T_0 \wedge R_n) = \tilde{P}(\{\zeta > T_0 \wedge R_n\} \cap \{T_\infty > R_n\}) = \mathbb{E}^P[1_{\{\zeta > T_0 \wedge R_n\}} Z_{R_n}] = 0$$

because $\{\zeta > T_0 \wedge R_n\}$ is an \mathcal{F}_{R_n} measurable event. The last equality holds because $P(\zeta > T_0 \wedge R_n) = 0$. Then, by monotone convergence,

$$\tilde{P}(\zeta > T_0 \wedge T_\infty) = \lim_{n \rightarrow \infty} \tilde{P}(\zeta > T_0 \wedge R_n) = 0.$$

□

In view of Lemma 2.9 and the definition of Z_t in (2.4), there are only three possibilities, almost surely, under the measures P and \tilde{P} :

$$\zeta = T_0 < T_\infty = \tau \text{ or } \zeta = T_\infty < T_0 = \tau \text{ or } \zeta < T_0 = T_\infty = \tau.$$

In order to verify $\mathbb{E}^P[Z_T] = 1$ for $T \in [0, \infty]$, the equivalent condition in Proposition 2.6, (3) can be transformed into a condition related to integral functionals of Y under \tilde{P} as shown in the following proposition:

PROPOSITION 2.10. Assume⁶ conditions (2.2) and (2.6), and $T \in [0, \infty]$. Then, Z_t is an UI P -martingale for $t \in [0, T]$, i.e., $E^P[Z_T] = 1$, if and only if $\tilde{P}(\int_0^{T \wedge \zeta} b^2(Y_u)du < \infty) = 1$.

⁶A similar result for the general setting of multidimensional diffusions appears in theorem 1 of Ruf (2013a).

Proof. By Proposition 2.6(3), we have an UI martingale satisfying $\mathbb{E}^P[Z_T] = 1$ if and only if

$$\tilde{P}(T_\infty > T) = \tilde{P}(0 < \inf_{t \in [0, T]} \tilde{Z}_t) = 1.$$

But by Proposition 2.7(2), under the measure \tilde{P}

$$\tilde{Z}_t = \mathcal{E}(-\tilde{L}_t) = \exp \left\{ - \int_0^{t \wedge \zeta} b(Y_u) dW_u^{(1)} + \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u) du \right\}$$

is a continuous local martingale and for a stopping time $\tau = T \wedge T_\infty \wedge \zeta = T \wedge \zeta$, by Lemma 2.4, $\{\tilde{Z}_\tau > 0\} = \{\int_0^{\tau \wedge \zeta} b^2(Y_u) du < \infty\}$. The result follows. \square

REMARK 2.11. As $\tilde{P}(\int_0^{T \wedge \zeta} b^2(Y_u) du < \infty) = \lim_{q \rightarrow 0} E^{\tilde{P}}[e^{-q \int_0^{T \wedge \zeta} b^2(Y_u) du}]$ is the right limit of the Laplace transform $\mathcal{L}(q)$ of $\int_0^{T \wedge \zeta} b^2(Y_u) du$ at 0 under the measure \tilde{P} (and defining $\mathcal{L}(0) = 1$), we have the alternative formulation of Proposition 2.10 that Z_t is a (UI) P -martingale on $[0, T]$, i.e., $\mathbb{E}^P[Z_T] = 1$, if and only if $\mathcal{L}(q)$ is right continuous at 0.

3. CLASSIFICATION OF CONVERGENCE PROPERTIES OF INTEGRAL FUNCTIONALS OF TIME-HOMOGENEOUS DIFFUSIONS

The Engelbert–Schmidt zero-one law was initially proved in the Brownian motion case (see Engelbert and Schmidt 1981 or proposition 3.6.27, page 216 of Karatzas and Shreve 1991). Engelbert and Tittel (2002) obtain a generalized Engelbert–Schmidt type zero-one law for the integral functional $\int_0^t f(X_s) ds$, where f is a nonnegative Borel function and X is a strong Markov continuous local martingale. In an expository paper, Mijatović and Urusov (2012a) consider the case of a one-dimensional time-homogeneous diffusion and the zero-one law is given in their theorem 2.11. They provide two proofs that circumvent the use of Jeulin’s lemma.⁷ Through stochastic time Cui (2014) proposes a new proof under a slightly stronger assumption.

Recall the scale function $s(\cdot)$ defined in (2.7), and introduce the following test functions for $x \in \bar{J}$, with a constant $c \in J$:

$$(3.1) \quad \begin{aligned} v(x) &:= \int_c^x (s(x) - s(y)) \frac{2}{s'(y)\sigma^2(y)} dy, \\ v_b(x) &:= \int_c^x (s(x) - s(y)) \frac{2b^2(y)}{s'(y)\sigma^2(y)} dy. \end{aligned}$$

Note that if $s(\infty) = \infty$, then $v(\infty) = \infty$ and $v_b(\infty) = \infty$ by the definition in (3.1). Define $\tilde{s}(\cdot)$, $\tilde{v}(\cdot)$, and $\tilde{v}_b(\cdot)$ similarly based on the SDE (2.13) under \tilde{P} . Throughout this section, we assume that $\lambda(x \in (\ell, r) : b^2(x) > 0) > 0$, which is assumed in Mijatović and Urusov (2012a).

We have the following Engelbert–Schmidt type zero-one law for the SDE (2.1) under P , which is theorem 2.11 of Mijatović and Urusov (2012a) with $f(\cdot) = b^2(\cdot)$ using our notation.

⁷The first proof is based on William’s theorem (ch. VII, corollary 4.6, page 317, Revuz and Yor 1999). The second proof is based on the first Ray–Knight theorem (ch. XI, theorem 2.2, page 455, Revuz and Yor 1999).

PROPOSITION 3.1 (*Engelbert–Schmidt type zero-one law for a time-homogeneous diffusion, theorem 2.11 of Mijatović and Urusov 2012a*).

Assume conditions (2.2), (2.6) and $s(r) < \infty$.

- (i) If $v_b(r) < \infty$, then $\int_0^\zeta b^2(Y_u)du < \infty$, P -a.s. on $\{\lim_{t \rightarrow \zeta} Y_t = r\}$.
- (ii) If $v_b(r) = \infty$, then $\int_0^\zeta b^2(Y_u)du = \infty$, P -a.s. on $\{\lim_{t \rightarrow \zeta} Y_t = r\}$.

Analogous results on the set $\{\lim_{t \rightarrow \zeta} Y_t = \ell\}$ can be similarly stated. Clearly, the above proposition has a counterpart for the SDE (2.13) under \tilde{P} for the end points r and ℓ .

The following result is proposition 5.5.22 on page 345 of Karatzas and Shreve (1991) using our notation. It classifies possible exit behaviors of the process Y at the boundaries of its state space J under P .

PROPOSITION 3.2. (proposition 5.5.22, Karatzas and Shreve 1991). Assume condition (2.2). Let Y be a weak solution of (2.1) in J under P , with nonrandom initial condition $Y_0 = x_0 \in J$. Distinguish four cases:

- (a) If $s(\ell) = -\infty$ and $s(r) = \infty$, $P(\zeta = \infty) = P(\sup_{0 \leq t < \infty} Y_t = r) = P(\inf_{0 \leq t < \infty} Y_t = \ell) = 1$.
- (b) If $s(\ell) > -\infty$ and $s(r) = \infty$, $P(\lim_{t \rightarrow \zeta} Y_t = \ell) = P(\sup_{0 \leq t < \zeta} Y_t < r) = 1$.
- (c) If $s(\ell) = -\infty$ and $s(r) < \infty$, $P(\lim_{t \rightarrow \zeta} Y_t = r) = P(\inf_{0 \leq t < \zeta} Y_t > \ell) = 1$.
- (d) If $s(\ell) > -\infty$ and $s(r) < \infty$, $P(\lim_{t \rightarrow \zeta} Y_t = \ell) = 1 - P(\lim_{t \rightarrow \zeta} Y_t = r) = \frac{s(r) - s(x_0)}{s(r) - s(\ell)}$. Note that $0 < \frac{s(r) - s(x_0)}{s(r) - s(\ell)} < 1$.

Analogous results also hold for the SDE (2.13) under \tilde{P} .

REMARK 3.3. In the conditions (b), (c), and (d) above, we make no claim concerning the finiteness of ζ . See remark 5.5.23 on page 345 of Karatzas and Shreve (1991). Note that conditions (b) and (c) are consequences of the expression in condition (d), by letting either $s(r) = \infty$ or $s(\ell) = -\infty$.

Similar to the statements in Proposition 3.2, for the study of the convergence or divergence properties of integral functionals of time-homogeneous diffusions, we distinguish the following four exhaustive and disjoint cases under P :

- Case (1): $s(\ell) = -\infty$, $s(r) = \infty$.
- Case (2): $s(\ell) = -\infty$, $s(r) < \infty$.
- Case (3): $s(\ell) > -\infty$, $s(r) = \infty$.
- Case (4): $s(\ell) > -\infty$, $s(r) < \infty$.

Further divide each case above into the following subcases based on the finiteness of $v_b(r)$ and $v_b(\ell)$ as defined in (3.1):

- Case (2)(i): $s(\ell) = -\infty$, $s(r) < \infty$, $v_b(r) = \infty$.
- Case (2)(ii): $s(\ell) = -\infty$, $s(r) < \infty$, $v_b(r) < \infty$.
- Case (3)(i): $s(\ell) > -\infty$, $s(r) = \infty$, $v_b(\ell) = \infty$.
- Case (3)(ii): $s(\ell) > -\infty$, $s(r) = \infty$, $v_b(\ell) < \infty$.
- Case (4)(i): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) = \infty$, $v_b(\ell) = \infty$.
- Case (4)(ii): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) < \infty$, $v_b(\ell) = \infty$.
- Case (4)(iii): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) = \infty$, $v_b(\ell) < \infty$.
- Case (4)(iv): $s(\ell) > -\infty$, $s(r) < \infty$, $v_b(r) < \infty$, $v_b(\ell) < \infty$.

Define

$$(3.2) \quad \varphi_t := \int_0^t b^2(Y_u) du,$$

for $t \in [0, \zeta]$. Recall that $b^2(\cdot)$ is a nonnegative Borel function, thus φ_t is a nondecreasing function for $t \in [0, \zeta]$. Because φ_t is an integral, it is continuous for $t \in [0, \zeta)$, and is left continuous at $t = \zeta$. We now apply the Engelbert–Schmidt type zero-one law under P as in Proposition 3.1 to determine whether $P(\varphi_\zeta < \infty) = 1$ or $P(\varphi_\zeta = \infty) = 1$ in each of the cases above. We first prove two lemmas.

LEMMA 3.4. *Assume⁸ conditions (2.2) and (2.6), then “ $v_b(\ell) = \infty$ and $v_b(r) = \infty$ ” are necessary and sufficient for $P(\varphi_\zeta = \infty) = 1$.*

Proof. For the sufficiency, assume $v_b(r) = \infty$ and $v_b(\ell) = \infty$ and consider the following four distinct cases:

- Case (1): $s(\ell) = -\infty$, $s(r) = \infty$. From Proposition 3.2(a), we have $P(\zeta = \infty) = 1$. This, combined with Lemma 2.3 implies $P(\varphi_\zeta = \infty) = 1$.
- Case (2): $s(\ell) = -\infty$, $s(r) < \infty$. From Proposition 3.2(c), $P(\lim_{t \rightarrow \zeta} Y_t = r) = 1$. As $v_b(r) = \infty$, then from Proposition 3.1 $P(\varphi_\zeta = \infty) = P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r)$ and from Proposition 3.2, $P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) = P(\lim_{t \rightarrow \zeta} Y_t = r) = 1$.
- Case (3): $s(\ell) > -\infty$, $s(r) = \infty$. The proof is similar to Case (2) above by switching the roles of ℓ and r , and applying Proposition 3.2(b) and Proposition 3.1.
- Case (4): $s(\ell) > -\infty$, $s(r) < \infty$. From Proposition 3.2(d), $0 < p = P(\lim_{t \rightarrow \zeta} Y_t = r) < 1$. Since $v_b(r) = \infty$ and $v_b(\ell) = \infty$, from Proposition 3.1

$$\begin{aligned} P(\varphi_\zeta = \infty) &= P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) + P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) \\ &= P(\lim_{t \rightarrow \zeta} Y_t = r) + P(\lim_{t \rightarrow \zeta} Y_t = \ell) = 1. \end{aligned}$$

For the necessity, we only need to prove the contrapositive statement: “If at least one of $v_b(\ell)$ or $v_b(r)$ is finite, then $P(\varphi_\zeta = \infty) < 1$.” Note that case (a) of Proposition 3.2 is ruled out here so that we are assured that $P(\lim_{t \rightarrow \zeta} Y_t = r) + P(\lim_{t \rightarrow \zeta} Y_t = \ell) = 1$. Without loss of generality, assume that $v_b(\ell) < \infty$, because the case $v_b(r) < \infty$ can be similarly proved. Then,

$$\begin{aligned} P(\varphi_\zeta = \infty) &= P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) + P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) \\ &= P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) \\ &\leq P(\lim_{t \rightarrow \zeta} Y_t = r), \end{aligned}$$

where the second line follows because from Proposition 3.1, $P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) = 0$. There are now two possibilities for $s(r)$. If $s(r) = \infty$, as $s(\ell) > -\infty$, we have $P(\lim_{t \rightarrow \zeta} Y_t = r) = 0$ from Proposition 3.2(b). Alternatively, if $s(r) < \infty$, as also $s(\ell) >$

⁸Lemma 5.7, page 149 of Cherny and Urusov (2006) is a one-sided version of the current result, namely, “if $s(\ell) = \infty$ and $s(r) = \infty$ (which implies $v_b(\ell) = \infty$ and $v_b(r) = \infty$), then $P(\varphi_\zeta = \infty) = 1$.”

$-\infty$, we have from Proposition 3.2(d), $0 < p = P(\lim_{t \rightarrow \zeta} Y_t = r) < 1$. In both cases $P(\lim_{t \rightarrow \zeta} Y_t = r) < 1$, thus $P(\varphi_\zeta = \infty) < 1$, and the necessity follows. \square

LEMMA 3.5. Assume⁹ conditions (2.2) and (2.6), and $s(\ell) > -\infty$, $s(r) < \infty$, then “ $v_b(\ell) < \infty$ and $v_b(r) < \infty$ ” are necessary and sufficient for $P(\varphi_\zeta < \infty) = 1$.

Proof. With $s(\ell) > -\infty$ and $s(r) < \infty$, denote $p = P(\lim_{t \rightarrow \zeta} Y_t = r) = 1 - P(\lim_{t \rightarrow \zeta} Y_t = \ell)$. From Proposition 3.2(d), $0 < p < 1$.

For the sufficiency, assume that $v_b(\ell) < \infty$ and $v_b(r) < \infty$ hold. We aim to prove that $P(\varphi_\zeta < \infty) = 1$ where $\phi_\zeta = \int_0^\zeta b^2(Y_u) du$ according to its definition (3.2).

From Proposition 3.1, $P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = r) = P(\lim_{t \rightarrow \zeta} Y_t = r)$ and $P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) = P(\lim_{t \rightarrow \zeta} Y_t = \ell)$. Then,

$$\begin{aligned} P(\varphi_\zeta < \infty) &= P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = r) + P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) \\ &= P(\lim_{t \rightarrow \zeta} Y_t = r) + P(\lim_{t \rightarrow \zeta} Y_t = \ell) = 1. \end{aligned}$$

For the necessity, we only need to prove the contrapositive argument: “If at least one of $v_b(\ell)$ and $v_b(r)$ is infinite, then $P(\varphi_\zeta < \infty) < 1$.” Without loss of generality, assume that $v_b(r) = \infty$, because the case $v_b(\ell) = \infty$ can be similarly proved. From Proposition 3.1, $P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = r) = 0$, and

$$\begin{aligned} P(\varphi_\zeta < \infty) &= P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = r) + P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) \\ &= P(\varphi_\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) \\ &\leq P(\lim_{t \rightarrow \zeta} Y_t = \ell) < 1, \quad \text{from Proposition 3.2.} \end{aligned}$$

Thus, the necessity follows. \square

We now give a detailed study of the function φ_t , $t \in [0, \zeta]$ under P using the Engelbert–Schmidt type zero-one law. Theorem 3.6 completely characterizes the convergence or divergence property of φ_t , $t \in [0, \zeta]$, and several results from the literature are one-sided versions of it: Theorem 3.6(i) is Lemma 2.2, which is stated and proved after equation (9) on page 5 of Mijatović and Urusov (2012c). Theorem 2 on page 3 of Khoshnevisan et al. (2006) provides¹⁰ the necessary and sufficient conditions for $P(\varphi_\zeta < \infty) = 1$, which corresponds to Theorem 3.6(ii). However, they make use of the stochastic time change and Itô’s lemma in their proof, and thus need to assume the twice differentiability of a function $g(\cdot)$ defined in their paper. Our proof is based on Engelbert–Schmidt type zero-one laws of Mijatović and Urusov (2012a), and our weaker assumptions concern the local integrability of certain deterministic functions. Under these assumptions, Mijatović and Urusov (2012a) give a result similar to Theorem 3.6(ii) (in their theorem 2.11). In a parallel paper, Engelbert and Tittel (2002) consider a strong Markov continuous local martingale and are broader in scope. As a comparison, their proposition 3.7 gives necessary and sufficient conditions for the integral functional to be convergent or divergent, but assume in proposition 3.7 that the process X has exactly one absorbing point whereas in our setting and that of Mijatović and Urusov (2012a), it is assumed that the process Y can be absorbed at either boundary ℓ or r .

⁹Theorem 2.11 on page 61 of Mijatović and Urusov (2012a) gives a similar result.

¹⁰Salminen and Yor (2006) give similar conditions for a Brownian motion with drift, and Khoshnevisan et al. (2006) extend it to time-homogeneous diffusions.

TABLE 3.1

Table Indicating the Positivity of the Stock Price and the Finiteness of φ_ζ (* indicates that the probability lies in the open interval (0,1))

Case		$s(\ell)$	$s(r)$	$v_b(\ell)$	$v_b(r)$	$P(\varphi_\zeta < \infty)$	$P(Z_\infty > 0)$
(1)		$-\infty$	∞	∞	∞	0	0
(2)	(i)	$-\infty$	$< \infty$	∞	∞	0	0
	(ii)	$-\infty$	$< \infty$	∞	$< \infty$	1	1
(3)	(i)	$> -\infty$	∞	∞	∞	0	0
	(ii)	$> -\infty$	∞	$< \infty$	∞	1	1
(4)	(i)	$> -\infty$	$< \infty$	∞	∞	0	0
	(ii)	$> -\infty$	$< \infty$	∞	$< \infty$	$(0, 1)^*$	$(0, 1)^*$
	(iii)	$> -\infty$	$< \infty$	$< \infty$	∞	$(0, 1)^*$	$(0, 1)^*$
	(iv)	$> -\infty$	$< \infty$	$< \infty$	$< \infty$	1	1

THEOREM 3.6. Under conditions (2.2) and (2.6), the following properties for φ_t , $t \in [0, \zeta]$ hold:

- (i) $\varphi_t < \infty$ P -a.s. on $\{0 \leq t < \zeta\}$.
- (ii) $P(\varphi_\zeta < \infty) = 1$ if and only if at least one of the following conditions is satisfied:
 - (a) $v_b(r) < \infty$ and $s(\ell) = -\infty$,
 - (b) $v_b(\ell) < \infty$ and $s(r) = \infty$,
 - (c) $v_b(r) < \infty$ and $v_b(\ell) < \infty$.
- (iii) $P(\varphi_\zeta = \infty) = 1$ if and only if $v_b(r) = \infty$ and $v_b(\ell) = \infty$.

We summarize the results of Theorem 3.6 in Table 3.1 hereafter. Note that $P(\varphi_\zeta < \infty) = P(Z_\infty > 0)$ always holds by taking $\tau = \infty$ in Lemma 2.4, and the last two columns in Table 3.1 agree.

Proof. Statement (i) follows from Lemma 2.2. For statement (ii), the detailed proof for each of the cases in Table 3.1 is as follows:

- In Case (1), $s(\ell) = -\infty$ and $s(r) = \infty$ and so from Lemma 2.3, $P(\varphi_\zeta = \infty) = 1$.
- In Case (2), $s(\ell) = -\infty$ and $s(r) < \infty$ and so from Proposition 3.2, $P(\lim_{t \rightarrow \zeta} Y_t = r) = 1$. There are two possible subcases. First, in Case (2)(i), $v_b(r) < \infty$ and it follows from Lemma 2.2 that $P(\varphi_\zeta = \infty) = 1$. In Case (2)(ii), as $v_b(r) < \infty$, we have from Lemma 3.4 that $\varphi_\zeta < \infty$ a.s. on the set $\{\lim_{t \rightarrow \zeta} Y_t = r\}$. Moreover, from Proposition 3.2, $P(\lim_{t \rightarrow \zeta} Y_t = r) = 1$. It follows that $P(\varphi_\zeta < \infty) = 1$.
- In Case (3), $s(\ell) > -\infty$ and $s(r) = \infty$ and so from Proposition 3.2, $P(\lim_{t \rightarrow \zeta} Y_t = \ell) = 1$. Again, there are two possible subcases, but they are the reverse of cases in (2); Case (3)(i) is exactly the reverse of (2)(i) with ℓ and r interchanged and similarly, Case (3)(ii) is exactly the reverse of (2)(ii) so the proofs in Case (2) suffice.
- In Case (4): $s(\ell) > -\infty$ and $s(r) < \infty$. Then, from Proposition 3.2, $1 > p = P(\lim_{t \rightarrow \zeta} Y_t = r) = 1 - P(\lim_{t \rightarrow \zeta} Y_t = \ell) > 0$. For individual subcases, in Case 4(i), Lemma 3.4 implies $P(\varphi_\zeta = \infty) = 1$. In Case (4)(ii), Proposition 3.1 implies that $P(\varphi_\zeta = \infty) < 1$ so that $P(\varphi_\zeta < \infty) > 0$. By Lemma 3.5, we have $P(\varphi_\zeta < \infty) < 1$.

TABLE 3.2
Table Indicating $\mathbb{E}^P(\mathbf{Z}_\infty)$ and the Uniform Integrability of Z (*indicates that the probability lies in the open interval $(0, 1)$)

Case		$\tilde{s}(\ell)$	$\tilde{s}(r)$	$\tilde{v}_b(\ell)$	$\tilde{v}_b(r)$	$\tilde{P}(\varphi_\zeta < \infty)$	$\mathbb{E}^P(\mathbf{Z}_\infty)$	UI Mart.
(1)		$-\infty$	∞	∞	∞	0	< 1	No
(2)	(i)	$-\infty$	$< \infty$	∞	∞	0	< 1	No
	(ii)	$-\infty$	$< \infty$	∞	$< \infty$	1	1	Yes
(3)	(i)	$> -\infty$	∞	∞	∞	0	< 1	No
	(ii)	$> -\infty$	∞	$< \infty$	∞	1	1	Yes
(4)	(i)	$> -\infty$	$< \infty$	∞	∞	0	< 1	No
	(ii)	$> -\infty$	$< \infty$	∞	$< \infty$	$(0, 1)^*$	< 1	No
	(iii)	$> -\infty$	$< \infty$	$< \infty$	∞	$(0, 1)^*$	< 1	No
	(iv)	$> -\infty$	$< \infty$	$< \infty$	$< \infty$	1	1	Yes

Case (4)(iii) is exactly the reverse of (4)(ii) with ℓ and r interchanged so the proof follows using this substitution. And finally, for Case (4)(iv), $P(\varphi_\zeta < \infty) = 1$ follows from Lemma 3.5. Therefore, we have the three distinct behaviors for $P(\varphi_\zeta < \infty)$ as outlined in Table 3.1. The necessity follows by examination of Table 3.1.

Similar results as Theorem 3.6 hold under \tilde{P} , and the results are summarized in Table 3.2. Note that $\mathbb{E}^P[\mathbf{Z}_\infty] = \tilde{P}(\varphi_\zeta < \infty)$ from Proposition 2.10, and the second-to-last and third-to-last columns in Table 3.2 are equal.

The following result provides necessary and sufficient conditions for $P(\varphi_{\zeta \wedge T} < \infty) = 1$, for $T \in (0, \infty)$:

THEOREM 3.7. *Assume conditions (2.2) and (2.6).*

$$P(\varphi_{\zeta \wedge T} < \infty) = P\left(\int_0^{\zeta \wedge T} b^2(Y_u) du < \infty\right) = 1$$

for all $T \in (0, \infty)$ if and only if at least one of the following conditions is satisfied:

- (a) $v(\ell) = v(r) = \infty$,
- (b) $v_b(r) < \infty$ and $v(\ell) = \infty$,
- (c) $v_b(\ell) < \infty$ and $v(r) = \infty$,
- (d) $v_b(r) < \infty$ and $v_b(\ell) < \infty$.

Proof. The conditions state that $(\{v(\ell) = \infty\} \text{ or } \{v_b(\ell) < \infty\})$ and $(\{v(r) = \infty\} \text{ or } \{v_b(r) < \infty\})$. For a given $T < \infty$, define the events $A_T = \{\varphi_{\zeta \wedge T} < \infty\}$, $A = \{\varphi_\zeta < \infty\}$, and $B = \{\zeta < \infty\}$. Notice that the sets $A_T \cap B$ form a decreasing sequence of sets (as $T \rightarrow \infty$ through a countable set) so that $\bigcap_T (A_T \cap B) = A \cap B$. Therefore,

$$(3.3) \quad P(A_T \cap B) \downarrow P(A \cap B) \text{ as } T \rightarrow \infty.$$

TABLE 3.3
Correspondence between Conditions (b), (c), and (d) for the Sufficiency Case and Cases in Table 3.1

Case	Implies	Cases in Table 3.1
(b) $v_b(r) < \infty$ and $v(\ell) = \infty$	$s(r) < \infty$	2(ii), 4(ii), 4(iv)
(c) $v_b(\ell) < \infty$ and $v(r) = \infty$	$s(\ell) > -\infty$	3(ii), 4(iii), 4(iv)
(d) $v_b(r) < \infty$ and $v_b(\ell) < \infty$	$s(r) < \infty, s(\ell) > -\infty$	4(iv)

Moreover, from Theorem 3.6(i), for each $T < \infty$,

$$(3.4) \quad P(A_T \cap \bar{B}) = P(\bar{B}).$$

We wish to find necessary and sufficient conditions for $P(A_T) = 1$ for all $T < \infty$. In view of (3.3) and (3.4), this is equivalent to the condition

$$(3.5) \quad \begin{aligned} &P(A_T \cap B) + P(\bar{B}) = 1 \text{ for all } T \text{ or} \\ &P(A \cap B) + P(\bar{B}) = 1 \text{ or } P(B \cap \bar{A}) = 0. \end{aligned}$$

In other words, we seek necessary and sufficient conditions to ensure that

$$(3.6) \quad P(\zeta < \infty, \varphi_\zeta = \infty) = 0.$$

We first show the “sufficiency” of the above conditions. Condition (a) and Feller’s test for explosions implies $P(\zeta < \infty) = 0$ and so (3.6) follows. $P(\varphi_\zeta = \infty) = 0$ is implied in Cases 2(ii), 3(ii), or 4(iv) of Table 3.1. These conditions are special cases of conditions (b), (c), and (d) as indicated in Table 3.3.

It remains to show (3.6) in Case 4(ii), i.e., $v(\ell) = \infty, s(\ell) > -\infty, v_b(r) < \infty, s(r) < \infty$ and in Case 4(iii), i.e., $v_b(\ell) < \infty, s(\ell) > -\infty, v(r) = \infty, s(r) < \infty$. By interchanging the role of ℓ and r , it suffices to show the first of these. By Proposition 3.1, $\varphi_\zeta < \infty$ P -a.s. on the set $\{\lim_{t \rightarrow \zeta} Y_t = r\}$ or

$$P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) = 0.$$

From Feller’s test of explosions, $v(\ell) < \infty$ if and only if $P(\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) > 0$, and so in this case $v(\ell) = \infty$ implies

$$P(\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) = 0.$$

It follows that

$$(3.7) \quad \begin{aligned} P(\zeta < \infty, \varphi_\zeta = \infty) &= P(\zeta < \infty, \varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) + P(\zeta < \infty, \varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) \\ &\leq P(\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) + P(\varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = r) = 0. \end{aligned}$$

TABLE 3.4
Correspondence between the Two Conditions from the Contrapositive Case and Cases in Table 3.1

Contrapositive case	Implies		Cases in Table 3.1
$\{v(\ell) < \infty\}$ and $\{v_b(\ell) = \infty\}$	$s(\ell) > -\infty$	Consistent with	3(i),4(i),4(ii)
$\{v(r) < \infty\}$ and $\{v_b(r) = \infty\}$	$s(r) < \infty$	Consistent with	2(i),4(i),4(iii)

For the “necessity,” we wish to show the contrapositive: if $\{v(\ell) < \infty$ and $v_b(\ell) = \infty\}$ OR $\{v(r) < \infty$ and $v_b(r) = \infty\}$ (i.e., at least at one of the two boundaries, v is finite and v_b infinite), then (3.6) fails, that is

$$P(\zeta < \infty, \varphi_\zeta = \infty) > 0.$$

The contrapositive is consistent with Table 3.1, Cases 2(i), 3(i), 4(i), 4(ii), 4(iii) as indicated in Table 3.4.

Consider the first row above when $v(\ell) < \infty$, $v_b(\ell) = \infty$, $s(\ell) > -\infty$. By Feller’s test, $v(\ell) < \infty$ implies $P(\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) > 0$ and by Proposition 3.1, because $v_b(\ell) = \infty$, $\varphi_\zeta = \infty$ P -a.s. on the set $\{\lim_{t \rightarrow \zeta} Y_t = \ell\}$ and

$$P(\zeta < \infty, \varphi_\zeta = \infty) \geq P(\zeta < \infty, \varphi_\zeta = \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) = P(\zeta < \infty, \lim_{t \rightarrow \zeta} Y_t = \ell) > 0.$$

The proof in the second case $v(r) < \infty$, $v_b(r) = \infty$ follows once again by interchanging the roles of ℓ and r . \square

Similarly, statements as Theorem 3.7 hold under \tilde{P} with SDE (2.13).

4. GENERALIZATION OF SOME RESULTS IN MIJATOVIĆ AND URUSOV

In this section, we generalize the main results in Mijatović and Urusov (2012b, c) and provide new unified proofs without the concept of “separating times.” Note that Mijatović and Urusov (2012b, c) work in the $\rho = 1$ case, and we generalize it to the arbitrary correlation case.

Consider the stochastic exponential Z defined in (2.4). The following proposition provides the necessary and sufficient condition for Z_T to be a P -martingale for all $T \in (0, \infty)$, when $-1 \leq \rho \leq 1$. Note that theorem 2.1 in Mijatović and Urusov (2012c) is the case $\rho = 1$ of the following proposition:

PROPOSITION 4.1. *Assume conditions (2.2) and (2.6), then for all $T \in (0, \infty)$, $\mathbb{E}^P[Z_T] = 1$ if and only if at least one of the conditions (1)–(4) below is satisfied:*

- (1) $\tilde{v}(\ell) = \tilde{v}(r) = \infty$,
- (2) $\tilde{v}_b(r) < \infty$ and $\tilde{v}(\ell) = \infty$,
- (3) $\tilde{v}_b(\ell) < \infty$ and $\tilde{v}(r) = \infty$,
- (4) $\tilde{v}_b(r) < \infty$ and $\tilde{v}_b(\ell) < \infty$.

Proof. From Proposition 2.10, for all $T \in (0, \infty)$, $\mathbb{E}^P[Z_T] = 1$ if and only if $\tilde{P}(\int_0^{\zeta \wedge T} b^2(Y_u) du < \infty) = 1$. Then, the statement follows from Theorem 3.7 applied to \tilde{P} . \square

We have the following necessary and sufficient condition for Z to be an UI P -martingale on $[0, \infty]$, when $-1 \leq \rho \leq 1$. Note that theorem 2.3 of Mijatović and Urusov (2012c) proves the case $\rho = 1$ of the following proposition:

PROPOSITION 4.2. *Assume conditions (2.2) and (2.6), then $\mathbb{E}^P[Z_\infty] = 1$ if and only if at least one of the conditions (A') – (D') below is satisfied:*

- (A') $b = 0$ a.e. on J with respect to the Lebesgue measure,
- (B') $\tilde{v}_b(r) < \infty$ and $\tilde{s}(\ell) = -\infty$,
- (C') $\tilde{v}_b(\ell) < \infty$ and $\tilde{s}(r) = \infty$,
- (D') $\tilde{v}_b(r) < \infty$ and $\tilde{v}_b(\ell) < \infty$.

Proof. From Proposition 2.10, $\mathbb{E}^P[Z_\infty] = 1$ if and only if $\tilde{P}(\int_0^\zeta b^2(Y_u) du < \infty) = 1$. Condition (A') is a trivial case and it is easy to verify. From Theorem 3.6 applied to \tilde{P} and the classification in Table 3.2, $\mathbb{E}^P[Z_\infty] = 1$ if and only if at least one of the conditions (B'), (C'), or (D') holds. \square

Here, we generalize some results in Mijatović and Urusov (2012b) to the arbitrary correlation case and provide new proofs without the concept of *separating times*. Precisely, theorem 2.1 of Mijatović and Urusov (2012b) is the case $\rho = 1$ of the following proposition:

PROPOSITION 4.3. *Assume conditions (2.2) and (2.6), then for all $T \in (0, \infty)$, $Z_T > 0$ P -a.s. if and only if at least one of the conditions¹¹ (1)–(4) below is satisfied:*

- (1) $v(\ell) = v(r) = \infty$,
- (2) $v_b(r) < \infty$ and $v(\ell) = \infty$,
- (3) $v_b(\ell) < \infty$ and $v(r) = \infty$,
- (4) $v_b(r) < \infty$ and $v_b(\ell) < \infty$.

Proof. From Lemma 2.4, for all $T \in (0, \infty)$, $Z_T > 0$, P -a.s. if and only if $P(\int_0^{\zeta \wedge T} b^2(Y_u) du < \infty) = 1$. Then the statement follows from Theorem 3.7. \square

Note that theorem 2.3 of Mijatović and Urusov (2012b) proves the case $\rho = 1$ of the following proposition:

PROPOSITION 4.4. *Let the functions μ , σ , and b satisfy conditions (2.1), (2.3), and (2.5) of Mijatović and Urusov (2012b) (equivalently conditions (2.2) and (2.6) in this paper), and let Y be a (possibly explosive) solution of the SDE (2.1) under P , with Z defined in (2.4). Then, $Z_\infty > 0$, P -a.s. if and only if at least one of the conditions (I)–(IV) below is satisfied:*

- I. $b = 0$ a.e. on J with respect to the Lebesgue measure,
- II. $v_b(r) < \infty$ and $s(\ell) = -\infty$,
- III. $v_b(\ell) < \infty$ and $s(r) = \infty$,
- IV. $v_b(r) < \infty$ and $v_b(\ell) < \infty$.

¹¹Note that conditions (1)–(4) in Proposition 4.3 do not depend on the correlation ρ , which means that the positivity of the (discounted) stock price does not depend on the correlation. Similar remarks hold for Propositions 4.4 and 4.5.

Proof. Condition (I) is a trivial case and it is easy to verify. From Lemma 2.4, $Z_\infty > 0$, P -a.s. if and only if $P(\int_0^\zeta b^2(Y_s)ds < \infty) = 1$. Then the proof follows from Theorem 3.6 and the classification in Table 3.1. \square

Note that theorem 2.5 of Mijatović and Urusov (2012b) is a special case of the following proposition when $\rho = 1$:

PROPOSITION 4.5. *Let the functions μ , σ , and b satisfy conditions (2.1), (2.3), and (2.5) of Mijatović and Urusov (2012b) (equivalently conditions (2.2) and (2.6) in this paper), and let Y be a (possibly explosive) solution of the SDE (2.1) under P , with Z defined in (2.4). Then, $Z_\infty = 0$, P -a.s. if and only if both conditions (i) and (ii) below are satisfied:*

- (i) b is not identically zero with respect to Lebesgue measure on (ℓ, r) ,
- (ii) $v_b(\ell) = v_b(r) = \infty$.

Proof. Condition (i) is a trivial case and it is easy to verify. From Lemma 2.4, $Z_\infty = 0$, P -a.s. if and only if $P(\int_0^\zeta b^2(Y_u)du = \infty) = P(\varphi_\zeta = \infty) = 1$. From Theorem 3.6(iii), this is equivalent to checking the condition (ii) here. \square

5. EXAMPLES OF CORRELATED STOCHASTIC VOLATILITY MODELS

In this section, we apply the results in Section 4 to the study of martingale properties of (discounted) stock prices¹² in four popular correlated stochastic volatility models: the (stopped) Heston,¹³ the 3/2, the Schöbel–Zhu, and the Hull–White models. The results are summarized in Tables 5.10 and 5.11.

5.1. Stopped Heston Stochastic Volatility Model

Suppose that under a probability measure P , the (correlated) stopped Heston stochastic volatility model has the following diffusive dynamics:

$$(5.1) \quad \begin{aligned} dS_t &= S_t \sqrt{Y_t} 1_{t \in [0, \zeta)} dW_t^{(1)}, \quad S_0 = 1, \\ dY_t &= \kappa(\theta - Y_t) 1_{t \in [0, \zeta)} dt + \xi \sqrt{Y_t} 1_{t \in [0, \zeta)} dW_t, \quad Y_0 = x_0 > 0, \end{aligned}$$

with $\mathbb{E}^P[dW_t^{(1)} dW_t] = \rho dt$, $-1 \leq \rho \leq 1$, $\kappa > 0$, $\theta > 0$, $\xi > 0$. The natural state space for Y is $J = (\ell, r) = (0, \infty)$. ζ is the possible exit time of the process Y from its state space J . The model (5.1) belongs to the general stochastic volatility model considered in (2.5) with $\mu(x) = \kappa(\theta - x)$, $\sigma(x) = \xi\sqrt{x}$, and $b(x) = \sqrt{x}$. Clearly, $\sigma(x) = \xi\sqrt{x} \neq 0$, $x \in J$, $\frac{1}{\sigma^2(x)} = \frac{1}{\xi^2 x} \in L_{loc}^1(J)$, $\frac{\mu(x)}{\sigma^2(x)} = \frac{\kappa(\theta - x)}{\xi^2 x} \in L_{loc}^1(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{1}{\xi^2} \in L_{loc}^1(J)$ are satisfied. Thus, the conditions (2.2) and (2.6) are satisfied. From Proposition 2.8, under \tilde{P} , the diffusion Y satisfies the following SDE:

$$dY_t = \tilde{\kappa}(\tilde{\theta} - Y_t) 1_{t \in [0, \zeta)} dt + \xi \sqrt{Y_t} 1_{t \in [0, \zeta)} d\tilde{W}_t, \quad Y_0 = x_0 > 0,$$

where $\tilde{\kappa} = \kappa - \rho\xi$ and $\tilde{\theta} = \frac{\kappa\theta}{\kappa - \rho\xi}$.

¹²Equivalently, we may assume that the risk-free interest rate is zero.

¹³The volatility is stopped whenever it hits the boundary 0. When $2\kappa\theta > \xi^2$ (zero is unattainable), our model coincides with the usual Heston model.

TABLE 5.1
First Classification Table for the Heston Model

Case	$\tilde{v}(\ell)$	$\tilde{v}(r)$	$\tilde{v}_b(\ell)$	$\tilde{v}_b(r)$
$\alpha \geq 1$	∞	∞	∞	∞
$\alpha < 1$	$< \infty$	∞	$< \infty$	∞

For a constant $c \in J$, the scale functions of the SDEs (2.1) and (2.13) are, respectively,

$$(5.2) \quad \begin{aligned} s(x) &= e^{\frac{2\kappa c}{\xi^2}} c^{\frac{2\kappa\theta}{\xi^2}} \int_c^x y^{-\frac{2\kappa\theta}{\xi^2}} e^{\frac{2\kappa y}{\xi^2}} dy = C_1 \int_c^x y^{-\alpha} e^{\beta y} dy, \\ \tilde{s}(x) &= e^{\frac{2\tilde{\kappa}c}{\xi^2}} c^{\frac{2\tilde{\kappa}\tilde{\theta}}{\xi^2}} \int_c^x y^{-\frac{2\tilde{\kappa}\tilde{\theta}}{\xi^2}} e^{\frac{2\tilde{\kappa}y}{\xi^2}} dy = C_2 \int_c^x y^{-\alpha} e^{\gamma y} dy, \end{aligned}$$

with $\alpha = \frac{2\kappa\theta}{\xi^2}$, $\beta = \frac{2\kappa}{\xi^2} > 0$, $\gamma = \frac{2\kappa}{\xi^2} - \frac{2\rho}{\xi}$, and the constant terms are $C_1 = e^{\frac{2\kappa c}{\xi^2}} c^{\frac{2\kappa\theta}{\xi^2}} > 0$ and $C_2 = e^{\frac{2\tilde{\kappa}c}{\xi^2} - \frac{2\rho c}{\xi}} c^{\frac{2\tilde{\kappa}\tilde{\theta}}{\xi^2}} > 0$. Under \tilde{P} , we have the following test functions for $x \in \tilde{J}$:

$$\tilde{v}(x) = \frac{2}{\xi^2} \int_c^x \frac{\int_y^x z^{-\alpha} e^{\gamma z} dz}{y^{1-\alpha} e^{\gamma y}} dy, \quad \tilde{v}_b(x) = \frac{2}{\xi^2} \int_c^x \frac{\int_y^x z^{-\alpha} e^{\gamma z} dz}{y^{-\alpha} e^{\gamma y}} dy.$$

PROPOSITION 5.1. For¹⁴ the stopped Heston model (5.1), the underlying stock price $(S_t)_{0 \leq t < \infty}$ is a true P -martingale.

Proof. The proof of Proposition 5.1 is elementary and details are given in Supporting Information, an online appendix, available from the corresponding author's website. To prove it, we check the conditions of Proposition 4.1: the results are summarized in Table 5.1.

From Table 5.1 and Proposition 4.1, $(S_t)_{0 \leq t < \infty}$ is a true P -martingale. \square

PROPOSITION 5.2. For the stopped Heston model (5.1), the underlying stock price $(S_t)_{0 \leq t \leq \infty}$ is an UI P -martingale if and only if $\rho\xi \leq \kappa < \frac{\xi^2}{2\theta}$.

Note that the Feller condition has to be violated in order to have an UI martingale.

Proof. The proof of Proposition 5.2 is elementary and details are given in Supporting Information. To prove it, we check the conditions of Proposition 4.2: the results are summarized in Table 5.2.

From Table 5.2 and Proposition 4.2, $(S_t)_{0 \leq t \leq \infty}$ is an UI P -martingale if and only if $\alpha = \frac{2\kappa\theta}{\xi^2} < 1$, and $\gamma = \frac{2(\kappa - \rho\xi)}{\xi^2} \geq 0$, which is equivalent to $\rho\xi \leq \kappa < \frac{\xi^2}{2\theta}$. \square

Under P , we have the following result on the positivity of the stock price in the stopped Heston model:

PROPOSITION 5.3. For the stopped Heston model (5.1),

- (1) $P(S_T > 0) = 1$ for all $T \in (0, \infty)$,
- (2) $P(S_\infty > 0) = 1$ if and only if $\kappa < \frac{\xi^2}{2\theta}$.

¹⁴Proposition 5.1 is consistent with proposition 2.5, page 34 of Andersen and Piterbarg (2007), also see remark 4.2, page 2052 of Del Baño Rollin, Ferreiro-Castilla, and Utzet (2010)

TABLE 5.2
Second Classification Table for the Heston Model

Case		$\tilde{s}(\ell)$	$\tilde{s}(r)$	$\tilde{v}(\ell)$	$\tilde{v}(r)$	$\tilde{v}_b(\ell)$	$\tilde{v}_b(r)$
$\alpha > 1$	$\gamma < 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
	$\gamma = 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
	$\gamma > 0$	$-\infty$	∞	∞	∞	∞	∞
$\alpha = 1$	$\gamma < 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
	$\gamma = 0$	$-\infty$	∞	∞	∞	∞	∞
	$\gamma > 0$	$-\infty$	∞	∞	∞	∞	∞
$\alpha < 1$	$\gamma < 0$	$> -\infty$	$< \infty$	$< \infty$	∞	$< \infty$	∞
	$\gamma = 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞
	$\gamma > 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞

TABLE 5.3
Third Classification Table for the Heston Model

Case	$s(\ell)$	$s(r)$	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$\alpha > 1$	$-\infty$	∞	∞	∞	∞	∞
$\alpha = 1$	$-\infty$	∞	∞	∞	∞	∞
$\alpha < 1$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞

Proof. Similar to the proofs of Propositions 5.1 and 5.2 with γ replaced by $\beta > 0$ and C_2 by C_1 , we obtain the classification in Table 5.3.

Based on Table 5.3, from Propositions 4.3 and 4.4, we obtain the desired results. \square

5.2. 3/2 Stochastic Volatility Model

Under P , the (correlated) 3/2 stochastic volatility model has the following diffusive dynamics:

$$\begin{aligned}
 dS_t &= S_t \sqrt{Y_t} 1_{t \in [0, \zeta)} dW_t^{(1)}, \quad S_0 = 1, \\
 (5.3) \quad dY_t &= (\omega Y_t - \theta Y_t^2) 1_{t \in [0, \zeta)} dt + \xi Y_t^{\frac{3}{2}} 1_{t \in [0, \zeta)} dW_t, \quad Y_0 = x_0 > 0,
 \end{aligned}$$

where $\mathbb{E}^P[dW_t^{(1)} dW_t] = \rho dt$, $-1 \leq \rho \leq 1$, $\omega > 0$, $\xi > 0$, $\theta \in \mathbb{R}$. The natural state space is given by $J = (\ell, r) = (0, \infty)$. ζ is the possible exit time of the process Y from its state space J . The model (5.3) belongs to the general stochastic volatility model considered in (2.5) with $\mu(x) = \omega x - \theta x^2$, $\sigma(x) = \xi x^{3/2}$, and $b(x) = \sqrt{x}$. Clearly $\sigma(x) = \xi x^{3/2} \neq 0$, $x \in J$, $\frac{1}{\sigma^2(x)} = \frac{1}{\xi^2 x^3} \in L_{loc}^1(J)$, $\frac{\mu(x)}{\sigma^2(x)} = \frac{\omega - \theta x}{\xi^2 x^2} \in L_{loc}^1(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{1}{\xi^2 x^2} \in L_{loc}^1(J)$ are satisfied.

Thus, the conditions (2.2) and (2.6) are satisfied. From Proposition 2.8, under \tilde{P} , the diffusion Y satisfies the following SDE:

$$dY_t = (\omega Y_t - \tilde{\theta} Y_t^2) 1_{t \in [0, \xi)} dt + \xi Y_t^{\frac{3}{2}} 1_{t \in [0, \xi)} d\tilde{W}_t, \quad Y_0 = x_0 > 0,$$

where $\tilde{\theta} = \theta - \rho\xi$. For a constant $c \in J$, the scale functions of the SDEs (2.1) and (2.13) are, respectively,

$$(5.4) \quad s(x) = \frac{b}{c^a} \int_c^x y^a \exp\left(\frac{d}{y}\right) dy, \quad \tilde{s}(x) = \frac{b}{c^{\tilde{a}}} \int_c^x y^{\tilde{a}} \exp\left(\frac{d}{y}\right) dy, \quad x \in \bar{J},$$

where $a = \frac{2\theta}{\xi^2}$, $b = \exp(-\frac{2\omega}{c\xi^2})$, $d = \frac{2\omega}{\xi^2}$, and $\tilde{a} = a - \frac{2\rho}{\xi}$. Because the only difference between $s(\cdot)$ and $\tilde{s}(\cdot)$ is in the parameters a and \tilde{a} , the analysis under \tilde{P} is similar to the analysis under P , except with a change of the parameter from a to \tilde{a} . Thus, we only need the results under P . We have the following test functions:

$$(5.5) \quad v(x) = \frac{2}{\xi^2} \int_c^x \frac{1}{y^{a+3} \exp\left(\frac{d}{y}\right)} \left(\int_y^x z^a \exp\left(\frac{d}{z}\right) dz \right) dy,$$

$$(5.6) \quad v_b(x) = \frac{2}{\xi^2} \int_c^x \frac{1}{y^{a+2} \exp\left(\frac{d}{y}\right)} \left(\int_y^x z^a \exp\left(\frac{d}{z}\right) dz \right) dy.$$

LEMMA 5.4. *With $\omega > 0$, the following properties are satisfied:*

- (i) $a < -1 \iff v(r) < \infty$, (v) $\tilde{a} < -1 \iff \tilde{v}(r) < \infty$.
- (ii) $\forall a \in \mathbb{R}, v_b(r) = \infty$, (vi) $\forall \tilde{a} \in \mathbb{R}, \tilde{v}_b(r) = \infty$.
- (iii) $\forall a \in \mathbb{R}, v(\ell) = \infty$, (vii) $\forall \tilde{a} \in \mathbb{R}, \tilde{v}(\ell) = \infty$.
- (iv) $\forall a \in \mathbb{R}, v_b(\ell) = \infty$, (viii) $\forall \tilde{a} \in \mathbb{R}, \tilde{v}_b(\ell) = \infty$.

Proof. Details of the derivations can be found in Supporting Information. \square

PROPOSITION 5.5. *For¹⁵ the 3/2 model (5.3), the underlying stock price $(S_t)_{0 \leq t < \infty}$ is a true P -martingale if and only if $\xi^2 - 2\rho\xi + 2\theta \geq 0$.*

Proof. From Lemma 5.4 and Proposition 4.1, $(S_t)_{0 \leq t < \infty}$ is a true P -martingale if and only if $\tilde{a} \geq -1$, which is equivalent to $\xi^2 - 2\rho\xi + 2\theta \geq 0$ after some simplifications. \square

PROPOSITION 5.6. *For the 3/2 model (5.3), the underlying stock price $(S_t)_{0 \leq t \leq \infty}$ is not an UI P -martingale.*

Proof. From Lemma 5.4, for all $\tilde{a} \in \mathbb{R}$, $\tilde{v}_b(r) = \infty$ and $\tilde{v}_b(l) = \infty$ hold. From Proposition 4.2, $(S_t)_{0 \leq t \leq \infty}$ is not an UI P -martingale. \square

Under P , we have the following result on the positivity of the stock price in the 3/2 model:

PROPOSITION 5.7. *For the 3/2 model (5.3),*

- (1) $P(S_T > 0) = 1$ for all $T \in (0, \infty)$ if and only if $\xi^2 + 2\theta \geq 0$,
- (2) $P(S_\infty > 0) < 1$.

¹⁵Theorem 3, page 110 of Carr and Sun (2007) proves sufficiency. See also Lewis (2000).

TABLE 5.4
Classification Table for the 3/2 Model

Case	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$a < -1$	∞	$< \infty$	∞	∞
$a \geq -1$	∞	∞	∞	∞

Proof. Similar to the proofs of Propositions 5.5 and 5.6 with \tilde{a} replaced by a , we obtain the classification in Table 5.4.

Based on Table 5.4, Propositions 4.3 and 4.4, we have the desired results. Note that $a \geq -1$ is equivalent to $\xi^2 + 2\theta \geq 0$. \square

5.3. Schöbel–Zhu Stochastic Volatility Model

Under P , the correlated Schöbel–Zhu stochastic volatility model¹⁶ (see Schöbel and Zhu 1999) can be described by the following diffusive dynamics:

$$(5.7) \quad \begin{aligned} dS_t &= S_t Y_t 1_{t \in [0, \zeta)} dt dW_t^{(1)}, \quad S_0 = 1, \\ dY_t &= \kappa(\theta - Y_t) 1_{t \in [0, \zeta)} dt + \gamma 1_{t \in [0, \zeta)} dW_t, \quad Y_0 = x_0, \end{aligned}$$

where $\mathbb{E}[dW_t^{(1)} dW_t] = \rho dt$, $-1 \leq \rho \leq 1$, $\kappa > 0$, $\theta > 0$, $\gamma > 0$. The process Y is an Ornstein–Uhlenbeck process, and this implies that its natural state space is $J = (\ell, r) = (-\infty, \infty)$. ζ is the possible exit time of the process Y from its state space J . The model (5.7) belongs to the general stochastic volatility model considered in (2.5) with $\mu(x) = \kappa(\theta - x)$, $\sigma(x) = \gamma$, and $b(x) = x$. Clearly $\sigma(x) = \gamma \neq 0$, $x \in J$, then $\frac{1}{\sigma(x)^2} = \frac{1}{\gamma^2} \in L_{loc}^1(J)$, $\frac{\mu(x)}{\sigma(x)^2} = \frac{\kappa(\theta - x)}{\gamma^2} \in L_{loc}^1(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{x^2}{\gamma^2} \in L_{loc}^1(J)$ are satisfied. Thus, the conditions (2.2) and (2.6) are satisfied.

From Proposition 2.8, under \tilde{P} , the diffusion Y satisfies the following SDE:

$$dY_t = (\kappa\theta - (\kappa - \rho\gamma)Y_t) 1_{t \in [0, \zeta)} dt + \gamma 1_{t \in [0, \zeta)} d\tilde{W}_t, \quad Y_0 = x_0.$$

For a positive constant $c \in J$, denote $\alpha = \kappa - \rho\gamma$, and compute the scale functions, respectively, of the SDEs (2.1) and (2.13)

$$\begin{aligned} s(x) &= \int_c^x e^{\frac{\kappa(y-\theta)^2 - \kappa(c-\theta)^2}{\gamma^2}} dy = C_1 \int_c^x e^{\frac{\kappa(y-\theta)^2}{\gamma^2}} dy, \\ \tilde{s}(x) &= \int_c^x e^{\frac{\alpha y^2 - 2\kappa\theta y + 2\kappa\theta c - \alpha c^2}{\gamma^2}} dy = \begin{cases} C_2 \int_c^x e^{\frac{\alpha(y-\frac{\kappa\theta}{\alpha})^2}{\gamma^2}} dy, & \text{if } \alpha \neq 0, \\ C_3 \left(e^{-\frac{2\kappa\theta c}{\gamma^2}} - e^{-\frac{2\kappa\theta}{\gamma^2}x} \right), & \text{if } \alpha = 0, \end{cases} \end{aligned}$$

¹⁶It is the correlated version of the Stein and Stein (1991) model. In Rheinländer (2005), the minimal entropy martingale measure is studied in detail for this model, and its proposition 3.1 gives a necessary and sufficient condition such that the associated stochastic exponential is a true martingale. Here, we provide deterministic criteria.

TABLE 5.5
First Classification Table for the Schöbel–Zhu Model

Case	$\tilde{s}(\ell)$	$\tilde{s}(r)$	$\tilde{v}(\ell)$	$\tilde{v}(r)$	$\tilde{v}_b(\ell)$	$\tilde{v}_b(r)$
$\alpha \leq 0$	$> -\infty$	$< \infty$	$< \infty$	∞	$< \infty$	∞
$\alpha > 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞

TABLE 5.6
Second Classification Table for the Schöbel–Zhu Model

Case	$s(\ell)$	$s(r)$	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
$\alpha > 0$	$> -\infty$	∞	$< \infty$	∞	$< \infty$	∞

with constants $C_1 = e^{-\kappa(c-\theta)^2/\gamma^2} > 0$, $C_2 = e^{(-\kappa^2\theta^2/\alpha + 2\kappa\theta c - \alpha c^2)/\gamma^2} > 0$ for $\alpha \neq 0$, and the constant $C_3 = e^{2\kappa\theta c/\gamma^2} \frac{\gamma^2}{2\kappa\theta} > 0$ for $\alpha = 0$. As $\kappa > 0$ by assumption, $e^{\frac{\kappa(y-\theta)^2}{\gamma^2}} \geq 1$ for any $y \in [c, x]$, with $c \in J$, $x \in J$, then $s(r) = s(\infty) = \infty$ always holds, and consequently $v(r) = v(\infty) = \infty$.

PROPOSITION 5.8. *For the Schöbel–Zhu model (5.7), the underlying stock price $(S_t)_{0 \leq t < \infty}$ is a true P -martingale.*

Proof. We now check the conditions in Proposition 4.1. For the case of the right endpoint r , depending on the sign of $\alpha = \kappa - \rho\gamma$, we obtain the following classification:

$$\tilde{s}(\infty) \begin{cases} < \infty, & \text{if } \alpha \leq 0, \\ = \infty, & \text{if } \alpha > 0. \end{cases}$$

Details can be found in Supporting Information. Above all, we can summarize the results in Table 5.5. From Proposition 4.1(3), $(S_t)_{0 \leq t < \infty}$ is a true P -martingale. \square

PROPOSITION 5.9. *For the Schöbel–Zhu model (5.7), the underlying stock price $(S_t)_{0 \leq t \leq \infty}$ is an UI P -martingale if and only if $\kappa > \rho\gamma$.*

Proof. From Table 5.5 and Proposition 4.2, it follows that $(S_t)_{0 \leq t \leq \infty}$ is an UI P -martingale if and only if $\alpha > 0$, or equivalently $\kappa > \rho\gamma$. \square

Under P , we obtain the following result on the positivity of the stock price in the Schöbel–Zhu model:

- PROPOSITION 5.10.** *For the Schöbel–Zhu model (5.7),*
- (1) $P(S_T > 0) = 1$ for all $T \in (0, \infty)$,
 - (2) $P(S_\infty > 0) = 1$.

Proof. Similar to the proofs of Propositions 5.8 and 5.9 with α replaced by $\kappa > 0$, we obtain the classification given in Table 5.6. \square

5.4. Hull–White Stochastic Volatility Model

Under P , the correlated Hull–White stochastic volatility model (see Hull and White 1987) can be described by the following diffusive dynamics:

$$(5.8) \quad \begin{aligned} dS_t &= S_t \sqrt{Y_t} 1_{t \in [0, \zeta)} dW_t^{(1)}, \quad S_0 = 1, \\ dY_t &= \mu Y_t 1_{t \in [0, \zeta)} dt + \sigma Y_t 1_{t \in [0, \zeta)} dW_t, \quad Y_0 = x_0 > 0, \end{aligned}$$

where $\mathbb{E}[dW_t^{(1)} dW_t] = \rho dt$, $-1 \leq \rho \leq 1$, $\mu > 0$, and $\sigma > 0$. The process Y is a geometric Brownian motion process, and this implies that its natural state space is $J = (\ell, r) = (0, \infty)$. ζ is the possible exit time of the process Y from its state space J . The model (5.8) belongs to the general stochastic volatility model considered in (2.5) with $\mu(x) = \mu x$, $\sigma(x) = \sigma x$, and $b(x) = \sqrt{x}$. Clearly, $\sigma(x) = \sigma x \neq 0$, $x \in J$, $\frac{1}{\sigma(x)^2} = \frac{1}{\sigma^2 x^2} \in L_{loc}^1(J)$, $\frac{\mu(x)}{\sigma(x)^2} = \frac{\mu}{\sigma^2 x} \in L_{loc}^1(J)$, and $\frac{b^2(x)}{\sigma^2(x)} = \frac{1}{\sigma^2 x} \in L_{loc}^1(J)$ are satisfied. Thus, the conditions (2.2) and (2.6) are satisfied.

From Proposition 2.8, under \tilde{P} , the diffusion Y satisfies the following SDE:

$$(5.9) \quad dY_t = \left(\mu Y_t + \rho \sigma Y_t^{\frac{3}{2}} \right) 1_{t \in [0, \zeta)} dt + \sigma Y_t 1_{t \in [0, \zeta)} d\tilde{W}_t, \quad Y_0 = x_0 > 0.$$

Denote $\alpha = \frac{4\mu}{\sigma^2} - 1$ and $\gamma = \frac{4\rho}{\sigma}$. For a constant $c \in J$, compute the scale functions of the SDE (2.13)

$$(5.10) \quad \tilde{s}(x) = \int_c^x e^{-\int_c^y \frac{2\mu u + 2\rho \sigma u^{3/2}}{\sigma^2 u^2} du} dy = C_1 \int_c^x y^{-\frac{\alpha+1}{2}} e^{-\gamma \sqrt{y}} dy, \quad x \in \bar{J},$$

where $C_1 = c^{\frac{2\mu}{\sigma^2}} e^{\frac{4\rho}{\sigma} \sqrt{c}}$ is a positive constant. From the definition in (3.1) and the scale function in (5.10)

$$(5.11) \quad \tilde{v}(x) = \int_c^x \frac{2(\tilde{s}(x) - \tilde{s}(y))}{\tilde{s}'(y) \tilde{\sigma}^2(y)} dy = \frac{2}{\sigma^2} \int_c^x y^{\frac{\alpha-3}{2}} e^{\gamma \sqrt{y}} \left(\int_y^x z^{-\frac{\alpha+1}{2}} e^{-\gamma \sqrt{z}} dz \right) dy,$$

and

$$(5.12) \quad \tilde{v}_b(x) = \frac{2}{\sigma^2} \int_c^x y^{\frac{\alpha-1}{2}} e^{\gamma \sqrt{y}} \left(\int_y^x z^{-\frac{\alpha+1}{2}} e^{-\gamma \sqrt{z}} dz \right) dy.$$

PROPOSITION 5.11. *For¹⁷ the Hull–White model (5.8), the underlying stock price $(S_t)_{0 \leq t < \infty}$ is a true P -martingale if and only if $\rho \leq 0$.*

Proof. We distinguish three situations: (I): $\mu > \frac{1}{2}\sigma^2$, (II): $\mu = \frac{1}{2}\sigma^2$ and (III): $\mu < \frac{1}{2}\sigma^2$. Results are summarized in Table 5.7. Details can be found in Supporting Information. The results in Table 5.7, combined with Proposition 4.1 allow us to conclude if $(S_t)_{0 \leq t \leq T}$, $T \in (0, \infty)$ is a true P -martingale. For $2\mu/\sigma^2 > 1$ ($\alpha > 1$), $(S_t)_{0 \leq t \leq T}$, $T \in (0, \infty)$ is a true P -martingale if and only if $\tilde{v}(r) = \infty$. This is equivalent to $\gamma \leq 0$, and further equivalent to $\rho \leq 0$ from the definition of γ . When $2\mu/\sigma^2 = 1$ ($\alpha = 1$), $(S_t)_{0 \leq t \leq T}$, $T \in (0, \infty)$ is a true P -martingale if and only if $\tilde{v}(r) = \infty$, equivalently $\gamma \leq 0$,

¹⁷Proposition 5.11 is consistent with theorem 1 of Jourdain (2004), and proposition 2.5, page 34 of Andersen and Piterbarg (2007).

TABLE 5.7
First Classification Table for the Hull–White Model

Case		$\tilde{v}(\ell)$	$\tilde{v}(r)$	$\tilde{v}_b(\ell)$	$\tilde{v}_b(r)$
(I) $\mu > \frac{\sigma^2}{2}$	$\gamma \leq 0$	∞	∞	∞	∞
	$\gamma > 0$	∞	$< \infty$	∞	∞
(II) $\mu = \frac{\sigma^2}{2}$	$\gamma \leq 0$	∞	∞	∞	∞
	$\gamma > 0$	∞	$< \infty$	∞	∞
(III) $\mu < \frac{\sigma^2}{2}$	$\gamma \leq 0$	∞	∞	$< \infty$	∞
	$\gamma > 0$	∞	$< \infty$	$< \infty$	∞

TABLE 5.8
Second Classification Table for the Hull–White Model

Case		$\tilde{s}(\ell)$	$\tilde{s}(r)$	$\tilde{v}(\ell)$	$\tilde{v}(r)$	$\tilde{v}_b(\ell)$	$\tilde{v}_b(r)$
(I) $\mu > \frac{\sigma^2}{2}$	$\gamma > 0$	$-\infty$	$< \infty$	∞	$< \infty$	∞	∞
	$\gamma = 0$	$-\infty$	$< \infty$	∞	∞	∞	∞
	$\gamma < 0$	$-\infty$	∞	∞	∞	∞	∞
(II) $\mu = \frac{\sigma^2}{2}$	$\gamma > 0$	$-\infty$	$< \infty$	∞	$< \infty$	∞	∞
	$\gamma = 0$	$-\infty$	∞	∞	∞	∞	∞
	$\gamma < 0$	$-\infty$	∞	∞	∞	∞	∞
(III) $\mu < \frac{\sigma^2}{2}$	$\gamma > 0$	$> -\infty$	$< \infty$	∞	$< \infty$	$< \infty$	∞
	$\gamma = 0$	$> -\infty$	∞	∞	∞	$< \infty$	∞
	$\gamma < 0$	$> -\infty$	∞	∞	∞	$< \infty$	∞

TABLE 5.9
Third Classification Table for the Hull–White Model

Case	$s(\ell)$	$s(r)$	$v(\ell)$	$v(r)$	$v_b(\ell)$	$v_b(r)$
(I) $\mu > \frac{\sigma^2}{2}$	$-\infty$	$< \infty$	∞	∞	∞	∞
(II) $\mu = \frac{\sigma^2}{2}$	$-\infty$	∞	∞	∞	∞	∞
(III) $\mu < \frac{\sigma^2}{2}$	$> -\infty$	∞	∞	∞	$< \infty$	∞

that is $\rho \leq 0$. When $2\mu/\sigma^2 < 1$ ($\alpha < 1$), $(S_t)_{0 \leq t \leq T}$, $T \in (0, \infty)$ is a true P -martingale if and only if $\tilde{v}(r) = \infty$, equivalently $\gamma \leq 0$, that is $\rho \leq 0$. \square

PROPOSITION 5.12. *For the Hull–White model (5.8), the underlying stock price $(S_t)_{0 \leq t \leq \infty}$ is an UI P -martingale if and only if $\mu < \frac{1}{2}\sigma^2$ and $\rho \leq 0$.*

TABLE 5.10
Summary of Conditions for (Uniformly Integrable) Martingales

Model	True martingale on $(0, \infty)$	UI martingale on $[0, \infty]$
Heston model (5.1)	Under Feller condition $\kappa > \frac{\xi^2}{2\theta}$	Never under Feller condition $\rho\xi \leq \kappa < \frac{\xi^2}{2\theta}$
3/2 model (5.3)	$\xi^2 + 2\theta \geq \max(0, 2\rho\xi)$	Never
Schöbel–Zhu model (5.7)	Always	When $\kappa > \rho\gamma$
Hull–White model (5.8)	$\rho \leq 0$	$\mu < \frac{\sigma^2}{2}$ and $\rho \leq 0$

TABLE 5.11
Summary of Conditions for Positivity of Stock Prices

Model	S_T positive P -a.s.	S_∞ positive P -a.s.
Heston model (5.1)	Always	Never under Feller condition $\kappa < \frac{\xi^2}{2\theta}$
3/2 model (5.3)	$\xi^2 + 2\theta \geq 0$	Never
Schöbel–Zhu model (5.7)	Always	Always
Hull–White model (5.8)	Always	$\mu < \frac{\sigma^2}{2}$

Proof. The proof of Proposition 5.12 requires the same three cases as Proposition 5.11. Results are summarized in Table 5.8. Details can be found in Supporting Information. \square

Under P , we have the following result on the positivity of the stock price in the Hull–White model:

PROPOSITION 5.13. *For the Hull–White model (5.8),*
(1) $P(S_T > 0) = 1$ for all $T \in (0, \infty)$, (2) $P(S_\infty > 0) = 1$ if and only if $\frac{2\mu}{\sigma^2} < 1$.

Proof. Similar to the proofs of Propositions 5.11 and 5.12 with $\gamma = 0$, we have the classification in Table 5.9.

From Table 5.9, Propositions 4.3 and 4.4, we obtain the desired results. \square

5.5. Summary of the Examples

Tables 5.10 and 5.11 summarize the results obtained throughout Section 5. In all cases, we study the “stopped” price process as we assume that there are two absorbing barriers at ℓ and r . Conditions for UI martingales are stronger than those for a true martingale on $(0, \infty)$. Similar remarks hold for the positivity of S_T and S_∞ , where $0 < T < \infty$.

6. CONCLUDING REMARKS

This paper generalizes some results of Mijatović and Urusov (2012b, c) concerning the (UI) martingale property of the asset price from the case $\rho = 1$ to the case $-1 \leq \rho \leq 1$, and provides new direct proofs without using the concept of “separating times.” We also obtain deterministic criteria for the convergence or divergence of both perpetual and capped integral functionals of time-homogeneous diffusions. Explicit deterministic criteria for checking the (UI) martingale properties for four stochastic volatility models are provided. Future research directions include finding necessary and sufficient deterministic conditions for the martingale property of time-changed Lévy processes with nonzero correlation (Carr and Wu 2004), of which the time-homogeneous stochastic volatility models considered in this paper are special cases.

APPENDIX: TECHNICAL CONDITIONS ON THE PROBABILITY SPACE AND FILTRATION

Throughout the paper, we assume a space accommodating all four processes $(Y, Z, W, W^{(1)})$ in (2.5). This is described below, following closely the presentation in appendix B of Carr et al. (2014). For a fixed time horizon $T \in (0, \infty]$, we require a stochastic basis $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. As in Carr et al. (2014) and Föllmer (1972), page 156, for any stopping time τ , we define $\mathcal{F}_\tau := \{A \in \mathcal{F}_T \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in [0, T]\}$ and $\mathcal{F}_{\tau-} := \sigma(\{A \cap \{\tau > t\} \in \mathcal{F}_T \mid A \in \mathcal{F}_t \text{ for some } t \in [0, T] \cup \mathcal{F}_0\})$. In general, nonnegative random variables are permitted to take values in the set $[0, \infty]$ and stopping times τ are permitted to take values in the set $[0, \infty] \cup T$ for some transfinite time $T > T$. In special cases, we may restrict the range of stopping times.

Let Ω_1 denote the space of continuous paths $\omega_1 : [0, \infty) \rightarrow \bar{J}$ with $\omega_1(0) \in J$. Define $\zeta(\omega_1) := \inf\{t \in [0, T] : \omega_1(t) \notin J\}$ with the convention $\inf \emptyset = T$. Assume that ω_1 stays at either ℓ or r once it hits it, i.e., that $\omega_1(\zeta + s) = \omega_1(\zeta)$ for all $s > 0$ on the set $\{\zeta < \infty\}$. Let Ω_2 denote the space of continuous¹⁸ paths $\omega_2 : [0, \infty) \rightarrow [0, \infty]$ with $\omega_2(0) = 1$. As in Carr et al. (2014), define for all $i \in N$, $R_i := \inf\{t \in [0, T] : \omega_2(t) > i\}$, and $S_i := \inf\{t \in [0, T] : \omega_2(t) < \frac{1}{i}\}$. Then, $R := \lim_{i \uparrow \infty} R_i$ and $S := \lim_{i \uparrow \infty} S_i$ are, respectively, the first hitting time of infinity and zero by ω_2 , with the convention $\inf \emptyset = T$. Assume that $\omega_2(R + s) = \omega_2(R)$ for all $s > 0$ on $\{R < \infty\}$ and similarly $\omega_2(S + s) = \omega_2(S)$ for all $s > 0$ on $\{S < \infty\}$, so that ω_2 stays at zero or infinity once it hits it. Let Ω_3 denote the space of continuous paths $\omega_3 : [0, \infty) \rightarrow R$ with $\omega_3(0) = 0$. Similarly, let Ω_4 denote the space of continuous paths $\omega_4 : [0, \infty) \rightarrow R$ with $\omega_4(0) = 0$.

Denote $\Omega = \prod_{i=1}^4 \Omega_i$ and $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$. As in appendix B of Carr et al. (2014), we require that $\{\mathcal{F}_{R_i-}\}_{i \in N}$ is a standard system, see remark 6.1.1 of Föllmer (1972), so that in the proof of Proposition 2.6, the extension theorem V4.1 of Parthasarathy (1967) can be applied, and any probability measure on \mathcal{F}_{R-} has a (possibly nonunique) extension to a probability measure on \mathcal{F}_T . Such a canonical filtration can be constructed as in appendix B of Carr et al. (2014).

Given the canonical space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]})$, the processes $(Y, Z, W, W^{(1)})$ in (2.5) correspond, respectively, to the four components of ω and are formally functions of ω .

¹⁸Continuity where the function takes the value ∞ is defined as usual through a compactification: if $\lim_{t \rightarrow t_0} \omega_2(t) = \infty$, then ω_2 is continuous at t_0 .

We assume that processes Y, Z are adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, as are $W, W^{(1)}$, which are assumed to be Brownian motions with respect to the same filtration.

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