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PRICE-ADMISSIBILITY CONDITIONS FOR ARBITRAGE-FREE LINEAR PRICE FUNCTION MODELS FOR THE TERM STRUCTURE OF INTEREST RATES

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To assure price admissibility—that all bond prices, yields, and forward rates remain positive—we show how to control the state variables within the class of arbitrage-free linear price function models for the evolution of interest rate yield curves over time. Price admissibility is necessary to preclude cash-and-carry arbitrage, a market imperfection that can happen even with a risk-neutral diffusion process and positive bond prices. We assure price admissibility by (i) defining the state variables to be scaled partial sums of weighted coefficients of the exponential terms in the bond pricing function, (ii) identifying a simplex within which these state variables remain price admissible, and (iii) choosing a general functional form for the diffusion that selectively diminishes near the simplex boundary. By assuring that prices, yields, and forward rates remain positive with tractable diffusions for the physical and risk-neutral measures, an obstacle is removed from the wider acceptance of interest rate methods that are linear in prices.

KEY WORDS: term structure, interest rates, yields, bonds, linear price model.

1. INTRODUCTION

Considerable attention has been given to modeling the behavior of yield curves over time—the term structure of interest rates—due to their fundamental role in determining prices of both riskless and risky assets and their association with the larger economy. The class of linear price models represents the price of a zero-coupon bond as a linear combination of decaying exponential functions of the bond's maturity (where the faster decaying terms may be viewed as short-to-moderate-term perturbations that can create a variety of yield curve shapes) and the linear coefficients evolve over time according to a risk-neutral measure in a manner that avoids the possibility of arbitrage, a requirement of efficient markets.

The linear price model first appeared with state variables representing investors' posterior beliefs in either the inflation regime by Yared (1999a, 1999b) or in the consumption regime by Veronesi and Yared (1999) and with a one- or two-dimensional diffusion representing unexpected shocks to the state variables; bond prices are sums of linear combinations of state variables multiplying decaying exponential functions of maturity; in particular Yared noted the variety of yield curve shapes (including the hump shape) available with these models that admit a risk-neutral measure. Fundamental extensions

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are presented in Gabaix's (2009) comprehensive work on linearity-generating processes, which includes linear bond price models with fully dimensional volatility, in Cheridito and Gabaix (2008) who propose methods for diminishing the variability of the diffusion near a boundary for the purpose of maintaining positive prices, and in Carr, Gabaix, and Wu (2011) who propose a family of stochastic volatility models that involve nonlinear transformations of the state variables (so that bond prices are a linear combination of exponential functions of maturity, but whose coefficients depend nonlinearly on the state variables). Related asset pricing models presented in Menzly, Santos, and Veronesi (2004) and in Santos and Veronesi (2006) involve state variables restricted to a simplex that represents nonnegative shares of consumption that sum to 1, and a note by Santos and Veronesi (2005) shows how to model the diffusion in such a situation. One common feature of all linear price models is the drift of the risk-neutral dynamics under $\mathbb{Q}$ of the short rate and of the coefficients that multiply the price basis functions (the decaying exponential functions of time to maturity), which are constrained by the nature of risk-neutral price evolution as shown in Remark 3.13; features that distinguish linear price models from one another (features that are much less constrained) involve the detailed choice of diffusion model (which can serve the dual purposes of controlling stability of the model and introducing stochastic volatility) and the market price of risk specification as expressed by the physical dynamics under $\mathbb{P}$.

Arbitrage-free models for the term structure have been available since Vasicek (1977), Cox, Ingersoll, and Ross (1985), and Hull and White (1990). These are examples of the class of affine models that express the yield to maturity (the interest rate) as a linear combination, in terms of time-varying weights applied to a fixed set of functions of the term to maturity. The affine class has been extensively studied and is widely used, e.g., Duffie and Kan (1996); Dai and Singleton (2000). In contrast, with a linear price model, it is the price (not the yield) that is a linear combination of the state variables. While the yield and the price of a bond contain the same information, a linear model for yields is not a linear model in prices, due to the nonlinear transformation (log or exponential) from one to the other, although both classes of models satisfy the no-arbitrage property of admitting a risk-neutral measure. Additional work on positive interest rates includes Flesaker and Hughston (1996) and Rogers (1997).

Linear price models have an advantage in flexibility of the volatility specification, allowing diffusion to be specified separately from drift, a property not generally shared by affine yield models (for which Itô's Lemma will ensure that a change in the diffusion specification will necessitate a change in the drift, due to the nonlinear relationship between prices and yields). With linear price models, the specification of the diffusion of bond prices can be changed without a nonlinear Itô's Lemma term appearing in the drift (because the no-arbitrage condition involves a restriction on prices, which are modeled directly). In particular, this property of linear price models shows that there is no intrinsic connection between drift and diffusion in the term structure. Another advantage of linear price models is nonlinearity in the drift of the short-term rate, for which empirical evidence was found by Aït-Sahalia (1996) and Stanton (1997).

Distinct models (neither affine linear yield models nor linear price models) with quadratic terms have been introduced by Ahn, Dittmar, and Gallant (2002), who define the short-term interest rate to be a quadratic function of the state variables, which have linear drift. Their general quadratic model includes that of Constantinides (1992), Beaglehole and Tenney's (1992) modification of Longstaff's (1989) nonlinear model, as well as a special case of Cox et al. (1985). The model of Leippold and Wu (2002) is also of this type, while Ahn and Gao (1999) assume a drift for the short-term rate that is a quadratic function of itself. Linear price function models are not nested within these classes because, while the linear price function model's drift of the short-term rate is a quadratic function of the state variables (but not a function of just the short rate) the short-term rate is itself a linear function of the state variables and the drift of the state variables is quadratic.

To be fully arbitrage free, it is not enough merely for a term-structure model to admit a risk-neutral measure with positive bond prices. In addition, yields and forward rates must be positive for all bonds in order to avoid cash-and-carry arbitrage. When its yield is negative, a bond costs more than it will pay; the arbitrage opportunity is to short-sell the bond for cash, carrying enough to meet its payoff obligation at maturity while keeping the difference. When a forward rate is negative, there is a similar opportunity involving simultaneously buying and selling bonds of different maturities and carrying cash between their maturity dates.

The main contribution of this paper is a novel tractable closed-form specification of the state variables and their dynamics that guarantees price admissibility of the yield curves, where price admissibility is defined as positive bond prices, positive yields, and positive forward rates. The state variables are scaled partial sums of weighted coefficients of the exponential terms in the bond pricing function—a nontrivial linear transformation of the coefficients as usually specified for linear price models. Within this new representation, we show that the canonical open simplex (positive state variables whose sum is less than one) is a price-admissible region for the model. To maintain price admissibility as the process evolves over time, we choose a tractable diffusion specification that selectively diminishes the variability as the state variables approach a boundary of the simplex, while maintaining variability in directions that would not leave the simplex.

The paper is organized as follows: Section 2 introduces the proposed model by specifying the state variables and their dynamics, along with formulas for the short-term rate and for bond prices. Results are presented in Section 3, including that there exists a unique strong solution that perpetuates the Simplex Condition throughout the future; that the bond price formula satisfies the no-arbitrage requirement with the Proposed Dynamics defining a full-rank-diffusion risk-neutral measure; that the Simplex Condition implies that the yield curve is price admissible in the sense that bond prices are positive, that bond yields are positive (including the short rate r) and forward rates are positive for all maturities; and that a family of equivalent physical measures may be specified by introducing a market-price-of-risk vector that ensures price-admissible yield curves. Section 4 includes discussion and summary, while detailed proofs are in the Appendix.

2. SPECIFYING AND EXPLORING THE PROPOSED MODEL

This section specifies and explores the model while postponing important technical results (such as existence of solutions) to Section 3. We are given $n+1$-ordered positive constants $0 < \rho_0 < \rho_1 < \cdots < \rho_n$, let $a_1, a_2, \ldots, a_n$ denote the n state variables, and define the short-term interest rate as an affine linear sum of the state variables

$$r \equiv \rho_n - \rho_n \sum_{k=1}^{n} a_k. \tag{2.1}$$

DEFINITION 2.1. The Simplex Condition is satisfied by variables $a_1, a_2, \ldots, a_n$ if they are positive with sum less than 1:

$$a_1 > 0,\ a_2 > 0, \cdots, a_n > 0, \text{ and } \sum_{k=1}^{n} a_k < 1. \tag{2.2}$$

The Simplex Condition (2.2) ensures that the short rate r is positive, and also imposes a maximum value such that $r \in (0, \rho_n)$. We posit the following risk-neutral dynamics for the state variables for all $k = 1, \ldots, n$:

$$da_k = \left[a_k (r - \rho_{k-1}) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} a_i \right] dt + a_k \left(B_k - \sum_{i=1}^{n} a_i B_i \right) dZ, \tag{2.3}$$

where $\boldsymbol{B}$ is an $n \times n$ matrix (whose purpose is to specify correlation and variability of the state variables) whose kth row is B_k, and Z is standard n-dimensional Brownian motion with its usual filtration. For now, we assume that $\boldsymbol{B}$ is a fixed matrix; extension to any bounded measurable Lipschitz matrix with additional state variables is straightforward (for the purpose of introducing stochastic volatility and/or correlation) as is extension to any measurable matrix for which solutions and expectations exist. We adopt the convention in (2.3) that a sum whose index extends from 1 to 0 is identically zero, so that the dynamics for a_1 are $da_1 = a_1(r - \rho_0)dt + a_1(B_1 dZ - \sum_{i=1}^{n} a_i B_i dZ)$. Dynamics from (2.3) may be stacked into a single vector equation as follows:

DEFINITION 2.2. The Proposed Dynamics for the state variables are given by

$$\begin{aligned}
\begin{pmatrix} da_1 \\ da_2 \\ da_3 \\ \vdots \\ da_n \end{pmatrix} &= \left\{ \begin{pmatrix} a_1 (r - \rho_0) \\ a_2 (r - \rho_1) \\ a_3 (r - \rho_2) \\ \vdots \\ a_n (r - \rho_{n-1}) \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} 0 \\ (\rho_2 - \rho_1)\, \rho_0 \rho_1 a_1 / (\rho_1 \rho_2) \\ (\rho_3 - \rho_2)(\rho_0 \rho_1 a_1 + \rho_1 \rho_2 a_2) / (\rho_2 \rho_3) \\ \vdots \\ (\rho_n - \rho_{n-1}) \sum_{i=1}^{n-1} \rho_i \rho_{i-1} a_i / (\rho_{n-1} \rho_n) \end{pmatrix} \right\} dt + \left(\boldsymbol{D} - AA' \right) \boldsymbol{B}\, dZ
\end{aligned} \tag{2.4}$$

where $\boldsymbol{D} \equiv diag(a_1, a_2, \ldots, a_n)$ denotes the diagonal $n \times n$ matrix formed from the state variables, and $A \equiv (a_1, a_2, \ldots, a_n)'$ denotes the column of state variables. With this notation, (2.3) may equivalently be written for all $k = 1, \ldots, n$ as

$$da_k = \left[a_k (r - \rho_{k-1}) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} a_i \right] dt + a_k (B_k - A' \boldsymbol{B}) dZ. \tag{2.5}$$

To ensure uniqueness of the diffusion specification, we may require that the matrix $\boldsymbol{B}$ be upper (or lower) triangular with nonnegative diagonal elements because the instantaneous variance of the state variables $(\boldsymbol{D} - AA')\boldsymbol{B}\boldsymbol{B}'(\boldsymbol{D} - AA')dt = (\boldsymbol{D} - AA')(\boldsymbol{B}\boldsymbol{U})(\boldsymbol{B}\boldsymbol{U})'(\boldsymbol{D} - AA')dt$ remains unchanged with any orthogonal matrix $\boldsymbol{U}$. We will show in Lemma 3.4 that the dynamics of the short rate are

$$dr = \left[r\,(r - \rho_n) + \sum_{k=1}^{n} \rho_k \rho_{k-1} a_k \right] dt - r\,A'\boldsymbol{B}dZ. \tag{2.6}$$

We define the price of a zero-coupon bond at time t that matures, paying \$1 at a later time T, to be an affine linear combination using the state variables

$$P_{t,T} \equiv e^{-\rho_n(T-t)} + \sum_{k=1}^{n} a_k \left[\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1} e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}} - e^{-\rho_n(T-t)} \right]. \tag{2.7}$$

We note that the maturing bond has price $P_{t,t} = 1$, that we may alternatively represent $P_{t,T}$ using the short rate as

$$P_{t,T} = \frac{r}{\rho_n} e^{-\rho_n(T-t)} + \sum_{k=1}^{n} a_k \frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1} e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}} \tag{2.8}$$

and also note that the representation

$$P_{t,T} = e^{-\rho_n(T-t)} + \sum_{k=1}^{n} a_k g_k\,(T - t) \tag{2.9}$$

shows that prices are an affine linear combination of the basis functions

$$\begin{aligned} g_k\,(m) &\equiv \frac{\rho_k e^{-\rho_{k-1} m} - \rho_{k-1} e^{-\rho_k m}}{\rho_k - \rho_{k-1}} - e^{-\rho_n m} \\ &= \frac{\rho_k \left(e^{-\rho_{k-1} m} - e^{-\rho_n m} \right) - \rho_{k-1} \left(e^{-\rho_k m} - e^{-\rho_n m} \right)}{\rho_k - \rho_{k-1}}. \end{aligned} \tag{2.10}$$

Interest rates (yields) implied by these prices are given by

$$Y_{t,T} = -\frac{\ln P_{t,T}}{T - t} \tag{2.11}$$

because the price represents exponential discounting $P_{t,T} = e^{-(T-t)Y_{t,T}}$ at this yield over this time period.

REMARK 2.3. Note that there is parameter separation in the dynamics of the state variables with this specification: the matrix $\boldsymbol{B}$ appears only in the diffusion function and not in the drift, while the constants $\rho_0, \ldots, \rho_n$ appear only in the drift and not in the diffusion. In addition, ρ_n is the only parameter that appears in r itself. For the dynamics of r, the constants $\rho_0, \ldots, \rho_{n-1}$ appear only in the drift (and not the diffusion), $\boldsymbol{B}$ appears only in the diffusion (not the drift), while ρ_n is represented in both drift and diffusion (indirectly, through r itself). All $\rho_0, \ldots, \rho_n$ (but not $\boldsymbol{B}$) appear in bond prices themselves, and we will see in Theorem 3.8 that $\boldsymbol{B}$ is absent from the drift of bond prices.

To see that the state variables are scaled partial sums of coefficients of the individual exponential terms in the bond pricing function, note that (2.7) implies that bond prices

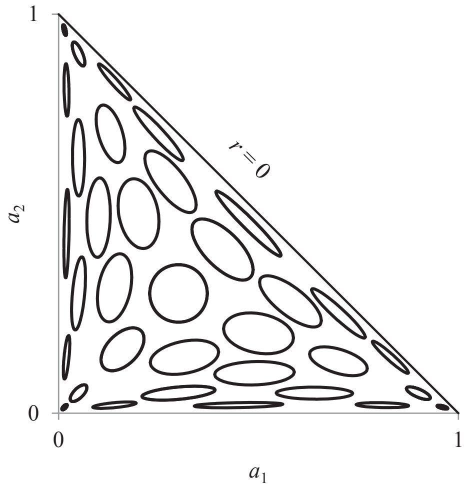

FIGURE 2.1. Ellipses of constant instantaneous probability density, each enclosing equal probability, centered at various parts of the state space show that the volatility selectively decreases as the boundary of the simplex is approached, while allowing full variability otherwise. For illustration, this example uses $n = 2$ where the simplex defined by (2.2) is represented by the triangle, and uses an approximation by scaling the instantaneous volatility. The ellipses become appropriately flatter near the triangle boundaries, adopting positive or negative correlation as needed. The matrix $\boldsymbol{B}$ has entries $b_{11} = 1$, $b_{12} = 1.05$, $b_{21} = 0$, and $b_{22} = 1.45$ chosen to obtain a circle near the central region. Other choices for $\boldsymbol{B}$ can produce either positive or negative correlation in the central region while maintaining low variability near the boundaries. The volatility term $(\boldsymbol{D} - AA')\,\boldsymbol{B}$ in (2.4) implies an instantaneous covariance matrix for (da_1, da_2) of $(\boldsymbol{D} - AA')\,\boldsymbol{BB}'\,(\boldsymbol{D} - AA')$; each contour in the figure shows the ellipse of points (x_1, x_2) for which the resulting Mahalanobis Distance $(x_1 - a_1, x_2 - a_2)\left[(\boldsymbol{D} - AA')\,\boldsymbol{BB}'\,(\boldsymbol{D} - AA')\right]^{-1}(x_1 - a_1, x_2 - a_2)' = 0.1$ is constant. Hence the bivariate probability density is constant on each ellipse, as is the probability enclosed by each ellipse, where these probabilities are bivariate Gaussian approximations that match this instantaneous covariance structure for various states (a_1, a_2) in the simplex.

may be expressed directly as a weighted sum of distinct exponential terms with weights $w_0, w_1, \ldots, w_n$ in the form

$$P_{t,T} = \sum_{k=0}^{n} w_k e^{-\rho_k(T-t)}, \tag{2.12}$$

where the weights are $w_0 = \frac{1}{\rho_0}(\frac{\rho_1\rho_0}{\rho_1-\rho_0}a_1)$, $w_k = \frac{1}{\rho_k}(\frac{\rho_{k+1}\rho_k}{\rho_{k+1}-\rho_k}a_{k+1} - \frac{\rho_k\rho_{k-1}}{\rho_k-\rho_{k-1}}a_k)$ for $k = 1, \ldots, n-1$, and $w_n = 1 - \sum_{k=1}^{n} a_k - \frac{\rho_{n-1}}{\rho_n-\rho_{n-1}}a_n$, and it may be verified by direct calculation that $\sum_{k=0}^{n} w_k = 1$ as required by the maturing bond condition. That the state

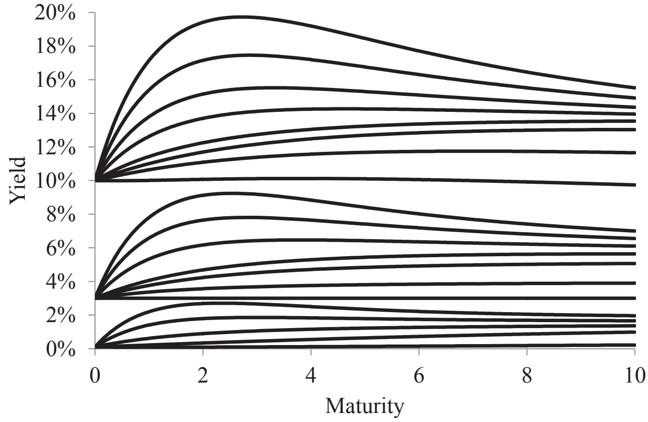

FIGURE 2.2. A sample of yield curves that are either flat or end above where they began, possibly with a hump in between, to show the diversity available with state variables in the simplex within the linear price function family. These curves use $n = 4$ with $\rho_0 = 0.01$, $\rho_1 = 0.05$, $\rho_2 = 0.15$, $\rho_3 = 0.90$, and $\rho_4 = 1.00$.

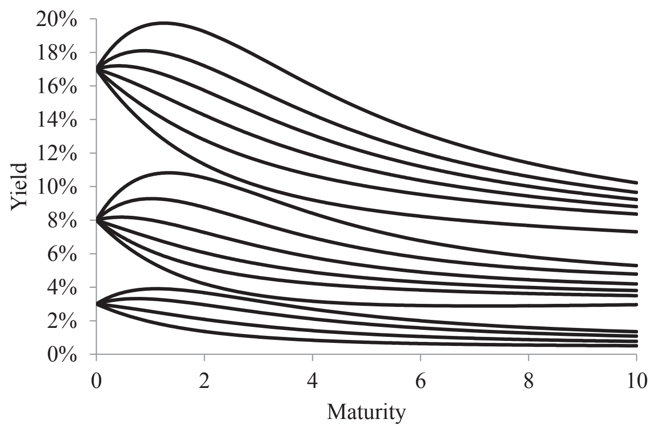

FIGURE 2.3. A sample of yield curves that end below where they began, possibly with a hump in between, to show the diversity available with state variables in the simplex within the linear price function family. These curves also use $n = 4$ with $\rho_0 = 0.01$, $\rho_1 = 0.05$, $\rho_2 = 0.15$, $\rho_3 = 0.90$, and $\rho_4 = 1.00$.

variables $a_1, \ldots, a_n$ are scaled partial sums of these weighted coefficients (the w_k, weighted by the ρ_k) may be seen from the representation

$$a_k = \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=0}^{k-1} \rho_i w_i . \tag{2.13}$$

The instantaneous volatility component $(\boldsymbol{D} - AA')\boldsymbol{B}dZ$ selectively decreases as the boundary of the simplex is approached, without compromising the ability of the model to explore the remaining parts of the space, as visualized in Figure 2.1, which provides an example with $n = 2$ where the simplex defined by the Simplex Condition (2.2) (that $a_1 > 0$,

$a_2 > 0$, and $a_1 + a_2 < 1$) is represented by the triangle. In particular, for small values of the short rate (near the diagonal) the ellipses of constant probability density flatten, keeping the short rate positive while permitting substantial variation in other aspects of the yield curve. More generally, the ellipses become appropriately flatter near the three triangular boundaries, adopting positive or negative correlation as needed (positive correlation for smaller nearly equal values near the lower left, negative correlation near the short-rate constraint along the diagonal). In this way, the state variables have low variability with respect to the constraint(s) while maintaining full variability otherwise. In this particular case, the fixed matrix $\boldsymbol{B}$ was chosen so that variances are equal with correlation zero near the central region, represented by the circle. Other choices for $\boldsymbol{B}$ can produce positive or negative correlation in the center, and would otherwise adjust near the boundaries to enforce the Simplex Constraints.

To demonstrate the flexibility of yield curve shapes available within the simplex using a fixed set of constants while changing the state variables, Figure 2.2 shows a sample of yield curves that are either flat or end above where they began, possibly with a hump in between. Figure 2.3 shows a sample of yield curves that end below where they began, possibly with a hump in between.

REMARK 2.4. It is possible to use $(a_1, \ldots, a_{n-1}, r)$ as the state variables, instead of $(a_1, \ldots, a_n)$. To do this, we would use $A'^{*} = (a_1, \ldots, a_{n-1}, 1 - \sum_{k=1}^{n-1} a_k - \frac{r}{\rho_n})$ instead of $A' = (a_1, \ldots, a_n)$ for the diffusion terms of $a_1, \ldots, a_{n-1}$ in (2.4), and similarly for $\boldsymbol{D}$, because $a_n = 1 - \sum_{k=1}^{n-1} a_k - \frac{r}{\rho_n}$ from (2.1). The dynamics of r would then be written $dr = [\rho_n \rho_{n-1} + r(r - \rho_n - \rho_{n-1}) + \sum_{k=1}^{n-1} (\rho_k \rho_{k-1} - \rho_n \rho_{n-1}) a_k] dt - r\, A'^{*} \boldsymbol{B} dZ$. The Simplex Condition would then need to also include the requirement that $0 < r < \rho_n - \rho_n \sum_{k=1}^{n-1} a_k$ in addition to $0 < a_1, \ldots, a_{n-1} < 1$.

3. RESULTS

Important features of this model (including existence of solutions to the Proposed Dynamics) are verified in the theorems of this section: that the stochastic differential equation (SDE) has a unique strong solution that maintains the Simplex Condition (thereby guaranteeing positive prices, yields, and forward rates throughout the future), that the model satisfies the arbitrage-free requirement of a risk-neutral measure, that the Simplex Condition (2.2) implies price admissibility of the yield curve implied by bond prices (2.7), that the dynamics (2.6) as reported earlier for r are correct, and that the diffusion for the state variables in (2.4) has full rank. Proofs are in the Appendix; we assume throughout that constants satisfying $0 < \rho_0 < \rho_1 < \ldots < \rho_n$ are given, along with a fixed $n \times n$ matrix $\boldsymbol{B}$.

To establish that there exists a unique strong solution to the Proposed Dynamics (2.4) for the state variables, we begin with a Restricted SDE (for the purpose of imposing boundedness of drift and diffusion) for which four lemmas establish existence and properties of a unique strong solution. These lemmas then lead to the first theorem, which establishes existence and properties of a unique strong solution to the (original, unrestricted) Proposed Dynamics (2.4) when initialized within the Simplex because the solution to the Restricted SDE is also a solution to the Proposed Dynamics (2.4) for the state variables.

DEFINITION 3.1. The Restricted SDE, using the bounded function $\varphi(a_k) \equiv \max[0, \min(a_k, 1)] \in [0, 1]$, is defined as

$$\begin{aligned} da_k &= \left[\varphi(a_k)\left(r_\varphi - \rho_{k-1}\right) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} \varphi(a_i)\right] dt \\ &\quad + \varphi(a_k)\left[B_k - \sum_{i=1}^{n} \varphi(a_i)\, B_i\right] dZ \\ &= \left[\varphi(a_k)\left(r_\varphi - \rho_{k-1}\right) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} \varphi(a_i)\right] dt \\ &\quad + \varphi(a_k)\left(B_k - A_\varphi' \boldsymbol{B}\right) dZ \end{aligned} \tag{3.1}$$

for $k = 1, \ldots, n$, where we also define the restricted short rate $r_\varphi \equiv \rho_n - \rho_n \sum_{k=1}^{n} \varphi(a_k)$, the column of restricted variables $A_\varphi \equiv [\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)]'$, and the diagonal matrix of restricted variables $\boldsymbol{D}_\varphi \equiv diag[\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)]$; we may write the Restricted SDE (3.1) as

$$\begin{aligned} \begin{pmatrix} da_1 \\ da_2 \\ da_3 \\ \vdots \\ da_n \end{pmatrix} &= \left\{ \begin{pmatrix} \varphi(a_1)\left(r_\varphi - \rho_0\right) \\ \varphi(a_2)\left(r_\varphi - \rho_1\right) \\ \varphi(a_3)\left(r_\varphi - \rho_2\right) \\ \vdots \\ \varphi(a_n)\left(r_\varphi - \rho_{n-1}\right) \end{pmatrix} \right. \\ &\quad + \left. \begin{pmatrix} 0 \\ (\rho_2 - \rho_1)\, \rho_0 \rho_1 \varphi(a_1) / (\rho_1 \rho_2) \\ (\rho_3 - \rho_2)\left[\rho_0 \rho_1 \varphi(a_1) + \rho_1 \rho_2 \varphi(a_2)\right] / (\rho_2 \rho_3) \\ \vdots \\ (\rho_n - \rho_{n-1}) \sum_{i=1}^{n-1} \rho_i \rho_{i-1} \varphi(a_i) / (\rho_{n-1} \rho_n) \end{pmatrix} \right\} dt + \left(\boldsymbol{D}_\varphi - A_\varphi A_\varphi'\right) \boldsymbol{B} dZ. \end{aligned} \tag{3.2}$$

LEMMA 3.2. *There is a unique strong solution to the Restricted SDE (3.2) for any given initial value* $(a_1, \ldots, a_n)$.

LEMMA 3.3. *If* $(a_1, \ldots, a_n)$ *initially satisfies the Simplex Condition (2.2), then these variables remain positive as they evolve under the Restricted SDE (3.2).*

LEMMA 3.4. *The dynamics of* r *(defined as* $\rho_n - \rho_n \sum_{k=1}^{n} a_k$*) implied by the Proposed Dynamics (2.4), are given by*

$$dr = \left[r\left(r - \rho_n\right) + \sum_{k=1}^{n} \rho_k \rho_{k-1} a_k\right] dt - r \sum_{k=1}^{n} a_k B_k dZ$$

$$= \left[r\,(r - \rho_n) + \sum_{k=1}^{n} \rho_k \rho_{k-1} a_k \right] dt - r\,A' \boldsymbol{B} dZ. \tag{3.3}$$

LEMMA 3.5. *If* $(a_1, \ldots, a_n)$ *initially satisfies the Simplex Condition (2.2), then these variables continue to satisfy the Simplex Condition as they evolve under the Restricted SDE (3.2).*

THEOREM 3.6. *If the state variables* $(a_1, \ldots, a_n)$ *initially satisfy the Simplex Condition (2.2), then there exists a unique strong solution under the (unrestricted) Proposed Dynamics (2.4) for which the Simplex Condition continues to hold throughout the future and* $r = \rho_n - \rho_n \sum_{k=1}^{n} a_k$ *remains positive.*

REMARK 3.7. Intuition for positivity of the state variables and r comes from inspecting the dynamics for each of $a_1, \ldots, a_n$ from (2.5) and r from (3.3) which guarantee that if one of these variables comes close to zero, its diffusion tends to zero quickly enough (thus preventing negativity due to randomness) while its drift is positive (thus pushing the variable away from zero). The detailed proof is presented in the Appendix.

THEOREM 3.8. *Let* A *be as specified in Theorem 3.6, where the Brownian motion* Z *is defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$. *Then the dynamics of bond prices, as defined by (2.7), are given by*

$$dP_{t,T} = r\,P_{t,T} dt - \left(P_{t,T} A' \boldsymbol{B} - \sum_{k=1}^{n} \frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1} e^{-\rho_k (T-t)}}{\rho_k - \rho_{k-1}} a_k B_k \right) dZ \tag{3.4}$$

and $e^{-\int_0^t r_s ds} P_{t,T}$ *is a martingale. Therefore, the probability measure* $\mathbb{Q}$ *is a risk-neutral measure for the model and there is no arbitrage in the bond market.*

The next theorem establishes that the Simplex Condition for state variables is sufficient for price admissibility of the yield curve (implying positive bond prices, yields, and forward rates) and is proven by establishing that the bond price function is strictly monotonically decreasing from 1 to 0 as maturity increases from 0 to ∞ for state variables within the simplex.

THEOREM 3.9. *If the Simplex Condition (2.2) holds, so that the state variables* $a_1, \ldots, a_n$ *are all positive and* $\sum_{k=1}^{n} a_k < 1$, *then the yield curve implied by the bond prices of (2.7) is price admissible: bond prices are positive, bond yields are positive (including the short rate* r*) and forward rates are positive for all maturities.*

It is important that the volatility component $(\boldsymbol{D} - AA')\,\boldsymbol{B}$ of the Proposed Dynamics (2.4) of the state variables permit full exploration of the space of yield curve shapes in the interior of the simplex, despite its important property of approaching singularity near the boundary in order to prevent escape from the simplex. This is verified in the next theorem.

THEOREM 3.10. *If* $\boldsymbol{B}$ *has full rank and the Simplex Condition (2.2) holds, then the* $n \times n$ *diffusion matrix* $(\boldsymbol{D} - AA')\,\boldsymbol{B}$ *in the Proposed Dynamics (2.4) has full rank.*

REMARK 3.11. A family of physical measures (equivalent to the risk-neutral measure $\mathbb{Q}$ and ensuring price admissibility) may be specified by introducing a market-price-of-risk vector $\eta = (\eta_1, \ldots, \eta_n)$ and modifying the Proposed Dynamics (2.4) accordingly to obtain, for the state variables,

$$da_k = \left[a_k (r - \rho_{k-1}) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} a_i + a_k \left(B_k - A' \boldsymbol{B} \right) \eta \right] dt$$
$$+ a_k \left(B_k - A' \boldsymbol{B} \right) dZ^{\mathbb{P}}, \tag{3.5}$$

for the short rate,

$$dr = \left[r (r - \rho_n) + \sum_{k=1}^{n} \rho_k \rho_{k-1} a_k - r A' \boldsymbol{B} \eta \right] dt - r A' \boldsymbol{B} dZ^{\mathbb{P}}, \tag{3.6}$$

and, for bond prices $P_{t,T}$ from (2.8) we find the following additional term in their drift that characterizes the bond's risk premium and its sensitivity to each component of η:

$$\left(- P_{t,T} A' \boldsymbol{B} + \sum_{k=1}^{n} \frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1} e^{-\rho_k (T-t)}}{\rho_k - \rho_{k-1}} a_k B_k \right) \eta. \tag{3.7}$$

Because the additional drift terms for a_k and for r are proportional (to a_k and to r, respectively, with bounded multiple) the proofs of Lemmas 3.2–3.5 and Theorem 3.6 can be generalized to conclude that the physical measure retains the Simplex Condition with price-admissible yield curves.

REMARK 3.12. In the deterministic case $\boldsymbol{B} = \boldsymbol{0}$, there is a steady-state configuration of the state variables that corresponds to a flat yield curve with interest rate ρ_0 with state variables $a_k = \frac{\rho_0}{\rho_{k-1}}(1 - \frac{\rho_{k-1}}{\rho_k})$ for $k = 1, \cdots, n$ that satisfy the Simplex Condition due to the ordering $0 < \rho_0 < \rho_1 < \cdots < \rho_n$. With probability one, any randomly selected yield curve chosen from the simplex will (with deterministic dynamics) approach this steady-state configuration in the limit as $t \to \infty$ because the terms $e^{-\rho_k(T-t)}$ with $k \geq 1$ will decay more quickly than $e^{-\rho_0(T-t)}$; intuition for this result stems from the fact that continuous deterministic no-arbitrage simply rescales (with a single multiple) the initial price curve at each later time t so that the maturing bond has price 1. Figure 3.1 illustrates this deterministic evolution, $da_1 = [-\rho_2 a_1^2 + (\rho_2 - \rho_0) a_1 - \rho_2 a_1 a_2] dt$ and $da_2 = [(\rho_2 - \rho_1)\rho_0 a_1/\rho_2 + (\rho_2 - \rho_1) a_2 - \rho_2 a_1 a_2 - \rho_2 a_2^2] dt$, for the case $n = 2$ with $\rho_0 = 5\%$, $\rho_1 = 10\%$, and $\rho_2 = 20\%$, for which the steady-state fixed point is $(0.5, 0.25)$ by showing the deterministic no-arbitrage paths for the state variables that begin at various points in the simplex.

REMARK 3.13. One common feature of all linear bond-price models is the drift of the risk-neutral dynamics under $\mathbb{Q}$ of the short rate and of the coefficients that multiply the price basis functions (the decaying exponential functions of time to maturity) which are constrained by the nature of risk-neutral price evolution. To see this, begin by writing bond prices as linear combinations of distinct decaying exponential functions of the time to maturity, with weights w_k (the coefficients that multiply the price-basis functions) as follows:

$$P_{t,T} = \sum_{k=0}^{n} w_k e^{-\rho_k (T-t)}. \tag{3.8}$$

Note that this representation can be used with any model for which bond prices can be written as linear combinations of decaying exponentials in the time to maturity (including

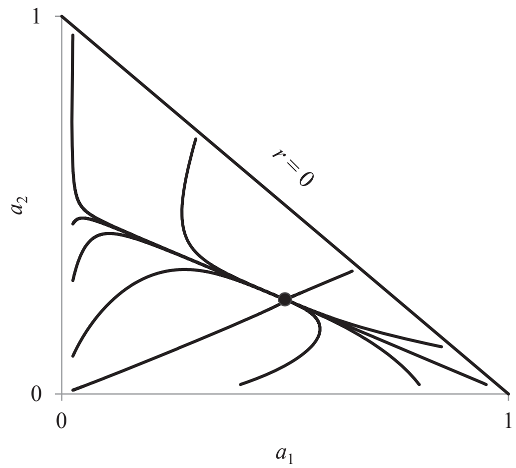

FIGURE 3.1. Deterministic ($\boldsymbol{B} = \mathbf{0}$, drift only) evolution of state variables (starting at various locations near the edge of the simplex, which is here represented by the triangle) showing their convergence to a steady-state fixed point. This example uses $n = 2$ with $\rho_0 = 5\%$, $\rho_1 = 10\%$, and $\rho_2 = 20\%$, for which the steady-state fixed point is (0.5, 0.25).

models for which the weights w_k are nonlinear functions of the original state variables). We must have $\sum_{k=0}^{n} w_k = 1$ to ensure that the maturing bond is priced correctly, and, taking the limit of the yield $-\ln(P_{t,T})/(T-t)$ as maturity $T - t \to 0$, we find that the short rate must be $r = \sum_{k=0}^{n} w_k \rho_k$. The no-arbitrage requirement under the risk-neutral measure $\mathbb{Q}$, that the drift of bond price dynamics must be $\mathit{drift}\,(dP_{t,T}) = rP_{t,T}$, implies that

$$\begin{aligned} \mathit{drift}\,(dP_{t,T}) &= \mathit{drift}\left[d\left(\sum_{k=0}^{n} w_k e^{-\rho_k(T-t)}\right)\right] \\ &= \sum_{k=0}^{n} w_k \rho_k e^{-\rho_k(T-t)} + \sum_{k=0}^{n} e^{-\rho_k(T-t)} \mathit{drift}\,(dw_k) = r\,P_{t,T} \\ &= r \sum_{k=0}^{n} w_k e^{-\rho_k(T-t)} \end{aligned} \tag{3.9}$$

and thus $\sum_{k=0}^{n} [w_k \rho_k + \mathit{drift}\,(dw_k) - rw_k]\, e^{-\rho_k(T-t)} = 0$ must hold at any fixed time t for all maturities $T \geq t$. Because the weights w_k do not depend on bond maturities, and using linear independence of exponential functions with distinct decay rates, this forces $w_k \rho_k + \mathit{drift}\,(dw_k) - rw_k = 0$ and thus $\mathit{drift}\,(dw_k) = (r - \rho_k)\, w_k$ for each k. Multiplying by the fixed coefficients and summing, we find that $\mathit{drift}\,(dr) = \mathit{drift}[d(\sum_{k=0}^{n} w_k \rho_k)] = \sum_{k=0}^{n} \rho_k w_k\,(r - \rho_k)$, completing the demonstration that the functional form of the risk-neutral drift of the dynamics of the short rate r (and of the coefficients w_k that multiply the price-basis functions) is fixed within the class of linear price function models for bond prices.

4. DISCUSSION AND SUMMARY

The class of linear price models represents the price of a zero-coupon bond as a linear combination of decaying exponential functions of the bond's maturity, creating a variety of yield curve shapes while satisfying the no-arbitrage requirement that a risk-neutral measure exist. Linear price models enable the separation of the functional form of the state variables' diffusion from both their drift and from the functional form of the spaces of price functions and yield curves, in contrast to the widely used family of affine linear yield models. This has the potential to increase the flexibility of the modeler by allowing dynamically modified unanticipated stochastic volatility and stochastic correlation without the need to rederive the space of yield curve shapes permitted by the model.

The main contribution of this paper is a novel tractable closed-form specification of the state variables that guarantees price admissibility by assuring positive bond prices, positive yields, and positive forward rates at all maturities throughout the future of the process, while also admitting equivalent price-admissible physical measures with a market-price-of-risk vector. The state variables are scaled partial sums of weighted coefficients of the exponential terms in the bond pricing function. The open simplex of the state variables is proven to be a persistent and price-admissible region. The full-rank diffusion specification selectively diminishes the variability as the state variables approach a boundary of the simplex, while maintaining variability in directions that would not leave the simplex (and the diffusion specification can be extended from a fixed matrix $\boldsymbol{B}$ to any bounded measurable Lipschitz matrix, perhaps with additional state variables). Existence of a unique strong solution is established. Because the model is linear in prices, the no-arbitrage requirement—that a risk-neutral measure exists—is verified by examining the drift of the state variables. However, to be fully arbitrage free, it is not enough merely for a term-structure model to admit a risk-neutral measure with positive bond prices. In addition, yields and forward rates must be positive in order to avoid cash-and-carry arbitrage. The proposed dynamics satisfy these considerations.

APPENDIX: PROOFS

Proof of Lemma 3.2. To see that there is a unique strong solution to the Restricted SDE (3.2), first note that each drift coefficient $\varphi(a_k)(r_\varphi - \rho_{k-1}) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} \varphi(a_i)$ is bounded (because each $\varphi(a_k)$ is bounded, as is $r_\varphi = \rho_n - \rho_n \sum_{k=1}^{n} \varphi(a_k)$, with all ρ_i constant) and that the diffusion coefficient vector $(\boldsymbol{D}_\varphi - A_\varphi A'_\varphi)\boldsymbol{B}$ in (3.2) is similarly bounded. Next, note that the drift and diffusion coefficients of the Restricted SDE (3.2) are either quadratic functions of $a_1, \ldots, a_n$ (when restrictions are not binding) or constants and therefore satisfy the Lipschitz condition. A unique strong solution to the Restricted SDE (3.2) now follows, e.g., from the SDE Proposition in appendix E of Duffie (2001) because the growth condition follows from boundedness of the coefficients. □

Proof of Lemma 3.3. To see that the components of the solution to the Restricted SDE, $a_1, \ldots, a_n$, remain positive throughout the future, we proceed indirectly to show that the contrary assumption (that they do not remain positive) leads to a contradiction on any particular path. Let $\mathcal{T}_0$ denote the first time on this path that any state variable reaches zero, let this be the Jth variable a_J, and let $\mathcal{T}_1$ denote the most recent previous time this variable was equal to 1, where $\mathcal{T}_1 = 0$ if 1 has not yet been reached by a_J. Thus $\mathcal{T}_1 < \mathcal{T}_0$ and from $\mathcal{T}_1$ to $\mathcal{T}_0$ we have $\varphi(a_J) = a_J$ and the restriction is not binding on this variable. We treat the cases $J = 1$ and $J > 1$ separately because a_1 is special in that its drift is simply $\varphi(a_1)(r_\varphi - \rho_0)$ without additional terms. If $J = 1$, then from $\mathcal{T}_1$ to $\mathcal{T}_0$ the Restricted SDE for a_1 is

$$da_1 = a_1(r_\varphi - \rho_0)dt + a_1(B_1 - A'_\varphi \boldsymbol{B})dZ \tag{A.1}$$

and we note that the dynamics of $\ln(a_1)$ in the closed interval $[\mathcal{T}_1, \mathcal{T}_0]$ are, using Itô's Lemma and $\varphi(a_1) = a_1$, given by

$$\begin{aligned}
d\ln(a_1) &= \frac{da_1}{a_1} - \frac{1}{2}\left[\left(B_1 - A'_\varphi \boldsymbol{B}\right)dZ\right]^2 \\
&= (r_\varphi - \rho_0)dt + \left(B_1 - A'_\varphi \boldsymbol{B}\right)dZ - \frac{1}{2}\left(B_1 - A'_\varphi \boldsymbol{B}\right)dZ(dZ)'\left(B'_1 - \boldsymbol{B}'A_\varphi\right) \\
&= \left(r_\varphi - \rho_0\right)dt - \frac{1}{2}\left(B_1 - A'_\varphi \boldsymbol{B}\right)\left(B'_1 - \boldsymbol{B}'A_\varphi\right)dt + \left(B_1 - A'_\varphi \boldsymbol{B}\right)dZ \\
&= \left[\left(r_\varphi - \rho_0\right) - \frac{1}{2}\left(B_1B'_1 - 2B_1\boldsymbol{B}'A_\varphi + A'_\varphi \boldsymbol{B}\boldsymbol{B}'A_\varphi\right)\right]dt + \left(B_1 - A'_\varphi \boldsymbol{B}\right)dZ
\end{aligned} \tag{A.2}$$

whose integral over $[\mathcal{T}_1, \mathcal{T}_0]$ is finite (because all terms are bounded) implying that $\ln(a_1)$ is finite, and leading to the contradictory conclusion that $a_1 = e^{\ln(a_1)}$ is positive at $\mathcal{T}_0$. Thus J cannot equal 1 and a_1 cannot be the first variable to reach zero.

Now suppose that $J \neq 1$, let k denote the observed value of J for this path, and note that $a_1 > 0, \ldots, a_{k-1} > 0$ while $a_k = 0$ at time $\mathcal{T}_0$. The dynamics for $\ln(a_k)$ in the closed interval $[\mathcal{T}_1, \mathcal{T}_0]$ are, using Itô's Lemma and $\varphi(a_k) = a_k$ in this interval, given by

$$\begin{aligned}
d\ln(a_k) &= \frac{da_k}{a_k} - \frac{1}{2}\left[\left(B_k - A'_\varphi \boldsymbol{B}\right)dZ\right]^2 \\
&= \left[\left(r_\varphi - \rho_{k-1}\right) + \frac{\rho_k - \rho_{k-1}}{a_k\rho_k\rho_{k-1}}\sum_{i=1}^{k-1}\rho_i\rho_{i-1}\varphi(a_i)\right]dt + \left(B_k - A'_\varphi \boldsymbol{B}\right)dZ \\
&\quad - \frac{1}{2}\left(B_k - A'_\varphi \boldsymbol{B}\right)dZ(dZ)'\left(B'_k - \boldsymbol{B}'A_\varphi\right) \\
&= \left[\left(r_\varphi - \rho_{k-1}\right) + \frac{\rho_k - \rho_{k-1}}{a_k\rho_k\rho_{k-1}}\sum_{i=1}^{k-1}\rho_i\rho_{i-1}\varphi(a_i)\right. \\
&\quad \left. - \frac{1}{2}\left(B_kB'_1 - 2B_k\boldsymbol{B}'A_\varphi + A'_\varphi \boldsymbol{B}\boldsymbol{B}'A_\varphi\right)\right]dt + \left(B_k - A'_\varphi \boldsymbol{B}\right)dZ
\end{aligned} \tag{A.3}$$

whose integral over this region satisfies

$$\begin{aligned}
\int d\ln(a_k) &= \int\left[\left(r_\varphi - \rho_{k-1}\right) + \frac{\rho_k - \rho_{k-1}}{a_k\rho_k\rho_{k-1}}\sum_{i=1}^{k-1}\rho_i\rho_{i-1}\varphi(a_i)\right. \\
&\quad \left. - \frac{1}{2}\left(B_kB'_1 - 2B_k\boldsymbol{B}'A_\varphi + A'_\varphi \boldsymbol{B}\boldsymbol{B}'A_\varphi\right)\right]dt + \int\left(B_k - A'_\varphi \boldsymbol{B}\right)dZ \\
&> \int\left[\left(r_\varphi - \rho_{k-1}\right) - \frac{1}{2}(B_kB'_1 - 2B_k\boldsymbol{B}'A_\varphi + A'_\varphi \boldsymbol{B}\boldsymbol{B}'A_\varphi)\right]dt \\
&\quad + \int\left(B_k - A'_\varphi \boldsymbol{B}\right)dZ,
\end{aligned} \tag{A.4}$$

where the possibly unbounded term is positive:

$$\int \frac{\rho_k - \rho_{k-1}}{a_k \rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} \varphi(a_i)\, dt > 0, \tag{A.5}$$

which follows from the fact that k is the first variable to reach zero on this path, so $a_i > 0$ and thus $\varphi(a_i) > 0$ for $i < k$, along with the ordering of the constants ρ_i. Thus $\ln(a_k)$ cannot approach $-\infty$ as we approach $\mathcal{T}_0$ from the left (because all remaining terms are bounded) and therefore $a_k = e^{\ln(a_k)}$ cannot be 0 at $\mathcal{T}_0$. Thus J cannot equal k and a_k cannot be the first variable to reach zero. Since no path can contain a value of zero for any variable, and the variables were initially positive due to the Simplex Condition, the variables must always remain positive. $\square$

Proof of Lemma 3.4. To evaluate the dynamics of $r = \rho_n - \rho_n \sum_{k=1}^{n} a_k$, we sum the Proposed Dynamics (2.4) and (2.5) of the state variables as follows:

$$dr = -\rho_n \sum_{k=1}^{n} da_k.$$

$$(\text{A.6}) = -\rho_n \sum_{k=1}^{n} \left\{ \left[a_k (r - \rho_{k-1}) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} a_i \right] dt + a_k \left(B_k - A' \boldsymbol{B}\right) dZ \right\}.$$

The drift term, using $\rho_n \sum_{k=1}^{n} a_k = \rho_n - r$, is

$$\begin{aligned} &-\rho_n \sum_{k=1}^{n} \left[a_k (r - \rho_{k-1}) + \frac{\rho_k - \rho_{k-1}}{\rho_k \rho_{k-1}} \sum_{i=1}^{k-1} \rho_i \rho_{i-1} a_i \right] \\ &= r(r - \rho_n) + \rho_n \sum_{k=1}^{n} \rho_{k-1} a_k - \rho_n \sum_{k=1}^{n} \sum_{i=1}^{k-1} \left(\frac{1}{\rho_{k-1}} - \frac{1}{\rho_k} \right) \rho_i \rho_{i-1} a_i \end{aligned} \tag{A.7}$$

for which we will reverse the order of summation in the final double sum, then recognize and simplify a telescoping series, to find

$$\begin{aligned} &r(r - \rho_n) + \rho_n \sum_{k=1}^{n} \rho_{k-1} a_k - \rho_n \sum_{i=1}^{n-1} \rho_i \rho_{i-1} a_i \sum_{k=i+1}^{n} \left(\frac{1}{\rho_{k-1}} - \frac{1}{\rho_k} \right) \\ &= r(r - \rho_n) + \rho_n \sum_{k=1}^{n} \rho_{k-1} a_k - \rho_n \sum_{i=1}^{n-1} \rho_i \rho_{i-1} a_i \left(\frac{1}{\rho_i} - \frac{1}{\rho_n} \right) \\ &= r(r - \rho_n) + \sum_{i=1}^{n} \rho_k \rho_{k-1} a_k \end{aligned} \tag{A.8}$$

establishing correctness of the drift term in (3.3). The diffusion term of r from (A.6) is

$$-\rho_n \sum_{k=1}^{n} a_k \left(B_k - A' \boldsymbol{B}\right) = -\rho_n \left(1 - \sum_{k=1}^{n} a_k\right) A' \boldsymbol{B} = -r\, A' \boldsymbol{B} \tag{A.9}$$

completing the proof. □

Proof of Lemma 3.5. It remains only to show that $\sum_{k=1}^{n} a_k < 1$ continues to hold, because the other requirement of the Simplex Condition (positivity of the variables) was established by Lemma 3.3. Thus it will suffice to show that $r > 0$ where $r = \rho_n - \rho_n \sum_{k=1}^{n} a_k$; note that this is satisfied initially by assumption. We focus now on r from time 0 until the first time $\mathcal{U}_0$ that $r = 0$, where $\mathcal{U}_0$ is assumed finite for the purpose of an indirect proof. Note that in the interval $[0, \mathcal{U}_0]$ $r \geq 0$ implies $\sum_{k=1}^{n} a_k \leq 1$ which (because each $a_k > 0$) in turn implies that all $0 < a_k \leq 1$ and thus $\varphi(a_k) = a_k$ and hence $r_\varphi = \rho_n - \rho_n \sum_{k=1}^{n} \varphi(a_k) = \rho_n - \rho_n \sum_{k=1}^{n} a_k = r$ in this interval, loosening the restriction on the variables. The dynamics of r in the interval $[0, \mathcal{U}_0]$ must therefore be identical to those derived in Lemma 3.4:

$$dr = \left[r(r - \rho_n) + \sum_{k=1}^{n} \rho_k \rho_{k-1} a_k \right] dt - r A' \boldsymbol{B} dZ. \tag{A.10}$$

We next find the dynamics of $\ln(r)$ over the closed interval $[0, \mathcal{U}_0]$

$$\begin{aligned} d\ln(r) &= \frac{dr}{r} - \frac{1}{2}\left(A'\boldsymbol{B}dZ\right)^2 \\ &= \left[(r - \rho_n) + \frac{1}{r}\sum_{k=1}^{n} \rho_k \rho_{k-1} a_k \right] dt - A'\boldsymbol{B}dZ - \frac{1}{2} A'\boldsymbol{B}dZ\,(dZ)'\,\boldsymbol{B}'A \\ &= \left[(r - \rho_n) + \frac{1}{r}\sum_{k=1}^{n} \rho_k \rho_{k-1} a_k - \frac{1}{2} A'\boldsymbol{B}\boldsymbol{B}'A \right] dt - A'\boldsymbol{B}dZ \end{aligned} \tag{A.11}$$

whose integral over this region satisfies

$$\begin{aligned} \int d\ln(r) &= \int \left[(r - \rho_n) + \frac{1}{r}\sum_{k=1}^{n} \rho_k \rho_{k-1} a_k - \frac{1}{2} A'\boldsymbol{B}\boldsymbol{B}'A \right] dt \\ &\quad - \int A'\boldsymbol{B}dZ > \int \left[(r - \rho_n) - \frac{1}{2} A'\boldsymbol{B}\boldsymbol{B}'A \right] dt - \int A'\boldsymbol{B}dZ \end{aligned} \tag{A.12}$$

where the possibly unbounded term is positive:

$$\int \frac{1}{r} \sum_{k=1}^{n} \rho_k \rho_{k-1} a_k dt > 0, \tag{A.13}$$

which follows from the fact that $r \geq 0$ because U_0 is the first time that r reaches zero, and the constants ρ_k and the state variables are all positive. Thus, $\ln(r)$ cannot approach $-\infty$ as we approach $\mathcal{U}_0$ from the left (because all remaining terms are bounded) and therefore $r = e^{\ln(r)}$ cannot be 0 at $\mathcal{U}_0$, completing the proof. □

Proof of Theorem 3.6. Lemma 3.5 establishes that, under the Restricted SDE, the Simplex Condition holds which, in turn, implies that variables always satisfy $0 < a_k < 1$. It now follows that $\varphi(a_k) = \max[0, \min(a_k, 1)] = a_k$ always holds and the Restricted SDE (3.2) is identical to the unrestricted Proposed Dynamics (2.4) and, therefore, must share the same solution from Lemma 3.2. Thus the unique strong solution to the Restricted SDE is a unique strong solution to the (unrestricted) Proposed Dynamics (2.4), the state variables must continue to satisfy the Simplex Condition throughout the future, and r remains always positive. □

Proof of Theorem 3.8. We start from the price function (2.8), using Itô's Lemma twice, together with dr from (3.3) and da_k from (2.5), to find

$$
\begin{aligned}
dP_{t,T} &= d\left(\frac{r}{\rho_n}e^{-\rho_n(T-t)}\right) + d\left(\sum_{k=1}^{n} a_k \frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}\right)\\
&= \left(re^{-\rho_n(T-t)} + \frac{1}{\rho_n}\left[r\,(r-\rho_n) + \sum_{k=1}^{n}\rho_k\rho_{k-1}a_k\right]e^{-\rho_n(T-t)}\right)dt\\
&\quad - r\,A'\boldsymbol{B}\left(\frac{1}{\rho_n}e^{-\rho_n(T-t)}\right)dZ\\
&\quad + \sum_{k=1}^{n} a_k\rho_{k-1}\rho_k\left(\frac{e^{-\rho_{k-1}(T-t)} - e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}\right)dt\\
&\quad + \sum_{k=1}^{n} a_k\,(r-\rho_{k-1})\left(\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}\right)dt\\
&\quad + \sum_{k=1}^{n}\left(\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k\rho_{k-1}}\right)\sum_{i=1}^{k-1}\rho_i\rho_{i-1}a_i\,dt\\
&\quad + \left(\sum_{k=1}^{n}\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}a_k\left(B_k - A'\boldsymbol{B}\right)\right)dZ.
\end{aligned}
\tag{A.14}
$$

Extracting, from the drift, only the terms that explicitly involve r, we find

$$
\begin{aligned}
&re^{-\rho_n(T-t)} + \frac{1}{\rho_n}r\,(r-\rho_n)\,e^{-\rho_n(T-t)} + \sum_{k=1}^{n} a_k r\left(\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}\right)\\
&= r\left(e^{-\rho_n(T-t)} + \frac{1}{\rho_n}(r-\rho_n)\,e^{-\rho_n(T-t)} + \sum_{k=1}^{n} a_k\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}\right)\\
&= r\left(\frac{r}{\rho_n}e^{-\rho_n(T-t)} + \sum_{k=1}^{n} a_k\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k - \rho_{k-1}}\right) = r\,P_{t,T}.
\end{aligned}
\tag{A.15}
$$

To see that the remaining terms from the drift of (A.14) sum to zero, note that (after some cancellation) these may be written as

$$
\frac{e^{-\rho_n(T-t)}}{\rho_n}\sum_{k=1}^{n} a_k\,\rho_k\,\rho_{k-1} - \sum_{k=1}^{n} a_k\rho_{k-1}e^{-\rho_k(T-t)} + \sum_{k=1}^{n}\sum_{i=1}^{k-1} a_i\,\rho_i\,\rho_{i-1}\left(\frac{e^{-\rho_{k-1}(T-t)}}{\rho_{k-1}} - \frac{e^{-\rho_k(T-t)}}{\rho_k}\right).
\tag{A.16}
$$

Reversing the order of summation in the final double sum, recognizing and simplifying a telescoping series, we indeed find zero:

$$\begin{aligned}
&\frac{e^{-\rho_n(T-t)}}{\rho_n}\sum_{k=1}^{n} a_k\rho_k\rho_{k-1} - \sum_{k=1}^{n} a_k\rho_{k-1}e^{-\rho_k(T-t)} \\
&+\sum_{i=1}^{n-1} a_i\rho_i\rho_{i-1}\sum_{k=i+1}^{n}\left(\frac{e^{-\rho_{k-1}(T-t)}}{\rho_{k-1}} - \frac{e^{-\rho_k(T-t)}}{\rho_k}\right) \\
&=\frac{e^{-\rho_n(T-t)}}{\rho_n}\sum_{k=1}^{n} a_k\rho_k\rho_{k-1} - \sum_{k=1}^{n} a_k\rho_{k-1}e^{-\rho_k(T-t)} \\
&+\sum_{i=1}^{n-1} a_i\rho_i\rho_{i-1}\left(\frac{e^{-\rho_i(T-t)}}{\rho_i} - \frac{e^{-\rho_n(T-t)}}{\rho_n}\right) = 0,
\end{aligned} \tag{A.17}$$

which, together with (A.15) completes the proof for the drift. The diffusion of $dP_{t,T}$ from (A.14) simplifies as follows:

$$\begin{aligned}
&-\frac{r}{\rho_n}e^{-\rho_n(T-t)}A'\boldsymbol{B} + \left(\sum_{k=1}^{n}\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k-\rho_{k-1}}a_k\left(B_k - A'\boldsymbol{B}\right)\right) \\
&=\sum_{k=1}^{n}\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k-\rho_{k-1}}a_k B_k \\
&-\frac{r}{\rho_n}e^{-\rho_n(T-t)}A'\boldsymbol{B} - \sum_{k=1}^{n}\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k-\rho_{k-1}}a_k A'\boldsymbol{B} \\
&=\sum_{k=1}^{n}\frac{\rho_k e^{-\rho_{k-1}(T-t)} - \rho_{k-1}e^{-\rho_k(T-t)}}{\rho_k-\rho_{k-1}}a_k B_k - P_{t,T}A'\boldsymbol{B}
\end{aligned} \tag{A.18}$$

completing the derivation of the dynamics of $P_{t,T}$. If we write the bond price as $P_{t,T} = G(T-t,\,A_t)$ in terms of a function G of time to maturity and the current state variables, then we have shown that the process $(e^{-\int_0^t r_s ds}G(T-t,\,A_t))_{t\in[0,T]}$ is a local martingale (it is a true martingale by boundedness of A and r). Thus the probability measure $\mathbb{Q}$ is a risk-neutral measure and there is no arbitrage in the bond market. ☐

Proof of Theorem 3.9. The condition $\sum_{k=1}^{n} a_k < 1$ immediately implies that the short rate is positive because the constant ρ_n is positive. To see that bond prices defined by (2.7) are positive, first note that $P(0) = 1$ and that $\lim_{m\to\infty} P(m) = 0$, where we represent prices here in terms of the maturity $m = T - t$ at a fixed time t. Positivity of prices will follow by showing that $P'(m) < 0$ for all $m > 0$, because this implies monotonicity of the price function as it decreases from 1 to 0. To show that $-P'(m) > 0$, calculate as follows:

$$\begin{aligned}
-P'(m) &= \rho_n e^{-\rho_n m} + \sum_{k=1}^{n} a_k\left(\frac{\rho_{k-1}\rho_k e^{-\rho_{k-1}m} - \rho_{k-1}\rho_k e^{-\rho_k m}}{\rho_k-\rho_{k-1}} - \rho_n e^{-\rho_n m}\right) \\
&= \rho_n e^{-\rho_n m}\left(1-\sum_{k=1}^{n} a_k\right) + \sum_{k=1}^{n}\rho_{k-1}\rho_k a_k\left(\frac{e^{-\rho_{k-1}m} - e^{-\rho_k m}}{\rho_k-\rho_{k-1}}\right) > 0,
\end{aligned} \tag{A.19}$$

which is positive because $\sum_{k=1}^{n} a_k < 1$, the constants ρ_k are all positive and ordered so that $\rho_{k-1} < \rho_k$, proving that all prices are positive. It now follows that yields

$Y(m) = -\ln[P(m)]/m$ are positive for $m > 0$ because prices fall monotonically from 1 to 0 and therefore have negative logarithm. Having established that $P(m) > 0$ and $P'(m) < 0$, it follows that forward rates $-P'(m)/P(m)$ are all positive, completing the proof. □

Proof of Theorem 3.10. Given that $\boldsymbol{B}$ has full rank and all $a_k > 0$ from the Simplex Condition, the diffusion matrix $(\boldsymbol{D} - AA')\,\boldsymbol{B}$ will also have full rank unless there exists a nonzero column vector X such that $X'(\boldsymbol{D} - AA') = 0$, which would be equivalent to $(a_1x_1, a_2x_2, \ldots, a_nx_n) = \left(\sum_{k=1}^{n} x_k a_k\right)(a_1, a_2, \ldots, a_n)$, which would in turn imply that $x_1 = x_2 = \cdots = x_n = \sum_{k=1}^{n} x_k a_k = x_1 \sum_{k=1}^{n} a_k$ with nonzero x_1. This would imply that $\sum_{k=1}^{n} a_k = 1$, in violation of the Simplex Condition (2.2), completing the proof. □

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