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MODEL-INDEPENDENT LOWER BOUND ON VARIANCE SWAPS

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It is well known that, under a continuity assumption on the price of a stock S, the realized variance of S for maturity T can be replicated by a portfolio of calls and puts maturing at T. This paper assumes that call prices on S maturing at T are known for all strikes but makes no continuity assumptions on S. We derive semiexplicit expressions for the supremum lower bound V_{inf} on the hedged payoff, at maturity T, of a long position in the realized variance of S. Equivalently, V_{inf} is the supremum strike K such that an investor with a long position in a variance swap with strike K can ensure a nonnegative payoff at T. We study examples with constant implied volatilities and with a volatility skew. In our examples, V_{inf} is close to the fair variance strike obtained under the continuity assumption.

KEY WORDS: model risk, hedging, subreplication, realized variance, variance swap.

1. INTRODUCTION

A variance swap with maturity T and strike K on a stock S is a contract that pays the realized variance minus K at time T. If the value of such a contract is null at inception, we say that K is *the fair variance strike* of the swap. Variance swaps can be used to speculate on future volatility levels and to hedge the volatility exposure of other positions. As noted by Demeterfi et al. (1999), whereas stock options provide exposure to both the stock price and its volatility, variance swaps provide pure exposure to the volatility level. Carr and Lee (2009) give a detailed history of the variance swaps and other volatility derivatives markets.

Unless otherwise specified, we assume that call prices on S for maturity T are known for all strikes. Under a continuity assumption on the stock price, the realized variance can be replicated (Dupire 1993; Neuberger 1994) by hedging a position in a *log contract* whose payoff at time T is $2T^{-1}\ln(F_0/S_T)$, where S_T is the price of S at T and F_0 is the forward price of the stock. Thus, the fair variance strike is equal to the forward price $V_{\log}$ of the log contract. On the other hand, the log contract can itself be statically replicated (Carr and Madan 1998; Demeterfi et al. 1999; Britten-Jones and A. Neuberger 2000) by a portfolio of calls and puts maturing at T. Furthermore,

$$V_{\log} = \frac{2}{T}\int_{(0,\infty)} \frac{C(K) - \max(0, F_0 - K)}{K^2}\,dK, \tag{1.1}$$

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where $C(K)$ is the undiscounted price of a call with maturity T and strike K. Equation (1.1) provides the basis for an analytic approximation (see Demeterfi et al. 1999) of $V_{\log}$ in the presence of a volatility skew and for the calculation of the VIX index by the Chicago Board Options Exchange.

Consider an investor who is long in a contract that pays the realized variance of S at time T. The investor is allowed to buy or sell at time 0 call options maturing at T and to take dynamic positions in S during the period $[0, T]$. The hedged payoff at T is the combined result of his positions. This paper answers the following question: what is the supremum lower bound $V_{\inf}$ on the hedged payoff, at T, of a long position in the realized variance? Equivalently, what is the supremum strike K such that a properly hedged long position in a variance swap with strike K is guaranteed to have a nonnegative payoff at T independently of the stock behavior? The answer to this question is $V_{\log}$, under the continuity assumption, because the price of a variance swap with strike $V_{\log}$ is null. This paper, however, makes no assumptions on the stock price process other than being strictly positive. Several authors (Demeterfi et al. 1999; Broadie and Jain 2008; Carr and Wu 2009; Carr, Lee, and Wu 2012) have shown that, in the presence of jumps, the price of a variance swap with strike $V_{\log}$ depends on the jumps sizes, their frequency, and can be significantly positive or negative.

In related work, model-independent bounds on lookback and barrier options have been derived (Hobson 1998; Brown, Hobson, and Rogers 2001) in terms of a continuum of calls with the same maturity. Bounds on prices of basket options (Hobson, Laurence, and Wang 2005a,b; Laurence and Wang 2005; d'Aspremont and El Ghaoui 2006) and spread options (Laurence and Wang 2008, 2009) have been established in terms of prices of vanilla options with the same maturity. Carr and Lee (2008) have shown that volatility derivatives can be perfectly replicated by vanilla options under a continuity assumption on the stock and an independence assumption on the volatility. Kahalé (2010) has proven that, under certain conditions, for any $\epsilon > 0$, the optimal super and subreplicating prices of a financial derivative in terms of other financial derivatives prices can be attained, within precision ϵ, by a model on a sample space of bounded size. Hobson and Neuberger (2012) have derived an optimal upper bound for forward start options in terms of vanilla options with corresponding maturities. Zhu and Lian (2011) have priced discretely sampled variance swaps under Heston's stochastic volatility model. Zheng and Kwok (2014) have presented a general framework for pricing discretely sampled generalized variance swaps under a class of stochastic volatility models with jumps. Under a continuity assumption on the stock price, Davis, Obloj, and Raval (2014) have given robust pricing bounds for a weighted variance swap in terms of prices of a finite number of comaturing put options.

The variance swaps considered in this paper are based on the squared log-return kernel. Hobson and Klimmek (2012) have recently extended our results to other kernels by using the Skorokhod embedding problem and have established asymptotically tight upper bounds as well as lower bounds on the prices of variance swaps. Their results complement ours: their derivation of the lower bound on variance swaps based on the log-squared returns assumes either that interest rates are null or that the time-partition is uniform and interest rates are constant. These assumptions are not needed in our paper. They are also not required for upper bounds or for other kernels considered in Hobson and Klimmek (2012). A detailed comparison of the two approaches can be found in Hobson and Klimmek (2012). More recently, Kahalé (2012) has given a numerical method based on convex programming to calculate optimal super and subreplicating prices and corresponding hedging strategies of a financial derivative in terms of other financial derivatives. The method can be used to calculate optimal upper and lower bounds on variance swaps with various kernels when the prices of vanilla options are known for a finite set of strikes and maturities. When call prices are known for maturities t_i, $1 \leq i \leq n+1$, and all strikes, optimal bounds on variance swaps monitored at t_i have been established in Henry-Labordere and Touzi (2013).

The rest of the paper is organized as follows: Section 2 states the remaining definitions and assumptions. Section 3 shows how to obtain lower bounds on variance swaps from a new class of functions, the *V-convex functions*. By building upon the results of Section 3, Section 4 gives a semiexplicit lower bound on variance swaps via jump processes. Section 5 shows that this lower bound is optimal whenever V_{inf} is finite, by constructing martingale measures that asymptotically attain it. It also gives a necessary and sufficient condition for the finiteness of V_{inf} and, under a regularity condition on the call function, a simplified expression for V_{inf}. Section 6 illustrates the results. In Section 6.1, numerical calculations and a simple approximation for V_{inf} are given, if the implied volatilities for maturity T are the same for all strikes. An example where implied volatilities for maturity T are given by the Merton jump-diffusion model is considered in Section 6.2, and an example with a discrete set of strikes and a volatility skew is treated in Section 6.3. Section 7 contains concluding remarks. Proofs of some of the results are in the Appendix.

2. DEFINITIONS AND ASSUMPTIONS

We assume that the short risk-free rate $r(t)$ is deterministic and bounded by κ for $t \in [0, T]$. To simplify the notation, we assume that S pays no dividends. It is easy to show that our results still hold when S pays dividends at a continuous, deterministic and bounded rate. More precisely, Theorems 3.5, 5.3, 5.7, and 6.1 still hold without any modification, with the same or slightly different proofs. Call options on S with maturity T are assumed to have arbitrage free prices and to be liquid for nonnegative strikes K. Thus, $C(K)$ is a convex and decreasing function of K (see, e.g., Davis and Hobson 2007), with $C(0) = F_0$. We further assume that $C'_+(0) = -1$ and that $C(y)$ goes to 0 as y goes to ∞. A standard result (Follmer and Schied 2004, Lemma 7.23) then implies the existence of a probability measure μ on $[0, \infty)$ such that $C(K) = \int_{(K,\infty)} (z - K)\, d\mu(z)$ for $K \geq 0$, and so $C'_+(K) = -\mu((K, \infty))$. Let $P(K)$ be the undiscounted price of a put with maturity T and strike K. By the put-call parity relation, $C(K) - P(K) = F_0 - K$. For simplicity of presentation, we exclude the trivial case where μ is the Dirac measure δ_{F_0}. Given a subdivision of $[0, T]$ with time-steps $0 = t_0 < \cdots < t_{n+1} = T$, denote by F_i the forward price of the stock at time t_i for maturity T. The realized variance of S over the time-period $[0, T]$ is defined as

$$V[\mathbf{t}] = \frac{1}{T} \sum_{i=0}^{n} \ln^2 \left(\frac{S_{i+1}}{S_i} \right),$$

where S_i is the value of the stock at time t_i. Let $V_{\text{inf}}[\mathbf{t}]$ be the supremum lower bound on the hedged payoff, at maturity T, of a long position in $V[\mathbf{t}]$. Thus, $V_{\text{inf}} = \inf_{\mathbf{t}} V_{\text{inf}}[\mathbf{t}]$.

3. VARIANCE SWAPS AND V-CONVEX FUNCTIONS

We introduce the class of V-convex functions and use them to derive lower bounds on V_{inf}. The definition and many properties of V-convex functions in this section and in Section 4 are inspired from those of convex functions.

DEFINITION 3.1. A real-valued function g defined on a subinterval J of $(0, \infty)$ is V-convex if, for $x, y, z \in J$ with $x < z < y$,

$$\frac{g(x) + \ln^2(x/z) - g(z)}{x - z} \leq \frac{g(y) + \ln^2(y/z) - g(z)}{y - z}. \tag{3.1}$$

LEMMA 3.2. *Let g be a real-valued function on an open sub-interval J of $(0, \infty)$. The following conditions are equivalent:*

1. *g is V-convex.*
2. *g is continuous, has a right derivative g'_+ and, for $x, y \in J$ with $x < y$,*

$$g'_+(x)(y - x) \leq g(y) + \ln^2(y/x) - g(x). \tag{3.2}$$

3. *For $x > 0$ and $y \in J$,*

$$\delta_{x,y;g} = \inf_{v \in J, v > y} \frac{g(v) + \ln^2(v/x) - g(y)}{v - y}$$

is finite and, for $z \in J$,

$$g(y) - g(z) \leq \ln^2(z/x) + \delta_{x,y;g}(y - z). \tag{3.3}$$

Proof. See the Appendix. □

Note that if we omit the $\ln^2$ terms in Definition 3.1 and Lemma 3.2, we obtain classical properties of convex functions.

REMARK 3.3. *The proof of Lemma 3.2 shows that Condition 2 can be replaced by the following one: g is continuous and, for $x \in J$ except for a finite number of points, $g'_+(x)$ exists and equation (3.2) holds for $y \in J$ with $x < y$.*

The following variant of results proven in Breeden and Litzenberger (1978), Carr and Madan (1998), Demeterfi et al. (1999), shows how to approximately super-replicate a European derivative meeting certain conditions, with standard call options. The idea behind the proof is to super-replicate the derivative payoff with a piece-wise affine function, and to observe that such a function is, up to a constant, equal to the payoff of a portfolio of calls.

LEMMA 3.4 (Breeden Litzenberger). *Consider a Lipschitz function f on $[0, \infty)$. Then $\int_{(0,\infty)} |f|\, d\mu < \infty$ and, for any $\epsilon > 0$, a European option that pays $f(S_T)$ at maturity T can be super-replicated by a portfolio of a zero-coupon bond and of call options maturing at T, with forward price at most $\epsilon + \int_{(0,\infty)} f\, d\mu$.*

Proof. Let $k > 0$ be such that f is k-Lipschitz. Since $|f(z)| \leq |f(0)| + kz$, the integral $\int_{(0,\infty)} |f|\, d\mu$ is finite. Consider a strike $K > 0$ such that $C(K) < \epsilon/k$ and let $m = \lceil kK/\epsilon \rceil$. Define f_m as the unique function on $[0, \infty)$ that coincides with f at the points jK/m, $0 \leq j \leq m$, is affine on the intervals $[jK/m, (j+1)K/m]$, $0 \leq j \leq m-1$, and on $[K, \infty)$, with slope k on the latter interval. Then

$$f_m(z) - \epsilon - 2k \max(0, z - K) \leq f(z) \leq f_m(z) + \epsilon \tag{3.4}$$

for $z \geq 0$. As $f_m + \epsilon$ is piece-wise affine, it can be statically replicated by a portfolio P of a zero-coupon bond and of $m + 1$ call options (see, e.g., Demeterfi et al. 1999) with maturity T. Using equation (3.4), we conclude that the forward price $\epsilon + \int_{(0,\infty)} f_m \, d\mu$ of P is at most $4\epsilon + \int_{(0,\infty)} f \, d\mu$. Replacing ϵ with $\epsilon/4$ finishes the proof. □

THEOREM 3.5. *Let g be a Lipschitz function on* $[0, \infty)$ *which is V-convex on* $(0, \infty)$ *and such that $g(F_0) = 0$. Then $\int_{(0,\infty)} |g| \, d\mu < \infty$ and $V_{\inf} \geq -T^{-1} \int_{(0,\infty)} g \, d\mu$.*

Proof. Consider a subdivision of $[0, T]$ with time-steps $0 = t_0 < \cdots < t_{n+1} = T$. For $0 \leq i \leq n$, let $Z_i = \exp(-\int_{t_{i+1}}^{T} r(u) \, du)$. By Condition 3, $\xi_i = \delta_{S_i/Z_i, F_i; g}$ is finite. By equation (3.3) and the relation $Z_i F_{i+1} = S_{i+1}$,

$$g(F_i) - g(F_{i+1}) \leq \ln^2(S_{i+1}/S_i) + \xi_i(F_i - F_{i+1}), \tag{3.5}$$

and so

$$0 \leq g(S_T) + T \, V[\mathbf{t}] + \sum_{i=0}^{n} \xi_i(F_i - F_{i+1}). \tag{3.6}$$

Furthermore, the value of ξ_i is known at time t_i. Fix $\epsilon > 0$. By Lemma 3.4, there is a portfolio P of a zero-coupon bond and of call options maturing at T, with forward price at most $\epsilon + \int_{(0,\infty)} g \, d\mu$, that super-replicates g. A long position in the realized variance $V[\mathbf{t}]$ can be hedged by taking a long position in $T^{-1} P$ at time 0, together with a short position of $T^{-1}\xi_i$ stocks at time t_i, and unwinding the latter position at time t_{i+1}. By equation (3.6), the payoff at time T of the position in $V[\mathbf{t}]$, combined with the hedges, is lower bounded by $-T^{-1}(\epsilon + \int_{(0,\infty)} g \, d\mu)$. Thus, $V_{\inf} \geq -T^{-1} \int_{(0,\infty)} g \, d\mu$. □

EXAMPLE 3.6. Given $a \in (0, F_0]$, the function $g_a(x) = -\ln^2(x/a) 1_{x \geq a}$ is Lipschitz on $[0, \infty)$ and, by Condition 2 of Lemma 3.2, is V-convex on $(0, \infty)$. We conclude that $V_{\inf} \geq T^{-1}(\int_{(a,\infty)} \ln^2(x/a) \, d\mu(x) - \ln^2(F_0/a))$.

REMARK 3.7. It is easy to show that $\xi_i = g'_+(S_i)$ if interest rates are null.

4. A LOWER BOUND VIA JUMP PROCESSES

Throughout the rest of the paper, let $a = F_0$, $b = \max\{x \geq 0 : P(x) = 0\}$ and $I = \{y \geq a : C'_-(y) < 0\}$. For $0 < x \leq y$ and any function g with a right derivative at x, define

$$\hat{g}(x, y) = g(x) + g'_+(x)(y - x) - \ln^2(y/x).$$

This section establishes a lower bound on $V_{\inf}$, which will be shown to be optimal in Section 5 if $V_{\inf}$ is finite. The main idea behind our approach is to obtain the best lower bound on $V_{\inf}$ achievable through Theorem 3.5. This is a convex optimization problem because the set of V-convex functions is convex.

Assume first that g is known on $[0, a]$. We want to find the smallest V-convex extension (if it exists) of g to (a, ∞). As $\hat{g}(x, y) \leq g(y)$ for $0 < x < a < y$ by equation (3.2), a natural such extension for g is given by equation (4.1) below. Lemma 4.1 shows that, under certain assumptions on g, the proposed extension of g to (a, ∞) satisfies the conditions of Theorem 3.5.

LEMMA 4.1. *Let g be a V-convex and Lipschitz function on* $[0, a]$ *which is affine on* $[0, b']$, *for some $b' \in (0, a)$, with $g(a) = 0$. Define*

$$g(y) = \sup_{x \in (0,a)} \hat{g}(x, y) \tag{4.1}$$

for $y > a$. Then g is Lipschitz and V-convex on $[0, \infty)$.

Proof. As g is V-convex and Lipschitz on $[0, a]$, it follows from equation (3.2) that equation (4.1) holds for $y = a$. Thus, for $y \geq a$,

$$g(y) = \sup_{x \in (b'/2, a)} \hat{g}(x, y).$$

This implies that the restriction of g to $[a, \infty)$ is Lipschitz, since it is the supremum of functions with uniformly bounded derivatives. Thus, g is Lipschitz on $[0, \infty)$. We show that g is V-convex on $(0, \infty)$ by applying Remark 3.3. As g is V-convex on $[0, a]$, equation (3.2) holds for $0 < x < y \leq a$. By equation (4.1), it follows that equation (3.2) holds for $0 < x < a$ and $x < y$. On the other hand, Example 3.6 shows that the function $y \mapsto -\ln^2(y/x)$ is V-convex on $[a, \infty)$ if $0 < x \leq a$. As the set of V-convex functions on I is invariant by adding an affine function and by taking the supremum, it follows from equation (4.1) that g is V-convex on (a, ∞). Thus, equation (3.2) holds for $a < x < y$. We have thus shown that equation (3.2) holds for $0 < x < y$ with $x \neq a$. By Remark 3.3, it follows that g is V-convex on $[0, \infty)$. □

Consider now a decreasing function ϕ from $(b, a]$ to $[a, \infty)$ such that I is the interval spanned by $\phi((b, a])$. Let

$$g(x) = 2 \int_{(x,a)} (x - u) \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)}\, du, \tag{4.2}$$

for $b \leq x \leq a$, and

$$g(y) = \sup_{x \in (b,a)} \hat{g}(x, y), \tag{4.3}$$

for $y \in I - \{a\}$.

LEMMA 4.2. *The function g is finite and V-convex on $[b, a) \cup I$ and, for $x \in (b, a]$,*

$$g(\phi(x)) = \hat{g}(x, \phi(x)). \tag{4.4}$$

For $y \in I$,

$$-\ln^2(y/a) \leq g(y). \tag{4.5}$$

If $\int_{[b,a)\cup I} |g|\, d\mu < \infty$ then $V_{\inf} \geq -T^{-1} \int_{[b,a)\cup I} g\, d\mu$.

Proof. See the Appendix. □

To motivate equation (4.2), assume that a function g has a continuous second derivative on (b, a), is V-convex on $[b, a) \cup I$ and satisfies equation (4.4). Thus, the function $u \mapsto \hat{g}(u, \phi(x))$ attains its maximum at x. Taking the derivative with respect to u, it follows that $g''(x) = 2\ln(\phi(x)/x)/(x(x - \phi(x))$, which can be solved via equation (4.2). We now justify equation (4.4) by assuming that $b > 0$ and interest rates are null, and that a function g satisfies equation (4.4) and the conditions in Theorem 3.5. The assumptions made in this paragraph are for intuition and are not used elsewhere in the paper. Let $x_i = a(b/a)^{i/(n+1)}$. By Remark 3.7, if F_i follows a jump process such that either $F_{i+1} = F_i$ or $F_i = x_i$ and $F_{i+1} \in \{x_{i+1}, \phi(x_i)\}$, the difference between the right- and left-hand sides of equation (3.5) is at most $\ln^2(x_{i+1}/x_i)$. Thus, the right-hand side of equation (3.6) is bounded by a constant divided by n and $V_{\inf}$ equals $-T^{-1} \int_{[b,a)\cup I} g\, d\mu$. Hence, g would provide the best lower bound on $V_{\inf}$.

The process described above is not, in general, consistent with the undiscounted call price function C. Under martingale measures consistent with C that will be described in Section 5, the forward price follows a process similar to the above-described one for an appropriate function ϕ. The following lemma motivates the choice of ϕ.

LEMMA 4.3. *Given $i \in [0, n]$ and $0 < x < y$, assume that the forward price is a Q-martingale and that, either*

$$\begin{cases} F_i = x \quad \text{and } S_T \notin (x, y), \quad \text{or} \\ F_i \in (x, y] \quad \text{and } F_j = F_i \qquad \text{for } j \geq i, \end{cases} \tag{4.6}$$

Q almost surely. Assume that Q is consistent with C and let $C_i(K) = E_Q(\max(F_i - K, 0))$. Then

$$C_i(K) = C(K) - C(y) + \frac{P(x) - C(y)}{y - x}(K - y), \tag{4.7}$$

and

$$C'_-(y) \leq \frac{C(y) - P(x)}{y - x} \leq C'_+(y). \tag{4.8}$$

Furthermore, for any Borel subset A of $[0, \infty)$,

$$Q(F_{i+1} \in A \wedge F_i = x) = Q(F_{i+1} \in A) - Q(F_i \in A) + Q(F_i = x)\delta_x(A). \tag{4.9}$$

Proof. As F_i and S_T have the same distribution on (x, y) and $C_i(y) = 0$, for $K \in (x, y)$,

$$C_i(K) = C(K) - C(y) + \alpha(K - y),$$

where α is a constant. Since $C_i(x) = F_0 - x$, we conclude, using the put-call parity relation, that

$$\alpha = \frac{P(x) - C(y)}{y - x},$$

which yields equation (4.7). Denote by C'_{i-} (resp. C'_{i+}) the left (resp. right) derivative of C_i. The lower bound in equation (4.8) follows by observing that $C'_{i-}(y) = C'_-(y) + \alpha$. On the other hand, the relation $Q(F_i > x) \leq Q(S_T \in (x, y])$ implies that $C'_{i+}(x) \geq C'_+(x) - C'_+(y)$. Since $C'_{i+}(x) = C'_+(x) + \alpha$, this yields the upper bound in equation (4.8). Finally, equation (4.9) follows by observing that

$$\begin{aligned} Q(F_{i+1} \in A \wedge F_i = x) &= Q(F_{i+1} \in A) - Q(F_{i+1} \in A \wedge F_i \neq x) \\ &= Q(F_{i+1} \in A) - Q(F_i \in A \wedge F_i \neq x). \end{aligned}$$

□

Note that equation (4.8) implies that the line containing $(x, P(x))$ and $(y, C(y))$ is tangent at y to the function C curve. For $(x, y) \in [0, \infty)^2$, define $H(x, y) = C(y) + (x - y)C'_+(y) - P(x)$. As C is convex, the function $y \mapsto H(x, y)$ is decreasing and right-continuous on $[x, \infty)$. Lemma 4.3 motivates the following choice of ϕ, which will be used throughout the rest of the paper:

$$\phi(x) = \min\{y \geq a : H(x, y) \leq 0\}, \tag{4.10}$$

for $x \in (b, a]$. It is easy to see that equation (4.8) holds if $y = \phi(x)$. The function ϕ is decreasing on $(b, a]$ and, by the put-call parity relation, $\phi(a) = a$. Finally, set

$$G(x) = \sup_{y>a} \frac{P(x) - C(y)}{y - x}, \tag{4.11}$$

for $x \in [0, a]$.

LEMMA 4.4. *The interval I is the interval spanned by $\phi((b, a])$. The function G is increasing and continuous on* $[0, a]$. *For $x \in (b, a)$,*

$$G(x) = \frac{P(x) - C(\phi(x))}{\phi(x) - x}. \tag{4.12}$$

Furthermore, for any nonnegative Borel-measurable function γ on (a, ∞),

$$\int_{(b,a)} (\gamma \circ \phi)\, dG = \int_{(a,\infty)} \gamma\, d\mu, \tag{4.13}$$

where the integral in left-hand side is the Stieltjes integral with respect to G. For $x \in (b, a]$,

$$-C'_{+}(\phi(x)) \le G(x) \le -C'_{-}(\phi(x)). \tag{4.14}$$

Proof. See the Appendix. □

Note that, by equation (4.14), if C and ϕ are sufficiently smooth, equation (4.13) reduces to the change of variables theorem. Using equation (4.10), we conclude the section by an alternative expression for the lower bound in Lemma 4.2.

LEMMA 4.5. *The integral $\int_{[b,a)\cup I} |g|\, d\mu$ is finite and $V_{\inf}$ is lower-bounded by*

$$T^{-1} \int_{(b,a)} \ln^2(\phi(x)/x)\, dG(x) = -T^{-1} \int_{[b,a)\cup I} g\, d\mu.$$

Proof. See the Appendix. □

5. MARTINGALE MEASURES ATTAINING THE LOWER BOUND

Consider the subdivision of $[0, T]$ with time-steps $t_i = iT/(n+1)$, $0 \le i \le n+1$, and let $\Omega = \{F_0\} \times (0, \infty)^{n+1}$ be the set of possible values of the sequence $(F_0, F_1, \dots, F_{n+1})$. We construct martingale measures consistent with C and that attain, if $V_{\inf}$ is finite, the lower bound in Lemma 4.5 as n goes to infinity. Consider a strictly decreasing sequence x_i, $0 \le i \le n+1$, with $x_0 = a$ and $x_{n+1} = b$. We want F_i to be a Q-martingale such that, Q almost surely, for $0 \le i \le n$, either $F_{i+1} = F_i$, or $F_i = x_i$ and $F_{i+1} \in [x_{i+1}, x_i] \cup [\phi(x_i), \phi(x_{i+1})]$. Thus, condition (4.6) holds for $x = x_i$ and $y = \phi(x_i)$. We construct Q by reverse-engineering equation (4.7). For $0 \le i \le n$, define

$$C_i(K) = \begin{cases} F_0 - K & \text{if } 0 \le K < x_i \\ C(K) - C(\phi(x_i)) + G(x_i)(K - \phi(x_i)) & \text{if } x_i \le K < \phi(x_i) \\ 0 & \text{if } \phi(x_i) \le K, \end{cases} \tag{5.1}$$

and $C_{n+1} = C$. Using equation (4.14), it can be shown that C_i is convex and decreasing on $[0, \infty)$, with $C'_{i+}(0) = -1$. For $0 \le i \le n+1$, let μ_i be the probability measure induced by C_i. In other words, μ_i is the unique probability on $[0, \infty)$ such that $C_i(K) = \int_{(K,\infty)}(z - K)\, d\mu_i(z)$ for $K \ge 0$. Motivated by equation (4.9), define the signed measure

$$\nu_i = \delta_{x_i} + \frac{1}{\mu_i(\{x_i\})}(\mu_{i+1} - \mu_i), \tag{5.2}$$

for $0 \le i \le n$. Using equation (4.14) once again, it can be easily verified that $\mu_i(\{x_i\}) > 0$ and $\nu_i(A) \ge 0$ for any Borel subset A of $[0, \infty)$, and so ν_i is a probability measure on $[0, \infty)$. Let Q be the unique probability on Ω such that (F_i), $0 \le i \le n+1$, is a Q-Markov chain with initial probability δ_{F_0} and transition probabilities π_i, $0 \le i \le n$, defined as follows:

$$\pi_i(z) = \begin{cases} \nu_i & \text{if } z = x_i \\ \delta_z & \text{otherwise.} \end{cases} \tag{5.3}$$

LEMMA 5.1. *For $0 \le i \le n+1$ and any Borel subset A of $[0, \infty)$,*

$$Q(F_i \in A) = \mu_i(A). \tag{5.4}$$

Furthermore, (F_i), $0 \le i \le n+1$, is a Q-martingale.

Proof. We show equation (5.4) by induction on i. The base case clearly holds. If equation (5.4) is true for i then $Q(F_i = x_i) = \mu_i(\{x_i\})$ and, for any Borel subset A of $[0, \infty)$,

$$\begin{aligned} Q(F_{i+1} \in A) &= Q(F_i = x_i)\nu_i(A) + Q(F_i \ne x_i \wedge F_{i+1} \in A) \\ &= \mu_i(\{x_i\})\nu_i(A) + Q(F_i \ne x_i \wedge F_i \in A) \\ &= \mu_{i+1}(A) - \mu_i(A - \{x_i\}) + \mu_i(A - \{x_i\}) \\ &= \mu_{i+1}(A). \end{aligned}$$

Thus, equation (5.4) holds for $i + 1$.

On the other hand, by equations (5.3) and (5.4),

$$\begin{aligned} E_Q((F_{i+1} - F_i)1_{F_i = x_i}) &= E_Q(F_{i+1} - F_i) \\ &= C_{i+1}(0) - C_i(0) \\ &= 0. \end{aligned}$$

This shows that F_i is a Q-martingale. □

LEMMA 5.2. *For $0 \le i \le n+1$ and any nonnegative Borel-measurable function γ on (a, ∞),*

$$\int_{(x_i,a)} (\gamma \circ \phi)\, dG = \int_{(a,\infty)} \gamma\, d\mu_i. \tag{5.5}$$

Proof. See the Appendix. □

THEOREM 5.3. *If*

$$\int_{(0,\infty)} \ln^2\left(\frac{x}{S_0}\right) d\mu(x) = \infty \tag{5.6}$$

then $V_{\inf} = \infty$. Otherwise, $V_{\inf}$ is finite and equal to

$$-T^{-1}\int_{[b,a)\cup I} g\, d\mu = T^{-1}\int_{(b,a)} \ln^2\left(\frac{\phi(x)}{x}\right) dG(x). \tag{5.7}$$

Proof. Assume first that equation (5.6) holds. For $\epsilon > 0$, let

$$f_\epsilon(z) = \begin{cases} 0 & \text{if } z \in [0,\epsilon) \\ \ln^2(z/S_0) & \text{if } z \in [\epsilon,\infty). \end{cases}$$

By Lemma 3.4, a European option that pays $f(S_{n+1})$ at maturity T can be subreplicated by a portfolio of a zero-coupon bond and of call options maturing at T with forward price lower bounded by $\int_{(0,\infty)} f_\epsilon\, d\mu - \epsilon$. As, by the Cauchy-Schwartz inequality,

$$\ln^2 \frac{S_{n+1}}{S_0} \le (n+1)\sum_{i=0}^{n} \ln^2 \frac{S_{i+1}}{S_i},$$

we conclude that $T(n+1)V_{\inf}[\mathbf{t}] \ge \int_{(0,\infty)} f_\epsilon\, d\mu - \epsilon$. Letting ϵ go to 0 and using the monotone convergence theorem implies that $V_{\inf} = \infty$.

Assume now that $\int_{(0,\infty)} \ln^2(x/S_0)\, d\mu(x)$ is finite. By Lemma 4.5, the two sides of equation (5.7) are finite, equal, and are a lower bound to $V_{\inf}$. We show that this lower bound is optimal using the probability Q. By equation (5.3), for $0 \le i \le n$,

$$\begin{aligned} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) &= Q(F_i = x_i) E_Q\left(\ln^2 \frac{F_{i+1}}{x_i} | F_i = x_i\right) \\ &= \mu_i(\{x_i\}) \int_{(0,\infty)} \ln^2 \frac{z}{x_i}\, d\nu_i(z). \end{aligned}$$

As ν_i is null on $(0, x_{i+1}) \cup (x_i, a]$, we conclude, using Lemma 5.2, that

$$\begin{aligned} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) &\le \ln^2 \frac{x_{i+1}}{x_i} + \mu_i(\{x_i\}) \int_{(a,\infty)} \ln^2 \frac{z}{x_i}\, d\nu_i(z) \\ &= \ln^2 \frac{x_{i+1}}{x_i} + \int_{(x_{i+1},x_i]} \ln^2 \frac{\phi(x)}{x_i}\, dG(x), \end{aligned}$$

for $0 \le i \le n-1$, and

$$E_Q\left(\ln^2 \frac{F_{n+1}}{F_n}\right) = \int_{(0,x_n)} \ln^2 \frac{z}{x_n}\, d\mu(z) + \int_{(b,x_n]} \ln^2 \frac{\phi(x)}{x_n}\, dG(x).$$

We now set $x_i = a(1/n + (1-1/n)b/a)^{i/n}$ for $0 \le i \le n$. It follows that

$$\sum_{i=0}^{n} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) \le \frac{\alpha}{n} + \int_{(0,x_n)} \ln^2 \frac{z}{x_n}\, d\mu(z) + \int_{(b,a)} \ln^2 \frac{\phi(x)}{x}\, dG(x),$$

where α is a constant independent of n. Since $\int_{(0,\infty)} \ln^2(x/S_0)\, d\mu(x)$ is finite, so is $\int_{(0,\infty)} \ln^2(x/a)\, d\mu(x)$. By Lebesgue's dominated convergence theorem, we conclude that

$$\limsup_{n\to\infty} \sum_{i=0}^{n} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) \leq \int_{(b,a)} \ln^2 \frac{\phi(x)}{x}\, dG(x). \tag{5.8}$$

As $\ln(S_{i+1}/S_i) \leq \kappa T/(n+1) + \ln(F_{i+1}/F_i)$, it follows from the Cauchy-Schwarz inequality that

$$\sum_{i=0}^{n} \ln^2 \frac{S_{i+1}}{S_i} \leq \frac{(\kappa T)^2}{n} + \sum_{i=0}^{n} \ln^2 \frac{F_{i+1}}{F_i} + \frac{2\kappa T}{\sqrt{n}} \sqrt{\sum_{i=0}^{n} \ln^2 \frac{F_{i+1}}{F_i}}. \tag{5.9}$$

Combining equations (5.8) and (5.9) and using Jensen's inequality shows that

$$\limsup_{n\to\infty} \sum_{i=0}^{n} E_Q\left(\ln^2 \frac{S_{i+1}}{S_i}\right) \leq \int_{(b,a)} \ln^2 \frac{\phi(x)}{x}\, dG(x).$$

As Q is risk-neutral, this implies that $V_{\inf} \leq T^{-1} \int_{(b,a)} \ln^2(\phi(x)/x)\, dG(x)$. □

REMARK 5.4. It follows from equations (4.2) and (4.5) that $g(z) \geq g_{\log}(z)$ for $z \in [b,a) \cup I$, where $g_{\log}(z) = 2(\ln(z/a) - z/a + 1)$. Hence, when $V_{\inf}$ is finite, $V_{\inf} \leq V_{\log}$ since $V_{\log} = -T^{-1} \int_{[b,a)\cup I} g_{\log}\, d\mu$.

REMARK 5.5. For $y \in I$, let $\psi(y)$ be the unique solution to the equation $C(y) + (x - y)C'_-(y) = P(x)$. It is easy to show that $\psi(y) \in (b,a]$. Furthermore,

$$\phi(u) \leq y \text{ for } u \in (\psi(y), a] \text{ and } \phi(u) \geq y \text{ for } u \in (b, \psi(y)).$$

By a proof similar to that of equation (4.4), we conclude that $g(y) = \hat{g}(\psi(y), y)$.

REMARK 5.6. Assume that $V_{\inf}$ is finite. Using the proofs and notation of Theorem 3.5 and of Lemma 4.2, it follows that, for any $\epsilon > 0$, there is $b' > b$ such that an investor with a long position in the realized variance of S can guarantee a payoff greater than $V_{\inf} - \epsilon$ at T by buying or selling at time 0 a finite number of calls maturing at T and dynamically maintaining a short position of $T^{-1}\delta_{S_i/Z_i, F_i; g_{b'}}$ stocks at t_i.

THEOREM 5.7. *Assume that the function C is differentiable on (a,∞) and that $V_{\inf}$ is finite. Then,*

$$V_{\inf} = T^{-1} \int_I \ln^2 \frac{y}{\psi(y)}\, d\mu(y). \tag{5.10}$$

Proof. Consider $x \in (b,a)$. Equation (4.10) shows that $H(x, \phi(x)) \leq 0$ and that $H(x,y) > 0$ for $y \in (a, \phi(x))$. As C is convex and differentiable on (a,∞), C' is continuous on (a,∞). We conclude that $H(x,\phi(x)) = 0$. Equivalently, $\psi(\phi(x)) = x$ for $x \in (b,a)$. By applying equation (4.13) to the function γ defined on (a,∞) with

$$\gamma(y) = \begin{cases} \ln^2(y/\psi(y)) & \text{if } y \in I - \{a\} \\ 0 & \text{otherwise,} \end{cases}$$

TABLE 6.1
$\sqrt{V_{\text{inf}}}$ and Its Approximation Obtained via Theorems 5.7 and 6.1 When $T = 0.25$

σ	10%	15%	20%	25%	30%	35%
$\sqrt{V_{\text{inf}}}$	9.782%	14.510%	19.129%	23.641%	28.044%	32.340%
$\sigma - \frac{c}{2}\sigma^2\sqrt{T}$	9.782%	14.509%	19.128%	23.637%	28.038%	32.329%

it follows that

$$\int_{(b,a)} \ln^2 \frac{\phi(x)}{x}\, dG(x) = \int_{I-\{a\}} \ln^2 \frac{y}{\psi(y)}\, d\mu(y).$$

We conclude the proof by noting that $\psi(a) = a$ and using Theorem 5.3. □

Equation (5.10) has a simple intuitive interpretation: when n goes to infinity, the forward price follows a process where it may jump from $\psi(y)$ to y.

6. EXAMPLES

We study an example with constant implied volatilities and two examples with a volatility skew.

6.1. A Constant Implied Volatilities Example

This subsection assumes that the implied volatilities for maturity T and all strikes are equal to a constant σ. For $u \geq 0$, denote by $\theta(u)$ the unique solution to the equation $N'(u) - N'(\theta) = \theta(N(\theta) + N(-u))$ in the interval $[-u, 0]$.

THEOREM 6.1. *If, for any strike, the call price for maturity T is equal to the Black-Scholes price with volatility σ, then $V_{\text{inf}} = \sigma^2 - c\sigma^3\sqrt{T} + O(\sigma^4)$ as $\sigma \to 0$, where*

$$c = \frac{1}{2}\int_0^\infty u(u - \theta(u))^2 N'(u)\, du \approx 0.8721.$$

Proof. See the Appendix. □

It follows from Theorem 6.1 that $\sqrt{V_{\text{inf}}} = \sigma - c\sigma^2\sqrt{T}/2 + O(\sigma^3)$ as $\sigma \to 0$. Table 6.1 shows values of $\sqrt{V_{\text{inf}}}$ calculated numerically via Theorem 5.7 when $T = 0.25$ and compares them with the approximation $\sigma - c\sigma^2\sqrt{T}/2$.

6.2. A Jump-Diffusion Example

This subsection assumes that call prices for maturity T and all strikes are given by the Merton jump-diffusion model introduced in Merton (1976). The risk-neutral process for the model is:

$$\frac{dS(t)}{S(t-)} = (r - \lambda m)dt + \sigma dW(t) + dJ(t), \tag{6.1}$$

TABLE 6.2
The Merton Jump-Diffusion Model Parameters and Corresponding Values of $\sqrt{K_{var}^*}$ and $\sqrt{V_{\log}}$

σ	λ	β	γ	$\sqrt{K_{var}^*}$	$\sqrt{V_{\log}}$
20%	0.1	−1	0.5	40.620%	35.124%

TABLE 6.3
$\sqrt{V_{\inf}}$ for Different Maturities When Call Prices Are Calculated from the Merton Jump-Diffusion Model

T	0.08333	0.16667	0.25	0.5
$\sqrt{V_{\inf}}$	33.173%	32.728%	32.388%	31.619%

where W is a Brownian motion, $J(t) = \sum_{j=1}^{N(t)}(Y_j - 1)$, and $N(t)$ is a Poisson process with rate λ. If a jump occurs at time τ_j, then $S(\tau_j+) = S(\tau_j-)Y_j$, where $\ln(Y_j)$ is a Gaussian random variable with mean β and standard deviation γ. The model parameters satisfy the equation: $m + 1 = e^{\beta+\gamma^2/2}$. We assume that W, N, and the Y_j's are independent. The fair variance strike under Merton's model of a discrete variance swap monitored at $t_i = iT/(n+1)$, $1 \le i \le n+1$, is equal to $K_{var}^* + O(1/n)$ (see Broadie and Jain 2008), where

$$K_{var}^* = \sigma^2 + \lambda(\beta^2 + \gamma^2). \tag{6.2}$$

On the other hand, it follows from Broadie and Jain (2008, equation A.4) that the forward price of a log-contract under Merton's model is

$$V_{\log} = \sigma^2 + 2\lambda(m - \beta). \tag{6.3}$$

Undiscounted call prices under Merton's model are given by (see, e.g., Glasserman 2004)

$$C(K) = e^{rT}\sum_{i=0}^{\infty} e^{-\lambda' T}\frac{(\lambda' T)^i}{i!} BS(S_0, \sigma_i, T, r_i, K), \tag{6.4}$$

where $\lambda' = \lambda(1+m)$, $\sigma_i^2 = \sigma^2 + \gamma^2 i/T$, $r_i = r - \lambda m + i\log(1+m)/T$, and $BS(\cdot)$ is the Black-Scholes call option formula with initial spot S_0, volatility σ_i, maturity T, interest rate r_i, and strike K. Theorem 5.7 and equation (6.4) imply that $V_{\inf}$ depends only on λ, σ, β, γ, and T. Table 6.2 gives values for λ, σ, β, and γ close to the ones calculated in Andersen and Andreasen (2000), by calibrating the Merton jump diffusion model to market option prices on the S&P 500 index.

Corresponding values of $\sqrt{K_{var}^*}$ and $\sqrt{V_{\log}}$ are calculated from equations (6.2) and (6.3), respectively. Table 6.3 shows values of $\sqrt{V_{\inf}}$ calculated numerically via Theorem 5.7 and equation (6.4) for different maturities using the parameters in Table 6.2.

We note that the calculation of $V_{\inf}$ assumes the knowledge of all call prices for maturity T, which are given by Merton's model in this example, but does not make the assumption that the stock price actually follows Merton's model, or any other particular model. As the fair variance strike under Merton's model of a discrete variance swap monitored at $t_i = iT/(n+1)$, $1 \le i \le n+1$, is equal to $K^*_{var} + O(1/n)$, it follows that $V_{\inf}[\mathbf{t}] \le K^*_{var} + O(1/n)$, and so $V_{\inf} \le K^*_{var}$ for any given values of Merton's model parameters. Furthermore, by Remark 5.4, $V_{\inf} \le V_{\log}$. The sign of $K^*_{var} - V_{\log}$ depends however (see Broadie and Jain 2008) on the model parameters.

6.3. A Discrete Set of Strikes Example

In practice, European call (or, equivalently, put) options prices with a given maturity T are known only for a finite set of positive strikes $K_1 < \cdots < K_l$. Demeterfi et al. (1999) have calculated $V_{\log}$ in this case by approximating the function $g_{\log}$ with a piecewise affine function. This is computationally equivalent to extending the call prices to all strikes via a piecewise affine interpolation. We use a similar approach to calculate $V_{\inf}$. Assume that the sequence $C(0), C(K_1), \ldots, C(K_l)$ is strictly convex and strictly decreasing with respect to the strike, where $C(0) = F_0$, and that $C(K_1) \ge F_0 - K_1$. By the necessary and sufficient conditions for the absence of arbitrage among call prices established by Davis and Hobson (2007), it can be shown that there are two strikes $0 < K_0 < K_1$ and $K_{l+1} > K_l$ such that the following extension of C to all strikes is arbitrage-free: $C(K) = F_0 - K$ for $K \le K_0$, $C(K) = 0$ for $K \ge K_{l+1}$, and

$$C(K) = C(K_i) + \frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i}(K - K_i),$$

for $K_i < K < K_{i+1}$. Furthermore, the support of μ is included in the set $\{K_0, \ldots, K_{l+1}\}$, with

$$\mu(K_0) = 1 - \frac{C(K_1) - C(K_0)}{K_1 - K_0},$$

$$\mu(K_i) = \frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i} - \frac{C(K_i) - C(K_{i-1})}{K_i - K_{i-1}},$$

for $1 \le i \le l$, and

$$\mu(K_{l+1}) = \frac{C(K_l)}{K_{l+1} - K_l}.$$

We now apply Theorem 5.3 to calculate $V_{\inf}$. The function ϕ can be calculated by inspection via equation (4.10), which implies that $\mathrm{Im}(\phi) \subseteq \{F_0\} \cup \mathrm{Supp}(\mu)$. On the other hand, equation (4.13) shows that, for $y \in \mathrm{Supp}(\mu) \cap (F_0, \infty)$, the set $\phi^{-1}(\{y\})$ is a subinterval of $(K_0, F_0]$ with a nonempty interior. Thus, an element x of $\phi^{-1}(\{y\})$ can be found using the bisection method. We can then calculate $g(y)$ via equation (4.4).

In our numerical example in Table 6.4, we assume that $K_i = 35 + 5i$ for $1 \le i \le l$, with $l = 22$. We chose $K_0 = 35$ and $K_{l+1} = 150$. The function g has been calculated on $\mathrm{Supp}(\mu) = \{K_0, \ldots, K_{l+1}\}$ via equations (4.2) and (4.3), and G has been evaluated via equation (4.12). The integrals in equations (4.2), (4.3), and (5.7) have been calculated using the midpoint method with 100, 000 points. The difference between the right- and left-hand sides of equation (5.7) is of order 10^{-7}. Equation (5.7) shows that $V_{\inf} = (24.263\%)^2$.

TABLE 6.4
The Spot Price Is \$100, Implied Volatilities Are Given for European Call Options with Maturity $T = 0.25$ and Strikes Ranging from \$40 to \$145, Spaced \$5 Apart. There Are No Dividends and the Continuously Compounded Risk-free Interest Rate with Maturity T is 2%

Strike	Implied Volatility	Call Price	μ	g	$g_{\log}$
35			0.000000	−0.5770	−0.8062
40	37%	60.199502	0.000002	−0.4725	−0.6386
45	36%	55.224448	0.000015	−0.3832	−0.5025
50	35%	50.249468	0.000088	−0.3066	−0.3913
55	34%	45.274923	0.000394	−0.2411	−0.3002
60	33%	40.302340	0.001411	−0.1855	−0.2257
65	32%	35.336777	0.004169	−0.1387	−0.1651
70	31%	30.391954	0.010436	−0.0998	−0.1164
75	30%	25.499053	0.022568	−0.0681	−0.0779
80	29%	20.718432	0.042685	−0.0431	−0.0483
85	28%	16.150173	0.071128	−0.0242	−0.0266
90	27%	11.935778	0.104687	−0.0109	−0.0117
95	26%	8.242208	0.135842	−0.0030	−0.0031
100	25%	5.224458	0.154436	0.0000	0.0000
105	24%	2.975040	0.152160	−0.0019	−0.0019
110	23%	1.482628	0.127838	−0.0081	−0.0084
115	22%	0.626216	0.089568	−0.0179	−0.0190
120	21%	0.215409	0.050812	−0.0304	−0.0334
125	20%	0.057392	0.022458	−0.0447	−0.0512
130	19%	0.011107	0.007359	−0.0597	−0.0723
135	18%	0.001437	0.001677	−0.0743	−0.0963
140	17%	0.000111	0.000245	−0.0865	−0.1231
145	16%	0.000004	0.000021	−0.0947	−0.1524
150			0.000001	−0.0970	−0.1841

TABLE 6.5
Variance Rates in the Presence of Skew. Strikes Range in the Interval [40, 200]

ΔK	0.1	1	5
$V_{\inf}$	$(23.955\%)^2$	$(23.967\%)^2$	$(24.263\%)^2$
$V_{\log}$	$(25.267\%)^2$	$(25.280\%)^2$	$(25.608\%)^2$

In comparison, we have calculated $V_{\log}$ via the equation $V_{\log} = \sum_{i=0}^{l+1} \mu(K_i) g_{\log}(K_i)$ and have found that $V_{\log} = (25.608\%)^2$.

Table 6.5 compares the values of $V_{\inf}$ and of $V_{\log}$ using the same implied volatility function $\sigma_{\mathrm{imp}}(K) = 0.45 - 0.002K$ as in Table 6.4 and different values for the strikes spacing. We do the same comparison in Table 6.6 using a constant implied volatility function $\sigma_{\mathrm{imp}}(K) = 0.25$. In the limit case $\Delta K = 0$, $V_{\inf}$ has been extracted from Table 6.1.

TABLE 6.6
Variance Rates with Flat Volatility. Strikes Range in the Interval [40, 200]

ΔK	0	0.1	1	5
V_{inf}	$(23.641\%)^2$	$(23.641\%)^2$	$(23.653\%)^2$	$(23.951\%)^2$
$V_{\log}$	$(25.000\%)^2$	$(25.000\%)^2$	$(25.014\%)^2$	$(25.344\%)^2$

7. CONCLUDING REMARKS

Alternative semiexplicit expressions for the optimal lower bound V_{inf} on the hedged payoff of a long position in the realized variance have been given. We have assumed that call prices on S with maturity T are known for all strikes but have not made any continuity assumptions on S. When call prices are known for a finite number of strikes, we need to choose an interpolation/extrapolation scheme to extend call prices for all strikes in order to calculate V_{inf}. Such a choice is not required by Davis et al. (2014) who, on the other hand, assume the stock price to be continuous.

When V_{inf} is finite, for any $\epsilon > 0$, an explicit hedging scheme guarantees that a long position in the floating leg of a variance swap pays at least $V_{\text{inf}} - \epsilon$ at T. Furthermore, under an explicitly constructed martingale measure consistent with the call prices, the expected realized variance is at most $V_{\text{inf}} + \epsilon$.

Numerical values and an approximation for V_{inf} have been given when the implied volatilities for maturity T are constant. We have also treated a numerical example based on the Merton jump diffusion model and another one with a discrete set of strikes and a volatility skew. Quite surprisingly, even though the models we have used to attain V_{inf} are jump processes, the values of V_{inf} in our examples are close to the fair variance strikes obtained under the continuity assumption.

APPENDIX

Proof of Lemma 3.2. $1 \Rightarrow 2$. Assume that g is V-convex. As the function $u \mapsto \ln^2 u + 4\ln u$ is concave on an open interval containing 1,

$$\frac{\ln^2(y/z) + 4\ln(y/z)}{y - z} \leq \frac{\ln^2(x/z) + 4\ln(x/z)}{x - z} \tag{A.1}$$

for $0 < (1+\epsilon)^{-1}z < x < z < y < (1+\epsilon)z$, where ϵ is a sufficiently small positive constant. Combining equations (3.1) and (A.1) shows that, for any $z \in J$, the function $g(v) - 4\ln v$ is convex on the interval $(z, (1+\epsilon)z) \cap J$. Consequently, g is continuous and has a right derivative. Consider now $x, y \in J$, with $x < y$. Equation (3.2) follows by taking limits as $z \to x^+$ in equation (3.1).

$2 \Rightarrow 3$. Assume that 2. holds. We first show that, for $x, y, z \in J$ with $x < z < y$,

$$\frac{g(z) - g(x)}{z - x} \leq \frac{g(y) - g(z)}{y - z} + \frac{\ln^2(y/x)}{y - x}. \tag{A.2}$$

Fix $x, y \in J$, with $x < y$. For $z \in [x, y]$, let

$$h(z) = g(z) - g(x) - \frac{g(y) - g(x)}{y - x}(z - x) + \frac{\ln^2(y/x)}{(y - x)^2}(z - y)(z - x).$$

As the map $u \mapsto (y - u)^{-1}\ln(y/u)$ is decreasing on $(0, y)$,

$$\begin{aligned} h(z) + h'_+(z)(y - z) &= g(z) + g'_+(z)(y - z) - g(y) - \frac{\ln^2(y/x)}{(y - x)^2}(y - z)^2 \\ &\le \ln^2(y/z) - \frac{\ln^2(y/x)}{(y - x)^2}(y - z)^2 \\ &\le 0. \end{aligned} \tag{A.3}$$

Let v be the first point in the interval $[x, y]$ where h attains its maximum. If $h(v) > 0$ then h is positive on an interval $[u, v]$, where $u \in (x, v)$. By equation (A.3), the function h is decreasing on $[u, v]$, leading to a contradiction. We conclude that $h(z) \le 0$ for $x \le z \le y$ which is, after some simplifications, equivalent to equation (A.2).

Now, for $x > 0$ and $u, y, v \in J$ with $u < y < v$, the Cauchy-Schwarz inequality

$$\left(\sqrt{v - y}\frac{\ln(v/x)}{\sqrt{v - y}} + \sqrt{y - u}\frac{\ln(x/u)}{\sqrt{y - u}}\right)^2 \le (v - u)\left(\frac{\ln^2(v/x)}{v - y} + \frac{\ln^2(x/u)}{y - u}\right)$$

can be rewritten as

$$\frac{\ln^2(v/u)}{v - u} \le \frac{\ln^2(v/x)}{v - y} + \frac{\ln^2(u/x)}{y - u}. \tag{A.4}$$

Combining equation (A.2), applied to (u, y, v), and equation (A.4) yields

$$\frac{g(y) - g(u) - \ln^2(u/x)}{y - u} \le \frac{g(v) - g(y) + \ln^2(v/x)}{v - y}.$$

This implies that $\delta_{x,y;g}$ is finite and that equation (3.3) holds.

$3 \Rightarrow 1$. Assume that 3. holds. Consider $x, z, y \in J$ with $x < z < y$. It follows from equation (3.3) that

$$\frac{g(x) + \ln^2(x/z) - g(z)}{x - z} \le \delta_{z,z;g}.$$

As

$$\delta_{z,z;g} \le \frac{g(y) + \ln^2(y/z) - g(z)}{y - z},$$

we conclude that equation (3.1) holds. □

Proof of Lemma 4.2. We first observe that $\phi(a) = a$. Equation (4.2) shows that g is continuous on $[b, a]$ and that, for $x \in (b, a)$,

$$g'(x) = 2\int_{(x,a)} \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)}du. \tag{A.5}$$

It follows from equations (4.2) and (A.5) that, for $b < x < z < a$ and $y \geq z$,

$$\hat{g}(z, y) - \hat{g}(x, y) = 2 \int_{(x,z)} \frac{y-u}{u} \left(\frac{\ln(y/u)}{y-u} - \frac{\ln(\phi(u)/u)}{\phi(u)-u} \right) du. \tag{A.6}$$

As the map $v \mapsto \frac{\ln v}{v-1}$ is decreasing on $(1, \infty)$, equation (A.6) implies that $\hat{g}(x, y) \leq \hat{g}(y, y)$ for $b < x < y < a$, and so equation (3.2) holds for $b < x < y < a$. Thus, g is V-convex on $[b, a]$. As ϕ is decreasing, equation (A.6) implies that $\hat{g}(x, y) \leq \hat{g}(z, y)$ for $b < x < z < a \leq y \leq \phi(z)$, and so $g(y)$ is finite. Equation (4.4) follows from equation (A.6) by a similar argument.

Given $b' \in (b, a)$, let

$$g_{b'}(z) = \begin{cases} g(b') + g'(b')(z - b') & \text{for } z \in [0, b') \\ g(z) & \text{for } z \in [b', a] \\ \sup_{x \in [b', a)} \hat{g}(x, z) & \text{for } z \in (a, \infty). \end{cases}$$

By equation (A.5), the function $g_{b'}$ is Lipschitz on $[0, a]$. On the other hand, by Condition 2 of Lemma 3.2, and since g is V-convex on $[b, a]$, the function $g_{b'}$ is V-convex on $[0, a]$. By Lemma 4.1, it follows that $g_{b'}$ is Lipschitz and V-convex on $[0, \infty)$ and so, by Theorem 3.5,

$$V_{\text{inf}} \geq -T^{-1} \int_{[b,a) \cup I} g_{b'} \, d\mu. \tag{A.7}$$

On the other hand, by equation (A.5), g is concave and increasing on (b, a), and so

$$g(x) \leq g_{b'}(x) \leq 0 \tag{A.8}$$

for $b \leq x \leq a$. Also, since $g'(x)$ goes to 0 as x goes to a, $x < a$,

$$-\ln^2(y/a) \leq g_{b'}(y) \leq g(y)$$

for $y \in I$. Thus, the functions $g_{b'}$ are dominated by a μ-integrable function on $[b, a) \cup I$. Furthermore, from equation (A.8) and the bound $g_{b'}(b) \leq g(b')$, the function $g_{b'}$ converges pointwise to g on $[b, a) \cup I$ as b' goes to b. The lemma thus follows from equation (A.7) and Lebesgue's dominated convergence theorem. □

Proof of Lemma 4.4. The function G is increasing and continuous on $[0, a]$ since it is the supremum of increasing and continuous functions on $[0, a]$ with uniformly bounded right derivatives on any closed subinterval of $[0, a)$. Fix $x \in (b, a)$. Equation (4.12) follows from equation (4.10) by observing that the right derivative of the function $y \mapsto \frac{P(x)-C(y)}{y-x}$ is positive for $a < y < \phi(x)$, and is at most 0 for $y > \phi(x)$.

We first show equation (4.13) when $\gamma = 1_{(v,\infty)}$, with $v \geq a$. Let

$$u = \min\{x \geq 0 : H(x, v) = 0\}. \tag{A.9}$$

By the convexity of C, the function $y \mapsto C(y) + (u - y)C'_+(v)$ attains its minimum at v, and $u \leq a$. Thus, by equation (A.9),

$$\min_{y \geq a} C(y) + (u - y)C'_+(v) = P(u),$$

and so

$$G(u) = -C'_+(v). \tag{A.10}$$

On the other hand, it follows from equations (4.10) and (A.9) that, for $x \in (b, a)$,

$$\begin{aligned} v < \phi(x) &\Leftrightarrow H(x, v) > 0 \\ &\Leftrightarrow x < u. \end{aligned}$$

Thus, by equation (A.10) and since $G(b) = 0$, equation (4.13) holds when $\gamma = 1_{(v,\infty)}$. Consequently, equation (4.13) holds when γ is the indicator function of any Borel subset of (a, ∞). As any nonnegative Borel-measurable function on (a, ∞) is a pointwise limit of an increasing sequence of nonnegative simple functions on (a, ∞), this concludes the proof of equation (4.13).

Equation (4.14) follows by applying equation (4.13) with $\gamma = 1_{(\phi(x),\infty)}$ or $\gamma = 1_{[\phi(x),\infty)}$.

We now prove the first assertion in the lemma. Consider $y \in I - \{a\}$. Since the right-hand side of equation (4.13) is $-C'_{-}(y)$ when $\gamma = 1_{[y,\infty)}$, there is $x \in (b, a)$ with $y \le \phi(x)$. Thus, y belongs to the interval spanned by $\phi((b, a])$. Conversely, consider $x \in (b, a)$ and assume for contradiction that $C'_{-}(\phi(x)) = 0$. By equations (4.14) and (4.12), we conclude that $P(x) = C(\phi(x)) = 0$, leading to a contradiction. Thus, I contains the interval spanned by $\phi((b, a])$. □

Proof of Lemma 4.5. By Lemma 4.4, for $u \in (b, a)$,

$$\begin{aligned} \int_{(b,u)} (\phi(x) - \phi(u))\, dG(x) &= \int_{(b,a)} \max(0, \phi(x) - \phi(u))\, dG(x) \\ &= \int_{(a,\infty)} \max(0, y - \phi(u))\, d\mu(y) \\ &= C(\phi(u)). \end{aligned}$$

Since

$$\int_{(b,u)} (\phi(x) - u) dG(x) = \int_{(b,u)} (\phi(x) - \phi(u)) dG(x) + (\phi(u) - u)G(u),$$

we conclude, using equation (4.12), that

$$\int_{(b,u)} (\phi(x) - u) dG(x) = P(u).$$

Equivalently,

$$\text{(A.11)} \qquad \int_{(b,a)} (p_u + (\phi - \mathcal{I}) p'_{u+}) dG = -\int_{[b,a)} p_u\, d\mu,$$

where $p_u(x) = \max(0, u - x)$, p'_{u+} is the right derivative of p_u with respect to x, and $\mathcal{I}$ is the identity function. Since, for $x \in (b, a)$,

$$g(x) = \int_{(b,a)} \omega(u) p_u(x)\, du,$$

and

$$g'(x) = \int_{(b,a)} \omega(u) p'_{u+}(x)\, du,$$

where $\omega(u) = -2\frac{\ln(\phi(u)/u)}{u(\phi(u)-u)}$, it follows from equation (A.11) and Fubini's theorem that

$$\int_{(b,a)} (g + (\phi - \mathcal{I})g')\, dG = \int_{[b,a)} -g\, d\mu. \tag{A.12}$$

Both integrands in equation (A.12) are positive and, since g is continuous, $\int_{[b,a)} -g\, d\mu$ is finite.

Equations (A.12) and (4.4) show that the integral $\int_{(b,a)} \max(0, g \circ \phi)\, dG$ is finite. Also, by setting $g(y) = 0$ for $y \in (a, \infty) - I$, it follows from equation (4.5) that the integral $\int_{(a,\infty)} \max(0, -g)\, d\mu$ is finite. Thus, by Lemma 4.4,

$$\int_{(b,a)} (g \circ \phi)\, dG = \int_{(a,\infty)} g\, d\mu. \tag{A.13}$$

Combining equations (A.12), (4.4), and (A.13) concludes the proof. □

Proof of Lemma 5.2. The proof is similar to that of equation (4.13), which is identical to equation (5.5) when $i = n + 1$. Assume now that $0 \le i \le n$. We first observe that both sides of equation (5.5) are null when $\gamma = 1_{(v,\infty)}$, with $v \ge \phi(x_i)$. Similarly, when $\gamma = 1_{(a,v]}$, with $v < \phi(x_i)$, then by equation (4.13),

$$\begin{aligned}\int_{(x_i,a)} (\gamma \circ \phi)\, dG &= \int_{(b,a)} (\gamma \circ \phi)\, dG \\ &= \mu((a, v]) \\ &= \mu_i((a, v]) \\ &= \int_{(a,\infty)} \gamma\, d\mu_i.\end{aligned}$$

Thus, equation (5.5) also holds in this case. Finally, since $C(a) = P(a)$ and C is convex, it follows from equation (4.11) that $G(a) = -C'_+(a)$, and so equation (5.5) holds when γ is constant and equal to 1 on (a, ∞). Consequently, equation (5.5) holds if γ is the indicator function of any Borel subset of (a, ∞), and so it holds for any nonnegative Borel-measurable function γ on (a, ∞). □

Proof of Theorem 6.1. Assume for simplicity that $F_0 = T = 1$. Then, for $v \le 0 \le u$ and $\sigma \ge 0$,

$$H(e^{\sigma v}, e^{\sigma u}) = N\left(-u + \frac{\sigma}{2}\right) + N\left(v - \frac{\sigma}{2}\right) - e^{\sigma v}\left(N\left(-u - \frac{\sigma}{2}\right) + N\left(v + \frac{\sigma}{2}\right)\right), \tag{A.14}$$

where $N(z) = (2\pi)^{-1/2} \int_{-\infty}^{z} \exp(-w^2/2)\, dw$. It follows that $H(1, e^{\sigma u}) \le 0 \le H(e^{-\sigma u}, e^{\sigma u})$, and so

$$\psi(e^{\sigma u}) \in [e^{-\sigma u}, 1]. \tag{A.15}$$

Let

$$h(v, u) = N'(u) - N'(v) - v N(-u) - v N(v),$$

and

$$\eta(\sigma; v, u) = e^{-\sigma v/2} H(e^{\sigma v}, e^{\sigma u}).$$

Equation (A.14) shows that the definitions of H and η can be extended to any real number σ and the function η is odd with respect to σ. On the other hand, $\partial\eta(\sigma; v, u)/\partial\sigma = h(v, u)$ at $\sigma = 0$. Furthermore, an easy calculation shows that there is a constant $\gamma > 0$ such $|\partial^3\eta(\sigma; v, u)/\partial\sigma^3| \le \gamma e^{2u} N'(v)$ for $-u \le v \le 0 \le u$ and $0 \le \sigma \le 1$ and so, by the Taylor-Lagrange theorem,

$$|\eta(\sigma; v, u) - \sigma h(v, u)| \le \gamma\sigma^3 e^{2u} N'(v)/6. \tag{A.16}$$

Let us now assume that $H(e^{\sigma v}, e^{\sigma u}) = 0$ and $\gamma\sigma^2 e^{4u} \le 1$, with $v \le 0 \le u$ and $0 \le \sigma \le \min(1, 1/\gamma)$. By equation (A.15), $-u \le v \le 0$ and so, by equation (A.16), $|h(v, u)| \le \gamma\sigma^2 e^{2u} N'(v)/6$. We show that

$$|v - \theta(u)| \le \gamma\sigma^2 e^{3u}. \tag{A.17}$$

As the function $N(z) - e^z N'(z)$ has a positive derivative and goes to 0 as z goes to $-\infty$,

$$e^v N'(v) \le N(v). \tag{A.18}$$

We now distinguish two cases:

- $v \le \theta(u)$: since $\partial h(v, u)/\partial v = -N(v) - N(-u)$ and h is concave with respect to v,

$$h(\theta(u), u) - h(v, u) \le -(\theta(u) - v)(N(v) + N(-u)).$$

  Thus $(\theta(u) - v)N(v) \le \gamma\sigma^2 e^{2u} N'(v)/6$. By equation (A.18), we conclude that equation (A.17) holds in this case.
- $v > \theta(u)$: let $z = v - \gamma\sigma^2 e^{3u}$. By the Taylor-Lagrange theorem and equation (A.18),

$$\begin{aligned} h(z, u) &\ge h(v, u) + (v - z)(N(v) + N(-u)) - \frac{(v - z)^2}{2} N'(v) \\ &\ge N'(v)\left(-\gamma\sigma^2 e^{2u}/6 + (v - z)e^{-u} - \frac{(v - z)^2}{2}\right) \\ &\ge 0 \\ &\ge h(v, u). \end{aligned}$$

  Thus equation (A.17) holds again in this case.

Equation (A.17) implies that, for $0 \le \sigma \le \min(1, 1/\gamma)$ and $0 \le u \le -\ln(\gamma\sigma^2)/4$,

$$\left|\ln^2 \frac{e^{\sigma u}}{\psi(e^{\sigma u})} - \sigma^2(u - \theta(u))^2\right| \le 4\gamma\sigma^4 u e^{3u}. \tag{A.19}$$

By Theorem 5.7, equations (A.15) and (A.19),

$$\begin{aligned} V_{\inf} &= \int_1^\infty \ln^2 \frac{y}{\psi(y)} C''(y)\, dy \\ &= \int_0^\infty \ln^2 \frac{e^{\sigma u}}{\psi(e^{\sigma u})} N'(u + \sigma/2)\, du \\ &= \int_0^{-\ln(\gamma\sigma^2)/4} \ln^2 \frac{e^{\sigma u}}{\psi(e^{\sigma u})} N'(u + \sigma/2)\, du + O(\sigma^4) \end{aligned}$$

$$= \sigma^2 \int_0^\infty (u - \theta(u))^2 N'(u + \sigma/2)\, du + O(\sigma^4)$$
$$= \sigma^2 \int_0^\infty (u - \theta(u))^2 N'(u)\, du - c\sigma^3 + O(\sigma^4).$$

On the other hand, by the implicit function theorem, the function θ has a continuous derivative for $u > 0$. Thus,

$$\begin{aligned}
\int_0^\infty (u - \theta(u))^2 N'(u)\, du &= \frac{1}{2} + 2\int_0^\infty \theta(u) N''(u)\, du + \int_0^\infty \theta(u)^2 N'(-u)\, du \\
&= \frac{1}{2} - 2\int_0^\infty \theta'(u)(N'(u) - \theta(u)N(-u))\, du \\
&= \frac{1}{2} - 2\int_0^\infty \theta'(u)(N'(\theta(u)) + \theta(u)N(\theta(u)))\, du \\
&= \frac{1}{2} + 2\int_{-\infty}^0 (N'(z) + zN(z))\, dz \\
&= 1.
\end{aligned}$$

This concludes the proof. □

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