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ASYMPTOTICALLY OPTIMAL PORTFOLIOS

FARSHID JAMSHIDIAN [1]

Fuji International Finance PLC, 7–11 Finsbury Circus, London EC2M 7NT, UK

This paper extends to continuous time the concept of universal portfolio introduced by Cover (1991). Being a performance weighted average of constant rebalanced portfolios, the universal portfolio outperforms constant rebalanced and buy-and-hold portfolios exponentially over the long run. An asymptotic formula summarizing its long-term performance is reported that supplements the one given by Cover. A criterion in terms of long-term averages of instantaneous stock drifts and covariances is found which determines the particular form of the asymptotic growth. A formula for the expected universal wealth is given.

KEY WORDS: rebalancing, constant rebalanced portfolios, optimal growth, quadratic programming, performance weighting, universal portfolio

1. INTRODUCTION

Cover (1991) sets forth the long-term investment objective of achieving wealth that grows nearly as fast as the best constant rebalanced portfolio (termed here the *target optimal portfolio*) which can be formed with the knowledge of future stock prices ahead of time. Working in discrete time in full generality, he remarkably constructs a certain "universal portfolio," depending only on past stock prices, which achieves this goal. In the long run, the universal portfolio exponentially outperforms all constant rebalanced (and buy-and-hold) portfolios, except of course the target optimal portfolio itself, which it succeeds in tracking to first order in the exponent.

Cover's main result assumes that the model is "asymptotically active," meaning basically that in the long run all stocks are present in the target optimal portfolio with a high probability. His "generalized universal portfolio" lifts this restriction. But there still remain the interesting problems of classifying asymptotically active models and investigating the asymptotic growth of the universal portfolio in general. Working in continuous time, we pursue these questions.

Central to the theme of this paper is the existence, under what is called here a "weak regularity" condition, of a unique constant portfolio weight vector, termed the *asymptotically optimal constant weight,* toward which the weight vector of the target optimal portfolio converges. By definition, the constant rebalanced portfolio with this weight (termed the *asymptotically optimal constant rebalanced portfolio*) has the highest long-term expected rate of return among all constant rebalanced portfolios. It outperforms the universal portfolio like a power of time, and both outperform all other constant rebalanced portfolios exponentially.

The asymptotically optimal constant weight is the solution of a quadratic programming problem whose quadratic and linear terms are the long-term averages of, respectively, the instantaneous stock covariances and expected returns in the future. Unfortunately, this means that in practice it cannot be determined, as such long-term future

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knowledge is lacking. This makes the universal portfolio all the more significant because its construction depends only on the realized path of stock prices. Cover's idea of averaging constant portfolio weights by their performance ensures that the optimal constant rebalanced weight is securely captured against its suboptimal neighbors.

To bring this discussion to sharp focus, consider the simplest possible example, that of two stocks with bivariate geometric random walk prices. Let μ_1 and μ_2 denote their drifts, σ_1 and σ_2 their volatilities, and ρ their correlation. Let $b^*(t) = (b_1^*(t), b_2^*(t))$ denote the weight vector of the target optimal portfolio at time t ($b_1^*(t) + b_2^*(t) = 1$, $b_i^*(t) \geq 0$). Set

$$\eta_i = \mu_i - \frac{1}{2}\sigma_i^2 \qquad (i = 1, 2);$$

$$J = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

If $|\eta_2 - \eta_1| < J/2$, it turns out that the model is asymptotically active. In this case, the asymptotically optimal constant weight is

$$b^\infty = \left(\frac{1}{2} + \frac{\eta_1 - \eta_2}{J}, \frac{1}{2} + \frac{\eta_2 - \eta_1}{J}\right) \qquad \left(|\eta_2 - \eta_1| < \frac{J}{2}\right).$$

We will see $b^*(t)$ converges to b^∞ with probability 1. Also, with $W^*(t)$ and $\hat{W}(t)$ respectively denoting the optimal and universal wealths, as in equation (5.44) of Cover (1991),

$$\frac{\hat{W}(t)}{W^*(t)} \sim \sqrt{\frac{2\pi}{Jt}} \qquad \left(|\eta_2 - \eta_1| < \frac{J}{2}\right).$$

Moreover, the ratio of the constant rebalanced wealth with weight b^∞ over the target optimal wealth is convergent in distribution, so the former asymptotically outperforms the universal portfolio. Now suppose $|\eta_2 - \eta_1| > J/2$, say, $\eta_2 > \eta_1 + J/2$. Then $b^\infty = (0, 1)$, the probability that the optimal portfolio consists only of stock 2 converges to 1, and

$$\frac{\hat{W}(t)}{W^*(t)} \sim \frac{1}{(|\eta_2 - \eta_1| - J/2)t} \qquad \left(|\eta_2 - \eta_1| > \frac{J}{2}\right).$$

In both cases, all constant rebalanced portfolios with weight different than b^∞ underperform the target optimal portfolio and the universal portfolio exponentially.

The aim of this paper is to develop these ideas for arbitrary number of stocks and more general stochastic processes. Additionally, we discuss examples, primarily the cases of geometric Wiener and "mean-reverting" stock prices, study connections to log-optimum portfolios, produce a formula for the expected universal wealth and bounds for it, show the universal portfolio is a "mildly profit taking" strategy, and point out a counterexample.

The continuous-time framework simplifies the analysis because Itô's lemma serves to establish a formula for the constant rebalanced wealth which is quadratic (exponential) in the weights. This implies that at each state of the world the target optimal portfolio is the solution of a quadratic programming problem, one which closely resembles the mean-variance portfolio optimization of Markowitz (1959). This is not the case in discrete time.

2. DEFINITIONS AND STATEMENTS OF MAIN RESULTS

We work in continuous time; accordingly, our formalism differs somewhat from Cover's, though the concepts are analogous. Let $S_i(t)$ be the price of stock i at time t, and $S(t) = (S_1(t), \ldots, S_m(t))'$ denote the price vector. The initial prices $S_i(0)$ are assumed known and given. Assume that $S(t)$ follows the Itô process

$$\text{(2.1)} \qquad \frac{dS_i(t)}{S_i(t)} = \mu_i(t)\,dt + \sum_{j=1}^{m} \sigma_{ij}(t)\,dz_j(t) \qquad (i = 1, \ldots, m),$$

where $z_i(t)$ are independent Brownian motions. The drift vector $\mu(t) = (\mu_1(t), \ldots, \mu_m(t))'$ and the matrix $\sigma(t) = (\sigma_{ij}(t))$ are, in general, unknown. The matrix of instantaneous covariances is

$$\text{(2.2)} \qquad \Sigma(t) \equiv \sigma(t)\sigma(t)^{\mathrm{t}}.$$

A *portfolio* of long stocks at any time t is identified by its total wealth (value) $W(t)$ and its weight vector process $b(t) \in B$, where

$$B \equiv \{b \in \mathbb{R}^m : b_i \geq 0, \sum_{i=1}^{m} b_i = 1\}.$$

The weights $b_i(t) \geq 0$ are the fraction of wealth $W(t)$ invested in stock i at time t. A *rebalanced portfolio* (also known as a *self-financing portfolio* if there are no dividends[2]) is a portfolio whose wealth process $W(t)$ satisfies

$$\text{(2.3)} \qquad \frac{dW(t)}{W(t)} = \sum_{i=1}^{m} b_i(t)\,\frac{dS_i(t)}{S_i(t)}.$$

A *constant rebalanced portfolio* is a rebalanced portfolio with constant weights b_i. Denote its wealth by $W(t; b)$ and the initial wealth by $W(0)$.

PROPOSITION 2.1. *In general,*

$$\text{(2.4)} \qquad W(t; b) = W(0) \prod_{i=0}^{m} \left[\frac{S_i(t)}{S_i(0)}\right]^{b_i} \exp\left(-\frac{1}{2}\, b^{\mathrm{t}}\Lambda(t)b + \frac{1}{2}\sum_{i=0}^{m} \Lambda_{ii}(t) b_i\right),$$

$$\text{(2.5)} \qquad \Lambda(t) \equiv \int_0^t \Sigma(s)\,ds.$$

[2] If dividends are continuous and reinvested in the stocks, then "rebalanced" is the same as "self-financing," provided dividend yields are added to the stock drifts in (2.3).

Proof. Equation (2.4) follows by integrating (2.3) with the aid of Itô's lemma applied to the logarithm function. □

Cover proposes a non-self-financing *target optimal portfolio* as the benchmark to measure long-term growth of rebalanced portfolios. At any time t, it is the best constant rebalanced portfolio that can be formed at time 0 with the benefit of hindsight. The target optimal wealth is defined by

$$W^*(t) \equiv W(t;\, b^*(t)) \equiv \max_{b \in B} W(t;\, b), \tag{2.6}$$

where $b^*(t)$ denotes a point where the maximum is reached.[3] By (2.4), $b^*(t)$ is an optimal solution of the quadratic programming problem

$$\max_{b \in B} \; -\frac{1}{2} b^{\mathrm{t}} \Lambda(t) b + \sum_{i=0}^{m} \left(\log \frac{S_i(t)}{S_i(0)} + \frac{1}{2} \Lambda_{ii}(t) \right) b_i. \tag{2.7}$$

The condition for the uniqueness of the optimal solution is given in

PROPOSITION 2.2. *The following properties are equivalent for any m by m symmetric positive semidefinite matrix Λ.*

(a) *The null space of Λ does not contain any nonzero vector x with $x_1 + \cdots + x_m = 0$.*
(b) *The (convex) function $f(b) = b^{\mathrm{t}} \Lambda b$ restricted to B is strictly convex.*
(c) *The $m - 1$ by $m - 1$ (symmetric positive semidefinite) matrix $V = (V_{ij})$ is positive definite, where $V_{ij} = \Lambda_{ij} - \Lambda_{im} - \Lambda_{jm} + \Lambda_{mm}$, $1 \le i, j \le m - 1$.*

Moreover, if the above properties hold, then (and only then) for any $\alpha \in \mathbb{R}^m$, the function $-b^{\mathrm{t}} \Lambda b/2 + \alpha^{\mathrm{t}} b$ on B has a unique maximum point in B.

Proof. The equivalence of (a) and (b) is an immediate consequence of the easily verified fact that for any vectors x and y and scalar a, $f(ax + (1-a)y) = af(x) + (1-a)f(y) - a(1-a)f(x-y)$. The equivalence (b) with (c) follows by observing that for $b \in B$, $f(b)$ equals $b'^{\mathrm{t}} V b'$ plus a linear term in $b' \equiv (b_1, \ldots, b_{m-1})^{\mathrm{t}}$. The final statement about uniqueness is a direct consequence of (b). □

Property (a) is satisfied, and consequently $b^*(t)$ is unique, if $\sigma(t)$ (and hence $\Lambda(t)$) is nonsingular, or if one stock, say stock 1, represents accumulating cash in a money market account, in the sense that $\sigma_{1j}(t) = 0$ (and hence $\Lambda_{1j}(t) = 0$) for all $1 \le j \le m$ and all t, and the remaining $m - 1$ stocks have a nondegenerate instantaneous covariance matrix.

If $b_i^*(t) > 0$, then, following Cover, we say stock i is *active* at time t. Similarly, call

[3] It should be emphasized that the target optimal portfolio maximizes the wealth among *constant* rebalanced portfolios (*not* among *all* rebalanced portfolios). At each state of the world at time t, the target optimal portfolio represents a constant rebalanced portfolio (the one that at all times s ($0 \le s \le t$) has weight $b^*(t)$). But, at a different state of the world (or at a different time), it represents a different constant rebalanced portfolio. The target optimal wealth is not achievable by any self-financing trading strategy. One can only hope to approximate its growth. As we shall see, the universal portfolio achieves this goal to first order in the exponent.

stock i *inactive* if $b_i^*(t) = 0$, and *optimal* if $b_i^*(t) = 1$ (then $W^*(t) = W(0)S_i(t)/S_i(0)$).

Call a (vector) stochastic process $X(t)$ (possibly deterministic) *weakly regular* if $|E[X(t)]| < \infty$ for all t, $\theta = \lim E[X(t)]/t$ exists, and $X(t)/t \to \theta$ in probability as $t \to \infty$. The stock market model of (2.1) is called a *weakly regular model* if both $\Lambda(t)$ and $\log(S(t))$ are weakly regular, and both $\Lambda(t)$ and $\lim \Lambda(t)/t$ satisfy the equivalent properties in Proposition 2.2. In a weakly regular model we set

$$\Sigma^\infty \equiv \lim \frac{E[\Lambda(t)]}{t}, \qquad \eta^\infty \equiv \lim \frac{E[\log S(t)]}{t}. \tag{2.8}$$

In a weakly regular model,[4]

$$\mu^\infty \equiv \lim \frac{1}{t}\left(E\left[\int_0^t \mu_i(s)\, ds\right]\right) = \left(\eta_i^\infty + \frac{1}{2}\Sigma_{ii}^\infty\right) \in \mathbb{R}^m, \tag{2.9}$$

and by (2.4), $\log(W(t;\, b))$ is weakly regular and

$$r(b) \equiv \lim \frac{1}{t} E[\log W(t;\, b)] = -\frac{1}{2} b^{\mathrm{t}} \Sigma^\infty b + b^{\mathrm{t}} \mu^\infty. \tag{2.10}$$

In a weakly regular model, define the *asymptotically optimal constant weight* $b^\infty \in B$ to be the unique weight with the highest constant rebalanced expected rate of return:

$$r(b^\infty) \equiv \max_{b \in B} r(b).$$

For a stochastic process $X(t)$, write $X(t) \approx 0$, if there is a $\delta > 0$ such that $X(t)\exp(\delta t) \to 0$ in probability. A rebalanced portfolio is called *asymptotically suboptimal* if $W(t)/W^*(t) \approx 0$.

THEOREM 2.1. *In a weakly regular model if* $b \neq b^\infty$ *then* $W(t;\, b)/W(t;\, b^\infty) \approx 0$; *so any constant rebalanced portfolio with weight* $b \neq b^\infty$ *is asymptotically suboptimal.*

Proof. Set $X(t) = \log(W(t;\, b^\infty)/W(t;\, b))$ and $c = r(b^\infty) - r(b) > 0$. Since $W(t;\, b)$ and $W(t;\, b^\infty)$ are weakly regular, $X(t)/t \to c$. Let $0 < \delta < c$. Choose $0 < \varepsilon < c - \delta$. Then $\mathrm{Prob}\{\delta t - X(t) < -\varepsilon t\} \to 1$. Exponentiating implies $\exp(\delta t - X(t)) \to 0$ in probability. Hence, $\exp(-X(t)) \approx 0$. □

Assume the model is weakly regular. Then stock i is called *asymptotically active* if $b_i^\infty > 0$. If all stocks are asymptotically active, we say the *model* is asymptotically active. For $1 \le k \le m-1$, we say the model is *asymptotically k-inactive* if stocks $k+1, \ldots, m$ are asymptotically active, $b_i^\infty = 0$ for $1 \le i \le k$, and the Lagrange multipliers of these

[4] Since the weak regularity of $\Lambda(t)$ entails $E[\Lambda_{ii}(t)] < \infty$ for all i, one gets $E[\int_0^t \sigma(s)\, dz(s)] = 0$. Therefore in a weakly regular model, by (2.3) and Itô's lemma, $E[\int_0^t \mu_i(s)\, ds] = E[\Lambda_{ii}(t)]/2 + E[\log S_i(t)/S_i(0)]$. If we divide by t and take the limit, (2.9) follows. (Also, from Lemma 4.1(a) it follows that if $\Lambda(t)$ is weakly regular, then $\log(S(t))$ is weakly regular if and only if the process $\int_0^t \mu(s)\, ds$ is weakly regular.)

k constraints are all positive.[5] In an asymptotically $(m-1)$-inactive model, stock m is called an *asymptotically strongly optimal stock* (then $b_m^\infty = 1$ and $W(t;\, b^\infty) = W(0)S_m(t)/S_m(0)$). We will see this occur if and only if $\eta_m^\infty > \eta_i^\infty + (\Sigma_{ii}^\infty - 2\Sigma_{im}^\infty + \Sigma_{mm}^\infty)/2$ for all $1 \le i \le m-1$. Call a weakly regular model *stable* if it is either asymptotically active or, after possibly a renumbering of stocks, asymptotically k-inactive for some $1 \le k \le m-1$.

Let $X(t)$ and $Y(t)$ be stochastic processes (possibly deterministic). Write $X(t) \succ Y(t)$ or $Y(t) \prec X(t)$ if there is an $\varepsilon > 0$ such that $\text{Prob}\{X(t) > \varepsilon Y(t)\} \to 1$ as $t \to \infty$. If $X(t) > 0$ and $Y(t) > 0$, call $X(t)$ *asymptotic* to $Y(t)$ and write $X(t) \sim Y(t)$ if $X(t)/Y(t) \to 1$ in probability. We say $X(t)$ is *bounded in distribution* if for every $\varepsilon > 0$ there is an $M > 0$ such that, for all t, $\text{Prob}\{|X(t)| > M\} < \varepsilon$. If $X(t)$ and $Y(t)$ are vector processes, we use the above notation to mean it is satisfied componentwise.

A rebalanced portfolio is called *asymptotically weakly optimal* if, for some $c > 0$, $W(t)/W^*(t) \succ t^{-c}$, *asymptotically optimal* if $W^*(t)/W(t)$ is bounded in distribution (then $W(t)/W^*(t) \succ t^{-\varepsilon}$ for any $\varepsilon > 0$), and *asymptotically strongly optimal* if $W(t)/W^*(t) \sim 1$.[6]

Theorems 2.2–2.6 will be proved in the next section after the solution of the quadratic programming problem (2.7) is provided.

THEOREM 2.2. *In a stable model $b^*(t) \to b^\infty$ in probability. In addition:*

(a) *The model is asymptotically active if and only if $b^*(t) \succ 1$.*

(b) *The model is asymptotically k-inactive $(1 \le k \le m-1)$ if and only if $b_i^*(t) \succ 1$ for all $k+1 \le i \le m$, $\text{Prob}\{b_i^*(t) = 0\} \to 1$ for all $1 \le i \le k$, and all k of the Lagrange multipliers of these k constraints are $\succ t$. Moreover, if stock m is asymptotically strongly optimal, then $\text{Prob}\{b^*(t) = b^\infty\} = \text{Prob}\{W^*(t) = W(t;\, b^\infty)\} \to 1$.*

A weakly regular process $X(t)$ with $\theta = \lim X(t)/t$ is called *regular* if $X(t)/\sqrt{t} - \sqrt{t}\theta$ is bounded in distribution, and *strongly regular* if $X(t)/\sqrt{t} - \sqrt{t}\theta \to 0$ in probability.

THEOREM 2.3. *In a stable model, if both $\log(S(t))$ and $\Lambda(t)$ are (strongly) regular, then the constant rebalanced portfolio with weight b^∞ is asymptotically (strongly) optimal.*

Despite their mathematical interest, Theorems 2.1–2.3 are of little practical use for investment because the construction of b^∞ requires the long-term averages of future instantaneous expected returns and covariances. But Cover's idea is that performance weighting remedies this. The *universal portfolio* is defined as the rebalanced portfolio with weights

$$\hat{b}_i(t) \equiv \frac{\int_B b_i W(t;\, b)\, db}{\int_B W(t;\, b)\, db}. \tag{2.11}$$

[5]In our convention, the Lagrange multipliers corresponding to the inequality constraints $b_i \ge 0$ are given a nonnegative sign for convenience.

[6]Basically, a weakly optimal portfolio underperforms the target optimal portfolio only like a power of time, while a suboptimal portfolio underperforms the latter exponentially. This is put in perspective by considering the rates of return $r(t) \equiv (\log W(t))/t$ and $r^*(t) \equiv (\log W^*(t))/t$. Then weak optimality states that $r(t) = r^*(t) - O((\log t)/t)$. In particular, $r^*(t) - r(t) \to 0$, that is, $W(t)$ approximates $W^*(t)$ to first order in the exponent. This is not the case for suboptimal portfolios.

Since $W(0; b) = W(0)$, $\hat{b}_i(0) = 1/m$. As in Lemma 2.5 of Cover (1991), we have

PROPOSITION 2.3. *At any time t, the universal wealth $\hat{W}(t)$ is*

$$\hat{W}(t) = \frac{\int_B W(t; b)\, db}{\int_B db}. \tag{2.12}$$

Proof. Momentarily let $W(t)$ denote the right side of (2.12). Then by (2.3) and (2.11),

$$\frac{dW}{W} = \frac{\int dW(t; b)\, db}{\int W(t; b)\, db} = \frac{\int \Sigma\, W(t; b) b_i dS_i/S_i\, db}{\int W(t; b)\, db} = \Sigma\, \hat{b}_i(t)\frac{dS_i}{S_i} = \frac{d\hat{W}}{\hat{W}}.$$

In addition, $W(0) = \hat{W}(0)$. Hence, $W(t) = \hat{W}(t)$ for all t. □

In a weakly regular model, define the $(m-1)$ by $(m-1)$ symmetric positive definite matrix

$$J^\infty = (J_{ij}^\infty) \equiv (\Sigma_{ij}^\infty - \Sigma_{im}^\infty - \Sigma_{jm}^\infty + \Sigma_{mm}^\infty) \qquad (1 \le i, j \le m-1). \tag{2.13}$$

Our continuous-time analog of Cover's equation (6.1) is

THEOREM 2.4. *If the model is asymptotically active, then*

$$\frac{\hat{W}(t)}{W^*(t)} \sim \frac{(m-1)!}{|J^\infty|^{1/2}} \left(\frac{2\pi}{t}\right)^{(m-1)/2} \tag{2.14}$$

We now give asymptotic formulae for stable models in general.

THEOREM 2.5. *If stock m is asymptotically strongly optimal (i.e., $\eta_m^\infty > \eta_i^\infty + J_{ii}^\infty/2$), then*

$$\frac{\hat{W}(t)}{W^*(t)} \sim \frac{(m-1)!}{t^{m-1}} \prod_{i=1}^{m-1} \frac{1}{\eta_m^\infty - \eta_i^\infty - J_{ii}^\infty/2}. \tag{2.15}$$

More generally, in asymptotically k-inactive models, (2.14) and (2.15) combine into a single asymptotic formula. For $1 \le k \le m-2$, set $J^{(k)} = (J_{jl}^\infty)$, $k+1 \le j, l \le m-1$, and

$$\gamma_i^\infty \equiv \eta_m^\infty - \eta_i^\infty - J_{ii}^\infty/2, \qquad \gamma^\infty \equiv (\gamma_1^\infty, \ldots, \gamma_{m-1}^\infty)^t; \tag{2.16}$$

$$\gamma_i^{(k)} \equiv \gamma_i^\infty + \sum_{j=k+1}^{m-1} J_{ij}^\infty \beta_j^{(k)} \qquad (1 \le i \le k \le m-2), \tag{2.17}$$

where the $(m-k-1)$-vector $\beta^{(k)}$ is the solution of the linear equation

$$\sum_{l=k+1}^{m-1} J_{jl}^\infty \beta_l^{(k)} = -\gamma_j^\infty \qquad (k+1 \le j \le m-1). \tag{2.18}$$

We will see that in an asymptotically k-inactive model, $b_j^{\times} = \beta_j^{(k)}$ for all $k+1 \le j \le m-1$, and $\gamma_i^{(k)} > 0$ $(1 \le i \le k)$ are the Lagrange multipliers.

THEOREM 2.6. *If the model is asymptotically k-inactive* $(1 \le k \le m-2)$, *then*

$$\frac{\hat{W}(t)}{W^*(t)} \sim (m-1)! \prod_{i=1}^{k} \frac{1}{\gamma_i^{(k)}} \sqrt{\frac{(2\pi)^{m-1-k}}{|J^{(k)}|}} \frac{1}{t^{(m+k-1)/2}}. \tag{2.19}$$

Combining Theorems 2.4–2.6, we conclude

COROLLARY 2.1. *The universal portfolio is asymptotically weakly optimal in stable models.*

3. PROOFS OF THE MAIN RESULTS

An explicit solution for the optimization problem (2.7) obtains by eliminating a variable (say the mth) from the unity constraint which governs the notion of a portfolio. Let $V(t)$ denote the $m-1$ by $m-1$ symmetric positive semidefinite matrix with components

$$V_{ij}(t) \equiv \Lambda_{ij}(t) - \Lambda_{im}(t) - \Lambda_{jm}(t) + \Lambda_{mm}(t) \qquad (1 \le i, j \le m-1). \tag{3.1}$$

If $\Lambda(t)$ is weakly regular, then so is $V(t)$, and $\lim V(t)/t = J^{\times}$. Set

$$\lambda_i(t) \equiv \log\left(\frac{S_m(t)}{S_m(0)}\right) - \log\left(\frac{S_i(t)}{S_i(0)}\right) - \frac{V_{ii}(t)}{2}, \tag{3.2}$$

$$\lambda(t) \equiv (\lambda_1(t), \ldots, \lambda_{m-1}(t))^{\mathrm{t}};$$

$$\beta^*(t) \equiv -V^{-1}(t)\lambda(t) \in \mathbb{R}^{m-1}; \tag{3.3}$$

$$\beta^{\times} \equiv -(J^{\times})^{-1}\gamma^{\times} \in \mathbb{R}^{m-1}. \tag{3.4}$$

For any $b \in B$, write $b = (b', b_m)$. Set $B' \equiv \{b' \in \mathbb{R}^{m-1}\colon b_i' > 0,\ b_1' + \cdots + b_{m-1}' < 1\}$.

PROPOSITION 3.1. *Suppose* $V(t)$ *is nonsingular (so that* $b^*(t)$ *is unique). Then*

(a)

$$W(t; b) = W(0)\frac{S_m(t)}{S_m(0)} \exp[-\tfrac{1}{2}b'^{\mathrm{t}}V(t)b' - \lambda(t)^{\mathrm{t}}b'], \tag{3.5}$$

$$W(t; b) = W(0)\frac{S_m(t)}{S_m(0)}\, e^{\beta^*(t)^{\mathrm{t}}V(t)\beta^*(t)/2} \times \exp[-\tfrac{1}{2}(b' - \beta^*(t))^{\mathrm{t}}V(t)(b' - \beta^*(t))], \tag{3.6}$$

$$W^*(t) = W(0)\frac{S_m(t)}{S_m(0)}e^{b^*(t)'^tV(t)b^*(t)'/2}. \tag{3.7}$$

(b) *All stocks are active at time t if and only if $\beta^*(t) \in B'$. Then $b^*(t)' = \beta^*(t)$, and*

$$\frac{W(t;\, b)}{W^*(t)} = \exp[-\tfrac{1}{2}(b' - \beta^*(t))^tV(t)(b' - \beta^*(t))]. \tag{3.8}$$

(c) *Stock m is optimal at time t, i.e., $b^*(t)' = 0$, $b^*_m(t) = 1$ and $W^*(t) = W(0)S_m(t)/S_m(0)$, if and only if $S_m(t)/S_m(0) \geq S_i(t)/S_i(0)\exp(V_{ii}(t))/2$, for all $1 \leq i \leq m-1$, i.e., if and only if $\lambda_i(t) \geq 0$. In this case, $\lambda_i(t)$ are the Lagrange multipliers, and*

$$\frac{W(t;\, b)}{W^*(t)} = e^{-b'^tV(t)b'/2\; -\; \lambda(t)^tb'}. \tag{3.9}$$

(d) *Define the $(m-k-1)$-vector $\beta^{*(k)}(t)$ to be the solution of*

$$\sum_{l=k+1}^{m-1} V_{jl}(t)\beta^{*(k)}_l(t) = -\lambda_j(t) \qquad (k+1 \leq j \leq m-1). \tag{3.10}$$

Then the model is k-inactive at time t; i.e., $b^*_1(t) = \cdots = b^*_k(t) = 0$, and $b^*_j(t) > 0$ for $k+1 \leq j \leq m$ if and only if $(0, \beta^{*(k)}(t)) \in B'$ and

$$\lambda^{(k)}_i(t) \equiv \lambda_i(t) + \sum_{j=k+1}^{m-1} V_{ij}(t)\beta^{*(k)}_j(t) \geq 0 \qquad (1 \leq i \leq k \leq m-2). \tag{3.11}$$

In this case, $b^*(t)' = (0, \beta^{*(k)}(t))$, $\lambda^{(k)}_i(t)$ are the Lagrange multipliers, and

$$\frac{W(t;\, b)}{W^*(t)} = \exp\left[-\frac{1}{2}(b' - b^*(t)')^tV(t)(b' - b^*(t)') - \sum_{i=1}^{k}\lambda^{(k)}_i(t)b_i\right]. \tag{3.12}$$

(e) *In a weakly regular model, similar statements and formulae hold for b^∞.*

Proof. Substituting $b_m = 1 - b_1 - \cdots - b_{m-1}$ in (2.4), (3.5) follows, and completing the square yields (3.6). The global maximum of $W(t;\, b)$ now visibly occurs at $\beta^*(t)$, which will then be optimal and all stocks will be active if and only if $\beta^*(t) \in B'$. Equation (3.7) for this case now follows from (3.6), and combining it again with (3.6) gives (3.8). From (3.5), the Lagrange multiplier at $b' = 0$ is $\lambda(t)$. So $b^*(t)' = 0$ if and only if $\lambda(t) \geq 0$. Equation (3.7) is clear in this case, as is (3.9) from (3.5). More generally, part (d) states the Kuhn-Tucker conditions for the objective function (3.5) on B'. Equation (3.7) for this case and (3.12) follow by easy calculations. Part (e) is clear. □

We use the following elementary properties, which are stated without proof.

LEMMA 3.1.

(a) "$\sim$" *is an equivalence relation. If* $X(t) \sim f(t)$ *and* $Y(t) \sim g(t)$, *then* $X(t)Y(t) \sim f(t)g(t)$, $X(t) + Y(t) \sim f(t) + g(t)$, *and* $X(t)/Y(t) \sim f(t)/g(t)$.
(b) *If* $X(t) \sim Y(t) < 1$, *then* $X(t) - Y(t) \to 0$ *in probability, while if* $X(t) - Y(t) \to 0$ *in probability and* $Y(t) > 1$, *then* $X(t) \sim Y(t)$.
(c) *If* $A(t)$ *is a weakly regular n by n positive definite symmetric matrix process and* $\lim A(t)/t = M$ *is positive definite, then* $\operatorname{tr}(A(t)) \sim t \operatorname{tr}(M)$, *and* $|A(t)| \sim t^n|M|$.
(d) *If* $M(t) \to M$ *in probability and* $X(t)$ *is bounded in distribution, then so is* $M(t)X(t)$.
(e) *If* $X(t)$ *is bounded in distribution and* $M(t) \to 0$ *in probability, then so does* $M(t)X(t)$.

Proof of Theorem 2.2. Since the model is weakly regular,

$$\beta^*(t) = -\left(\frac{V(t)}{t}\right)^{-1}\left(\frac{\lambda(t)}{t}\right) \to -(J^\infty)^{-1}\gamma^\infty = \beta^\infty. \tag{3.13}$$

Now assume the model is asymptotically active. Then by part (e) version part (b) of Proposition 3.1, $\beta^\infty = b^{\infty\prime} > 0$. Thus, $\beta^*(t) > 1$. But then Proposition 3.1(b) implies $\operatorname{Prob}\{\beta^*(t) = b^*(t)'\} \to 1$. Therefore $b^*(t) \to b^\infty > 0$. The converse of part (a) is similar. Next, for $1 \le k \le m-2$, since the model is weakly regular, as in (3.13) one gets

$$\beta^{*(k)}(t) \to \beta^{(k)}, \qquad \frac{\lambda^{(k)}(t)}{t} \to \gamma^{(k)}.$$

In particular, $\gamma^{(k)} > 0$ if and only if $\lambda^{(k)}(t) > t$. Part (b) for this case now follows from Proposition 3.1(d), (e), and for $k = m-1$ similarly, using part (c). $\square$

Proof of Theorem 2.3. First assume the model asymptotically active. By (3.8), it suffices to show $(b^\infty - b^*(t))^t V(t)(b^\infty - b^*(t))$ is bounded in distribution (convergent to zero in probability in the strongly regular case), which is the case if $V^{1/2}(t)(b^\infty - b^*(t))$ is so. But

$$\begin{aligned} V^{1/2}(t)(b^\infty - b^*(t)) &= V^{1/2}(t)[V^{-1}(t)\lambda(t) - (J^\infty)^{-1}\gamma^\infty] \\ &= \left(\frac{V(t)}{t}\right)^{-1/2}\left[\left(\frac{\lambda(t)}{\sqrt{t}} - \sqrt{t}\gamma^\infty\right)\right. \\ &\qquad \left. - \left(\frac{V(t)}{\sqrt{t}} - \sqrt{t}J^\infty\right)(J^\infty)^{-1}\gamma^\infty\right]. \end{aligned}$$

($V^{1/2}(t)$ denotes the symmetric square root of $V(t)$.) If the model is (strongly) regular, then so is $\lambda(t)$; hence, the right side is bounded in distribution (converges to zero) and the desired result is proved. For asymptotically k-inactive models, (3.12) is used, observing that the linear term vanishes because $b_i^* = 0$ for $i \leq k$, while for the same reason, in the quadratic part, $V(t)$ can be replaced by the matrix $V^{(k)}(t)$ consisting of the last $m-k-1$ rows and columns of $V(t)$. So the same argument as above is applicable. $\square$

Proof of Theorem 2.4. Equations (2.12) and (3.8) and the change of variable $x = V^{1/2}(t)(b' - b^{*\prime}(t))$ yield

$$\frac{\hat{W}(t)}{W^*(t)} = \frac{(m-1)!}{|V(t)|^{1/2}} \int_{\Delta_t} e^{-|x|^2/2}\, dx,$$

where $\Delta_t = V^{1/2}(t)(B' - b^{*\prime}(t))$, and we used $\mathrm{vol}(B') = (m-1)!$. Let $U = \{x \in \mathbb{R}^{m-1}: |x| < 1\}$. Since the model is asymptotically active, by Theorem 2.2 there is an $\varepsilon > 0$ such that $\mathrm{Prob}\{B' - b^{*\prime}(t) \supset \varepsilon U\} \to 1$. Recall for any matrix A, $\|A\|^2 \leq \mathrm{tr}(A^t A)$, and if A is invertible then for any vector y, $|Ay| \geq |y|/\|A^{-1}\|$. Applying these to $y = b^*(t)' - b'$ and $A = V^{1/2}(t)$ yields

$$|x|^2 \geq |b^*(t)' - b'|^2/\mathrm{tr}(V^{-1}(t)) \sim |b^*(t)' - b'|^2 t/\mathrm{tr}(J^*)^{-1} > t.$$

It follows that for any $c > 0$, $\mathrm{Prob}\{\Delta_t \supset cU\} \to 1$. So the integral above approaches in probability the integral of $\exp(-|x|^2/2)$ over $\mathbb{R}^{m-1}$, which is $(2\pi)^{(m-1)/2}$, while $|V(t)| \sim t^{m-1}|J^*|$. $\square$

The main step in the proofs of Theorems 2.5 and 2.6 is the following.

LEMMA 3.2.

(a) *Let J be an n by n symmetric positive definite matrix, and $\gamma \in \mathbb{R}^n$ with $\gamma_i > 0$ for all i. (In our applications $n = m-1$.) Then, for any $\varepsilon > 0$,*

$$\int_{[0,\varepsilon]^n} \exp\left(-\frac{t}{2}x^t J x - t\gamma^t x\right) dx \sim \prod_{i=1}^{n} \frac{1}{\gamma_i t}. \tag{3.14}$$

(b) *Let J be as above, and $\gamma \in \mathbb{R}^k$, $\gamma_i > 0$ for all i. Write $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^{n-k}$. Similarly decompose J. Then, for any $\varepsilon > 0$,*

$$I \equiv \int_{[0,\varepsilon]^k \times [-\varepsilon,\varepsilon]^{n-k}} \exp\left(-\frac{t}{2}x^t J x - t\gamma^t x_1\right) dx \sim |J_{22}|^{-1/2}\left(\frac{2\pi}{t}\right)^{(n-k)/2} \prod_{i=1}^{k} \frac{1}{\gamma_i t}. \tag{3.15}$$

(c) *If $U \subset \mathbb{R}^n$, and there is an $\varepsilon > 0$ such that* $[0, \varepsilon]^n \subset U \subset [0, 1/\varepsilon]^n$, *then* (3.14) *holds with the integral taken over U. Similarly, if* $[0, \varepsilon]^k \times [-\varepsilon, \varepsilon]^{n-k} \subset U \subset [0, 1/\varepsilon]^k \times [-1/\varepsilon, 1/\varepsilon]^{n-k}$, *then* (3.15) *holds if the integral is taken over U.*

(d) *Equations* (3.14) *and* (3.15) *still hold if J is replaced by a stochastic process matrix $J(t)$ which converges in probability to a positive definite matrix and γ is replaced similarly.*

Proof. (a) Evidently,

$$\int_{[0,\varepsilon]^n} e^{-tx'Jx/2-t\gamma'x}\,dx < \int_{[0,\infty]^n} e^{-t\gamma'x} = \prod_{i=1}^{n} \frac{1}{\gamma_i t}.$$

Next, by Eq. (5.37) of Cover (1991),

$$1 - \frac{1}{c^2} < ce^{c^2/2} \int_c^\infty e^{-r^2/2}\,dr < 1 \qquad (c > 0).$$

Changing variables and completing the square yields

$$\int_0^\varepsilon e^{-tJx^2/2-t\gamma x}\,dx > \frac{1}{\gamma t}\left(1 - \frac{J}{\gamma^2 t}\right) - \frac{e^{-t\gamma^2/2J-t\varepsilon\gamma}}{(\gamma + \varepsilon J)t} \sim \frac{1}{\gamma t}.$$

This proves (3.14) for $n = 1$. For $n = 2$, choose $0 < \delta < \varepsilon$. Then

$$\int_{[0,\varepsilon]^2} e^{-tx'Jx/2-t\gamma'x}\,dx > \int_0^\delta e^{-tx_2J_{22}x_2/2-t\gamma_2x_2}\,dx_2 \int_0^\varepsilon e^{-tJ_{11}x_1^2/2-t(\gamma_1+\delta|J_{12}|)x_1}\,dx_1$$
$$\sim \frac{1}{\gamma_2(\gamma_1 + \delta|J_{12}|t^2}.$$

Since δ was arbitrary, the assertion follows for $n = 2$, and, in general, similarly.

(b) There are $C_i > 0$ such for any $0 < \delta < \varepsilon$ if $x_2 \in [-\delta, \delta]^{n-k}$ then $|(J_{12}x_2)_i| < \delta C_i$. Thus

$$I > \int_{[0,\varepsilon]^k} e^{-tx_1'J_{11}x_1/2-t(\gamma+\delta C)'x_1}\,dx_1 \int_{[-\delta,\delta]^{n-k}} e^{-tx_2'J_{22}x_2/2}dx_2$$
$$\sim |J_{22}|^{-1/2}\left(\frac{2\pi}{t}\right)^{(n-k)/2} \prod_{i=1}^{k} \frac{1}{(\gamma_i + \delta C_i)t}.$$

On the other hand, the integral in (3.15) over $|x| > \delta$ is exponentially decreasing. Hence,

$$I \sim \int_{[0,\varepsilon]^k\times[-\delta,\delta]^{n-k}} e^{-tx'Jx/2-t\gamma'x_1}\,dx < \int_{[0,\varepsilon]^k} e^{-tx_1'J_{11}x_1/2-t(\gamma-\delta C)'x_1}\,dx_1$$
$$\times \int_{[-\delta,\delta]^{n-k}} e^{-tx_2'J_{22}x_2/2}\,dx_2$$
$$\sim |J_{22}|^{-1/2}\left(\frac{2\pi}{t}\right)^{(n-k)/2} \prod_{i=1}^{k} \frac{1}{(\gamma_i - \delta C_i)t}.$$

Since δ was arbitrary, (b) follows. Part (c) is clear, because, as already used, if $f(t) \sim X(t) \leq Z(t) \leq Y(t) \sim f(t)$, then $Z(t) \sim f(t)$. Part (d) follows by reexamining that the proof still goes through as already illustrated in the proof of Theorem 2.4. □

Proof of Theorem 2.5. By assumption

$$\lambda(t) \sim \gamma^{*}t = \left(\eta_m^{*} - \eta_i^{*} - \frac{1}{2} J_{ii}^{*}\right)t.$$

The theorem now follows from (2.12), (3.9), and parts (a), (c), and (d) of Lemma 3.2. □

Proof of Theorem 2.6. By assumption

$$\lambda^{(k)}(t) \sim \gamma^{(k)}t, \qquad |V^{(k)}(t)|^{-1/2} \sim |J^{(k)}|^{-1/2}/t^{(m-k-1)/2}.$$

Now use (2.12) and (3.12) together with the change of variable $x = b' - b^{*}(t)'$. By assumption, $U \equiv B' - b^{*\prime}$ is a set as in Lemma 3.2(c). The same is therefore true for $B' - b^{*}(t)'$ in probability. So Lemma 3.2(b) is still applicable. □

4. OTHER RESULTS

4.1. Examples

In order for a specific model to satisfy the conclusions of Theorems 2.1, 2.2, 2.4, 2.5, and 2.6, it is required that $\log(S(t))$ and $\Lambda(t)$ be weakly regular. Theorem 2.3 also requires (strong) regularity. The other needed condition is the positivity of the Lagrange multipliers of the quadratic problem (2.10). But this is an open and dense condition, satisfied for all coefficients (the covariance matrix of the quadratic term and the vector of the linear term) outside an algebraic variety of positive codimension.

A necessary condition for a process $X(t)$ to be (weakly/strongly) regular is that $E[X(t)]$ is so. Appropriate conditions on $\mathrm{var}[X(t)]$ make this sufficient.

LEMMA 4.1. *Let $X(t)$ be a stochastic process such that $\theta = \lim E[X(t)]/t$ exists.*

(a) *If $E[X(t)]/\sqrt{t} - \sqrt{t}\theta$ and $\mathrm{var}[X(t)]/t$ are bounded, then $X(t)$ is regular. (In particular, any martingale of the form $\int_0^t f(s)\,dz(s)$ is regular if $E[\int_0^t |f(s)|^2\,ds]/t$ is bounded.)*

(b) *If $E[X(t)]/\sqrt{t} - \sqrt{t}\theta$ and $\mathrm{var}[X(t)]/t$ converge to zero, then $X(t)$ is strongly regular.*

(c) *If $\mathrm{var}[X(t)]/t^2 \to 0$, then $X(t)$ is weakly regular.*

Proof. We only prove (a) since (b) and (c) are similar. Since a linear combination of regular processes is regular and $E[X(t)]$ is regular by assumption, it suffices to show $Y(t) = X(t) - E[X(t)]$ is regular with $\theta = 0$. Let $\varepsilon > 0$. Let M be such that $M^2 > \sup \mathrm{var}[X(t)]/t\varepsilon$. Then, by Chebyshev's inequality, $\mathrm{Prob}\{|Y(t)/\sqrt{t}| > M\} < \mathrm{var}[Y(t)]/tM^2 < \varepsilon$. □

In particular, in a geometric Wiener model, i.e., one in which μ and σ are constants, $\Sigma(t) = \Sigma^\infty = \Lambda/t$ are constants, and $\log(S(t))$ is regular. More strongly, by the strong law of large numbers, $\log(S(t))/t \to \eta$ with probability 1. Also, for any time t, $\log(S(t)/S(0))/\sqrt{t} - \sqrt{t}\eta$ is multivariate normal with mean 0 and constant covariance matrix Σ. So it is not only bounded but convergent in distribution. By (3.3), at any time t, $\beta^*(t)$ is multivariately normally distributed with mean β^∞, and the covariance matrix of $\beta^*(t)$ equals $(J^\infty)^{-1}/t$, which approaches zero. Moreover, $\beta^*(t) \to \beta^\infty$ with probability 1. Similar properties hold for $\beta^{*(k)}(t)$ and $\beta^{(k)}$. It follows that $b^*(t) \to b^\infty$ with probability 1. Similar statements can be made if $\mu(t)$ and $\sigma(t)$ are time dependent but deterministic and appropriately well behaved at infinity.

As a contrasting example, assume σ is constant and $\mu(t) = b(t) - \kappa(t)\log(S(t))$ for a deterministic and bounded vector $b(t)$ and deterministic matrix $\kappa(t)$ with positive eigenvalues bounded away from zero. Then the model is strongly regular. In fact, now $\eta^\infty = 0$, and $E[\log(S(t))]$ and $\mathrm{var}[\log(S(t))]$ are bounded, so Lemma 4.1(b) applies. Thus, for such models $W(t;\, b^\infty) \sim W^*(t)$. This time, again, $\beta^*(t)$ is multivariately normally distributed with mean β^∞, but its covariance matrix approaches zero like $1/t^2$. Again, these statements generalize if $\Lambda(t)$ is deterministic and strongly regular. For $m = 2$, weak regularity is sufficient. Then $b^\infty = (\frac{1}{2}, \frac{1}{2})$, and the model is asymptotically active.

4.2. Log-Optimum Rebalanced Portfolios

The asymptotically optimal constant rebalanced portfolio maximizes the long-term expected rate of return among all constant rebalanced portfolios. In discrete time, Algoet and Cover (1988) define the *log-optimum portfolio* as the rebalanced portfolio whose weight at each period maximizes the conditional expectation of the logarithm of portfolio value at the end of the period. They show that in the long run it outgrows any other nonanticipating, time-varying rebalanced portfolio. By analogy one may formulate continuous-time log optimality as the continuous maximization over B of

$$E_t[d(\log W(t;\, b))]/dt = -\frac{1}{2}\, b^t\Sigma(t)b + b^t\mu(t).$$

The weight of the log-optimum portfolio at time t is defined to be the solution $b^{**}(t)$ of this quadratic programming problem. In continuous time, the log-optimum portfolio maximizes the expected rate of return globally as well as locally in time.

PROPOSITION 4.1. *Let $T > 0$. Suppose $E[\mathrm{tr}\,\Lambda(T)] < \infty$ and $E[\int_0^T |\mu_i(t)|\, dt] < \infty$ for all i. Then the log-optimum portfolio maximizes $E[\log(W(T))]$ among all rebalanced portfolios.*

Proof. Clearly, $\int_0^t |b(s)^t\sigma(s)|^2\, ds \le \mathrm{tr}(\Lambda(t))$ for $b(t) \in B$. The assumption $E[\mathrm{tr}\,\Lambda(t)] < \infty$ (which is equivalent to $E[\int_0^t \sigma_{ij}^2(s)\, ds] < \infty$ for all i, j) thus implies that $E[\int_0^t b(s)^t\sigma(s)\, dz(s)] = 0$. Hence, by the assumptions, (2.3), and Itô's lemma, for any rebalanced portfolio with wealth $W(t)$ and weight $b(t)$ we have

$$E\left[\log \frac{W(T)}{W(0)}\right] = E\int_0^T \left[b^t(t)\mu(t) - \frac{1}{2}\, b^t(t)\Sigma(t)b(t)\right] dt.$$

The claim follows because by definition $b^{**}(t)$ maximizes the integrand for all t. □

In a geometric Wiener model (i.e., one with constant μ and σ), the log-optimum portfolio is evidently the same as the asymptotically optimal constant rebalanced portfolio; that is, the log-optimum weight $b^{**}(t)$ equals b^{∞}. More generally, if $\Sigma(t)$ and $\mu(t)$ have limits, then $\Lambda(t)$ and $\log(S(t))$ are weakly regular, and these limits respectively equal Σ^{∞} and μ^{∞}.[7] Then, as in the proof of Theorem 2.2, one can then show that $b^{**}(t) \to b^{\infty}$.

When short selling is allowed, then, under suitable boundedness conditions, $W(t)/W^{**}(t)$ is a martingale for all rebalanced wealth $W(t)$. In our setting with short-sale prohibition, the following special case is more readily formulated; its proof generalizes.

PROPOSITION 4.2. *Suppose μ and σ are constants and the model is asymptotically active (so b^{∞} is interior). Then for any rebalanced portfolio with wealth $W(t)$ and all $0 \le t \le T$,*

$$W(t) = W(t;\, b^{\infty})E_t[W(T)/W(T;\, b^{\infty})].$$

In particular, for all $t > 0$, $E[W(t)/W(t;\, b^{\infty})] = 1$ (despite the fact that $W(t;\, b)/W(t;\, b^{\infty}) \approx 0$), which in turn implies (since $1/x$ is convex) $E[W(t;\, b^{\infty})/W(t)] > 1$.

Proof. The first-order conditions imply that $\Sigma b^{\infty} - \mu = \alpha\mathbf{1}$ for some scalar α, where $\mathbf{1} = (1, \ldots, 1)^{t}$. Hence, $(b^{\infty} - b(t))^{t}(\Sigma b^{\infty} - \mu) = 0$. For any Itô process $X(t) > 0$, write $dX/X = \mu_X\, dt + \sigma_X\, dz$. Then by Itô's division rule, $\mu_{X/Y} = \mu_X - \mu_Y + (\sigma_Y - \sigma_X)\sigma_Y^{t}$. Thus, by (2.3)

$$\mu_{W(t)/W(t;b^{\infty})} = (b(t) - b^{\infty})^{t}\mu + (b^{\infty t}\sigma - b(t)^{t}\sigma)\sigma^{t}b^{\infty} = 0,$$

and $\sigma_{W(t)/W(t;b^{\infty})} = (b(t) - b^{\infty})^{t}\sigma$, which is bounded. It follows that $W(t)/W(t;\, b^{\infty})$ is a martingale. □

4.3. Expected Universal Wealth

PROPOSITION 4.3. *Suppose the drifts $\mu_i(t)$ are deterministic and $E[\exp(\frac{1}{2}\operatorname{tr}(\Lambda(t)))] < \infty$. Set*

$$u_i(t) \equiv \int_0^t \mu_i(s)\, ds. \tag{4.1}$$

The following hold.

(a)

$$E[W(t;\, b)] = W(0)\exp(\Sigma\, b_i u_i(t)). \tag{4.2}$$

(b) *If at time t, $u_i(t)$ are all equal, then the expected values of all self-financing portfolios are the same, namely, $E[W(t)] = W(0)\exp(u_m(t))$.*

(c) *If the drift of (say) stock m dominates the rest, i.e., if $\mu_m(t) > \mu_i(t)$ for $i < m$,*

[7] The existence of these limits is not a requirement of weak regularity. For example, the volatility matrix $\sigma(t) = \sigma_0 + (\sin t)\alpha$ (with σ_0 and α constant matrices) oscillates without a limit. Yet $\int_0^t \sigma(s)\, ds/t \to \sigma_0$, $\Lambda(t)$ is weakly regular, and $\Sigma^{\infty} = \sigma_0\sigma_0^{t} + \alpha\alpha^{t}/2$.

then the expected wealth from buying and holding stock m exceeds the expected wealth from any other self-financing portfolio; i.e., $E[W(t)] < W(0)\exp(u_m(t))$.

(d) *Unless* $u_1(t) = \cdots = u_m(t)$ *(in which case* (b) *applies), the expected evenly diversified buy-and-hold wealth is greater than the expected universal wealth, which in turn is greater than the expected equally weighted constant rebalanced wealth; i.e.,*

$$\exp\left(\frac{1}{m}\sum_{i=1}^{m} u_i(t)\right) < \frac{E[\hat{W}(t)]}{W(0)} < \frac{1}{m}\sum_{i=1}^{m} \exp(u_i(t)). \tag{4.3}$$

(e) *The expected universal wealth for any holding period t is*

$$E[\hat{W}(t)] = W(0)(m-1)! \sum_{i=1}^{m} \frac{e^{u_i(t)}}{\prod_{j \neq i}(u_i(t) - u_j(t))}. \tag{4.4}$$

In particular, for $m = 2$ *and* $m = 3$ *we have*

$$E[\hat{W}(t)] = W(0)\,\frac{e^{u_1(t)} - e^{u_2(t)}}{u_1(t) - u_2(t)} \qquad (m = 2);$$

$$E[\hat{W}(t)] = 2W(0)\left[\frac{e^{u_1}}{(u_1 - u_2)(u_1 - u_3)} + \frac{e^{u_2}}{(u_2 - u_1)(u_2 - u_3)} + \frac{e^{u_3}}{(u_3 - u_1)(u_3 - u_2)}\right] \qquad (m = 3).$$

Proof. For any Itô process $X(t) > 0$ set $dX/X = \mu_X\, dt + \sigma_X\, dz$. Then $Y(t) \equiv \exp(-\int_0^t \mu_X(s)\, ds)X(t)$ satisfies $dY/Y = \sigma_X\, dz$ and is a martingale on $[0, T]$ if $E[\exp(\frac{1}{2}\int_0^T |\sigma_X(t)|^2\, dt] < \infty$. One then gets $X(0) = E[\exp(-\int_0^t \mu_X(s)\, ds)X(t)]$. In particular, if μ_X is deterministic, then $E[X(t)] = X(0)\exp(\int_0^t \mu_X(s)\, ds)$. Parts (a) and (b) follow. Part (c) follows since $\mu_W < \mu_m$; therefore,

$$\begin{aligned} W(0) &= E[\exp(-\int_0^t \mu_w(s)\, ds)W(t)] > E[\exp(-u_m(t))W(t)] \\ &= \exp(-u_m(t))E[W(t)]. \end{aligned}$$

To prove (d), note by (2.12) and (4.2),

$$E[\hat{W}(t)] = W(0) \int_B \exp\left(\sum_{i=0}^{m} b_i u_i(t)\right) db \Big/ \int_B db.$$

Since the exponential function is convex, $\exp(\Sigma b_i u_i(t)) < \Sigma b_i \exp(u_i(t))$.[8] The second inequality in (4.3) follows. The first inequality follows similarly by Jensen's inequality. Finally, let $E_m(u_1, \ldots, u_m)$ denote the right side of the above equation. One shows

[8] This shows that (for deterministic drifts) the expected wealth of a constant rebalanced portfolio is lower than that of the buy-and-hold portfolio initially having the same weight.

$$E_m(u_1, \ldots, u_m)$$
$$= (m - 1)e^{u_m} \int_0^1 x^{m-2} E_{m-1}(x(u_1 - u_m), \ldots, x(u_{m-1} - u_m))\, dx.$$

Equation (4.4) now follows easily from this recursive relation by induction, the identity

$$\sum_{i=1}^{m} \frac{1}{\prod_{j \neq i} (u_i - u_j)} = 0,$$

and checking that for $m = 2$, $E_2(u_1, u_2)$ is indeed as given in part (e). □

4.4. Generalized Universal Portfolio

As asserted by Cover, and as Theorem 2.6 shows, the long-term performance of the universal portfolio relative to the target optimal portfolio in asymptotically k-inactive models is worse than that of asymptotically active models by the factor $t^{k/2}$. Cover's *generalized universal portfolio* improves the performance by this factor. Its weights are defined as in (2.11), except that the Euclidean surface element db is replaced by the probability measure on B which is the normalized sum of the surface elements on each of the $2^m - 1$ faces of the simplex B. Then a formula similar to (2.12) holds for the generalized universal wealth $\hat{S}(t)$. The generalized universal portfolio is an evenly diversified buy-and-hold portfolio of universal portfolios corresponding to each (nonempty) subset of the stocks (each of which is of course rebalanced). Now, if the model is asymptotically k-inactive, then by Theorems 2.4–2.6, the largest asymptotic contribution relative to the target optimal portfolio is made by the subset consisting of the last $m - k$ stocks. It follows, as in (9.4) of Cover (1991) that (with k there replaced by $m - k$ here)

$$\frac{\hat{S}(t)}{W^*(t)} \sim \frac{(m - k - 1)!}{2^m - 1} |(J^{(k)}|^{-1/2} \left(\frac{2\pi}{t}\right)^{(m-k-1)/2} \tag{4.5}$$

Proposition 4.3 and the symmetry of the situation imply that, if the drifts are deterministic, then just like the universal wealth, the expected generalized universal wealth is less than the expected wealth from the evenly diversified buy-and-hold portfolio and greater than the expected wealth of the equally weighted constant rebalanced portfolio.

4.5. Portfolios of Two Stocks

PROPOSITION 4.4. *For $m = 2$, set*

$$v^2(t) \equiv V(t) = \Lambda_{11}(t) + \Lambda_{22}(t) - 2\Lambda_{12}(t),$$

$$h(t) \equiv \frac{\log(S_2(t)/S_2(0)) - \log(S_1(t)/S_1(0))}{v(t)}.$$

Then, with $\Phi(x)$ denoting the standard normal distribution function, the following hold:

$$W\left(t; \frac{1}{2}\right) = W(0) \sqrt{\frac{S_1(t)S_2(t)}{S_1(0)S_2(0)}}\, e^{v^2(t)/8}, \tag{4.6}$$

$$\hat{W}(t) = W\left(t; \frac{1}{2}\right) \frac{\sqrt{2\pi}}{v(t)} e^{h^2(t)/2} \left[\Phi\left(h(t) + \frac{v(t)}{2}\right) - \Phi\left(h(t) - \frac{v(t)}{2}\right)\right], \tag{4.7}$$

$$\hat{b}(t) = \left(\frac{1}{2} - B(t), \frac{1}{2} + B(t)\right), \tag{4.8}$$

$$B(t) \equiv \frac{1}{v}\left[h + \frac{e^{-(h+v/2)^2/2} - e^{-(h-v/2)^2/2}}{\sqrt{2\pi}\,(\Phi(h + v/2) - \Phi(h - v/2))}\right]$$

$$= \frac{h(t)}{v(t)} + \frac{S_1(t)/S_1(0) - S_2(t)/S_2(0)}{v^2(t)\hat{W}(t)}.$$

Moreover, both stocks are active if and only if $|h(t)| < v(t)/2$, *in which case*

$$b^*(t) = (\tfrac{1}{2} - h(t)/v(t), \tfrac{1}{2} + h(t)/v(t)),$$

and

$$W^*(t) = W\left(t; \frac{1}{2}\right) e^{h^2(t)/2}$$

$$= W(0) \frac{S_1(t)}{S_1(0)} e^{(h(t)+v(t)/2)^2/2} = W(0) \frac{S_2(t)}{S_2(0)} e^{(h(t)-v(t)/2)^2/2}.$$

Proof. In view of Proposition 3.1, it only remains to prove (4.8). But this follows by calculating $d(\log(\hat{W}(t)))$ from (4.7) and looking at the coefficient of dS_i, which is $\hat{b}_i(t)$. □

Equation (4.7) easily implies that if $|h(t)| < 1$, then $\hat{W}(t) < W(t;\ 1/2)$, while for $|h(t)| > 1 + v(t)/2$, $\hat{W}(t) > W(t;\ 1/2)$. Moreover, for fixed $h(t)$ with $|h(t)| > 1$, the ratio $\hat{W}(t)/W(t;\ 1/2) > 1$ for small values of $v(t)$, and it keeps increasing as $v(t)$ increases, reaches a maximum, and then decreases, eventually approaching zero for large $v(t)$.

The function $B(t)$ above is an odd function of $h(t)$. In particular, $B(t) = 0$ if $h(t) = 0$. This means that if at any time the two stocks have appreciated equally, then the universal portfolio is equally weighted in the two stocks at that time. It can be shown that $B(t)$ is an increasing function of $h(t)$. In particular, it is positive if $h(t) > 0$, i.e., if $S_2(t)/S_2(0) > S_1(t)/S_1(0)$. In other words, in the universal portfolio, more investment is made in the stock that has grown more (percentagewise). Furthermore, letting $n_i(t) = \hat{b}_i(t)S_i(0)/S_i(t)$ denote the "normalized number of shares" in stock i (that is the number of shares in stock i if initially both stocks had unit prices), it is the case that if $S_2(t)/S_2(0) > S_1(t)/S_1(0)$ then $n_2(t) < n_1(t)$. In other words, in the universal portfolio there are fewer normalized shares of the stock that has appreciated more. Thus, as one stock outperforms the other, there is some profit taking, but not to the extent of leaving less money invested in the stock that has performed better.

4.6. Example of an Unstable Model

Let $m = 2$ and assume stock prices follow a geometric random walk. We write $J = J^\infty$, etc. Assume $\eta_2 = \eta_1 + J/2$. Then $b^\infty = (0, 1)$. But $\gamma^{(1)} = 0$, so the model is not stable. In this case still $b^*(t) \to b^\infty$ in probability, but $\mathrm{Prob}\{b^*(t) = b^\infty\}$ does not converge to 1, and buy-and-hold in the second stock is not an asymptotically strongly optimal portfolio (though it is an asymptotically optimal portfolio). The asymptotic formulae in Theorems 2.4 and 2.5 do not hold either. Indeed, $W^*(t)/(\sqrt{t}\hat{W}(t))$ does not converge to a constant as in asymptotically active models. However, it remains bounded in distribution, so the universal portfolio is still asymptotically weakly optimal.

5. CONCLUSION

We showed that a unique asymptotically optimal constant rebalanced portfolio exists, but its determination requires the knowledge of the long-term averages of future instantaneous expected returns and covariances of stock prices (which are assumed to have limits under the condition of weak regularity). As such long-term knowledge is not available in practice, the universal portfolio (or the generalized universal portfolio) is the best long-term bet because it is always asymptotically weakly optimal, underperforming the target optimal portfolio only by a power of time, and outperforming exponentially all constant rebalanced portfolios (except if the asymptotically optimal constant rebalanced portfolio was chosen by a fluke).

As explained by Cover (1991), the universal portfolio succeeds because it is performance weighted; i.e., it rebalances its weight according to the past performance of constant rebalanced portfolios, putting more weight on the better-performing ones. For weakly regular models, our results elucidate this point further. Indeed, if there is a constant weight that will in the long run outperform its neighbors exponentially, it is not necessary to know its exact location, but it is sufficient to average all weights by their past performance, recognizing that the superior trend of the asymptotically optimal constant weight causes it to be weighed heavily.

The universal portfolio enjoys an optimal growth rate, but its expected wealth is not necessarily so high. This is not a material disadvantage. Indeed, due to compounding, the wealth behaves in a certain way that is unrelated to expected wealth. It is the expected rate of return rather than expected wealth that provides a reliable indicator of the portfolio growth rate.

The assumption of weak regularity is convenient for stating asymptotic results, but in the final analysis, performance weighting works even when the time averages of stock drifts and instantaneous covariances have no limits. In general, the universal portfolio tracks the running values of these time averages.[9] It would be interesting to investigate ways to relax the requirement of weak regularity.

[9] For example, the proof of Theorem 2.4 yields the following generalization of (2.14): If $b^*(t) > 1$, then $\hat{W}(t)/W^*(t) \sim (m-1)!(2\pi)^{(m-1)/2}|V(t)|^{1/2}$, provided only that $\mathrm{tr}((V(t)/t)^{-1}) < 1$. (This more general statement is more in the spirit of Cover's Theorem 6.1, which, by focusing on the empirical distribution of stock prices, requires no distributional assumptions and imposes only a boundedness condition on the market sequence.) The usefulness of weak regularity lies in providing a criterion for such boundedness away from zero requirements as $b^*(t) > 1$, which is based explicitly on the drift and volatility parameters $\mu(t)$ and $\sigma(t)$ of the model (2.1) (namely, on the position of b^∞).

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