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STABILITY OF THE EXPONENTIAL UTILITY MAXIMIZATION PROBLEM WITH RESPECT TO PREFERENCES

HAO XING

London School of Economics and Political Science

This paper studies stability of the exponential utility maximization when there are small variations on agent's utility function. Two settings are considered. First, in a general semimartingale model where random endowments are present, a sequence of utilities defined on $\mathbb{R}$ converges to the exponential utility. Under a uniform condition on their marginal utilities, convergence of value functions, optimal payoffs, and optimal investment strategies are obtained, their rate of convergence is also determined. Stability of utility-based pricing is studied as an application. Second, a sequence of utilities defined on $\mathbb{R}_+$ converges to the exponential utility after shifting and scaling. Their associated optimal strategies, after appropriate scaling, converge to the optimal strategy for the exponential hedging problem. This complements Theorem 3.2 in [Nutz, M. (2012): Risk aversion asymptotics for power utility maximization. *Probab. Theory & Relat. Fields* 152, 703–749], which establishes the convergence for a sequence of power utilities.

KEY WORDS: utility maximization, exponential utility, stability, semimartingales, utility-based prices.

1. INTRODUCTION

This paper considers an optimal investment problem where an agent, whose preference is described by a utility function, seeks to maximize expected utility of her wealth from investment and a random endowment (illiquid asset) at an investment horizon $T \in \mathbb{R}_+$. Given two problem primitives, utility function and market structure, the goal is to identify the optimal investment strategy that the agent undertakes. When the utility has constant absolute risk aversion, Delbaen et al. (2002) give an elegant solution to this problem. We study in this paper stability of the optimal investment strategy when agent's utility deviates from exponential utility. In particular, we are interested in a *quantitative* measure on how far the optimal strategy deviates when there are small variations on agent's preference.

Two settings are studied. First, consider a sequence of utility functions $(U_\delta)_{\delta>0}$, each of which is defined on $\mathbb{R}$, such that it converges pointwise to U_0, which has unit absolute risk aversion. Deviation is measured by two components: (i) the ratio of marginal utilities $\mathfrak{R}_\delta$ between U_δ and an exponential utility $\widetilde{U}_\delta$ with absolute risk aversion α_δ; and (ii) the difference between α_δ and 1. The first component measures how far U_δ is away from an exponential utility; while the second component describes how far this exponential utility is away from the exponential utility with unit risk aversion. When $\mathfrak{R}_\delta$ is bounded

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Address correspondence to the author at Department of Statistics, London School of Economics and Political Science, 10 Houghton St, London, WC2A 2AE, UK; e-mail:h.xing@lse.ac.uk.

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from above and away from zero, uniformly in δ, our first main result, Theorem 2.8, states the convergence of the optimal payoffs and value functions in a general semimartingale model; moreover the convergence of optimal strategies also follows, when asset price processes are continuous. Beyond these continuity results, the rate of convergence is determined in Corollary 2.11. The aforementioned two components of variations impact deviation of the optimal payoff (hence the optimal strategy) at different rates: the convergence of absolute risk aversions has *first-order* impact, while the convergence of $\mathfrak{R}_\delta$ has *second-order* effect. Stability of utility-based prices, Davis price, and indifference price, with respect to agent's preference is also discussed as an application; cf. Corollaries 2.13 and 2.14.

The stability problem studied in the first setting is similar to Carassus and Rásonyi (2007), where the problem is formulated in a discrete time setting and asset price processes are assumed to be bounded. For utilities defined on $\mathbb{R}_+$, the aforementioned stability problem has been extensively studied. Jouini and Napp (2004) consider an Itô process model. Larsen (2009) extends the analysis to continuous semimartingale models. Kardaras and Žitković (2011) allow simultaneous variations on preferences and subjective probabilities. More recently, Mocha and Westray (2013) focus on the power utility maximization problem and investigate stability respect to relative risk aversion, market price of risk, and investment constraints.

In Larsen (2009) and Kardaras and Žitković (2011), convergence in probability of optimal payoffs is obtained under an uniform integrability assumption. One can prove that the optimal investment strategies also converge; cf. Remark 2.9. Our uniform bound on the ratio of marginal utilities implies an analogous integrability condition; cf. Remark 2.10. However the additional structure imposed here allows us to obtain more precise information on how fast the convergence takes place.

A different type of stability problem is studied in Larsen and Žitković (2007). Therein stability of the optimal payoff with respect to market variations is studied while a utility defined on $\mathbb{R}_+$ is fixed. This type of stability problem has recently been investigated in Frei (2013) and Bayraktar and Kravitz (2013) for the exponential utility maximization problem.

In the second setting, we consider a sequence of utility random fields $(\mathcal{U}_p)_{p<0}$, each of which is of the form $\mathcal{U}_p = D\, U_p$ for a positive random variable D and a utility function U_p defined on $\mathbb{R}_+$. For each U_p, the ratio of its marginal utility with respect to x^{p-1} is bounded from above and away from zero. In this sense, U_p is comparable to power utility $\widetilde{U}_p = x^p/p$ with constant relative risk aversion $1-p$. As the ratio of marginal utilities going to 1, $(U_p)_{p<0}$ approaches $(\widetilde{U}_p)_{p<0}$, which converges to exponential utility, with appropriate domain shift, as $p \downarrow -\infty$ (cf. Nutz 2012, remark 3.3).

Our second main result, Theorem 2.20, states that, when the ratio of marginal utilities converges to 1 at a rate at least as fast as the relative risk aversion going to infinity, then the optimal proportion invested in risky assets, scaled by $1-p$, converges to the optimal monetary value invested in risky assets in the exponential hedging problem. Therein $(1-p)^{-1}$ can be regarded as the rate of convergence. This result is first obtained in Nutz (2012), where $(U_p)_{p<0}$ is a sequence of power utilities. We complement Nutz's result by allowing deviation from power utility and analyze the impact on the convergence from the ratio of marginal utilities. On the dual side, the stability problem formulated here is related to the convergence of optimal martingale measures, which is studied in Grandits and Rheinländer (2002), Mania and Tevzadze (2003), and Santacroce (2005).

The starting point of our proofs in both settings is the following key result from the *duality theory*: the optimal wealth process is a *martingale* after multiplied by the optimal dual process, and a *supermartingale* after multiplied by any other processes in the dual domain. When random endowment presents, the aforementioned properties have been proved in Owen and Žitković (2009) for utility defined on $\mathbb{R}$ and in Karatzas and Žitković (2003) for utility defined on $\mathbb{R}_+$. This property, combined with scaling properties of exponential (resp. power) utility, leads to an estimate on the difference (resp. ratio) of optimal payoffs for U_δ (resp. U_p) and exponential (resp. power) utility. The remaining proof does not depend on the market specifications. Therefore methods in this paper could potentially be applied to other market settings where the aforementioned property on the optimal wealth process holds, for example, markets with transaction cost, see Cvitanić and Karatzas (1996), and the utility maximization with forward criteria, see Musiela and Zariphopoulou (2009).

The structure of the paper is simple. After this introduction, Section 2 describes the problems and states main results, while all proofs are given in Sections 3 and 4.

2. MAIN RESULTS

We consider a financial market of d-risky assets whose discounted prices are modeled by a locally bounded $\mathbb{R}^d$-valued semimartingale $(S_t)_{t\in[0,T]}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, in which $\mathcal{F}_0$ coincides with the family of $\mathbb{P}$-null sets and $(\mathcal{F}_t)_{t\in[0,T]}$ is right continuous. When price processes are nonlocally bounded, we refer reader to Biagini and Frittelli (2005, 2007).

2.1. Utilities Defined on $\mathbb{R}$

Consider a sequence of standard utility functions[1] $U_\delta : \mathbb{R} \to \mathbb{R}$, indexed by $\delta \geq 0$, converging in the following sense:

ASSUMPTION 2.1. $\lim_{\delta\downarrow 0} U_\delta(x) = U_0(x)$ *for* $x \in \mathbb{R}$, *where* $U_0(x) = -\exp(-x)$.[2]

The pointwise convergence of utility functions is widely used in the literature, for example, Jouini and Napp (2004) and Larsen (2009). The pointwise convergence, restricted to the class of concave functions (utility functions), implies a more economic meaningful mode of convergence: the pointwise (and hence locally uniformly) convergence of their derivatives (marginal utilities); see Rockafellar (1970, p. 90 and p. 248). However the pointwise convergence is not enough for the stability of utility maximization problem; see an counterexample in Larsen (2009). We further restrict each U_δ to a class of utilities, which are comparable to the exponential utility $-\frac{1}{\alpha_\delta}\exp(-\alpha_\delta x)$.

ASSUMPTION 2.2. *There exist constants* $0 < \underline{\mathfrak{R}} \leq 1 \leq \overline{\mathfrak{R}}$ *and* $(\alpha_\delta)_{\delta>0}$ *with* $\lim_{\delta\downarrow 0}\alpha_\delta = 1$ *such that*

$$\underline{\mathfrak{R}} \leq \mathfrak{R}_\delta(x) := \frac{U_\delta'(x)}{\exp(-\alpha_\delta x)} \leq \overline{\mathfrak{R}}, \qquad \text{for all } \delta > 0 \text{ and } x \in \mathbb{R}.$$

REMARK 2.3. This assumption implies that each U_δ is bounded from above. Indeed, integrating $\underline{\mathfrak{R}}\exp(-\alpha_\delta x) \leq U_\delta'(x) \leq \overline{\mathfrak{R}}\exp(-\alpha_\delta x)$ on $(0,\infty)$ yields $\underline{\mathfrak{R}}/\alpha_\delta + U_\delta(0) \leq U_\delta(\infty) \leq \overline{\mathfrak{R}}/\alpha_\delta + U_\delta(0)$.[3] Moreover, U_δ is sandwiched between two utilities with constant

[1]A standard utility function is strictly increasing, strictly concave, and continuously differentiable.

[2]After appropriate scaling all results in this paper hold when U_0 has other value of absolute risk aversion.

[3]These bounds can be made uniform in δ, since $\lim_{\delta\downarrow 0} U_\delta(0) = -1$ and $\lim_{\delta\downarrow 0}\alpha_\delta = 1$.

absolute risk aversion α_δ. To see this, integrating the previous bounds for $U'_\delta(x)$ on (x,∞) induces $U_\delta(\infty) - \frac{1}{\alpha_\delta}\overline{\mathfrak{R}}\exp(-\alpha_\delta x) \le U_\delta(x) \le U_\delta(\infty) - \frac{1}{\alpha_\delta}\underline{\mathfrak{R}}\exp(-\alpha_\delta x)$ for any $x \in \mathbb{R}$. One can also derive from Assumption 2.2 that each U_δ satisfies the Inada conditions, that is, $\lim_{x\downarrow-\infty} U'_\delta(x) = \infty$ and $\lim_{x\uparrow\infty} U'_\delta(x) = 0$, and U_δ has reasonable asymptotic elasticity, that is,

$$AE_{-\infty}(U_\delta) := \liminf_{x\downarrow-\infty} \frac{xU'_\delta(x)}{U_\delta(x)} > 1 \quad \text{and} \quad AE_{\infty}(U_\delta) := \limsup_{x\uparrow\infty} \frac{xU'_\delta(x)}{U(x)} < 1.$$

Hence each U_δ is reasonable risk averse at high and low wealth limit; cf. Kramkov and Schachermayer (1999, 2003).

To introduce the utility maximization problem considered, we denote by M^a (resp. M^e) the class of probability measures $\widetilde{\mathbb{P}} \ll \mathbb{P}$ (resp. $\widetilde{\mathbb{P}} \sim \mathbb{P}$) such that S is a local martingale under $\widetilde{\mathbb{P}}$. Consider the convex conjugate $V_\delta : (0,\infty) \to \mathbb{R}$ defined by $V_\delta(y) := \sup_{x\in\mathbb{R}}(U_\delta(x) - xy)$. The *generalized entropy* of $\widetilde{\mathbb{P}} \in M^a$ relative to $\mathbb{P}$ is defined as $\mathbb{E}_{\mathbb{P}}[V_\delta(d\widetilde{\mathbb{P}}/d\mathbb{P})] \in (0,\infty]$. We denote by $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) the set of probability measures $\widetilde{\mathbb{P}} \in M^a$ (resp. $\widetilde{\mathbb{P}} \in M^e$) with finite generalized entropy. Even though definition of $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) depends on V_δ, Lemma 3.1 shows that all $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) are the same for $\delta \ge 0$ under Assumption 2.2. Henceforth we drop the subscript δ and write $\mathcal{M}^a$ (resp. $\mathcal{M}^e$) instead.

There is an agent whose preference is described by one of the utility function U_δ. She is able to trade in the financial market and has a random endowment ξ_δ, which is an $\mathcal{F}_T$-measurable random variable. Following Owen and Žitković (2009), we assume that ξ_δ is potentially unbounded but can be superhedged.

ASSUMPTION 2.4. *There exist $x_\delta, \widetilde{x}_\delta \in \mathbb{R}$ and a predictable S-integrable process G_δ such that*

$$x_\delta \le \xi_\delta \le \widetilde{x}_\delta + G_\delta \cdot S_T, \qquad \textit{for each } \delta \ge 0,$$

where $G_\delta \cdot S$ is $\mathbb{P}$-a.s. uniformly bounded from below by a constant and $G_\delta \cdot S_T$ stands for $\int_0^T G_{\delta,t} dS_t$.

When the utility function is defined on $\mathbb{R}$, the class of wealth processes with uniform lower bound is not large enough for the problem considered below; cf. Schachermayer (2001). Therefore we recall the following class of permissible strategies from Owen and Žitković (2009): H is a *permissible trading strategy* if it is inside[4]

$$\mathcal{H}^{perm} := \left\{ H : \begin{array}{c} H \text{ is a predictable, } S\text{-integrable process such that} \\ H \cdot S \text{ is a } \widetilde{\mathbb{P}}\text{-supermartingale for all } \widetilde{\mathbb{P}} \in \mathcal{M}^a \end{array} \right\}.$$

Our agent chooses permissible strategies to maximize her utility on wealth and endowment at an investment horizon T:

$$u_\delta := \sup_{H\in\mathcal{H}^{perm}} \mathbb{E}_{\mathbb{P}}\left[U_\delta\left(H \cdot S_T + \xi_\delta\right)\right]. \tag{2.1}$$

In order to ensure the existence of the optimal strategy, we impose

[4]Since $\mathcal{M}^a$ is the same for different δ, $\mathcal{H}^{perm}$ is independent of δ as well. Therefore even though the utility of the agent may change with respect to δ, she always chooses trading strategy from the same permissible class.

ASSUMPTION 2.5. $\mathcal{M}^e \neq \emptyset$.

When U_δ has reasonable asymptotic elasticity, $\mathcal{M}^a \neq \emptyset$, and Assumption 2.4 holds, Assumption 2.5 is actually the necessary and sufficient condition for the existence of optimal strategy for equation (2.1); cf. Owen and Žitković (2009, Theorem 1.9). We further recall the following result from Owen and Žitković (2009).

PROPOSITION 2.6 (Owen–Žitković). *Let U_δ be of reasonable asymptotic elastic and Assumptions 2.4 and 2.5 hold. Then there exists an optimal strategy $H_\delta \in \mathcal{H}^{perm}$ for equation* (2.1) *such that $H_\delta \cdot S$ is a $\widetilde{\mathbb{P}}$-supermartingale for all $\widetilde{\mathbb{P}} \in \mathcal{M}^a$ and a $\mathbb{Q}_\delta$-martingale for some $\mathbb{Q}_\delta \in \mathcal{M}^e$, whose density $d\mathbb{Q}_\delta/d\mathbb{P}$ satisfies*

$$y_\delta \frac{d\mathbb{Q}_\delta}{d\mathbb{P}} = U'_\delta\left(H_\delta \cdot S_T + \xi_\delta\right), \quad \textit{for some positive constant } y_\delta.$$

In the previous result, $\mathbb{Q}_0$ is the the *minimal entropy measure* that minimizes $\mathbb{E}_{\mathbb{P}}[V_0(d\widetilde{\mathbb{P}}/d\mathbb{P})]$, with $V_0(y) = y \log y - y$, among all $\widetilde{\mathbb{P}} \in M^a$. To simplify notation, we drop the subscript 0 and denote the minimal entropy measure by $\mathbb{Q}$. In order to investigate the convergence of equation (2.1) and its optimal strategy as $\delta \downarrow 0$. We assume the following convergence of random endowments:

ASSUMPTION 2.7. *There exists a constant $C \in \mathbb{R}_+$ such that $\alpha_\delta \xi_\delta - \xi_0 \geq -C$, $\mathbb{P}$-a.s. for all $\delta > 0$. Moreover* $\lim_{\delta \downarrow 0} \mathbb{E}_{\mathbb{Q}}[|\alpha_\delta \xi_\delta - \xi_0|] = 0$.

The previous assumption clearly holds when $(\xi_\delta)_{\delta \geq 0}$ is uniformly bounded and $\mathbb{Q} - \lim_{\delta \downarrow 0} \xi_\delta = \xi_0$, where $\mathbb{Q} - \lim$ represents convergence in probability $\mathbb{Q}$. Denote the optimal payoff by $X^\delta_T = H_\delta \cdot S_T$ for $\delta \geq 0$. The first main result states the convergence of X^δ_T, its associated strategy, and u_δ, as $\delta \downarrow 0$.

THEOREM 2.8. *Let Assumptions 2.1, 2.2, 2.4, 2.5, and 2.7 hold. Then the following statements hold:*

(i) $\lim_{\delta \downarrow 0} \mathbb{E}_{\mathbb{Q}}[|X^\delta_T - X^0_T|] = 0$;
(ii) $\lim_{\delta \downarrow 0} u_\delta = u_0$;
(iii) If S is continuous then

$$\lim_{\delta \downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (H_\delta - H_0)_t^\top d\langle S\rangle_t (H_\delta - H_0)_t\right)^{p/2}\right] = 0, \quad \textit{for any } p \in (0,1).$$

REMARK 2.9. When $(U_\delta)_{\delta \geq 0}$ are defined on $\mathbb{R}_+$, the analogue result has been proved in Larsen (2009) and Kardaras and Žitković (2011). Therein $\mathbb{P} - \lim_{\delta \downarrow 0} X^\delta_T / X^0_T = 1$ and $\lim_{\delta \downarrow 0} u_\delta = u_0$ are proved. Define $\overline{\mathbb{P}}$ via $d\overline{\mathbb{P}}/d\mathbb{P} = c U'_0(X^0_T) X^0_T$ for a normalization constant c. Then X^0 has the numéraire property under $\overline{\mathbb{P}}$,that is, X^δ / X^0 is a $\overline{\mathbb{P}}$-supermartingale. Then $\lim_{\delta \downarrow 0} \mathbb{E}_{\overline{\mathbb{P}}}[|X^\delta_T / X^0_T - 1|] = 0$ and the convergence of the associated strategies follow from Kardaras (2010, Theorem 2.5).

REMARK 2.10. In Larsen (2009) and Kardaras and Žitković (2011), a uniform integrability assumption is the key to stability. Assumption 2.2 implies an analog condition is satisfied. Indeed, Remark 2.3 implies that $(U_\delta)_{\delta \geq 0}$ is uniformly bounded from above by

$$U_*(x) := \frac{\overline{\mathfrak{R}}}{\alpha_*} + \frac{\underline{\mathfrak{R}}}{\alpha_*}\left(1 - \exp(-\alpha_* x)\right), \quad \text{where } \alpha_* = \min_{\delta \geq 0} \alpha_\delta.$$

Since $\mathbb{E}_{\mathbb{P}}[V_*(d\mathbb{Q}/d\mathbb{P})] < \infty$, where V_* is the convex conjugate of U_* and dominates all V_δ, $\{V_\delta(d\mathbb{Q}/d\mathbb{P})\}_{\delta\geq 0}$ is then clearly uniformly integrable under $\mathbb{P}$. However, the additional structure in Assumption 2.2 allows us to discuss the rate of convergence in what follows.

Let us describe rates of convergence for the ratio of marginal utilities, absolute risk aversion, and random endowments via

$$f(\delta) := \sup_{x\in\mathbb{R}} |\mathfrak{R}_\delta(x) - 1|, \quad g(\delta) := |\alpha_\delta - 1|, \quad \text{and} \quad h(\delta) := \mathbb{E}_{\mathbb{Q}}\left[|\xi_\delta - \xi_0|^2\right], \quad \text{for } \delta \geq 0.$$

COROLLARY 2.11. *Let Assumptions 2.1, 2.2, and 2.5 hold. Suppose that ξ_δ is bounded uniform in δ, moreover $\lim_{\delta\downarrow 0} f(\delta) = \lim_{\delta\downarrow 0} g(\delta) = \lim_{\delta\downarrow 0} h(\delta) = 0$. Then*

$$\mathbb{E}_{\mathbb{Q}}\left[|X_T^\delta - X_T^0|\right] \sim O\left(f(\delta)^2 + g(\delta) + h(\delta)\right), \quad \textit{for sufficiently small } \delta.$$

REMARK 2.12. When U_δ is the exponential utility with risk aversion a_δ and no random endowment presents, it is clear that $X_T^\delta = X_T^0/\alpha_\delta$ converges to X_T^0 at the rate of $g(\delta)$. When U_δ deviates from exponential utility and random endowment presents, the rate of convergence for the optimal payoff is determined by three components: convergence of the ratio of marginal utilities, convergence of absolute risk aversions, and convergence of random endowments. Corollary 2.11 shows that the rate of convergence is at least second order on the first component, first order on the second and third components. This provides a quantitative measure on how far X_T^δ is away from X_T^0.

The convergence rate for optimal strategies can also be determined. When S is continuous, Corollary 2.11 and Burkholder–Davis–Gundy inequality combined imply

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (H_\delta - H_0)_t^\top d\langle S\rangle_t (H_\delta - H_0)_t\right)^{p/2}\right] \sim O\left(f(\delta)^2 + g(\delta) + h(\delta)\right),$$

for any $p \in (0, 1)$ and small δ; see Lemma 3.4 and Corollary 3.5 for more details. Here H_0 is the hedging strategy in the exponential hedging problem (cf. Delbaen et al. 2002; Kabanov and Stricker 2002).

Another application of Theorem 2.8 is the stability of utility-based prices with respect to agent's preference. Consider a contingent claim $B \in L^\infty(\mathcal{F}_T)$. An agent, endowed with utility U_δ and endowment ξ_δ, takes her preference into account to price the claim B as

$$\mathbb{E}_{\mathbb{Q}_\delta}[B],$$

where $\mathbb{Q}_\delta$ is introduced in Proposition 2.6. This price is called *fair price* (Davis price), cf. Davis (1997). Theorem 2.8 implies the continuity of fair price with respect to agent's preference.

COROLLARY 2.13. *Let Assumptions 2.1, 2.2, 2.4, 2.5, and 2.7 hold. Then*

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}_\delta}[B] = \mathbb{E}_{\mathbb{Q}}[B].$$

Another utility-based pricing is the *indifference price* introduced into mathematical finance by Hodges and Neuberger (1989); see Carmona (2009) and references therein for recent development on this topic. Given an agent endowed with utility U_δ and an initial wealth $x_0 \in \mathbb{R}$, her *indifference buyer's price*, $p_\delta = p(B, x, U_\delta)$, of B is defined as the solution to the equation

$$u_\delta(x_0 + B - p_\delta) = u_\delta(x_0),$$

where $u_\delta(\zeta)$ is defined in equation (2.1) with $\xi_\delta = \zeta$. The existence and uniqueness of p_δ is proved in Owen and Žitković (2009, proposition 7.2). Theorem 2.8 (ii) allows us to establish the following stability property of the indifference buyer's price with respect to agent's preference.

COROLLARY 2.14. *Let Assumptions 2.1, 2.2, and 2.5 hold. Then* $\lim_{\delta\downarrow 0} p_\delta = p_0$.

REMARK 2.15. The continuity of Davis prices and indifference prices with respect to agent's preference has been investigated in Carassus and Rásonyi (2007) in a discrete time market with bounded stock price processes.

2.2. Utilities Defined on $\mathbb{R}_+$

We continue with our second main result, which concerns the convergence of problems with utilities defined on $\mathbb{R}_+$ to the exponential utility maximization problem. Consider a sequence of utility random fields $\mathcal{U}_p : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, indexed by $p < 0$, each of which is of the form

$$\mathcal{U}_p(x) = D\, U_p(x), \quad x \in \mathbb{R}_+,$$

where D is a $\mathcal{F}_T$-measurable positive random variable and $U_p : \mathbb{R}_+ \to \mathbb{R}$ is a standard utility function. We assume that each U_p is comparable to power utility x^p/p in the following sense:

ASSUMPTION 2.16. *There exist constants* $0 < \underline{\mathfrak{R}}_p \leq 1 \leq \overline{\mathfrak{R}}_p$ *such that*

$$\underline{\mathfrak{R}}_p \leq \mathfrak{R}_p(x) := \frac{U_p'(x)}{x^{p-1}} \leq \overline{\mathfrak{R}}_p, \quad \textit{for all } x \in \mathbb{R}_+.$$

REMARK 2.17. The previous assumption implies that each U_p is bounded from above. Indeed, integrating $U_p'(x) \leq \overline{\mathfrak{R}}_p\, x^{p-1}$ on $(1, x)$ yields $U_p(x) \leq U_p(1) + \overline{\mathfrak{R}}_p(x^p/p - 1/p) \leq U_p(1) - \overline{\mathfrak{R}}_p/p$ for $x \geq 1$ and $p < 0$. Moreover U_p is sandwiched between two utilities with relative risk aversion $1 - p$. To see this, integrating $\underline{\mathfrak{R}}_p x^{p-1} \leq U_p'(x) \leq \overline{\mathfrak{R}}_p x^{p-1}$ on $(1, x)$ when $x \geq 1$ or on $(x, 1)$ when $x < 1$ yields $c_p(x^p/p - 1/p) + U_p(1) \leq U_p(x) \leq C_p(x^p/p - 1/p) + U_p(1)$, for $x > 0$ and some constants c_p and C_p. Furthermore each U_p satisfies the Inada condition, that is, $\lim_{x\downarrow 0} U_p'(x) = \infty$ and $\lim_{x\uparrow\infty} U_p'(x) = 0$, and U_δ has reasonable asymptotic elasticity, that is, $AE_\infty(U_\delta) < 1$.

The discounted prices of risky assets are specified to be stochastic exponential $S = (\mathcal{E}(R^1), \cdots, \mathcal{E}(R^d))$, where R is an $\mathbb{R}^d$-valued càdlàg locally bounded semimartingale with $R_0 = 0$. The agent is endowed with the utility random field $\mathcal{U}_p$ and an initial capital $x_0 \in \mathbb{R}_+$. A trading strategy is a predictable R-integrable $\mathbb{R}^d$-valued process π whose i-th component π^i represents the fraction of current wealth invested in the i-th risky asset. Then the associated wealth process $X(\pi)$ satisfies

$$X_t = x_0 + \int_0^t X_{s-}\pi_s\, dR_s, \quad 0 \leq t \leq T.$$

A trading strategy is *admissible* if the associated wealth process is strictly positive. We denote by $\mathcal{A}(x_0)$ the class of admissible trading strategies. For an admissible strategy π, $H^i := \pi^i X/S^i_- \mathbb{I}_{\{S^i_- \neq 0\}}$ corresponds to the number of shares invested in the i-th asset.

The agent chooses admissible trading strategies to maximize her utility of payoff:

$$u_p(x_0) := \sup_{\pi \in \mathcal{A}(x_0)} \mathbb{E}_{\mathbb{P}} \left[DU_p(X_T(\pi)) \right]. \tag{2.2}$$

The dependence of u_p on x_0 will be omitted if no confusion is caused. Since U_p is bounded from above, $u_p(x_0) < \infty$ whenever D_T has finite $\mathbb{P}$-expectation. We recall the following version of theorem 3.10 from Karatzas and Žitković (2003).

PROPOSITION 2.18 (Karatzas–Žitković). *Assume that the set of equivalent local martingale measures for S is not empty, moreover there exist constants $0 < k_1 \leq k_2 < \infty$ such that $k_1 \leq D \leq k_2$. Then for each $p < 0$ there exists an optimal strategy $\pi_p \in \mathcal{A}(x_0)$ for equation* (2.2). *The associated wealth process $X^{(p)}$ satisfies*

$$y_p Y_T^{(p)} = DU_p'(X_T^{(p)}),$$

where $y_p = u_p'(x_0)$ and $Y^{(p)}$ is some supermartingale deflator with $Y_0^{(p)} = 1$. Moreover

$$y_p x_0 = \mathbb{E}_{\mathbb{P}} \left[DU_p'(X_T^{(p)}) X_T^{(p)} \right] \geq \mathbb{E}_{\mathbb{P}} \left[DU_p'(X_T^{(p)}) X_T \right],$$

for any admissible wealth process X.

To state our second main result, let us recall the exponential hedging problem. Given a contingent claim $B \in L^\infty(\mathcal{F}_T)$, the agent chooses a permissible strategy to maximize the expected exponential utility of the terminal wealth including the claim

$$\sup_{\vartheta \text{ permissible}} \mathbb{E}_{\mathbb{P}} [-\exp(B - x_0 - \vartheta \cdot R_T)]. \tag{2.3}$$

Here ϑ is the monetary value invested in the risky assets. Its corresponding number of shares is $H^i := \vartheta^i/S^i_- \mathbb{I}_{\{S^i_- \neq 0\}}$, which satisfies $H \cdot S = \vartheta \cdot R$. The strategy ϑ is permissible if its corresponding $H \in \mathcal{H}^{perm}$. When S is locally bounded, equation (2.3) admits an optimal strategy $\hat{\vartheta}$; cf. Kabanov and Stricker (2002, theorem 2.1).

We impose the following assumption on filtration, which is satisfied for the Brownian filtration.

ASSUMPTION 2.19. *The filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is continuous, that is, all $\mathcal{F}$-local martingales are continuous.*

The previous assumption implies that S is continuous. Hence R satisfies the *structure condition*:

$$R = M + \int d\langle M \rangle \lambda,$$

where M is a continuous local martingale with $M_0 = 0$ and $\lambda \in L^2_{loc}(M)$; cf. Schweizer (1995).

Our second main result studies the asymptotic behavior of the optimal strategy π_p for equation (2.1) as $p \downarrow -\infty$.

THEOREM 2.20. *Let Assumptions 2.5, 2.16, and 2.19 hold. Set* $D = \exp(B)$ *for* $B \in L^\infty(\mathcal{F}_T)$. *If* $\underline{\mathfrak{R}}_p$ *and* $\overline{\mathfrak{R}}_p$ *in Assumption 2.16 satisfy*

$$\limsup_{p\downarrow-\infty}(1-p)(\overline{\mathfrak{R}}_p - 1) < \infty \quad \textit{and} \quad \limsup_{p\downarrow-\infty}(1-p)(1-\underline{\mathfrak{R}}_p) < \infty, \tag{2.4}$$

then

$$\mathbb{P} - \lim_{p\downarrow-\infty}\int_0^T \Big((1-p)\pi_p - \hat{\vartheta}\Big)_t^\top d\langle M\rangle_t \Big((1-p)\pi_p - \hat{\vartheta}\Big)_t = 0.$$

This result states that whenever the ratio of marginal utilities converges to 1 at least as fast as the relative risk aversion converging to infinity, the optimal fraction invested in risky assets in the power-type problem, after scaled by $1-p$, converges to the optimal monetary value invested in the exponential hedging problem. Here $(1-p)^{-1}$ can be considered as the rate of convergence.

REMARK 2.21. Given a utility function U such that

$$\underline{\mathfrak{R}} \leq \frac{U'(x)}{x^{p_0-1}} \leq \overline{\mathfrak{R}}, \qquad \text{for all } x > 0,$$

where $0 < \underline{\mathfrak{R}} \leq 1 \leq \overline{\mathfrak{R}}$ and $p_0 < 0$, there exists a family of utilities $(U_p)_{p\leq p_0}$ such that $U_{p_0} = U$ and equation (2.4) is satisfied for some sequences $(\overline{\mathfrak{R}}_p)_{p\leq p_0}$ and $(\underline{\mathfrak{R}}_p)_{p\leq p_0}$. Indeed, take any function $f:(-\infty,0)\to(0,1)$ such that $f(p_0)=1$ and $\limsup_{p\downarrow-\infty}(1-p)\,f(p) < \infty$. Set

$$U_p'(x) = f(p)x^{p-p_0}U'(x) + (1-f(p))x^{p-1}, \qquad \text{for } p \leq p_0.$$

One can check that U_p is a standard utility function and

$$\underline{\mathfrak{R}}_p := f(p)(\underline{\mathfrak{R}}-1)+1 \leq \frac{U_p'(x)}{x^{p-1}} \leq f(p)(\overline{\mathfrak{R}}-1)+1 =: \overline{\mathfrak{R}}_p,$$

where both $\limsup_{p\downarrow-\infty}(1-p)(1-\underline{\mathfrak{R}}_p)$ and $\limsup_{p\downarrow-\infty}(1-p)(\overline{\mathfrak{R}}_p - 1)$ are finite.

REMARK 2.22. Denote by $\widetilde{\pi}_p$ the optimal strategy for equation (2.2) when $U_p = x^p/p$. Nutz proved a remarkable result in Nutz (2012, theorem 3.2) that $(1-p)\widetilde{\pi}_p \to \hat{\vartheta}$ in $L^2_{loc}(M)$; cf. Nutz (2012, lemma A.3) for characterization of this convergence. In particular the previous convergence implies

$$\mathbb{P} - \lim_{p\downarrow-\infty}\int_0^T ((1-p)\widetilde{\pi}_p - \hat{\vartheta})_t^\top d\langle M\rangle_t ((1-p)\widetilde{\pi}_p - \hat{\vartheta})_t = 0. \tag{2.5}$$

Therefore $\widetilde{\pi}_p$ converges to $\hat{\vartheta}$ at the rate of $(1-p)^{-1}$. We complement Nutz's result by showing that $\pi_p - \widetilde{\pi}_p$ converges to 0 at the rate $(1-p)^{-1}$, when the ratio of marginal utilities converges to 1 at least at the same rate. In particular, we prove

$$\mathbb{P} - \lim_{p\downarrow-\infty}\int_0^T (1-p)(\widetilde{\pi}_p - \pi_p)_t^\top d\langle M\rangle_t (1-p)(\widetilde{\pi}_p - \pi_p)_t = 0. \tag{2.6}$$

Then Theorem 2.20 follows from combining the previous two convergences.

REMARK 2.23. One can assume that both S and the opportunity processes $(L^{(p)})_{p<0}$, recalled in Section 4, are continuous instead of Assumption 2.19, which is the most important and easy to check sufficient condition for the continuity of S and $(L^{(p)})_{p<0}$. Only the continuity of S is used to prove equation (2.6), continuity of both S and $L^{(p)}$ for all $p < 0$ are needed for equation (2.5).

3. STABILITY FOR UTILITIES DEFINED ON $\mathbb{R}$

Theorem 2.8 and its corollaries will be proved in this section. Let us start with the following property on the family $(\mathcal{M}^a_\delta)_{\delta\geq 0}$.

LEMMA 3.1. *Under Assumption 2.2, all $\mathcal{M}^a_\delta$ (resp. $\mathcal{M}^e_\delta$) are the same for $\delta \geq 0$.*

Proof. Denote $\widetilde{U}_\delta(x) = -\frac{1}{\alpha_\delta}\exp(-\alpha_\delta x)$ and $\widetilde{V}_\delta(y) = \frac{1}{\alpha_\delta} y\log y - \frac{y}{\alpha_\delta}$ to be its convex conjugate. Here α_δ converges to $a_0 := 1$ as $\delta \downarrow 0$. Set $y = U'_\delta(x)$, which can take arbitrary value in $(0,\infty)$ as x varies in $\mathbb{R}$. It follows from Assumption 2.2 that $y/\overline{\mathfrak{R}} \leq \widetilde{U}'_\delta(-V'_\delta(y)) \leq y/\underline{\mathfrak{R}}$, which implies $\widetilde{V}'_\delta(y/\overline{\mathfrak{R}}) \leq V'_\delta(y) \leq \widetilde{V}'_\delta(y/\underline{\mathfrak{R}})$ for any $y \in (0,\infty)$. Integrating the previous inequalities on $(0, y)$ and utilizing $\widetilde{V}_\delta(0) = \widetilde{U}_\delta(\infty) = 0$, we obtain

$$\overline{\mathfrak{R}}\,\widetilde{V}_\delta(y/\overline{\mathfrak{R}}) + V_\delta(0) \leq V_\delta(y) \leq \underline{\mathfrak{R}}\widetilde{V}_\delta(y/\underline{\mathfrak{R}}) + V_\delta(0).$$

Recall from Remark 2.3 that $(U_\delta(\infty))_{\delta>0}$ is uniformly bounded. Then there exists N such that $-N \leq V_\delta(0) = U_\delta(\infty) \leq N$ for any δ. The previous two inequalities combined yield

$$\frac{1}{\alpha_\delta}\widetilde{V}_0(y) - \frac{1}{\alpha_\delta} y\log\overline{\mathfrak{R}} - N \leq V_\delta(y) \leq \frac{1}{\alpha_\delta}\widetilde{V}_0(y) - \frac{1}{\alpha_\delta} y\log\underline{\mathfrak{R}} + N, \qquad \text{for any } y.$$

Therefore $\mathbb{E}_{\mathbb{P}}[V_\delta(d\widetilde{\mathbb{P}}/d\mathbb{P})] < \infty$ if and only if $\mathbb{E}_{\mathbb{P}}[\widetilde{V}_0(d\widetilde{\mathbb{P}}/d\mathbb{P})] < \infty$. □

To prove Theorem 2.8, observe that, without loss of generality all $(\alpha_\delta)_{\delta\geq 0}$ in Assumption 2.2 can be assumed to be 1. Indeed, define $\overline{U}_\delta(x) := \alpha_\delta U_\delta(x/\alpha_\delta)$. Assumption 2.2 implies

$$\underline{\mathfrak{R}} \leq \frac{\overline{U}'_\delta(x)}{\exp(-x)} \leq \overline{\mathfrak{R}}, \qquad \text{for any } x \in \mathbb{R}.$$

Moreover, $\overline{U}(x)$ converges to $-\exp(-x)$ pointwise, since α_δ converges to 1 and $U_\delta(x)$ converges to $-\exp(-x)$ locally uniformly; see Rockafellar (1970, p. 90). Therefore equation (2.1) can be rewritten as

$$u_\delta = \frac{1}{\alpha_\delta}\sup_{H\in\mathcal{H}^{perm}} \mathbb{E}_{\mathbb{P}}\left[\overline{U}_\delta\left(\alpha_\delta H\cdot S_T + \alpha_\delta\xi_\delta\right)\right] = \frac{1}{\alpha_\delta}\sup_{\overline{H}\in\mathcal{H}^{perm}} \mathbb{E}_{\mathbb{P}}\left[\overline{U}_\delta\left(\overline{H}\cdot S_T + \overline{\xi}_\delta\right)\right],$$

where $\overline{\xi}_\delta := \alpha_\delta\xi_\delta$. Therefore the optimal strategy H_δ for equation (2.1) is exactly $\overline{H}_\delta/\alpha_\delta$, where $\overline{H}_\delta$ maximizes the rightmost problem. Hence we can consider equation (2.1) with utility $\overline{U}_\delta$ and the random endowment $\overline{\xi}_\delta$. In this case Assumption 2.2 holds with $\alpha_\delta = 1$ for all $\delta \geq 0$.

Now suppose that Theorem 2.8 holds for $\overline{U}_\delta$, then the same statements hold for U_δ as well. For example, if $\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\left|(\overline{H}_\delta - \overline{H}_0)\cdot S_T\right|\right] = 0$, then

$$\begin{aligned}
(3.1)\quad \mathbb{E}_{\mathbb{Q}}\left[|(H_\delta - H_0)\cdot S_T|\right] &= \frac{1}{\alpha_\delta}\mathbb{E}_{\mathbb{Q}}\left[\left|(\overline{H}_\delta - \alpha_\delta H_0)\cdot S_T\right|\right] \\
&\leq \frac{1}{\alpha_\delta}\mathbb{E}_{\mathbb{Q}}\left[\left|(\overline{H}_\delta - \overline{H}_0)\cdot S_T\right|\right] + \frac{|\alpha_\delta - 1|}{\alpha_\delta}\mathbb{E}_{\mathbb{Q}}\left[|H_0\cdot S_T|\right] \\
&\to 0, \qquad \text{as } \delta \downarrow 0,
\end{aligned}$$

where $\overline{H}_0 = H_0$ and $\mathbb{E}_{\mathbb{Q}}[|H_0 \cdot S_T|] < \infty$ since $H_0 \cdot S$ is a $\mathbb{Q}$-martingale. Therefore, due to the previous change of variable, it suffices to prove Theorem 2.8 when

$$\alpha_\delta = 1, \qquad \text{for all } \delta \geq 0. \tag{3.2}$$

To this end, Theorem 2.8 (i) will be proved in Corollary 3.3, (ii) in Proposition 3.7, and (iii) in Corollary 3.5. In the rest of this section, Assumptions 2.1, 2.2, 2.4, 2.5, and 2.7 are enforced. To simplify notation, we introduce

$$X^\delta := H_\delta \cdot S, \quad \mathcal{X}^\delta := X^\delta + \xi_\delta, \quad \Delta\xi_\delta := \xi_\delta - \xi_0, \quad \text{and} \quad \Delta\mathcal{X}^\delta := \mathcal{X}^\delta - \mathcal{X}^0,$$
$$\text{for } \delta \geq 0.$$

Proof of Theorem 2.8 (i) starts with the following estimate.

LEMMA 3.2. *It holds that*

$$\lim_{\delta \downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right)\right| \left|\Delta X^\delta_T\right|\right] = 0.$$

Proof. Recall from Proposition 2.6 that X^0 is a $\mathbb{Q}_\delta$-supermartingale and X^δ is a $\mathbb{Q}_\delta$-martingale, where the density $d\mathbb{Q}_\delta/d\mathbb{P}$ is $U'_\delta(\mathcal{X}^\delta_T)$ up to a constant. Therefore $U'_\delta(\mathcal{X}^\delta_T)X^0_\cdot$ is a $\mathbb{P}$-supermartingale and $U'_\delta(\mathcal{X}^\delta_T)X^\delta_\cdot$ is a $\mathbb{P}$-martingale. Since both these two processes have initial value zero, therefore $\mathbb{E}_{\mathbb{P}}\left[U'_\delta(\mathcal{X}^\delta_T)X^0_T\right] \leq 0 = \mathbb{E}_{\mathbb{P}}\left[U'_\delta(\mathcal{X}^\delta_T)X^\delta_T\right]$, which induces

$$\mathbb{E}_{\mathbb{P}}\left[U'_\delta(\mathcal{X}^\delta_T)\,(X^0_T - X^\delta_T)\right] \leq 0.$$

Similarly, the previous argument applied to $\mathbb{Q}$ gives

$$\mathbb{E}_{\mathbb{P}}\left[U'_0(\mathcal{X}^0_T)\,(X^\delta_T - X^0_T)\right] \leq 0.$$

Summing up the previous two inequalities and changing to the measure $\mathbb{Q}$ whose density is $U'_0(\mathcal{X}^0_T)$ up to a constant, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\left(1 - \frac{U'_\delta(\mathcal{X}^\delta_T)}{U'_0(\mathcal{X}^0_T)}\right)\left(X^\delta_T - X^0_T\right)\right] \leq 0.$$

Observe that the random variable in the expectation of the previous inequality is negative only when $X^0_T \geq X^\delta_T \geq I_\delta\left(U'_0(\mathcal{X}^0_T)\right) - \xi_\delta$ or $I_\delta\left(U'_0(\mathcal{X}^0_T)\right) - \xi_\delta \geq X^\delta_T \geq X^0_T$, where $I_\delta = (U'_\delta)^{-1}$. In either cases,

$$\left(\left(1 - \frac{U'_\delta(\mathcal{X}^\delta_T)}{U'_0(\mathcal{X}^0_T)}\right)\left(X^\delta_T - X^0_T\right)\right)_- \leq \left(\frac{U'_\delta(X^0_T + \xi_\delta)}{U'_0(X^0_T + \xi_0)} - 1\right)\left(I_\delta\left(U'_0(\mathcal{X}^0_T)\right) - \xi_\delta - X^0_T\right),$$

where $(\cdot)_-$ represents the negative part. Utilizing the fact that $\mathbb{E}_{\mathbb{Q}}[|A|] \leq 2\mathbb{E}_{\mathbb{Q}}[A_-]$ for any random variable A with $\mathbb{E}_{\mathbb{Q}}[A] \leq 0$, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\left|\left(1 - \frac{U'_\delta(\mathcal{X}^\delta_T)}{U'_0(\mathcal{X}^0_T)}\right)\left(X^\delta_T - X^0_T\right)\right|\right] \\
&\leq 2\,\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{U'_\delta(X^0_T + \xi_\delta)}{U'_0(X^0_T + \xi_0)} - 1\right)\left(I_\delta\left(U'_0(\mathcal{X}^0_T)\right) - \xi_\delta - X^0_T\right)\right].
\end{aligned}$$

Note that the left side of the previous inequality is $\mathbb{E}_{\mathbb{Q}}\left[\left|1-\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\right||\Delta X_T^\delta|\right]$. The statement follows once the expectation on the right side converges to zero as $\delta\downarrow 0$.

To prove the desired convergence, let us first estimate the upper bound of $|I_\delta(U_0'(x))-x|$ on $\mathbb{R}$. Set $y=U_0'(x)$. It follows

$$\begin{aligned} I_\delta\left(U_0'(x)\right)-x &= I_\delta(y)-I_0(y) = -\log\left[\exp\left(-\left(I_\delta(y)-I_0(y)\right)\right)\right] \\ &= -\log\left[\tfrac{\exp(-I_\delta(y))}{y}\right] = \log\left[\tfrac{U_\delta'(I_\delta(y))}{U_0'(I_\delta(y))}\right]. \end{aligned}$$

Assumption 2.2 then implies

$$\sup_{x\in\mathbb{R}}\left|I_\delta(U_0'(x))-x\right| \le \max\{\log\overline{\mathfrak{R}}, \log 1/\underline{\mathfrak{R}}\} =: \eta.$$

As a result, $\left|I_\delta(U_0'(\mathcal{X}_T^0))-X_T^0-\xi_\delta\right|\le \eta+|\Delta\xi_\delta|$. Assumptions 2.2 and 2.7 combined imply that

$$\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)} = \mathfrak{R}_\delta(X_T^0+\xi_\delta)\exp(-\Delta\xi_\delta)\le\overline{\mathfrak{R}}e^C.$$

The previous two estimates combined yield

$$\left|\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)}-1\right|\left|I_\delta\left(U_0'(\mathcal{X}_T^0)-\xi_\delta-X_T^0\right)\right|\le(\overline{\mathfrak{R}}e^C+1)(\eta+|\Delta\xi_\delta|), \tag{3.3}$$

where the right side is uniformly integrable in δ under $\mathbb{Q}$ thanks to $\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}[|\Delta\xi_\delta|]=0$ in Assumption 2.7. On the other hand, the term on the left side of equation (3.3) converges to 0 in probability $\mathbb{Q}$. This follows from facts that $\limsup_{\delta\downarrow 0}|I_\delta(U_0'(\mathcal{X}_T^0))-\xi_\delta-X_T^0|$ is bounded and $\mathbb{Q}-\lim_{\delta\downarrow 0}\mathfrak{R}_\delta(X_T^0+\xi_\delta)\exp(-\Delta\xi_\delta)=1$. The previous convergence follows from

$$\begin{aligned} &\mathbb{Q}\left(|\mathfrak{R}_\delta(X_T^0+\xi_\delta)\ \exp\left(-\Delta\xi_\delta\right)-1|\ge\epsilon\right) \\ &\le\mathbb{Q}\left(|\mathfrak{R}_\delta(X_T^0+\xi_\delta)\ \exp\left(-\Delta\xi_\delta\right)-1|\ge\epsilon, |\xi_\delta|\le N, |X_T^0|\le N\right)+\mathbb{Q}(|\xi_\delta|>N) \\ &\quad+\mathbb{Q}(|X_T^0|>N), \end{aligned}$$

where the first term on the right converges to 0 as $\delta\downarrow 0$ since $\mathfrak{R}_\delta$ converges to 1 locally uniformly and $\mathbb{Q}-\lim_{\delta\downarrow 0}\Delta\xi_\delta=0$, both second and third terms can be made arbitrarily small for sufficiently large N. The uniform integrability and convergence in probability combined imply

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\left|\frac{U_\delta'(X_T^0+\xi_\delta)}{U_0'(X_T^0+\xi_0)}-1\right|\left|I_\delta\left(U_0'(\mathcal{X}_T^0)-\xi_\delta-X_T^0\right)\right|\right]=0,$$

hence the statement. □

The previous result provides a handle to study the $\mathbb{L}^1(\mathbb{Q})$ convergence of $X_T^\delta-X_T^0$.

COROLLARY 3.3. *It holds that*

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\right]=0.$$

Proof. We will first prove

$$\lim_{\delta\downarrow 0} \mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T| \geq \epsilon, |\mathcal{X}^\delta_T| \leq N\right) = 0, \quad \text{for any } \epsilon, N > 0. \tag{3.4}$$

To this end, for fixed ϵ and N, $\exp(-\Delta\mathcal{X}^\delta_T) \leq e^{-\epsilon}$ when $\Delta\mathcal{X}^\delta_T \geq \epsilon$. Since U'_δ converges to U'_0 locally uniformly, there exists a sufficiently small δ such that $e^{-\epsilon/2} \leq \mathfrak{R}_\delta(\mathcal{X}^\delta_T) \leq e^{\epsilon/2}$ for $|\mathcal{X}^\delta_T| \leq N$. On the other hand, $|\Delta X^\delta_T| \geq \epsilon/2$ when $|\Delta\xi_\delta| \leq \epsilon/2$ and $|\Delta\mathcal{X}^\delta_T| \geq \epsilon$. The previous estimates combined imply that on $\{\Delta\mathcal{X}^\delta_T \geq \epsilon, |\Delta\xi_\delta| \leq \epsilon/2, |\mathcal{X}^\delta_T| \leq N\}$,

$$\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right)\right| |\Delta X^\delta_T| \geq (1 - e^{\epsilon/2}e^{-\epsilon})\epsilon/2 > 0, \quad \text{for sufficiently small } \delta.$$

Similarly, on $\{\Delta\mathcal{X}^\delta_T \leq -\epsilon, |\Delta\xi_\delta| \leq \epsilon/2, |\mathcal{X}^\delta_T| \leq N\}$,

$$\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right)\right| |\Delta X^\delta_T| \geq (e^{-\epsilon/2}e^{\epsilon} - 1)\epsilon/2 > 0, \quad \text{for sufficiently small } \delta.$$

Set $\eta = \min\{1 - e^{-\epsilon/2}, e^{\epsilon/2} - 1\} \cdot \epsilon/2 > 0$. Previous two inequalities and Lemma 3.2 combined yield

$$\eta \cdot \mathbb{Q}\left(|\Delta\mathcal{X}^\delta_T| \geq \epsilon, |\Delta\xi_\delta| \leq \epsilon/2, |\mathcal{X}^\delta_T| \leq N\right) \leq \mathbb{E}_\mathbb{Q}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right)\right| |\Delta X^\delta_T|\right] \to 0, \quad \text{as } \delta \downarrow 0.$$

Therefore equation (3.4) follows from the previous inequality and $\lim_{\delta\downarrow 0}\mathbb{Q}(|\Delta\xi_\delta| > \epsilon/2) = 0$. Second, we will prove

$$\lim_{\delta\downarrow 0} \mathbb{Q}(|\Delta\mathcal{X}^\delta_T| \geq \epsilon) = 0. \tag{3.5}$$

To this end, note that

$$\begin{aligned} \mathbb{Q}(|\mathcal{X}^\delta_T| \geq N) &\leq \mathbb{Q}(|\mathcal{X}^\delta_T| \geq N, |\mathcal{X}^0_T| \leq N/2) + \mathbb{Q}(|\mathcal{X}^0_T| \geq N/2) \\ &\leq \mathbb{Q}(|\Delta\mathcal{X}^\delta_T| \geq N/2) + \mathbb{Q}(|\mathcal{X}^0_T| \geq N/2), \quad \text{for any } N. \end{aligned} \tag{3.6}$$

Let us prove in what follows

$$\lim_{\delta\downarrow 0} \mathbb{Q}(|\Delta\mathcal{X}^\delta_T| \geq N/2) = 0, \quad \text{for sufficiently large } N. \tag{3.7}$$

Take $N/2 > \max\{2, \log 1/\underline{\mathfrak{R}}, \log\overline{\mathfrak{R}}\}$ and set $M^\delta = N/2 \vee (|\Delta\xi_\delta| + 1)$. On $\{\Delta\mathcal{X}^\delta_T \leq -M^\delta\}$, $\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right) \geq \underline{\mathfrak{R}}\, \exp\left(N/2\right) > 1$ and $|\Delta X^\delta_T| = |\Delta\mathcal{X}^\delta_T - \Delta\xi_\delta| \geq 1$. Hence on the same set,

$$\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right)\right| |\Delta X^\delta_T| \geq \underline{\mathfrak{R}}\, \exp\,(N/2) - 1.$$

On $\{\Delta\mathcal{X}^\delta_T \geq M^\delta\}$, $\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right) \leq \overline{\mathfrak{R}}\, \exp\left(-N/2\right) < 1$ and $|\Delta X^\delta_T| \geq 1$. Hence on the same set,

$$\left|1 - \mathfrak{R}_\delta(\mathcal{X}^\delta_T)\, \exp\left(-\Delta\mathcal{X}^\delta_T\right)\right| |\Delta X^\delta_T| \geq 1 - \overline{\mathfrak{R}}\, \exp\,(-N/2).$$

Set $\eta = \min\{\underline{\mathfrak{R}}\exp(N/2) - 1, 1 - \overline{\mathfrak{R}}\exp(-N/2)\} > 0$. The previous two inequalities combined yield

$$
\begin{aligned}
(3.8) \qquad \eta \cdot \mathbb{Q}\left(|\Delta \mathcal{X}_T^\delta| \geq M^\delta\right) &\leq \eta\, \mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\, \mathbb{I}_{\{|\Delta \mathcal{X}_T^\delta| \geq M^\delta\}}\right] \\
&\leq \mathbb{E}_{\mathbb{Q}}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta \mathcal{X}_T^\delta)\right| |\Delta X_T^\delta|\, \mathbb{I}_{\{\Delta \mathcal{X}_T^\delta \geq M^\delta\}}\right] \\
&\to 0, \qquad \text{as } \delta \downarrow 0,
\end{aligned}
$$

where the convergence follows from Lemma 3.2. Therefore equation (3.7) follows from

$$
\begin{aligned}
\mathbb{Q}\left(|\Delta \mathcal{X}_T^\delta| \geq N/2\right) &\leq \mathbb{Q}\left(|\Delta \mathcal{X}_T^\delta| \geq N/2, |\Delta \xi_\delta| \leq 1\right) + \mathbb{Q}\left(|\Delta \xi_\delta| > 1\right) \\
&= \mathbb{Q}\left(|\Delta \mathcal{X}_T^\delta| \geq M^\delta, |\Delta \xi_\delta| \leq 1\right) + \mathbb{Q}\left(|\Delta \xi_\delta| > 1\right) \\
&\to 0, \qquad \text{as } \delta \downarrow 0.
\end{aligned}
$$

Switch our attention to $\mathbb{Q}(|\mathcal{X}_T^0| \geq N/2)$. Assumption 2.4 yields $x_0 \leq \mathbb{E}_{\mathbb{Q}}[\xi_0] \leq \widetilde{x}_0 + \mathbb{E}_{\mathbb{Q}}[G_0 \cdot S_T] \leq \widetilde{x}_0$, where $G_0 \cdot S$ is a $\mathbb{Q}$-local martingale bounded from below hence a $\mathbb{Q}$-supermartingale. Moreover recall that X^0 is a $\mathbb{Q}$-martingale. Therefore $\mathbb{Q}(|\mathcal{X}_T^0| \geq N/2) \leq 2\,\mathbb{E}_{\mathbb{Q}}[|\mathcal{X}_T^0|]/N$, which can be made arbitrarily small for sufficiently large N. The previous inequality combined with equations (3.6) and (3.7) yields that $\limsup_{\delta\downarrow 0} \mathbb{Q}(|\mathcal{X}_T^\delta| \geq N)$ is sufficiently small for large N. Hence equation (3.5) follows from combining the previous limit superior with equation (3.4).

Finally, let us prove

$$
\lim_{\delta \downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\right] = 0.
$$

To this end, we have seen in equation (3.8) that $\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\left|\Delta X_T^\delta\right|\, \mathbb{I}_{\{|\Delta \mathcal{X}_T^\delta| \geq M^\delta\}}\right] = 0$. On the other hand,

$$
\mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\, \mathbb{I}_{\{|\Delta X_T^\delta| < M^\delta\}}\right] \leq \mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\, \mathbb{I}_{\{|\Delta X_T^\delta| < M^\delta, |\Delta \xi_\delta| \leq 1\}}\right] + \mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\, \mathbb{I}_{\{|\Delta X_T^\delta| < M^\delta, |\Delta \xi_\delta| > 1\}}\right]
$$

Here the second term on the right is bounded from above by $\frac{N}{2}\mathbb{Q}(|\Delta \xi_\delta| > 1) + \mathbb{E}_{\mathbb{Q}}\left[(|\Delta \xi_\delta| + 1)\, \mathbb{I}_{\{|\Delta \xi_\delta| > 1\}}\right]$, which converges to 0 as $\delta \downarrow 0$ due to Assumption 2.7. The first term converges to 0 as well. Indeed, since $|\Delta X_T^\delta| \leq N/2 + 1$ when $|\Delta \mathcal{X}_T^\delta| < M^\delta$ and $|\Delta \xi_\delta| \leq 1$, the bounded convergence theorem implies that $\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[|\Delta X_T^\delta|\, \mathbb{I}_{\{|\Delta X_T^\delta| < M^\delta, |\Delta \xi_\delta| \leq 1\}}\right] = 0$ along any subsequence of δ such that ΔX_T^δ converges to 0 $\mathbb{Q}$-a.s. Since for any sequence, there is a subsequence along which ΔX_T^δ converges $\mathbb{Q}$-a.s., the previous convergence in expectation also holds along the entire sequence of δ. This argument, which combines convergence in probability with the bounded convergence theorem, will be used frequently later without being mentioned explicitly. □

Now Theorem 2.8 (iii) follows from Corollary 3.3 and the following result:

LEMMA 3.4. *For any supermartingale Z with $Z_0 = 0$,*[5]

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} |Z_t|^p\right] \leq \frac{1}{1-p} 2^p\, \mathbb{E}\left[|Z_T|\right]^p, \qquad \textit{for any } p \in (0,1).
$$

[5]This result holds for any probability measure, which is denoted by $\mathbb{P}$ in the proof.

Proof. It follows from Doob's maximal inequality (cf. Karatzas and Shreve 1991, chapter 1, theorem 3.8) that

$$\lambda\,\mathbb{P}\left(\sup_{0\le t\le T}|Z_t|\ge\lambda\right)\le 2\,\mathbb{E}[|Z_T|].$$

Set $Z_*=\sup_{0\le t\le T}|Z_t|$. It then follows

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Z_t|^p\right]&=\mathbb{E}\left[\int_0^\infty \mathbb{I}_{\{Z_*>x\}}px^{p-1}\,dx\right]=\int_0^\infty \mathbb{P}(Z_*>x)px^{p-1}\,dx\\&\le\int_0^\infty \min\left\{1,\frac{2\,\mathbb{E}[|Z_T|]}{x}\right\}p\,x^{p-1}\,dx=\frac{1}{1-p}2^p\,\mathbb{E}[|Z_T|]^p.\end{aligned}$$

Compare to the standard Doob's L^p-inequality where $p>1$, the only difference in proof is the last inequality. □

Applying the previous lemma to the $\mathbb{Q}$-supermartingale ΔX^δ and utilizing Corollary 3.3, we obtain $\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\le t\le T}|\Delta X_t^\delta|^p\right]=0$. Hence Theorem 2.8 (iii) follows from Burkholder–Davis–Gundy inequality, cf. Rogers and Williams (1987, chapter IV, theorem 42.1):

COROLLARY 3.5. *If S is continuous, then*

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\left[\Delta X^\delta,\Delta X^\delta\right]_T^{p/2}\right]=0,\quad \textit{for any } p\in(0,1).$$

The following result prepares the proof of Theorem 2.8 (ii).

LEMMA 3.6. *It holds that*

$$\lim_{\delta\downarrow 0}\frac{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\mathcal{X}_T^\delta\right)\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\mathcal{X}_T^0\right)\right]}=1.$$

Proof. Proposition 2.6 implies that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}=\frac{\exp(-\mathcal{X}_T^0)}{\mathbb{E}_{\mathbb{P}}[\exp(-\mathcal{X}_T^0)]}.$$

After changing to the measure $\mathbb{Q}$, the statement is equivalent to

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}[\exp(-\Delta\mathcal{X}_T^\delta)]=1. \tag{3.9}$$

Fix $N>\max\{C,\log 1/\underline{\mathfrak{R}}\}$, where C is the constant in Assumption 2.7. It follows from equation (3.5) that

$$\lim_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta\ge -N\}}\right]=1. \tag{3.10}$$

On the other hand, when $\Delta\mathcal{X}_T^\delta\le -N$, $\Delta X_T^\delta=\Delta\mathcal{X}_T^\delta-\Delta\xi_\delta\le -N+C<0$, then

$$\begin{aligned}\left|1-\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\right||\Delta X_T^\delta|&=\exp(-\Delta\mathcal{X}_T^\delta)\left|\exp(\Delta\mathcal{X}_T^\delta)-\mathfrak{R}_\delta(\mathcal{X}_T^\delta)\right||\Delta X_T^\delta|\\&\ge\exp(-\Delta\mathcal{X}_T^\delta)\,(\underline{\mathfrak{R}}-\exp(-N))\,(N-C).\end{aligned}$$

Set $\eta = (\underline{\mathfrak{R}} - \exp(-N))(N - C) > 0$. It then follows from Lemma 3.2 that

$$\eta \cdot \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta \le -N\}}\right] \le \mathbb{E}_{\mathbb{Q}}\left[\left|1 - \mathfrak{R}_\delta(\mathcal{X}_T^\delta)\exp(-\Delta\mathcal{X}_T^\delta)\right|\,|\Delta X_T^\delta|\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta \le -N\}}\right] \to 0,$$

$$\text{as } \delta \downarrow 0. \tag{3.11}$$

As a result, equation (3.9) follows from combining (3.10) and (3.11). □

Now we are ready to prove Theorem 2.8 (ii).

PROPOSITION 3.7. *It holds that*

$$\lim_{\delta\downarrow 0} u_\delta = u_0.$$

Proof. After changing to the measure $\mathbb{Q}$, the statement is equivalent to

$$1 = \lim_{\delta\downarrow 0} \frac{\mathbb{E}_{\mathbb{P}}\left[U_\delta(\mathcal{X}_T^\delta)\right]}{\mathbb{E}_{\mathbb{P}}\left[U_0(\mathcal{X}_T^0)\right]} = \lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\right].$$

In what follows, we will prove

$$\limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\right] \le 1; \tag{3.12}$$

while $\liminf_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\right] \ge 1$ can be proved similarly. To prove equation (3.12), we will estimate the limit superior of the expectation on sets $\{-N \le \mathcal{X}_T^\delta \le N\}$, $\{\mathcal{X}_T^\delta > N\}$, and $\{\mathcal{X}_T^\delta < -N\}$ separately, for a fixed sufficiently large N, in the following three steps:

Step 1: on $\{-N \le \mathcal{X}_T^\delta \le N\}$. For any $\epsilon, N > 0$, there exists $\delta_{\epsilon,N}$ such that $1 - \epsilon \le \frac{U_\delta'(x)}{U_0'(x)} \le 1 + \epsilon$ for $x \in (-N, N)$ and $\delta \le \delta_{\epsilon,N}$. Integrating $U_\delta'(x) \le (1+\epsilon)U_0'(x)$ on (x, N) gives $U_\delta(x) \ge (1+\epsilon)U_0(x) - (1+\epsilon)U_0(N) + U_\delta(N)$, which yields

$$\frac{U_\delta(x)}{U_0(x)} \le 1 + \epsilon + \frac{U_\delta(N) - (1+\epsilon)U_0(N)}{U_0(x)}, \qquad \text{for } x \in [-N, N] \text{ and } \delta \le \delta_{\epsilon,N}.$$

It then follows

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^\delta)}\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}\right] \\
&\le (1+\epsilon)\,\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}\right] + (U_\delta(N) - (1+\epsilon)U_0(N))\,\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}}{U_0(\mathcal{X}_T^0)}\right] \\
&= (1+\epsilon)\,\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}\right] + (U_\delta(N) - (1+\epsilon)U_0(N))\,\frac{\mathbb{P}(-N\le\mathcal{X}_T^\delta\le N)}{\mathbb{E}_{\mathbb{P}}\left[U_0(\mathcal{X}_T^0)\right]}.
\end{aligned} \tag{3.13}$$

In what follows the two terms on the right side of the previous inequality will be estimated separately.

Let us first prepare

$$\begin{aligned}
\mathbb{Q}(-N < \mathcal{X}_T^0 < N) &\le \liminf_{\delta\downarrow 0} \mathbb{Q}(-N \le \mathcal{X}_T^\delta \le N) \le \limsup_{\delta\downarrow 0} \mathbb{P}(-N \le \mathcal{X}_T^\delta \le N) \\
&\le \mathbb{Q}(-N \le \mathcal{X}_T^0 \le N).
\end{aligned} \tag{3.14}$$

Indeed, for any ϵ,

$$\mathbb{Q}(-N \le \mathcal{X}_T^\delta \le N) = \mathbb{Q}(-N - \Delta\mathcal{X}_T^\delta \le \mathcal{X}_T^0 \le N - \Delta\mathcal{X}_T^\delta, |\Delta\mathcal{X}_T^\delta| \le \epsilon) \\ +\mathbb{Q}(-N - \Delta\mathcal{X}_T^\delta \le \mathcal{X}_T^0 \le N - \Delta\mathcal{X}_T^\delta, |\Delta\mathcal{X}_T^\delta| > \epsilon).$$

Here the second term converges to 0 due to equation (3.5), and the first term is bounded from below by $\mathbb{Q}(-N+\epsilon \le \mathcal{X}_T^0 \le N-\epsilon, |\Delta\mathcal{X}_T^\delta| \le \epsilon)$ whose limit, as $\delta \downarrow 0$, is $\mathbb{Q}(-N+\epsilon \le \mathcal{X}_T^0 \le N-\epsilon)$. Hence the first inequality in equation (3.14) follows since ϵ is chosen arbitrarily. The third inequality in (3.14) can be proved similarly.

Now to estimate the first term on the right side of (3.13), note

$$\mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N, \mathcal{X}_T^0\le 2N\}}\right] \\ = \mathbb{E}_\mathbb{Q}\left[\left(\exp(-\Delta\mathcal{X}_T^\delta)-1\right)\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N, \mathcal{X}_T^0\le 2N\}}\right] + \mathbb{Q}\left(-N \le \mathcal{X}_T^\delta \le N, \mathcal{X}_T^0 \le 2N\right).$$

Here, since $\Delta\mathcal{X}_T^\delta \ge -3N$ when $-N \le \mathcal{X}_T^\delta \le N$ and $\mathcal{X}_T^0 \le 2N$, then the first term on the right-hand side converges to zero by the bounded convergence theorem and (3.5). For the second term, we employ the same estimate as in (3.14). Combining estimates for both terms, we obtain

$$\mathbb{Q}(-N < \mathcal{X}_T^0 < N) \le \liminf_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N, \mathcal{X}_T^0\le 2N\}}\right] \\ \le \limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N, \mathcal{X}_T^0\le 2N\}}\right] \le \mathbb{Q}(-N \le \mathcal{X}_T^0 \le N).$$

On the other hand, $\Delta\mathcal{X}_T^\delta \le -N$ when $-N \le \mathcal{X}_T^\delta \le N$ and $\mathcal{X}_T^0 > 2N$. Therefore

$$\limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N, \mathcal{X}_T^0> 2N\}}\right] \le \lim_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\Delta\mathcal{X}_T^\delta\le -N\}}\right] = 0, \\ \text{as } \delta \downarrow 0,$$

where the last convergence holds owing to (3.11). The previous two convergences combined imply

$$\mathbb{Q}(-N < \mathcal{X}_T^0 < N) \le \liminf_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}\right] \\ \le \limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{-N\le\mathcal{X}_T^\delta\le N\}}\right] \le \mathbb{Q}(-N \le \mathcal{X}_T^0 \le N). \tag{3.15}$$

To estimate the second term on the right of (3.13), note $U_\delta(N) - (1+\epsilon)U_0(N) < 0$, for sufficiently small δ, and $\mathbb{E}_\mathbb{P}[U_0(\mathcal{X}_T^0)] < 0$. The third inequality in (3.14) (where $\mathbb{Q}$ can be replaced by $\mathbb{P}$, since $\mathbb{Q} \sim \mathbb{P}$) yields

$$\limsup_{\delta\downarrow 0}\,(U_\delta(N) - (1+\epsilon)U_0(N))\,\frac{\mathbb{P}(-N \le \mathcal{X}_T^\delta \le N)}{\mathbb{E}_\mathbb{P}[U_0(\mathcal{X}_T^0)]} \\ \le -\epsilon\, U_0(N)\frac{\mathbb{P}(-N \le \mathcal{X}_T^0 \le N)}{\mathbb{E}_\mathbb{P}[U_0(\mathcal{X}_T^0)]}. \tag{3.16}$$

Step 2: on $\{\mathcal{X}_T^\delta > N\}$.Integrating $\underline{\mathfrak{R}} U_0'(x) \le U_\delta'(x)$ on (N, x) yields that $\underline{\mathfrak{R}} U_0(x) - \underline{\mathfrak{R}} U_0(N) + U_\delta(N) \le U_\delta(x)$ for $x > N$. This implies

$$\begin{aligned}(3.17)\qquad \mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\mathbb{I}_{\{\mathcal{X}_T^\delta > N\}}\right] &\le \underline{\mathfrak{R}}\,\mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\mathcal{X}_T^\delta > N\}}\right] \\ &+ (U_\delta(N) - \underline{\mathfrak{R}} U_0(N))\frac{\mathbb{P}(\mathcal{X}_T^\delta > N)}{\mathbb{E}_\mathbb{P}[U_0(\mathcal{X}_T^0)]}.\end{aligned}$$

Lemma 3.6 and the first inequality in (3.15) combined give

$$(3.18)\qquad \limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\mathcal{X}_T^\delta > N, \mathcal{X}_T^\delta < -N\}}\right] \le \mathbb{Q}(\mathcal{X}_T^0 \ge N, \mathcal{X}_T^0 \le -N).$$

On the other hand, $\overline{\mathfrak{R}} U_0(N) \le U_\delta(N) \le \underline{\mathfrak{R}} U_0(N) < 0$ for sufficiently small δ. Combining the previous inequality with $U_\delta(N) - \underline{\mathfrak{R}} U_0(N) \ge U_\delta(0) - \underline{\mathfrak{R}} U_0(0)$, we obtain $0 \ge U_\delta(N) - \underline{\mathfrak{R}} U_0(N) \ge U_\delta(0) - \underline{\mathfrak{R}} U_0(0)$, where the right side is bounded uniformly in δ. Utilizing the similar argument as in (3.14), we obtain $\limsup_{\delta\downarrow 0} \mathbb{P}(\mathcal{X}_T^\delta > N) \le \mathbb{P}(\mathcal{X}_T^0 \le N)$. Combining the above estimates for the right side of (3.17),

$$\begin{aligned}(3.19)\qquad &\limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\mathbb{I}_{\{\mathcal{X}_T^\delta > N\}}\right] \\ &\le \underline{\mathfrak{R}}\,\mathbb{Q}(\mathcal{X}_T^0 \ge N, \mathcal{X}_T^0 \le -N) + (1 - \underline{\mathfrak{R}})\, U_0(0)\frac{\mathbb{P}(\mathcal{X}_T^0 \ge N)}{\mathbb{E}_\mathbb{P}[U_0(\mathcal{X}_T^0)]}.\end{aligned}$$

Step 3: on $\{\mathcal{X}_T^\delta < -N\}$.Integrating $U_\delta'(x) \le \overline{\mathfrak{R}} U_0'(x)$ on $(x, -N)$ gives $U_\delta(x) \ge \overline{\mathfrak{R}} U_0(x) + U_\delta(-N) - \overline{\mathfrak{R}} U_0(-N) \ge \overline{\mathfrak{R}} U_0(x)$, where the second inequality holds since $U_\delta(-N) \ge \overline{\mathfrak{R}} U_0(-N)$ for sufficiently small δ. As a result, we have from (3.18) that

$$\begin{aligned}(3.20)\qquad \limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\frac{U_\delta(\mathcal{X}_T^\delta)}{U_0(\mathcal{X}_T^0)}\mathbb{I}_{\{\mathcal{X}_T^\delta < -N\}}\right] &\le \overline{\mathfrak{R}} \limsup_{\delta\downarrow 0} \mathbb{E}_\mathbb{Q}\left[\exp(-\Delta\mathcal{X}_T^\delta)\,\mathbb{I}_{\{\mathcal{X}_T^\delta < -N\}}\right] \\ &\le \overline{\mathfrak{R}}\,\mathbb{Q}(\mathcal{X}_T^0 \ge N, \mathcal{X}_T^0 \le -N).\end{aligned}$$

Finally combining (3.15), (3.16), (3.19), and (3.20), (3.12) follows after sending $\epsilon \downarrow 0$ then $N \uparrow \infty$. $\square$

Proof of Corollary 2.11. Following the discussion after Lemma 3.1, we consider problem (2.1) for $\overline{U}_\delta(\alpha_\delta x)$ and $\overline{\xi}_\delta = \alpha_\delta x_0$. After the previous change of variable, $f(\delta) = \sup_{x\in\mathbb{R}} |\overline{\mathfrak{R}}_\delta(x) - 1|$, where $\overline{\mathfrak{R}}_\delta(x) = \overline{U}_\delta'(x)/\exp(-x)$. In what follows, we add a bar to random variables and processes associated to the problem for $\overline{U}_\delta$. In the rest of the proof, C represents a constant which may be different in different places.

First, we utilize the argument in Lemma 3.2 to prove

$$(3.21)\qquad \mathbb{E}_\mathbb{Q}\left[\left|1 - \overline{\mathfrak{R}}_\delta(\overline{\mathcal{X}}_T^\delta)\exp(-\Delta\overline{\mathcal{X}}_T^\delta)\right| |\Delta\overline{\mathcal{X}}_T^\delta|\right] \le C\left(f(\delta)^2 + g(\delta)^2 + h(\delta)\right),$$

for sufficiently small δ .

To this end, we have seen in Lemma 3.2 that the left side is bounded from above by

$$(3.22)\qquad 2\,\mathbb{E}_\mathbb{Q}\left[\left|\overline{\mathfrak{R}}_\delta(\overline{X}_T^0 + \alpha_\delta\xi_\delta)\exp(-\Delta\overline{\xi}_\delta) - 1\right| \left|\overline{I}_\delta\left(\overline{U}_0'(\overline{\mathcal{X}}_T^0)\right) - \overline{\mathcal{X}}_T^0 - \Delta\overline{\xi}_\delta\right|\right],$$

where $\Delta\overline{\xi}_\delta = \alpha_\delta \xi_\delta - \xi_0$. To estimate the expectation above, note that $|\Delta\overline{\xi}_\delta| \leq Cg(\delta) + |\Delta\xi_\delta|$, where the constant C depends on the uniform bound of $|\Delta\xi_\delta|$ (cf. assumptions of Corollary 2.11). Then

$$1 - C(g(\delta) + |\Delta\xi_\delta|) \leq \exp(-Cg(\delta) - |\Delta\xi_\delta|) \leq \exp(-\Delta\overline{\xi}_\delta) \leq \exp(Cg(\delta) + |\Delta\xi_\delta|) \\ \leq 1 + C(g(\delta) + |\Delta\xi_\delta|),$$

where the first inequality follows from $e^{-y} \geq 1 - y$ for $y > 0$ and the fourth inequality holds due to $e^y = 1 + \int_0^y e^z\, dz \leq 1 + Cy$ when $e^y \leq C$. On the other hand, $1 - f(\delta) \leq \overline{\mathfrak{R}}_\delta \leq 1 + f(\delta)$ for sufficiently small δ. Therefore

$$\left|\overline{\mathfrak{R}}_\delta(\overline{X}^0_T + \alpha_\delta\xi_0)\exp(-\Delta\overline{\xi}_\delta) - 1\right| \leq f(\delta) + C(g(\delta) + |\Delta\xi_\delta|) + Cf(\delta)(g(\delta) + |\Delta\xi_\delta|) \\ \leq C(f(\delta) + g(\delta) + |\Delta\xi_\delta|), \quad \mathbb{Q} - a.s., \tag{3.23}$$

for sufficiently small δ. On the other hand, we have seen in Lemma 3.2 that $\overline{I}_\delta(\overline{U}'_0(\overline{x})) - \overline{x} = \log\overline{\mathfrak{R}}_\delta(\overline{I}_\delta(\overline{y}))$, where $\overline{y} = \overline{U}'_0(\overline{x})$. It then follows $-2f(\delta) \leq \overline{I}_\delta(\overline{U}'_0(\overline{x})) - \overline{x} \leq 2f(\delta)$, where we use $\log(1 - y) = -\int_{-y}^0 (1 + z)^{-1} dz \geq -2y$ for $0 < y < 1/2$ and $\log(1 + y) \leq y$ for $y > 0$. As a result

$$\left|\overline{I}_\delta\left(\overline{U}'_0(\overline{\mathcal{X}}^0_T)\right) - \overline{\mathcal{X}}^0_T - \Delta\overline{\xi}_\delta\right| \leq 2f(\delta) + Cg(\delta) + |\Delta\xi_\delta|, \quad \mathbb{Q} - a.s., \tag{3.24}$$

for sufficiently small δ. Combining equations (3.23) and (3.24), we obtain that the expectation in (3.22) is bounded from above by

$$C\,\mathbb{E}_\mathbb{Q}\left[(f(\delta) + g(\delta) + |\Delta\xi_\delta|)^2\right] \leq C\left(f(\delta)^2 + g(\delta)^2 + \mathbb{E}_\mathbb{Q}[|\Delta\xi_\delta|^2]\right),$$

for sufficiently small δ .

This confirms equation (3.21). In the next step, we will prove

$$\mathbb{E}_\mathbb{Q}\left[|\Delta\overline{X}^\delta_T|\right] \leq C\left(f(\delta)^2 + g(\delta)^2 + h(\delta)\right), \quad \text{for sufficiently small } \delta. \tag{3.25}$$

Indeed, an argument similar to that in Corollary 3.3 implies that there exists $N, \eta > 0$ such that

$$\eta\,\mathbb{E}_\mathbb{Q}\left[\left|\Delta\overline{X}^\delta_T\right|\,\mathbb{I}_{\{|\Delta\overline{X}^\delta_T| \geq M^\delta\}}\right] \leq \mathbb{E}_\mathbb{Q}\left[\left|1 - \overline{\mathfrak{R}}_\delta(\overline{\mathcal{X}}^\delta_T)\exp(-\Delta\overline{\mathcal{X}}^\delta_T)\right|\,|\Delta\overline{X}^\delta_T|\,\mathbb{I}_{\{|\Delta\overline{X}^\delta_T| \geq M^\delta\}}\right],$$

where $M^\delta = N/2 \vee (|\Delta\overline{\xi}_\delta| + 1)$. The previous inequality, combined with (3.21), yields

$$\mathbb{E}_\mathbb{Q}\left[|\Delta\overline{X}^\delta_T|\,\mathbb{I}_{\{|\Delta\overline{X}^\delta_T| \geq M^\delta\}}\right] \leq C\left(f(\delta)^2 + g(\delta)^2 + h(\delta)\right), \quad \text{for sufficiently small } \delta.$$

Now (3.25) follows after noticing $\mathbb{E}_\mathbb{Q}\left[|\Delta\overline{X}^\delta_T|\mathbb{I}_{\{|\Delta\overline{X}^\delta_T| \leq M^\delta\}}\right] \leq \mathbb{E}_\mathbb{Q}\left[|\Delta\overline{X}^\delta_T|\mathbb{I}_{\{|\Delta\overline{X}^\delta_T| \geq M^\delta\}}\right]$.

Finally, come back to the problem before changing of variable,

$$\mathbb{E}_\mathbb{Q}\left[|\Delta X^\delta_T|\right] \leq \frac{1}{\alpha_\delta}\mathbb{E}_\mathbb{Q}\left[|\Delta\overline{X}^\delta_T|\right] + \frac{|\alpha_\delta - 1|}{\alpha_\delta}\mathbb{E}_\mathbb{Q}[|X^0_T|] \\ \leq C\left[f(\delta)^2 + g(\delta)^2 + h(\delta) + g(\delta)\right] \\ \leq C\left(f(\delta)^2 + g(\delta) + h(\delta)\right), \quad \text{for sufficiently small } \delta.$$

□

Let us now prove implications of Theorem 2.8 on utility-based prices.

Proof of Corollary 2.13. Following the change of variable after Lemma 3.1, we can assume without loss of generality that $\alpha_\delta = 1$ for all $\delta \geq 0$ throughout this proof. Since $B \in L^\infty(\mathcal{F}_T)$, it suffices to prove $\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}[|d\mathbb{Q}_\delta/d\mathbb{Q} - 1|] = 0$, which follows from $\mathbb{Q} - \lim_{\delta\downarrow 0} d\mathbb{Q}_\delta/d\mathbb{Q} = 1$ in virtual by Scheffe's lemma.

To prove the convergence in probability, the following form of $d\mathbb{Q}_\delta/d\mathbb{Q}$ can be read from Proposition 2.6:

$$\frac{d\mathbb{Q}_\delta}{d\mathbb{Q}} = \frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)} \frac{\mathbb{E}_{\mathbb{P}}[U_0'(\mathcal{X}_T^0)]}{\mathbb{E}_{\mathbb{P}}[U_\delta'(\mathcal{X}_T^\delta)]}.$$

In what follows, both factors on the right side will be proved converging to 1. Let us estimate the first factor. For any given N and ϵ, there exists a sufficiently small δ such that $|\mathfrak{R}_\delta(x) - 1| \leq \epsilon$ for $|x| \leq N$. Then $\mathbb{Q}(|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \geq \epsilon, |\mathcal{X}_T^\delta| \leq N) = 0$ for sufficiently small δ. Hence

$$\begin{aligned}\limsup_{\delta\downarrow 0} \mathbb{Q}(|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \geq \epsilon) &\leq \limsup_{\delta\downarrow 0} \mathbb{Q}(|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \geq \epsilon, |\mathcal{X}_T^\delta| \leq N) \\ &\quad + \limsup_{\delta\downarrow 0} \mathbb{Q}(|\mathcal{X}_T^\delta| > N) \leq \mathbb{Q}(|\mathcal{X}_T^0| \geq N),\end{aligned}$$

which can be made arbitrarily small for sufficiently large N. Therefore $\mathbb{Q} - \lim_{\delta\downarrow 0} \mathfrak{R}_\delta(\mathcal{X}_T^\delta) = 1$, which combined $\mathbb{Q} - \lim_{\delta\downarrow 0} \exp(-\Delta\mathcal{X}_T^\delta) = 1$ from equation (3.5), implies

$$\mathbb{Q} - \lim_{\delta\downarrow 0} \frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)} = \mathbb{Q} - \lim_{\delta\downarrow 0} \mathfrak{R}_\delta(\mathcal{X}_T^\delta) \exp(-\Delta\mathcal{X}_T^\delta) = 1.$$

In this paragraph, we will prove

$$\lim_{\delta\downarrow 0} \frac{\mathbb{E}_{\mathbb{P}}[U_\delta'(\mathcal{X}_T^\delta)]}{\mathbb{E}_{\mathbb{P}}[U_0'(\mathcal{X}_T^0)]} = 1.$$

Changing to the measure $\mathbb{Q}$, the previous convergence is equivalent to

$$\lim_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)}\right] = 1, \tag{3.26}$$

which we will prove next. For any ϵ and N, there exists a sufficiently small δ such that $|\mathfrak{R}_\delta(\mathcal{X}_T^\delta) - 1| \leq \epsilon$ when $|\mathcal{X}_T^\delta| \leq N$. The previous inequality combined with (3.15) yields

$$\begin{aligned}\limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\frac{U_\delta'(\mathcal{X}_T^\delta)}{U_0'(\mathcal{X}_T^0)} \mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] &= \limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\mathfrak{R}_\delta(\mathcal{X}_T^\delta) \exp(-\Delta\mathcal{X}_T^\delta) \mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] \\ &\leq (1+\epsilon) \limsup_{\delta\downarrow 0} \mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}_T^\delta) \mathbb{I}_{\{|\mathcal{X}_T^\delta| \leq N\}}\right] \\ &\leq (1+\epsilon)\, \mathbb{Q}(|\mathcal{X}_T^0| \leq N).\end{aligned}$$

Similar argument also gives $\liminf_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[U'_\delta(\mathcal{X}^\delta_T)/U'_0(\mathcal{X}^0_T)\,\mathbb{I}_{\{|\mathcal{X}^\delta_T|\le N\}}\right]\ge (1-\epsilon)\,\mathbb{Q}(|\mathcal{X}^0_T|<N)$. On the other hand, it follows from (3.18) that

$$\begin{aligned}\limsup_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\mathfrak{R}_\delta(\mathcal{X}^\delta_T)\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{|\mathcal{X}^\delta_T|>N\}}\right] &\le \overline{\mathfrak{R}}\limsup_{\delta\downarrow 0}\mathbb{E}_{\mathbb{Q}}\left[\exp(-\Delta\mathcal{X}^\delta_T)\,\mathbb{I}_{\{|\mathcal{X}^\delta_T|>N\}}\right]\\ &\le \overline{\mathfrak{R}}\,\mathbb{Q}(|\mathcal{X}^0_T|\ge N).\end{aligned}$$

Combining the previous two convergences and sending $N\uparrow\infty$ then $\epsilon\downarrow 0$, we confirm (3.26), hence the statement of the corollary. □

Proof of Corollary 2.14. It follows from Owen and Žitković (2009, proposition 7.2 (i)) that $(p_\delta)_{\delta\ge 0}$ is uniformly bounded since $B\in L^\infty(\mathcal{F}_T)$. Therefore in every subsequence of $(p_\delta)_{\delta\ge 0}$ there exists a further subsequence $(p_{\delta_n})_{n\ge 0}$ converging to some limit, say $\widetilde{p}_0$. In the next paragraph, we will prove $\widetilde{p}_0=p_0$. This implies that the entire sequence of $(p_\delta)_{\delta\ge 0}$ converges to p_0 as well, since the choice of subsequence is arbitrary.

For the subsequence $(\delta_n)_{n\ge 0}$, Assumption 2.7 holds for $\xi_n=x_0+B-p_{\delta_n}$ and $\xi_0=x_0+B-\widetilde{p}_0$ when B is bounded. It then follows from Theorem 2.8 (ii) that

$$\lim_{\delta_n\downarrow 0}u_{\delta_n}(x_0+B-p_{\delta_n})=u_0(x_0+B-\widetilde{p}_0).$$

Apply Theorem 2.8 (ii) with $\xi_n=x_0$,

$$\lim_{\delta_n\downarrow 0}u_{\delta_n}(x_0)=u_0(x_0).$$

Since $u_{\delta_n}(x_0+B-p_{\delta_n})=u_{\delta_n}(x_0)$, the previous two convergence combined imply $u_0(x_0+B-\widetilde{p}_0)=u_0(x_0)$. Then $p_0=\widetilde{p}_0$ follows from the uniqueness of the indifference price p_0. □

4. STABILITY FOR UTILITIES DEFINED ON $\mathbb{R}_+$

We will prove Theorem 2.20 in this section. To this end, we can assume without loss of generality that $D=1$ $\mathbb{P}$-a.s. Otherwise, we can define $\mathbb{P}_D\sim\mathbb{P}$ via $d\mathbb{P}_D/d\mathbb{P}=D/\mathbb{E}_{\mathbb{P}}[D]$ and work with $\mathbb{P}_D$ instead of $\mathbb{P}$ throughout this section. Assumptions 2.5, 2.16, and 2.19 are enforced throughout this section, equation (2.4) is satisfied as well. To simplify notation, denote $\widetilde{U}_p(x)=x^p/p$ and $\widetilde{X}^{(p)}$, $\widetilde{Y}^{(p)}$, and $\widetilde{y}_p$ quantities in Proposition 2.18 when U_p is chosen as $\widetilde{U}_p$.

Denote the ratio of optimal wealth processes as

$$r^{(p)}=\frac{X^{(p)}}{\widetilde{X}^{(p)}},$$

and introduce a sequence of auxiliary probability measures $(\mathbb{P}_p)_{p<0}$ via

$$\frac{d\mathbb{P}_p}{d\mathbb{P}}=\frac{\left(\widetilde{X}^{(p)}_T\right)^p}{\mathbb{E}_{\mathbb{P}}\left[\left(\widetilde{X}^{(p)}_T\right)^p\right]},\qquad\text{for each } p<0.$$

It follows from Proposition 2.18 that $(X^{(p)}_T)^p>0$, $\mathbb{P}$-a.s., therefore $\mathbb{P}_p\sim\mathbb{P}$ for each $p<0$. This sequence of auxiliary measures will facilitate various estimates in this section. Another important observation is that $\widetilde{X}^{(p)}$ has the *numéraire property* under $\mathbb{P}_p$, that is, $\mathbb{E}_{\mathbb{P}_p}[X_T/\widetilde{X}_T^{(p)}] \leq 1$ for any admissible wealth process X. Indeed, Proposition 2.18 implies $\mathbb{E}_{\mathbb{P}}\left[(\widetilde{X}_T^{(p)})^{p-1}(X_T - \widetilde{X}_T^{(p)})\right] \leq 0$ for any admissible X. The claim then follows from changing the measure to $\mathbb{P}_p$ in the previous inequality. As a result, every admissible wealth process X deflated by $\widetilde{X}^{(p)}$ is a $\mathbb{P}_p$-supermartingale; (see Guasoni et al. 2014, equation (3.10)). In particular, $r^{(p)}$ is a $\mathbb{P}_p$-supermartingale.

As the last section, we start our analysis with the following estimate.

LEMMA 4.1. *It holds that*

$$\lim_{p\downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[|p|\left|\mathfrak{R}(X_T^{(p)})(r_T^{(p)})^{p-1} - 1\right|\left|1 - r_T^{(p)}\right|\right] = 0.$$

Proof. Throughout this proof we omit the superscript (p) in $X^{(p)}$, $\widetilde{X}^{(p)}$, and $r^{(p)}$ to simplify notation. Applying Proposition 2.18 to U_p and $\widetilde{U}_p$, respectively, yields

$$\mathbb{E}_{\mathbb{P}}\left[U_p'(X_T)(\widetilde{X}_T - X_T)\right] \leq 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}\left[\widetilde{X}_T^{p-1}(X_T - \widetilde{X}_T)\right] \leq 0.$$

Summing up the previous two inequalities and changing to the measure $\mathbb{P}_p$, we obtain

$$\mathbb{E}_{\mathbb{P}_p}\left[\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right] \leq 0.$$

Similar to Lemma 3.2, $(U_p'(X_T)\widetilde{X}_T^{1-p} - 1)(1 - X_T/\widetilde{X}_T) \leq 0$ only when $I_p(\widetilde{X}_T^{p-1}) \leq X_T \leq \widetilde{X}_T$ or $\widetilde{X}_T \leq X_T \leq I_p(\widetilde{X}^{p-1})$, where $I_p = (U_p')^{-1}$. In either cases,

$$\left(\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right)_- \leq \left(1 - \frac{U_p'(\widetilde{X}_T)}{\widetilde{X}_T^{p-1}}\right)\left(1 - \frac{I_p(\widetilde{X}_T^{p-1})}{\widetilde{X}_T}\right).$$

Therefore,

$$\mathbb{E}_{\mathbb{P}_p}\left[\left|\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right|\right] \leq 2\,\mathbb{E}_{\mathbb{P}_p}\left[\left(1 - \mathfrak{R}(\widetilde{X}_T)\right)\left(1 - \frac{I_p(\widetilde{X}_T^{p-1})}{\widetilde{X}_T}\right)\right].$$

Note that

$$\frac{I_p(x^{p-1})}{x} = \frac{I_p(y)}{y^{\frac{1}{p-1}}} = \left(\frac{I_p(y)^{p-1}}{U_p'(I_p(y))}\right)^{\frac{1}{p-1}} = \mathfrak{R}_p(I_p(y))^{\frac{1}{1-p}},$$

where $y = x^{p-1}$. Utilizing the previous identity, we obtain from the previous inequality and Assumption 2.16 that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}_p}\left[\left|\left(\frac{U_p'(X_T)}{\widetilde{X}_T^{p-1}} - 1\right)\left(1 - \frac{X_T}{\widetilde{X}_T}\right)\right|\right] \\ &\leq 2\,\max\left\{(\overline{\mathfrak{R}}_p - 1)(\overline{\mathfrak{R}}_p^{\frac{1}{1-p}} - 1), (1 - \underline{\mathfrak{R}}_p)(1 - \underline{\mathfrak{R}}_p^{\frac{1}{1-p}})\right\}. \end{aligned} \tag{4.1}$$

Since $\limsup_{p\downarrow-\infty} |p|(\overline{\mathfrak{R}}_p - 1) < \infty$ from equation (2.4), $\lim_{p\downarrow-\infty} \overline{\mathfrak{R}}_p^{\frac{1}{1-p}} = \lim_{p\downarrow-\infty} \exp(\frac{1}{1-p}\log\overline{\mathfrak{R}}_p) = 1$. Therefore the first term on the right side of equation (4.1), after multiplying by $|p|$, converges to 0 as $p \downarrow -\infty$. Similar argument applies to the second term as well. As a result, the left side expectation, after multiplying $|p|$, converges to 0 as $p \downarrow -\infty$. $\square$

The previous estimate induces the convergence of $r_T^{(p)}$ in the following sense:

COROLLARY 4.2. *It holds that*

$$\lim_{p\downarrow-\infty} \mathbb{P}_p\left(\left|(r_T^{(p)})^p - 1\right| \geq \epsilon\right) = 0, \quad \textit{for any } \epsilon > 0.$$

Proof. Throughout this proof we still omit the superscript (p). When $r_T^p \geq 1+\epsilon$, $1 - r_T \geq 1-(1+\epsilon)^{1/p}$. Note that $(1+\epsilon)^{1/p} = \exp(p^{-1}\log(1+\epsilon)) = 1 + p^{-1}\log(1+\epsilon) + o(p^{-1})$. Hence $\lim_{p\downarrow-\infty} -p(1-(1+\epsilon)^{1/p}) = \log(1+\epsilon) > 0$. Therefore when $r_T^p \geq 1+\epsilon$, $-p(1-r_T) \geq \frac{1}{2}\log(1+\epsilon) > 0$ for sufficiently small p. When $r_T^p \leq 1-\epsilon$, we can similarly obtain $-p(r_T - 1) \geq -\frac{1}{2}\log(1-\epsilon) > 0$ for sufficiently small p. Set $\eta = \min\{\frac{1}{2}\log(1+\epsilon), -\frac{1}{2}\log(1-\epsilon)\} > 0$. The previous two estimates combined yield

$$-p\,|r_T - 1| \geq \eta \quad \text{when } |r_T^p - 1| \geq \epsilon \text{ for sufficiently small } p.$$

On the other hand, when $r_T^p \geq 1+\epsilon$, $r_T^{p-1} \geq 1+\epsilon/2$ for sufficiently small p. Moreover equation (2.4) and Assumption 2.16 combined imply that $\mathfrak{R}_p(X_T) \geq \underline{\mathfrak{R}}_p \geq (1+\epsilon/2)^{-\frac{1}{2}}$ for sufficiently small p. As a result,

$$\mathfrak{R}_p(X_T) r_T^{p-1} - 1 \geq (1+\epsilon/2)^{-\frac{1}{2}}(1+\epsilon/2) - 1 = (1+\epsilon/2)^{\frac{1}{2}} - 1 > 0,$$

when $r_T^p - 1 \geq \epsilon$ for sufficiently small p.
Similarly,

$$1 - \mathfrak{R}(X_T) r_T^{p-1} \geq 1 - (1-\epsilon/2)^{\frac{1}{2}} > 0, \quad \text{when } r_T^p - 1 \leq -\epsilon, \text{ for sufficiently small } p.$$

Combining estimates in the last two paragraphs, we obtain

$$|p|\left|\mathfrak{R}(X_T) r_T^{p-1} - 1\right| |1 - r_T| \geq \eta \cdot \min\left\{(1+\epsilon/2)^{\frac{1}{2}} - 1, 1 - (1-\epsilon/2)^{\frac{1}{2}}\right\} > 0, \quad \text{when } |r_T^p - 1| \geq \epsilon,$$

for sufficiently small p. The statement then follows from the previous inequality and Lemma 4.1. $\square$

The previous convergence in probability implies that $(r_T^{(p)})^p$ converges to 1 in expectation.

PROPOSITION 4.3. *It holds that*

$$\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}\left[\left|\left(r_T^{(p)}\right)^p - 1\right|\right] = 0.$$

Proof. Throughout this proof we omit the superscript (p). The proof is split into two steps. The first step proves

$$\lim_{p\downarrow-\infty} \mathbb{E}_{\mathbb{P}_p}[r_T^p] = 1. \tag{4.2}$$

The second step confirms the statement.

Step 1: After the measure $\mathbb{P}_p$ is changed to $\mathbb{P}$, (4.2) is equivalent to

$$\lim_{p\downarrow-\infty} \frac{\mathbb{E}_{\mathbb{P}}[X_T^p]}{\mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p]} = 1, \tag{4.3}$$

which will be proved in this step. We have seen in Proposition 2.18 that

$$\frac{\underline{\mathfrak{R}}_p}{y_p}\mathbb{E}_{\mathbb{P}}[X_T^p] \le x_0 = \frac{1}{y_p}\mathbb{E}_{\mathbb{P}}\left[U_p'(X_T)X_T\right] \le \frac{\overline{\mathfrak{R}}_p}{y_p}\mathbb{E}_{\mathbb{P}}[X_T^p],$$

where Assumption 2.16 is used to obtain two inequalities. Sending $p \downarrow -\infty$ in previous inequalities, we obtain from $\underline{\mathfrak{R}}_p, \overline{\mathfrak{R}}_p \to 1$,

$$\lim_{p\downarrow-\infty} \frac{1}{y_p}\mathbb{E}_{\mathbb{P}}[X_T^p] = x_0.$$

The optimality of $\widetilde{X}$ gives $\mathbb{E}_{\mathbb{P}}[X_T^p]/p \le \mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p]/p = x_0\widetilde{y}_p/p$. The previous convergence and $p < 0$ then yields

$$\limsup_{p\downarrow-\infty} \frac{\widetilde{y}_p}{y_p} \le 1.$$

The reverse inequality on the limit inferior will be proved in the next paragraph. Note that $\frac{y}{I_p(y)^{p-1}} = \frac{U_p'(x)}{x^{p-1}}$ for $x = I_p(y)$. Then Assumption 2.16 gives $\underline{\mathfrak{R}}_p \le \frac{y}{I_p(y)^{p-1}} \le \overline{\mathfrak{R}}_p$, hence

$$\underline{\mathfrak{R}}_p^{\frac{1}{1-p}} \le \frac{I_p(y)}{y^{\frac{1}{p-1}}} \le \overline{\mathfrak{R}}_p^{\frac{1}{1-p}}, \qquad \text{for } y > 0.$$

Proposition 2.18 then yields

$$x_0 = \mathbb{E}_{\mathbb{P}}\left[Y_T I_p\left(y_p Y_T\right)\right] \le \overline{\mathfrak{R}}^{\frac{1}{1-p}}\mathbb{E}_{\mathbb{P}}\left[Y_T\left(y_p Y_T\right)^{\frac{1}{p-1}}\right] = \overline{\mathfrak{R}}^{\frac{1}{1-p}} y_p^{\frac{1}{p-1}}\mathbb{E}_{\mathbb{P}}\left[Y_T^q\right],$$

where $q := p/(p-1)$. Note $\mathbb{E}_{\mathbb{P}}[Y_T^q]^{1-p} \le \mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p/x_0^p]$ follows from $\mathbb{E}_{\mathbb{P}}[Y_T\widetilde{X}_T/x_0] \le 1$ and Hölder's inequality (see e.g., Guasoni and Robertson 2012, Lemma 5). The previous two inequalities combined yield $x_0 y_p \le \overline{\mathfrak{R}}_p\mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p] = \overline{\mathfrak{R}}_p\, x_0\widetilde{y}_p$. Sending $p \downarrow -\infty$ and utilizing $\lim_{p\downarrow-\infty}\overline{\mathfrak{R}}_p = 1$, we obtain from the previous inequality

$$\liminf_{p\downarrow-\infty} \frac{\widetilde{y}_p}{y_p} \ge 1.$$

Estimates from the last two paragraphs yield $\lim_{p\downarrow-\infty} y_p/\widetilde{y}_p = 1$, which is equivalent to

$$\lim_{p\downarrow-\infty} \frac{\mathbb{E}_{\mathbb{P}}\left[U_p'(X_T)X_T\right]}{\mathbb{E}_{\mathbb{P}}\left[\widetilde{X}_T^p\right]} = 1.$$

Since $\underline{\mathfrak{R}}_p \mathbb{E}_{\mathbb{P}}[X_T^p] \le \mathbb{E}_{\mathbb{P}}[U_p'(X_T)X_T] \le \overline{\mathfrak{R}}_p \mathbb{E}_{\mathbb{P}}[X_T^p]$, equation (4.3) follows from dividing by $\mathbb{E}_{\mathbb{P}}[\widetilde{X}_T^p]$ on both sides of the previous inequality and sending $p \downarrow -\infty$.

Step 2: For any $N > 1$, $\lim_{p\downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p \le N\}}\right] = 0$ is proved in this paragraph. To this end, for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p \le N\}}\right] &= \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p \le N, |r_T^p - 1| \le \epsilon\}}\right] \\ &+ \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p \le N, |r_T^p - 1| > \epsilon\}}\right] \\ &\le \epsilon + (N-1)\,\mathbb{P}_p(|r_T^p - 1| > \epsilon) \\ &\to \epsilon, \quad \text{as } p \downarrow -\infty,\end{aligned}$$

where the convergence follows from Corollary 4.2. Therefore the claim is confirmed since the choice of ϵ is arbitrary in the previous inequality.

Now $\lim_{p\downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p > N\}}\right] = 0$ is proved in this paragraph. Combining this convergence and the one in the last paragraph confirm $\lim_{p\downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\right] = 0$. To prove the claim,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_p}\left[|r_T^p - 1|\,\mathbb{I}_{\{r_T^p > N\}}\right] &\le \mathbb{E}_{\mathbb{P}_p}\left[r_T^p\,\mathbb{I}_{\{r_T^p > N\}}\right] \\ &= \mathbb{E}_{\mathbb{P}_p}[r_T^p] - \mathbb{E}_{\mathbb{P}_p}\left[(r_T^p - 1)\,\mathbb{I}_{\{r_T^p \le N\}}\right] - \mathbb{P}_p(r_T^p \le N) \\ &\to 1 - 0 - 1 = 0, \quad \text{as } p \downarrow -\infty,\end{aligned}$$

where the convergence of three terms follow from the result in Step 1, the result in the last paragraph, and Corollary 4.2, respectively.

□

The convergence of optimal payoffs in Proposition 4.3 implies the ratio of optimal wealth processes converges uniformly in probability. The proof of the following two results adapt arguments in Kardaras (2010, theorem 2.5) into our context.

COROLLARY 4.4. *It holds that*

$$\lim_{p\downarrow -\infty} \mathbb{P}_p\left(\sup_{t\in[0,T]} \left|(r_T^{(p)})^p - 1\right| \ge \epsilon\right) = 0.$$

Proof. The superscript (p) is still omitted throughout to simplify notation. Recall that r is a $\mathbb{P}_p$-supermartingale; see the discussion before Lemma 4.1. Then $p < 0$ implies that r^p is a $\mathbb{P}_p$-submartingale. Indeed, $\mathbb{E}_{\mathbb{P}_p}\left[r_t^p \,|\, \mathcal{F}_s\right] \ge \left(\mathbb{E}_{\mathbb{P}_p}[r_t \,|\, \mathcal{F}_s]\right)^p \ge r_s^p$ for any $s \le t$, where the Jensen's inequality is applied to obtain the first inequality. In the next two paragraphs, we will prove

$$\lim_{p\downarrow -\infty} \mathbb{P}_p\left(\left|\sup_{t\in[0,T]} r_t^p - 1\right| \ge \epsilon\right) = 0 \quad \text{and} \quad \lim_{p\downarrow -\infty} \mathbb{P}_p\left(\left|\inf_{t\in[0,T]} r_t^p - 1\right| \ge \epsilon\right) = 0, \tag{4.4}$$

for any fixed $\epsilon > 0$. These two convergence combined confirm the statement.

To prove the first convergence in (4.4), define $\tau_p := \inf\{t \geq 0 \mid r_t^p \geq 1 + \delta\} \wedge T$ for $p < 0$ and $\delta > 0$. It then suffices to prove

$$\lim_{p \downarrow -\infty} \mathbb{P}_p \left(\tau_p < T\right) = 0,$$

since δ is arbitrarily chosen. Suppose the previous convergence does not hold. Then there exists $\eta > 0$ and a subsequence, which we still denote by τ_p, such that $\lim_{p \downarrow -\infty} \mathbb{P}_p \left(\tau_p < T\right) = \eta$. It then follows from Proposition 4.3 that

$$\left|\mathbb{E}_{\mathbb{P}_p}\left[r_T^p \mathbb{I}_{\{\tau_p = T\}}\right] - \mathbb{P}_p(\tau_p = T)\right| = \left|\mathbb{E}_{\mathbb{P}_p}\left[(r_T^p - 1)\mathbb{I}_{\{\tau_p = T\}}\right]\right| \leq \mathbb{E}_{\mathbb{P}_p}[|r_T^p - 1|] \to 0, \quad \text{as } p \downarrow -\infty.$$

This implies $\lim_{p \downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_T^p \mathbb{I}_{\{\tau_p = T\}}\right] = 1 - \eta$. On the other hand, the $\mathbb{P}_p$-submartingale property of r^p implies

$$1 \leq \mathbb{E}_{\mathbb{P}_p}[r_{\tau_p}^p] \leq \mathbb{E}_{\mathbb{P}_p}\left[r_T^p\right] \to 1, \quad \text{as } p \downarrow -\infty,$$

where the last convergence follows from (4.2). Hence $\lim_{p \downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p\right] = 1$. Therefore

$$\begin{aligned} 1 = \lim_{p \downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}[r_{\tau_p}^p] &\geq \liminf_{p \downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p \mathbb{I}_{\{\tau_p < T\}}\right] + \lim_{p \downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}\left[r_{\tau_p}^p \mathbb{I}_{\{\tau_p = T\}}\right] \\ &\geq (1+\delta)\eta + (1-\eta) = 1 + \delta\eta > 1, \end{aligned}$$

which is a contradiction. The proof of the second convergence in (4.4) is similar. □

Our next goal is to pass from convergence of optimal payoffs to convergence of optimal strategies.

PROPOSITION 4.5. *If S is continuous, then the following statements hold for any $\epsilon > 0$:*

(i) $\lim_{p \downarrow -\infty} \mathbb{P}_p \left(\left[\left(r^{(p)}\right)^p, \left(r^{(p)}\right)^p\right]_T \geq \epsilon\right) = 0$;
(ii) $\lim_{p \downarrow -\infty} \mathbb{P}_p \left(\left[\mathcal{L}^{(p)}, \mathcal{L}^{(p)}\right]_T \geq \epsilon\right) = 0$, *where* $\mathcal{L}^{(p)} := \int_0^{\cdot} \left(1/(r_t^{(p)})^p\right) d(r_t^{(p)})^p$, *that is,* $\mathcal{L}^{(p)}$ *is the stochastic logarithm of* $(r^{(p)})^p$.

REMARK 4.6. Under the structure condition, $[\mathcal{L}^{(p)}, \mathcal{L}^{(p)}]_T = \int_0^T p(\pi_p - \widetilde{\pi}_p)_t \, d\langle M\rangle_t \, p(\pi_p - \widetilde{\pi}_p)_t$, which measures how far $p(\pi_p - \widetilde{\pi}_p)$ is away from 0.

Proof. The superscript (p) on r and $\mathcal{L}$ is omitted throughout this proof. Note that $[r^p, r^p]_\cdot = \int_0^\cdot |r^p|^2 \, d[\mathcal{L}, \mathcal{L}]_t$. Statement (ii) then follows from statement (i) and Corollary 4.2 directly. We will prove statement (i) in what follows.

Define $\tau_p = \inf\{t \geq 0 \mid r_t^p \geq 2\} \wedge T$. It follows from Corollary 4.4 that $\lim_{p \downarrow -\infty} \mathbb{P}_p \left(\tau_p = T\right) = 1$. Therefore it suffices to prove

$$\lim_{p \downarrow -\infty} \mathbb{P}_p \left([r^p, r^p]_{T \wedge \tau_p} \geq \epsilon\right) = 0. \tag{4.5}$$

Set $Z^{(p)} = r_{\cdot \wedge \tau_p}^p$. Since r^p is a $\mathbb{P}_p$-submartingale, so is $Z^{(p)}$. Therefore equation (4.2) induces $\lim_{p \downarrow -\infty} \mathbb{E}_{\mathbb{P}_p}[Z_T^{(p)}] = 1$. On the other hand, the continuity of S implies the continuity of r^p, hence $Z^{(p)}$ is bounded from above by 2 for all $p < 0$. The Doob–Meyer decomposition gives $Z^{(p)} = M^{(p)} + B^{(p)}$, where $M^{(p)}$ is a $\mathbb{P}_p$-martingale and $B^{(p)}$ is a continuous nondecreasing process with $B_0^{(p)} = 0$. The continuity of $B^{(p)}$ follows from Karatzas and

Shreve (1991, theorem 1.4.14). Note $sup_{t\in[0,T]}|Z_t^{(p)}-1| \leq \sup_{t\in[0,T]}|M_t^{(p)}-1| + B_T^{(p)}$. Hence

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|M_t^{(p)}-1|\right] &\leq \mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|Z_t^{(p)}-1|\right] + \mathbb{E}_{\mathbb{P}_p}[B_T^{(p)}]\\ &= \mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|Z_t^{(p)}-1|\right] + \mathbb{E}_{\mathbb{P}_p}[Z_T^{(p)}] - \mathbb{E}_{\mathbb{P}_p}[M_T^{(p)}]\\ &\to 0+1-1=0, \quad \text{ as } p\downarrow -\infty,\end{aligned}$$

where $\mathbb{E}_{\mathbb{P}_p}\left[\sup_{t\in[0,T]}|Z_t^{(p)}-1|\right]\to 0$ holds owing to $|Z^{(p)}-1|\leq 1$ and Corollary 4.4, $\mathbb{E}_{\mathbb{P}_p}[M_T^{(p)}]=1$ holds because $M^{(p)}$ is a $\mathbb{P}_p$-martingale. Therefore the Davis inequality yields $\lim_{p\downarrow-\infty}\mathbb{E}_{\mathbb{P}_p}[[M^{(p)}, M^{(p)}]_T^{1/2}]=0$, which implies $\lim_{p\downarrow-\infty}\mathbb{P}_p([M^{(p)}, M^{(p)}]_T\geq\epsilon)=0$. Hence (4.5) is confirmed, since $B^{(p)}$ is a continuous increasing process.

□

In the last step to prove Theorem 2.20, we are going to identify limit of $\mathbb{P}_p$ as $p\downarrow-\infty$. To this end, we recall the opportunity process for power utility. The càdlàg semimartingale $L^{(p)}$ is called the *opportunity process* for the power utility x^p/p if it satisfies

$$L_t^{(p)}\frac{1}{p}(X_t(\pi))^p = \mathrm{esssup}_{\widetilde{\pi}\in\mathcal{A}(\pi)}\mathbb{E}_{\mathbb{P}}\left[\frac{1}{p}(X(\widetilde{\pi})_T)^p\,\middle|\,\mathcal{F}_t\right],$$

for any $t\in[0,T]$ and $\pi\in\mathcal{A}$, where $\mathcal{A}(\pi)=\{\widetilde{\pi}\in\mathcal{A}:\widetilde{\pi}=\pi \text{ on } [0,t]\}$. The existence and uniqueness of $L^{(p)}$ have been proved in Nutz (2010, proposition 3.1). Thanks to the scaling property of power utility, $L^{(p)}$ can be viewed as a dynamic version of the reduced value function. In particular, the definition above implies that $L_0^{(p)}x_0^p/p=\widetilde{u}_p(x_0)$, where $\widetilde{u}_p(x_0)$ is defined in equation (2.2) with $U_p(x)=x^p/p$, and $L_0^{(p)}x_0^{p-1}=\widetilde{y}_p=\widetilde{u}'_p(x_0)$. As a result, the density of $\mathbb{P}_p$ can be rewritten as

$$\frac{d\mathbb{P}_p}{d\mathbb{P}} = \frac{\left(\widetilde{y}_p\widetilde{Y}_T^{(p)}\right)^q}{p\widetilde{u}_p(x_0)} = \frac{\left(L_0^{(p)}\widetilde{Y}_T^{(p)}\right)^q}{L_0^{(p)}} = \frac{\left(\widetilde{Y}_T^{(p)}\right)^q}{\left(L_0^{(p)}\right)^{1-q}},$$

where $q=p/(p-1)$. As $p\downarrow-\infty$, using convergence results in Nutz (2012), we will show that the denominator in the rightmost equality above converges to 1 and the numerator converges to the density of the minimal entropy measure $\mathbb{Q}$. Therefore convergence under the sequence of measures $(\mathbb{P}_p)_{p<0}$ in Proposition 4.5 can be replaced by convergence in probability $\mathbb{Q}$. This, combined with Nutz (2012, theorem 3.2), concludes the proof of Theorem 2.20.

Proof of Theorem 2.20. Let us first prove

$$\lim_{p\downarrow-\infty}\mathbb{E}_{\mathbb{P}}\left[\left|\frac{d\mathbb{P}_p}{d\mathbb{P}}-\frac{d\mathbb{Q}}{d\mathbb{P}}\right|\right]=0. \tag{4.6}$$

To this end, when S is continuous, it follows from Nutz (2012, theorem 6.6) that $\lim_{p\downarrow-\infty}L_0^{(p)}=L_0^{\exp}$, where $L^{\exp}$ is the opportunity process for exponential utility $-\exp(-x)$ defined in the similar fashion as that for power utility (cf. Nutz 2012, equation (6.3)). Since $q\to 1$ as $p\downarrow-\infty$, then $\lim_{p\downarrow-\infty}(L_0^{(p)})^{1-q}=1$. On the other hand, when S and $(L^{(p)})_{p<0}$ are continuous, Nutz (2012, proposition 6.13) proved that $\widetilde{Y}^{(p)}$

converges in the semimartingale topology to the density of $\mathbb{Q}$ as $p \downarrow -\infty$. In particular, $\mathbb{P}-\lim_{p\downarrow-\infty} \widetilde{Y}_T^{(p)} = d\mathbb{Q}/d\mathbb{P}$. Hence $\mathbb{P}-\lim_{p\downarrow-\infty}(\widetilde{Y}_T^{(p)})^q = d\mathbb{Q}/d\mathbb{P}$, which, after combined with $\lim_{p\downarrow-\infty}(L_0^{(p)})^{1-q} = 1$, implies

$$\mathbb{P}-\lim_{\delta\downarrow-\infty} \frac{d\mathbb{P}_p}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Hence the $\mathbb{L}^1(\mathbb{P})$ convergence in equation (4.6) follows from the previous convergence and Scheffe's lemma. The assumptions on the continuity of S and $(L^{(p)})_{p<0}$ are ensured by Assumption 2.19 (cf. Nutz 2012, remark 4.2).

Proposition 4.5 (ii) and equation (4.6) combined yield $\mathbb{Q}-\lim_{p\downarrow-\infty}\left[p(\pi_p-\widetilde{\pi}_p)\cdot R\right]_T = 0$, where $[Z] := [Z, Z]$ is the quadratic variation for the semimartingale Z. Hence

$$\mathbb{P}-\lim_{p\downarrow-\infty}\left[(1-p)(\pi_p-\widetilde{\pi}_p)\cdot R\right]_T = 0, \tag{4.7}$$

since $\mathbb{Q}\sim\mathbb{P}$. On the other hand, Nutz (2012, theorem 3.2) proved that $(1-p)\widetilde{\pi}_p \to \hat{\vartheta}$ in $L^2_{loc}(M)$ as $p\downarrow-\infty$. This implies $\mathbb{P}-\lim_{p\downarrow-\infty}[((1-p)\widetilde{\pi}_p-\hat{\vartheta})\cdot R]_{T\wedge\tau_n} = 0$, for a sequence of stopping time (τ_n) with $\lim_{n\uparrow\infty}\tau_n = \infty$ (cf. Nutz 2012, Lemma A.3). The previous convergence then yields

$$\mathbb{P}-\lim_{p\downarrow-\infty}\left[((1-p)\widetilde{\pi}_p-\hat{\vartheta})\cdot R\right]_T = 0. \tag{4.8}$$

Finally, the statement is confirmed via

$$\begin{aligned}\left[((1-p)\pi_p-\hat{\vartheta})\cdot R\right]_T &= \left[(1-p)(\pi_p-\widetilde{\pi}_p)\cdot R + ((1-p)\widetilde{\pi}_p-\hat{\vartheta})\cdot R\right]_T \\ &\le 2\left[(1-p)(\pi_p-\widetilde{\pi}_p)\cdot R\right]_T + 2\left[((1-p)\widetilde{\pi}_p-\hat{\vartheta})\cdot R\right]_T,\end{aligned}$$

where both terms in the right side converge in probability $\mathbb{P}$ to zero as we have seen in equations (4.7) and (4.8). □

REFERENCES

BAYRAKTAR, E., and R. KRAVITZ (2013): Stability of Exponential Utility Maximization with Respect to Market Perturbations, *Stochastic Process. Appl.* 123(5), 1671–1690.

BIAGINI, S., and M. FRITTELLI (2005): Utility Maximization in Incomplete Markets for Unbounded Processes, *Finance Stoch.* 9, 493–517.

BIAGINI, S., and M. FRITTELLI (2007): The Supermartingale Property for the Optimal Wealth Process for General Semimartingales, *Finance Stoch.* 11, 253–266.

CARASSUS, L., and M. RÁSONYI (2007): Optimal Strategies and Utility-Based Prices Converge When Agents' Preferences Do, *Math. Oper. Res.* 32(1), 102–117.

CARMONA, R., ed. (2009): Indifference Pricing: Theory and Applications, in *Princeton Series in Financial Engineering*, Princeton: Princeton University Press.

CVITANIĆ, J., and I. KARATZAS (1996): Hedging and Portfolio Optimization under Transaction Costs: A Martingale Approach, *Math. Finance* 6(2), 133–165.

DAVIS, M. H. A. (1997): Option Pricing in Incomplete Markets, in *Mathematics of Derivative Securities,* Volume 15 of Publications of the Newton Institute, M. A. H. Dempster and S. R. Pliska, eds. Cambridge: Cambridge University Press, pp. 216–226.

DELBAEN, F., P. GRANDITS, T. RHEINLÄNDER, D. SAMPERI, M. SCHWEIZER, and C. STRICKER (2002): Exponential Hedging and Entropic Penalties, *Math. Finance* 12(2), 99–123.

FREI, C. (2013): Convergence Results for the Indifference Value Based on the Stability of BSDEs, *Stochastics* 85, 464–488.

GRANDITS, P., and T. RHEINLÄNDER (2002): On the Minimal Entropy Martingale Measure, *Ann. Probab.* 30(3), 1003–1038.

GUASONI, P., and S. ROBERTSON (2012): Portfolios and Risk Permia for the Long Run, *Ann. Appl. Probab.* 22, 239–284.

GUASONI, P., C. KARDARAS, R. ROBERTSON, and H. XING (2014): Abstract, Classic, and Explicit Turnpikes, *Finance Stoch.* 18(1), 75–114.

HODGES, R., and K. NEUBERGER (1989): Optimal Replication of Contingent Claims under Transaction Costs, *Review of Futures Markets* 8, 222–239.

JOUINI, E., and C. NAPP (2004): Convergence of Utility Functions and Convergence of Optimal Strategies, *Finance Stoch.* 8(1), 133–144.

KABANOV, Y., and C. STRICKER (2002): On the Optimal Portfolio for the Exponential Utility Maximization: Remark to the Six-Author Paper, *Math. Finance* 12, 125–134.

KARATZAS, I., and S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus, Volume 113 of Graduate Texts in Mathematics*, 2nd ed., New York: Springer-Verlag.

KARATZAS, I., and G. ŽITKOVIĆ (2003): Optimal Consumption from Investment and Random Endowment in Incomplete Semimartingale Markets, *Ann. Probab*. 31(4), 1821–1858.

KARDARAS, C. (2010): The Continuous Behavior of Numéraire Portfolio under Small Changes in Information Structure, Probabilisitic Views and Investment Constraints, *Stochastic Process. Appl.* 120, 331–347.

KARDARAS, C., and G. ŽITKOVIĆ (2011): Stability of the Utility Maximization Problem with Random Endowment in Incomplete Markets, *Math. Finance* 21, 313–333.

KRAMKOV, D., and W. SCHACHERMAYER (1999): The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets, *Ann. Appl. Probab.* 9(3), 904–950.

KRAMKOV, D., and W. SCHACHERMAYER (2003): Necessary and Sufficient Conditions in the Problem of Optimal Investment in Incomplete Markets, *Ann. Appl. Probab.* 13(4), 1504–1516.

LARSEN, K. (2009): Continuity of Utility-Maximization with Respect to Preferences, *Math. Finance* 19(2), 237–250.

LARSEN, K., and G. ŽITKOVIĆ (2007): Stability of Utility-Maximization in Incomplete Markets, *Stochastic Process. Appl.* 117(11), 1642–1662.

MANIA, M., and R. TEVZADZE (2003): A Unified Characterization of q-Optimal and Minimal Entropy Martingale Measures by Semimartingale Backward Equations, *Georgian Math. J.* 10(2), 289–310. [Dedicated to the memory of Professor Revaz Chitashvili.]

MOCHA, M., and N. WESTRAY (2013): The Stability of the Constrained Utility Maximization Problem: A BSDE Approach, *SIAM J. Financial Math.* 4(1), 86–116.

MUSIELA, M., and T. ZARIPHOPOULOU (2009): Portfolio Choice under Dynamic Investment Performance Criteria, *Quant. Finance* 9(2), 161–170.

NUTZ, M. (2010): The Opportunity Process for Optimal Consumption and Investment with Power Utility, *Math. Finan. Econ.* 3(3), 139–159.

NUTZ, M. (2012): Risk Aversion Asymptotics for Power Utility Maximization, *Probab. Theory Relat. Fields* 152, 703–749.

OWEN, M., and G. ŽITKOVIĆ (2009): Optimal Investment with an Unbounded Random Endowment and Utility-Based Pricing, *Math. Finance* 19(1), 129–159.

ROCKAFELLAR, R. (1970): *Convex Analysis*, Princeton, NJ:Princeton University Press.

ROGERS, L., and D. WILLIAMS (1987): *Diffusions, Markov Processes, and Martingales, Volume 2: Itô Calculus*, New York: Wiley.

SANTACROCE, M. (2005): On the Convergence of the p-Optimal Martingale Measures to the Minimal Entropy Martingale Measure, *Stoch. Anal. Appl.* 23(1), 31–54.

SCHACHERMAYER, W. (2001): Optimal Investment in Incomplete Markets When Wealth May Become Negative, *Ann. Appl. Probab.* 11, 694–734.

SCHWEIZER, M. (1995): On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition, *Stoch. Anal. Appl.* 13, 573–599.