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HOPE, FEAR, AND ASPIRATIONS

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We propose a rank-dependent portfolio choice model in continuous time that captures the role in decision making of three emotions: hope, fear, and aspirations. Hope and fear are modeled through an inverse-S shaped probability weighting function and aspirations through a probabilistic constraint. By employing the recently developed approach of quantile formulation, we solve the portfolio choice problem both thoroughly and analytically. These solutions motivate us to introduce a fear index, a hope index, and a lottery-likeness index to quantify the impacts of three emotions, respectively, on investment behavior. We find that a sufficiently high level of fear endogenously necessitates portfolio insurance. On the other hand, hope is reflected in the agent's perspective on good states of the world: a higher level of hope causes the agent to include more scenarios under the notion of good states and leads to greater payoffs in sufficiently good states. Finally, an exceedingly high level of aspirations results in the construction of a lottery-type payoff, indicating that the agent needs to enter into a pure gamble in order to achieve his goal. We also conduct numerical experiments to demonstrate our findings.

KEY WORDS: portfolio choice, continuous time, rank-dependent utility, probability weighting, SP/A theory, quantile formulation, portfolio insurance.

1. INTRODUCTION

The failure of the classical expected utility theory (EUT) to describe many observed human behaviors has motivated economists to develop alternative models of choice. In

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the past few decades, a substantial amount of research has been conducted on this subject in two directions. In the first direction, economists have attempted to change or relax some of the axioms in EUT, hoping to find satisfactory models of choice. Examples of this approach include Yaari's *dual theory of choice* (Yaari 1987) and Quiggin's *rank-dependent utility theory* (RDUT; Quiggin 1982; Schmeidler 1989). In the second direction, which is inspired by behavioral psychology, researchers seek to model the processes that lead to choice. The most celebrated accomplishment along this line is Kahneman and Tversky's *cumulative prospect theory* (CPT; Kahneman and Tversky 1979; Tversky and Kahneman 1992). Another notable project is Lopes' security, potential, and aspiration theory (or the *SP/A theory*; Lopes 1987).

Inspired by these models, we propose to use in this paper a new model of choice that considers three emotions relevant to decision making—hope, fear, and aspirations—while examining a continuous-time financial portfolio choice problem in which an agent, whose preferences are represented by this model of choice, chooses the portfolio that will optimize his payoff at a given terminal time. Although seemingly contradictory psychological states, hope and fear are typically present simultaneously in the same individual. The former is characterized by overweighting outcomes in very good situations and the latter by overweighting outcomes in very bad situations. Aspirations, by contrast, are responsive to the exigencies and opportunities of each decision nexus. We therefore employ an inverse-S shaped probability weighting (or distortion) function to model hope and fear, and a probabilistic constraint to model aspirations. Notably, our model derives fundamental components from both RDUT and SP/A theory. Its connection with these two theories and the motivations behind our model are discussed in detail in Section 2.

In addition to the introduction of a new portfolio choice model, another contribution of this paper is to solve the portfolio choice problem analytically and explicitly. A probability weighting function is a nonlinear transformation applied to the underlying probability measure when risky choices are evaluated. The presence of such a weighting function ruins both the time-consistency that underlies the dynamic programming principle and the concavity of the preference functional that is essential for any optimization problem. For this reason, the classical approaches that have been developed to address expected utility maximization fail to solve the portfolio choice problem introduced in this paper.

A new approach, known as quantile formulation, has recently been developed to overcome the difficulty associated with probability weighting functions; see, e.g., Carlier and Dana (2006, 2011), Jin and Zhou (2008), and He and Zhou (2011b). This approach involves changing the decision variable from the random variable of the future payoff to the quantile function of the payoff. This change of variable, in general, recovers the concavity of the underlying preference measure, and thus one can employ either calculus of variations or a pointwise maximization/minimization approach to solve the optimization problem.

In particular, in Jin and Zhou (2008), as an important step in eventually solving a CPT portfolio selection problem, an RDUT problem is solved explicitly via the quantile approach under a monotonicity condition that involves the probability weighting function and the pricing kernel. Unfortunately, this monotonicity condition is not satisfied with some common weighting functions proposed in the literature together with a lognormal pricing kernel. In the context of the model presented in this paper, we will remove this condition which, however, leads to considerable technical challenges. Moreover, there is an additional probabilistic constraint representing the aspect of aspirations, which can be reformulated naturally in terms of the quantile function. This constraint introduces additional technical complications besides those caused by the absence of the aforementioned monotonicity condition, but we are able to derive the optimal terminal payoff of the agent explicitly after a careful and involved analysis.

The main contribution of this paper, however, is the introduction of the three indices—a fear index, a hope index, and a lottery-likeness index—that quantify hope, fear, and aspirations, as well as a comparative statics analysis that studies the impacts of these emotions on investing behavior. These indices are put forth naturally in the process of solving the underlying portfolio choice problem. The fear index is a quantity that, when sufficiently large, necessitates portfolio insurance in the optimal portfolio. This index is defined in terms of the curvatures (the second- and first-order derivatives) of the probability weighting function $w(p)$ near $p = 1$; thus, it is related to the exaggeration of small probabilities of extremely bad outcomes or, indeed, to the emotion of fear. On the other hand, the hope index is defined through the first-order derivative of $w(p)$ when p is close to zero, which is relevant to the exaggeration of small probabilities of extremely good outcomes or, in psychobehavioral terms, to the emotion of hope. We find that a higher level of hope makes the agent include more scenarios under the notion of good states. In addition, in choosing optimal portfolios, an agent with a higher level of hope sets his terminal wealth greater in each of the sufficiently good states than another agent with a lower level of hope. Finally, a sufficiently high level of aspirations forces the agent to construct a lottery-type payoff that has a discontinuity with respect to market conditions. More precisely, under good market conditions, characterized as the realization of a set of good states, the agent's payoff is exceedingly high, but when market conditions are not good, the agent's payoff is much lower. Because the payoff resulting from a high level of aspirations thus resembles that of a lottery ticket, we introduce a "lottery-likeness index" to capture its essential characteristic.

Decision-making behavior can be characterized as risk-averse or risk-seeking. In the model presented in this paper, both fear and a concave utility function lead to risk-averse behavior. Fear, as pointed out earlier, is an overweight on the left tail of a payoff distribution. The concave utility function, by contrast, captures the aversion of the agent to a mean-preserving spread. In the portfolio choice model presented here, these two elements play qualitatively distinct roles in deciding investing behavior. The presence of a certain level of fear determines whether one needs portfolio insurance. The level of insurance needed, however, is unaffected by the level of fear; rather, it is dependent on, among other factors, the utility function that one employs. On the other hand, both hope and aspirations lead to risk-seeking behavior in decision making, yet these elements have qualitatively different impacts on investing behavior in the portfolio selection model. A higher level of hope causes the agent to include more scenarios among "good" states of the world, implying that he will need to take more leverage in order to reap the payoffs from these states. However, hope and aspirations are qualitatively different, and the agent constructs a lottery-type payoff only when he has an exceedingly high level of aspirations, thus causing him to gamble outright.

Shefrin (2008, section 27.5) proposes a *CRRA-based SP/A* theory, which is related to our model of choice proposed here; see detailed discussions in Section 2. There, the author investigates the investment behavior suggested by his theory in a finite-state probability space with the complete market assumption. However, no formal portfolio choice model is set up and the optimal portfolio is demonstrated only graphically. By contrast, in the present paper we build a formal portfolio choice model in the continuous-time setting, solve the optimal portfolio explicitly, and study the impact of the three human emotions on investment behavior rigorously.

After completing a draft of this paper, we came across Carlier and Dana (2011), in which the authors examine an RDUT maximization problem. Although this work and the present paper share certain features, there are key differences in motivation, scope, and implications. First, Carlier and Dana set out to produce a normative description of a choice model, so their paper considers the RDUT maximization without the probabilistic constraint that is used to model aspirations in the present paper. By contrast, our goal is to investigate and capture the decision-making role of the three emotions—hope, fear, and aspirations—by solving a portfolio choice problem. Second, Carlier and Dana focus on obtaining some of the necessary and sufficient conditions for an optimal terminal payoff, whereas we focus on deriving solutions explicitly with a different approach. This, in turn, facilitates further analysis, such as comparative statics and numerical implementation. Finally, we introduce a quantitative index for each of the emotions considered in the paper, and we discuss the impact that each can have on trading behavior.

The paper is organized as follows: In Section 2, we review the SP/A theory and the RDUT, from which we derive key elements of our model. In Section 3, we pose our portfolio choice problem in continuous time and present its quantile formulation. Section 4 is devoted to studying the feasibility and well-posedness of the problem. The study of feasibility examines whether the investor's aspirations are too high, relative to his initial capital, for him to achieve. The study of well-posedness, on the other hand, investigates whether the investor would take infinite leverage on risky assets. In Section 5, we solve the portfolio choice problem thoroughly. Along the way, we introduce our indices for hope, fear, and aspirations and study the effects of these emotions on investing behavior. In Section 6, we provide an example in which we specify a particular probability weighting function and a power utility function and employ historical US equity and bond data to obtain some numerical results on the optimal portfolio. These numerical results confirm the theoretical results obtained in Section 5. Moreover, using this example we compare our model with the EUT portfolio selection model. Finally, Section 7 concludes the paper and the proof of the main theorem is provided in the Appendix.

2. SP/A THEORY AND RANK-DEPENDENT UTILITY THEORY

In this section, we explain the motivation for our choice of an inverse-S shaped probability weighting function to model hope and fear and of a probabilistic constraint to model aspirations. Let us start from the SP/A theory, which inspired our model.

SP/A theory is a two-factor theory developed by Lopes (1987). It uses both a *dispositional factor* and a *situational factor* to explain risky preferences and choices. The dispositional factor describes the *internal*, natural motives that dictate individuals' risk attitudes. In this regard, risk-aversion appears to be motivated by a desire for *security*, whereas risk-loving is rationalized by a desire to achieve or maximize *potential*. Individuals are found to inherently seek both security and potential when facing risky choices. The situational factor, by contrast, reflects *external*, specific needs or opportunities that the individual faces when making choices.

In the decision model proposed by Lopes (1987), the dispositional factor enters into the individual's objective function. A nonnegative prospect (random payoff) X is evaluated as

$$V(X) := \int_0^{+\infty} x d[-w(1 - F_X(x))], \tag{2.1}$$

where $F_X(\cdot)$ is the cumulative distribution function (CDF) of X. The nonlinear transformation $w(\cdot)$, in Lopes' terms, is called the *decumulative weighting function*, and the integral is in the Lebesgue-Stieltjes sense.[1] In the SP/A theory, $w(\cdot)$ takes the following form:

$$w(z) := \nu z^{q_s+1} + (1-\nu)[1-(1-z)^{q_p+1}], \tag{2.2}$$

where $0 \le \nu \le 1$ and $q_s, q_p \ge 0$. Clearly, z^{q_s+1} and $1-(1-z)^{q_p+1}$ are convex and concave functions, respectively; therefore, according to Yaari's theory, they imply risk-aversion and risk-loving, respectively.[2] The decumulative weighting function w is a mixture of a convex function and a concave function; thus, it seems to represent the individual's (somewhat conflicting) desire for both security and potential. In Lopes' words, individuals stand "*between* hope and fear." It is worth mentioning that (2.1) is a natural extension of the definition in Lopes and Oden (1999) that applies to purely discrete prospects.

The situational factor, on the other hand, is modeled by the probabilistic constraint

$$P(X \ge A) \ge \alpha, \tag{2.3}$$

where A is the *aspiration level* and $0 \le \alpha \le 1$ is the *confidence level*. See for instance Lopes and Oden (1999). The pair (A, α) represents individuals' aspirations responding to specific circumstances and opportunities. For instance, if an individual has a loan of amount A which will be due soon, he may be forced to set the loan amount to be the aspiration level. On the other hand, if an individual is prohibited from taking excess risk, then he may set up a low aspiration level with a high confidence level to achieve it.[3]

While it is quite reasonable to model aspirations (the situational factor) via a probabilistic constraint, it is not convincing to us that the dispositional factor should be modeled as (2.1) with the weighting function $w(\cdot)$ specified as (2.2). It is, in our view, neither justifiable nor adequate to explain the complex psychological interlacement of hope and fear as a mere simple convex combination of two completely opposite attitudes toward risk. Rather, it is more appropriate to attempt to capture hope and fear *simultaneously*. On the other hand, the use of a linear utility in (2.1) is also debatable. Linear utility functions are not favored in the risky choice literature. Moreover, we will show later that the preference functional (2.1) leads to an ill-posed portfolio choice model at least in the continuous-time setting, mainly due to the linearity of the utility function.

Thus, it is desirable to seek a more suitable preference representation to describe the dispositional factor. We turn therefore to the RDUT. RDUT was introduced by Quiggin (1982) and further developed by Schmeidler (1989). It can explain many paradoxes that EUT has failed to capture, and, at the same time, it provides mathematical tractability. As stated in Starmer (2000), "the rank-dependent model is likely to become more widely used," because it captures many robust empirical phenomena "in a model which is quite amenable to application within the framework of conventional economic analysis."

[1] If we define $\bar{w}(z) := 1 - w(1-z)$, the *dual* of w, then (2.1) can be written as $\int_0^{+\infty} x d\bar{w}(F_X(x))$ which has been used by some authors. Here, we follow the convention in Tversky and Kahneman (1992) to use w instead of $\bar{w}$.

[2] See theorem 2, Yaari (1987).

[3] Indeed, (2.3) can also be interpreted as a type of value-at-risk (VaR) constraint. Such a constraint is popular in the practice of risk management.

In RDUT, a prospect X is evaluated as

$$V(X) := \int_0^{+\infty} u(x)d[-w(1 - F_X(x))], \tag{2.4}$$

where $w(\cdot)$ is called a *probability weighting/distortion function*, and $u(\cdot)$ is the (outcome) utility function. Mathematically, the RDUT preference measure (2.4) generalizes the SP/A counterpart (2.1) by involving a nonlinear utility function $u(\cdot)$. However, the two measures have distinct economic interpretations. The SP/A preference measure (2.1) can be regarded as a mixture of two preference measures in Yaari's dual theory. As we argued earlier, hope and fear are not suitably modeled by this mixture. On the other hand, an RDUT preference measure, as we will see later, can provide a meaningful characterization of hope and fear.

Abundant work has been done to elicit the utility function and weighting function from experimental data, both using a parameter-based method (Tversky and Kahneman 1992; Tversky and Fox 1995) and using a parameter-free method (Abdellaoui 2000). The typical resulting weighting function is *inverse-S shaped*, showing that small probabilities of both very good and very bad events are overweighed. Meanwhile, the typical utility function is concave, reflecting the fact that people are less favorably disposed toward a risky gamble than to its mean payoff when only intermediate probabilities are involved.

Although RDUT is established in a normative way, i.e., via axiomatization, it shows deep psychological intuition. Because the typical weighting function is inverse-S shaped, its high end exhibits convexity, indicating that the individual pays too much attention to the worst outcomes[4] and hence shows pessimism or *fear*. At the same time, the weighting function is concave at the low end, reflecting the fact that the individual also pays too much attention to the best outcomes and shows optimism or *hope*. Therefore, RDUT—in particular, the inverse-S shaped weighting function—does indeed capture both fear and hope simultaneously.

The following three families of parameterized weighting functions are popular in the literature: the Tversky and Kahneman (1992) weighting function

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \tag{2.5}$$

with $0 < \gamma < 1$; the Tversky and Fox (1995) weighting function

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma} \tag{2.6}$$

with $\delta > 0$, $0 < \gamma < 1$; and the Prelec (1998) weighting function

$$w(p) = e^{-\delta(-\ln p)^\gamma} \tag{2.7}$$

with $\delta > 0$, $0 < \gamma < 1$. All of these weighting functions are inverse-S shaped.[5] Estimates of the parameters are available in many papers, such as Abdellaoui (2000), Wu and

[4]Assuming $w(\cdot)$ is differentiable, it follows from (2.4) that $V(X) = \int_0^{+\infty} w'(1 - F_X(x))u(x)dF_X(x)$, assuming that $F_X(\cdot)$ is continuous. Hence, $w'(\cdot)$ serves as a weight on the utility of the outcome. Clearly the high end of the weighting function corresponds to the low end of the outcome.

[5]The values of the parameters in (2.5) and (2.6) must be restricted to certain ranges to make the resulting weighting functions increasing. For instance, Ingersoll (2008) shows that for the Tversky–Kahneman weighting function (2.5) to be increasing, γ must be larger than 0.28. Luckily, all the estimates of those parameters in the literature lead to increasing weighting functions.

Gonzalez (1996), Tversky and Kahneman (1992), Camerer and Ho (1994), Bleichrodt and Pinto (2000), and Abdellaoui, Bleichrodt, and Paraschiv (2007).

Inspired by Jin and Zhou (2008) and Wang (2000), we will consider in this paper another class of weighting functions. Parameterized by $(a, b, \bar{z})$, this class of weighting functions is defined as follows:

$$w(z) = \begin{cases} ke^{(a+b)\Phi^{-1}(\bar{z})+\frac{a^2}{2}}\Phi(\Phi^{-1}(z)+a), & z \le \bar{z}, \\ A + ke^{\frac{b^2}{2}}\Phi(\Phi^{-1}(z)-b), & z \ge \bar{z}, \end{cases} \tag{2.8}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable, $a, b \ge 0$, and k and A are given as

$$k = \frac{1}{e^{\frac{b^2}{2}}\Phi(-\Phi^{-1}(\bar{z})+b) + e^{(a+b)\Phi^{-1}(\bar{z})+\frac{a^2}{2}}\Phi(\Phi^{-1}(\bar{z})+a)} > 0, \quad A = 1 - ke^{\frac{b^2}{2}},$$

respectively. Essentially, the weighting function (2.8) is obtained by pasting together smoothly the two one-parameter weighting functions in Wang (2000).[6] On the other hand, in contrast to Jin and Zhou (2008), where the values of a and b are restricted in a particular range in order to fulfill the monotonicity condition therein involving both the weighting function and the pricing kernel, we allow a and b to take any nonnegative values because that monotonicity condition is no longer needed in the present paper.

It is straightforward to compute that

$$w'(z) = \begin{cases} ke^{(a+b)\Phi^{-1}(\bar{z})-a\Phi^{-1}(z)}, & z \le \bar{z}, \\ ke^{b\Phi^{-1}(z)}, & z \ge \bar{z}. \end{cases}$$

Therefore, $w'(\cdot)$ is decreasing on $(0, \bar{z})$ and increasing on $(\bar{z}, 1)$, and consequently $w(\cdot)$ is inverse-S shaped. In addition, $\bar{z}$ is the inflection point of $w(\cdot)$.

Figure 2.1 depicts the various weighting functions (2.5)–(2.8). We use the parameter values estimated in Tversky and Kahneman (1992), Abdellaoui (2000), and Wu and Gonzalez (1996) for the first three weighting functions, i.e., $\gamma = 0.69$ for (2.5), $\delta = 0.65$, $\gamma = 0.6$ for (2.6), and $\gamma = 0.74$ for (2.7). For the Jin–Zhou weighting function (2.8), we choose the parameter values so that the resulting weighting function is graphically close to the other three weighting functions.[7] The specific parameter values are provided in Section 6.

It should be noted that the decumulative weighting function (2.2) used in the SP/A theory is not, in general, inverse-S shaped. It can generate an inverse-S shaped function with certain parameter values, e.g., $\nu = 0.3$, $q_s = 2$, and $q_p = 6$. However, there is an essential difference between (2.2) and the classical weighting functions such as (2.5)–(2.8). Extremely small probabilities, say 10^{-5} and 10^{-6}, are often indistinguishable to most people; hence, it is reasonable for a weighting function to have infinite sensitivity at both zero and one, like the ones in (2.5)–(2.8). However, for the decumulative weighting function $w(\cdot)$ in (2.2), $w'(0) = (1-\nu)(q_p + 1) < \infty$ and $w'(1) = \nu(q_s + 1) < \infty$.

The shape of the utility function marks the major difference between the decision-making model proposed here and the SP/A model. The linear utility in the SP/A theory

[6]Wang (2000) considers the probability weighting function $w(z) := \Phi(\Phi^{-1}(z) + \alpha)$, parameterized by α. It is straightforward to verify that this function is concave if $\alpha \ge 0$ and convex if $\alpha \le 0$.

[7]Unlike the other three weighting functions (2.5)–(2.7), the Jin–Zhou weighting function (2.8) has not been calibrated to real data. The main reason we use this weighting function here is because it allows for separate investigation of the effects of hope and fear on asset allocation; see Section 6.

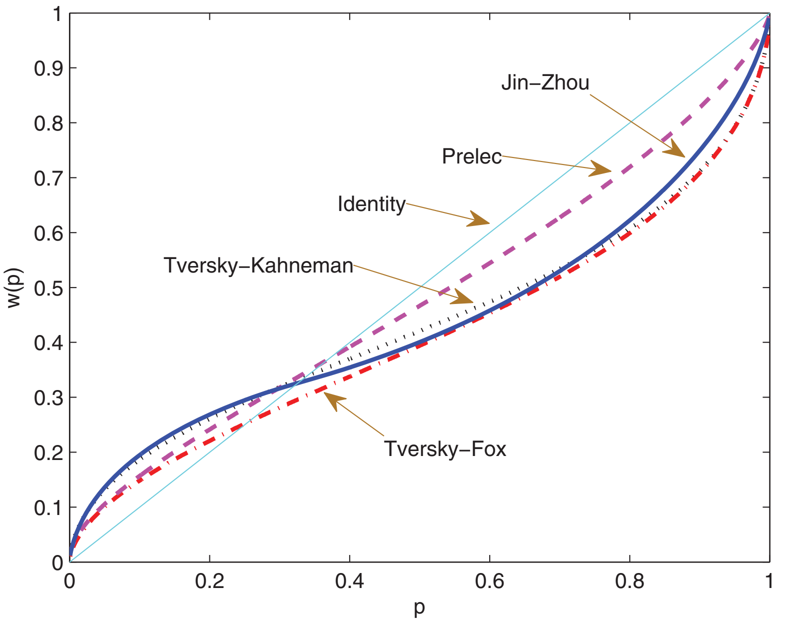

FIGURE 2.1. Graph of Kahneman–Tversky weighting function (2.5), Tversky–Fox weighting function (2.6), Prelec weighting function (2.7), Jin–Zhou weighting function (2.8), and the identity function.

makes it difficult to explain commonly observed risk-averse behavior when individuals face random gains of moderate probability; see for instance the experimental results in Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Indeed, the inverse-S shape of the weighting function implies that $w(p_0) = p_0$ for a moderate probability p_0. As a result, the preference of winning $\$p_0x$ over the gamble of winning $\$x$ with probability p_0 or \$0 with probability $1 - p_0$ implies that $u(x)p_0 + u(0)(1 - p_0) < u(p_0x)$, which is inconsistent with the linear utility setting. Lopes and Oden (1999, footnote 1) also note that the utility function does have mild concavity. Thus, the generalization of the SP/A theory here by considering a concave utility function can describe human behavior better than the original theory.[8]

On the other hand, the presence of the dispositional factor characterizes the marked difference of our model from one that is purely based on RDUT. RDUT alone fails to explain risk-seeking behavior when individuals face random losses of moderate probability; again see the experimental results in Kahneman and Tversky (1979) and Tversky and Kahneman (1992). This risk-seeking behavior, however, is well captured by choosing a proper aspiration level. Indeed, the aspiration constraint plays a similar role to the reference point in the CPT of Tversky and Kahneman (1992), in describing risk-seeking behavior. Lopes and Oden (1999) show that the SP/A theory can fit experimental data on risky choices as well as CPT. Thus, the model proposed in the present paper, which is a generalization of the SP/A theory, is able to explain many commonly observed patterns

[8]Shefrin (2008, section 27.5) proposes the CRRA-based SP/A theory, in which the author replaces the linear utility function in Lopes' SP/A theory with a power function. Thus, the model proposed in the present paper by using RDUT to model the dispositional factor is a further generalization of the CRRA-based SP/A theory.

in risky choices such as the fourfold pattern of risk attitudes (Tversky and Kahneman 1992, p. 306). Having said all these, we consider our model to be an alternative to, *not a replacement* of, CPT for modeling portfolio choice problems.

In the rest of this paper, we will formulate and solve our hope, fear, and aspiration (HF/A) model, a new portfolio selection model in continuous time. In this model, the objective function is of the form (2.4), which is taken from RDUT with an inverse-S shaped probability weighting function used to capture both hope and fear, whereas a probabilistic constraint of type (2.3) taken from SP/A theory is used to represent aspirations.[9]

3. FORMULATION OF HF/A PORTFOLIO CHOICE MODEL

Following Karatzas and Shreve (1998), we define a continuous-time financial market on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, where there lives a standard n-dimensional Brownian motion $W(t)$, $t \ge 0$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : 0 \le s \le t\}$, augmented by all the P-null sets. There is one risk-free asset with interest rate process $r(t)$, $t \ge 0$ and m risky stocks with expected growth rate vector process $\mu(t)$, $t \ge 0$ and volatility matrix process $\sigma(t)$, $t \ge 0$. All these processes are $(\mathcal{F}_t)$-progressively measurable.

In this market, the wealth process of an agent with an initial endowment x and a progressively measurable trading strategy represented by a *portfolio* $\pi(t)$, $t \ge 0$, satisfies

$$dX(t) = r(t)X(t)dt + \pi(t)^\top[(\mu(t) - r(t)\mathbf{1}_m)dt + \sigma(t)dW(t)], \quad t \ge 0; \quad X(0) = x,$$

where $\pi_i(t)$, the ith component of the m-dimensional vector $\pi(t)$, is the dollar amount invested in stock i at time t, $\mathbf{1}_m$ denotes the m-dimensional vector with all entries to be 1, and $A^\top$ denotes the transpose of matrix A.

We assume there exists a unique *market price of risk* $\theta(\cdot)$ satisfying $\sigma(t)\theta(t) = \mu(t) - r(t)\mathbf{1}_m$, $t \ge 0$. This assumption makes the market arbitrage-free and complete. We do not state explicitly here the market setting and the relevant technical conditions; but see He and Zhou (2011b, section 2.1). In addition, although we assume the completeness of the market here, the portfolio choice problem we are going to present can also be solved for some special incomplete markets; see the discussion in He and Zhou (2011b, section 4).

We now formulate the portfolio choice model featuring hope, fear, and aspirations in this financial market. At time 0, given an initial wealth x_0, the agent seeks the optimal trading strategy such that the wealth/payoff at a given terminal time T satisfies the probabilistic constraint that represents aspirations and the preference value of the payoff modeled as in RDUT that captures hope and fear is maximized. Therefore, the agent

[9] One may wonder why we do not apply the same probability weighting on the aspiration constraint. To answer this question, let us reiterate the different roles of hope, fear, and aspirations in affecting decision making. Hope and fear are two general psychological states that affect the individual's preference over prospects. The probability weighting function is merely a model of the two states. The individual's belief of the prospect X is still represented by the CDF $F_X(\cdot)$, rather than by $1 - w(1 - F_X(\cdot))$. On the other hand, aspirations are responsive to exigencies of each decision nexus and are unrelated to the inherent emotions of hope and fear. Thus, in the probabilistic constraint (2.3) that models aspirations, we should not apply any probability weighting function on $P(X \ge A)$. That said, applying the probability weighting in (2.3) would not mathematically change the model since the new constraint would be equivalent to (2.3) with a revised value of α.

faces the following HF/A portfolio choice problem:

(3.1)

$$\begin{aligned}
&\underset{\pi}{\text{Max}} && \int_0^\infty u(x)d[-w(1-F_{X(T)}(x))]\\
&\text{subject to} && dX(t) = r(t)X(t)dt + \pi(t)^\top[(\mu(t) - r(t)1_m)dt + \sigma(t)dW(t)],\ t \in [0, T];\\
& && X(0) = x_0, \quad P(X(T) \geq A) \geq \alpha, \quad X(t) \geq 0, \quad \forall\, t \in [0, T],
\end{aligned}$$

where $F_{X(T)}(\cdot)$ is the CDF of $X(T)$ viewed at time 0, and $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $w(\cdot) : [0, 1] \to [0, 1]$ are respectively the utility function and the probability weighting function.

When $w(\cdot)$ is the identity function and the aspiration constraint $P(X(T) \geq A)$ is absent, problem (3.1) becomes a classical portfolio choice problem under EUT, which has been mainly solved by dynamic programming approach in literature. In this approach, a *class* of problems under the same preference measure at different future times and states (t, x) are considered. Time consistency, which stipulates that the optimal strategy planned today must also be optimal in the future, provides a link among these problems. This leads to the dynamic programming principle, and to Hamilton-Jacobi-Bellman equations in Markovian settings, from which we can solve all the problems and, in particular, the one at time 0. With probability weighting, however, our problem (3.1) is inherently time inconsistent (i.e., the class of problems in the future using the same objective function as in (3.1) are time inconsistent), for which reason dynamic programming fails.[10]

REMARK 3.1. In literature, there are two ways to address the time-inconsistency. One is to consider so-called *equilibrium* strategies in lieu of optimal ones, in the context of all the agent's incarnations at different times playing games among themselves; see for instance Ekeland and Lazrak (2006), Björk, Murgoci, and Zhou (2014) and the references therein. The other is to consider *precommitted* optimal strategies, namely, the agent solves the underlying dynamic optimization problem at time 0 for an optimal strategy and commits himself to follow this strategy in the future. Due to the time-inconsistency of our problem, a precommitted strategy will not in general solve the portfolio choice problem under the same objective function at a future time. However, precommitted strategies are still important, since they are frequently applied in practice, sometimes with the help of certain commitment devices. For instance, Barberis (2012) finds that the precommitted strategy of a casino gambler is a stop-loss one (when the model parameters are in reasonable ranges). Many gamblers indeed follow this strategy by using some commitment devices, such as leaving ATM cards at home or bringing a little money to the casino; see Barberis (2012) for a full discussion. In the present paper, we will study only the precommitted strategies.

Since the dynamic programming approach fails for our model, we turn to the martingale approach. In this approach, we first determine the optimal terminal payoff and then replicate it using some feasible portfolio. Define the pricing kernel as follows:

$$\rho := \exp\left\{-\int_0^T \left[r(s) + \frac{1}{2}|\theta(s)|^2\right] ds - \int_0^T \theta(s)^\top dW(s)\right\}. \tag{3.2}$$

[10]Ekeland and Pirvu (2008) employ dynamic programming to solve an optimal investment-consumption problem with hyperbolic discounting at a fixed time by constructing a class of future problems having *different* objectives as in the original problem. For our problem (3.1), it seems impossible to do a similar construction.

Then, the HF/A portfolio choice problem can be reformulated as the following optimization problem:

$$
\begin{aligned}
(3.3)\qquad & \underset{X}{\text{Max}} && \int_0^\infty u(x)d[-w(1-F_X(x))] \\
& \text{subject to} && P(X \geq A) \geq \alpha, \\
& && E[\rho X] \leq x_0, \quad X \geq 0, \quad X \text{ is } \mathcal{F}_T \text{ measurable},
\end{aligned}
$$

where X represents the terminal payoff of certain portfolio.

The objective function in (3.3) is not concave due to the presence of the weighting function, so the convex dual method that is employed to solve portfolio selection problems under EUT cannot be applied. Here, we employ a new method—*the quantile formulation*—to cope with this difficulty. To use this method, we need to impose the following technical assumption throughout this paper:

ASSUMPTION 3.2. *ρ admits no atom, i.e., its CDF $F_\rho(\cdot)$ is continuous, and $E\rho < \infty$.*

If $(r(\cdot), \mu(\cdot), \sigma(\cdot))$ is deterministic and $\int_0^T |\theta(t)|^2 ds > 0$, then ρ is a lognormal random variable, which satisfies Assumption 3.2.

For a full account of the theory of quantile formulation, see He and Zhou (2011b). In this theory, roughly speaking, we deal with a generic portfolio choice model satisfying two basic assumptions: law-invariance and "the more the better." Law-invariance means that the preference measure and all of the constraints other than the budget constraint depend only on the probability distribution of the terminal payoff. "The more the better" means that if the agent has more initial budget, he can achieve higher objective value (see assumption 2.3 in He and Zhou (2011b) for a precise statement). This assumption is very natural for a sensible economic model. With these two assumptions, He and Zhou (2011b) demonstrated that the portfolio choice problem is equivalent to an optimization problem, i.e., the so-called quantile formulation, in which the optimal quantile function of the terminal payoff is to be found. The advantage of this formulation is that concavity is restored and hence traditional optimization techniques become applicable. Furthermore, there is a simple connection between the optimal solutions to the portfolio choice problem and its quantile formulation. If X^* is the optimal terminal payoff to the portfolio choice problem, then its quantile function is optimal to the quantile formulation. On the other hand, once the optimal quantile function $G^*(\cdot)$ is found, the optimal terminal payoff can be recovered by $X^* := G^*(Z_\rho)$ where $Z_\rho := 1 - F_\rho(\rho)$ is a particular uniformly distributed random variable.[11]

One can easily verify that the HF/A portfolio choice model (3.3) satisfies the aforementioned two basic assumptions, and hence it has the following quantile formulation:[12]

$$
\begin{aligned}
(3.4)\qquad & \underset{G(\cdot)}{\text{Max}} && U(G(\cdot)) = \int_0^1 u(G(z))w'(1-z)\,dz \\
& \text{subject to} && \int_0^1 F_\rho^{-1}(1-z)G(z)\,dz \leq x_0. \\
& && G(\cdot) \in \mathbb{G}, \quad G((1-\alpha)+) \geq A, \quad G(0+) \geq 0,
\end{aligned}
$$

[11] A general result derived from the quantile formulation is that the optimal payoff must be *anticomonotonic* with respect to the pricing kernel ρ. This property heavily relies on Assumption 3.2. Examples of violation of this property when Assumption 3.2 does not hold can be found in Ingersoll (2011).

[12] For a derivation one may follow the same procedure as described in section 2 of He and Zhou (2011b).

where $\mathbb{G}$ is the set of all lower-bounded quantile functions, i.e.,

$$\mathbb{G} = \{G(\cdot) : (0,1) \to \mathbb{R},\ \text{nondecreasing, left continuous, and } G(0+) > -\infty\}. \tag{3.5}$$

Assumption 3.2 will be in force in the reminder of this paper so that we could use quantile formulation to solve the HF/A portfolio choice problem.[13] In the remainder of this paper, we also assume throughout that $0 < \alpha < 1$ and $A \geq 0$,[14] as well as the following technical assumption on the weighting function $w(\cdot)$, which is satisfied by all the functions in (2.5)–(2.8):

ASSUMPTION 3.3. *$w(\cdot) : [0,1] \to [0,1]$ is continuous and strictly increasing with $w(0) = 0$, $w(1) = 1$. Furthermore, $w(\cdot)$ is continuously differentiable on $(0,1)$.*

4. FEASIBILITY AND WELL-POSEDNESS

An optimization problem is feasible if the set of decision variables satisfying *all* the constraints is nonempty. Feasibility is relevant only to the constraints, not the objective. For our model (3.3) the aspiration constraint $P(X \geq A) \geq \alpha$ gives rise to the feasibility issue that we should deal with first. It turns out that the feasibility issue can be addressed via a goal-reaching problem, which has been solved in the literature.

The following notation is used throughout this paper: A "feasible solution" to a constrained maximization problem satisfies all the constraints involved, and an "optimal solution," while being a feasible solution, achieves the maximum of the problem.

PROPOSITION 4.1. *Problem (3.3) is feasible if and only if $x_0 \geq AE[\rho \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}}]$. Furthermore, if $x_0 = AE[\rho \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}}]$, then there is only one feasible solution to (3.3), given as $X = A\mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}}$.*

Proof. Consider the following goal-reaching problem:

$$\begin{array}{ll}
\underset{X}{\text{Max}} & P(X \geq A) \\
\text{subject to} & E[\rho X] \leq x_0. \\
 & X \geq 0, \quad X \in \mathcal{F}_T.
\end{array}$$

Problem (3.3) is feasible if and only if the optimal value of the above goal-reaching problem is larger than α. From He and Zhou (2011b), theorem 1, we conclude that if $x_0 \geq AE[\rho]$, then the optimal value is 1. When $x_0 < AE[\rho]$, the optimal value is $F_\rho(c^*)$ where c^* is the value such that $AE[\rho \mathbf{1}_{\{\rho \leq c^*\}}] = x_0$. Therefore, problem (3.3) is feasible if and only if $x_0 \geq AE[\rho]$, or $x_0 < AE[\rho]$ and $F_\rho(c^*) \geq \alpha$, $AE[\rho \mathbf{1}_{\{\rho \leq c^*\}}] = x_0$ for some $c^* \in \mathbb{R}_+$. Equivalently, (3.3) is feasible if and only if $x_0 \geq AE[\rho \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}}]$. In particular, if $x_0 = AE[\rho \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}}]$, the goal-reaching problem has the unique optimal solution

[13] In the case of an incomplete market and/or in the presence of constraints on portfolios, quantile formulation does not work in general. However, when the investment opportunity set—consisting of $r(\cdot)$, $b(\cdot)$, and $\sigma(\cdot)$—is deterministic, the quantile formulation still works even with conic constraints imposed on portfolios. In this case, one needs to replace ρ in the quantile formulation (3.4) with the *minimal pricing kernel*. See He and Zhou (2011b, section 4) for details.

[14] If $\alpha = 0$, then the aspiration constraint is not in force. Therefore, the case in which $\alpha = 0$ can be covered by setting $A = 0$. If $\alpha = 1$, then the aspiration constraint becomes a uniformly lower bound on X, in which case we could shift the terminal payoff X by A and turn the portfolio choice model to a model without the aspiration constraint.

$A\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}$ and the optimal value is α. As a result, the only feasible solution to (3.3) is $X = A\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}$. $\square$

Proposition 4.1 shows that, with exogenously given aspiration level and confidence level, the agent must be sufficiently endowed in order for the level of aspirations to be at least feasible. Put differently, the aspiration level relative to the initial wealth cannot be set too high in the HF/A model. Generally speaking, the triplet (x_0, A, α) must be internally consistent so as to make the model minimally meaningful. In the remainder of this paper, to exclude the infeasibility and the trivial case, we assume that $x_0 > AE[\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}]$.

The next issue is the well-posedness of the portfolio choice problem. An optimization problem is considered well posed if its optimal value is finite; otherwise, it is ill posed. In an ill-posed model, optimality is achieved at the extremal points or boundary of the feasible domain. If the portfolio choice problem is ill posed, the agent is willing to take as much leverage as possible, leading to excessive risk-taking behavior. For detailed discussions on the ill-posedness issue arising in portfolio choice problems and its economic interpretations, see Jin and Zhou (2008) and He and Zhou (2011a).

We have argued that the preference measure (2.1) proposed in Lopes' SP/A does not capture hope and fear well. Here, we will show that, furthermore, the SP/A theory indeed leads to an ill-posed portfolio choice problem in the continuous-time setting.

THEOREM 4.2. *Let* $x_0 > AE[\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}]$. *Assume* $u(x)\equiv x$ *and* $\operatorname{essinf}\rho = 0$. *If* $\liminf_{z\downarrow 0} w'(z) > 0$, *then (3.3) is ill posed. In particular, if* $w(\cdot)$ *is given by (2.2) with* $\nu < 1$, *then (3.3) is ill posed.*

Proof. Let $v(x_0)$ be the optimal value of the problem (3.3) or (3.4). Let $\tilde{\mathbb{G}} := \{G(\cdot) \mid G(z) = (A + \tilde{G}(1 - \frac{1-z}{\alpha}))\mathbf{1}_{\{1-\alpha<z<1\}}, \tilde{G}(\cdot) \in \mathbb{G}, \tilde{G}(0+) \ge 0\} \subset \mathbb{G}$ and consider the following problem:

$$\begin{aligned} \underset{G(\cdot)}{\text{Max}} \quad & \int_0^1 G(z)w'(1-z)\,dz \\ \text{Subject to} \quad & G(\cdot) \in \tilde{\mathbb{G}}, \\ & \int_0^1 F_\rho^{-1}(1-z)G(z)\,dz \le x_0. \end{aligned}$$

This problem has a smaller optimal value than (3.3) because of the smaller feasible set. Rewrite the above problem as

$$\begin{aligned} \underset{\tilde{G}(\cdot)}{\text{Max}} \quad & Aw(\alpha) + \alpha\int_0^1 \tilde{G}(z)w'(\alpha(1-z))\,dz \\ \text{Subject to} \quad & \tilde{G}(\cdot) \in \mathbb{G},\ \tilde{G}(0+) \ge 0, \\ & \alpha\int_0^1 F_\rho^{-1}(\alpha(1-z))\tilde{G}(z)\,dz \le x_0 - AE\big[\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}\big]. \end{aligned}$$

This problem is identical to Yaari's dual model as given in He and Zhou (2011b). Recalling He and Zhou (2011b), theorem 3.4, and the fact that $\liminf_{z\downarrow 0}\frac{w'(\alpha z)}{F_\rho^{-1}(\alpha z)} = \liminf_{z\downarrow 0}\frac{w'(z)}{F_\rho^{-1}(z)} = +\infty$, we conclude that the above problem is ill posed. $\square$

Theorem 4.2 suggests that as long as the agent has hope for extremely satisfactory situations (i.e., $\nu < 1$) while the utility function is linear, he will take as high leverage as possible, leading to an ill-posed problem. The fear of possible catastrophic situations is insufficient to prevent him from taking excessive risky exposures. Therefore, the preference measure (2.1) in Lopes' SP/A theory is *generally* not suitable for portfolio choice problems in the continuous-time setting.

It is, however, worth mentioning that the continuous-time setting is not an essential reason for ill-posedness. Indeed, any continuous-time market can be considered as a single-period market in which agents trade a set of contingent claims whose prices are determined by a pricing kernel ρ. In the present paper, with the complete market assumption, the set of tradeable contingent claims is sufficiently large (i.e., all $\mathcal{F}_T$-measurable contingent claims are tradeable). Recall that $\rho(\omega)$ is the price of the contingent claim that pays \$1 in state ω and \$0 otherwise. Thus, with the assumption that $\operatorname{essinf} \rho = 0$, one could use \$1 to buy contingent claims that pay as much as possible in certain states of the world and pay nothing otherwise. It is those contingent claims that give rise to the ill-posedness of the SP/A portfolio choice problem in our setting. Therefore, the SP/A portfolio choice problem is likely to be well posed if the set of tradeable contingent claims is restricted. For instance, in a single-period complete market with *finite* states of the world, the portfolio choice problem under SP/A theory is well posed; see Shefrin and Statman (2000). On the other hand, even in a continuous-time market, the SP/A portfolio choice problem could be well posed if the market is sufficiently incomplete.

The finding that the portfolio choice problem under SP/A theory can be easily ill posed in the continuous-time complete market further enhances the argument for taking the RDUT preference instead of the SP/A one as the model for hope and fear. In the remainder of this paper, we impose the following diminishing marginality on the utility function being used:

ASSUMPTION 4.3. *$u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, differentiable, strictly concave, and satisfies the Inada condition, i.e., $u'(0+) = +\infty$ and $u'(+\infty) = 0$.*

REMARK 4.4. In CPT, utility functions are assumed to be S-shaped in order to model risk-seeking behavior regarding random losses of moderate probability. In the decision-making model proposed in the present paper, this risk-seeking behavior is modeled through the situational factor—the aspiration constraint. For this reason, we do not consider S-shaped utility functions in this paper.

5. OPTIMAL SOLUTIONS IN THE HF/A MODEL

In this section, we develop the procedure in finding optimal solutions to problem (3.3) by attacking its quantile formulation (3.4). Along the way we will introduce, rather naturally, various indices for quantifying the level of hope, fear, and aspirations and study their impacts on trading behavior.

5.1. Optimal Solutions under a Monotonicity Condition

Following the general solution scheme in He and Zhou (2011b), we start with applying the Lagrange dual method to (3.4). For any Lagrange multiplier $\lambda > 0$, we consider the following problem:

$$\begin{aligned} &\underset{G(\cdot)}{\operatorname{Max}} && U_\lambda(G(\cdot)) = \int_0^1 \left[u(G(z))w'(1-z) - \lambda G(z)F_\rho^{-1}(1-z)\right]dz \\ &\text{subject to} && G(\cdot) \in \mathbb{G}, \quad G((1-\alpha)+) \geq A, \quad G(0+) \geq 0. \end{aligned} \tag{5.1}$$

To simplify the notation, let us denote

$$f(x,z) := u(x)w'(1-z) - \lambda x F_{\rho}^{-1}(1-z), \quad 0 < z < 1. \tag{5.2}$$

Then, the objective function in (5.1) becomes

$$U_{\lambda}(G(\cdot)) = \int_0^1 f(G(z), z)\, dz.$$

Define the following function:

$$M(z) := \frac{w'(1-z)}{F_{\rho}^{-1}(1-z)}, \quad 0 < z < 1, \tag{5.3}$$

which plays an important role in finding optimal solutions.

Let us first ignore the constraint that $G(\cdot)$ must be a quantile function (its monotonicity being the key requirement), and consider the pointwise maximization of integrand $f(x,z)$ for each fixed z. When $z \in (0, 1-\alpha]$, the maximization problem is $\max_{x \geq 0} f(x,z)$. Clearly, the unique maximizer is $x^* = (u')^{-1}(\frac{\lambda}{M(z)})$. When $z \in (1-\alpha, 1)$, the maximization problem is $\max_{x \geq A} f(x,z)$ and the unique maximizer is $x^* = (u')^{-1}(\frac{\lambda}{M(z)}) \vee A$, where for any $a, b \in \mathbb{R}$, we denote $a \vee b := \max(a,b)$. This pointwise maximization procedure leads to the introduction of the following function:

$$G_{\lambda}^*(z) := (u')^{-1}\left(\frac{\lambda}{M(z)}\right)\mathbf{1}_{\{z \leq 1-\alpha\}} + \left[(u')^{-1}\left(\frac{\lambda}{M(z)}\right) \vee A\right]\mathbf{1}_{\{z > 1-\alpha\}}. \tag{5.4}$$

By this construction, $G_{\lambda}^*(\cdot)$ automatically satisfies the nonnegativity constraint and the aspiration constraint in (5.1). If, furthermore, $M(\cdot)$ turns out to be nondecreasing, then $G_{\lambda}^*(\cdot)$ is nondecreasing and hence a quntile function. In this case, $G_{\lambda}^*(\cdot)$ is optimal to (5.1).[15]

One can easily check that $M(\cdot)$ is nondecreasing when $w(z) \equiv z$ or in general when $w(\cdot)$ is concave. When $w(z) \equiv z$, there is no probability weighting and the rank-dependent utility degenerates into the classical expected utility. In general, the concavity of $w(\cdot)$ represents a risk-seeking attitude (Yaari 1987). In this case, our HF/A model incorporates a risk-averse attitude resulting from the concave utility function and a risk-loving attitude described by the probability weighting function, and an optimal payoff is to strike a balance between the two conflicting risk attitudes.[16]

We now present the solution to (3.3) assuming $M(\cdot)$ is nondecreasing. As in standard expected utility maximization problems, we also need the following integrability assumption:

ASSUMPTION 5.1. *There exists $c > 0$ such that for any $\lambda > 0$, $E[\rho(u')^{-1}(\frac{\lambda\rho}{w'(F_{\rho}(\rho))})\mathbf{1}_{\{\rho \leq c\}}] < +\infty$ and $E[u((u')^{-1}(\frac{\lambda\rho}{w'(F_{\rho}(\rho))}))w'(F_{\rho}(\rho))\mathbf{1}_{\{\rho \leq c\}}] < +\infty$.*

This assumption can be replaced with a weaker one that the asymptotic elasticity of $u(\cdot)$ is strictly less than one. See Jin, Xu, and Zhou (2007) and Kramkov and Schachermayer (1999) for detailed discussions. In classic expected utility maximization problems, i.e., in the case in which $w(z) \equiv z$, Assumption 5.1 holds in most of the interesting cases, e.g.,

[15] This argument is applied in Jin and Zhou (2008) to solve the gain part problem in a portfolio selection model under CPT.

[16] However, as discussed in Section 2, a concave weighting function does not appropriately capture hope and fear.

when ρ is lognormally distributed and $u(\cdot)$ is a power utility function. In the presence of the weighting function, one could check that this assumption still holds when ρ is lognormally distributed, $u(\cdot)$ is a power utility function and $w(\cdot)$ is taken as (2.5) and (2.6), or (2.7) when $\gamma > \frac{1}{2}$. Having said that, Assumption 5.1 is not always valid. For instance, if $u(\cdot)$ is a power function, ρ is lognormally distributed and $w(\cdot)$ is taken as (2.7) with a small γ, it is not difficult to show that Assumption 5.1 fails.

THEOREM 5.2. *Let Assumptions 4.3 and 5.1 hold, $x_0 > AE[\rho \mathbf{1}_{\{\rho\le F^{-1}(\alpha)\}}]$, and assume $M(\cdot)$ is nondecreasing. Then, the unique optimal solution to (3.3) is given as*

$$X^* = \left[(u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right)\vee A\right]\mathbf{1}_{\{\rho\le F^{-1}(\alpha)\}} + (u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right)\mathbf{1}_{\{\rho > F^{-1}(\alpha)\}},$$

where λ^* is the value such that the initial budget constraint binds, i.e., $E[\rho X^*] = x_0$.

Proof. For each fixed $\lambda > 0$, because $M(\cdot)$ is nondecreasing, $G^*_\lambda(\cdot)$ defined in (5.4) is also nondecreasing. As a result, $G^*_\lambda(\cdot)$ is optimal to (5.1). Let

$$\begin{aligned}\mathcal{X}(\lambda) :&= \int_0^1 F_\rho^{-1}(1-z)G^*_\lambda(z)\,dz \\ &= E\left[\rho\left((u')^{-1}\left(\frac{\lambda\rho}{w'(F_\rho(\rho))}\right)\vee A\right)\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}} + \rho(u')^{-1}\left(\frac{\lambda\rho}{w'(F_\rho(\rho))}\right)\mathbf{1}_{\{\rho > F_\rho^{-1}(\alpha)\}}\right].\end{aligned}$$

By Assumption 5.1 and the fact that $E\rho < \infty$, $\mathcal{X}(\cdot)$ is finite-valued and nonincreasing on $(0, +\infty)$. Furthermore, because ρ has no atom, by the monotone convergence theorem, $\mathcal{X}(\cdot)$ is continuous on $(0, +\infty)$ and

$$\lim_{\lambda\downarrow 0}\mathcal{X}(\lambda) = +\infty, \qquad \lim_{\lambda\uparrow +\infty}\mathcal{X}(\lambda) = AE[\rho\mathbf{1}_{\{\rho\le F^{-1}(\alpha)\}}].$$

Therefore, for each $x_0 > AE[\rho\mathbf{1}_{\{\rho\le F^{-1}(\alpha)\}}]$, there exists $\lambda^* > 0$ such that $\mathcal{X}(\lambda^*) = x_0$. As discussed in the general solution scheme in He and Zhou (2011b), $G^*_{\lambda^*}(\cdot)$ is optimal to (5.1) and consequently $X^* := G^*_{\lambda^*}(1 - F_\rho(\rho))$ is optimal to (3.3). The uniqueness follows easily. □

5.2. Fear, Portfolio Insurance, and Fear Index

Theorem 5.2 requires the monotonicity of $M(\cdot)$. However, in this subsection we show that $M(\cdot)$ is *not* nondecreasing for many weighting functions proposed in the literature together with a reasonable pricing kernel ρ. Thus, a different methodology needs to be developed to solve (5.1) without the monotonicity of $M(\cdot)$.

To start, it is straightforward to check that $M(\cdot)$ is nondecreasing if and only if

$$\frac{w''(z)}{w'(z)} \le \frac{\bar{F}'(z)}{\bar{F}(z)}, \quad 0 < z < 1 \tag{5.5}$$

where $\bar{F}(z) := F_\rho^{-1}(z)$, $0 < z < 1$.

The following proposition shows, however, that for the weighting functions in (2.5)–(2.8) together with a reasonable pricing kernel, (5.5) is violated.

PROPOSITION 5.3. *Suppose ρ is lognormally distributed, i.e.,*

$$F_\rho(x) = \Phi\left(\frac{\ln x - \mu_\rho}{\sigma_\rho}\right)$$

for some μ_ρ and $\sigma_\rho > 0$. For any weighting function in (2.5)–(2.7) with $0 < \gamma < 1$, there exists $\varepsilon > 0$ such that

$$\frac{w''(z)}{w'(z)} > \frac{\bar{F}'(z)}{\bar{F}(z)}, \quad 1 - \varepsilon < z < 1. \tag{5.6}$$

For the weighting function in (2.8), if $b > \sigma_\rho$, then

$$\frac{w''(z)}{w'(z)} > \frac{\bar{F}'(z)}{\bar{F}(z)}, \quad \bar{z} < z < 1. \tag{5.7}$$

Proof. Because $\bar{F}(z) = F_\rho^{-1}(z) = e^{\mu_\rho + \sigma_\rho \Phi^{-1}(z)}$, we have

$$\frac{\bar{F}'(z)}{\bar{F}(z)} = \frac{\sigma_\rho}{\Phi'(\Phi^{-1}(z))},$$

where $\Phi'(\cdot)$ is the probability density function (PDF) of a standard normal random variable. Thus, for the weighting functions (2.5)–(2.7), it is sufficient to show that

$$\frac{w''(\Phi(y))}{w'(\Phi(y))} > \frac{\sigma_\rho}{\Phi'(y)},$$

when y goes to $+\infty$. Noticing $1 - \Phi(y) < \Phi'(y)/y$, $y > 0$, we can show that

$$\liminf_{y \uparrow +\infty} \frac{w''(\Phi(y))}{w'(\Phi(y))} \Phi'(y) = +\infty$$

for all the weighting functions in (2.5)–(2.7), which shows that (5.6) holds for some $\varepsilon > 0$.

For the weighting function (2.8), it is straightforward to compute that

$$\frac{w''(z)}{w'(z)} = \begin{cases} -\dfrac{a}{\Phi'(\Phi^{-1}(z))}, & 0 < z < \bar{z}, \\ \dfrac{b}{\Phi'(\Phi^{-1}(z))}, & \bar{z} < z < 1. \end{cases}$$

So (5.7) follows easily. □

Proposition 5.3 stipulates that $M(\cdot)$ is not nondecreasing with some common weighting functions and a lognormally distributed pricing kernel. The following theorem shows that in this case the behavior of the optimal solution to (3.3) is changed drastically when compared with Theorem 5.2.

THEOREM 5.4. *Let Assumption 4.3 hold and $x_0 > AE[\rho \mathbf{1}_{\{\rho \le F_\rho^{-1}(\alpha)\}}]$. Suppose $w(\cdot)$ is twice differentiable and $\bar{F}(\cdot)$ is differentiable. If there exists $\varepsilon > 0$ such that*

$$\frac{w''(z)}{w'(z)} \ge \frac{\bar{F}'(z)}{\bar{F}(z)}, \quad 1 - \varepsilon < z < 1, \tag{5.8}$$

then for any optimal solution X^* to (3.3), it must hold that $\operatorname{essinf} X^* > 0$.

Proof. Let X^* be an optimal solution to (3.3) and let $G^*(\cdot)$ be its quantile function. Then, from the general theory of quantile formulation, $G^*(\cdot)$ is optimal to (3.4) and $X^* = G^*(1 - F_\rho(\rho))$ $P -$ a.s. Therefore, it is sufficient to prove that $G^*(0+) > 0$.

Since $G^*(\cdot)$ is optimal to (3.4), by convex duality, $G^*(\cdot)$ is also optimal to (5.1) for some $\lambda^* > 0$. Because $\frac{w''(z)}{w'(z)} \geq \frac{\bar{F}'(z)}{\bar{F}(z)}$, $1 - \varepsilon < z < 1$, $M(\cdot)$ is nonincreasing on $(0, \varepsilon)$. Suppose $G^*(0+) = 0$. Let $z_1 := \inf\{z \in (0,1) \mid G^*(z) > 0\}$. We must have $z_1 < 1$ because $G^*(\cdot) \not\equiv 0$. If $z_1 = 0$, there exists $0 < z_2 < \epsilon$ such that $G^*(z) < M(z)$ for $z \in (0, z_2]$. Define

$$\tilde{G}(z) := \begin{cases} G^*(z_2) & 0 < z \leq z_2 \\ G^*(z) & z_2 < z < 1. \end{cases}$$

If $z_1 > 0$, we pick $z_3 \in (z_1, 1)$ such that $G^*(z_3) > 0$. Define

$$\tilde{G}(z) := \begin{cases} G^*(z) \vee \delta & 0 < z \leq z_3 \\ G^*(z) & z_3 < z < 1 \end{cases}$$

for some $0 < \delta < \min\{G^*(z_3), \min_{0<z\leq z_3} M(z)\}$. In both cases, $\tilde{G}(\cdot)$ is feasible to (5.1). Furthermore, it is straightforward to check that $U_{\lambda^*}(\tilde{G}(\cdot)) > U_{\lambda^*}(G^*(\cdot))$, which is a contradiction. Therefore, we must have $G^*(0+) > 0$ and the proof is complete. □

This theorem shows that an optimal strategy needs to set a *strictly positive* deterministic floor (essinf $X^* > 0$) in wealth at the terminal time, in sharp contrast to the strategy presented in Theorem 5.2 when $M(\cdot)$ is monotone.[17] Such a deterministic floor is in line with the *portfolio insurance* commonly practiced in the asset management industry. Portfolio insurance is a risk management strategy by means of which a minimum level of wealth is guaranteed across the investment period. Portfolio insurance has been widely studied in the literature; see, e.g., Basak (1995) and Grossman and Zhou (1996). In most studies, the portfolio insurance level is imposed *exogenously*, and it is not clear how the floor changes with respect to various market parameters. By contrast, our model generates a portfolio insurance level *decisively and endogenously*.[18] Moreover, in the following subsection we will derive an explicit expression for this level.

The key condition that has prompted the need for portfolio insurance is (5.8), which is expressed in terms of the "curvature" of the probability weighting function near 1. Recall that $w(z)$ when z is close to 1 is relevant to the exaggeration of the (small) probabilities of very bad outcomes; hence, $\frac{w''(z)}{w'(z)}$ when z is near 1 is relevant to fear. Theorem 5.4 then implies that portfolio insurance is necessary when the agent is sufficiently fearful, which is quantified as $\frac{w''(z)}{w'(z)}$ exceeding a certain (moving) level when z is near 1.

The above discussion motivates us to define the following *fear index* for a weighting function $w(\cdot)$:

[17]For instance, if esssup $\rho = +\infty$ and there is no probability weighting, i.e., $w(z) \equiv z$, in which case the agent's preference is dictated by the EUT and $M(\cdot)$ is trivially nondecreasing, then the terminal wealth X^* has no strictly positive floor.

[18]The possibility of deriving an endogenous portfolio insurance level from a portfolio choice problem has also been illustrated in several papers. Carlier and Dana (2011) derive the optimal demand for contingent claims for an agent with rank-dependent utility and find that the optimal demand may have a flattening part, suggesting portfolio insurance. The flattening structure has also been discussed in Ingersoll (2011) where the author considers a portfolio choice problem faced by an agent whose preference is modeled by CPT.

$$\mathcal{I}_w(z) := \frac{w''(z)}{w'(z)}, \quad 0 < z < 1. \tag{5.9}$$

Note that this index is important and relevant only when z is sufficiently close to 1.

The fear index is clearly an analogue of the Arrow-Pratt measure of absolute risk-aversion for a utility function.[19] It can be used to measure the agent's level of fear. The higher this index, the more convex the weighting function is, and the more fear the agent has. We have shown that this index is critical in deciding the monotonicity of $M(\cdot)$ and the optimal behaviors of the agent following our HF/A model. In particular, by Theorem 5.4, a sufficiently high level of fear, which is characterized by the fear index exceeding a certain threshold, endogenously necessitates portfolio insurance.

We now compute the fear indices for the weighting functions in (2.5)–(2.8). For the Kahneman–Tversky weighting function (2.5), we have $\mathcal{I}_w(z) \approx 2(1-z)^{-1}$ as $z \uparrow 1$. For the Tversky–Fox weighting function (2.6), we have $\mathcal{I}_w(z) \approx (1-\gamma)(1-z)^{-1}$ as $z \uparrow 1$. For the Prelec weighting function (2.7), we have $\mathcal{I}_w(z) \approx (1-\gamma)(-\ln p)^{-1}$ as $z \uparrow 1$. For the Jin–Zhou weighting function (2.8), we have $\mathcal{I}_w(z) = \frac{b}{\Phi'(\Phi^{-1}(z))}$ as $z \uparrow 1$. Thus, for the Kahneman–Tversky weighting function, the degree of fear is independent of the parameter γ. For the Tversky–Fox and the Prelec weighting functions, a smaller γ leads to a higher degree of fear. For the Jin–Zhou's weighting function, b measures the degree of fear.

In view of Proposition 5.3, $M(\cdot)$ is typically *not* nondecreasing. From this point, we impose the following assumption in place of the monotonicity of $M(\cdot)$:

ASSUMPTION 5.5. *$M(\cdot)$ is continuously differentiable on $(0, 1)$ and there exists $0 < z_0 < 1$ such that $M(\cdot)$ is strictly decreasing on $(0, z_0)$ and strictly increasing on $(z_0, 1)$. Furthermore, $\lim_{z\uparrow 1} M(z) = +\infty$.*

Assumption 5.5 requires $M(\cdot)$ to be first decreasing and then increasing. From an economics point of view, this requirement aligns with a suitable combination of an inverse-S shaped weighting function and a market variable, namely, the agent is fearful of very bad market conditions and hopeful for very good ones. Not surprisingly, Assumption 5.5 holds with the weighting functions in (2.5)–(2.7) and a lognormally distributed market pricing kernel ρ, as shown in Figures 5.1–5.3.[20]

[19] Although we have been unable to find an explicit definition of the index (5.9) for probability weighting in the literature, there are works that imply such an index. For instance, Abdellaoui (2002), theorem 12, establishes that a weighting function w_2 is more "probabilistic risk averse" than another one w_1 if there exists a continuous, convex, and strictly increasing function θ such that $w_2 = \theta \circ w_1$. It is not difficult to show that the latter condition is further equivalent to $\mathcal{I}_{w_2}(z) \geq \mathcal{I}_{w_1}(z)$, $0 < z < 1$. On the other hand, when $u(x) \equiv x$, the RDUT measure (2.4) can be written as $V(X) = \int_0^\infty w(1 - F_X(x))dx$ by integration by parts. Thus, this preference functional admits the local utility $U(x; F) := -\int_0^x w'(1 - F(y))dy$ according to Machina (1982). The (generalized) Arrow-Pratt index for the local utility, $-U''(x; F)/U'(x; F)$, is $w''(1 - F(x))F'(x)/w'(1 - F(x))$. By theorem 4 in Machina (1982), the higher the Arrow-Pratt index is, the more risk averse the agent is. Thus, the fear index defined here is also related to the generalized Arrow-Pratt index of a preference functional.

[20] Here, we use the same market data as Mehra and Prescott (1985). More precisely, assume that there is only one stock, e.g., the S&P 500 index. The investment opportunity set (r, b, σ) is estimated from the data of real returns of the U.S. S&P 500 index and Treasury Bills during the period 1889–1978: $r = 1\%$, $b = 7\%$, and $\sigma = 16.55\%$. The investment period T is taken as 1 year. The pricing kernel can then be computed from (3.2). As for the weighting functions, the values of parameters are estimated in Tversky and Kahneman (1992), Abdellaoui (2000), and Wu and Gonzalez (1996). On the other hand, we can in fact prove analytically that Assumption 5.5 is satisfied for the Prelec weighting function (2.7).

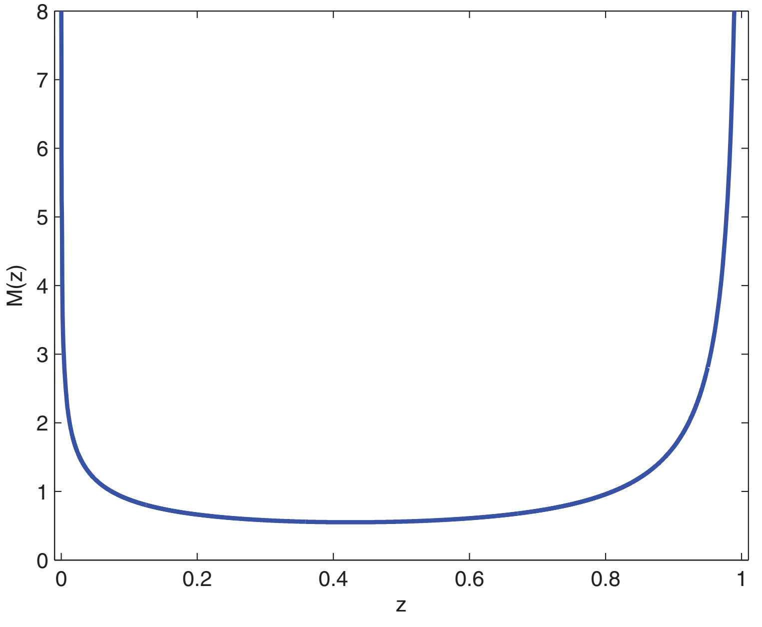

FIGURE 5.1. Graph of $M(z)$ with Tversky–Kahneman weighting function.

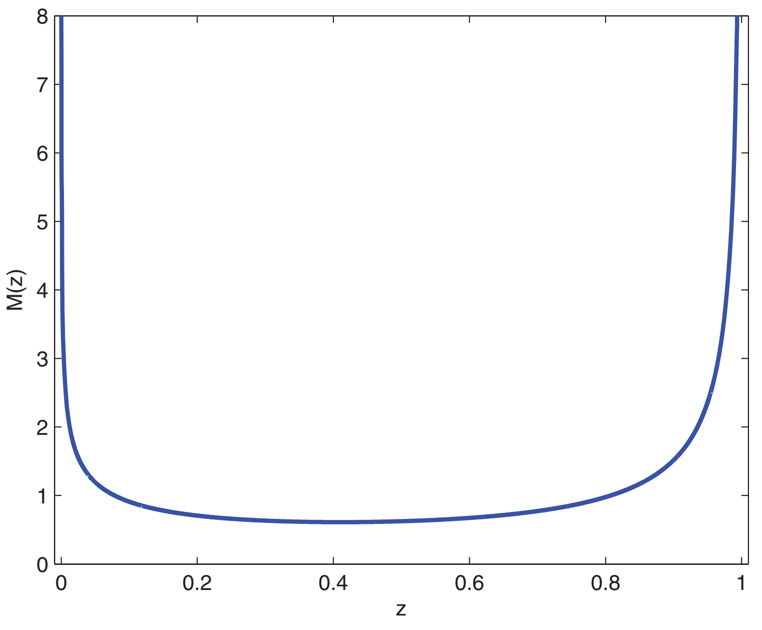

FIGURE 5.2. Graph of $M(z)$ with Tversky–Fox weighting function.

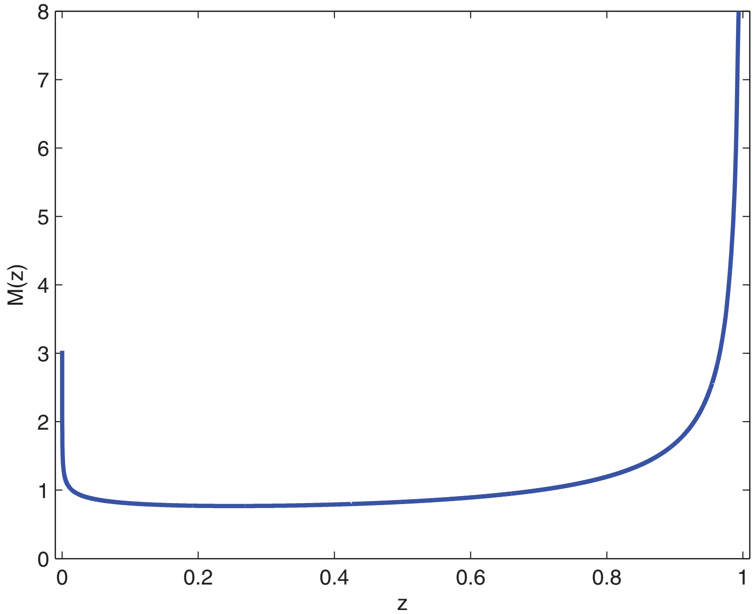

FIGURE 5.3. Graph of $M(z)$ with Prelec weighting function.

As for the Jin–Zhou weighting function (2.8), if ρ is lognormally distributed, i.e., $F_\rho(x) = \Phi(\frac{\ln x - \mu_\rho}{\sigma_\rho})$, then it is easy to compute that

$$M(z) = \begin{cases} ke^{-\mu_\rho + (b-\sigma_\rho)\Phi^{-1}(1-z)}, & 0 < z < 1 - \bar{z}. \\ ke^{(a+b)\Phi^{-1}(\bar{z}) - \mu_\rho - (a+\sigma_\rho)\Phi^{-1}(1-z)}, & 1 - \bar{z} < z < 1. \end{cases}$$

Thus, when $b > \sigma_\rho$, $M(\cdot)$ satisfies Assumption 5.5 with $z_0 = 1 - \bar{z}$. When $b \leq \sigma_\rho$, $M(\cdot)$ is nondecreasing. In this subsection, we are interested in the case in which $b > \sigma_\rho$.

We conclude this subsection by studying the impact of fear on the optimal terminal wealth when the aspiration constraint is absent.

PROPOSITION 5.6. *Under Assumption 4.3 and given that* $A = 0$, *suppose there are two functions,* $w_1(\cdot)$ *and* $w_2(\cdot)$, *satisfying Assumptions 3.3, 5.1, and 5.5 with the same* z_0. *Assume further that*

$$w_1(z) = w_2(z), \quad 0 < z < 1 - z_0.$$

Let X_1^* *and* X_2^* *be the optimal solutions corresponding to* $w_1(\cdot)$ *and* $w_2(\cdot)$, *respectively. Then,* $X_1^* = X_2^*$.

Since the proof of Proposition 5.6 uses a result provided in the following subsection, we defer it to the end of that subsection.

Proposition 5.6 states that as long as the agent is sufficiently fearful that portfolio insurance becomes necessary, the level of fear, quantified by $\mathcal{I}_w(z)$, $1 - z_0 < z < 1$, affects neither the optimal terminal wealth nor—perhaps more surprisingly—the level of portfolio insurance necessitated! This result can nevertheless be explained as follows: Recall that fear is described as the tendency of individuals to overweight extremely negative outcomes that occur with small probabilities. In other words, fear is triggered only by the possibility of *catastrophic events*. If the agent is sufficiently fearful, then he will choose to avoid catastrophic events by taking portfolio insurance. Once a portfolio insurance strategy is in place, the degree of fear no longer affects the optimal portfolio because loss due to any catastrophic event will be covered. In this case, the optimal portfolio depends only on the utility function, as well as on the agent's degree of hope and level of aspirations.

5.3. Optimal Solutions without the Monotonicity Condition

When $M(\cdot)$ is not nondecreasing, the earlier pointwise maximization argument does not work because $G^*_\lambda(\cdot)$, as defined in (5.4), is no longer an increasing function. In this case, a new technique is necessary to solve the HF/A portfolio choice problem (3.3).

Without the monotonicity condition on $M(\cdot)$, the function $G^*_\lambda(\cdot)$ defined by (5.4) no longer qualifies as a quantile function. Therefore, we have to modify $G^*_\lambda(\cdot)$ in some way so that the resulting function is nondecreasing and, hopefully, a candidate for the optimal solution to (5.1). To demonstrate the technique that we apply, we first consider the case in which there is no aspiration constraint, i.e., $A = 0$. Then the problem (5.1) becomes much simpler because the constraint $G((1-\alpha)+) \geq A$ is removed.[21]

In the following, we denote

$$\bar{G}(z) := (u')^{-1}(\lambda / M(z)), \quad 0 < z < 1. \tag{5.10}$$

By Assumption 5.5, $\bar{G}(\cdot)$ is continuous on $(0, 1)$, strictly decreasing on $(0, z_0)$ and strictly increasing on $(z_0, 1)$, and $\lim_{z\uparrow 1} \bar{G}(z) = +\infty$. Recall $f(x, z)$ defined in (5.2). For any fixed $0 < z < 1$, $f(\cdot, z)$ is strictly increasing on $(0, \bar{G}(z))$ and strictly decreasing on $(\bar{G}(z), +\infty)$.

Define

$$\mathbb{S}_\lambda := \{G(\cdot) \in \mathbb{G} \mid \exists y \in [z_0, 1) \text{ such that } G(z) = \bar{G}(y)\mathbf{1}_{\{0<z\leq y\}} + \bar{G}(z)\mathbf{1}_{\{y<z<1\}},\ 0 < z < 1\}.$$

We prove in the following proposition that one only needs to consider the type of quantile functions in $\mathbb{S}_\lambda$ in order to solve (5.1).

PROPOSITION 5.7. *Let Assumptions 4.3, 5.1, and 5.5 hold and $A = 0$. For any $G(\cdot) \in \mathbb{G}$ that is feasible to (5.1), there exists $\tilde{G}(\cdot) \in \mathbb{S}_\lambda$ such that $U_\lambda(\tilde{G}(\cdot)) \geq U_\lambda(G(\cdot))$ and the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$.*

Proof. For any $G(\cdot) \in \mathbb{G}$, let $z_1 := \inf\{z \in (0, z_0] \mid G(z) > \bar{G}(z)\}$ with the convention that $\inf \emptyset = z_0$. Define $z_2 := \inf\{z \in [z_0, 1)] \mid \bar{G}(z) > \bar{G}(z_1)\}$. Evidently, $z_0 \leq z_2 < 1$ because $\lim_{z\uparrow 1} \bar{G}(z) = +\infty$. Due to the continuity, $\bar{G}(z_2) = \bar{G}(z_1)$. Define $\tilde{G}(z) := \bar{G}(z_2)\mathbf{1}_{\{0<z\leq z_2\}} + \bar{G}(z)\mathbf{1}_{\{z_2<z<1\}}$. Clearly, $\tilde{G}(\cdot) \in \mathbb{S}_\lambda$. We are going to show that $U_\lambda(\tilde{G}(\cdot)) \geq U_\lambda(G(\cdot))$.

For $0 < z < z_1$, $G(z) \leq G(z_1) \leq \bar{G}(z_1) = \tilde{G}(z) \leq \bar{G}(z)$, where the last inequality is due to the fact that $\bar{G}(\cdot)$ is decreasing on $(0, z_0]$. Therefore, $f(G(z); z) \leq f(\tilde{G}(z); z)$. If $z_1 < z_2$, then $z_1 < z_0$. In the case $z_1 < z < z_2$, $G(z) \geq G(z_1+) \geq \bar{G}(z_1) = \bar{G}(z_2) = \tilde{G}(z) \geq \bar{G}(z)$ and thus $f(G(z), z) \leq f(\tilde{G}(z), z)$. For $z_2 < z < 1$, clearly, $f(G(z), z) \leq f(\bar{G}(z), z) =$

[21] The problem (3.4) is called a Choquet maximization problem in Jin and Zhou (2008). However, Jin and Zhou (2008) solved the Choquet maximization problem by assuming that $M(\cdot)$ was nondecreasing. So our approach in this paper can also be employed to solve an extension of the Choquet maximization problem and, consequently, that of the portfolio choice problem under CPT in Jin and Zhou (2008).

$f(\tilde{G}(z), z)$. Therefore, we have

$$U_\lambda(G(\cdot)) = \int_0^1 f(G(z), z)\,dz \le \int_0^1 f(\tilde{G}(z), z)\,dz = U_\lambda(\tilde{G}(\cdot))$$

and it is easy to see that the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$. □

In view of Proposition 5.7, we need only to consider the following problem:

$$(5.11)\qquad \begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U_\lambda(G(\cdot)) = \int_0^1 \left[u(G(z))w'(1-z) - \lambda G(z) F_\rho^{-1}(1-z)\right]dz \\ \text{subject to} & G(\cdot) \in \mathbb{S}_\lambda, \end{array}$$

which is essentially a one-dimensional optimization problem, in which one determines the optimal $y \in [z_0, 1)$. To solve this problem, we introduce the following function:

$$(5.12)\qquad \varphi(y) = \int_0^y w'(1-z)\,dz - M(y)\int_0^y F_\rho^{-1}(1-z)\,dz, \quad 0 < y < 1.$$

Because $M(\cdot)$ is strictly decreasing on $(0, z_0]$, we have, for any $y \in (0, z_0]$,

$$\frac{\int_0^y w'(1-z)\,dz}{\int_0^y F_\rho^{-1}(1-z)\,dz} = \frac{\int_0^y F_\rho^{-1}(1-z)M(z)\,dz}{\int_0^y F_\rho^{-1}(1-z)\,dz} > M(y).$$

Thus, we conclude that $\varphi(y) > 0$ on $(0, z_0]$. On the other hand, on $(z_0, 1)$,

$$\begin{aligned} \varphi'(y) &= w'(1-y) - M(y)F_\rho^{-1}(1-y) - M'(y)\int_0^y F_\rho^{-1}(1-z)\,dz \\ &= -M'(y)\int_0^y F_\rho^{-1}(1-z)\,dz < 0 \end{aligned}$$

and

$$\lim_{y \uparrow 1} \varphi(y) = -\infty.$$

So there exists a unique root $z^* \in (z_0, 1)$ of $\varphi(\cdot)$ such that $\varphi(z) > 0$ on $(0, z^*)$ and $\varphi(z) < 0$ on $(z^*, 1)$.

PROPOSITION 5.8. *Let Assumptions 4.3, 5.1, and 5.5 hold and $A = 0$. Let z^* be the unique root of $\varphi(\cdot)$ defined in (5.12). Then the unique optimal solution to (5.1) is*

$$(5.13)\qquad G_\lambda^*(z) = \bar{G}(z^*)\mathbf{1}_{\{0<z\le z^*\}} + \bar{G}(z)\mathbf{1}_{\{z^*<z<1\}}, \quad 0 < z < 1.$$

Proof. Let

$$\begin{aligned} V(y) &:= U_\lambda(\bar{G}(y)\mathbf{1}_{\{0<z\le y\}} + \bar{G}(z)\mathbf{1}_{\{y<z<1\}}), \\ &= u(\bar{G}(y))\int_0^y w'(1-z)\,dz - \lambda\bar{G}(y)\int_0^y F_\rho^{-1}(1-z)\,dz \\ &\quad + \int_y^1 u(\bar{G}(z))w'(1-z)\,dz - \lambda\int_y^1 \bar{G}(z)F_\rho^{-1}(1-z)\,dz. \end{aligned}$$

By Proposition 5.7, problem (5.1) is equivalent to

$$\max_{z_0 \le y < 1} V(y).$$

We check the first-order derivative of $V(\cdot)$,

$$V'(y) = \frac{\lambda \bar{G}'(y)}{M(y)} \varphi(y),$$

which implies directly that z^* is the unique optimizer. □

Proposition 5.8 shows that the optimal solution to (5.1) is a left truncation of $\bar{G}(\cdot)$. To the left of z^* the optimal quantile function is flat and to the right of z^* it follows $\bar{G}(\cdot)$. Surprisingly, it follows from (5.12) that the critical point z^* is independent of the multiplier λ—in fact, independent even of the utility function! It depends only on the investment opportunity $F_\rho(\cdot)$ and on the weighting function $w(\cdot)$.

THEOREM 5.9. *Let Assumptions 4.3, 5.1, and 5.5 hold and* $A = 0$. *Suppose that* $x_0 > 0$. *Then (3.3) has a unique optimal solution*

$$X^* = (u')^{-1}\left(\frac{\lambda^* \rho}{w'(F_\rho(\rho))}\right)\mathbf{1}_{\{\rho \le c^*\}} + (u')^{-1}\left(\frac{\lambda^* c^*}{w'(F_\rho(c^*))}\right)\mathbf{1}_{\{\rho > c^*\}}, \tag{5.14}$$

where c^* *is the unique root of*

$$\tilde{\varphi}(x) := x(1 - w(F_\rho(x))) - w'(F_\rho(x)) \int_x^\infty s\, dF_\rho(x) \tag{5.15}$$

on $(\operatorname{essinf} \rho, F_\rho^{-1}(1 - z_0))$ *and* $\lambda^* > 0$ *is the value such that* $E[\rho X^*] = x_0$.

Proof. Let

$$\begin{aligned}
\mathcal{X}(\lambda) &:= \int_0^1 F_\rho^{-1}(1 - z) G_\lambda^*(z)\, dz, \\
&= (u')^{-1}\left(\frac{\lambda F_\rho^{-1}(1 - z^*)}{w'(1 - z^*)}\right) \int_0^{z^*} F_\rho^{-1}(1 - z)\, dz \\
&\quad + \int_{z^*}^1 (u')^{-1}\left(\frac{\lambda F_\rho^{-1}(1 - z)}{w'(1 - z)}\right) F_\rho^{-1}(1 - z)\, dz.
\end{aligned}$$

From Assumption 5.1, it follows that $\mathcal{X}(\lambda)$ is finite and nonincreasing on $(0, +\infty)$. By means of the monotone convergence theorem, recalling that ρ is atomless, one can show that $\mathcal{X}(\cdot)$ is continuous and that

$$\lim_{\lambda \downarrow 0} \mathcal{X}(\lambda) = +\infty, \qquad \lim_{\lambda \uparrow +\infty} \mathcal{X}(\lambda) = 0.$$

Therefore, there exists λ^* such that $\mathcal{X}(\lambda^*) = x_0$ and, consequently, $G_{\lambda^*}^*(\cdot)$ is optimal to (3.4). Then, $X^* := G_{\lambda^*}^*(1 - F_\rho(\rho))$ is optimal to (3.3). If we let $c^* := F_\rho^{-1}(1 - z^*)$, then X^* is exactly as given in (5.14). Because z^* is the unique root of $\varphi(\cdot)$ on $(z_0, 1)$, by changing the variable, one can easily check that c^* is the unique root of $\tilde{\varphi}(\cdot)$ on $(\operatorname{essinf}\rho, F_\rho^{-1}(1-z_0))$. $\square$

With the optimal terminal wealth profile X^* given by (5.14), we can divide the states of the world into two classes: the class of "good" states (corresponding to the set $\{\rho \le c^*\}$) and the class of "bad" states (corresponding to the set $\{\rho > c^*\}$). In bad states of the world, the final wealth has a fixed, deterministic value, $(u')^{-1}(\frac{\lambda^* c^*}{w'(F_\rho(c^*))}) > 0$, which is irrelevant to which state the world is in. This value is exactly the insured wealth floor. Meanwhile the agent is hopeful for good states, in which the wealth is $(u')^{-1}(\frac{\lambda^* \rho}{w'(F_\rho(\rho))})$, the value of which depends on the specific state through ρ.[22]

Problem (3.3) without aspiration constraint has also been solved by Carlier and Dana (2011) using a different method. These authors employ the calculus of variation to derive a sufficient and necessary condition satisfied by the optimal quantile function; see Carlier and Dana (2011, proposition 3.2). Here, we find the solution by showing that the optimal quantile function must take some special form. Not surprisingly, one can verify that the solution in Theorem 5.9 satisfies the condition in Carlier and Dana (2011).

In general, in the presence of the aspiration constraint (i.e., $A > 0$), problem (3.3) has not been solved in the literature. By applying the similar technique to the one employed in the case of no aspiration constraint, we can derive the optimal terminal payoff explicitly. We defer the proof to Appendix A and state only the results here.

Introduce the following function:

$$\psi(z) := \frac{\displaystyle\int_0^z w'(1-s)ds}{\displaystyle\int_0^z F_\rho^{-1}(1-s)ds}, \quad 0 < z < 1. \tag{5.16}$$

On the one hand, compared with $\varphi(\cdot)$ in (5.12), we conclude that $\psi(z) > M(z)$ on $(0, z^*)$, $\psi(z) < M(z)$ on $(z^*, 1)$ and $\psi(z^*) = M(z^*)$. On the other hand, from $\psi'(z) = -\frac{F_\rho^{-1}(1-z)}{(\int_0^z F_\rho^{-1}(1-s)ds)^2}\varphi(z)$, it follows that $\psi(\cdot)$ is strictly decreasing on $(0, z^*)$ and strictly increasing on $(z^*, 1)$.

Let

$$x_r := E\left\{\rho\left[(u')^{-1}\left(\frac{u'(A)M(z^*)\rho}{w'(F_\rho(\rho))}\right)\mathbf{1}_{\{\rho\le F_\rho^{-1}(1-z^*)\}} + A\mathbf{1}_{\{\rho> F_\rho^{-1}(1-z^*)\}}\right]\right\}, \tag{5.17}$$

$$x_p := E\left\{\rho\left[\left((u')^{-1}\left(\frac{u'(A)\psi(1-\alpha)\rho}{w'(F_\rho(\rho))}\right)\vee A\right)\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}} + A\mathbf{1}_{\{\rho> F_\rho^{-1}(\alpha)\}}\right]\right\}. \tag{5.18}$$

The following theorem provides a complete solution to the HF/A portfolio choice problem (3.3).

THEOREM 5.10. *Let Assumptions 4.3, 5.1, and 5.5 hold and suppose* $x_0 > AE[\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}]$. *Let* c^* *be the unique root of* $\tilde{\varphi}(\cdot)$ *in (5.15).*

[22] Because $\lim_{z\uparrow 1} M(z) = +\infty$ due to Assumption 5.5 and $u'(+\infty) = 0$ due to Assumption 4.3, we have $\lim_{\rho\to 0}(u')^{-1}(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}) = \lim_{\rho\to 0}(u')^{-1}(\frac{\lambda^*}{M(1-F_\rho(\rho))}) = +\infty$. Thus, the wealth of the agent in good states is unbounded.

1. *Suppose* $F_\rho(c^*) < \alpha < 1$.
   (a) *If* $x_0 \geq x_r$, *then the unique optimal solution to (3.3) is*

$$X^* := (u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right)\mathbf{1}_{\{\rho\leq c^*\}} + (u')^{-1}\left(\frac{\lambda^* c^*}{w'(F_\rho(c^*))}\right)\mathbf{1}_{\{\rho>c^*\}} \tag{5.19}$$

   where $\lambda^* \leq u'(A)M(z^*)$ *is the value such that* $E[\rho X^*] = x_0$.
   (b) *If* $x_p \leq x_0 \leq x_r$, *then the unique optimal solution to (3.3) is*

$$X^* := \left[(u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right) \vee A\right]\mathbf{1}_{\{\rho\leq F_\rho^{-1}(1-z_0)\}} + A\mathbf{1}_{\{\rho>F_\rho^{-1}(1-z_0)\}} \tag{5.20}$$

   where $u'(A)M(z^*) \leq \lambda^* \leq u'(A)\psi(1-\alpha)$ *is the value such that* $E[\rho X^*] = x_0$.
   (c) *If* $x_0 \leq x_p$, *then the unique optimal solution to (3.3) is*

$$\begin{aligned} X^* := {} & \left[(u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right) \vee A\right]\mathbf{1}_{\{\rho\leq F_\rho^{-1}(\alpha)\}} \\ & + (u')^{-1}\left(\frac{\lambda^*}{\psi(1-\alpha)}\right)\mathbf{1}_{\{\rho>F_\rho^{-1}(\alpha)\}}, \end{aligned} \tag{5.21}$$

   where $\lambda^* \geq u'(A)\psi(1-\alpha)$ *is the value such that* $E[\rho X^*] = x_0$.

2. *Suppose* $0 < \alpha \leq F_\rho(c^*)$, *then the unique optimal solution to (3.3) is*

$$\begin{aligned} X^* : = {} & \left[(u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right) \vee A\right]\mathbf{1}_{\{\rho\leq F_\rho^{-1}(\alpha)\}} \\ & + (u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right)\mathbf{1}_{\{F_\rho^{-1}(\alpha)<\rho\leq c^*\}} + (u')^{-1}\left(\frac{\lambda^* c^*}{w'(F_\rho(c^*))}\right)\mathbf{1}_{\{\rho>c^*\}}, \end{aligned} \tag{5.22}$$

   where λ^* *is the multiplier such that* $E[\rho X^*] = x_0$.

As in the $A = 0$ case, an optimal strategy divides the states of the world into two or three classes, depending on the parameter values. In the worst states, which are represented by $\{\rho > c^*\}$, $\{\rho > F_\rho^{-1}(1-z_0)\}$ or $\{\rho > F_\rho^{-1}(\alpha)\}$, the terminal wealth has a positive floor. In the best states, which are represented by $\{\rho \leq c^*\}$, $\{\rho \leq F_\rho^{-1}(1-z_0)\}$, or $\{\rho \leq F_\rho^{-1}(\alpha)\}$, the terminal wealth is unbounded from above.

We close this subsection by providing a proof of Proposition 5.6.

Proof of Proposition 5.6. It follows from Theorem 5.9 that the optimal solution to (3.3) depends, in addition to the pricing kernel ρ, the initial wealth x_0, and the utility function $u(\cdot)$, only on $w(z)$, $0 < z < 1 - z_0$. This completes the proof. □

5.4. Hope, Good States of the World, and the Hope Index

We now introduce an index for the agent's level of hope and examine its impact on the optimal payoff. The introduction of the hope index is motivated by the following result:

PROPOSITION 5.11. *Let Assumption 4.3 hold. Suppose that there are two functions,* $w_1(\cdot)$ *and* $w_2(\cdot)$, *satisfying Assumptions 3.3, 5.1, and 5.5 with the same* z_0. *Let* c_i^* *be the critical* *point c^* determined as the root of (5.15) corresponding to $w_i(\cdot)$, $i = 1, 2$, respectively. If*

$$\frac{w_1'(z)}{1 - w_1(z)} \geq \frac{w_2'(z)}{1 - w_2(z)}, \quad 0 < z < 1 - z_0,$$

then $c_1^* \geq c_2^*$.

Proof. Let $\varphi_i(\cdot)$ be the function defined in (5.12) corresponding to $w_i(\cdot)$ and let z_i^* be its unique root in $(z_0, 1)$, $i = 1, 2$. Then, we have $c_i^* = F_\rho^{-1}(1 - z_i^*)$. Now we deduce

$$\begin{aligned}
0 = \varphi_1(z_1^*) &= w_1'(1 - z_1^*) \left[\frac{1 - w_1(1 - z_1^*)}{w_1'(1 - z_1^*)} - \frac{\int_0^{z_1^*} F_\rho^{-1}(1 - z)\, dz}{F_\rho^{-1}(1 - z_1^*)} \right] \\
&\leq w_1'(1 - z_1^*) \left[\frac{1 - w_2(1 - z_1^*)}{w_2'(1 - z_1^*)} - \frac{\int_0^{z_1^*} F_\rho^{-1}(1 - z)\, dz}{F_\rho^{-1}(1 - z_1^*)} \right], \\
&= \frac{w_1'(1 - z_1^*)}{w_2'(1 - z_1^*)} \varphi_2(z_1^*).
\end{aligned}$$

Thus, we have $\varphi_2(z_1^*) \geq 0$. Because $\varphi_2(\cdot)$ is strictly decreasing in $(z_0, 1)$, we conclude that $z_1^* \leq z_2^*$, and consequently $c_1^* \geq c_2^*$. □

Inspired by this result, we define

$$\mathcal{H}_w(z) := \frac{w'(z)}{1 - w(z)}, \quad 0 < z < 1 \tag{5.23}$$

for any given weighting function $w(\cdot)$. Theorem 5.11 can be restated as the critical point c^* being increasing with respect to $\mathcal{H}_w(z)$ when z is close to 0. According to Theorem 5.9, c^* divides between good and bad states of the world, and a greater c^* means that the agent includes more scenarios under good states, $\{\rho \leq c^*\}$. Thus, we call $\mathcal{H}_w(z)$ the *hope index*, since this index sheds light on how hope affects investing behavior: the higher the value of the index, the more hopeful the agent is about the future world, and hence the higher leverage he needs to take in order to reap the payoffs from more good states. Note also that the hope index, $\mathcal{H}_w(z)$, is, in terms of the curvature of the weighting function when z is close to 0, consistent with the notion that hope is relevant to the probability weighting of extremely good outcomes.

On the other hand, the following result stipulates that a significantly higher hope index leads to a higher payoff in sufficiently good scenarios than that with a lower hope index.

PROPOSITION 5.12. *Let Assumption 4.3 hold and suppose there are two functions, $w_1(\cdot)$ and $w_2(\cdot)$, satisfying Assumptions 3.3, 5.1, and 5.5. Suppose essinf $\rho = 0$ and the utility function is a power one $u(x) = \frac{x^{1-\eta} - 1}{1-\eta}$ where $\eta > 0$. Let the optimal payoffs corresponding to $w_1(\cdot)$ and $w_2(\cdot)$ be X_1^* and X_2^*, respectively. If*

$$\lim_{z \downarrow 0} \frac{\mathcal{H}_{w_1}(z)}{\mathcal{H}_{w_2}(z)} = +\infty,$$

then there exists $c > 0$ such that $X_1^* > X_2^*$ on $\{\rho \leq c\}$.

Proof. From Theorem 5.10, $X_1^* = (u')^{-1}(\frac{\lambda_1^*\rho}{w_1'(F_\rho(\rho))})$ and $X_2^* = (u')^{-1}(\frac{\lambda_2^*\rho}{w_2'(F_\rho(\rho))})$ for some $\lambda_1^*, \lambda_2^* > 0$ when ρ is sufficiently small. Noticing that

$$\lim_{\rho\downarrow 0}\frac{(u')^{-1}\left(\dfrac{\lambda_1^*\rho}{w_1'(F_\rho(\rho))}\right)}{(u')^{-1}\left(\dfrac{\lambda_2^*\rho}{w_2'(F_\rho(\rho))}\right)} = \lim_{\rho\downarrow 0}(u')^{-1}\left(\frac{w_2'(F_\rho(\rho))}{w_1'(F_\rho(\rho))}\frac{\lambda_1^*}{\lambda_2^*}\right) = (u')^{-1}\left(\lim_{\rho\downarrow 0}\frac{w_2'(F_\rho(\rho))}{w_1'(F_\rho(\rho))}\frac{\lambda_1^*}{\lambda_2^*}\right),$$

$$= (u')^{-1}\left(\lim_{\rho\downarrow 0}\frac{\mathcal{H}_{w_2}(F_\rho(\rho))}{\mathcal{H}_{w_1}(F_\rho(\rho))}\frac{\lambda_1^*}{\lambda_2^*}\right) = +\infty,$$

we deduce that there exists $c > 0$ such that $X_1^* > X_2^*$ on $\{\rho \le c\}$. ☐

Let us compute the hope indices for the weighting functions in (2.5)–(2.7). For the Tversky–Kahneman weighting (2.5), we have $\mathcal{H}_w(z) \approx \gamma z^{\gamma-1}$ as $z \downarrow 0$. For the Tversky–Fox weighting function (2.6), we have $\mathcal{H}_w(z) \approx \delta\gamma z^{\gamma-1}$ as $z \downarrow 0$. For the Prelec weighting function (2.7), we have $\mathcal{H}_w(z) \approx \delta\gamma\frac{1}{p}(-\ln p)^{\gamma-1}e^{-\delta(-\ln p)^\gamma}$ as $z \downarrow 0$. For each of these three weighting functions, if we have two weighting functions $w_1(\cdot)$ and $w_2(\cdot)$ with $\gamma_1 < \gamma_2$, then $\lim_{z\downarrow 0}\frac{\mathcal{H}_{w_1}(z)}{\mathcal{H}_{w_2}(z)} = +\infty$. In other words, $1-\gamma$ can measure the hope for good situations. From Proposition 5.12, the higher the value of $1-\gamma$ is, the higher the optimal payoff is for good scenarios.

For the Jin–Zhou weighting function (2.8), the hope index is $\mathcal{H}_w(z) \approx ke^{(a+b)\Phi^{-1}(\bar{z})-a\Phi^{-1}(z)}$ as $z \downarrow 0$. Furthermore, for any $a_1 > a_2$ and their associated weighting functions, $w_1(\cdot)$ and $w_2(\cdot)$, we have $\lim_{z\downarrow 0}\frac{\mathcal{H}_{w_1}(z)}{\mathcal{H}_{w_2}(z)} = +\infty$. Therefore, a measures the degree of hope. From Proposition 5.12, the higher the value of a is, the higher the optimal payoff is for good scenarios.

5.5. Aspirations, Gambles, and the Lottery-Likeness Index

In this subsection, we investigate the role that aspirations play in affecting optimal investing behavior, especially when the level of aspirations is exceedingly high. To simplify the discussion, we assume that the utility function is a power function, i.e.,

$$u(x) = \frac{x^{1-\eta}-1}{1-\eta}, \tag{5.24}$$

where $\eta > 0$. Nonetheless, all of the qualitative results in this subsection remain true for a general utility function.

Because the utility function is a power function, it is easy to see that the optimal terminal payoff in the HF/A portfolio choice problem (3.3) is proportional to the initial wealth x_0 if the aspiration level is set to be a fixed proportion of the initial wealth. Therefore, in the remainder of this subsection, we assume $x_0 = 1$ without loss of generality, and we consider A to be the aspiration level *relative* to the initial wealth.

Define

$$k_r := \frac{1}{E\left[\left(\dfrac{\rho}{w'(F_\rho(\rho))}\dfrac{w'(F_\rho(c^*))}{c^*}\right)^{-\frac{1}{\eta}}\rho\mathbf{1}_{\{\rho\le c^*\}} + \rho\mathbf{1}_{\{\rho>c^*\}}\right]}, \tag{5.25}$$

$$k_p := \frac{1}{E\left[\left(\left(\frac{\rho}{w'(F_\rho(\rho))}\psi(1-\alpha)\right)^{-\frac{1}{\eta}} \vee 1\right)\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}} + \rho\mathbf{1}_{\{\rho> F_\rho^{-1}(\alpha)\}}\right]}, \tag{5.26}$$

$$k_u := \frac{1}{E\left[\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}}\right]}. \tag{5.27}$$

To study the impact of aspirations on trading behavior, we first reproduce Theorem 5.10 when $u(\cdot)$ is a power function.

COROLLARY 5.13. *Suppose $u(\cdot)$ is given in (5.24), $x_0 = 1$ and let Assumptions 5.1 and 5.5 hold. Let c^* be the unique root of $\tilde{\varphi}(\cdot)$ in (5.15). If $A > k_u$, the problem (3.3) is infeasible. Otherwise, we have the following assertions:*

1. *Suppose $F_\rho(c^*) < \alpha < 1$.*
   (a) *When $0 \le A \le k_r$, the unique optimal solution to (3.3) is*

$$X^* = k_r\left(\frac{w'(F_\rho(c^*))}{c^*}\right)^{-\frac{1}{\eta}}\left[\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}\mathbf{1}_{\{\rho\le c^*\}} + \left(\frac{c^*}{w'(F_\rho(c^*))}\right)^{-\frac{1}{\eta}}\mathbf{1}_{\{\rho> c^*\}}\right].$$

   (b) *When $k_r \le A \le k_p$, the unique optimal solution to (3.3) is*

$$X^* = A\left[\left(\left(\lambda_1(A)\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}\right)\vee 1\right)\mathbf{1}_{\{\rho\le F_\rho^{-1}(1-z_0)\}} + \mathbf{1}_{\{\rho> F_\rho^{-1}(1-z_0)\}}\right]$$

   where $\lambda_1(A)$ is the unique number in the interval $[\psi(1-\alpha)^{-\frac{1}{\eta}}, (\frac{w'(F_\rho(c^*))}{c^*})^{-\frac{1}{\eta}}]$ such that

$$E\left[\left(\left(\lambda\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}\right)\vee 1\right)\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(1-z_0)\}} + \rho\mathbf{1}_{\{\rho> F_\rho^{-1}(1-z_0)\}}\right] = \frac{1}{A}.$$

   (c) *When $k_p \le A \le k_u$, the unique optimal solution to (3.3) is*

$$X^* = A\left[\left(\left(\lambda_2(A)\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}\right)\vee 1\right)\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}} + \lambda_2(A)\left(\frac{1}{\psi(1-\alpha)}\right)^{-\frac{1}{\eta}}\mathbf{1}_{\{\rho> F_\rho^{-1}(\alpha)\}}\right]$$

   where $\lambda_2(A)$ is the unique number in the interval $[0, \psi(1-\alpha)^{-\frac{1}{\eta}}]$ such that

$$E\left[\left(\left(\lambda\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}\right)\vee 1\right)\rho\mathbf{1}_{\{\rho\le F_\rho^{-1}(\alpha)\}} + \lambda\left(\frac{1}{\psi(1-\alpha)}\right)^{-\frac{1}{\eta}}\rho\mathbf{1}_{\{\rho> F_\rho^{-1}(\alpha)\}}\right] = \frac{1}{A}.$$

2. *Suppose* $0 < \alpha \leq F_\rho(c^*)$.

   (a) *When* $0 \leq A \leq (\frac{w'(F_\rho(c^*))F_\rho^{-1}(\alpha)}{c^* w'(\alpha)})^{-\frac{1}{\eta}} k_r$, *the unique optimal solution to (3.3) is*

$$X^* = k_r \left(\frac{w'(F_\rho(c^*))}{c^*}\right)^{-\frac{1}{\eta}} \left[\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}} \mathbf{1}_{\{\rho \leq c^*\}} + \left(\frac{c^*}{w'(F_\rho(c^*))}\right)^{-\frac{1}{\eta}} \mathbf{1}_{\{\rho > c^*\}}\right].$$

   (b) *When* $(\frac{w'(F_\rho(c^*))F_\rho^{-1}(\alpha)}{c^* w'(\alpha)})^{-\frac{1}{\eta}} k_r \leq A \leq k_u$, *the unique optimal solution to (3.3) is*

$$\begin{aligned} X^* := A\Bigg[& \left(\left(\lambda_3(A)\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}\right) \vee 1\right) \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}} \\ & + \lambda_3(A)\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}} \mathbf{1}_{\{F_\rho^{-1}(\alpha) < \rho \leq c^*\}} \\ & + \lambda_3(A)\left(\frac{c^*}{w'(F_\rho(c^*))}\right)^{-\frac{1}{\eta}} \mathbf{1}_{\{\rho > c^*\}} \Bigg] \end{aligned}$$

   where $\lambda_3(A)$ *is the unique number in the interval* $[0, (\frac{w'(\alpha)}{F_\rho^{-1}(\alpha)})^{-\frac{1}{\eta}}]$ *such that*

$$\begin{aligned} E\Bigg\{ & \left[\lambda\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}} \vee A\right] \rho \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}} \\ & + \lambda\left(\frac{\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}} \rho \mathbf{1}_{\{F_\rho^{-1}(\alpha) < \rho \leq c^*\}} + \lambda\left(\frac{c^*}{w'(F_\rho(c^*))}\right)^{-\frac{1}{\eta}} \rho \mathbf{1}_{\{\rho > c^*\}} \Bigg\} = \frac{1}{A}. \end{aligned}$$

Proof. Corollary 5.13 is the direct consequence of Theorem 5.10, taking into consideration the special form of $u(\cdot)$. (Note here we break down Case 2 of Theorem 5.10 into two subcases.) □

From Corollary 5.13-1(a), we can see that k_r, as defined in (5.25), is the portfolio insurance level when there are no aspirations. It is clear that the portfolio insurance level increases with respect to the relative risk aversion η.

Aspirations are responsive to immediate, specific needs or opportunities of the agent's decision nexus. Intuitively, a higher level of aspirations will drive the agent to be more aggressive and to take on more risk. An extremely high level of aspirations will *force* the agent to exhibit a lottery-buying type of investment behavior, namely, gambling on a big gain with a risk of losing everything. Since $X^* = G^*(1 - F_\rho(\rho)) \geq G^*(1 - \alpha) \geq A$, i.e., the aspirations are achieved when $\rho \leq F_\rho^{-1}(\alpha)$, we can regard $\{\rho \leq F_\rho^{-1}(\alpha)\}$ as "winning states." Similarly, $\{\rho > F_\rho^{-1}(\alpha)\}$ consists of "losing states" where the aspirations are not met. Motivated by this, we define the following *lottery-likeness index*:

$$\mathcal{L}(A) = \frac{\operatorname{essinf}\big(X^* \mid \rho \le F_\rho^{-1}(\alpha)\big)}{\operatorname{esssup}\big(X^* \mid \rho > F_\rho^{-1}(\alpha)\big)}, \tag{5.28}$$

which gives the ratio between the worst winning payoff and the best losing payoff. For a payoff like that of a lottery, the value of this index is expected to be extremely large.[23]

THEOREM 5.14. *Suppose $u(\cdot)$ is given in (5.24), $x_0 = 1$ and let Assumptions 5.1 and 5.5 hold. Let c^* be the unique root of $\tilde{\varphi}(\cdot)$ in (5.15). Suppose $A \le k_u$, and let $\lambda_2(A)$ and $\lambda_3(A)$ be defined in Corollary 5.13. When $F_\rho(c^*) < \alpha < 1$, we have*

$$\mathcal{L}(A) = \begin{cases} 1, & 0 \le A \le k_p, \\ \dfrac{\psi(1-\alpha)^{-\frac{1}{\eta}}}{\lambda_2(A)}, & k_p < A < k_u. \end{cases}$$

When $0 < \alpha \le F_\rho(c^*)$, we have

$$\mathcal{L}(A) = \begin{cases} 1, & 0 \le A \le \left(\dfrac{F_\rho^{-1}(\alpha) w'(F_\rho(c^*))}{w'(\alpha) c^*}\right)^{-\frac{1}{\eta}} k_r, \\ \dfrac{1}{\lambda_3(A)} \left(\dfrac{w'(\alpha)}{F_\rho^{-1}(\alpha)}\right)^{-\frac{1}{\eta}}, & \left(\dfrac{F_\rho^{-1}(\alpha) w'(F_\rho(c^*))}{w'(\alpha) c^*}\right)^{-\frac{1}{\eta}} k_r \le A < k_u. \end{cases}$$

Furthermore, $\mathcal{L}(A)$ is increasing in A, and

$$\lim_{A \uparrow k_u} \mathcal{L}(A) = +\infty.$$

Proof. The expression of $\mathcal{L}(A)$ can be calculated directly from Corollary 5.13. Furthermore, it is easy to see that $\lambda_2(A)$ and $\lambda_3(A)$ are decreasing in A and go to 0 as A goes to k_u. Consequently, $\mathcal{L}(A)$ is increasing in A and $\lim_{A \uparrow k_u} \mathcal{L}(A) = +\infty$. □

The fact that $\mathcal{L}(A) > 1$ for sufficiently large A indicates that there is a discontinuity of the terminal wealth $X^* \equiv X^*(\rho)$ at $\rho = F_\rho^{-1}(\alpha)$. This discontinuity suggests that the least payoff in "winning states" is greater than the best payoff in "losing states," which is a characteristic feature of payoffs of lottery tickets. Theorem 5.14 confirms that a higher level of aspirations will make the trading behavior more like that of buying into a lottery, which in turn justifies the introduction of the lottery-likeness index to quantify the impact of aspirations on trading behavior.

Recall that in our HF/A model, there are two factors that may drive the agent to be risk-taking: a high level of aspirations and a high level of hope, the latter being reflected by a large value of the hope index or by a steep curvature of $w(z)$ when z is close to 0. It is a natural question whether a high level of hope will also induce lottery-buying behavior. Assume for simplicity of the discussion that $A = 0$. According to Theorem 5.9, the winning states are given by $\{\rho \le c^*\}$, while the losing states are given by $\{\rho > c^*\}$. Then, an analogous definition of the index (5.28) is

$$\frac{\operatorname{essinf}(X^* \mid \rho \le c^*)}{\operatorname{esssup}(X^* \mid \rho > c^*)} \equiv 1.$$

[23] For example, if a jackpot in a mark six lottery is regarded as the winning state and all the other prizes (including no prize) as the losing states, then the corresponding index value (5.28) is extremely large.

In other words, even an extremely high level of hope will not lead to lottery-buying behavior.

6. NUMERICAL EXPERIMENTS

In this section, we report the results of our numerical experiments with the aim of demonstrating the analytical findings in the previous section. In our experiments, we assume a constant interest rate r, a constant stock expected return rate μ, and a constant stock volatility rate σ. As a result, the market price of risk is $\theta := \sigma^{-1}(\mu - r)$. Given a terminal time T, the pricing kernel ρ is lognormally distributed, i.e., $\ln \rho$ is a normal random variable with the mean and standard deviation

$$\mu_\rho = -\left(r + \frac{||\theta||^2}{2}\right) T, \quad \sigma_\rho = ||\theta||\sqrt{T}.$$

We use a power utility in (5.24) and the Jin–Zhou weighting function (2.8). We use the Jin–Zhou weighting instead of the ones in (2.5)–(2.7) for the following two reasons: First, as observed in Sections 5.2 and 5.4, for the Tversky–Kahneman weighting function, the fear index is independent of the parameter γ, whereas the hope index depends on γ. For the Tversky–Fox and Prelec weighting functions, both the degrees of hope and fear are measured by the same parameter γ. The Jin–Zhou weighting function is therefore the only one that has *separate* parameters, a and b, to measure the degrees of hope and fear, respectively. This allows us to study the impact of hope and fear on asset allocation separately by varying one parameter and fixing the other. Second, by using the Jin–Zhou weighting function, we can compute the optimal portfolios explicitly. As we will see shortly, such explicit solutions facilitate a comparison between the optimal allocation to risky assets in our model and that in the classical expected utility model.

We use the data set from Mehra and Prescott (1985) to provide estimates for r, μ, and σ. Calibrating to the real returns of Treasury Bills for the period 1889–1978, we set $r = 1\%$. We assume only one risky asset, which can be considered to be the market portfolio, and we use the S&P 500 index as a proxy for the market portfolio. Using the real returns of the S&P 500 index for the period 1889–1978, we set $\mu = 7\%$ and $\sigma = 15.34\%$. The terminal time T, which measures how frequently investors evaluate their portfolios, is set at one year in light of the argument presented in Benartzi and Thaler (1995). As a result, we have $\theta = 39.11\%$, $\mu_\rho = -8.65\%$, and $\sigma_\rho = 39.11\%$.

Lucas (1994) claims that the reasonable range of the relative risk aversion η should be between 1 and 2.5. Thus, we set $\eta = 1.5$ by taking a value in the middle of the range. Because the Jin–Zhou weighting function has not been calibrated to real data in the literature, we choose parameter values such that the resulting weighting function is graphically close to the weighting functions (2.5)–(2.7), which are based on estimates available in the literature. We set $\bar{z} = \frac{1}{3}$ because it has been reported in the literature that the inflection point of the estimated weighting functions is near 1/3; see, e.g., Abdellaoui (2000). On the other hand, we set $a = 3\sigma_\rho$ and $b = 2.2\sigma_\rho$. The resulting weighting function is graphed in Figure 2.1 in Section 2 in comparison with the three classical weighting functions. These values of a and b are fixed as a benchmark. Recall that a measures the degree of hope and b measures the degree of fear. When studying the impact of hope and fear on asset allocation, we will vary the values of a and b, respectively.

Finally, we set the initial wealth $x_0 = 1$ without loss of generality.

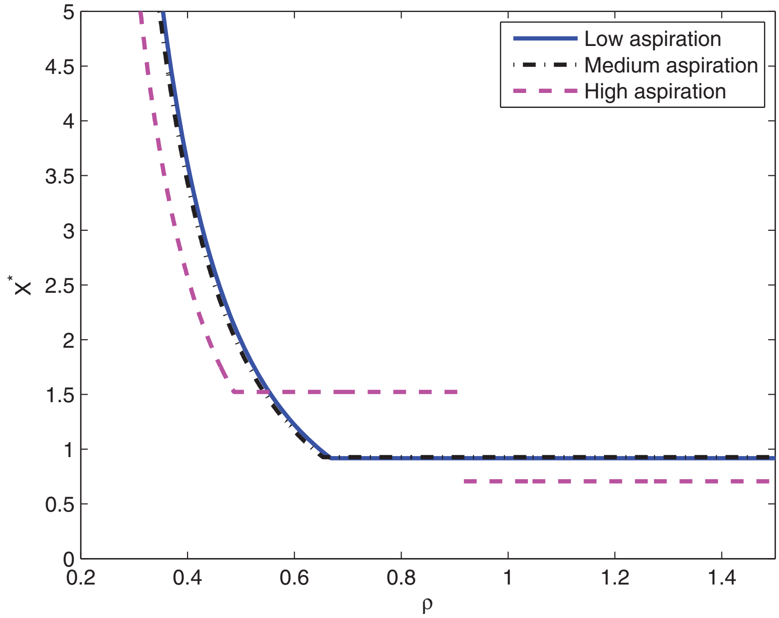

FIGURE 6.1. Graph of optimal terminal payoff X^* as a function of ρ with different aspiration levels A and a given confidence level $\alpha > F_\rho(c^*)$.

Figures 6.1 and 6.2 below show the optimal terminal payoffs as functions of the pricing kernel with different confidence and aspiration levels. In Figure 6.1, α is set at 0.5 and the aspiration level A is given as 0.8, 0.91, and 1.5. In Figure 6.2, α is 0.1 and the aspiration level A is given as 1 and 3.4, respectively. We see that in both figures the functions with high aspiration levels are discontinuous at points that divide winning and losing states, suggesting that the agent is forced to construct a lottery-type payoff. This observation is further confirmed in Figure 6.3, where we vary the aspiration level A from 0 to k_u and plot the lottery-likeness index $\mathcal{L}$ (again, we consider two confidence levels α, which in this case are set at 0.5 and 0.1, respectively). As predicted by Theorem 5.14, this index increases when the aspiration level goes up, and it increases sharply when the aspiration level becomes sufficiently high.

Next, we study the impact of hope and fear by varying the values of a and b while fixing the aspiration level at $A = 0$. Figure 6.4 provides the optimal terminal payoffs with a at 0, $2\sigma_\rho$, $3\sigma_\rho$, and $4\sigma_\rho$, respectively, while b is set at $2.2\sigma_\rho$. Figure 6.5 depicts the optimal payoffs with b at $1.5\sigma_\rho$, $2.2\sigma_\rho$, $3\sigma_\rho$, and $4\sigma_\rho$ while $a = 3\sigma_\rho$. From these two cross-sections, we can see that a higher a or a lower b leads to a lower portfolio insurance level, as well as to a higher payoff in good scenarios.[24] To further investigate the impact of hope and fear on portfolio insurance, we take different values of a and b and compute the corresponding portfolio insurance levels. The results are shown in Figure 6.6. We can see that the portfolio insurance level increases when b increases or when a decreases.

[24]Proposition 5.6 shows that the optimal payoff does not depend on the degree of fear if the investor is already sufficiently fearful. This result only holds true when the probability weighting function does not change at the high end, i.e., when the investor's degree of hope is fixed. Here, when the value of b changes, the whole shape of the probability weighting function changes, leading to a change in the optimal payoff.

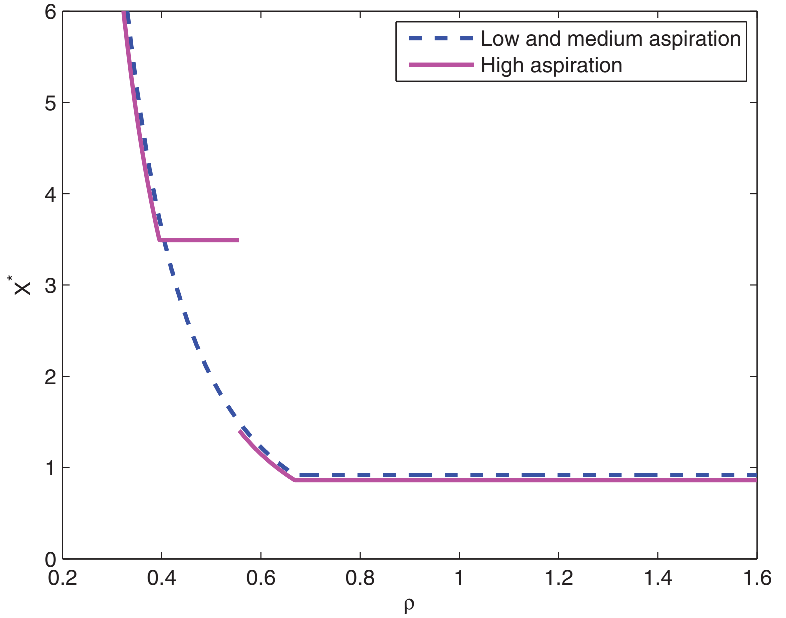

FIGURE 6.2. Graph of optimal terminal payoff X^* as a function of ρ with different aspiration levels A and a given confidence level $\alpha < F_\rho(c^*)$.

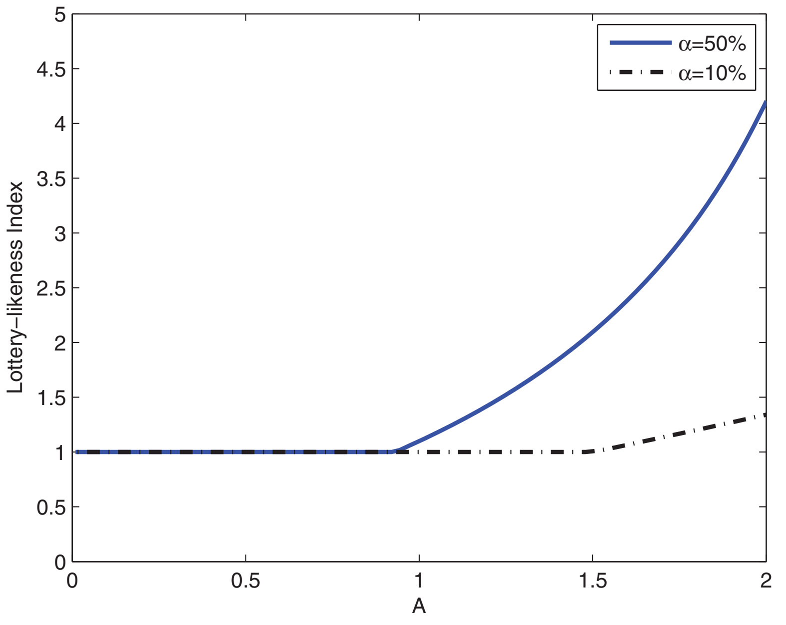

FIGURE 6.3. Increase in lottery-likeness index $\mathcal{L}$ with respect to aspiration level A.

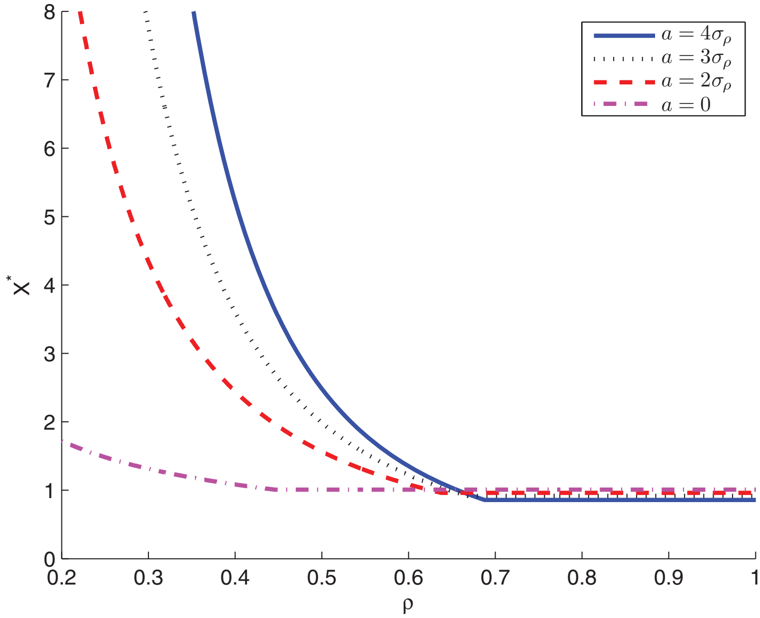

FIGURE 6.4. Optimal terminal payoff X^* for different values of a without aspiration.

Interestingly, the portfolio insurance level is always higher than 75% of the initial wealth, and, with a high value of b, it is over 95% of the initial wealth.

Finally, we set out to find the optimal dynamic portfolio and compare our model with the classic expected utility maximization model. The particular structure of the Jin–Zhou weighting function allows us to calculate the optimal portfolio explicitly. Recall that r is the constant risk-free rate and θ is the constant market price of risk. Let

$$\rho(t) := e^{-\left(r+\frac{1}{2}||\theta||^2\right)t-\theta^\top W(t)}, \quad 0 \le t \le T.$$

Recall that $\Phi(\cdot)$ and $\Phi'(\cdot)$ are the CDF and PDF of the standard normal, respectively. The following theorem provides the optimal portfolio and optimal wealth process under this market setting.[25]

THEOREM 6.1. *Suppose $A = 0$, $u(\cdot)$ is given by (5.24), $x_0 = 1$ and $w(\cdot)$ is given in (2.8) with $a \ge 0$, $b \ge \sigma_\rho$. Let c^* be the unique root of $\tilde{\varphi}(\cdot)$ in (5.15). Then the optimal wealth process and the corresponding portfolio process are given as*

$$X(t) = \Gamma(t, \rho(t)), \quad \pi(t) = \Delta(t, \rho(t))\frac{1}{\eta}(\sigma^\top)^{-1}\theta X(t), \quad 0 \le t < T, \tag{6.1}$$

[25] In this theorem, we assume $A = 0$ for ease of exposition in order to be able to compare our results with those of the utility maximization model. We can also compute the optimal portfolio explicitly with a general aspiration level A.

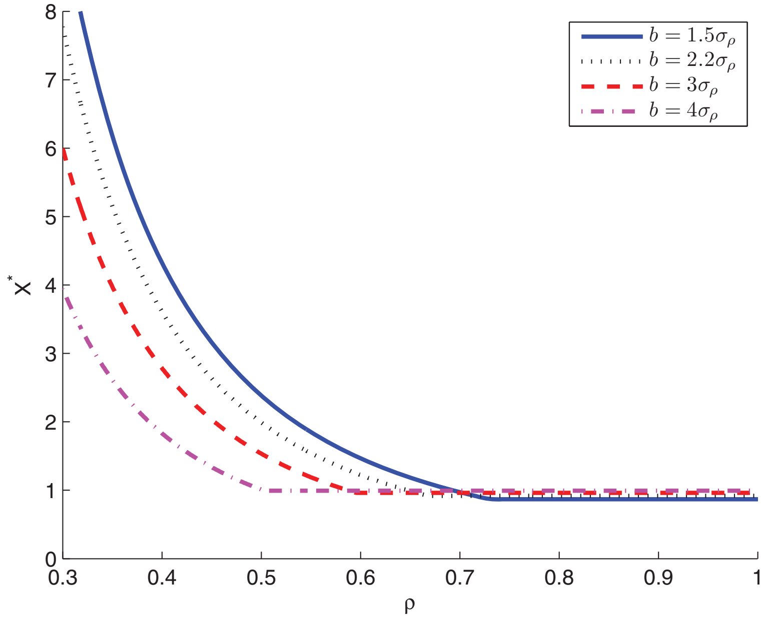

FIGURE 6.5. Optimal terminal payoff X^* for different values of b without aspiration.

where

$$\Gamma(t,\rho) := k_r \Bigg[(c^*)^{\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)} e^{((\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1)r+\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)(\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1)\frac{||\theta||^2}{2})(T-t)} \rho^{-\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)}$$
$$\times \Phi\left(\frac{\ln c^* + \left(r + \left(\frac{1}{\eta}\left(\frac{a}{\sigma_\rho}+1\right) - \frac{1}{2}\right)||\theta||^2\right)(T-t) - \ln\rho}{||\theta||\sqrt{T-t}}\right)$$
$$+ e^{-r(T-t)}\left(1 - \Phi\left(\frac{\ln c^* + \left(r - \frac{1}{2}||\theta||^2\right)(T-t) - \ln\rho}{||\theta||\sqrt{T-t}}\right)\right)\Bigg],$$

and

$$\Delta(t,\rho) := \left(\frac{a}{\sigma_\rho}+1\right)\left[1 - \frac{k_r e^{-r(T-t)}\left(1 - \Phi\left(\frac{\ln c^* + \left(r - \frac{1}{2}||\theta||^2\right)(T-t) - \ln\rho}{||\theta||\sqrt{T-t}}\right)\right)}{\Gamma(t,\rho)}\right].$$

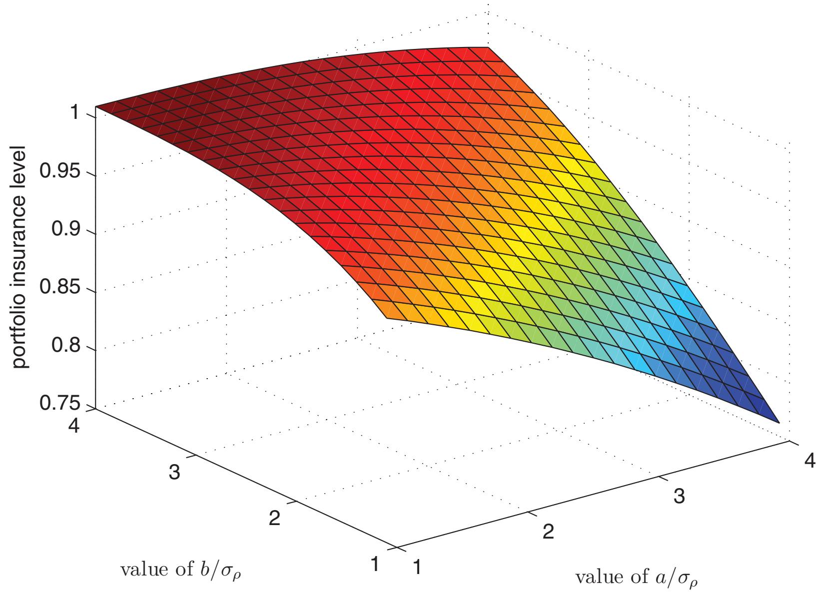

FIGURE 6.6. Portfolio insurance level for different values of a and b and zero aspiration level.

Furthermore, for each fixed $t < T$, $\Gamma(t, \rho)$ and $\Delta(t, \rho)$ are strictly decreasing in ρ and

(6.2)

$$\lim_{\rho\downarrow 0}\Gamma(t,\rho)=+\infty,\quad \lim_{\rho\uparrow\infty}\Gamma(t,\rho)=e^{-r(T-t)}k_r,\quad \lim_{\rho\downarrow 0}\Delta(t,\rho)=\frac{a}{\sigma_\rho}+1,\quad \lim_{\rho\uparrow\infty}\Delta(t,\rho)=0.$$

Proof. It is easy to compute that

$$\frac{w'(F_\rho(x))}{x}=ke^{(a+b)\Phi^{-1}(1-z_0)+a\frac{\mu_\rho}{\sigma_\rho}}x^{-(\frac{a}{\sigma_\rho}+1)},\qquad x\le F_\rho^{-1}(\bar{z}).$$

Thus, from Corollary 5.13-1(a), the optimal terminal payoff is given as

$$X^*=k_r(c^*)^{\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)}\rho^{-\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)}\mathbf{1}_{\{\rho\le c^*\}}+k_r\mathbf{1}_{\{\rho>c^*\}}.$$

By theorem E.1 in Jin and Zhou (2008), the portfolio replicating $\rho^{-\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)}\mathbf{1}_{\{\rho\le c^*\}}$ and its wealth process are

$$\pi_1(t)=\left[\frac{1}{\eta}\left(\frac{a}{\sigma_\rho}+1\right)X_1(t)+\frac{1}{||\theta||\sqrt{T-t}\rho(t)}(c^*)^{1-\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)}\right.$$
$$\left.\times\Phi'\left(\frac{\ln c^*+\left(r+\frac{||\theta||^2}{2}\right)(T-t)-\ln\rho(t)}{||\theta||\sqrt{T-t}}\right)\right](\sigma^\top)^{-1}\theta,$$

$$
\begin{aligned}
X_1(t) &= \rho(t)^{-\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)} e^{[(\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1)r+\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)(\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1)\frac{||\theta||^2}{2}](T-t)} \\
&\quad \times \Phi\left(\frac{\ln c^* + \left(r + \left(\frac{1}{\eta}\left(\frac{a}{\sigma_\rho}+1\right) - \frac{1}{2}\right)||\theta||^2\right)(T-t) - \ln \rho(t)}{||\theta||\sqrt{T-t}}\right).
\end{aligned}
$$

Similarly, the portfolio replicating $\mathbf{1}_{\{\rho > c^*\}}$ is

$$
\pi_2(t) = \left[-\frac{1}{||\theta||\sqrt{T-t}\rho(t)} c^* \Phi'\left(\frac{\ln c^* + \left(r + \frac{||\theta||^2}{2}\right)(T-t) - \ln \rho(t)}{||\theta||\sqrt{T-t}}\right)\right](\sigma^\top)^{-1}\theta,
$$

$$
X_2(t) = e^{-r(T-t)}\left[1 - \Phi\left(\frac{\ln c^* + \left(r - \frac{1}{2}||\theta||^2\right)(T-t) - \ln \rho(t)}{||\theta||\sqrt{T-t}}\right)\right].
$$

As a result, we derive (6.1).

On the other hand, we have

$$
\begin{aligned}
\frac{\partial \Gamma}{\partial \rho}(t, \rho) &= k_r (c^*)^{\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)} e^{((\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1)r+\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)(\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1)\frac{||\theta||^2}{2})(T-t)} \left(-\frac{1}{\eta}\left(\frac{a}{\sigma_\rho}+1\right)\right) \\
&\quad \times \rho^{-\frac{1}{\eta}(\frac{a}{\sigma_\rho}+1)-1} \Phi\left(\frac{\ln c^* + \left(r + \left(\frac{1}{\eta}\left(\frac{a}{\sigma_\rho}+1\right) - \frac{1}{2}\right)||\theta||^2\right)(T-t) - \ln \rho}{||\theta||\sqrt{T-t}}\right) < 0.
\end{aligned}
$$

Thus, $\Gamma(t, \rho)$ is strictly decreasing in ρ. Because

$$
k_r e^{-r(T-t)}\left(1 - \Phi\left(\frac{\ln c^* + \left(r - \frac{1}{2}||\theta||^2\right)(T-t) - \ln \rho}{||\theta||\sqrt{T-t}}\right)\right)
$$

is strictly increasing in ρ, we can deduct that $\Delta(t, \rho)$ is strictly decreasing in ρ. Finally, all of the limits in (6.2) can be derived easily. $\square$

From the monotonicity of $\Gamma(t, \rho)$ in ρ and (6.2), it follows that the optimal wealth process always lies above the portfolio insurance level, $e^{-r(T-t)}k_r$. On the other hand, $\Delta(t, \rho)$ is an interesting quantity. Recall that in classical expected utility maximization the optimal portfolio is given in the feedback form $\frac{1}{\eta}(\sigma^\top)^{-1}\theta X(t)$, i.e., the well-known Merton strategy. Thus, $\Delta(t, \rho)$ measures the deviation from the Merton strategy due to the presence of the probability weighting function. In view of (6.2), when ρ is very small (i.e., when the market is very good), $\Delta(t, \rho) \approx \frac{a}{\sigma_\rho} + 1 > 1$, indicating that the agent takes higher leverage than the Merton strategy does. Moreover, the magnitude of $\Delta(t, \rho)$ is (roughly) proportional to a, a parameter that measures the degree of hope. When ρ is very large (i.e., the market is very bad), $\Delta(t, \rho) \approx 0$, indicating that the agent takes little

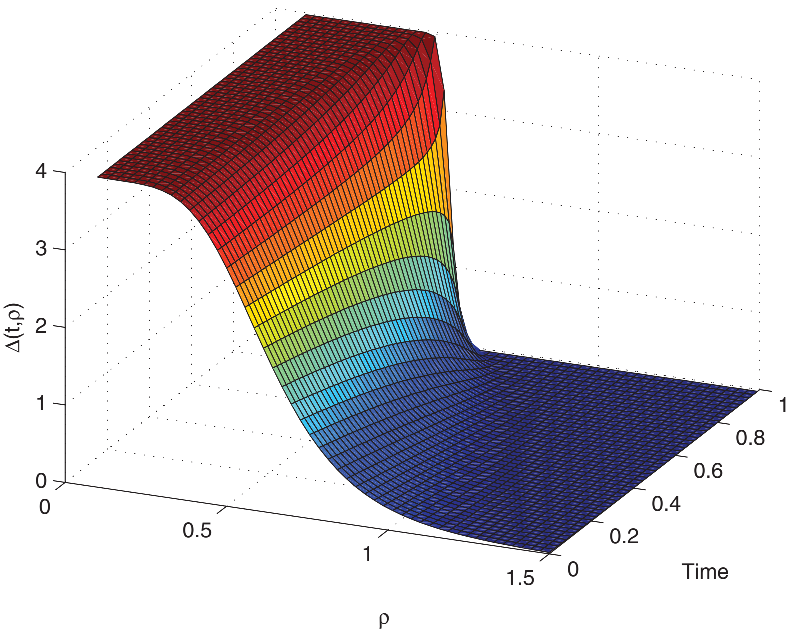

FIGURE 6.7. The quantity $\Delta(t, \rho)$ at different time t and pricing kernel value ρ.

risky exposure due to fear and indicating the resulting portfolio insurance requirement. Figure 6.7 depicts $\Delta(t, \rho)$ with the parameters taking the values specified at the beginning of the present section.

7. CONCLUDING REMARKS

In this paper, we have introduced and formulated a new portfolio choice model—the HF/A model—in continuous time. This model considers the role in decision making of three emotions: hope, fear, and aspirations. Hope and fear are modeled through an inverse-S shaped probability weighting function from Quiggin's RDUT and aspirations are modeled by a probabilistic constraint from Lopes' SP/A theory. Motivated by a comparative statics analysis on the optimal strategies in our model, these emotions have been further quantified via respective indices. In particular, the hope and fear indices are related to the curvatures of the probability weighting function, whereas the level of aspirations can be measured by the degree to which the optimal payoff resembles that of a lottery ticket.

The SP/A theory motivated us to study the portfolio choice problem featuring hope, fear, and aspirations. However, the ill-posedness result (Theorem 4.2) implies that the preference measure in SP/A theory is inappropriate in continuous-time portfolio choice modeling because of the linearity of the utility function. We have resolved this problem by introducing a concave utility function, a component taken from RDUT. We have found, in our HF/A model, that both fear and the dislike of mean-preserving spreads lead to risk-averse behavior and that both hope and aspirations lead to risk-seeking behavior. However, we have shown that each of these four component plays its own distinct role in influencing investing behavior.

APPENDIX A: PROOF OF THEOREM 5.10

The key step in the proof of Theorem 5.10 is to solve (5.1) for each $\lambda > 0$. First, we characterize the subset of feasible quantile functions in which the optimal solution to (5.1) must lie. Following the notation in Section 5.3, define

$$\mathbb{S}_\lambda^A := \{G(\cdot) \in \mathbb{G} \mid G(z) = b\mathbf{1}_{\{0<z\le z_0\}} + [b \vee \bar{G}_A(z)]\mathbf{1}_{\{z_0<z<1\}}, b \ge \bar{G}(z_0)\}, \tag{A.1}$$

if $1-\alpha \ge z_0$; and

$$\begin{aligned}\mathbb{S}_\lambda^A := \{G(\cdot) \in \mathbb{G} \mid G(z) = b\mathbf{1}_{\{0<z\le 1-\alpha\}} + [b \vee A]\mathbf{1}_{\{1-\alpha<z\le z_0\}} \\ + [b \vee \bar{G}_A(z)]\mathbf{1}_{\{z_0<z<1\}}, b \ge \bar{G}(z_0)\},\end{aligned} \tag{A.2}$$

if $1-\alpha \le z_0$, where

$$\bar{G}_A(z) := \bar{G}(z)\mathbf{1}_{\{0<z\le 1-\alpha\}} + [\bar{G}(z) \vee A]\mathbf{1}_{\{1-\alpha<z<1\}}, \quad 0 < z < 1$$

and $\bar{G}(\cdot)$ is defined as in (5.10).

PROPOSITION A.1. *Let Assumptions 4.3 and 5.5 hold. For any $G(\cdot) \in \mathbb{G}$, there exists a $\tilde{G}(\cdot) \in \mathbb{S}_\lambda^A$ such that $U_\lambda(\tilde{G}(\cdot)) \ge U_\lambda(G(\cdot))$ and the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$.*

Proof. First, consider the case in which $1-\alpha \ge z_0$. For any feasible $G(\cdot)$, let $z_1 := \inf\{z \in (0, z_0] \mid G(z) > \bar{G}(z)\}$ with $\inf\emptyset = z_0$. Let $b = \bar{G}(z_1)$ and $z_2 := \inf\{z \in [z_0, 1) \mid \bar{G}(z) > b\}$. Clearly, $z_0 \le z_2 < 1$ and $\bar{G}(z_2) = \bar{G}(z_1) = b$. If $z_2 \le 1-\alpha$, define

$$\tilde{G}(z) := \begin{cases} b & 0 < z \le z_2, \\ \bar{G}(z) & z_2 < z \le 1-\alpha, \\ \bar{G}_A(z) & 1-\alpha < z < 1. \end{cases}$$

If $z_2 > 1-\alpha$, define

$$\tilde{G}(z) := \begin{cases} b & 0 < z \le 1-\alpha, \\ b \vee \bar{G}_A(z) & 1-\alpha < z < 1. \end{cases}$$

In both cases, we have $\tilde{G}(\cdot) \in \mathbb{S}_\lambda^A$.

If $z_2 \le 1-\alpha$, we have

$$\begin{aligned} U_\lambda(G(\cdot)) &= \int_0^1 f(G(z), z)\,dz \\ &= \int_0^{z_1} f(G(z), z)\,dz + \int_{z_1}^{z_2} f(G(z), z)\,dz \\ &\quad + \int_{z_2}^{1-\alpha} f(G(z), z)\,dz + \int_{1-\alpha}^1 f(G(z), z)\,dz \\ &\le \int_0^{z_1} f(\bar{G}(z_1), z)\,dz + \int_{z_1}^{z_2} f(\bar{G}(z_1), z)\,dz \\ &\quad + \int_{z_2}^{1-\alpha} f(\bar{G}(z), z)\,dz + \int_{1-\alpha}^1 f(\bar{G}_A(z), z)\,dz, \\ &= \int_0^1 f(\tilde{G}(z), z)\,dz, \end{aligned}$$

and the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$. If $z_2 > 1 - \alpha$, we have

$$\begin{aligned}
U_\lambda(G(\cdot)) &= \int_0^1 f(G(z); z)\, dz \\
&= \int_0^{z_1} f(G(z), z)\, dz + \int_{z_1}^{1-\alpha} f(G(z), z)\, dz + \int_{1-\alpha}^1 f(G(z), z)\, dz \\
&\le \int_0^{z_1} f(\bar{G}(z_1), z)\, dz + \int_{z_1}^{1-\alpha} f(\bar{G}(z_1), z)\, dz + \int_{1-\alpha}^1 f(b \vee \bar{G}_A(z), z)\, dz, \\
&= \int_0^1 f(\tilde{G}(z), z)\, dz,
\end{aligned}$$

and the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$.

Next, we consider the case in which $1 - \alpha < z_0$. For any feasible $G(\cdot)$, again let $z_1 := \inf\{z \in (0, z_0] \mid G(z) > \bar{G}(z)\}$ with $\inf \emptyset = z_0$. Let $b = \bar{G}(z_1)$. If $b \ge A$, define $z_2 := \inf\{z \in [z_0, 1) \mid \bar{G}(z) > b\}$, and

$$\tilde{G}(z) := \begin{cases} b & 0 < z \le z_2, \\ \bar{G}(z) & z_2 < z < 1. \end{cases}$$

If $b < A$, then the feasibility of $G(\cdot)$ implies $z_1 \le 1 - \alpha$. Define

$$\tilde{G}(z) := \begin{cases} b & 0 < z \le 1 - \alpha, \\ \bar{G}_A(z) & 1 - \alpha < z < 1. \end{cases}$$

In both cases, $\tilde{G}(z) \in \mathbb{S}_\lambda^A$. If $b \ge A$, we have

$$\begin{aligned}
U_\lambda(G(\cdot)) &= \int_0^1 f(G(z), z)\, dz, \\
&= \int_0^{z_1} f(G(z), z)\, dz + \int_{z_1}^{z_2} f(G(z), z)\, dz + \int_{z_2}^1 f(G(z), z)\, dz \\
&\le \int_0^{z_1} f(\bar{G}(z_1), z)\, dz + \int_{z_1}^{z_2} f(\bar{G}(z_1), z)\, dz + \int_{z_2}^1 f(\bar{G}(z), z)\, dz, \\
&= U_\lambda(\tilde{G}(\cdot)),
\end{aligned}$$

and the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$. If $b < A$, we have

$$\begin{aligned}
U_\lambda(G(\cdot)) &= \int_0^1 f(G(z), z)\, dz, \\
&= \int_0^{z_1} f(G(z), z)\, dz + \int_{z_1}^{1-\alpha} f(G(z), z)\, dz + \int_{1-\alpha}^1 f(G(z), z)\, dz \\
&\le \int_0^{z_1} f(\bar{G}(z_1), z)\, dz + \int_{z_1}^{1-\alpha} f(\bar{G}(z_1), z)\, dz + \int_{1-\alpha}^1 f(\bar{G}_A(z), z)\, dz, \\
&= U_\lambda(\tilde{G}(\cdot)),
\end{aligned}$$

and the inequality becomes equality if and only if $G(\cdot) = \tilde{G}(\cdot)$. $\square$

In view of Proposition A.1, we need only consider the following problem:

$$
\text{(A.3)}\quad \begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U_\lambda(G(\cdot)) = \int_0^1 \left[u(G(z))w'(1-z) - \lambda G(z)F_\rho^{-1}(1-z)\right]dz, \\ \text{subject to} & G(\cdot) \in \mathbb{S}_\lambda^A, \quad G((1-\alpha)+) \geq A, \quad G(0+) \geq 0, \end{array}
$$

which is an optimization problem over the real line. The following proposition provides the result.

Before we state the proposition and its proof, however, let us recall $\varphi(\cdot)$ in (5.12) and $\psi(\cdot)$ in (5.16) and their properties. $\varphi(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$. Furthermore, $\varphi(\cdot)$ is positive on $(0, z^*)$ and negative on $(z^*, 1)$. $\psi(\cdot)$ is strictly decreasing on $(0, z^*)$ and strictly increasing on $(z^*, 1)$. In addition, $\psi(z) > M(z)$ on $(0, z^*)$, $\psi(z) < M(z)$ on $(z^*, 1)$, and $\psi(z^*) = M(z^*)$.

PROPOSITION A.2. *Let Assumptions 4.3, 5.1, and 5.5 hold. Let z^* be the unique root of (5.12) and recall $\psi(\cdot)$ in (5.16).*

1. *If $0 < 1-\alpha < z^*$, then $\psi(1-\alpha) > \psi(z^*) = M(z^*)$, and the unique optimal solution to (5.1) is given as*

$$
\text{(A.4)}\quad G_\lambda^*(z) = \begin{cases} \bar{G}(z^*)\mathbf{1}_{\{0<z\leq z^*\}} + \bar{G}(z)\mathbf{1}_{\{z^*<z\leq 1\}} & \dfrac{\lambda}{u'(A)} \leq M(z^*), \\ A\mathbf{1}_{\{0<z\leq z_0\}} + [A \vee \bar{G}_A(z)]\mathbf{1}_{\{z_0<z\leq 1\}} & M(z^*) \leq \dfrac{\lambda}{u'(A)} \leq \psi(1-\alpha), \\ (u')^{-1}\left(\dfrac{\lambda}{\psi(1-\alpha)}\right)\mathbf{1}_{\{0<z\leq 1-\alpha\}} & \dfrac{\lambda}{u'(A)} \geq \psi(1-\alpha). \\ \qquad + \bar{G}_A(z)\mathbf{1}_{\{1-\alpha<z\leq 1\}} & \end{cases}
$$

2. *If $z^* \leq 1-\alpha < 1$, then the unique optimal solution to (5.1) is given as*

$$
\text{(A.5)}\quad G_\lambda^*(z) = \bar{G}(z^*)\mathbf{1}_{\{0<z\leq z^*\}} + \bar{G}_A(z)\mathbf{1}_{\{z^*<z\leq 1\}}.
$$

Proof. The proof is split into three cases: (i) $1-\alpha \geq z^*$; (ii) $z_0 < 1-\alpha < z^*$; and (iii) $1-\alpha < z_0$.

(i) First, consider the case in which $1-\alpha \geq z^*$. If $A \leq \bar{G}(1-\alpha)$, i.e., $\frac{\lambda}{u'(A)} \leq M(1-\alpha)$, then the optimal solution to problem (5.1) when $A = 0$ in Proposition 5.8 automatically satisfies the additional aspiration constraint $G((1-\alpha)+) \geq A$, and therefore it is also optimal in the presence of A. It is easy to check that the optimal solution in (5.13) coincides with the one in (A.5) when $A \leq \bar{G}(1-\alpha)$. Thus, we can assume that $A \geq \bar{G}(1-\alpha)$, i.e., $\frac{\lambda}{u'(A)} \geq M(1-\alpha)$. Define

$$
\tilde{V}(b) := U_\lambda(b\mathbf{1}_{\{0<z\leq z_0\}} + [b \vee \bar{G}_A(z)]\mathbf{1}_{\{z_0<z<1\}}), \quad b \geq \bar{G}(z_0).
$$

Consider $\tilde{V}(\cdot)$ in three different domains. When $\bar{G}(z_0) \leq b \leq \bar{G}(1-\alpha)$, let $y := \bar{G}^{-1}(b)$, where $\bar{G}^{-1}(\cdot)$ is the inverse function of $\bar{G}(\cdot)$ when restricted on $[z_0, 1)$. We have

$$\tilde{V}(b) = u(b)\int_0^y w'(1-z)\,dz - \lambda b\int_0^y F_\rho^{-1}(1-z)\,dz$$
$$+\int_y^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_y^1 \bar{G}_A(z)F_\rho^{-1}(1-z)\,dz.$$

Clearly,

$$\tilde{V}'(b) = u'(b)\int_0^y w'(1-z)\,dz - \lambda\int_0^y F_\rho^{-1}(1-z)\,dz = \frac{\lambda}{M(y)}\varphi(y).$$

Because $\bar{G}(z_0) \le b \le \bar{G}(1-\alpha)$, $z_0 \le y \le 1-\alpha$. Recalling that $z_0 < z^* \le 1-\alpha$, we conclude that $b_1^* := \bar{G}(z^*)$ uniquely maximizes $\tilde{V}(\cdot)$ in the interval $[\bar{G}(z_0), \bar{G}(1-\alpha)]$.

When $\bar{G}(1-\alpha) < b < A$, we have

$$\tilde{V}(b) = u(b)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda b\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz$$
$$+\int_{1-\alpha}^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_{1-\alpha}^1 \bar{G}_A(z)F_\rho^{-1}(1-z)\,dz,$$

and

$$\tilde{V}'(b) = u'(b)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz$$
$$< \lambda\left[\frac{1}{M(1-\alpha)}\int_0^{1-\alpha} w'(1-z)\,dz - \int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz\right] \le 0,$$

where the first inequality is the case because $b > \bar{G}(1-\alpha)$ and the last inequality is the case because $1-\alpha \ge z^*$ and $\varphi(z) < 0$ on $(z^*, 1)$. Therefore, $\tilde{V}(\cdot)$ is strictly decreasing in the interval $[\bar{G}(1-\alpha), A]$.

When $b \ge A$, let $y := \bar{G}^{-1}(b) \ge 1-\alpha$. Then, we have

$$\tilde{V}(b) = u(b)\int_0^y w'(1-z)\,dz - \lambda b\int_0^y F_\rho^{-1}(1-z)\,dz$$
$$+\int_y^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_y^1 (\bar{G}_A(z))F_\rho^{-1}(1-z)\,dz,$$

and

$$\tilde{V}'(b) = u'(b)\int_0^y w'(1-z)\,dz - \lambda\int_0^y F_\rho^{-1}(1-z)\,dz = \frac{\lambda}{M(y)}\varphi(y) \le 0,$$

where the last inequality is due to $z^* \le 1-\alpha$ and $\varphi(z) < 0$ on $(z^*, 1)$. Therefore, $\tilde{V}(\cdot)$ is decreasing on $(A, +\infty)$.

To summarize, we conclude that $\tilde{V}(\cdot)$ obtains its maximum value at $b_1^* = \bar{G}(z^*)$, and therefore the optimal quantile function is given in (A.5).

(ii) Next, consider the case in which $z_0 < 1-\alpha < z^*$. Using the same argument as for case (i), we can assume that $A \ge \bar{G}(1-\alpha)$, i.e., $\frac{\lambda}{u'(A)} \ge M(1-\alpha)$. Because

$1-\alpha < z^*$, we have $M(1-\alpha) < \psi(1-\alpha)$. Again, define

$$\tilde{V}(b) := U_\lambda(b\mathbf{1}_{\{0<z\le z_0\}} + [b \vee \bar{G}_A(z)]\mathbf{1}_{\{z_0<z<1\}}), \quad b \ge \bar{G}(z_0).$$

When $\bar{G}(z_0) \le b \le \bar{G}(1-\alpha)$, letting $y := \bar{G}^{-1}(b) \le 1-\alpha$, we have

$$\begin{aligned}\tilde{V}(b) &= u(b)\int_0^y w'(1-z)\,dz - \lambda b\int_0^y F_\rho^{-1}(1-z)\,dz \\ &\quad + \int_y^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_y^1 (\bar{G}_A(z))F_\rho^{-1}(1-z)\,dz.\end{aligned}$$

Clearly,

$$\tilde{V}'(b) = u'(b)\int_0^y w'(1-z)\,dz - \lambda\int_0^y F_\rho^{-1}(1-z)\,dz = \frac{\lambda}{M(y)}\varphi(y) > 0,$$

where the last inequality is the case because $1-\alpha < z^*$ and $\varphi(z) > 0$ on $(0, z^*)$. Therefore, $\tilde{V}(\cdot)$ is strictly increasing in the interval $[\bar{G}(z_0), \bar{G}(1-\alpha)]$.

When $\bar{G}(1-\alpha) < b < A$, we have

$$\begin{aligned}\tilde{V}(b) &= u(b)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda b\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz \\ &\quad + \int_{1-\alpha}^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_{1-\alpha}^1 \bar{G}_A(z)F_\rho^{-1}(1-z)\,dz,\end{aligned}$$

and

$$\tilde{V}'(b) = u'(b)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz.$$

If $M(1-\alpha) \le \frac{\lambda}{u'(A)} \le \psi(1-\alpha)$, we have

$$\begin{aligned}\tilde{V}'(b) &> u'(A)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz \\ &= u'(A)\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz\left[\psi(1-\alpha) - \frac{\lambda}{u'(A)}\right] \ge 0,\end{aligned}$$

and, consequently, $\tilde{V}(\cdot)$ is strictly increasing in the interval $[\bar{G}(1-\alpha), A]$. If $\frac{\lambda}{u'(A)} \ge \psi(1-\alpha)$, then it is easy to see that the unique optimizer of $\tilde{V}(\cdot)$ in this interval is $b_2^* := (u')^{-1}(\frac{\lambda}{\psi(1-\alpha)})$.

When $b \ge A$, let $y := \bar{G}^{-1}(b)$. Then, we have

$$\begin{aligned}\tilde{V}(b) &= u(b)\int_0^y w'(1-z)\,dz - \lambda b\int_0^y F_\rho^{-1}(1-z)\,dz \\ &\quad + \int_y^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_y^1 \bar{G}_A(z)F_\rho^{-1}(1-z)\,dz,\end{aligned}$$

and

$$\tilde{V}'(b) = u'(b)\int_0^y w'(1-z)\,dz - \lambda\int_0^y F_\rho^{-1}(1-z)\,dz = \frac{\lambda}{M(y)}\varphi(y).$$

If $\frac{\lambda}{u'(A)} \geq M(z^*)$, then $\bar{G}^{-1}(A) \geq z^*$. Consequently, $y > \bar{G}^{-1}(A) \geq z^*$ and $\tilde{V}'(b) < 0$ because $\varphi(\cdot)$ is negative on $(z^*, 1)$. In other words, $\tilde{V}(\cdot)$ is strictly decreasing on this interval. If $\frac{\lambda}{u'(A)} \leq M(z^*)$, then $\bar{G}^{-1}(A) \leq z^*$ and, consequently, the unique maximizer $b_3^* = (u')^{-1}(\frac{\lambda}{M(z^*)})$.

Finally, because $z_0 \leq 1-\alpha < z^*$, $M(1-\alpha) < M(z^*) = \psi(z^*) < \psi(1-\alpha)$. Then, we can sum up the results and conclude that the optimizer of $\tilde{V}(\cdot)$ is $(u')^{-1}(\frac{\lambda}{\psi(1-\alpha)})$ if $\frac{\lambda}{u'(A)} \geq \psi(1-\alpha)$, is A if $M(z^*) \leq \frac{\lambda}{u'(A)} \leq \psi(1-\alpha)$, and is $(u')^{-1}(\frac{\lambda}{M(z^*)})$ if $M(1-\alpha) \leq \frac{\lambda}{u'(A)} \leq M(z^*)$. Therefore, the optimal quantile function is given in (A.4).

(iii) Finally, consider the case in which $1-\alpha \leq z_0$. Using the same arguments as in (i) and (ii), we can assume that $A \geq \bar{G}(z_0)$, i.e., $\frac{\lambda}{u'(A)} \geq M(z_0)$. Again, we let

$$\tilde{V}(b) := U_\lambda(b\mathbf{1}_{\{0<z\leq 1-\alpha\}} + [b \vee A]\mathbf{1}_{\{1-\alpha<z\leq z_0\}} + [b \vee \bar{G}_A(z)]\mathbf{1}_{\{z_0<z<1\}}), \quad b \geq M(z_0).$$

We first optimize $\tilde{V}(\cdot)$ in $[A, +\infty)$. In this interval, we have

$$\tilde{V}(b) = U(b\mathbf{1}_{\{0<z\leq z_0\}} + [b \vee \bar{G}(z)]\mathbf{1}_{\{z_0<z<1\}}),$$

which is the same as the function $\tilde{V}(\cdot)$ in the proof of Proposition 5.8 after making the transformation $y = \bar{G}^{-1}(b)$. From the proof of Proposition 5.8, we can deduce that the maximizer of $\tilde{V}(\cdot)$ on $[A, +\infty)$ is $\bar{G}(z^*)$ if $\frac{\lambda}{u'(A)} \leq M(z^*)$ and is A if $\frac{\lambda}{u'(A)} \geq M(z^*)$.

Next, we consider $\tilde{V}(b)$ for other possible b. If $A \leq \bar{G}(1-\alpha)$, i.e., $\frac{\lambda}{u'(A)} \leq M(1-\alpha)$, then it is easy to see that $\tilde{V}(\cdot)$ is strictly increasing on $[\bar{G}(z_0), A]$. If $A \geq \bar{G}(1-\alpha)$, i.e., $\frac{\lambda}{u'(A)} \geq M(1-\alpha)$, then it is easy to check that $\tilde{V}(\cdot)$ is strictly increasing in $[\bar{G}(z_0), \bar{G}(1-\alpha)]$. Thus, we need only consider the case in which $A > \bar{G}(1-\alpha)$, i.e., $\frac{\lambda}{u'(A)} > M(1-\alpha)$, and check $\tilde{V}(b)$ for $\bar{G}(1-\alpha) < b < A$. In this case, we have

$$\begin{aligned}\tilde{V}(b) = u(b)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda b\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz \\ + \int_{1-\alpha}^1 u(\bar{G}_A(z))w'(1-z)\,dz - \lambda\int_{1-\alpha}^1 \bar{G}_A(z)F_\rho^{-1}(1-z)\,dz,\end{aligned}$$

and

$$\tilde{V}'(b) = u'(b)\int_0^{1-\alpha} w'(1-z)\,dz - \lambda\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz.$$

Because $1-\alpha \leq z_0 < z^*$, we have $\psi(1-\alpha) > M(1-\alpha)$. Now, when $M(1-\alpha) \leq \frac{\lambda}{u'(A)} \leq \psi(1-\alpha)$,

$$\tilde{V}'(b) > u'(A)\left(\int_0^{1-\alpha} F_\rho^{-1}(1-z)\,dz\right)\left[\psi(1-\alpha) - \frac{\lambda}{u'(A)}\right] \geq 0,$$

where the first inequality is due to $b < A$. Consequently, $\tilde{V}(\cdot)$ is strictly increasing in $[\bar{G}(1-\alpha), A]$. When $\frac{\lambda}{u'(A)} \geq \psi(1-\alpha)$, it is easy to see that the optimizer of $\tilde{V}(\cdot)$ in this interval is $(u')^{-1}(\frac{\lambda}{\psi(1-\alpha)})$.

To summarize, the unique optimizer of $V(\cdot)$ is $\bar{G}(z^*)$ if $\frac{\lambda}{u'(A)} \leq M(z^*)$, is A if $M(z^*) \leq \frac{\lambda}{u'(A)} \leq \psi(1-\alpha)$, and is $(u')^{-1}(\frac{\lambda}{\psi(1-\alpha)})$ if $\frac{\lambda}{u'(A)} \geq \psi(1-\alpha)$. Therefore, the optimal quantile function is given in (A.4).

Ultimately, the uniqueness of the optimal solution can be easily derived from the above proof. □

Proof of Theorem 5.10. Let

$$\mathcal{X}(\lambda) = \int_0^1 G_\lambda^*(z) F_\rho^{-1}(1-z)\, dz, \quad \lambda > 0$$

where $G_\lambda^*(\cdot)$ is as given in (A.4) or (A.5), depending on the value of α. By Assumption 5.1, $\mathcal{X}(\cdot)$ is finite and decreasing on $(0, +\infty)$. Furthermore, because ρ is atomless, by applying the monotone convergence theorem, we conclude that $\mathcal{X}(\cdot)$ is continuous and

$$\lim_{\lambda \uparrow +\infty} \mathcal{X}(\lambda) = AE\big[\rho \mathbf{1}_{\{\rho \leq F_\rho^{-1}(\alpha)\}}\big], \quad \lim_{\lambda \downarrow 0} \mathcal{X}(\lambda) = +\infty.$$

Therefore, we can find λ^* such that $\mathcal{X}(\lambda^*) = x_0$ and, consequently, $G_{\lambda^*}^*(\cdot)$ is optimal to (3.4). Noticing that $\mathcal{X}(u'(A)M(z^*)) = x_r$ and that $\mathcal{X}(u'(A)\psi(1-\alpha)) = x_p$, the optimal solution to (3.3), $X^* := G_{\lambda^*}^*(1 - F_\rho(\rho))$, is exactly as given in (5.19)–(5.22). □

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