

MULTIVARIATE RISK MEASURES: A CONSTRUCTIVE APPROACH BASED ON SELECTIONS

ILYA MOLCHANOV

University of Bern

IGNACIO CASCOS

Universidad Carlos III de Madrid

Since risky positions in multivariate portfolios can be offset by various choices of capital requirements that depend on the exchange rules and related transaction costs, it is natural to assume that the risk measures of random vectors are set-valued. Furthermore, it is reasonable to include the exchange rules in the argument of the risk measure and so consider risk measures of set-valued portfolios. This situation includes the classical Kabanov's transaction costs model, where the set-valued portfolio is given by the sum of a random vector and an exchange cone, but also a number of further cases of additional liquidity constraints. We suggest a definition of the risk measure based on calling a set-valued portfolio acceptable if it possesses a selection with all individually acceptable marginals. The obtained selection risk measure is coherent (or convex), law invariant, and has values being upper convex closed sets. We describe the dual representation of the selection risk measure and suggest efficient ways of approximating it from below and from above. In the case of Kabanov's exchange cone model, it is shown how the selection risk measure relates to the set-valued risk measures considered by Kulikov (2008, *Theory Probab. Appl.* 52, 614–635), Hamel and Heyde (2010, *SIAM J. Financ. Math.* 1, 66–95), and Hamel, Heyde, and Rudloff (2013, *Math. Financ. Econ.* 5, 1–28).

KEY WORDS: exchange cone, random set, selection, set-valued portfolio, set-valued risk measure, transaction costs.

1. INTRODUCTION

Since the seminal papers by Artzner et al. (1999) and Delbaen (2002), most studies of risk measures deal with the univariate case, where the gains or liabilities are expressed by a random variable. We refer to Föllmer and Schied (2004) and McNeil, Frey, and

The authors are grateful to Leonid Hanin for his advice on duality results in Lipschitz spaces. IM acknowledges the hospitality of the University Carlos III de Madrid. IM has benefited from discussions with Qiyu Li, Michael Schmutz, and Irina Sikharulidze at various stages of this work. The second version of this preprint was greatly inspired by insightful comments of Birgit Rudloff concerning her recent work on multivariate risk measures. The authors are grateful to the Associate Editor and the referees for thoughtful comments and encouragement that led to a greatly improved paper. IC supported by the Spanish Ministry of Science and Innovation Grants No. MTM2011-22993 and ECO2011-25706. IM supported by the Chair of Excellence Programme of the Universidad Carlos III de Madrid and Banco Santander and the Swiss National Foundation Grant No. 200021-137527.

Manuscript received February 2013; final revision received May 2014.

Address correspondence to Ilya Molchanov, Institute of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland; e-mail: ilya.molchanov@stat.unibe.ch.

Embrechts (2005) for a thorough treatment of univariate risk measures and to Acciaio and Penner (2011) for a recent survey of the dynamic univariate setting.

Multiasset portfolios in practice are often represented by their total monetary value in a fixed currency with the subsequent calculation of univariate risk measures that can be used to determine the overall capital requirements. The main emphasis is put on the dependency structure of the various components of the portfolio, see Burgert and Rüschendorf (2006), Embrechts and Puccetti (2006). Numerical risk measures for a multivariate portfolio X have been also studied in Ekeland, Galichon, and Henry (2012) and Rüschendorf (2006). The key idea is to consider the expected scalar product of $(-X)$ with a random vector Z and take supremum over all random vectors that share the same distribution with X and possibly over a family of random vectors Z . Farkas, Koch-Medina, and Munari (2014) defined the scalar risk as the infimum of the payoff associated with a random vector that being added to the portfolio renders it acceptable. A vector-valued variant of the value-at-risk has been suggested in Cousin and Di Bernardino (2013).

However, in many natural applications it is necessary to assess the risk of a vector X in \mathbb{R}^d whose components represent different currencies or gains from various business lines, where profits/losses from one line or currency cannot be used directly to offset the position in a different one. Even in the absence of transaction costs, the exchange rates fluctuate and so may influence the overall risk assessment. Also the regulatory requirements may be very different for different lines (e.g., in the case of several states within the same currency area), and moving assets may be subject to transaction costs, taxes, or other restrictions. For such cases, it is important to determine the necessary reserves that should be allocated in each line (currency or component) of X in order to make the overall position acceptable. The simplest solution would be to treat each component separately and allocate reserves accordingly, which is not in the interest of (financial) agents who might want to use profits from one line to compensate for eventual losses in other ones. Thus, in addition to assessing the risk of the original vector X , one can also evaluate the risk of any other portfolio that may be obtained from X by allowed transactions. In view of this, it is natural to assume that the acceptability may be achieved by several (and not directly comparable) choices of capital requirements that form a set of possible values for the risk measure. This suggests the idea of working with *set-valued* risk measures.

Since the first work on multivariate risk measures (Jouini, Meddeb, and Touzi 2004), by now it is accepted that multiasset risk measures can be naturally considered as taking values in the space of sets, see Ben Tahar and Lepinette (2014), Cascos and Molchanov (2007), Hamel and Heyde (2010), Hamel, Heyde, and Rudloff (2011), Kulikov (2008). The risk measures of random vectors are mostly considered in relation to Kabanov's transaction costs model, whose main ingredient is a cone \mathbf{K} of portfolios available at price zero at the chosen time horizon (also called the *exchange cone*), while the central symmetric variant of \mathbf{K} is a solvency cone. If X is the terminal gain, then each random vector with values in $X + \mathbf{K}$ is possible to obtain by converting X following the rules determined by \mathbf{K} . In other words, instead of measuring the risk of X we consider the whole family of random vectors taking values in $X + \mathbf{K}$. In relation to this, note that families of random vectors representing attainable gains are often considered in the financial studies of transaction costs models, see, e.g., Schachermayer (2004).

The set-valued portfolios framework can be related to the classical setting by replacing a univariate random gain X with half-line $(-\infty, X]$ and measuring risks of all random variables dominated by X . The monotonicity property of a chosen risk measure r implies that all these risks build the set $\rho(X) = [r(X), \infty)$. If r is subadditive, then $\rho(X + Y) \supset$

$\rho(X) + \rho(Y)$. Furthermore, $r(X) \leq 0$ if and only if $\rho(X)$ contains the origin. While in the univariate case this construction leads to half-lines, in the multivariate situation it naturally gives rise to so-called upper convex sets, see Hamel and Heyde (2010), Hamel et al. (2011), Kulikov (2008). A portfolio is acceptable if its risk measure contains the origin and the value of the risk measure is the set of all $a \in \mathbb{R}^d$ such that the portfolio becomes acceptable if capital a is added to it.

Note that in the setting of real-valued risk measures adapted in Ekeland et al. (2012), the family of all $a \in \mathbb{R}^d$ that make $X + a$ acceptable is a half-space, which apparently only partially reflects the nature of cone-based transaction costs models. Farkas et al. (2014) establish relation between families of real-valued risks and set-valued risks from Hamel et al. (2011). The setting of Riesz spaces (partially ordered linear spaces), in particular Fréchet lattices and Orlicz spaces, has become already common in the theory of risk measures, see Biagini and Frittelli (2008), Cheridito and Li (2009). However, these spaces are mostly used to describe the arguments of risk measures whose values belong to the (extended) real line. Furthermore, the space of sets is no longer a Riesz space—while the addition is well defined, the matching subtraction does not exist. The recent study of risk preferences (Drapeau and Kupper 2013) also concentrates on the case of vector spaces for arguments of risk measures.

The dual representation for risk measures of random vectors in the case of a deterministic exchange cone is obtained in Hamel and Heyde (2010) and for the random case in Hamel et al. (2011), see also Kulikov (2008) who considers both deterministic and random exchange cones. However, in the case of a random exchange cone, it does not produce law-invariant risk measures—the risk measure in Hamel and Heyde (2010), Hamel et al. (2011), Hamel et al. (2013), Kulikov (2008) is defined as a function of a random vector X representing the gain, while identically distributed gains might exhibit different properties in relation to the random exchange cone. Although the dual representations from Hamel and Heyde (2010), Hamel et al. (2011), Kulikov (2008) are general, they are rather difficult to use in order to calculate risks for given portfolios, since they are given as intersections of half-spaces determined by a rather rich family of random vectors from the dual space. Recent advances in vector optimization have led to a substantial progress in computation of set-valued risk measures, see Hamel et al. (2013). However, the dual approximation also does not explicitly yield the relevant trading (or exchange) strategy that determines transactions suitable to compensate for risks. The construction of set-valued risk measures from Cascos and Molchanov (2007) is based on the concept of the depth-trimmed region, and their values are easy to calculate numerically or analytically, but it only applies for deterministic exchange cones and often results in marginalized risks (so that the risk measure is a translate of the solvency cone).

In order to come up with a law invariant risk measure and also cover the case of random exchange cones, we assume that the argument of a risk measure is a *random closed set* that consists of all attainable portfolios. This random set may be the sum $X + K$ of a random vector X and the exchange cone K (which has been the most important example so far) or may be defined otherwise. For instance, if only a linear space of portfolios M is available for compensation, then the set of attainable portfolios is the intersection of $X + K$ with M . In any such case we speak about a set-valued portfolio X . This guiding idea makes it possible to work out the law-invariance property of risk measures and naturally arrive at set-valued risks.

In this paper we suggest a rather simple and intuitive way to measure risks for set-valued portfolios based on considering the family of all terminal gains that may be attained after some exchanges are performed. The crucial step is to consider all random vectors taking values in a random set X (*selections* of X) as possible gains and regard

the random set acceptable if it possesses a selection with all acceptable components. In view of this, we do not only determine the necessary capital reserves, but also the way of converting the terminal value of the portfolio into an acceptable one.

In the case of exchange cones, we relate our construction to the dual representation from Hamel and Heyde (2010) and Kulikov (2008). Throughout the paper we concentrate on the coherent case and one-period setting, but occasionally comment on noncoherent generalizations. Feinstein and Rudloff (2014a) thoroughly analyze and compare various approaches, including one from this paper, in view of defining multiperiod set-valued risks, see also Feinstein and Rudloff (2014b).

EXAMPLE 1.1. Let $X = (X_1, X_2)$ represent terminal gains on two business lines expressed in two different currencies. Assume that X_1 and X_2 are i.i.d. normally distributed with mean 0.5 and variance 1. Assume that the exchanges between currencies are free from transaction costs with the initial exchange rate $\pi_0 = 1.5$ (number of units of the second currency to buy one unit of the first one), the terminal exchange rate π is lognormal with mean 1.5 and volatility 0.4, and π is independent of X . The set-valued portfolio X is a half-plane with the boundary passing through X and normal $(\pi, 1)$.

Assume that necessary capital reserves are determined using the expected shortfall (ES) $\text{ES}_{0.05}$ at level 0.05, see Acerbi and Tasche (2002). If compensation between business lines is not allowed, the necessary capital reserves at time zero are given by $\text{ES}_{0.05}(X_1) + \pi_0^{-1} \text{ES}_{0.05}(X_2) \approx 2.6045$ (all numbers are given in the units of the first currency). If the terminal gains are transferred to one currency, then the needed reserves are given by $\text{ES}_{0.05}(X_1 + \pi^{-1} X_2) \approx 1.801$ and $\pi_0^{-1} \text{ES}_{0.05}(\pi X_1 + X_2) \approx 1.784$, respectively, in the case of transfers to the first and the second currency. These values correspond to evaluating the risks of selections of X located at the points of intersection of the boundary of X with coordinate axes.

However, it is possible to choose a selection that further reduces the required capital requirements. After transferring to the second asset $(X_1 - \pi X_2)/(1 + \pi^2)$ units of the first currency, we arrive at the selection of X given by

$$(1.1) \quad \xi = (\xi_1, \xi_2) = \left(\frac{\pi(X_1\pi + X_2)}{1 + \pi^2}, \frac{X_1\pi + X_2}{1 + \pi^2} \right)$$

obtained by projecting the origin onto the boundary of X . In this case the needed reserves are $\text{ES}_{0.05}(\xi_1) + \pi_0^{-1} \text{ES}_{0.05}(\xi_2) \approx 1.661$. The situation of random frictionless exchanges is also considered analytically in Example 5.1 and numerically in Examples 8.3 and 8.4.

Assume that now liquidity restrictions are imposed meaning that at most one unit of each currency may be obtained after conversion from the other. This framework corresponds to dealing with a nonconical set-valued portfolio $Y = X \cap (X + (1, 1) + \mathbb{R}_+^2)$. A reasonable strategy would be to use selection ξ from (1.1) if $(\pi X_2 - X_1)/(1 + \pi^2) \leq 1$ and $(\pi(X_1 - \pi X_2))/(1 + \pi^2) \leq 1$ and otherwise choose the nearest point to ξ from the extreme points of Y . The corresponding selection is given by

$$(1.2) \quad \eta = (\eta_1, \eta_2) = \begin{cases} (X_1 + 1, X_2 - \pi) & \text{if } \pi X_2 - X_1 > 1 + \pi^2, \\ (X_1 - \pi^{-1}, X_2 + 1) & \text{if } \pi(X_1 - \pi X_2) > 1 + \pi^2, \\ \xi & \text{otherwise,} \end{cases}$$

and the needed reserves are $\text{ES}_{0.05}(\eta_1) + \pi_0^{-1} \text{ES}_{0.05}(\eta_2) \approx 1.735$, which is higher than those corresponding to the choice of ξ in view of imposed liquidity restrictions.

Before describing the structure of the paper, we would like to point out that our approach is *constructive* in the sense that instead of starting with an axiomatic definition

of a set-valued risk measure we explicitly construct one based on selections of a random portfolio and univariate marginal risk measures. Then we show that the constructed risk measure indeed satisfies the desired properties of set-valued risk measures, in particular, the coherency and the Fatou properties, instead of imposing them. We show how to approximate the values of the risk measure from below (which is the aim of the market regulator) and from above (as the agent would aim to do). The suggested bounds provide a feasible alternative to exact calculations of risk. Furthermore, the computational burden is passed to the agent who aims to increase the family of selections in order to obtain a tighter approximation from above and so reduce the capital requirements, quite differently to the dual constructions of Hamel et al. (2011) and Kulikov (2008), where the market regulator faces the task of making the acceptance criterion more stringent by approximating from below the exact value of the risk measure. It should be noted that our approach constitutes just one possible way to construct multivariate risk measures (and the corresponding acceptance sets) that satisfy the axioms of set-valued coherent risk measures from Hamel et al. (2011).

Section 2 introduces the concept of set-valued portfolios and the definition of set-valued risk measures for set-valued portfolios adapted from Hamel et al. (2011), where the conical setting was considered. Section 3 defines the *selection risk measure*, which relies on d univariate risk measures applied to the components of selections for a set-valued portfolio. In particular, the coherency of the selection risk measure is established in Theorem 3.4. While throughout the paper we work with coherent risk measures defined on L^p spaces with $p \in [1, \infty]$, the construction can be also based on convex noncoherent and nonconvex univariate risk measures, so that it yields their noncoherent set-valued analogues, such as the value-at-risk.

Section 4 derives lower and upper bounds for risk measures. Section 5 is devoted to the setting of *exchange cones* (or conical market models) that has been at the center of attention in all other works on multiasset risks. It is shown that, for the exchange cones setting, the lower bound corresponds to the dual representation of risk measures from Hamel and Heyde (2010), Hamel et al. (2011), and Kulikov (2008). For deterministic exchange cones the bounds become even simpler and in the case of comonotonic portfolios the risk measure admits an easy expression.

We briefly comment on scalarization issues in Section 6, i.e., explain relationships to univariate risk measures constructed for set-valued portfolios, which in the case of a deterministic exchange cone are related to those considered in Ekeland et al. (2012), Rüschendorf (2006), and Farkas et al. (2014).

Section 7 establishes the dual representation of the selection risk measures. While the idea is to handle set-valued portfolios through their support functions, the key difficulty consists in dealing with possibly unbounded values of the support functions. For this, we introduce the Lipschitz space of random sets and specify the weak-star convergence in this space in order to come up with a general dual representation for set-valued risk measures with the Fatou property. Theorems 7.4, 7.5, and 7.7 establish the Fatou property of the selection risk measure and so yield the closedness of its values and the validity of the dual representation. In the deterministic exchange cone model and for random exchange cones with $p \in [1, \infty)$, the selection risk measure has the same dual representation as in Hamel and Heyde (2010), Kulikov (2008).

Section 8 presents several numerical examples of set-valued risk measures covering the exchange cone setting, the frictionless case, and liquidity restrictions. The algorithms used to approximate risk measures are transparent and easy to implement in comparison with a considerably more sophisticated set-optimization approach from Löhne (2011) used in Hamel et al. (2013) in order to come up with exact values of set-valued risk

measures in conical models. A particular computational advantage is due to the use of the primal representation of selection risk measures in order to compute upper bounds, while utilizing the dual representation to arrive at lower bounds.

2. SET-VALUED PORTFOLIOS AND RISK MEASURES

2.1. Operations with Sets

In order to handle set-valued portfolios, we need to define several important operations with sets in \mathbb{R}^d . The closure of a set M is denoted by $\text{cl}(M)$. Further,

$$\check{M} = \{-x : x \in M\}$$

denotes the centrally symmetric set to M . The sum $M + L$ of two (deterministic) sets M and L in a linear space is defined as the set $\{x + y : x \in M, y \in L\}$. If one of the summands is compact and the other is closed, the set of pairwise sums is also closed. In particular, the sum $x + M$ of a point and a set is given by $\{x + y : y \in M\}$. For instance, $x + \mathbb{R}_-^d$ is the set of points dominated by x , where $\mathbb{R}_-^d = (-\infty, 0]^d$. Denote $\mathbb{R}_+^d = [0, \infty)^d$.

The *norm* of a set M is defined as $\|M\| = \sup\{\|x\| : x \in M\}$, where $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^d$. A set M is said to be *upper*, if $x \in M$ and $x \leq y$ imply that $y \in M$, where all inequalities between vectors are understood coordinatewisely. Inclusions of sets are always understood in the nonstrict sense, i.e., $M \subset L$ allows for the equality $M = L$.

The ε -envelope M^ε of a closed set M is defined as the set of all points x such that the distance between x and the nearest point of M is at most ε . The *Hausdorff distance* $\mathfrak{d}_H(M_1, M_2)$ between two closed sets M_1 and M_2 in \mathbb{R}^d is the smallest $\varepsilon \geq 0$ such that $M_1 \subset M_2^\varepsilon$ and $M_2 \subset M_1^\varepsilon$. The Hausdorff distance metrises the family of compact sets, while it can be infinite for unbounded sets.

The *support function* (see (Schneider 1993, sec. 1.7)) of a set M in \mathbb{R}^d is defined as

$$h_M(u) = \sup\{\langle u, x \rangle : x \in M\}, \quad u \in \mathbb{R}^d,$$

where $\langle u, x \rangle$ denotes the scalar product. The support function may take infinite values if M is not bounded. Denote by

$$M' = \{u : |h_M(u)| \neq \infty\}$$

the *effective domain* of the support function of M . The set M' is always a convex cone in \mathbb{R}^d . If K is a cone in \mathbb{R}^d , then K' equals the *dual cone* to K defined as

$$(2.1) \quad K' = \{u \in \mathbb{R}^d : \langle u, x \rangle \leq 0 \text{ for all } x \in K\}.$$

2.2. Set-Valued Portfolios

Let X be an almost surely nonempty *random closed convex set* in \mathbb{R}^d (shortly called random set) that represents all feasible terminal gains on d assets expressed in physical units. The random set X is called *set-valued portfolio*. Assume that X is defined on a complete nonatomic probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Any attainable terminal gain is a random vector ξ that almost surely takes values from X , i.e., $\xi \in X$ a.s., and such ξ is called a *selection* of X . We refer to Molchanov (2005) for the modern mathematical theory of random sets.

Since the free disposal of assets is allowed, with each point x , the set X also contains all points dominated by x coordinatewisely and so X is a *lower* set in \mathbb{R}^d . The *efficient part* of X is the set $\partial^+ X$ of all points $x \in X$ such that no other point of X dominates x in the coordinatewise order. While X itself is never bounded, X is called *quasi-bounded* if $\partial^+ X$ is a.s. bounded.

Fix $p \in [1, \infty]$ and consider the space $L^p(\mathbb{R}^d)$ of p -integrable random vectors in \mathbb{R}^d defined on $(\Omega, \mathcal{F}, \mathbf{P})$. The reciprocal q is defined from $p^{-1} + q^{-1} = 1$. The L^p -norm of ξ is denoted by $\|\xi\|_p$. Furthermore, the family of p -integrable selections of X is denoted by $L^p(X)$, and $L^\infty(X)$ is the family of all essentially bounded selections.

In the following we assume that X is *p-integrable*, i.e., X possesses at least one p -integrable selection. A random closed set is called *p-integrably bounded* if its norm is p -integrable (a.s. bounded if $p = \infty$). In the case of set-valued portfolios, this property is considered for $\partial^+ X$.

The closed sum $X + Y$ is defined as the random set being the closure of $X(\omega) + Y(\omega)$ for $\omega \in \Omega$. It is shown in Hiai and Umegaki (1977, th. 1.4) (see also, Molchanov 2005, prop. 2.1.4) that the set of p -integrable selections of $X + Y$ coincides with the norm closure of $L^p(X) + L^p(Y)$ if $p \in [1, \infty)$. If $p = \infty$, then $L^\infty(X + Y)$ is a norm closed set that contains the closed sum $L^\infty(X) + L^\infty(Y)$.

2.3. Examples of Set-Valued Portfolios

EXAMPLE 2.1 (Univariate portfolios). If $d = 1$, then $X = (-\infty, X]$ is a half-line and the monotonicity of risks implies that it suffices to consider only its upper bound X as in the classical theory of risk measures.

EXAMPLE 2.2 (Exchange cones). Let $X \in L^p(\mathbb{R}^d)$ represent gains from d assets. Furthermore, let $K \supset \mathbb{R}_-^d$ be a convex (distinct from the whole space and possibly random) *exchange cone* representing the family of portfolios available at price zero. Its symmetric variant \check{K} is the solvency cone, while the dual cone K' contains all consistent price systems, see Kabanov and Safarian (2009) and Schachermayer (2004). Formally, K is a random closed set with values being cones. Define $X = X + K$, so that selections of X correspond to portfolios that are possible to obtain from X following the exchange rules determined by K .

If K does not contain any line (and is called a *proper cone*), then the market has an *efficient friction*. Otherwise, some exchanges are free from transaction costs. In this case different random vectors X yield the same portfolio $X + K$, which is also an argument in favor of working directly with set-valued portfolios. The cone K is a half-space if and only if all exchanges do not involve transaction costs. In difference to Ben Tahar and Lepinette (2014), our setting does not require that the exchange cone is proper.

If K is deterministic, then we denote it by K . If $K = \mathbb{R}_-^d$, no exchanges are allowed.

EXAMPLE 2.3 (Cones generated by bid-ask matrix). In the case of d currencies, the cone K is usually generated by a bid-ask matrix, as in Kabanov's transaction costs model, see Kabanov (1999) and Schachermayer (2004). Let $\Pi = (\pi^{(ij)})$ be a (possibly random) matrix of exchange rates, so that $\pi^{(ij)}$ is the number of units of currency i needed to buy one unit of currency j . It is assumed that the elements of Π are positive, the diagonal elements are all one and $\pi^{(ij)} \leq \pi^{(ik)}\pi^{(kj)}$ meaning that a direct exchange is always cheaper than a chain of exchanges. The cone K describes the family of portfolios available at price zero, so that K is spanned by vectors $-e_i$ and $e_j - \pi^{(ij)}e_i$ for $i, j = 1, \dots, d$, where

e_1, \dots, e_d are standard basis vectors in \mathbb{R}^d . If the gain X contains derivatives drawn on the exchange rates, then we arrive at the situation when X and the exchange cone K are dependent.

EXAMPLE 2.4 (Conical setting with constraints). It is possible to modify the conical setting by requiring that all positions acquired after trading are subject to some linear or other constraints, see, e.g., Farkas et al. (2014). This amounts to considering the intersection of $X = X + K$ with a linear subspace or a more general subset of \mathbb{R}^d , that (if a.s. nonempty) results in a possibly nonconical set-valued portfolio and provides another motivation for working with set-valued portfolios.

Pennanen and Penner (2010) study in depth not necessarily conical transaction costs models in view of the no-arbitrage property, see also Kaval and Molchanov (2006). One of the most important examples is the model of currency markets with liquidity costs or exchange constraints.

The following examples describe several nonconical models that yield quasi-bounded set-valued portfolios. Despite the fact that some of them are generated by random vectors, it is essential to treat these portfolios as random sets, e.g., for possible diversification effects. The latter means that a sum of such set-valued portfolios is not necessarily equal to the set-valued portfolio generated by the sum of the generating random vectors.

EXAMPLE 2.5 (Restricted liquidity). Let $X = X + K$, where $K = \{x : \sum x_i \leq 0, x_i \leq 1, i = 1, \dots, d\}$. Then the exchanges up to the unit volume are at the unit rate free from transaction costs while other exchanges are not allowed. A similar example with transaction costs and a random exchange cone K can be constructed as $X = X + (K \cap (a + \mathbb{R}_+^d))$ for some $a \in \mathbb{R}_+^d$. A more general variant from Pennanen and Penner (2010, ex. 2.4) models liquidity costs depending on the transaction's volume.

EXAMPLE 2.6. Let $X^{(1)}, \dots, X^{(n)}$ be random vectors in \mathbb{R}^d that represent terminal gains in d lines (e.g., currencies) of n investments. The random set X is defined as the set of all points in \mathbb{R}^d dominated by at least one convex combination of the gains. In other words, X is the sum of \mathbb{R}_-^d and the convex hull of $X^{(1)}, \dots, X^{(n)}$.

If $X^{(1)} = (X_1, X_2)$ and $X^{(2)} = (X_2, X_1)$ for $d = 2$ and a bivariate random vector (X_1, X_2) , then X describes an arbitrary profit allocation between two different lines without transaction costs up to the amount $|X_1 - X_2|$.

EXAMPLE 2.7. Assume that $X = X + B_R + \mathbb{R}_-^d$, where B_R is the ball of fixed radius R centered at the origin. This model corresponds to the case, when infinitesimally small transactions are free to exchange at the rate that depends on the balance between the portfolio components.

EXAMPLE 2.8 (Transactions maintaining solvency). Let K be an exchange cone from Example 2.2 and let X be the value of a portfolio. Define X to be the set of points coordinatewisely dominated by a point from $(X + K) \cap \check{K}$ if X belongs to the solvency cone $\check{K} = \{-x : x \in K\}$, and $X = X + \mathbb{R}_-^d$ if $X \notin \check{K}$. In this case no transactions are allowed in the nonsolvent case and otherwise all transactions should maintain the solvency of the portfolio.

2.4. Set-Valued Risk Measure

The following definition is adapted from Hamel and Heyde (2010), Hamel et al. (2011), and Kulikov (2008), where it appears in the exchange cones setting.

DEFINITION 2.9. A function $\rho(X)$ defined on p -integrable set-valued portfolios is called a *set-valued coherent risk measure* if it takes values being upper convex sets and satisfies the following conditions.

- (1) $\rho(X + a) = \rho(X) - a$ for all $a \in \mathbb{R}^d$ (cash invariance).
- (2) If $X \subset Y$ a.s., then $\rho(X) \subset \rho(Y)$ (monotonicity).
- (3) $\rho(cX) = c\rho(X)$ for all $c > 0$ (homogeneity).
- (4) $\rho(X + Y) \supset \rho(X) + \rho(Y)$ (subadditivity).

The risk measure ρ is said to be *closed-valued* if its values are closed sets. Furthermore, ρ is said to be a *convex* set-valued risk measure if the homogeneity and subadditivity conditions are replaced by

$$\rho(tX + (1 - t)Y) \supset t\rho(X) + (1 - t)\rho(Y).$$

The names for the subadditivity and convexity properties are justified by the fact that sets can be ordered by the reverse inclusion; we follow Hamel and Heyde (2010), Hamel et al. (2011) in this respect. Definition 2.9 appears in Kulikov (2008) and Hamel and Heyde (2010), Hamel et al. (2011), with the argument of ρ being a random vector X and for a fixed exchange cone K , which in our formulation means that the argument of ρ is the random set $X + K$.

The set-valued portfolio X is *acceptable* if $0 \in \rho(X)$. The subadditivity of ρ means that the acceptability of X and Y entails the acceptability of $X + Y$, as in the classical case of coherent risk measures. In the univariate case, $X = (-\infty, X]$ and $\rho(X) = [r(X), \infty)$ for a coherent risk measure r , so that X is acceptable if and only if $r(X) \leq 0$. If $0 \in X$ a.s., then X is acceptable under any closed-valued coherent risk measure. Indeed, then $\rho(X) \supset \rho(\mathbb{R}_-^d)$, while $\rho(\mathbb{R}_-^d)$ contains the origin by the homogeneity and subadditivity properties and the closedness of the values for ρ . For a noncoherent ρ , it is sensible to extra impose the normalization condition $\rho(K) = \check{K}$ for each deterministic exchange cone K .

EXAMPLE 2.3. (cont.) (Capital requirements in the exchange cones setting). The value of the risk measure $\rho(X)$ determines capital amounts $a \in \mathbb{R}^d$ that make $X + a$ acceptable. The necessary capital should be allocated at time zero, when the exchange rules are determined by a nonrandom exchange cone K_0 . Thus, the initial capital x should be chosen so that $x + K_0$ intersects $\rho(X + K)$, and

$$A_0 = \rho(X + K) + \check{K}_0$$

is the family of all possible initial capital requirements. Optimal capital requirements are given by the extremal points from A_0 in the order generated by the cone K_0 . If $K = K_0$ is not random, $A_0 = \rho(X + K_0)$. If K_0 is a half-space, meaning that the initial exchanges are free from transaction costs, then A_0 is a half-space too. In this case, the sensible initial capital is given by the tangent point to $\rho(X + K)$ in direction of the normal to K_0 , see Example 1.1

If A_0 is the whole space, which might be the case, for instance, if \mathbf{K} and K_0 are two different half-spaces, then it is possible to release an infinite capital from the position, and this situation should be excluded for the modeling purposes.

3. SELECTION RISK MEASURE FOR SET-VALUED PORTFOLIOS

3.1. Acceptability of Set-Valued Portfolios

Below we explicitly construct set-valued risk measures based on selections of \mathbf{X} . Let r_1, \dots, r_d be law invariant coherent risk measures defined on the space $L^p(\mathbb{R})$ with values in $\mathbb{R} \cup \{\infty\}$. Furthermore, assume that each r_i satisfies the Fatou property, which for $p = \infty$ follows from the law invariance (Jouini, Schachermayer, and Touzi 2006) and for $p \in [1, \infty)$ is always the case if r_i takes only finite values, see Kaina and Rüschendorf (2009, th. 3.1). For a random vector $\xi = (\xi_1, \dots, \xi_d) \in L^p(\mathbb{R}^d)$ write

$$\mathbf{r}(\xi) = (r_1(\xi_1), \dots, r_d(\xi_d)).$$

Random vector ξ is said to be *acceptable* if $\mathbf{r}(\xi) \leq 0$, i.e., $r_i(\xi_i) \leq 0$ for all $i = 1, \dots, d$. This is exactly the case if portfolio $\mathbf{X} = \xi + \mathbb{R}_-^d$ is acceptable with respect to the set-valued measure

$$\rho(\mathbf{X}) = \mathbf{r}(\mathbf{X}) + \mathbb{R}_+^d = \times_{i=1}^d [r_i(X_i), \infty).$$

It is a special case of the *regulator risk measure* considered in Hamel et al. (2013). The following definition suggests a possible acceptability criterion for set-valued portfolios that leads to a risk measure satisfying the axioms from Definition 2.9.

DEFINITION 3.1. A p -integrable set-valued portfolio \mathbf{X} is said to be *selection acceptable* (in the following simply called *acceptable*) if $\mathbf{r}(\xi) \leq 0$ for at least one selection $\xi \in L^p(\mathbf{X})$.

The monotonicity property of univariate risk measures r_1, \dots, r_d implies that \mathbf{X} is acceptable if and only if its efficient part $\partial^+ \mathbf{X}$ admits an acceptable selection.

EXAMPLE 2.3 (cont.) The acceptability of $\mathbf{X} = X + \mathbf{K}$ means that it is possible to transfer the assets given by the components of X according to the exchange rules determined by \mathbf{K} , so that the resulting random vector $X + \eta$ with $\eta \in L^p(\mathbf{K})$ has all acceptable components.

REMARK 3.2 (Generalizations). The acceptability of selections can be judged using any other multivariate coherent risk measure, e.g., considered in Hamel et al. (2011), that are not necessarily of point plus cone type, or numerical multivariate risk measures from Burgert and Rüschendorf (2006), Ekeland et al. (2012), and Farkas et al. (2014). Furthermore, it is possible to consider acceptability of a general convex subset of $L^p(\mathbb{R}^d)$ that might not be interpreted as the family of selections of a set-valued portfolio. This may be of advantage in the dynamic setting (Feinstein and Rudloff 2014a) or for considering uncertainty models as in Bion-Nadal and Kervarec (2012).

3.2. Coherency of Selection Risk Measure

DEFINITION 3.3. The *selection risk measure* of X is defined as the set of deterministic portfolios x that make $X + x$ acceptable, i.e.,

$$(3.1) \quad \rho_{s,0}(X) = \{x \in \mathbb{R}^d : X + x \text{ is acceptable}\}.$$

Its closed-valued variant is $\rho_s(X) = \text{cl}\rho_{s,0}(X)$.

THEOREM 3.4. The selection risk measure $\rho_{s,0}$ defined by (3.1) and its closed-valued variant ρ_s are law invariant set-valued coherent risk measures, and

$$(3.2) \quad \rho_{s,0}(X) = \bigcup_{\xi \in L^p(X)} (\mathbf{r}(\xi) + \mathbb{R}_+^d).$$

Proof. We show first that $\rho_{s,0}(X)$ is an upper set. Let $x \in \rho_{s,0}(X)$ and $y \geq x$. If $\xi + x$ is acceptable for some $\xi \in L^p(X)$, then also $\xi + y$ is acceptable because of the monotonicity of the components of \mathbf{r} . Hence, $y \in \rho_{s,0}(X)$. If $x \in \rho_s(X)$ and $y \geq x$, then $x_n \rightarrow x$ for a sequence $x_n \in \rho_{s,0}(X)$. Consider any $y' > y$, then $x_n \leq y'$ for sufficiently large n , so that $y' \in \rho_{s,0}(X)$. Letting y' decrease to y yields that $y \in \rho_{s,0}(X)$, so that $\rho_s(X)$ is an upper set.

In order to confirm the convexity of $\rho_{s,0}(X)$, assume that $x, y \in \rho_{s,0}(X)$ with $\mathbf{r}(\xi + x) \leq 0$ and $\mathbf{r}(\eta + y) \leq 0$ and take any $\lambda \in (0, 1)$. The subadditivity of components of \mathbf{r} implies that

$$\begin{aligned} \mathbf{r}(\lambda\xi + (1-\lambda)\eta + \lambda x + (1-\lambda)y) &= \mathbf{r}(\lambda(\xi + x) + (1-\lambda)(\eta + y)) \\ &\leq \lambda\mathbf{r}(\xi + x) + (1-\lambda)\mathbf{r}(\eta + y) \leq 0. \end{aligned}$$

It remains to note that $\lambda\xi + (1-\lambda)\eta$ is a selection of X in view of the convexity of X . Then, $\rho_s(X)$ is convex as the closure of a convex set.

The law invariance property is not immediate, since identically distributed random closed sets might have rather different families of selections, see Molchanov (2005, p. 32). Denote by \mathfrak{F}_X the σ -algebra generated by the random closed set X , see Molchanov (2005, def. 1.2.4). If X is acceptable, then $\mathbf{r}(\xi) \leq 0$ for some $\xi \in L^p(X)$. The dilatation monotonicity of law invariant numerical coherent risk measures on a nonatomic probability space (see Cherny and Grigoriev 2007) implies that

$$\mathbf{r}(\mathbf{E}(\xi|\mathfrak{F}_X)) \leq \mathbf{r}(\xi) \leq 0.$$

Therefore, the conditional expectation $\eta = \mathbf{E}(\xi|\mathfrak{F}_X)$ is also acceptable. The convexity of X implies that η is a p -integrable \mathfrak{F}_X -measurable selection of X . Therefore, X is acceptable if and only if it has an acceptable \mathfrak{F}_X -measurable selection. It remains to note that two identically distributed random sets have the same families of selections which are measurable with respect to the minimal σ -algebras generated by these sets, see Molchanov (2005, prop. 1.2.18). In particular, the intersections of these families with $L^p(\mathbb{R}^d)$ are identical. Thus, $\rho_{s,0}$ and ρ_s are law invariant.

Representation (3.2) follows from the fact that $\rho_{s,0}(X)$ is the union of $\{x : \mathbf{r}(\xi + x) \leq 0\} = \mathbf{r}(\xi) + \mathbb{R}_+^d$ for $\xi \in L^p(X)$.

The first two properties of coherent risk measures follow directly from the definition of $\rho_{s,0}$. The homogeneity and subadditivity follow from the fact that all acceptable random sets build a cone. Indeed, if $\xi \in L^p(X)$ is acceptable, then $c\xi$ is an acceptable selection

of cX and so cX is acceptable. If $\xi \in L^p(X)$ and $\eta \in L^p(Y)$ are acceptable, then $\xi + \eta$ is acceptable because the components of \mathbf{r} are coherent risk measures and so $L^p(X + Y)$ contains an acceptable random vector. Thus, $\rho_{s,0}(X + Y) \supset \rho_{s,0}(X) + \rho_{s,0}(Y)$ and by passing to the closure we arrive at the subadditivity property of ρ_s . \square

Representation (3.2) was used in Hamel et al. (2013) to define the *market extension* of a regulator risk measure in the conical models setting.

REMARK 3.5 (Eligible portfolios). In order to simplify the presentation it is assumed throughout that all portfolios $a \in \mathbb{R}^d$ can be used to offset the risk. Following the setting of Hamel and Heyde (2010), Hamel et al. (2011), Hamel et al. (2013), it is possible to assume that the set of eligible portfolios is a proper linear subspace M of \mathbb{R}^d . The corresponding set-valued risk measure is $\rho_s(X) \cap M$, which equals the union of $(\mathbf{r}(\xi) + \mathbb{R}_+^d) \cap M$ for all $\xi \in L^p(X)$.

3.3. Properties of Selection Risk Measures

Conditions for closedness of set-valued risk measures in the exchange cone setting were obtained in Feinstein and Rudloff (2014b). The following result establishes the closedness of $\rho_{s,0}(X)$ for portfolios with p -integrably bounded essential part. Further results concerning closedness of $\rho_{s,0}$ are presented in Corollaries 7.6 and 7.8.

THEOREM 3.6. *If $\partial^+ X$ is p -integrably bounded, then the selection risk measure $\rho_{s,0}(X)$ is closed.*

Proof. Let $\xi_n + x_n$ be acceptable and $x_n \rightarrow x$. Note that $L^p(\partial^+ X)$ is weak compact for $p = 1$ by Molchanov (2005, th. 2.1.19) and for $p \in (1, \infty)$ by its boundedness in view of the reflexivity of $L^p(\mathbb{R}^d)$. By passing to subsequences we can assume that ξ_n weakly converges to ξ in $L^p(\mathbb{R}^d)$. The dual representation of coherent risk measures yields that for a random variable $\eta_n \in L^p(\mathbb{R})$, $r_i(\eta_n)$ equals the supremum of $\mathbf{E}(\eta_n \zeta)$ over a family of random variables ζ in $L^q(\mathbb{R})$. If η_n weakly converges to η in $L^p(\mathbb{R})$, then $\mathbf{E}(\eta_n \zeta) \rightarrow \mathbf{E}(\eta \zeta)$, so that $r_i(\eta) \leq \liminf r_i(\eta_n)$. Applying this argument to the components of ξ_n we obtain that

$$(3.3) \quad \mathbf{r}(\xi + x) \leq \liminf \mathbf{r}(\xi_n + x_n) \leq 0,$$

whence $x \in \rho_{s,0}(\partial^+ X)$. If $p = \infty$, then $\{\xi_n\}$ are uniformly bounded and so have a subsequence that converges in distribution and so can be realized on the same probability space as an almost surely convergent sequence. The Fatou property of the components of \mathbf{r} yields (3.3). \square

THEOREM 3.7 (Lipschitz property). *Assume that all components of \mathbf{r} take finite values on $L^p(\mathbb{R})$. Then there exists a constant $C > 0$ such that $\mathfrak{d}_H(\rho_s(X), \rho_s(Y)) \leq C \|\mathfrak{d}_H(X, Y)\|_p$ for all p -integrable set-valued portfolios X and Y .*

Proof. For each $\xi \in L^p(X)$ and $\varepsilon > \|\mathfrak{d}_H(X, Y)\|_p$, there exists $\eta \in L^p(Y)$ such that $\|\xi - \eta\|_p \leq \varepsilon$. The Lipschitz property of L^p -risk measures (see Föllmer and Schied 2004, lemma 4.3 for $p = \infty$ and Kaina and Rüschendorf 2009 for $p \in [1, \infty)$) implies that $\|\mathbf{r}(\xi) - \mathbf{r}(\eta)\| \leq C\varepsilon$ for a constant C . By (3.2), the Hausdorff distance between $\rho_s(X)$ and $\rho_s(Y)$ is bounded by $C\varepsilon$. \square

EXAMPLE 3.8 (Selection expectation). If $\mathbf{r}(\xi) = \mathbf{E}(-\xi)$ is the expectation of $-\xi$, then

$$\rho_s(X) = \mathbf{E}\check{X} + \mathbb{R}_+^d,$$

where $\mathbf{E}\check{X}$ is the *selection expectation* of \check{X} , i.e., the closure of the set of expectations $\mathbf{E}\xi$ for all integrable selections $\xi \in L^1(X)$, see Molchanov (2005, sec. 2.1). Thus, the selection risk measure yields a subadditive generalization of the selection expectation.

EXAMPLE 3.9. Assume that $p = \infty$ and all components of \mathbf{r} are given by $r_i(\xi_i) = -\text{essinf } \xi_i$. Then X is acceptable if and only if $X \cap \mathbb{R}_+^d$ is almost surely nonempty, and $\rho_s(X)$ is the set of points that belong to \check{X} with probability one.

EXAMPLE 3.10. If $X = K$ is an exchange cone, then $\rho_s(K)$ is a deterministic convex cone that contains \mathbb{R}_+^d . If $K = K$ is deterministic, then $\rho_s(K) = \check{K}$.

REMARK 3.11. Selection risk measures have a number of further properties.

- I. Assume that $\mathbf{r} = (r, \dots, r)$ has all identical components. If X is acceptable, then its orthogonal projection on the linear subspace \mathbb{H} of \mathbb{R}^d generated by any $u_1, \dots, u_k \in \mathbb{R}_+^d$ is also acceptable, and so the risk of the projected X contains the projection of $\rho_s(X)$.
- II. The conditional expectation of random set X with respect to a σ -algebra \mathfrak{B} is defined as the closure of the set of conditional expectations for all its integrable selections, see Molchanov (2005, sec. 2.1.6). The dilatation monotonicity property of components of \mathbf{r} implies that if ξ is acceptable, then $\mathbf{E}(\xi|\mathfrak{B})$ is acceptable. Therefore, ρ_s is also dilatation monotone meaning that

$$\rho_s(\mathbf{E}(X|\mathfrak{B})) \supset \rho_s(X).$$

In particular, $\rho_s(X) \subset \rho_s(\mathbf{E}X) = \mathbf{E}\check{X}$. Therefore, the integrability of the support function $h_X(u)$ for at least one u provides an easy condition that guarantees that $\rho_s(X)$ is not equal to the whole space.

REMARK 3.12. In the setting of Example 2.2, $\rho_s(X + K)$ written as a function of X only becomes a centrally symmetric variant of the coherent utility function considered in Kulikov (2008, def. 2.1). It should be noted that the utility function from Kulikov (2008) and risk measures from Hamel et al. (2011) depend on both X and K and on the dependency structure between them and so are not law invariant as function of X only, if K is random.

If the components of \mathbf{r} are *convex* risk measures, so that the homogeneity assumption is dropped, then $\rho_s(X)$ is a convex set-valued risk measure, which is not necessarily homogeneous. If the components of \mathbf{r} are not law invariant, then ρ_s is a possibly not law invariant set-valued risk measure. The following two examples mention *nonconvex* risk measures, which are also defined using selections.

EXAMPLE 3.13. Assume that the components of \mathbf{r} are general cash invariant risk measures without imposing any convexity properties, e.g., values-at-risk at the level α , bearing in mind that the resulting selection risk measure is no longer coherent and not necessarily law invariant. Then X is acceptable if and only if there exists a selection ξ of X such that $\mathbf{P}\{\xi_i \geq 0\} \geq \alpha$ for all i .

EXAMPLE 3.14. Let K_0 be a deterministic exchange cone and fix some acceptance level α . Call random vector ξ acceptable if $\mathbf{P}\{\xi \in \check{K}_0\} \geq \alpha$ and note that this condition differs from requiring that $\mathbf{P}\{\xi_i \geq 0\} \geq \alpha$ for all i . Then a set-valued portfolio X is acceptable if and only if $\mathbf{P}\{X \cap \check{K}_0 \neq \emptyset\} \geq \alpha$. If $X = X + \mathbb{R}_-^d$ and $K_0 = \mathbb{R}^d$, then $\{x : \mathbf{P}\{X \geq -x\} \geq \alpha\}$ is sometimes termed a multivariate quantile or the value-at-risk of X , see Embrechts and Puccetti (2006) and Hamel and Heyde (2010).

4. BOUNDS FOR SELECTION RISK MEASURES

4.1. Upper Bound

The family of selections of a random set is typically very rich. An *upper bound* for $\rho_s(X)$ can be obtained by restricting the choice of possible selections. The convexity property of $\rho_s(X)$ implies that it contains the convex hull of the union of $\mathbf{r}(\xi) + \mathbb{R}_+^d$ for the chosen selections ξ . This convex hull corresponds to the case of a higher risk than $\rho_s(X)$. Making the upper bound tighter by considering a larger family of selections is in the interest of the agent in order to reduce the capital requirements.

At first, it is possible to consider deterministic selections, also called fixed points of X , i.e., the points which belong to X with probability one. If $a \in X$ a.s., then $\rho_s(X) \supset -a + \mathbb{R}_+^d$. However, this set of fixed points is typically rather poor to reflect essential features related to the variability of X .

Another possibility would be to consider selections of X of the form $\xi + a$ for a fixed random vector ξ and a deterministic a . If $X \supset \xi + M$ for a deterministic set M (which always can be chosen to be convex in view of the convexity of X), then

$$(4.1) \quad \rho_s(X) \supset \mathbf{r}(\xi) + \mathbb{R}_+^d + \check{M}.$$

It is possible to tighten the bound by taking the convex hull for the union of the right-hand side for several ξ . The inclusion in (4.1) can be strict even if $X = \xi + M$, since taking *random* selections of M makes it possible to offset the risks as the following example shows.

EXAMPLE 4.1. Let $X = \xi + M$, where M is the unit ball and $X = (X_1, X_2)$ is the standard bivariate normal vector. Consider the risk measure \mathbf{r} with two identical components being ESs at level 0.05. Then $\mathbf{r}(X) + M$ is the upper set generated by the ball of radius one centered at $\mathbf{r}(X) = (2.063, 2.063)$. Consider the selection of M given by $\xi = (\mathbf{1}_{X_1 < X_2}, \mathbf{1}_{X_1 > X_2})$. By numerical calculation of the risks, it is easily seen that $\mathbf{r}(X + \xi) = (1.22, 1.22)$, which does not belong to $\mathbf{r}(X) + M$.

4.2. Lower Bound

Below we describe a *lower bound* for $\rho_s(X)$, which is a superset of $\rho_s(X)$ and is also a set-valued coherent risk measure itself. For $x, y \in \mathbb{R}^d$, xy (resp. x/y) denote the vectors composed of pairwise products (resp. ratios) of the coordinates of x and y . If M is a set in \mathbb{R}^d , then $M\dot{y} = \{xy : x \in M\}$. By agreement, let $\{0\}/0 = \mathbb{R}$ and $0/0 = -\infty$.

Let $\mathcal{Z} \subset L^q(\mathbb{R}^d)$ be a nonempty family of nonnegative q -integrable random vectors $Z = (\zeta_1, \dots, \zeta_d)$ in \mathbb{R}^d . Recall that $\mathbf{E}(\check{X}Z)$ denotes the selection expectation of \check{X} with the coordinates scaled according to the components of Z , see Example 3.8. It exists, since X is assumed to possess at least one p -integrable selection and so $L^1(\check{X}Z) \neq \emptyset$. Define the set-valued risk measure

$$(4.2) \quad \rho_{\mathcal{Z}}(X) = \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}(\check{X}Z)}{\mathbf{E}Z},$$

which is similar to the classical dual representation of coherent risk measures, see Delbaen (2002) and also corresponds to the (Q, w) -representation of risk measure in conical models from Hamel et al. (2011, th. 4.2) that is similar to (4.3) from the following theorem.

THEOREM 4.2. Assume that \mathcal{Z} is a nonempty family of nonnegative q -integrable random vectors. The functional $\rho_{\mathcal{Z}}(X)$ is a closed-valued coherent risk measure, and

$$(4.3) \quad \rho_{\mathcal{Z}}(X) = \bigcap_{Z \in \mathcal{Z}, u \in \mathbb{R}_+^d} \{x : \mathbf{E}\langle x, uZ \rangle \geq -\mathbf{E}h_X(uZ)\}.$$

Proof. The closedness and convexity of $\rho_{\mathcal{Z}}(X)$ follow from the fact that it is intersection of half-spaces; it is an upper set since the normals to these half-spaces belong to \mathbb{R}_+^d . It is evident that $\rho_{\mathcal{Z}}(X)$ is monotonic, cash invariant, and homogeneous. In order to check the subadditivity, note that

$$\begin{aligned} \rho_{\mathcal{Z}}(X + Y) &= \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}[(\check{X} + \check{Y})Z]}{\mathbf{E}Z} \supset \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}(\check{X}Z)}{\mathbf{E}Z} + \frac{\mathbf{E}(\check{Y}Z)}{\mathbf{E}Z} \\ &\supset \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}(\check{X}Z)}{\mathbf{E}Z} + \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}(\check{Y}Z)}{\mathbf{E}Z}. \end{aligned}$$

Recall that the support function of the expectation of a set equals the expected value of the support function. Since $\mathbf{E}(\check{X}Z)$ is an upper set,

$$\begin{aligned} \rho_{\mathcal{Z}}(X) &= \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}(\check{X}Z)}{\mathbf{E}Z} = \bigcap_{Z \in \mathcal{Z}} \bigcap_{u \in \mathbb{R}_+^d} \{x : \mathbf{E}h_{\check{X}Z}(u) \geq \langle x, u\mathbf{E}Z \rangle\} \\ &= \bigcap_{Z \in \mathcal{Z}} \bigcap_{u \in \mathbb{R}_+^d} \{x : \mathbf{E}h_X(-Zu) \geq \mathbf{E}\langle x, uZ \rangle\}, \end{aligned}$$

and we arrive at (4.3) by replacing u with $-u$. \square

The components of $\mathbf{r} = (r_1, \dots, r_d)$ admit the dual representations

$$(4.4) \quad r_i(\xi_i) = \sup_{\zeta_i \in \mathcal{Z}_i} \frac{\mathbf{E}(-\xi_i \zeta_i)}{\mathbf{E}\zeta_i}, \quad i = 1, \dots, d,$$

where, for each $i = 1, \dots, d$, \mathcal{Z}_i is the dual cone to the family of random variables $\xi \in L^p(\mathbb{R})$ such that $r_i(\xi) \leq 0$, i.e., \mathcal{Z}_i consists of nonnegative q -integrable random variables ζ such that $\mathbf{E}(\zeta \xi) \geq 0$ for all ξ with $r_i(\xi) \leq 0$, see Delbaen (2002) and Föllmer and Schied (2004). Note that $\mathcal{Z}_1, \dots, \mathcal{Z}_d$ are the maximal families that provide the dual representation (4.4). Despite \mathcal{Z}_i contains a.s. vanishing random variables, letting $0/0 = -\infty$ ensures the validity of (4.4).

THEOREM 4.3. Assume that the components of $\mathbf{r} = (r_1, \dots, r_d)$ admit the dual representations (4.4). Then $\rho_s(X) \subset \rho_{\mathcal{Z}}(X)$ for any family \mathcal{Z} of q -integrable random vectors $Z = (\zeta_1, \dots, \zeta_d)$ such that $\zeta_i \in \mathcal{Z}_i$, $i = 1, \dots, d$.

Proof. In view of (4.4),

$$[r_i(\xi_i), \infty) = \bigcap_{\zeta_i \in \mathcal{Z}_i} \left[\frac{\mathbf{E}(-\xi_i \zeta_i)}{\mathbf{E}\zeta_i}, \infty \right),$$

so that

$$\mathbf{r}(\xi) + \mathbb{R}_+^d \subset \bigcap_{Z \in \mathcal{Z}} \left(\frac{\mathbf{E}(-\xi Z)}{\mathbf{E}Z} + \mathbb{R}_+^d \right).$$

By (3.2),

$$\begin{aligned}\rho_s(X) &\subset \text{cl} \bigcup_{\xi \in L^p(X)} \bigcap_{Z \in \mathcal{Z}} \left(\frac{\mathbf{E}(-\xi Z)}{\mathbf{E}Z} + \mathbb{R}_+^d \right) \\ &\subset \text{cl} \bigcap_{Z \in \mathcal{Z}} \bigcup_{\xi \in L^p(X)} \left(\frac{\mathbf{E}(-\xi Z)}{\mathbf{E}Z} + \mathbb{R}_+^d \right) \subset \bigcap_{Z \in \mathcal{Z}} \frac{\mathbf{E}(\check{X}Z)}{\mathbf{E}Z}.\end{aligned}$$

Note that for each $\xi \in L^p(X)$ and $a \in \mathbb{R}_+^d$, the random vector $(-\xi + a)Z$ is an integrable selection of $\check{X}Z$. The closure is omitted, since the selection expectation is already closed by definition. \square

COROLLARY 4.4. *The selection risk measure $\rho_s(X)$ is not equal to the whole space if $h_X(Zu)$ is integrable for some $u \in \mathbb{R}_+^d$ and $Z = (\zeta_1, \dots, \zeta_d)$ with $\zeta_i \in \mathcal{Z}_i$, $i = 1, \dots, d$.*

It should be noted that the acceptability of X under the risk measure $\rho_{\mathcal{Z}}(X)$ does not necessarily imply the existence of a selection with all acceptable marginals.

REMARK 4.5. The risk measure $\rho_{\mathcal{Z}}$ is not law invariant in general. It is possible to construct a law invariant (and also tighter) lower bound for the selection risk measure by extending \mathcal{Z} to $\tilde{\mathcal{Z}}$, so that, with each Z , the family $\tilde{\mathcal{Z}}$ contains all random vectors \tilde{Z} that share the distribution with Z .

PROPOSITION 4.6. *Let all components of $\mathbf{r} = (r, \dots, r)$ be identical univariate risk measures whose dual representation (4.4) involves the same family $\mathcal{Z} = \mathcal{Z}_i$ of a.s. nonnegative random variables. Consider the family \mathcal{Z}_0 that consists of all $Z = (\zeta, \dots, \zeta)/\mathbf{E}\zeta$ for all $\zeta \in \mathcal{Z}$. Then*

$$(4.5) \quad \rho_{\mathcal{Z}_0}(X) = \bigcap_{u \in \mathbb{R}_+^d} \{x : \langle x, u \rangle \geq r(h_X(u))\},$$

where $r(h_X(u)) = -\infty$ if $h_X(u) = \infty$ with positive probability.

Proof. By (4.3),

$$\begin{aligned}\rho_{\mathcal{Z}_0}(X) &= \bigcap_{\zeta \in \mathcal{Z}, u \in \mathbb{R}_+^d} \{x : \langle x, u \rangle \mathbf{E}\zeta \geq -\mathbf{E}(h_X(u)\zeta)\} \\ &= \bigcap_{u \in \mathbb{R}_+^d} \left\{ x : \langle x, u \rangle \geq \sup_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(-h_X(u)\zeta)}{\mathbf{E}\zeta} \right\},\end{aligned}$$

so it remains to identify the supremum as the dual representation for $r(h_X(u))$. \square

The bounds for selection measures for set-valued portfolios determined by exchange cones are considered in the subsequent sections. Below we illustrate Proposition 4.6 on two examples of quasi-bounded portfolios.

EXAMPLE 4.7. Consider portfolio X with $\partial^+ X$ being the segment in the plane with end-points $X^{(1)}$ and $X^{(2)}$. Then

$$\rho_{\mathcal{Z}_0}(X) = \bigcap_{u \in \mathbb{R}_+^2} \{x : \langle x, u \rangle \geq r(\max(\langle X^{(1)}, u \rangle, \langle X^{(2)}, u \rangle))\}.$$

EXAMPLE 4.8. Consider portfolio X from Example 2.7. If all components of \mathbf{r} are identical, then

$$\rho_{\mathcal{Z}_0}(X) = \bigcap_{u \in \mathbb{R}_+^d} \{x : \langle x, u \rangle \geq r((X, u)) - R\|u\|\}.$$

5. CONICAL MARKET MODELS

5.1. Random Exchange Cones

Let $X = X + \mathbf{K}$ for $X \in L^p(\mathbb{R}^d)$ and a (possibly random) exchange cone \mathbf{K} , see Example 2.2. Then

$$(5.1) \quad \mathbf{r}(X) + \rho_s(\mathbf{K}) \subset \rho_s(X + \mathbf{K}) \subset \rho_s(\mathbf{E}(X|\mathbf{K}) + \mathbf{K}),$$

where the first inclusion relation is due to the subadditivity of $\rho_s((X + \mathbb{R}_+^d) + \mathbf{K})$ and the second one follows from the dilatation monotonicity of law invariant risk measures. If X and \mathbf{K} are independent, the lower bound becomes $-\mathbf{E}X + \rho_s(\mathbf{K})$.

Since $\mathbf{E}h_X(uZ)$ is infinite unless uZ almost surely belongs to the dual cone \mathbf{K}' and $h_X(uZ) = \langle X, uZ \rangle$ for $uZ \in \mathbf{K}'$, the lower bound (4.3) turns into

$$(5.2) \quad \rho_{\mathcal{Z}}(X + \mathbf{K}) = \bigcap_{u \in \mathbb{R}_+^d, Z \in \mathcal{Z}, uZ \in \mathbf{K}' \text{ a.s.}} \{x : \mathbf{E}\langle x, uZ \rangle \geq -\mathbf{E}\langle X, uZ \rangle\}.$$

The right-hand side of (5.2) corresponds to the dual representation for set-valued risk measures from Kulikov (2008) and Hamel et al. (2011), where it is written as function of X only.

Consider now the frictionless case, where \mathbf{K} is a random half-space, so that $\mathbf{K}' = \{tv : t \geq 0\}$ for a random direction $v \in \mathbb{R}_+^d$. Then

$$\rho_{\mathcal{Z}}(X + \mathbf{K}) = \bigcap_{\zeta \in \mathcal{Z}} \{x : \mathbf{E}(\langle x, v \rangle \zeta) \geq -\mathbf{E}(\langle X, v \rangle \zeta)\},$$

where \mathcal{Z} is a family of nonnegative q -integrable random variables ζ such that $v_i \zeta \in \mathcal{Z}_i$ for all $i = 1, \dots, d$.

EXAMPLE 5.1 (Bivariate frictionless random exchanges). Consider two currencies exchangeable at random rate $\pi = \pi^{(21)}$ without transaction costs, see Example 2.3, so that the exchange cone \mathbf{K} is the half-plane with normal $(\pi, 1)$. Assume that $\mathbf{r} = (r, r)$ for two identical L^p -risk measures. The selection risk measure of $X = X + \mathbf{K}$ is the closure of the set of $(r(X_1 + \eta), r(X_2 - \eta\pi))$ for $\eta, \pi\eta \in L^p(\mathbb{R})$, see Example 8.3 for a numerical illustration.

The value of $\rho_s(\mathbf{K})$ is useful to bound the selection risk measure of $X = X + \mathbf{K}$, see (5.1). Assume that both π and π^{-1} are p -integrable. For the purpose of computation of $\rho_s(\mathbf{K})$ it suffices to consider selections of the form $(\eta, -\eta\pi)$, where $\eta, \eta\pi \in L^p(\mathbb{R})$. Furthermore, it suffices to consider separately almost surely positive and almost surely negative η . Then $\rho_s(\mathbf{K})$ is a cone in \mathbb{R}^2 bounded by the two half-lines with slopes

$$\gamma_1 = \sup_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \frac{r(-\eta\pi)}{r(\eta)}, \quad \gamma_2 = \inf_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \frac{r(\eta\pi)}{r(-\eta)}.$$

The canonical choices $\eta = 1$ and $\eta = 1/\pi$ yield that

$$\frac{1}{r(1/\pi)} = \max \left(-r(-\pi), \frac{1}{r(1/\pi)} \right) \leq \gamma_1 \leq \gamma_2 \leq \min \left(r(\pi), \frac{-1}{r(-1/\pi)} \right) = r(\pi),$$

where the maximal and minimal elements above are obtained after applying the Jensen inequality to the dual representation of a univariate risk measure (observe that $f(x) = 1/x$ is convex on the positive half-line). Therefore, $\rho_s(\mathbf{K})$ contains the cone with slopes given by $(r(1/\pi))^{-1}$ and $r(\pi)$.

A lower bound for $\rho_s(\mathbf{K})$ relies on a lower bound for γ_2 and an upper bound for γ_1 . Using the dual representation of r as $r(\xi) = \sup_{\zeta \in \mathcal{Z}} \mathbf{E}(-\zeta\xi)/\mathbf{E}\zeta$ for a family $\mathcal{Z} \subset L^q(\mathbb{R}_+)$ of random variables with positive expectation, for (positive) η, π we have

$$\frac{r(\eta\pi)}{r(-\eta)} = \frac{\sup_{\zeta_1 \in \mathcal{Z}} \mathbf{E}(-\eta\pi\zeta_1/\mathbf{E}\zeta_1)}{\sup_{\zeta_2 \in \mathcal{Z}} \mathbf{E}(\eta\zeta_2/\mathbf{E}\zeta_2)} = - \inf_{\zeta_1, \zeta_2 \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta_1/\mathbf{E}\zeta_1)}{\mathbf{E}(\eta\zeta_2/\mathbf{E}\zeta_2)} \geq - \inf_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)},$$

where the inequality follows from setting $\zeta_1 = \zeta_2$. Then

$$\gamma_2 \geq \inf_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \left\{ - \inf_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)} \right\} \geq - \inf_{\zeta \in \mathcal{Z}} \sup_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)}.$$

If r is the ES_α , so that \mathcal{Z} consists of all random variables taking value 1 with probability α and 0 otherwise, and if π has continuous distribution with a convex support, then

$$\begin{aligned} \gamma_2 &\geq - \inf_{\zeta \in \mathcal{Z}} \sup_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \frac{\mathbf{E}(\pi\zeta)(\eta\zeta)}{\mathbf{E}(\eta\zeta)} \\ &\geq - \inf_{\zeta \in \mathcal{Z}} \sup_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \frac{\mathbf{E}(\pi\zeta\eta)}{\mathbf{E}(\eta)} = - \inf_{\zeta \in \mathcal{Z}} \text{esssup}(\pi\zeta) = \text{VaR}_\alpha(\pi), \end{aligned}$$

where VaR_α denotes the value-at-risk. With a similar argument,

$$\gamma_1 \leq - \sup_{\zeta \in \mathcal{Z}} \inf_{\eta, \eta\pi \in L^p(\mathbb{R}_+)} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)} \leq \text{VaR}_{1-\alpha}(\pi),$$

see Figure 8.3 for the corresponding lower and upper bounds. For any $\alpha \leq 1/2$ the upper bound for γ_1 is not greater than the lower bound for γ_2 meaning that $\rho_s(\mathbf{K})$ is distinct from the whole space.

The arguments presented in Example 5.1 and (5.1) yield the following result.

PROPOSITION 5.2. *Let $d = 2$. Consider the selection measure ρ_s generated by \mathbf{r} with all components being ES_α at level $\alpha \leq 1/2$. If $\mathbf{X} \subset \mathbf{X} + \mathbf{K}$, where \mathbf{K} is a half-plane with normal $(\pi, 1)$ such that $\pi, \pi^{-1} \in L^p(\mathbb{R}_+)$, π has convex support, and \mathbf{X} and π are independent, then $\rho_s(\mathbf{X})$ is distinct from the whole plane.*

5.2. Deterministic Exchange Cones

Assume that $\mathbf{X} = \mathbf{X} + \mathbf{K}$ for a deterministic exchange cone \mathbf{K} and a p -integrable random vector \mathbf{X} . Since $\rho_s(\mathbf{K}) = \check{\mathbf{K}}$, (5.1) yields that

$$\mathbf{r}(\mathbf{X}) + \check{\mathbf{K}} \subset \rho_s(\mathbf{X} + \mathbf{K}) \subset -\mathbf{E}\mathbf{X} + \check{\mathbf{K}}.$$

PROPOSITION 5.3. Assume that \mathbf{r} has all identical components being r . Then

$$(5.3) \quad \rho_{\mathcal{Z}_0}(X + K) = \bigcap_{u \in K'} \{x : r(\langle X, u \rangle) \leq \langle x, u \rangle\}.$$

Proof. The result follows from Proposition 4.6 and the fact that $h_{X+K}(u) = \langle X, u \rangle$ if $u \in K'$ and otherwise the support function is infinite. \square

A risk measure of $X + K$ for a deterministic cone K is said to *marginalize* if its values are translates of \check{K} for all $X \in L^p(\mathbb{R}^d)$. The coherency of r implies that the intersection in (5.3) can be taken over all u being extreme elements of K' , which in dimension 2, yields that $\rho_{\mathcal{Z}_0}(X + K)$ is a translate of cone \check{K} . In general dimension, if K is a *Riesz cone* (i.e., \mathbb{R}^d with the order generated by K is a Riesz space), then $\rho_{\mathcal{Z}_0}(X + K) = a + \check{K}$ with $a \in \mathbb{R}^d$ being the supremum of $\mathbf{E}(-X\zeta)/\mathbf{E}\zeta$ in the order generated by K . Similar risk measures were proposed in Cascos and Molchanov (2007, ex. 6.6), where instead of $\{\mathbf{E}(-X\zeta)/\mathbf{E}\zeta : \zeta \in \mathcal{Z}\}$ a depth-trimmed region was considered.

EXAMPLE 5.4. Let all components of \mathbf{r} be univariate ES_α at level α , so that \mathcal{Z} is the family of indicator random variables $\zeta = \mathbf{1}_A$ for all measurable $A \subset \Omega$ with $\mathbf{P}(A) > \alpha$ for some fixed $\alpha \in (0, 1)$. Then $\rho_{\mathcal{Z}_0}(X + K)$ becomes the vector-valued worst conditional expectation (WCE) of X introduced in Jouini et al. (2004, ex. 2.5). Furthermore, (5.3) yields the set-valued ES defined as

$$\text{ES}_\alpha(X + K) = \bigcap_{u \in K'} \{x : \text{ES}_\alpha(\langle X, u \rangle) \leq \langle x, u \rangle\}.$$

Its explicit expression if K is a Riesz cone is given in Cascos and Molchanov (2007, eq. 7.1)—in that case $\text{ES}_\alpha(X + K)$ is a translate of \check{K} . Since the univariate ES and WCE coincide for nonatomic random variables, their multivariate versions coincide by Proposition 4.6 if X has a nonatomic distribution. It should be noted that $\text{ES}_\alpha(X + K)$ is only a lower bound for the selection risk measure defined by applying ES_α to the individual components of selections of $X + K$. In particular, the acceptability of $X + K$ under $\text{ES}_\alpha(X + K)$ does not necessarily imply the existence of an acceptable selection of $X + K$.

The following result deals with the *comonotonic* case, see Föllmer and Schied (2004) for the definition of comonotonicity and comonotonically additive risk measures.

THEOREM 5.5. Assume that \mathbf{r} has all identical comonotonic additive components r . If the components of X are comonotonic and K is a deterministic exchange cone, then $\rho_s(X + K) = \mathbf{r}(X) + \check{K}$.

Proof. Since r is comonotonic additive and X is comonotonic, $r(\langle X, a \rangle) = \langle \mathbf{r}(X), a \rangle$ for any $a \in K' \subset \mathbb{R}_+^d$. Then

$$\mathbf{r}(X) + \check{K} = \bigcap_{a \in K'} \{x : \langle \mathbf{r}(X), a \rangle \leq \langle x, a \rangle\},$$

and consequently $\rho_s(X + K) = \mathbf{r}(X) + \check{K} = \rho_{\mathcal{Z}_0}(X + K)$. \square

Notice that $\mathbf{r}(X)$ is solely determined by the marginal distributions of X . Consequently, if \tilde{X} is a comonotonic rearrangement of X (i.e., a random vector with the same marginal distributions and comonotonic coordinates), then

$$\rho_s(X + K) \supset \mathbf{r}(X) + \check{K} = \mathbf{r}(\tilde{X}) + \check{K} = \rho_s(\tilde{X} + K).$$

The following result shows that $\rho_s(X + K)$ does not change if instead of all selections of $X + K$ one uses only those being deterministic functions of X .

THEOREM 5.6. *Set-valued portfolio $X = X + K$ is acceptable if and only if $\mathbf{r}(X + \eta(X)) \leq 0$ for a selection $\eta(X) \in K$, which is a deterministic function of X .*

Proof. If $\mathbf{r}(X + \eta(X)) \leq 0$, then X is acceptable. For the reverse implication, the dilatation monotonicity of the components of \mathbf{r} yields that

$$\mathbf{r}(X + \mathbf{E}(\eta|X)) \leq \mathbf{r}(X + \eta) \leq 0.$$

It remains to note that the conditional expectation $\mathbf{E}(\eta|X)$ is a function of X taking values from K , since K is a deterministic convex cone. \square

EXAMPLE 5.7 (Frictionless market with deterministic exchange rates). Consider a vector X of terminal gains on d currencies that can be exchanged without transaction costs, so that K is a half-space with normal $u = (1, \pi^{(21)}, \dots, \pi^{(d1)})$, where $\pi^{(j1)}$ is the number of units of currency j needed to buy one unit of currency one, see Example 2.3.

Let $\mathbf{r} = (r_1, \dots, r_d)$ with possibly different components whose maximal dual representations involve families of q -integrable random variables $\mathcal{Z}_1, \dots, \mathcal{Z}_d$ from (4.4). It will be shown later on in Corollaries 7.6 and 7.8 that $\rho_{\mathcal{Z}}(X + K) = \rho_s(X + K)$ for some family \mathcal{Z} satisfying the condition of Theorem 4.3.

Then $Z = (\zeta_1, \dots, \zeta_d) \in K' = \{\lambda u : \lambda \geq 0\}$ if and only if $\zeta_i = u_i \zeta$, $i = 1, \dots, d$, for a random variable ζ from $\mathcal{Z}_* = \cap_i \mathcal{Z}_i$. The family \mathcal{Z}_* determines the coherent risk measure r_* called the convex convolution of r_1, \dots, r_d , see Delbaen (2012). Then X is acceptable under ρ_s if and only if $r_*(\langle X, u \rangle) \leq 0$ and

$$\rho_s(X + K) = \rho_{\mathcal{Z}}(X + K) = \{x : \langle x, u \rangle \geq r_*(\langle X, u \rangle)\}.$$

If all components of X represent the same currency, but the regulator requirements applied to each component are different, e.g., because they represent gains in different countries, then $u = (1, \dots, 1)$ and X is acceptable if and only if $r_*(X_1 + \dots + X_d) \leq 0$.

While the above examples and Theorem 5.5 provide $\rho_s(X + K)$ of the marginalized form $\mathbf{r}(X) + \check{K}$, this is not always the case, as Example 8.1 confirms.

The calculation of ρ_s for $d = 2$ and $p = \infty$ can be facilitated by using the following result. It shows that $\rho_s(X + K)$ coincides with its upper bound $\rho_{\mathcal{Z}_0}(X + K)$ sufficiently far away from the origin.

PROPOSITION 5.8. *Assume that $p = \infty$ and \mathbf{r} has all identical comonotonically additive components r . If $X = X + K$ for a deterministic cone K and an essentially bounded random vector X in dimension $d = 2$, then for all x with sufficiently large norm, $x \in \rho_{\mathcal{Z}_0}(X + K)$ implies that $x \in \rho_s(X + K)$.*

Proof. In dimension 2, we can always assume that the cone K is generated by a bid-ask matrix (see Example 2.3), so that K and its dual are given by

$$K = \{\lambda b_1 + \delta b_2 : \lambda, \delta \geq 0\}, \quad K' = \{\lambda a_1 + \delta a_2 : \lambda, \delta \geq 0\},$$

where

$$\begin{aligned} b_1 &= (1, -\pi^{(21)}), & b_2 &= (-\pi^{(12)}, 1), \\ a_1 &= (\pi^{(21)}, 1), & a_2 &= (1, \pi^{(12)}). \end{aligned}$$

By Proposition 5.3 and by the coherence of r ,

$$(5.4) \quad \rho_{\mathcal{Z}_0}(X + K) = \bigcap_{i=1,2} \{x : r(\langle X, a_i \rangle) \leq \langle x, a_i \rangle\}.$$

In order to obtain a point lying on the boundaries of both $\rho_s(X + K)$ and $\rho_{\mathcal{Z}_0}(X + K)$, consider the positive random variable

$$\zeta = (X_1 - \text{essinf } X_1) / \pi^{(12)}.$$

Then

$$X + \zeta b_2 = (\text{essinf } X_1, X_2 + (X_1 - \text{essinf } X_1) / \pi^{(12)})$$

has the first a.s. deterministic coordinate and so its components are comonotonic. Since ζ is a.s. nonnegative, $X + \zeta b_2$ is a selection of $X + K$. Define

$$x_1 = \mathbf{r}(X + \zeta b_2) = \left(-\text{essinf } X_1, \frac{r(\langle X, a_2 \rangle) + \text{essinf } X_1}{\pi^{(12)}} \right).$$

The fact that $X + \zeta b_2 \in X + K$ guarantees that $x_1 \in \rho_s(X + K) \subset \rho_{\mathcal{Z}_0}(X + K)$. Since $a_2 \in \mathbb{R}_+^2$ and r is comonotonic additive, and the components of $X + \zeta b_2$ are comonotonic,

$$(5.5) \quad \langle x_1, a_2 \rangle = r(\langle X + \zeta b_2, a_2 \rangle) = r(\langle X, a_2 \rangle),$$

where the last equality holds because a_2 and b_2 are orthogonal. Because of (5.4) and (5.5), x_1 lies on the supporting line of $\rho_{\mathcal{Z}_0}(X + K)$ which is normal to a_2 , whence it lies on the boundaries of $\rho_s(X + K)$ and $\rho_{\mathcal{Z}_0}(X + K)$.

By a similar argument,

$$x_2 = \left(\frac{r(\langle X, a_1 \rangle) + \text{essinf } X_2}{\pi^{(21)}}, -\text{essinf } X_2 \right)$$

lies on the supporting line of $\rho_{\mathcal{Z}_0}(X + K)$ which is normal to a_1 , so on the boundaries of $\rho_s(X + K)$ and $\rho_{\mathcal{Z}_0}(X + K)$.

Let b be the radius of a closed ball centered at the origin and containing the triangle with vertices at x_1 , x_2 , and the vertex of cone $\rho_{\mathcal{Z}_0}(X + K)$. If $x \in \rho_{\mathcal{Z}_0}(X + K)$ with $\|x\| \geq b$, then clearly $x \in \lambda x_1 + (1 - \lambda)x_2 + \check{K}$ for some $0 \leq \lambda \leq 1$, which by the convexity of $\rho_s(X + K)$ and the fact that $\rho_s(X + K) = \rho_s(X + K) + \check{K}$ guarantees that $x \in \rho_s(X + K)$. \square

6. NUMERICAL RISK MEASURES FOR SET-VALUED PORTFOLIOS

The set-valued risk measure ρ_s gives rise to several *numerical* coherent risk measures, i.e., functionals $\mathbf{r}(X)$ with values in $\mathbb{R} \cup \{\infty\}$ that satisfy the following properties.

1. There exists $u \in \mathbb{R}^d$ such that $\mathbf{r}(X + a) = \mathbf{r}(X) - \langle a, u \rangle$ for all $a \in \mathbb{R}^d$.

2. If $X \subset Y$, then $r(X) \geq r(Y)$.
3. $r(cX) = cr(X)$ for all $c > 0$.
4. $r(X + Y) \leq r(X) + r(Y)$.

The canonical scalarization construction relies on the support function of $\rho_s(X)$. Namely,

$$r_u(X) = -h_{\rho_s(X)}(-u) = \inf\{\langle u, x \rangle : x \in \rho_s(X)\}$$

is a law invariant coherent risk measure for each $u \in \mathbb{R}_+^d$. Furthermore, X is acceptable under ρ_s , i.e., $0 \in \rho_s(X)$, if and only if X is acceptable under r_u for all $u \in \mathbb{R}_+^d$. If the exchange cone is trivial, i.e., $X = X + \mathbb{R}_-^d$, then $r_u(X) = \langle r(X), u \rangle$ is given by a linear combination of the marginal risks.

Let $Z \in L^q(\mathbb{R}^d)$. The *max-correlation* risk measure of p -integrable random vector X is defined as

$$\Phi_Z(X) = \sup_{\tilde{Z} \sim Z} \frac{\mathbf{E}\langle -X, \tilde{Z} \rangle}{\mathbf{E}Z},$$

where the supremum is taken over all random vectors \tilde{Z} distributed as Z , see Burgert and Rüschendorf (2006) and Rüschendorf (2006). A general coherent numerical risk measure of X can be represented as the supremum of $\Phi_Z(X)$ over a family $Z \in \mathcal{Z} \subset L^q(\mathbb{R}^d)$. Then the set of $x \in \mathbb{R}^d$ that make X acceptable is given by

$$\{x \in \mathbb{R}^d : \mathbf{E}\langle x, Z \rangle \geq -\mathbf{E}\langle X, \tilde{Z} \rangle, \tilde{Z} \sim Z, Z \in \mathcal{Z}\},$$

which is $\rho_{\mathcal{Z}}$ given by (5.2).

In the spirit of Farkas et al. (2014), it is possible to define the numerical risk of X as infimum of the payoffs $f(\eta)$ for eligible portfolios η , such that $X + \eta$ is acceptable, i.e., contains a selection with all acceptable components. In the conical market model and selection risk measures setting, this can be rephrased as infimum of $f(\eta)$ for η from $\mathcal{A} + L^p(\check{X})$, where \mathcal{A} is the subset of $L^p(\mathbb{R}^d)$ that consists of random vectors with all individually acceptable components.

7. DUAL REPRESENTATION

The representation of the selection risk measure by (3.2) can be regarded as its primal representation. In this section we arrive at the dual representations of the selection risk measure and also general set-valued coherent risk measures of set-valued portfolios.

7.1. Lipschitz Space of Random Sets

In general, the support function $h_X(u)$ at a deterministic u may be infinite with a positive probability. In order to handle this situation, we identify a random set-valued portfolio with its support function evaluated at random directions and consider some special families of set-valued portfolios.

Fix a (possibly random) closed convex cone $\mathbb{G} \subset \mathbb{R}_+^d$, define $\mathbb{G}_1 = \{x \in \mathbb{G} : \|x\| = 1\}$. Since \mathbb{G}_1 is compact, all its selections are bounded. Note that $L^\infty(\mathbb{G}_1)$ is endowed with the L^∞ -metric. Let $\text{Lip}^p(\mathbb{G})$ be the family of all functions $h : L^\infty(\mathbb{G}_1) \mapsto L^p(\mathbb{R})$ such that h is uniformly p -integrable on $L^\infty(\mathbb{G}_1)$ and is Lipschitz with p -integrable Lipschitz constant

L_h . The *Lipschitz space* is $\text{Lip}^p(\mathbb{G})$ with the norm $\|h\|_L$ defined as the maximum of the L^p -norm of L_h and the L^p -norm of the supremum of $|h(Y)|$ over all $Y \in L^\infty(\mathbb{G}_1)$.

A set-valued portfolio X is said to belong to the space $\text{Lip}^p(\mathbb{G})$ if the effective domain of its support function $h_X(\cdot)$ is \mathbb{G} , and h_X belongs to the space $\text{Lip}^p(\mathbb{G})$. Since

$$h_{X+Y}(Y) = h_X(Y) + h_Y(Y),$$

h_X provides a linear embedding of such set-valued portfolios into $\text{Lip}^p(\mathbb{G})$.

LEMMA 7.1. *Assume that \mathbb{G} is the effective domain of h_X for a set-valued portfolio X . Then X belongs to the space $\text{Lip}^p(\mathbb{G})$ if and only if $\zeta = \sup_{u \in \mathbb{G}_1} |h_X(u)|$ is p -integrable. In this case $\|h_X\|_L = \|\zeta\|_p$.*

Proof. By modifying the argument from Molchanov (2005, th. F.1) it is immediately seen that $h_X(u)$, $u \in \mathbb{G}_1$, is a.s. Lipschitz with the Lipschitz constant at most ζ . For any $Y, Y' \in L^\infty(\mathbb{G}_1)$,

$$|h_X(Y) - h_X(Y')| \leq \zeta \|Y - Y'\|_\infty,$$

whence

$$\|h_X(Y) - h_X(Y')\|_p \leq \|\zeta\|_p \|Y - Y'\|_\infty,$$

meaning that $h_X : L^\infty(\mathbb{G}_1) \mapsto L^p(\mathbb{R})$ is Lipschitz. It remains to note that

$$\zeta = \sup_{Y \in L^\infty(\mathbb{G}_1)} |h_X(Y)|. \quad \square$$

A quasi-bounded portfolio X belongs to $\text{Lip}^p(\mathbb{R}_+^d)$ if $\partial^+ X$ is p -integrably bounded. If $M = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1, x_2 \geq 1\}$, then $X = X + M$ does not belong to $\text{Lip}^p(\mathbb{R}_+^2)$, no matter what X is.

EXAMPLE 2.3 (cont.) The random set $X = X + K$ for an exchange cone K with an efficient friction belongs to $\text{Lip}^p(K')$ if X is p -integrable, and $\|h_X\|_L$ is bounded by the L^p -norm of $\|X\|$. In the case of some frictionless exchanges the choice of X is not unique, and the above statement holds with X chosen from the intersection of X and the linear hull of K' .

Considering portfolios of the form $X + K$ from the space $\text{Lip}^p(K')$ means that all of them share the same exchange cone K . This is reasonable, since the diversification effects affect only the portfolio components, while the (possibly random) exchange cone remains the same for all portfolios. This setting also corresponds to one adapted in Hamel et al. (2011) and Kulikov (2008) by assuming that the conical models share the same exchange cone.

Consider linear functionals acting on $\text{Lip}^p(\mathbb{G})$ as

$$(7.1) \quad \langle \mu, h \rangle = \mathbf{E} \int h(u) \mu(du) = \mathbf{E} \sum_{i=1}^n \eta_i h(Y_i),$$

where μ is a random signed measure with q -integrable weights η_1, \dots, η_n assigned to its atoms $Y_1, \dots, Y_n \in L^\infty(\mathbb{G}_1)$ and any $n \geq 1$. Note that (7.1) does not change if μ is a signed measure attaching weights 1 or -1 to $Z_i = \eta_i Y_i \in L^q(\mathbb{G})$, $i = 1, \dots, n$. These functionals form a linear space and build a complete family, in particular, the values of $\langle \mu, h_X \rangle$ uniquely identify the distribution of X . Indeed, by the Cramér–Wold device, it

suffices to take μ with atoms located at selections of \mathbb{G} scaled by q -integrable random variables in order to determine the joint distribution of the values of h_X at these selections.

7.2. Weak-Star Convergence and Dual Representation

It is known (Johnson 1970, th. 4.3) that a sequence of functions from a Lipschitz space with values in a Banach space E weak-star converges if and only if the norms of the functions are uniformly bounded and their values at each given argument weak-star converge in E . Note that the weak convergence in Lipschitz spaces has not yet been characterized, see Hanin (1997). In our case $E = L^p(\mathbb{R})$. In view of Lemma 7.1, the sequence X_n weak-star converges to X if and only if

- i) $\|\sup_{u \in \mathbb{G}_1} |h_{X_n}(u)|\|_p$ are uniformly bounded in n ;
- ii) for each $Y \in L^\infty(\mathbb{G}_1)$, $h_{X_n}(Y)$ converges to $h_X(Y)$ weakly in $L^p(\mathbb{R})$ if $p \in [1, \infty)$ and in probability if $p = \infty$.

LEMMA 7.2. *Let $X, X_n, n \geq 1$ belong to the Lipschitz space $\text{Lip}^p(\mathbb{G})$. Then X_n weak-star converges to X in the Lipschitz space $\text{Lip}^p(\mathbb{G})$ if and only if the Hausdorff distance $\mathfrak{d}_H(X_n, X)$ converges to zero weakly in $L^p(\mathbb{R})$ if $p \in [1, \infty)$, and in probability with $X_n \subset M + \mathbb{G}'$ a.s. for a deterministic compact set M and all n if $p = \infty$.*

Proof. Sufficiency. If $\mathfrak{d}_H(X_n, X)$ weakly converges to zero in $L^p(\mathbb{R})$, then

$$|\mathbf{E}(h_{X_n}(Y)Z) - \mathbf{E}(h_X(Y)Z)| \leq \mathbf{E}(\mathfrak{d}_H(X_n, X)Z) \rightarrow 0$$

for each $Z \in L^q(\mathbb{R})$. Furthermore, $\|\mathfrak{d}_H(X_n, X)\|_p, n \geq 1$, is bounded, whence $X_n \subset X + B_\zeta$ for a ball B_ζ with p -integrable radius ζ . Therefore,

$$\|h_{X_n}(u)\| \leq 2\|\mathfrak{d}_H(X_n, X)\|_p + \|\zeta\|_p,$$

implying that the norms $\|h_{X_n}\|_L$ are uniformly bounded. A similar argument applies for $p = \infty$ with $|h_{X_n}(u)|$ bounded by $|h_M(u)|$ and confirms the weak-star convergence $X_n \rightarrow X$ in the Lipschitz space.

Necessity. Assume that $X_n \rightarrow X$ in the Lipschitz space, so that $\|h_{X_n}\|_L \leq c$ for a constant c and all n and $\mathbf{E}(h_{X_n}(Y)Z) \rightarrow \mathbf{E}(h_X(Y)Z)$ for all $Y \in L^\infty(\mathbb{G}_1)$ and $Z \in L^q(\mathbb{R})$. Let Y_1, \dots, Y_k be an ε -net in $L^\infty(\mathbb{G}_1)$. Then

$$\begin{aligned} \mathfrak{d}_H(X_n, X) &= \sup_{Y \in L^\infty(\mathbb{G}_1)} |h_{X_n}(Y) - h_X(Y)| \\ &\leq \sum_{i=1}^k |h_{X_n}(Y_i) - h_X(Y_i)| + \varepsilon(\|h_{X_n}\|_L + \|h_X\|_L). \end{aligned}$$

If $Z \in L^q(\mathbb{R}_+)$, then

$$|\mathbf{E}\mathfrak{d}_H(X_n, X)Z| \leq \sum_{i=1}^k \mathbf{E}|h_{X_n}(Y_i)Z - h_X(Y_i)Z| + \varepsilon(\|h_{X_n}\|_p + \|h_X\|_p)\|Z\|_q,$$

so that the right-hand side can be made arbitrarily small. The proof for $p = \infty$ is similar. \square

Consider a general set-valued coherent risk measure ρ from Definition 2.9. If the family

$$\mathcal{A}_\rho = \{X \in \text{Lip}^p(\mathbb{G}) : \rho(X) \ni 0\}$$

of acceptable set-valued portfolios is weak-star closed, the risk measure ρ is said to satisfy the *Fatou property*. In view of the cash invariance property, this formulation of the Fatou property is equivalent to $\rho(X) \supset \limsup \rho(X_n)$ if X_n weak-star converges to X in $\text{Lip}^p(\mathbb{G})$, cf. Kulikov (2008). Recall that the *upper limit* of $\rho(X_n)$ is the set of limits for all convergent subsequences of $\{x_n\}$, where $x_n \in \rho(X_n)$, $n \geq 1$, see Molchanov (2005, def. B.4).

THEOREM 7.3 (Dual representation). *A function ρ on random sets from $\text{Lip}^p(\mathbb{G})$ with values being convex closed upper sets is a set-valued coherent risk measure with the Fatou property if and only if $\rho(X) = \rho_{\mathcal{Z}}(X)$ given by (4.3) for a certain family $\mathcal{Z} \subset L^q(\mathbb{G})$.*

Proof. *Necessity.* Note first that (4.3) can be equivalently written as

$$(7.2) \quad \rho_{\mathcal{Z}}(X) = \left\{ x : \langle x, \mathbf{E} \int u \mu(du) \rangle \geq -\mathbf{E} \int h_X(u) \mu(du), \mu \in \mathcal{M} \right\}$$

for \mathcal{M} being the family of counting measures with atoms from \mathcal{Z} .

The dual cone \mathcal{M} to \mathcal{A}_ρ is the family of signed measures μ on \mathbb{G} with q -integrable total variation such that $\langle \mu, h_X \rangle \geq 0$ for all $X \in \mathcal{A}_\rho$. Since $X = Y + \mathbb{G}'$ is acceptable for each random compact convex set Y containing the origin, each $\mu \in \mathcal{M}$ is nonnegative. By the bipolar theorem, \mathcal{A}_ρ is the dual cone to \mathcal{M} .

Sufficiency. By Theorem 4.2, $\rho_{\mathcal{Z}}$ is a set-valued coherent risk measure. By an application of the dominated convergence theorem and noticing that the weak-star convergence implies the uniform boundedness of the norms, its acceptance set is weak-star closed. \square

EXAMPLE 2.3 (cont.) If $X = X + K$ for a random exchange cone K and p -integrable random vector X , then $X \in \text{Lip}^p(\mathbb{G})$ with $\mathbb{G} = K'$. By Lemma 7.2, the weak-star convergence $X_n = X_n + K \rightarrow X = X + K$ in this Lipschitz space means that X_n converges to X weakly in $L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$ and in probability for $p = \infty$ with the additional requirement that $\|X_n\|$ are uniformly bounded for $p = \infty$. In the case of (partially) frictionless models the random vectors X_n and X should be chosen appropriately from the intersections of the corresponding set-valued portfolios with the linear hull of K' . Since the acceptance set is convex, its closedness in the weak topology in L^p is equivalent to its closedness in the norm topology. By Theorem 7.3, set-valued coherent risk measures defined on portfolios $X = X + K$ and satisfying the Fatou property can be represented by (4.3), which is exactly the dual representation Kulikov (2008) for $p = \infty$ and from Hamel et al. (2011) for a general p .

7.3. Fatou Property of Selection Risk Measure

Since the selection risk measure ρ_s is a special case of a general set-valued coherent risk measure, $\rho_s = \rho_{\mathcal{Z}}$ for a suitable (and possibly nonunique) family \mathcal{Z} provided ρ_s satisfies the Fatou property. Note that the Fatou property of ρ_s is weaker than the Fatou property of $\rho_{s,0}$, which is established in the following theorems. Even for the simplest selection risk measure being the selection expectation (see Example 3.8) the Fatou property is a rather delicate result proved in Balder and Hess (1995).

THEOREM 7.4. *For $p \in [1, \infty)$, the selection risk measure $\rho_{s,0}$ satisfies the Fatou property on random sets from $\text{Lip}^p(\mathbb{G})$.*

Proof. Assume that X_n weak-star converges to X , so that $L_{X_n} \leq c$ a.s. for a constant c and $h_{X_n}(Y)$ weakly converges in L^p for each $Y \in L^\infty(\mathbb{G}_1)$.

Fix any $\varepsilon > 0$ and choose a finite set of selections $Y = \{Y_1, \dots, Y_N\} \subset L^\infty(\mathbb{G}_1)$ such that the (ε/c) -neighborhood of Y covers \mathbb{G}_1 with probability one. Since the uniform norm of $h_{X_n}(Y_i)$ is bounded by c , it is possible to find a subsequence $n(k)$ such that the joint distribution of $h_{X_{n(k)}}(Y_1), \dots, h_{X_{n(k)}}(Y_N)$ converges to the distribution of $h_X(Y_1), \dots, h_X(Y_N)$. By the extended Skorohod coupling (see Kallenberg 2002, cor. 6.12), it is possible to realize the random sets $X_{n(k)}$ on the same probability space in order to ensure the convergence almost surely.

The dominated convergence theorem implies that $\|h_{X_{n(k)}}(Y_i) - h_X(Y_i)\|_p \rightarrow 0$ as $k \rightarrow \infty$. Since the Lipschitz constants of $h_{X_{n(k)}}$ are bounded by c ,

$$\sup_{u \in \mathbb{G}_1} \|h_{X_{n(k)}}(u) - h_X(u)\|_p \leq \frac{\varepsilon}{2} + \max_{i=1, \dots, N} \|h_{X_{n(k)}}(Y_i) - h_X(Y_i)\|_p.$$

Let ξ_n be an acceptable selection of X_n , $n \geq 1$. The uniform convergence of the support functions of $X_{n(k)}$ to the support function of X implies that for each $\varepsilon > 0$ there exists a selection $\xi \in L^p(X)$ such that $\|\xi_n - \xi\|_p \leq \varepsilon$. The monotonicity property of univariate risk measures implies that $\mathbf{r}(\xi) \leq (\varepsilon, \dots, \varepsilon)$, whence ξ is acceptable and so X is acceptable under $\rho_{s,0}$. \square

THEOREM 7.5. *For $p = \infty$, the selection risk measure $\rho_{s,0}$ satisfies the Fatou property on quasi-bounded set-valued portfolios.*

Proof. If necessary by passing to a subsequence, assume that $x_n \in \rho_{s,0}(X_n)$ and $x_n \rightarrow 0$. Then, for all n there exists $\xi_n \in L^\infty(\partial^+ X_n)$ such that $\mathbf{r}(\xi_n + x_n) \leq 0$.

By Lemma 7.2, the weak-star convergence of X_n in the Lipschitz space $\text{Lip}^\infty(\mathbb{R}_+^d)$ implies that $\partial^+ X_n$, $n \geq 1$, are all subsets of a fixed compact set M . Furthermore,

$$\mathfrak{d}_H(X_n, X) = \sup_{u \in \mathbb{R}_+^d, \|u\|=1} |h_{X_n}(u) - h_X(u)| \rightarrow 0$$

in probability as $n \rightarrow \infty$, which is seen by taking a finite family of u 's and using the fact that the Lipschitz constants of h_{X_n} and h_X are bounded by $\|M\|$. By Molchanov (2005, th. 1.6.21), X_n converges to X in probability as random closed sets. The convergence in probability implies the weak convergence of random closed sets, see Molchanov (2005, cor. 1.6.22), so that $\partial^+ X_n$ weakly converges to $\partial^+ X$ because the map ∂^+ is continuous on quasi-bounded portfolios. Since the family of convex subsets of a compact set is compact in the Hausdorff metric and $\xi_n \in M$, it is possible to find a subsequence of $(\partial^+ X_n, \xi_n)$ that weakly converges to $(\partial^+ X, \xi)$. Pass to the chosen subsequence and realize the pairs $(\partial^+ X_n, \xi_n)$ on the same probability space, so that $\partial^+ X_n \rightarrow \partial^+ X$ in the Hausdorff metric and $\xi_n \rightarrow \xi$ a.s. with $\xi \in \partial^+ X$ a.s.

Since the components of \mathbf{r} satisfy the Fatou property and $\{\xi_n\}$ are uniformly bounded,

$$\mathbf{r}(\xi) \leq \liminf \mathbf{r}(\xi_n + x_n) \leq 0.$$

Thus, $0 \in \rho_{s,0}(X)$. \square

Recall that $\mathcal{Z}_1, \dots, \mathcal{Z}_d$ denote the subsets of $L^q(\mathbb{R}_+)$ that yield the maximal dual representations of the components of $\mathbf{r} = (r_1, \dots, r_d)$, see (4.4).

COROLLARY 7.6. *The selection risk measure $\rho_s(X)$ on set-valued portfolios X from $\text{Lip}^p(\mathbb{G})$ for $p \in [1, \infty)$ and on quasi-bounded portfolios for $p = \infty$ equals $\rho_{s,0}(X)$ and admits representation as $\rho_Z(X)$ for a family $\mathcal{Z} \subset L^q(\mathbb{R}^d)$ such that each $Z = (\zeta_1, \dots, \zeta_d) \in \mathcal{Z}$ satisfies $\zeta_i \in \mathcal{Z}_i$ for all $i = 1, \dots, d$.*

Proof. Consider the sequence $X_n = X$, $n \geq 1$. Then the Fatou property of $\rho_{s,0}$ established in Theorems 7.4 and 7.5 implies that the upper limit of $\rho_{s,0}(X)$ (being the closure of $\rho_{s,0}(X)$) is a subset of $\rho_{s,0}(X)$, so that $\rho_{s,0}(X)$ is closed.

Consider any $\xi = (\xi_1, \dots, \xi_d) \in L^p(\mathbb{R}^d)$ with $\mathbf{r}(\xi) \leq 0$. Then $\xi + \mathbb{G}'$ is an acceptable set-valued portfolio, so that $0 \in \rho_Z(\xi + \mathbb{G}')$. By (4.3), this is the case if and only if $\mathbf{E}(\xi, Z) \geq 0$ for all $Z \in \mathcal{Z}$. In turn, this is equivalent to $\mathbf{E}(\xi_i \zeta_i) \geq 0$ for all ξ_i such that $r_i(\xi_i) \leq 0$ and ζ_i being the i th component of Z . Since \mathcal{Z}_i provides the maximal dual representation of r_i , it necessarily contains ξ_i . \square

The following result establishes the Fatou property for the selection risk measure for $p = \infty$ and portfolios obtained as $X = X + K$ for a deterministic exchange cone K .

THEOREM 7.7. *For $p = \infty$, the selection risk measure $\rho_{s,0}$ satisfies the Fatou property on the family of portfolios obtained as $X + K$ for an essentially bounded random vector X and a fixed deterministic exchange cone K .*

Proof. In the case of a fixed deterministic exchange cone K , the weak-star convergence of $X_n = X_n + K$ means that X_n , $n \geq 1$, converge in probability and are uniformly bounded. For each $u \in K'$, the numerical risk measure $-h_{\rho_{s,0}(X+K)}(-u)$ defined as a function of a random vector X is law invariant and coherent, so that it satisfies the Fatou property by Ekeland and Schachermayer (2011, th. 2.6) which also applies if the cash invariance property is relaxed as in Burgert and Rüschendorf (2006). Thus, if X_n converges to X in probability and $\|X_n\| \leq 1$ a.s. for all n , then

$$h_{\rho_{s,0}(X+K)}(u) \geq \limsup h_{\rho_{s,0}(X_n+K)}(u).$$

If $x \in \limsup \rho_{s,0}(X_n + K)$, then $x = \lim x_{n_k}$ for a certain sequence $x_{n_k} \in \rho_{s,0}(X_{n_k} + K)$. Therefore,

$$\langle x, u \rangle = \lim \langle x_{n_k}, u \rangle \leq \lim h_{\rho_{s,0}(X_{n_k}+K)}(u) \leq \limsup h_{\rho_{s,0}(X_n+K)}(u).$$

Thus,

$$h_{\rho_{s,0}(X+K)}(u) \geq h_{\limsup \rho_{s,0}(X_n+K)}(u), \quad u \in K',$$

so that $\rho_{s,0}(X + K)$ contains the upper limit of $\rho_{s,0}(X_n + K)$. \square

COROLLARY 7.8. *If $p = \infty$, the selection risk measure $\rho_{s,0}(X)$ for set-valued portfolios $X = X + K$ with a fixed deterministic exchange cone K has closed values and is equal to ρ_Z for a certain family of integrable random vectors \mathcal{Z} such that each $Z = (\zeta_1, \dots, \zeta_d) \in \mathcal{Z}$ satisfies $\zeta_i \in \mathcal{Z}_i$ for all $i = 1, \dots, d$.*

Proof. It is shown in Hamel et al. (2011) and Kulikov (2008) that a set-valued risk measure with $p = \infty$ satisfying the Fatou property has the dual representation as ρ_Z , and so ρ_s admits exactly the same dual representation. By repeating the argument from the proof of Corollary 7.6, it is seen that $\zeta_i \in \mathcal{Z}_i$. \square

It should be noted that not all risk measures ρ_Z are selection risk measures, and so the acceptability of X under ρ_Z does not immediately imply the existence of an acceptable

selection and so does not guarantee the existence of a trading strategy that eliminates the risk. While the calculation of the risk measure $\rho_{\mathcal{Z}}$ may be rather complicated, its primal representation (3.2) as the selection risk measure opens a possibility for an approximation of its values from above by exploring selections of $X + K$.

8. COMPUTATION AND APPROXIMATION OF RISK MEASURES

The evaluation of $\rho_s(X)$ involves calculation of $\mathbf{r}(\xi)$ for all selections $\xi \in L^p(X)$. The family of such selections is immense, and in application only several possible selections can be considered. A wider choice of selections is in the interest of the agent in order to better approximate $\rho_s(X)$ from above and so reduce the required capital reserves. For lower bounds, one can use the risk measure $\rho_{\mathcal{Z}}(X)$ or its superset obtained by restricting the family \mathcal{Z} , e.g., $\rho_{\mathcal{Z}_0}(X)$ if \mathbf{r} has all identical components.

In view of (3.2) and the convexity of its values, $\rho_s(X)$ contains the convex hull of the union of $\mathbf{r}(\xi) + \mathbb{R}_+^d$ for any collection of selections $\xi \in L^p(X)$. It should be noted that this convex hull is not subadditive in general (unless the family of selections builds a cone) and so itself cannot be used as a risk measure, while providing a reasonable approximation for it. It is possible to start with some “natural” selections ξ' and ξ'' and consider all their convex combinations or combine them as $\xi'1_A + \xi''1_{A^c}$ for events A .

For the deterministic exchange cone model $X = X + K$, it is sensible to consider selections $X + t\eta$ for all $t \geq 0$ and a selection η taking values from the boundary of K . By Theorem 5.6, it suffices to work with selections of K which are functions of X , and write them as $\eta(X)$. Since the aim is to minimize the risk, it is natural to choose η which is a sort of “countermonotonic” with respect to X , while choosing comonotonic X and η does not yield any gain in risk for their sum.

Assume that the components of \mathbf{r} are ESs at level α . Then in order to approximate ρ_s , it is possible to use a “favorable” selection $\eta_* \in K$ constructed by projecting X onto K following the two-step procedure. First, X is translated by subtracting the vector of univariate α -quantiles in order to obtain random vector Y whose univariate α -quantiles are zero. Then Y is projected onto the boundary of the solvency cone \tilde{K} and $\eta_*(X)$ is defined as the opposite of such projection. If Y belongs to the centrally symmetric cone to K' , it is mapped to the origin and no compensation will be applied in this case. Consider all selections of the form $X + t\eta_*(X)$ for $t > 0$. With this choice, the components of X assuming small values are partially compensated by the remaining components.

An alternative procedure is to modify the projection rule, so that only some part of the boundary of the solvency cone is used for compensation. In dimension $d = 2$, η_1 and η_2 are defined by projecting Y onto either one of the two half-lines that form the boundary of \tilde{K} .

EXAMPLE 8.1. Consider the random vector X taking values $(-2, 4)$ and $(4, -2)$ with equal probabilities. Let K be the cone with the points $(-5, 1)$ and $(1, -5)$ on its boundary, so the corresponding bid-ask matrix has entries $\pi^{(12)} = \pi^{(21)} = 5$, see Example 2.3. Assume that \mathbf{r} consists of two identical components being the ES_α at level $\alpha = 3/4$. Observe that $\text{ES}_\alpha(X) = (0, 0)$ and for

$$\eta_1 = \begin{cases} (1.2, -6) & \text{if } X = (-2, 4), \\ (0, 0) & \text{if } X = (4, -2), \end{cases} \quad \eta_2 = \begin{cases} (0, 0) & \text{if } X = (-2, 4), \\ (-6, 1.2) & \text{if } X = (4, -2), \end{cases}$$

we obtain $\text{ES}_\alpha(X + \eta_1) = (-0.8, 2)$ and $\text{ES}_\alpha(X + \eta_2) = (2, -0.8)$.

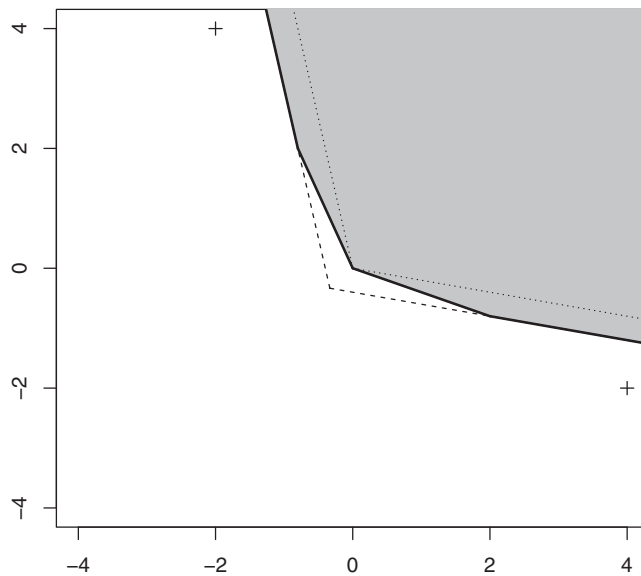


FIGURE 8.1. Two values for X , selection risk measure (shaded region) and its bounds from below and from above for Example 8.1.

By Theorem 5.6, the boundary of $\rho_s(X + K)$ is given by $\mathbf{r}(X + \eta(X))$ with $\eta(X)$ belonging to the boundary of K . In order to compensate the risks of X it is natural to choose η as function of X such that $\eta(-2, 4) = t(-5, 1)$ or $\eta(-2, 4) = t(1, -5)$ and $\eta(4, -2) = s(-5, 1)$ or $\eta(4, -2) = s(1, -5)$ for $t, s > 0$. The minimization problem over t and s can be easily solved analytically (or numerically) and yields the boundary of $\rho_s(X + K)$. In Figure 8.1, the boundary of $\rho_{\mathcal{Z}_0}(X + K)$ is shown as dashed line, the boundary of $\mathbf{r}(X) + \check{K}$ is dotted line, while $\rho_s(X + K)$ is the shaded region.

EXAMPLE 8.2. Consider the cone K with the points $(-1.5, 1)$ and $(1, -1.5)$ on its boundary, so the corresponding bid-ask matrix has entries $\pi^{(12)} = \pi^{(21)} = 1.5$. Let $\mathbf{r} = (\text{ES}_{0.05}, \text{ES}_{0.05})$ consist of two identical components. Let $X^{(n)}$ follow the empirical distribution for a sample of n observations from the bivariate normal distribution with i.i.d. components of mean 0.5 and variance 1.

In order to approximate $\rho_s(X^{(n)} + K)$ from above (i.e., construct a subset of $\rho_s(X^{(n)} + K)$), we first determine the set A consisting of $\mathbf{r}(\xi)$ for the selections ξ of $X + K$ obtained as $X^{(n)} + t\eta_*(X^{(n)})$, $X^{(n)} + t\eta_1(X^{(n)})$, and $X^{(n)} + t\eta_2(X^{(n)})$, for $t > 0$ and η_* , η_1 , η_2 described above using projection on \check{K} and the two half-lines from its boundary. Then we determine the convex hull of A and the points x_1 and x_2 described in the proof to Proposition 5.8 and finally add \check{K} to this convex hull.

Figure 8.2(a) shows a sample of $n = 1000$ observations of a standard bivariate normal distribution and the approximation to the true value of $\rho_s(X^{(n)} + K)$ described above, on the right panel a detail of the same plot is presented. The constructed approximation to $\rho_s(X^{(n)} + K)$ is the shaded region, the boundary of $\rho_{\mathcal{Z}_0}(X^{(n)} + K)$ obtained as described in Proposition 5.3 is plotted as dashed line, and the boundary of $\mathbf{r}(X^{(n)}) + \check{K}$ is plotted as dotted line.

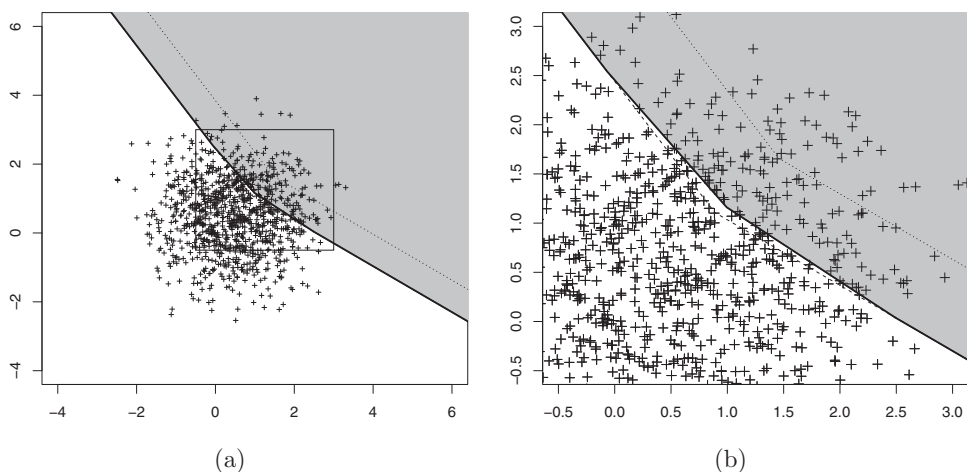


FIGURE 8.2. (a) An approximation to the selection risk measure and its bounds for a sample of normally distributed gains and a deterministic exchange cone; (b) enlarged part of the plot.

EXAMPLE 8.3. Consider the random frictionless exchange of two currencies described in Examples 1.1 and 5.1. Transactions that diminish the risk of $X = (X_1, X_2)$ can be constructed by projecting X onto the boundary line to half-plane \mathbf{K} with normal $(\pi, 1)$ and then subtracting the scaled projection from X . This leads to the family of selections $X + t\eta_*$ for $t \geq 0$ and $\eta_* \in \mathbf{K}$ given by

$$(8.1) \quad \eta_* = - \left(\frac{X_1 - \pi X_2}{1 + \pi^2}, \frac{\pi^2 X_2 - \pi X_1}{1 + \pi^2} \right).$$

Assume that the exchange rate $\pi = \pi^{(21)}$ is log-normally distributed with mean π_0 being the initial exchange rate and volatility σ . We approximate the risk of $X + \mathbf{K}$ with X having a bivariate standard normal distribution and independent of π (equivalently independent of \mathbf{K}) for \mathbf{r} with two identical components being ES_α . Observe that $\text{ES}_\alpha(\pi) = -\alpha^{-1} \Phi(\Phi^{-1}(\alpha) - \sigma) \pi_0$ and $\text{ES}_\alpha(1/\pi) = -\alpha^{-1} e^{\sigma^2} (1 - \Phi(\Phi^{-1}(1 - \alpha) + \sigma)) \pi_0^{-1}$, where Φ is the cumulative distribution function of a standard normal random variable and Φ^{-1} its quantile function.

Fix $\alpha = 0.05$, $\pi_0 = 1.5$ and the volatility $\sigma = 0.4$. The bounds on $\rho_s(\mathbf{K})$ are given by two cones C_1 and C_2 whose boundaries are determined in Example 5.1, see Figure 8.3. Namely C_1 is bounded by half-lines with slopes $\text{ES}_{0.05}(\pi) = -0.613$ and $1/\text{ES}_{0.05}(1/\pi) = -3.123$, while the half-lines determining C_2 have slopes $\text{VaR}_{0.05}(\pi) = -0.717$ and $\text{VaR}_{0.95}(\pi) = -2.674$, see Example 5.1.

In order to approximate the risk of $X + \mathbf{K}$, we take a sample of n observations of (X, π) . Denote by $(X^{(n)}, \pi^{(n)})$ a random vector whose distribution is the empirical distribution of the sample of (X, π) and by $\mathbf{K}^{(n)}$ a random half-space with normal $(\pi^{(n)}, 1)$. Note that the empirical distribution of the exchange rate is bounded away from the origin and infinity.

Figure 8.4(a) shows a sample of $n = 1000$ observations of a standard bivariate normal distribution and an approximation to the true value of $\rho_s(X^{(n)} + \mathbf{K}^{(n)})$ obtained as the sum of the convex hull of the risks of the selections of $X^{(n)} + \mathbf{K}^{(n)}$ described in (8.1) and C_1 (solid line). The boundary of $\mathbf{r}(X^{(n)}) + C_2$ is shown as dotted line. The tip of the solid

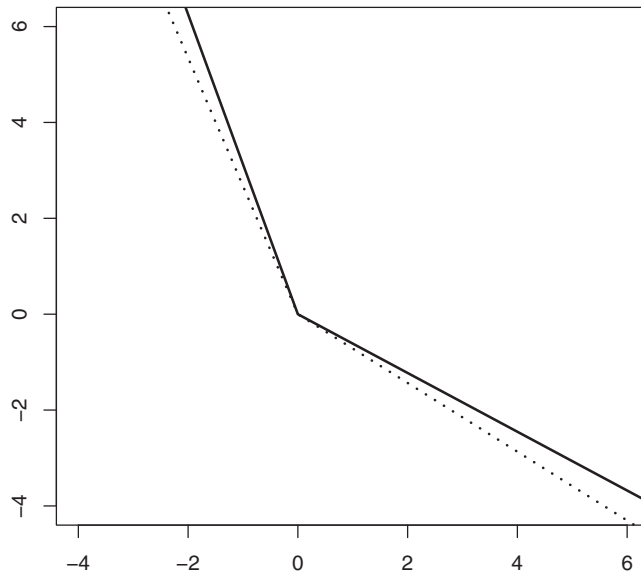


FIGURE 8.3. The boundaries of two cones C_1 and C_2 such that $C_1 \subset \rho_s(\mathbf{K}) \subset C_2$.

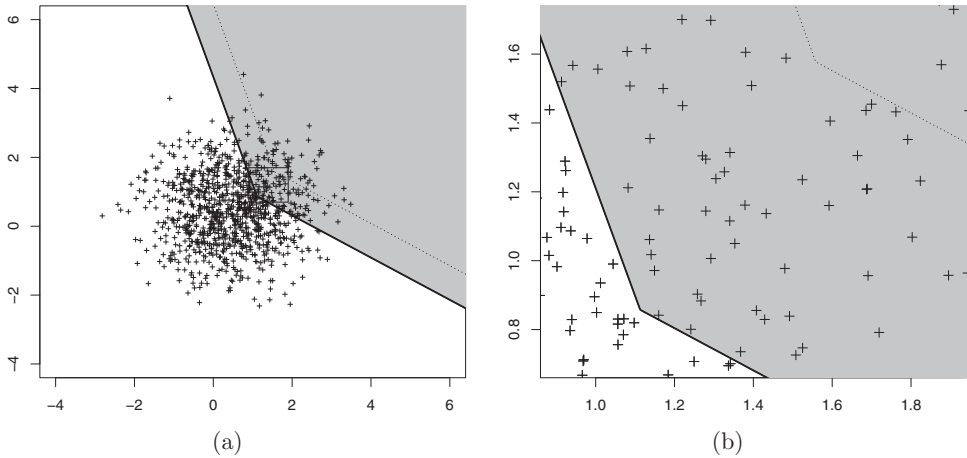


FIGURE 8.4. (a) An approximation to the selection risk measure and its upper bound for a normal sample in a random frictionless exchange case, (b) enlarged part of the plot.

cone is approximately at $(\text{ES}_{0.05}(\xi_1), \text{ES}_{0.05}(\xi_2)) = (1.1085, 0.829)$, where $\xi = X + \eta_*$ is the selection from (1.1). Since the tip of the cone is the tangent point to $\rho_s(X^{(n)} + \mathbf{K}^{(n)})$ in direction $-(0, \pi_0)$, it yields the minimal initial capital requirement of 1.661 units of the first currency, as mentioned in Example 1.1

EXAMPLE 8.4. Consider the restricted liquidity situation from Example 2.5. Let $\mathbf{X} = X + (\mathbf{K} \cap ((1, 1) + \mathbb{R}_-^2))$, where X follows a bivariate standard normal distribution and \mathbf{K} is the half-plane as in Examples 1.1 and 8.3. Assume that $\mathbf{r} = (\text{ES}_{0.05}, \text{ES}_{0.05})$. Transactions that diminish the risk of \mathbf{X} can be constructed by projecting $X = (X_1, X_2)$

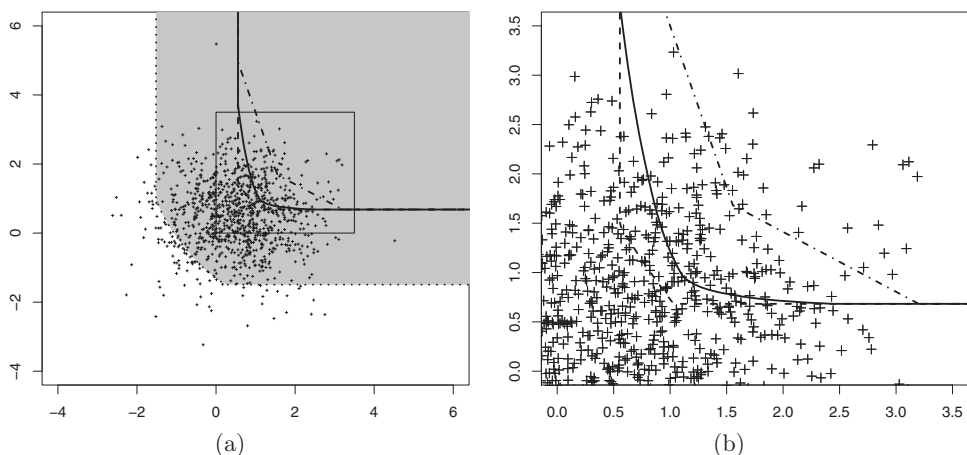


FIGURE 8.5. (a) An approximation to the selection risk measure and bounds for a normal sample in a random frictionless exchange case with restricted liquidity, (b) enlarged part of the plot.

onto the boundary line to half-plane K with normal $(\pi, 1)$ and then subtracting the projection from X . If the obtained point lies out of the line segment with end-points $X + (1, -\pi)$ and $X + (-1/\pi, 1)$, we take the nearest of the two end-points. This leads to a selection η of X given by (1.2). Other relevant selections of X are $X + (1, -\pi)$ and $X + (-1/\pi, 1)$.

In order to approximate the risk of X , we take a sample of $n = 1000$ observations of (X, π) . Denote by $(X^{(n)}, \pi^{(n)})$ a random vector whose distribution is the empirical distribution of the sample, by $K^{(n)}$ a random half-space with normal $(\pi^{(n)}, 1)$, and let $X^{(n)} = X^{(n)} + (K^{(n)} \cap ((1, 1) + \mathbb{R}_-^2))$.

Figure 8.5(a) shows a sample of $n = 1000$ observations of X and an approximation to the true value of $\rho_s(X^{(n)})$ obtained by calculating risks of all convex combinations of the selection η of X_n defined by (1.2) and the selections $X^{(n)} + (1, -\pi^{(n)})$, $X^{(n)} + (-1/\pi^{(n)}, 1)$. The shaded region with boundary plotted as a dotted line is $\mathbf{E}X^{(n)}$, the reflected selection expectation of $X^{(n)}$. The boundary of $\rho_{\mathcal{Z}_0}(X^{(n)})$ is plotted as a dashed line, and the boundary of $\mathbf{r}(X^{(n)}) + \rho_s(K^{(n)} \cap (\mathbb{R}_-^2 + (1, 1)))$ as a dash-dot line. The point $(\text{ES}_{0.05}(\eta_1), \text{ES}_{0.05}(\eta_2)) = (1.125, 0.915)$ belongs to the solid line and yields the capital reserves of 1.735 mentioned in Example 1.1

REFERENCES

- ACCIAIO, B., and I. PENNER (2011): Dynamic Risk Measures, in *Advanced Mathematical Methods for Finance*, G. Di Nunno and B. Øksendal, eds. Heidelberg: Springer, pp. 1–34.
- ACERBI, C., and D. TASCHE (2002): On the Coherence of Expected Shortfall, *J. Bank. Financ.* 26, 1487–1503.
- ARTZNER, P., F. DELBAEN, J.-M. EBER, and D. HEATH (1999): Coherent Measures of Risk, *Math. Financ.* 9, 203–228.
- BALDER, E. J., and C. HESS (1995): Fatou's Lemma for Multifunctions with Unbounded Values, *Math. Oper. Res.* 20, 175–188.

- BEN, TAHAR, I., and E. LÉPINETTE (2014): Vector-Valued Risk Measure Processes, *Int. J. Theor. Appl. Financ.* 17, 1450011.
- BIAGINI, S., and M. FRITELLI (2008): A Unified Framework for Utility Maximization Problems: An Orlicz Space Approach, *Ann. Appl. Probab.* 18, 929–966.
- BION-NADAL, J., and M. KERVAREC (2012): Risk Measuring under Model Uncertainty, *Ann. Appl. Probab.* 22, 213–238.
- BURGERT, C., and L. RÜSCHENDORF (2006): Consistent Risk Measures for Portfolio Vectors, *Insur. Math. Econom.* 38, 289–297.
- CASCOS, I., and I. MOLCHANOV (2007): Multivariate Risks and Depth-Trimmed Regions, *Financ. Stochas.* 11, 373–397.
- CHERIDITO, P., and T. LI (2009): Risk Measures on Orlicz Hearts, *Math. Financ.* 19, 189–214.
- CHERNY, A. S., and P. G. GRIGORIEV (2007): Dilatation Monotone Risk Measures Are Law Invariant, *Financ. Stochas.* 11, 291–298.
- COUSIN, A., and E. DI BERNARDINO (2013): On Multivariate Extension of Value-at-Risk, *J. Multivariate Anal.* 119, 32–46.
- DELBAEN, F. (2002): Coherent Risk Measures on General Probability Spaces, in *Advances in Finance and Stochastics*, K. Sandmann and P. J. Schönbucher, eds., Berlin: Springer, pp. 1–37.
- DELBAEN, F. (2012): *Monetary Utility Functions*, Osaka: Osaka University Press.
- DRAPEAU, S., and M. KUPPER (2013): Risk Preferences and Their Robust Representation, *Math. Oper. Res.* 38, 28–62.
- EKELAND, I., and W. SCHACHERMAYER (2011): Law Invariant Risk Measures on $L^\infty(\mathbb{R}^d)$, *Statist. Risk Model.* 28, 195–225.
- EKELAND, I., A. GALICHON, and M. HENRY (2012): Comonotonic Measures of Multivariate Risks, *Math. Financ.* 22, 109–132.
- EMBRECHTS, P., and G. PUCCETTI (2006): Bounds for Functions of Multivariate Risks, *J. Multivariate Anal.* 97, 526–547.
- FARKAS, W., P. KOCH-MEDINA, and C.-A. MUNARI (2014): Measuring Risk with Multiple Eligible Assets. Technical report, arXiv:1308.3331v2 [q-fin.RM].
- FEINSTEIN, Z., and B. RUDLOFF (2014a): A Comparison of Techniques for Dynamic Risk Measures with Transaction Costs, in *Set Optimization and Applications in Finance*, A. Hamel, F. Heyde, C. Löhne, and C. Schrage, eds., Springer. To appear, arXiv:1305.2151v1 [q-fin.RM].
- FEINSTEIN, Z., and B. RUDLOFF (2014b): Multi-Portfolio Time Consistency for Set-Valued Convex and Coherent Risk Measures, *Financ. Stochas.* To appear, arXiv:1212.5563v1 [q-fin.RM].
- FÖLLMER, H., and A. SCHIED (2004): *Stochastic Finance. An Introduction in Discrete Time*, 2nd ed., Berlin: De Gruyter.
- HAMEL, A. H., and F. HEYDE (2010): Duality for Set-Valued Measures of Risk, *SIAM J. Financ. Math.* 1, 66–95.
- HAMEL, A. H., F. HEYDE, and B. RUDLOFF (2011): Set-Valued Risk Measures for Conical Market Models, *Math. Financ. Econ.* 5, 1–28.
- HAMEL, A. H., B. RUDLOFF, and M. YANKOVA (2013): Set-Valued Average Value at Risk and Its Computation, *Math. Financ. Econ.* 7, 229–246.
- HANIN, L. G. (1997): Duality for General Lipschitz Classes and Applications, *Proc. London Math. Soc.* 75, 134–156.
- HAJAL, F., and H. UMEGAKI (1977): Integrals, Conditional Expectations, and Martingales of Multivalued Functions, *J. Multivariate Anal.* 7, 149–182.

- JOHNSON, J. A. (1970): Banach Spaces of Lipschitz Functions and Vector-Valued Lipschitz Functions, *Trans. Am. Math. Soc.* 148, 147–169.
- JOUINI, E., M. MEDDEB, and N. TOUZI (2004): Vector-Valued Coherent Risk Measures, *Financ. Stochas.* 8, 531–552.
- JOUINI, E., W. SCHACHERMAYER, and N. TOUZI (2006): Law Invariant Risk Measures Have the Fatou Property, *Adv. Math. Econ.* 9, 49–71.
- KABANOV, Y. M. (1999): Hedging and Liquidation under Transaction Costs in Currency Markets, *Financ. Stochas.* 3, 237–248.
- KABANOV, Y. M., and M. SAFARIAN (2009): *Markets with Transaction Costs. Mathematical Theory*, Berlin: Springer.
- KAINA, M., and L. RÜSCHENDORF (2009): On Convex Risk Measures on L^p -Spaces, *Math. Meth. Oper. Res.* 69, 475–495.
- KALLENBERG, O. (2002): *Foundations of Modern Probability*, 2nd ed., New York: Springer.
- KAVAL, K., and I. MOLCHANOV (2006): Link-Save Trading, *J. Math. Econ.* 42, 710–728.
- KULIKOV, A. V. (2008): Multidimensional Coherent and Convex Risk Measures, *Theory Probab. Appl.* 52, 614–635.
- LÖHNE, A. (2011): *Vector Optimization with Infimum and Supremum*, Berlin: Springer.
- MCNEIL, A. J., R. FREY, and P. EMBRECHTS (2005): *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton, NJ: Princeton Univ. Press.
- MOLCHANOV, I. (2005): *Theory of Random Sets*, London: Springer.
- PENNANEN, T., and I. PENNER (2010): Hedging of Claims with Physical Delivery under Convex Transaction Costs, *SIAM J. Financial Math.* 1, 158–178.
- RÜSCHENDORF, L. (2006): Law Invariant Convex Risk Measures for Portfolio Vectors, *Statist. Decisions* 24, 97–108.
- SCHACHERMAYER, W. (2004): The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time, *Math. Financ.* 14, 19–48.
- SCHNEIDER, R. (1993): *Convex Bodies. The Brunn–Minkowski Theory*, Cambridge: Cambridge University Press.