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PATH PROPERTIES OF THE LINEAR MULTIFRACTIONAL STABLE MOTION

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Abstract

The *linear multifractional stable motion* (LMSM) processes $Y = \{Y(t)\}_{t\in\mathbb{R}}$ is an α-stable ($0 < \alpha < 2$) stochastic process, which exhibits *local* self-similarity, has heavy tails and can have skewed distributions. The process Y is obtained from the well-known class of *linear fractional stable motion* (LFSM) processes by replacing their self-similarity parameter H by a function of time $H(t)$.

We show that the paths of $Y(t)$ are bounded on bounded intervals only if $1/\alpha \le H(t) < 1$, $t \in \mathbb{R}$. In particular, if $0 < \alpha \le 1$, then Y has everywhere discontinuous paths, with probability one. On the other hand, Y has a version with continuous paths if $H(t)$ is sufficiently regular and $1/\alpha < H(t)$, $t \in \mathbb{R}$. We study the Hölder regularity of the sample paths when these are continuous and establish almost sure bounds on the *pointwise* and *uniform pointwise Hölder exponents* of the (random) function $Y(t,\omega)$, $t \in \mathbb{R}$, in terms of the function $H(t)$ and its corresponding Hölder exponents.

The Gaussian *multifractional Brownian motion* (MBM) processes are LMSM processes when $\alpha = 2$. We obtain some new results on the Hölder regularity of their paths.

Keywords: Path Continuity; Hölder Regularity; Linear Fractional Stable Motion; Self-Similarity; Multifractional Brownian Motion; Local Self-Similarity; Heavy Tails.

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1. INTRODUCTION

We focus on a class of α-stable processes ($0 < \alpha \le 2$), called *linear multifractional stable motions*, which exhibit *local* self-similarity. The linear multifractional stable motion process (LMSM), was introduced in Stoev and Taqqu[1] following the ideas of Peltier and Lévy-Vehel[2] and Benassi *et al.*[3] in the Gaussian case (see also Ayache and Lévy-Vehel,[4] Ayache[5] and Ayache and Taqqu,[6] for other results in the Gaussian context). The LMSM process is as an extension of the well-known class of linear fractional stable motion (LFSM) processes.

We start by recalling the definition of the *fractional Brownian motion* (FBM) and the *multifractional Brownian motion* (MBM) processes, which are special cases of LMSMs. The FBM process $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ is a Gaussian stochastic process with mean zero and stationary increments, which is self-similar with self-similarity exponent $H \in (0,1)$. Recall that a process $X = \{X(t)\}_{t\in\mathbb{R}}$ is said to be *self-similar with self-similarity parameter* $H > 0$ (H*-self-similar*, in short) if, for all $c > 0$, $\{X(ct)\}_{t\in\mathbb{R}} =_d \{c^H X(t)\}_{t\in\mathbb{R}}$. Here $=_d$ means equal marginal *and* finite-dimensional distributions, that is, $\{X(t)\}_{t\in\mathbb{R}} =_d \{Y(t)\}_{t\in\mathbb{R}}$, if

$$\mathbb{P}\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}$$
$$= \mathbb{P}\{Y(t_1) \le x_1, \ldots, Y(t_n) \le x_n\},$$

for all t_j, $x_j \in \mathbb{R}$, $j = 1, \ldots, n$, $n \in \mathbb{N}$. When $H = 1/2$, the fractional Brownian motion process has independent increments and it becomes the classical Brownian motion process. The increments of B_H however are generally dependent — they are positively correlated when $H \in (1/2, 1)$ and negatively correlated when $0 < H < 1/2$ (see Fig. 1). The FBM processes ($1/2 < H < 1$) are important in modeling the long-range dependence

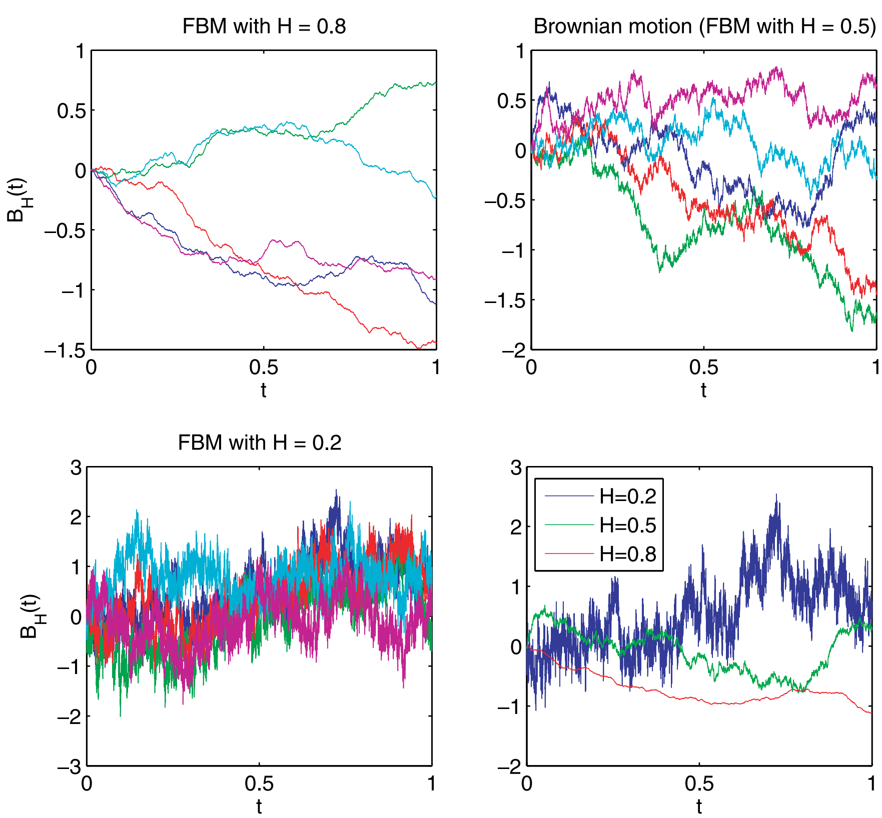

Fig. 1 The top-left plot displays five simulated paths of the fractional Brownian motion process $B_H(t)$, $t \in [0,1]$ with $H = 0.8$, the top-right plot displays five paths of the Brownian motion (that is, FBM with $H = 0.5$) and the bottom-left shows five paths of the FBM process with $H = 0.2$. On the bottom-right plot three paths are displayed, with $H = 0.8$, $H = 0.5$ and $H = 0.2$, respectively. All sample paths were obtained by simulating 10,000 observations at an uniform grid in the interval $[0,1]$. When $H = 0.8$, the paths are "more regular" than in the cases $H = 0.5$ and 0.2. The case $H = 0.2$ yields "most irregular" paths, since their pointwise Hölder exponent (equal to $H = 0.2$) is the lowest.

phenomenon.[7] The processes B_H have the following stochastic integral representation

$$B_H(t) = \int_{\mathbb{R}} \Big\{ a^+((t+s)_+^{H-1/2} - (s)_+^{H-1/2}) + a^-((t+s)_-^{H-1/2} - (s)_-^{H-1/2}) \Big\} W(ds), \qquad (1.1)$$

where $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and where

$$(x)_+^a := \begin{cases} x^a, & \text{when } x > 0, \\ 0, & \text{when } x \le 0 \end{cases} \quad \text{and} \quad x_-^a := (-x)_+^a. \qquad (1.2)$$

Physically, $W(ds)$, $s \in \mathbb{R}$ can be viewed as a collection of iid (independent and identically distributed) Gaussian random variables. More precisely, $W(ds)$, $s \in \mathbb{R}$ denotes a standard Gaussian random measure on $\mathbb{R}$, which assigns to each Borel set $A \subset \mathbb{R}$ (of finite Lebesgue measure $|A|$) a Gaussian random variable $W(A)$ with mean 0 and variance $|A|$. The measure $W(\cdot)$ is also independently scattered, that is, $W(A_1), \dots, W(A_n)$ are independent, if the sets A_j, $j = 1, \dots, n$ are *disjoint*. In general, the stochastic integral in (1.1) converges because its corresponding kernel function belongs to the space $L^2(ds) = \{g(x)\colon \int_{\mathbb{R}} g(x)^2\,dx < \infty\}$, for all $t \in \mathbb{R}$ and $H \in (0,1)$.

The paths of the FBMs can be made continuous, with probability one, and in this case, one can show that they are also Hölder — continuous and have pointwise Hölder exponents equal to H, almost surely (see, e.g. Secs. 4 and 5, below). Hölder exponents measure the degree of smoothness of the paths and since $0 < H < 1$, the paths are continuous but not differentiable. For more details on the stochastic and path properties of the fractional Brownian motion, see e.g. Embrechts and Maejima.[8]

The paths of the fractional Brownian motion $B_H(t)$ have the *same* pointwise Hölder exponent at all times $t \in \mathbb{R}$, which is also equal to its self-similarity parameter H. It is interesting to construct a process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ whose pointwise Hölder exponents may be a function of time. Let $H(t) \in (0,1)$, $t \in \mathbb{R}$ be a non-random function and replace H in (1.1) by $H(t)$. The so-obtained process

$$Y(t) = \int_{\mathbb{R}} \Big\{ a^+((t+s)_+^{H(t)-1/2} - (s)_+^{H(t)-1/2}) + a^-((t+s)_-^{H(t)-1/2} - (s)_-^{H(t)-1/2}) \Big\} W(ds)$$

is called a *multifractional Brownian motion* (MBM) process. The MBMs with $(a^+, a^-) = (1,0)$ were first introduced by Peltier and Lévy-Vehel.[2] Benassi *et al.*[3] have also introduced and studied a closely related process starting from a frequency domain stochastic integral representation of the FBMs. These processes are equivalent to the MBMs with parameters $(a^+, a^-) = (1,1)$, up to a multiplicative deterministic function.[9] Although the FBMs with different pairs of parameters (a^+, a^-) are equivalent in finite-dimensional distributions (up to a multiplicative factor), this is not the case for the MBMs with different (a^+, a^-)s when the function $H(t)$ is non-constant. In fact, the class of MBMs turns out to be very rich and their correlation structures depend non-trivially on the pairs of parameters (a^+, a^-).[10]

We shall now define the linear fractional stable motion (LFSM) and linear multifractional stable motion (LMSM) processes. We will do so by replacing the Gaussian measure $W(ds)$ in (1.1) by an α-stable random measure $M_{\alpha,\beta}(ds)$, $0 < \alpha \le 2$, and by modifying the kernel function in (1.1). Physically, $M_{\alpha,\beta}(ds)$, $s \in \mathbb{R}$ can be viewed as a collection of iid random variables with a stable distribution with parameters α and $\beta(s)$. More precisely, let $M_{\alpha,\beta}(ds)$ in (1.7) be a (strictly) α-stable, independently scattered random measure with control measure ds, and skewness intensity function $\beta(s) \in [-1,1]$, $s \in \mathbb{R}$. This means, in particular, that if $A_1, \dots, A_n \subset \mathbb{R}$ are disjoint Borel sets, then the random variables $M_{\alpha,\beta}(A_1), \dots, M_{\alpha,\beta}(A_n)$ are independent. Moreover, for any Borel set $A \subset \mathbb{R}$, the characteristic function of the random variable $M_{\alpha,\beta}(A)$ is given by

$$\mathbb{E}e^{i\theta M_{\alpha,\beta}(A)} = \begin{cases} \exp\Big\{ -|A||\theta|^\alpha \times \Big(1 - i\beta_A \operatorname{sign}(\theta) \tan\dfrac{\pi\alpha}{2}\Big)\Big\}, & \text{if } \alpha \ne 1 \\ \exp\{-|A||\theta|\}, & \text{if } \alpha = 1, \end{cases} \qquad (1.3)$$

where $|A| := \int_A ds$ and where $\beta_A := \int_A \beta(s)\,ds/|A|$. If the distribution of the α-stable random measure $M_{\alpha,\beta}(ds)$ is symmetric, then $\beta(\cdot) \equiv 0$ and

$$\mathbb{E}e^{i\theta M_{\alpha,\beta}(A)} = e^{-|A||\theta|^\alpha}.$$

For simplicity, we take the distribution of the α-stable random measure $M_{\alpha,\beta}(ds)$ to be symmetric if $\alpha = 1$. In the case $\alpha = 2$, $\beta(\cdot)$ is irrelevant and Relation (1.3) reduces to $\mathbb{E}e^{i\theta M_{\alpha,\beta}(A)} = e^{-|A||\theta|^2}$, which is the characteristic function of a Gaussian random variable with mean 0 and variance $2|A|$.

That is, when $\alpha = 2$, the measure $M_{\alpha,\beta}(ds)$ becomes $\sqrt{2}W(ds)$ and we recover the Gaussian case.

Definition 1.1. Let $0 < \alpha \le 2$, $\beta(s) \equiv \beta \in [-1, 1]$ be constant and let $0 < H < 1$. The *linear fractional stable motion* (LFSM) process $X_{H,\alpha}(t)$ is defined as

$$X_{H,\alpha}(t) := \int_{\mathbb{R}} f(t, H, s) M_{\alpha,\beta}(ds), \tag{1.4}$$

where $M_{\alpha,\beta}(ds)$ is an α-stable random measure as in (1.3) and where

$$\begin{aligned} f(u, v, s) := a^{+}((u+s)_{+}^{v-1/\alpha} - (s)_{+}^{v-1/\alpha}) \\ + a^{-}((u+s)_{-}^{v-1/\alpha} - (s)_{-}^{v-1/\alpha}), \end{aligned} \tag{1.5}$$

with $a^{+}, a^{-} \in \mathbb{R}$, $|a^{+}| + |a^{-}| > 0$.

The stochastic integral in (1.4) is well-defined since for any fixed $u \in \mathbb{R}$ and $v \in (0, 1)$, $f(u, v, s) \in L^{\alpha}(ds) := \{g(s)\colon \int_{\mathbb{R}} |g(s)|^{\alpha} ds < \infty\}$. Because $M_{\alpha,\beta}(ds)$ is stable, so is $X_{H,\alpha}(t)$, and consequently $\mathbb{E}|X_{H,\alpha}(t)|^p = \infty$, if $p \ge \alpha$, with $\alpha < 2$ and $t \ne 0$. Thus, $X_{H,\alpha}(t)$ has infinite variance, for $\alpha < 2$ and infinite mean, for $\alpha \le 1$. Also, $X_{H,\alpha}(t)$ has heavy tails, that is, for all $t \ne 0$,

$$\mathbb{P}\{|X_{H,\alpha}(t)| > x\} \sim \text{const}\, x^{-\alpha}, \quad \text{as } x \to \infty.$$

For more details on stable distributions and stable stochastic integrals, see e.g. Samorodnitsky and Taqqu.[11]

The LFSM processes $X_{H,\alpha}$ are also H-self-similar and have stationary increments. They can be viewed as α-stable analogs of the FBMs B_H (see Chap. 7 in Samorodnitsky and Taqqu[11]). As in the Gaussian case, we now define the LMSM process Y by replacing the self-similarity parameter H in (1.4) by a non-constant deterministic function:

$$0 < H(t) < 1, \quad t \in \mathbb{R}. \tag{1.6}$$

Definition 1.2. Let $0 < \alpha \le 2$ and $M_{\alpha,\beta}(ds)$ be an α-stable random measure as in (1.3), with skewness intensity function $\beta(s) \in [-1, 1]$, $s \in \mathbb{R}$. The *linear multifractional stable motion* (LMSM) process $Y(t)$ is defined by the stochastic integral,

$$Y(t) := \int_{\mathbb{R}} f(t, H(t), s) M_{\alpha,\beta}(ds), \tag{1.7}$$

where $f(u, v, s)$ is as in (1.5). Observe that in Definition 1.1, we supposed that $\beta(s) \equiv \text{const}$ but here we let $\beta(s) \in [-1, 1]$, $s \in \mathbb{R}$ be arbitrary.

Our goal here is to establish the behavior of the paths of the LMSM processes. To get a feeling about the relationship between the LMSM process Y and the LFSM processes $X_{H,\alpha}$ suppose that the skewness intensity is zero, $\beta(s) \equiv 0$. If the function $H(t) \in (0, 1)$ is non-constant, but is continuous and sufficiently regular, we expect the processes $Y(t)$ and $X_{H(t_0),\alpha}(t)$ to have very similar finite-dimensional distributions, as t approaches t_0, because their corresponding integral representations will be similar [see (1.7) and (1.4)]. Thus, since the LFSM $X_{H(t_0),\alpha}(t)$, is $H(t_0)$-self-similar with stationary increments, we expect as t approaches t_0, the LMSM $Y(t)$ to be locally self-similar, with self-similarity parameter $H(t_0)$. Indeed, as shown in Stoev and Taqqu[1] (Theorem 5.1), under certain regularity conditions on the function H, the process Y is *locally equivalent* to a LFSM process. Namely, that for all $t_0 \in \mathbb{R}$, as $\lambda \downarrow 0$,

$$\begin{aligned} \{\lambda^{-H(t_0)}(Y(\lambda t + t_0) - Y(t_0))\}_{t\in\mathbb{R}} \\ \to \{X_{H(t_0),\alpha}(t)\}_{t\in\mathbb{R}}, \end{aligned} \tag{1.8}$$

in the sense of finite-dimensional distributions. We thus call $H(t)$, $t \in \mathbb{R}$ the *local self-similarity* (or *scaling*) *exponent* function associated with the LMSM process Y. The limit process $X_{H(t_0),\alpha}$ in (1.8) can be interpreted as the *tangent* process of Y at t_0.

In Stoev and Taqqu,[1] we also established necessary and sufficient conditions for the continuity in probability of the process Y, in terms of the local scaling exponent function H. A process $\{Y(t)\}_{t\in\mathbb{R}}$ is said to be continuous in probability at $t_0 \in \mathbb{R}$, if for all $\epsilon > 0$, $\mathbb{P}\{|Y(t) - Y(t_0)| > \epsilon\} \to 0$, as $t \to t_0$. The continuity in probability of Y was established by using a representation of the LMSM process $Y(t)$ in terms of the α-stable random field $X = \{X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\}$,

$$X(u, v) := \int_{\mathbb{R}} f(u, v, s) M_{\alpha,\beta}(ds), \tag{1.9}$$

where $f(u, v, s)$ is given in (1.5) and $M_{\alpha,\beta}(ds)$ is the α-stable measure appearing in (1.7) and (1.4). The random field X allows us to view u (that is, t) and v (that is, H) as variables. Observe that, by (1.7),

$$Y(t) \overset{\text{a.s.}}{=} X(t, H(t)), \quad \text{for all } t \in \mathbb{R}. \tag{1.10}$$

The field $X(u, v)$ is continuous in probability in $(u, v) \in \mathbb{R} \times (0, 1)$ and, in fact, differentiable in v, in the sense of convergence in probability. Namely, for all $u \in \mathbb{R}$, there exists a random variable $\xi(u, v)$, such that

$$\frac{X(u, v+h) - X(u, v)}{h} \xrightarrow{P} \xi(u, v), \quad \text{as } h \to 0,$$

where $\xrightarrow{P}$ denotes convergence in probability. We will denote $\xi(u, v)$ by $\partial_v X(u, v)$. Section 2 below, contains some results on the field X, used in the sequel. By using the local properties of the field $X(u, v)$ and those of the function $H(t)$, $t \in \mathbb{R}$, one can establish the local behavior of the LMSM process $Y(t)$, $t \in \mathbb{R}$ and in particular its path properties.

Observe that when $\alpha = 2$, the random measure $M_{\alpha,\beta}(ds)$, $s \in \mathbb{R}$ becomes Gaussian and the linear fractional stable motion process $X_{H,\alpha}$ becomes the fractional Brownian motion (FBM) process. If $a^+ = 1$ and $a^- = 0$, for example, then the linear multifractional stable motion process Y coincides with the multifractional Brownian motion (MBM) process introduced by Peltier and Lévy-Vehel.[2] A number of other Gaussian and/or finite-variance extensions to the MBM processes have also been studied.[3,4,9,12–14] Some properties of the Gaussian MBM processes and the infinite-variance LMSM process are very similar, for example, continuity in probability and local asymptotic self-similarity.[1,2] However, as we will see below, the path properties of the MBM and the LMSM ($0 < \alpha < 2$) processes can be quite different (see Fig. 2). We assume throughout $0 < \alpha < 2$, with the exception of Sec. 5, where we provide some new results for the Gaussian MBM processes.

The paper is organized as follows. We present in Sec. 2 some technical results on the α-stable field $X(u, v)$ and its derivatives, which are used in the rest of the paper. In Sec. 3, we study the almost sure boundedness and continuity of the paths of Y. We show that the LMSM process $Y(t)$ can have bounded paths on bounded intervals only if $1/\alpha \leq H(t)$. We also obtain sufficient conditions

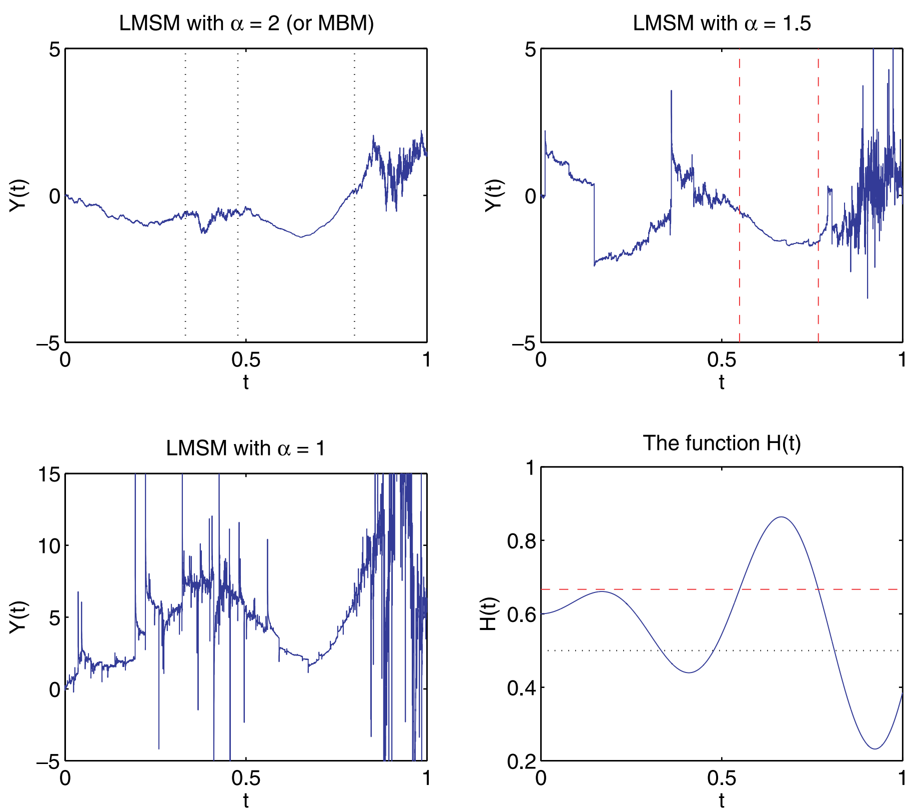

Fig. 2 The top-left plot displays a simulated path of the multifractional Brownian motion (MBM), that is, a linear multifractional stable motion (LMSM) with $\alpha = 2$. The top-right plot displays one path of the LMSM with $\alpha = 1.5$ and the bottom-left shows a path of the LMSM process with $\alpha = 1$. All sample paths were obtained by using the *same* local scaling function $H(t)$ (shown on the bottom right plot) and by simulating 10,000 observations at an uniform grid in the interval $[0, 1]$. The vertical dotted lines in the top-left plot indicate times when $H(t) = 1/\alpha = 0.5$ and the vertical dashed lines in the top-right plot indicate times when $H(t) = 1/\alpha = 1/1.5$. Observe that in the bottom-left plot we always have $H(t) < 1/\alpha = 1$. Note that in the regions where $1/\alpha < H(t) < 1$, the paths of LMSM are "smoother" as compared to regions with $H(t) < 1/\alpha$. Also the lower the value of α, the more irregular the paths. The paths are not continuous when $H(t) < 1/\alpha$ and $\alpha < 2$. In the Gaussian case $\alpha = 2$, the paths are continuous, but their regularity varies with $H(t)$.

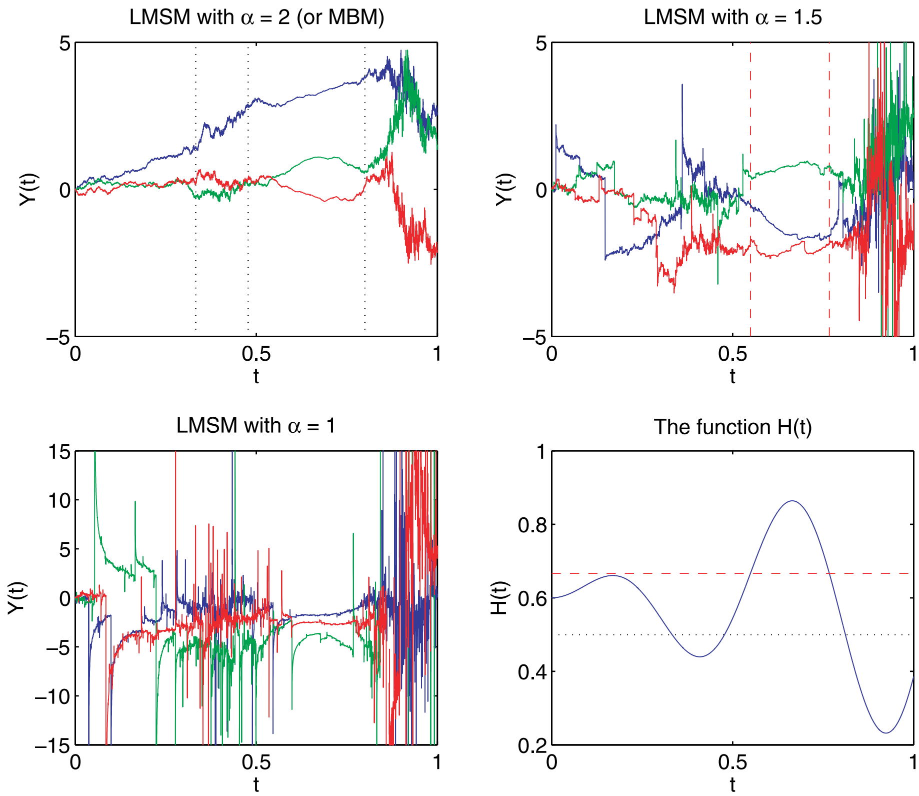

Fig. 3 The display is the same as in Fig. 2 but here three random paths of the linear multifractional stable motion (LMSM) are shown for each value of α. They all have the same local scaling function $H(t)$, which is displayed in the bottom-right plot.

for the almost sure sample path continuity of Y when $1 < \alpha < 2$. Section 4 contains results on the almost sure Hölder regularity of the paths $Y(t, \omega)$, $t \in \mathbb{R}$, $\omega \in \Omega$ of the process Y. These results relate the pointwise and uniform pointwise Hölder exponents of the (random) function $Y(t)$, $t \in \mathbb{R}$ to $H(t)$ and to the pointwise and uniform pointwise Hölder exponents of $H(t)$. In Sec. 5, we extend some existing results on the Hölder regularity of the Gaussian MBM processes. Section 6 contains some auxiliary lemmas.

2. TOOLS: THE FIELDS $X(u, v)$ AND $\partial_v^n X(u, v)$

We present here some basic properties of the field $X(u, v)$ and its corresponding derivative fields $\partial_v^n X(u, v)$, $n \in \mathbb{N}$. The results in this section are based on Stoev and Taqqu[1] (see Sec. 2 therein).

Let $0 < \alpha \le 2$ and $n \in \mathbb{N}$. For all $u \in \mathbb{R}$ and $v \in (0, 1)$, define

$$\partial_v^n X(u, v) := \int_{\mathbb{R}} \partial_v^n f(u, v, s) M_{\alpha,\beta}(ds), \tag{2.1}$$

where $M_{\alpha,\beta}(ds)$ is the α-stable measure in (1.9) and (1.7) and where

$$\begin{aligned}\partial_v^n f(u, v, s) &= \frac{\partial^n}{\partial v^n} f(u, v, s)\\ &= a^+ \big(\ln^n (u+s)_+ (u+s)_+^{v-1/\alpha}\\ &\quad - \ln^n (s)_+ (s)_+^{v-1/\alpha} \big)\\ &\quad + a^- \big(\ln^n (u+s)_- (u+s)_-^{v-1/\alpha}\\ &\quad - \ln^n (s)_- (s)_-^{v-1/\alpha} \big)\end{aligned} \tag{2.2}$$

denotes the nth partial derivative of the kernel function $f(u, v, s)$ with respect to v [see (1.2)]. One has $\int_{\mathbb{R}} |\partial_v^n f(u, v, s)|^\alpha ds < \infty$, for all $n \in \mathbb{N}$, $u \in \mathbb{R}$ and $v \in (0, 1)$, and therefore the α-stable integrals in (2.1) are well-defined.

We denote by $\|\xi\|_\alpha$ the scale coefficient in the characteristic function $\varphi_\xi(\theta) = \mathbb{E} \exp\{i\xi\theta\}$, $\theta \in \mathbb{R}$ of the α-stable ($0 < \alpha \le 2$) random variable ξ, namely,

$$\varphi_\xi(\theta) = \begin{cases} \exp\Big\{ -\|\xi\|_\alpha^\alpha |\theta|^\alpha \\ \quad \times \Big(1 - i\beta_\xi \operatorname{sign}(\theta) \tan \dfrac{\pi\alpha}{2} \Big) \Big\}, & \text{if } \alpha \ne 1 \\ \exp\{-\|\xi\|_1 |\theta|\}, & \text{if } \alpha = 1, \end{cases}$$

where $\beta_\xi \in [-1,1]$ denotes the skewness coefficient of the random variable ξ.

A collection of random variables are said to be *jointly* (strictly) α-stable if all their finite linear combinations are (strictly) α-stable. The functional $\|\cdot\|_\alpha^{1\wedge\alpha}$ metrizes the convergence in probability in the linear spaces of jointly α-stable random variables we consider. (Recall that when $\alpha = 1$, we consider only symmetric 1-stable random variables — see also Proposition 3.5.1 in Samorodnitsky and Taqqu,[11] for example.) That is, if ξ and $\xi_n,\ n \in \mathbb{N}$ are jointly α-stable random variables, then

$$\xi_n \xrightarrow{P} \xi, \quad \text{as } n \to \infty, \\ \Leftrightarrow \|\xi_n - \xi\|_\alpha \to 0, \quad \text{as } n \to \infty,$$

where $\xrightarrow{P}$ denotes convergence in probability. Furthermore, if ξ and η are jointly α-stable, then the following triangle inequality holds

$$\|\xi + \eta\|_\alpha^{1\wedge\alpha} \le \|\xi\|_\alpha^{1\wedge\alpha} + \|\eta\|_\alpha^{1\wedge\alpha}, \tag{2.3}$$

where $1 \wedge \alpha$ denotes $\min\{1,\alpha\}$. When $1 \le \alpha \le 2$, $\|\cdot\|_\alpha^{1\wedge\alpha} = \|\cdot\|_\alpha$ becomes a norm.

For all $g(s) \in L^\alpha(ds)$, the scale coefficient $\|\xi\|_\alpha$ of the stochastic integral $\xi = \int_{\mathbb{R}} g(s) M_{\alpha,\beta}(ds)$ is given by

$$\|\xi\|_\alpha = \left(\int_{\mathbb{R}} |g(s)|^\alpha\, ds\right)^{1/\alpha} = \|g(s)\|_{L^\alpha(ds)}.$$

In fact, the α-stable stochastic integral defines a linear isometry between the complete metric space of jointly α-stable random variables with the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$ and the complete metric space of functions $L^\alpha(ds)$ with the metric $\|\cdot\|_{L^\alpha(ds)}^{1\wedge\alpha}$. Observe that the scale coefficient $\|\xi\|_\alpha$ of the random variable ξ does *not* depend on the skewness intensity $\beta(s),\ s \in \mathbb{R}$ of the α-stable measure $M_{\alpha,\beta}(ds)$. For more details, see for example, the book of Samorodnitsky and Taqqu.[11]

In the sequel, we shall add a subscript β or sometimes $\beta(\cdot)$ to X and Y to indicate the skewness intensity of the underlying α-stable measure. For example, $\partial_v^n X_\beta(u,v),\ u \in \mathbb{R},\ v \in (0,1),\ n \in \mathbb{N}$ denotes the α-stable field, defined by (2.1), and $Y_\beta(t),\ t \in \mathbb{R}$ denotes the LMSM process, defined in (1.7). We omit the subscript β or $\beta(\cdot)$ when it is not necessary to indicate explicitly the skewness intensity of the underlying α-stable measure.

The fields $\partial_v^0 X(u,v) := X(u,v)$ and $\partial_v^n X(u,v),\ n \in \mathbb{N}$ have the following properties.

Proposition 2.1. *Let $\alpha \in (0,2]$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then:*

(a) *For all $v \in (0,1)$, $u \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have $\partial_v^n f(u,v,s) \in L^\alpha(ds)$.*
(b) *For all $n \in \mathbb{N}_0$, $\partial_v^n X_\beta(0,v) = 0$, almost surely, and for all $h > 0$,*

$$\begin{aligned}\{(\partial_v^n X_{\beta(\cdot)}(u+h,v) - \partial_v^n X_{\beta(\cdot)}(h,v)),\\ (n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\}\\ =_d \{\partial_v^n X_{\beta(\cdot - h)}(u,v),\ (n,u,v) \in \mathbb{N}_0\\ \times \mathbb{R} \times (0,1)\},\end{aligned} \tag{2.4}$$

where $=_d$ denotes the equality of the finite-dimensional distributions.
(c) *For all $c > 0$,*

$$\begin{aligned}&\{\partial_v^n X_{\beta(\cdot)}(cu,v),\ (n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\}\\ &=_d \left\{c^v \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} \partial_v^k X_{\beta(c\cdot)}(u,v),\right.\\ &\qquad \left.(n,u,v) \in \mathbb{N}_0 \times \mathbb{R} \times (0,1)\right\}.\end{aligned} \tag{2.5}$$

(d) *The field $\partial_v^n X = \{\partial_v^n X(u,v),\ u \in \mathbb{R},\ v \in (0,1)\}$ is continuous in the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$, and hence in probability, with respect to u and v.*
(e) *For all $u > 0$ and $v,\ v_0 \in (0,1)$, we have*

$$\begin{aligned}X(u,v) = X(u,v_0) + (v - v_0)\partial_v X(u,v_0)\\ + (v - v_0)^2 \xi_2(u,v,v_0).\end{aligned} \tag{2.6}$$

The remainder term $\xi_2(u,v,v_0)$ is an α-stable variable, such that

$$\sup_{(u,v)\in[a,b]\times[v_0-\epsilon,v_0+\epsilon]} \|\xi_2(u,v,v_0)\|_\alpha \\ =: C_\alpha(a,b,\epsilon,v_0) < \infty,$$

for all $[a,b] \times [v_0 - \epsilon, v_0 + \epsilon] \subset (0,\infty) \times (0,1)$. Furthermore, the function $C_\alpha(a,b,\epsilon,v_0)$ is continuous in v_0, for all $v_0 \in (\epsilon, 1-\epsilon)$.

Proposition 2.1 follows from Theorem 2.1, Corollary 2.1 and Theorem 2.2 in Stoev and Taqqu.[1]

Suppose that the skewness intensity $\beta(s),\ s \in \mathbb{R}$ of the α-stable measure $M_{\alpha,\beta}(ds)$ is constant. Then parts (b) and (c) of Proposition 2.1, imply that the fields $\partial_v^n X_\beta(u,v),\ n \in \mathbb{N}_0$ have *stationary increments* and some type of *scaling property.* By setting $n = 0$ in (2.4)–(2.5) they imply in particular the stationarity of the increments and the self-similarity and of the linear fractional stable motion process $X_{H,\alpha,\beta}$.

The following result will be used extensively.

Theorem 2.1. *Let $\alpha \in (0,2]$ and $K \subset \mathbb{R} \times (0,1)$ be a compact set. For all (u',v'), $(u'',v'') \in K$, we have*

$$\|X_\beta(u',v') - X_\beta(u'',v'')\|_\alpha^{1\wedge\alpha} \le C_K(|u'-u''|^{(1\wedge\alpha)v'} + |v'-v''|^{(1\wedge\alpha)}), \quad (2.7)$$

for some constant $C_K > 0$.

Proof. It suffices to prove (2.7) for an arbitrary compact set $K = [-T,T] \times [\epsilon, 1-\epsilon]$, where $T \in (0,\infty)$ and $\epsilon \in (0,1/2)$. By adding and subtracting the term $X_\beta(u'',v')$, using the triangle inequality (2.3) for the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$ and Proposition 2.1(b), we get

$$\begin{aligned}\|X_\beta(u',v') - X_\beta(u'',v'')\|_\alpha^{1\wedge\alpha} &\le \|X_\beta(u',v') - X_\beta(u'',v')\|_\alpha^{1\wedge\alpha} \\ &\quad + \|X_\beta(u'',v') - X_\beta(u'',v'')\|_\alpha^{1\wedge\alpha} \\ &\le \|X_\beta(u',v') - X_\beta(u'',v')\|_\alpha^{1\wedge\alpha} \\ &\quad + \|X_\beta(u''+2T,v') - X_\beta(u''+2T,v'')\|_\alpha^{1\wedge\alpha} \\ &\quad + \|X_\beta(2T,v') - X_\beta(2T,v'')\|_\alpha^{1\wedge\alpha} \\ &=: A + B + C. \end{aligned} \quad (2.8)$$

By Proposition 2.1(b) and 2.1(c), we get that the term A in the right-hand-side of (2.8) equals

$$\begin{aligned}&\|X_{\beta(\cdot-u'')}(u'-u'',v')\|_\alpha^{1\wedge\alpha} \\ &\quad = |u'-u''|^{(1\wedge\alpha)v'} \|X_{\beta(|u'-u''|\cdot-u'')} \\ &\qquad \times (\operatorname{sign}(u'-u''),v')\|_\alpha^{1\wedge\alpha} \\ &\quad = |u'-u''|^{(1\wedge\alpha)v'} \|X_{\beta(\cdot)}(1,v')\|_\alpha^{1\wedge\alpha},\end{aligned}$$

where the last equality follows by the fact that the scale coefficient $\|\cdot\|_\alpha$ does not depend on the skewness intensity $\beta(s)$, $s \in \mathbb{R}$. By using the continuity in probability of the field $X_\beta(u,v)$ [Proposition 2.1(d)], we obtain, for all (u',v'), $(u'',v'') \in K$,

$$\begin{aligned}A &= \|X_\beta(u',v') - X_\beta(u'',v')\|_\alpha^{1\wedge\alpha} \\ &\le C_{1,\epsilon}|u'-u''|^{(1\wedge\alpha)v'},\end{aligned} \quad (2.9)$$

where $C_{1,\epsilon} := \sup_{v\in[\epsilon,1-\epsilon]} \|X_\beta(1,v)\|_\alpha^{1\wedge\alpha}$ is finite, since the interval $[\epsilon, 1-\epsilon]$ is compact.

We now focus on bounding the terms B and C in (2.8). By applying Proposition 2.1(e) and using (2.3), we obtain that for all v', $v'' \in [\epsilon, 1-\epsilon]$, $|v'-v''| \le \epsilon/2$ and $u'' \in [-T,T]$,

$$\begin{aligned}B + C &\le |v'-v''|^{1\wedge\alpha}(\|\partial_v X_\beta(u''+2T,v'')\|_\alpha^{1\wedge\alpha} \\ &\qquad + \|\partial_v X_\beta(2T,v'')\|_\alpha^{1\wedge\alpha}) \\ &\qquad + 2|v'-v''|^{2(1\wedge\alpha)} C_\alpha^{1\wedge\alpha}(T,3T,\epsilon/2,v'') \\ &\le C_{2,T,\epsilon}|v'-v''|^{1\wedge\alpha},\end{aligned} \quad (2.10)$$

where

$$\begin{aligned}C_{2,T,\epsilon} := &\sup_{u''\in[-T,T],\ v''\in[\epsilon,1-\epsilon]} \\ &\times (\|\partial_v X_\beta(u''+2T,v'')\|_\alpha^{1\wedge\alpha} \\ &+ \|\partial_v X_\beta(2T,v'')\|_\alpha^{1\wedge\alpha}) \\ &+ 2 \sup_{v''\in[\epsilon,1-\epsilon]} C_\alpha^{1\wedge\alpha}(T,3T,\epsilon/2,v''). \end{aligned} \quad (2.11)$$

The constant $C_{2,T,\epsilon}$, above, is finite. Indeed, the first supremum in (2.11) is finite because the field $\partial_v X_\beta(u,v)$ is continuous in the metric $\|\cdot\|_\alpha^{1\wedge\alpha}$ for all $u \in \mathbb{R}$, $v, v_0 \in (0,1)$ [Proposition 2.1(d)]. The second supremum is also finite, because the function $C_\alpha(T,3T,\epsilon/2,v'')$ is continuous in v'', for all $v'' \in [\epsilon, 1-\epsilon]$.

When $|v'-v''| > \epsilon/2$, by using the continuity of the field $X_\beta(u,v)$ and the compactness of the set K, one can trivially bound above the terms B and C by a constant $C_{3,\epsilon}$, for example. Therefore, Relation (2.10) implies that

$$\begin{aligned}B + C &\le C_{2,T,\epsilon}|v'-v''|^{1\wedge\alpha} + C_{3,\epsilon} \\ &\le \left(C_{2,T,\epsilon} + \frac{C_{3,\epsilon}}{(\epsilon/2)^{1\wedge\alpha}}\right)|v'-v''|^{1\wedge\alpha}.\end{aligned}$$

This inequality, (2.9) and (2.8) yield (2.7), which completes the proof of the theorem. □

3. ALMOST SURE BOUNDEDNESS AND CONTINUITY OF LMSM

We focus here on the boundedness and continuity properties of the paths of the linear multifractional stable motion process Y defined in (1.7). Our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$ and we assume that the σ-algebra $\mathcal{F}$ is complete, that is, it contains all subsets of $\mathbb{P}$-null sets. We also assume, without loss of generality, that the stochastic processes appearing in the sequel are *separable* in the sense of Doob[15] (or, equivalently, *strongly separable* in the sense of Definition 9.2.3 in Samorodnitsky and Taqqu[11] — see also Theorem 9.2.5 therein). We will also say that a process $\tilde{X} = \{\tilde{X}(t)\}_{t\in(a,b)}$, $(a,b) \subset \mathbb{R}$, $a < b$, is a *version* (or *modification*) of the process $X = \{X(t)\}_{t\in(a,b)}$, if both $\tilde{X}$ and X are defined on the same probability space and, if

$$\mathbb{P}(\{\omega : \tilde{X}(\omega,t) \ne X(\omega,t)\}) = 0, \quad \text{for all } t \in (a,b).$$

The behavior of the paths of the linear fractional stable motion process $X_{H,\alpha} = \{X_{H,\alpha}(t),\ t \in \mathbb{R}\}$ is well known. When $H \ne 1/\alpha$, that is, when $X_{H,\alpha}$ is *not* a Lévy motion, we have the following dichotomy:

1. If $0 < H < \min\{1, 1/\alpha\}$, then the paths of any version of the LFSM process $X_{H,\alpha}(t)$, $t \in \mathbb{R}$ are *unbounded* on every finite interval $t \in (a, b)$, $a < b$, with probability one.
2. If $1/\alpha < H < 1$, then there exists a version of the process $X_{H,\alpha}$, which has continuous paths.

The proofs of these two results can be found, for example, in Chap. 10 of Samorodnitsky and Taqqu.[11] The following two theorems give the corresponding results for the LMSM process Y.

Theorem 3.1. *Let* $0 < \alpha < 2$ *and let* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *be the LMSM process defined in* (1.7), *where the function* $H(t)$, $t \in \mathbb{R}$ *satisfies* (1.6). *Suppose that, for some* $(a, b) \subset \mathbb{R}$, $a < b$,

$$d^* := \sup_{t\in(a,b)} H(t) - 1/\alpha < 0. \tag{3.1}$$

Then, for any separable version $\tilde{Y} = \{\tilde{Y}(t),\ t \in (a, b)\}$ *of the process* Y, *we have that*

$$\mathbb{P}\Big(\Big\{\omega\colon \sup_{t\in(a',b')} |\tilde{Y}(t, \omega)| = \infty\Big\}\Big) = 1,$$
$$\text{for all } (a', b') \subset (a, b),\ \ a' < b'.$$

That is, every version of the process $Y = \{Y(t),\ t \in (a, b)\}$ *has unbounded paths on any sub-interval* $(a', b') \subset (a, b)$ *of positive length.*

Proof. We will apply Theorem 10.2.3 in Samorodnitsky and Taqqu.[11] Indeed, let $(a', b') \subset (a, b)$, $a' < b'$, be an arbitrary sub-interval of positive length. Consider the countable set $T^* = \mathbb{Q} \cap (a', b')$, where $\mathbb{Q}$ denotes the set of rational numbers. Introduce the function

$$f^*(T^*; s) := \sup_{t\in T^*} |f(t, H(t), s)|, \quad s \in \mathbb{R},$$

where $f(u, v, s)$ is given in (1.5).

Without loss of generality, we assume that $a^+ \neq 0$. Relation (3.1) implies that $f^*(T^*; s) = \infty$, for all $s \in (-b', -a') \setminus \{0\}$. Indeed, let $s \in (-b', -a') \setminus \{0\}$. Since T^* is dense in (a', b'), there exists a sequence $t_n \in T^*$, $n \in \mathbb{N}$ such that $0 < t_n + s < 1$, $n \in \mathbb{N}$ and $t_n + s \to 0$, as $n \to \infty$. Now by (3.1), we have that $H(t_n) - 1/\alpha \le d^* < 0$ and since $0 < s + t_n < 1$, $n \in \mathbb{N}$, we have $(t_n + s)_-^{H(t)-1/\alpha} = 0$. Using the triangle inequality, we get

$$\begin{aligned} |f(t_n, H(t_n), s)| &\ge |a^+|(s + t_n)_+^{H(t_n)-1/\alpha} \\ &\quad - (|a^+| + |a^-|)|s|^{H(t_n)-1/\alpha} \\ &\ge |a^+|(s + t_n)_+^{d^*} \\ &\quad - (|a^+| + |a^-|)|s|^{H(t_n)-1/\alpha}. \end{aligned}$$

For all $s \in (-b', -a') \setminus \{0\}$, the last expression converges to infinity, as $n \to \infty$, because $t_n + s \downarrow 0$ and $d^* < 0$.

Therefore $f^*(T^*; s) = \infty$, $s \in (-b', -a') \setminus \{0\}$ and hence $\int_{\mathbb{R}} (f^*(T^*; s))^\alpha ds = \infty$, which shows that Condition (10.2.14) of Theorem 10.2.3 in Samorodnitsky and Taqqu[11] is violated and hence the process Y does not have a version with bounded paths on the interval (a', b'). This, in view of the zero-one law for sample boundedness of α-stable processes (Corollary 9.5.5 in Samorodnitsky and Taqqu[11]) means that any version of the process Y will have unbounded paths on (a', b'), with probability one. This completes the proof of the theorem, since the interval $(a', b') \subset (a, b)$ was arbitrary. □

In the following theorem, we give conditions for the almost sure continuity of the paths of the LMSM process Y. Since the continuity of $H(t)$, $t \in \mathbb{R}$ is necessary and sufficient for the continuity in probability of the LMSM process Y (Theorem 3.1 in Stoev and Taqqu[1]), the paths of Y can be continuous only if the function $H(t)$, $t \in \mathbb{R}$ is continuous.

Theorem 3.2. *Let* $\alpha \in (1, 2)$ *and* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *be the LMSM process defined in* (1.7), *where the function* $H(t) \in (0, 1)$, $t \in \mathbb{R}$ *is continuous on the interval* $t \in (a, b) \subset \mathbb{R}$, $a < b$. *Suppose that* $1/\alpha < H(t) < 1$, $t \in (a, b)$ *and that, for all* $t', t'' \in (a, b)$,

$$|H(t') - H(t'')| \le C|t' - t''|^\rho, \quad \text{with } 1/\alpha < \rho, \tag{3.2}$$

where $C > 0$ *does not depend on* t' *and* t''.

Then, the process Y *has a version* $\tilde{Y}$ *which has continuous paths on the interval* (a, b). *Furthermore, for any* $[a', b'] \subset (a, b)$ *and*

$$0 < r < \left(\rho \wedge \min_{t\in[a',b']} H(t)\right) - 1/\alpha, \tag{3.3}$$

we have that, for almost all $\omega \in \Omega$,

$$|\tilde{Y}(t', \omega) - \tilde{Y}(t'', \omega)| \le R(\omega)|t' - t''|^r, \tag{3.4}$$

for all $t', t'' \in [a', b']$. *Equivalently, for all* r *satisfying* (3.3),

$$\mathbb{P}\left(\left\{\omega : \lim_{\epsilon\to0,\epsilon>0} \sup_{|t'-t''|<\epsilon,t',t''\in[a',b']} \times \frac{|\tilde{Y}(t', \omega) - \tilde{Y}(t'', \omega)|}{|t' - t''|^r} = 0\right\}\right) = 1. \tag{3.5}$$

Proof. Let $[a', b']$, $a' < b'$ be an arbitrary closed sub-interval of (a, b). We will first show that for all $\gamma \in (1, \alpha)$ and t', $t'' \in [a', b']$,

$$\mathbb{E}|Y_\beta(t') - Y_\beta(t'')|^\gamma \leq C(\gamma)|t' - t''|^{(r^*+1/\alpha)\gamma}, \quad (3.6)$$

for some constant $C(\gamma) > 0$, where

$$r^* := \Big(\rho \wedge \min_{t\in[a',b']} H(t)\Big) - 1/\alpha.$$

Observe that by Lemma 6.2, the left-hand-side of (3.6) is bounded above by $C_{\alpha,\gamma}\|Y_\beta(t') - Y_\beta(t'')\|_\alpha^\gamma$. Therefore, in view of (1.10), Theorem 2.1 implies that, for all $\gamma \in (0, \alpha)$ and t', $t'' \in [a', b']$,

$$\mathbb{E}|Y_\beta(t') - Y_\beta(t'')|^\gamma \leq C_{\alpha,\gamma}\tilde{C}_K(|t' - t''|^{(1\wedge\alpha)H(t')} + |H(t') - H(t'')|^{(1\wedge\alpha)})^{\gamma/(1\wedge\alpha)}, \quad (3.7)$$

where $\tilde{C}_K = C_K^{\gamma/(1\wedge\alpha)}$ and where the set $K := \{(t, H(t))\}_{t\in[a',b']}$ is compact because $[a', b']$ is compact and $H(t)$ is continuous.

For all t', $t'' \in [a', b']$, and x, $y > 0$, we have

$$\begin{aligned} |t' - t''|^x &= |b' - a'|^x \left(\frac{|t' - t''|}{|b' - a'|}\right)^x \\ &\leq |b' - a'|^{(x - x\wedge y)}|t' - t''|^{x\wedge y} \\ &\leq (1 \vee |b' - a'|^{x\vee y})|t' - t''|^{x\wedge y}. \quad (3.8) \end{aligned}$$

Using $1\wedge\alpha = 1$ and Relations (3.2), (3.7) and (3.8), we get

$$\begin{aligned} \mathbb{E}|Y_\beta(t') - Y_\beta(t'')|^\gamma &\leq C_{\alpha,\gamma}\tilde{C}_K((1 \vee |b' - a'|) \\ &\quad \times |t' - t''|^{H_*} + C|t' - t''|^\rho)^\gamma \\ &\leq C(\gamma)|t' - t''|^{(H_*\wedge\rho)\gamma} \\ &= C(\gamma)|t' - t''|^{(r_*+1/\alpha)\gamma}, \end{aligned}$$

where $H_* := \min_{t\in[a',b']} H(t)$. This proves (3.6).

We will now show that (3.6) implies the statement of the theorem. Since $r^* > 0$ one can choose $\gamma \in (1, \alpha)$ close enough to α, so that $(r^*+1/\alpha)\gamma > 1$. Then, by (3.6), the Kolmogorov's continuity criterion applies and therefore, the process Y has a version $\tilde{Y}$ with continuous paths on the interval $[a', b']$. Since the open interval (a, b) can be approached by a countable sequence of closed intervals $[a_n', b_n'] \subset (a, b)$, we conclude, using Lemma 6.1, that Y has a version $\tilde{Y}$ with continuous paths on (a, b).

We now focus on the second statement of the theorem. Let $\tilde{Y}$ be a version of the process Y with continuous paths on the interval (a, b). By the strong version of the Kolmogorov's continuity criterion (see, for example, Theorem 3.3.16 in Stroock[16]), we have that for all r, $0 < r < ((r^* + 1/\alpha)\gamma - 1)/\gamma = r^* + 1/\alpha - 1/\gamma$ and $R > 0$,

$$\mathbb{P}\left(\left\{\omega : \sup_{t',t''\in[a',b']} \frac{|\tilde{Y}(t', \omega) - \tilde{Y}(t'', \omega)|}{|t' - t''|^r} \geq R\right\}\right) \leq \frac{KC(\gamma)}{R^\gamma},$$

where $K = K(\gamma, (r^* + 1/\alpha)\gamma - 1, h, b' - a') < \infty$. Borel-Cantelli lemma ($\gamma > 1$) implies that for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$|\tilde{Y}(t', \omega) - \tilde{Y}(t'', \omega)| \leq R_h(\omega)|t' - t''|^r, \quad (3.9)$$

for all t', $t'' \in [a', b']$ and some $R_h(\omega) < \infty$. Since (3.9) is valid with probability one, for all $0 < r < r^* + 1/\alpha - 1/\gamma$ and since $\gamma \in (1, \alpha)$ can be chosen arbitrarily close to α, we obtain that Relation (3.4) holds for all $0 < r < r^*$ and hence that Relation (3.5) holds as well. This completes the proof of the theorem. □

Remarks.

1. Theorem 3.2 states that the Hölder condition (3.2) on H implies that the paths of the process Y are Hölder-continuous in the sense of (3.4). We conjecture that if the regularity condition (3.2) on $H(t)$ is dropped, then the process $\{Y(t), t \in (a, b)\}$ will have a version with continuous paths, if the local scaling exponent function $H(t)$ is continuous and if $1/\alpha < H(t) < 1$, for all $t \in (a, b)$. This is true in the Gaussian case (see, for example, Ayache and Taqqu[6]).
2. Theorem 3.1 states that the paths of LMSM cannot be continuous in a neighborhood of t_0 if $H(t_0) < 1/\alpha$ and hence the condition $H(t_0) \geq 1/\alpha$ is necessary for continuity. Can one sharpen this to $H(t_0) > 1/\alpha$? In other words, is the more restrictive condition $H(t_0) > 1/\alpha$ necessary for continuity? It is so in the case of LFSM, since if $H(t)$ is constant and equal to $1/\alpha$, then the process coincides with the α-stable $(1 < \alpha < 2)$ Lévy motion which does not have continuous paths.
3. Relation (3.5) implies $\rho_{\tilde{Y}}^{\text{unif}}([a', b']) \geq r$, a.s. where the ρ_f^{unif} is the uniform Hölder exponent defined in the next section.

4. HÖLDER REGULARITY OF LMSM

In this section, we suppose that the paths of the LMSM process Y are continuous and establish upper and lower bounds on their *pointwise* and *uniform pointwise Hölder exponents*. These Hölder exponents characterize the regularity of the paths and are defined as follows.

Definition 4.1. Let $f\colon (a,b) \mapsto \mathbb{R}$ be a continuous function. The exponent

$$\rho_f(t) := \sup\left\{\rho \geq 0 :\ \lim_{h\to 0} \frac{f(t+h)-f(t)}{|h|^\rho} = 0\right\} \tag{4.1}$$

is called the *pointwise Hölder exponent* of the function f at t.

Definition 4.2. Let $f\colon (a,b) \mapsto \mathbb{R}$ be a continuous function. The exponent

$$\rho_f^{\text{unif}}(t) := \sup\left\{\rho \geq 0 :\ \lim_{t',t''\to t} \frac{f(t')-f(t'')}{|t'-t''|^\rho} = 0\right\} \tag{4.2}$$

is called the *uniform pointwise Hölder exponent* of the function f at t.

Definition 4.3. Let $f\colon (a,b) \mapsto \mathbb{R}$ be a continuous function and let $J \subset (a,b)$. The exponent

$$\rho_f^{\text{unif}}(J) := \sup\left\{\rho \geq 0\colon\ \sup_{s',s''\in J} \frac{|f(s')-f(s'')|}{|s'-s''|^\rho} < \infty\right\} \tag{4.3}$$

is called the *uniform Hölder exponent* of the function f over the set J.

One always has

$$0 \leq \rho_f^{\text{unif}}((a,b)) \leq \rho_f^{\text{unif}}(t) \leq \rho_f(t) \leq \infty, \tag{4.4}$$

for all $t \in (a,b) \subset \mathbb{R}$.

Definitions 4.1 and 4.2 above coincide with the corresponding definitions of pointwise and uniform pointwise Hölder exponents given in Seuret and Lévy-Vehel[17] when the function $f(t)$ is not differentiable at t (see Definitions 2.1 and 2.2, therein). In the sequel we deal only with stochastic processes which have non-differentiable sample paths.

Observe that if $\rho_f^{\text{unif}}(t) > 0$, then (4.2) implies that for all $\rho \in (0, \rho_f^{\text{unif}}(t))$, there exist $\epsilon > 0$ and $C < \infty$, such that

$$|f(t')-f(t'')| \leq C|t'-t''|^\rho, \quad \text{for all } t',\ t'' \in (t-\epsilon, t+\epsilon). \tag{4.5}$$

The notions of uniform Hölder exponent and uniform pointwise Hölder exponent given above are closely related. Indeed:

Lemma 4.1. *Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. For any a, $b \in \mathbb{R}$, $a < b$, we have*

$$\inf_{t\in[a,b]} \rho_f^{\text{unif}}(t) \leq \rho_f^{\text{unif}}([a,b]) \leq \rho_f^{\text{unif}}((a,b)) \leq \inf_{t\in(a,b)} \rho_f^{\text{unif}}(t). \tag{4.6}$$

The proof of this lemma is given in Sec. 6 below. The following example shows that the exponents ρ_f and ρ_f^{unif} are not necessarily equal.

Example. Let

$$f(t) := |t|^{1/2}\sin(1/t),\ t \in (-1,0)\cup(0,1),$$

and $f(0) := 0$. The function f is continuous in $(-1,1)$, but oscillates violently around 0. Seuret and Lévy-Vehel[17] state that

$$\rho_f^{\text{unif}}(0) = 1/4 < 1/2 = \rho_f(0).$$

We will show here that $\rho_f^{\text{unif}}(0) \leq 1/4$ and $1/2 \leq \rho_f(0)$. The inequality $1/2 \leq \rho_f(0)$ follows from the fact that $f(t) = \mathcal{O}(|t|^{1/2})$, $t \to 0$. To show that $\rho_f^{\text{unif}}(0) \leq 1/4$, choose t_k' and t_k'', such that $\sin(1/t_k') = 1$ and $\sin(1/t_k'') = -1$, for example, $t_k' := 2/(\pi+4k\pi)$ and $t_k'' := 2/(3\pi+4k\pi)$, $k \in \mathbb{N}$. Because of the opposite signs,

$$\begin{aligned}\frac{|f(t_k')-f(t_k'')|}{|t_k'-t_k''|^{1/4}} &= \frac{\sqrt{t_k'}+\sqrt{t_k''}}{(\pi t_k' t_k'')^{1/4}} \\ &= \frac{1}{\pi^{1/4}}((t_k'/t_k'')^{1/4} + (t_k''/t_k')^{1/4}) \to \frac{2}{\pi^{1/4}},\end{aligned}$$

as $k \to \infty$, since $t_k'/t_k'' \to 1$. Since t_k', $t_k'' \to 0$, as $k \to \infty$, (4.2) implies that $\rho_f^{\text{unif}}(0) \leq 1/4$.

The above example involves a function, which has an *oscillation* or *chirp-like* singularity at 0.[18] The difference between the notions of pointwise and uniform pointwise Hölder exponents of a deterministic function can be studied in the context of the so-called two-microlocal spaces.[17] These spaces provide a more detailed description of the local behavior of a deterministic function.[19,20]

We will use the following notation.

Definition 4.4. For two separable processes $\xi(t)$ and $\eta(t)$, $t \in (a,b)$, we write $\xi(t) \leq^{w.p.1} \eta(t)$, $t \in (a,b)$ (with probability 1) if

$$\mathbb{P}(\{\omega\colon \xi(t,\omega) \leq \eta(t,\omega),\ t \in (a,b)\}) = 1 \tag{4.7}$$

and $\xi(t) \overset{\text{a.s.}}{\leq} \eta(t)$, $t \in (a,b)$ if the weaker relation

$$\mathbb{P}(\{\omega : \xi(t,\omega) \leq \eta(t,\omega)\}) = 1, \quad \text{for all } t \in (a,b) \tag{4.8}$$

holds.

The next result shows that, in general, the (random) pointwise Hölder exponent $\rho_Y(t)$ of the paths of the LMSM process Y is not greater than the smaller of $H(t)$ and its pointwise Hölder exponent $\rho_H(t)$ (see Definition 4.2).

Proposition 4.1. *Let* $1 < \alpha \leq 2$ *and* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *be the LMSM process, defined in* (1.7), *where the function* H *satisfies* (1.6) *and is continuous. Suppose that the paths of the process* Y *are continuous. Then, for all* $t \neq 0$ *and* $Y(t) \neq 0$, *a.s. we have that*

$$\rho_Y(t,\omega) \overset{\text{a.s.}}{\leq} \rho_H(t) \wedge H(t), \tag{4.9}$$

where $\rho_Y(t,\omega) := \rho_{Y(\cdot,\omega)}(t)$, $\omega \in \Omega$ *denotes the pointwise Hölder exponent of the function* $Y(t,\omega)$, $t \in \mathbb{R}$, *defined in* (4.1).

Proof. Fix $t \neq 0$ and suppose $Y(t) \neq 0$. By (1.10), we have that

$$\begin{aligned} Y(t+h) - Y(t) &= (X(t+h, H(t+h)) \\ &\quad - X(t, H(t+h))) \\ &\quad + (X(t, H(t+h)) - X(t, H(t))) \\ &=: A_t(h) + B_t(h). \end{aligned} \tag{4.10}$$

We will show that for all $\rho > \rho_H(t) \wedge H(t)$,

$$\limsup_{h\to 0} \frac{|Y(t+h) - Y(t)|}{|h|^\rho} = \infty, \quad \text{almost surely,} \tag{4.11}$$

which will complete the proof of the statement. We consider separately the cases (i) $H(t) < \rho_H(t)$, (ii) $\rho_H(t) < H(t)$ and (iii) $H(t) = \rho_H(t)$.

Observe first that (4.11) is equivalent to

$$\limsup_{h\to 0} \frac{\|Y(t+h) - Y(t)\|_\alpha}{|h|^\rho} = \infty. \tag{4.12}$$

Indeed, (4.11) implies (4.12), since the almost sure convergence implies the convergence in probability, which, for strictly α-stable random variables, is equivalent to the convergence in the metric $\|\cdot\|_\alpha^{1\wedge\alpha} = \|\cdot\|_\alpha$. Conversely, if (4.12) holds, then there is a sequence $h_n \to 0$, $h_n \neq 0$, as $n \to \infty$, such that $|Y(t+h_n) - Y(t)|/|h_n|^\rho \to^P \infty$, and therefore there is a sub-sequence $h_{n_k} \to 0$, such that $|Y(t+h_{n_k}) - Y(t)|/|h_{n_k}|^\rho \to^{\text{a.s.}} \infty$, as $n_k \to \infty$. This implies (4.11).

(i) Suppose that $H(t) < \rho_H(t)$ and let $\rho \in (H(t), \rho_H(t))$ be arbitrary. By using Proposition 2.1(b) and (c), we obtain that

$$\begin{aligned} \frac{1}{|h|^\rho}\|A_t(h)\|_\alpha &= \frac{1}{|h|^\rho}\|X_{\beta(\cdot - t)}(h, H(t+h))\|_\alpha \\ &= \frac{|h|^{H(t+h)}}{|h|^\rho}\|X_\beta(\operatorname{sign}(h), H(t+h))\|_\alpha \\ &= |h|^{H(t+h)-\rho}\|X_{\beta(\cdot)}(1, H(t+h))\|_\alpha, \end{aligned} \tag{4.13}$$

where the last equality follows by the facts that the scale coefficient $\|\cdot\|_\alpha$ does not depend on the skewness intensity $\beta(s)$, $s \in \mathbb{R}$ and since the kernel $f(-1, v, s) = -f(1, v, s-1)$, $s \in \mathbb{R}$, $v \in (0,1)$.

The right-hand-side of (4.13) converges to infinity, as $h \to 0$. Indeed, the function $H(\cdot)$ is continuous and $H(t) < \rho$, thus $|h|^{H(t+h)-\rho} \to \infty$, as $h \to 0$. Also the continuity in probability of the field $X_\beta(u,v)$ [Proposition 2.1(d)] implies that $\|X_\beta(1, H(t+h))\|_\alpha \to \|X_\beta(1, H(t))\|_\alpha$, as $h \to 0$, where $\|X_\beta(1, H(t))\|_\alpha = |t|^{-H(t)}\|X_\beta(t, H(t))\|_\alpha$ is nonzero since by assumption $Y_\beta(t) = X_\beta(t, H(t)) \neq 0$.

On the other hand, by applying Proposition 2.1(e) to the second term in (4.10), we obtain

$$\begin{aligned} \frac{1}{|h|^\rho}|B_t(h)| &= \frac{|H(t+h) - H(t)|}{|h|^\rho}(|\partial_v X_\beta(t, H(t))| \\ &\quad + \mathcal{O}_P(H(t+h) - H(t))) \\ &\xrightarrow{P} 0, \quad \text{as } h \to 0. \end{aligned} \tag{4.14}$$

The last convergence follows from the fact that $H(t+h) - H(t) = o(|h|^\rho)$, as $h \to 0$, since $\rho < \rho_H(t)$ [see (4.1)].

Using the triangle inequality for the metric $\|\cdot\|_\alpha^{1\wedge\alpha} = \|\cdot\|_\alpha$ and Relations (4.10), (4.13) and (4.14), we obtain

$$\begin{aligned} &\limsup_{h\to 0} \frac{\|Y(t+h) - Y(t)\|_\alpha}{|h|^\rho} \\ &\quad \geq \limsup_{h\to 0} \left| \frac{\|A_t(h)\|_\alpha}{|h|^\rho} - \frac{\|B_t(h)\|_\alpha}{|h|^\rho} \right| = \infty, \end{aligned}$$

and hence the convergence in (4.11) holds.

(ii) Suppose now that $\rho_H(t) < H(t)$ and let $\rho \in (\rho_H(t), H(t))$. By using similar arguments as in Part (i) above, we now obtain that

$$\lim_{h\to 0} \frac{\|A_t(h)\|_\alpha}{|h|^\rho} = 0 \quad \text{and}$$
$$\limsup_{h\to 0} \frac{\|B_t(h)\|_\alpha}{|h|^\rho} = \infty. \tag{4.15}$$

Indeed, the first convergence in (4.15) follows by (4.13) and the fact that $\rho < H(t)$. The second convergence in (4.15) follows by Relation (4.14) and the fact that $\partial_v X(t, H(t))$, $t \neq 0$ is a non-trivial α-stable random variable and thus the remainder term in (4.14) is negligible with respect to $\partial_v X(t, H(t))$, as $h \to 0$. Hence (4.11) holds as in case (i).

(iii) Assume that $\rho_H(t) = H(t)$. Suppose, *ad absurdum*, that for some $\rho \in (\rho_H(t), 2\rho_H(t)) \equiv (H(t), 2H(t)) \subset (0,2)$, Relation (4.11) and, equivalently, (4.12) do not hold. Hence, there exists a constant $C > 0$, such that

$$\|Y(t+h) - Y(t)\|_\alpha \leq C|h|^\rho, \quad \text{for all } |h| \leq \epsilon, \tag{4.16}$$

for some $\epsilon > 0$.

Relation (1.9) implies that the first term in (4.10) can be expressed as follows:

$$A_t(h) = |h|^{H(t+h)} \int_{\mathbb{R}} g_h(t,s) M_{\alpha,\beta}(ds). \tag{4.17}$$

As in Relation (4.14), by using Proposition 2.1(e) and the continuity of the function H at t, for the second term in (4.10), we obtain

$$\begin{aligned} B_t(h) = {}& (H(t+h) - H(t)) \\ & \times \int_{\mathbb{R}} \partial_v f(t, H(t), s) M_{\alpha,\beta}(ds) \\ & + (H(t+h) - H(t))^2 \mathcal{O}_P(1), \end{aligned} \tag{4.18}$$

as $h \to 0$. Therefore, by using the triangle inequality (2.3) for the metric $\|\cdot\|_\alpha$ and (4.10), (4.17) and (4.18), one gets

$$\begin{aligned} &\|Y(t+h) - Y(t)\|_\alpha + |H(t+h) - H(t)|^2 \mathcal{O}(1) \\ &\geq \Bigg\| |h|^{H(t+h)} \int_{\mathbb{R}} \Bigg\{ g_h(t,s) + \frac{(H(t+h) - H(t))}{|h|^{H(t+h)}} \\ &\quad \times \partial_v f(t, H(t), s) \Bigg\} M_{\alpha,\beta}(ds) \Bigg\|_\alpha, \quad \text{as } h \to 0. \end{aligned} \tag{4.19}$$

Since $\rho < 2\rho_H(t)$, we have that

$$|H(t+h) - H(t)|^2 \leq \text{const}\, |h|^\rho,$$

as $h \to 0$ [see (4.1)]. Therefore, by dividing both sides of (4.19) by $|h|^{H(t+h)}$ and using (4.16), we obtain that, as $h \to 0$,

$$\begin{aligned} &\Bigg\| \int_{\mathbb{R}} \Bigg\{ g_h(t,s) + \frac{(H(t+h) - H(t))}{|h|^{H(t+h)}} \\ &\times \partial_v f(t, H(t), s) \Bigg\} M_{\alpha,\beta}(ds) \Bigg\|_\alpha \leq \text{const}\, |h|^{\rho - H(t+h)}. \end{aligned}$$

Hence, since H is continuous and $H(t) < \rho$, we get

$$\begin{aligned} &\Bigg(\int_{\mathbb{R}} \Bigg| g_h(t,s) + \frac{(H(t+h) - H(t))}{|h|^{H(t+h)}} \\ &\quad \times \partial_v f(t, H(t), s) \Bigg|^\alpha ds \Bigg)^{1/\alpha} \to 0. \end{aligned} \tag{4.20}$$

We will now show that the convergence in (4.20) is impossible, which will complete the proof of the proposition.

By using the triangle inequality for the metric $\|\cdot\|^{1\wedge\alpha}_{L^\alpha(ds)}$, we obtain that the left-hand-side of (4.20) is greater than or equal to

$$\begin{aligned} &\Bigg| \left(\int_{\mathbb{R}} |g_h(t,s)|^\alpha ds \right)^{1/\alpha} - \frac{|H(t+h) - H(t)|}{|h|^{H(t+h)}} \\ &\quad \times \left(\int_{\mathbb{R}} |\partial_v f(t, H(t), s)|^\alpha ds \right)^{1/\alpha} \Bigg| \to 0, \end{aligned} \tag{4.21}$$

as $h \to 0$. The second integral in (4.21) does not vanish since $t \neq 0$ [see (2.2)]. Also, in view of (4.23), by making a change of variables, we get that the first integral in (4.21) is equal to

$$\left(\int_{\mathbb{R}} |g_h(t,s)|^\alpha ds \right)^{1/\alpha} = \|X(1, H(t+h))\|_\alpha,$$

where X is defined in (1.9). Proposition 2.1(d) and the continuity of H at t imply that the last expression converges to $\|X(1, H(t))\|_\alpha$, which, as argued above, is positive. Therefore, the convergence in (4.21) is impossible, if $|H(t+h) - H(t)|/|h|^{H(t+h)}$ tends either to 0 or to ∞, as $h \to 0$.

Suppose now that

$$(H(t+h_n) - H(t))/|h_n|^{H(t+h)} \to C,$$

where $C \in \mathbb{R}$, $C \neq 0$, for some sequence $h_n \neq 0$, $h_n \to 0$, $n \to \infty$. Then, Relation (4.20) implies that the integrand in (4.20) converges in measure to zero, as $h_n \to 0$, and therefore, there exists a subsequence

$\tilde{h}_k := h_{n_k} \to 0$, as $k \to \infty$, such that

$$g_{\tilde{h}_k}(t,s) \to C\partial_v f(t, H(t), s), \quad \text{a.e. } ds, \text{ as } k \to \infty. \qquad (4.22)$$

We will now show that this is impossible. Observe that

$$g_h(t,s) := |h|^{-1/\alpha} \times \sum_{\kappa\in\{+,-\}} a^\kappa((1+(t+s)/|h|)_\kappa^{H(t+h)-1/\alpha} - ((t+s)/|h|)_\kappa^{H(t+h)-1/\alpha}). \qquad (4.23)$$

Without loss of generality, we can assume that $a^+ \neq 0$. Then, for all $s > |t|$, the terms in (4.23) involving a^- vanish, and by applying the mean value theorem for the function $g(x) = x^d$, $x > 0$, we get that

$$\begin{aligned} g_{\tilde{h}_k}(t,s) &= a^+|\tilde{h}_k|^{-1/\alpha}(H(t+\tilde{h}_k) - 1/\alpha) \times \left(\frac{(s+t)}{|\tilde{h}_k|} + \theta\right)_+^{H(t+\tilde{h}_k)-1/\alpha-1}, \\ &= a^+(H(t+\tilde{h}_k) - 1/\alpha)|\tilde{h}_k|^{1-H(t+\tilde{h}_k)} \times (s+t+\theta|\tilde{h}_k|)^{H(t+\tilde{h}_k)-1/\alpha-1}, \end{aligned} \qquad (4.24)$$

for some $\theta = \theta(s, \tilde{h}_k) \in [0,1]$. Since $s + t > 0$, and $H(t+\tilde{h}_k) \to H(t)$, $k \to \infty$, we have that

$$(s+t+\theta|\tilde{h}_k|)^{H(t+\tilde{h}_k)-1/\alpha-1} \to (s+t)^{H(t)-1/\alpha-1} > 0, \quad \text{as } k \to \infty.$$

However, $|H(t+\tilde{h}_k) - 1/\alpha||\tilde{h}_k|^{1-H(t+\tilde{h}_k)} \to 0$, as $k \to \infty$, since the function H is bounded and since $H(t+\tilde{h}_k) \to H(t) < 1$, $k \to \infty$. Therefore the last relation implies that the right-hand-side of (4.24) converges to zero and thus, for all $s > |t|$, $g_{\tilde{h}_k}(t,s) \to 0$, as $k \to \infty$. This contradicts (4.22), since

$$C\partial_v f(t, H(t), s) = Ca^+(\ln(t+s)(t+s)^{H(t)-1/\alpha} - \ln(s)(s)^{H(t)-1/\alpha}) \neq 0,$$

for $s > |t|$. If $a^+ = 0$, then $a^- \neq 0$ and one can similarly show that (4.22) is impossible by considering all $s < -|t|$.

We have thus shown that Relation (4.11) holds, for all $\rho \in (\rho_H(t), 2\rho_H(t)) = (H(t), 2H(t))$, which completes the proof of case (iii). This concludes the proof of the proposition since in cases (i)–(iii), one has $\rho_Y(t) \leq \rho_H(t) \wedge H(t)$, a.s. □

Remarks.

1. In Propositions 4.1 and 4.2, we consider LMSM processes $Y(t)$ defined for all $t \in \mathbb{R}$, but the results obviously apply when $Y(t)$ is restricted to the interval $t \in (a,b)$.
2. Proposition 4.1 involves the assumptions that $t \neq 0$ and $Y(t) \neq 0$, a.s. If $t = 0$, then $Y(t) = 0$, a.s. and then the pointwise Hölder exponent $\rho_Y(0)$ of the paths of Y may be greater than the bound $\rho_H(0) \wedge H(0)$ given in (4.9). In Proposition 4.2(a) below, we give another upper bound on $\rho_Y(0)$.
3. The case $t \neq 0$ and $Y(t) \equiv 0$, a.s. is special. As shown in Lemma 4.1 in Stoev and Taqqu,[1] for $t \neq 0$, $Y(t) = 0$, a.s. if and only if $H(t) = 1/\alpha$ and $a^+ = a^-$. In this case, Proposition 4.2(b) below, shows that $\rho_Y(t)$ is less than or equal to $\rho_H(t)$, almost surely.

The next proposition involves cases omitted in Proposition 4.1.

Proposition 4.2. *Let $1 < \alpha \leq 2$ and $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be the LMSM process, defined in* (1.7), *where the function H satisfies* (1.6) *and is continuous. Suppose that the paths of the process Y are continuous. Then*:

(a) *If $t = 0$, then $Y(t) = 0$, a.s. and*

$$\rho_Y(0,\omega) \equiv \rho_{Y(\cdot,\omega)}(0) \overset{\text{a.s.}}{\leq} \begin{cases} H(0), & \text{if } H(0) \neq 1/\alpha \text{ or } a^+ \neq a^- \\ 1/\alpha + \rho_H(0), & \text{if } H(0) = 1/\alpha \text{ and } a^+ = a^- \end{cases} \qquad (4.25)$$

(b) *If $t \neq 0$ and $Y(t) = 0$ a.s.* (*implying $H(t) = 1/\alpha$ and $a^+ = a^-$*), *then*

$$\rho_Y(t,\omega) \equiv \rho_{Y(\cdot,\omega)}(t) \overset{\text{a.s.}}{\leq} \rho_H(t). \qquad (4.26)$$

Here $\rho_{Y(\cdot,\omega)}(t)$, $\omega \in \Omega$ denotes the pointwise Hölder exponent of the function $Y(t,\omega)$, $t \in \mathbb{R}$, defined in (4.1).

Proof. We will first show Part (a). Consider the case $H(0) = 1/\alpha$ and $a^+ = a^-$. As argued in the proof of Proposition 4.1 [see (4.11) and (4.12), above], it suffices to show that

$$\limsup_{h\to 0} \frac{\|Y_\beta(h) - Y_\beta(0)\|_\alpha}{|h|^\rho} = \limsup_{h\to 0} \frac{\|Y_\beta(h)\|_\alpha}{|h|^\rho} = \infty, \qquad (4.27)$$

for all $\rho > H(0) + \rho_H(0) = 1/\alpha + \rho_H(0)$. (Since the scale coefficient $\|\cdot\|_\alpha$ does not depend on the skewness intensity, we can omit the subscript β in the rest of the proof.) Proposition 2.1(c) implies

$$\begin{aligned}\|Y(h)\|_\alpha &= |h|^{H(h)}\|X(1,H(h))\|_\alpha \qquad (4.28)\\ &= |h|^{H(h)}\|X(1,H(h)) - X(1,H(0))\|_\alpha, \qquad (4.29)\end{aligned}$$

where the last equality follows by the fact that $X(1,H(0)) = 0$, a.s. because $H(0) = 1/\alpha$ and $a^+ = a^-$ [see (1.5)].

Using Proposition 2.1(e) and the continuity of the function H, we obtain

$$\begin{aligned}&X(1,H(h)) - X(1,H(0))\\ &\quad = (H(h) - H(0))\partial_v X(1,H(0))\\ &\qquad + \mathcal{O}_P((H(h) - H(0))^2),\end{aligned}$$

as $h \to 0$. Therefore, the continuity of the field $\partial_v X(u,v)$ [Proposition 2.1(d)] and Relation (4.29) imply that

$$\|Y(h)\|_\alpha \sim |h|^{H(h)}|H(h) - H(0)|\|\partial_v X(1,H(0))\|_\alpha, \quad \text{as } h \to 0,$$

where $\|\partial_v X(1,H(0))\|_\alpha > 0$ and where $a_n \sim b_n$, $n \to \infty$ means $a_n/b_n \to 1$, $n \to \infty$. This, in view of (4.1) and the continuity of the function $H(t)$, yields (4.27), for all $\rho > H(0) + \rho_H(0)$. Indeed, for all $\rho > H(0) + \rho_H(0)$, we have that $\limsup_{h\to 0} |h|^{H(h)-\rho}|H(h)-H(0)| = \infty$, because, as $h \to 0$, $\rho - H(h) \to \rho - H(0) > \rho_H(0)$, and because for all $\gamma > \rho_H(0)$ one has $\limsup_{h\to 0} |h|^{-\gamma}|H(h) - H(0)| = \infty$.

If $H(t) \neq 1/\alpha$ *or* $a^+ \neq a^-$, then $\|X(1,H(0))\|_\alpha$ is positive. Thus, since the term $\|X(1,H(h))\|_\alpha$ in (4.28) converges, as $h \to 0$, to $\|X(1,H(0))\|_\alpha$, the step (4.29) does not apply and Relation (4.28) now implies (4.27) only for all $\rho > H(0)$. This proves Part (a).

The proof of Part (b) is similar. Since $Y(t) = 0$, a.s. and since $t \neq 0$, we have that $H(t) = 1/\alpha$ *and* $a^+ = a^-$ [see (1.5) above or Lemma 4.1 in Stoev and Taqqu[1]]. This implies that $X(t+h,H(t)) = 0$, a.s. for all $h \in \mathbb{R}$ and therefore, since $Y(t) = 0$,

$$\begin{aligned}&\|Y(t+h) - Y(t)\|_\alpha\\ &\quad = \|X(t+h,H(t+h))\|_\alpha\\ &\quad = \|X(t+h,H(t+h)) - X(t+h,H(t))\|_\alpha. \qquad (4.30)\end{aligned}$$

As in the proof of Part (a) above, by using Proposition 2.1(e) and the continuity of the function H, we get that the right-hand-side of (4.30) is asymptotically equivalent, as $h \to 0$, to

$$|H(t+h) - H(t)|\|\partial_v X(t,H(t))\|_\alpha,$$

since the field $\partial_v X(u,v)$ is continuous and since $\|\partial_v X(t,H(t))\|_\alpha > 0$, for all $t \neq 0$. This, in view of (4.1), implies

$$\limsup_{h\to 0} \|Y(t+h) - Y(t)\|_\alpha/|h|^\rho = \infty,$$

for all $\rho > \rho_H(t)$, which completes the proof of the proposition. □

Theorem 4.1. *Let $\alpha \in (1,2)$ and $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be the LMSM process, defined in (1.7), where the function $H(t)$, $t \in \mathbb{R}$ is continuous. Suppose that*

$$\begin{aligned}&1/\alpha < H(t) < 1, \quad \textit{and}\\ &1/\alpha < \rho_H^{\mathrm{unif}}(t), \quad \textit{for all } t \in \mathbb{R}, \qquad (4.31)\end{aligned}$$

where $\rho_H^{\mathrm{unif}}(t)$ denotes the uniform pointwise Hölder exponent at t of the function H. Then, the process Y has a version $\tilde{Y}$ with continuous paths, and for all $t \in \mathbb{R}$, $t \neq 0$,

$$\begin{aligned}\rho_H^{\mathrm{unif}}(t) \wedge H(t) - 1/\alpha &\overset{w.p.1}{\le} \rho_{\tilde{Y}}^{\mathrm{unif}}(t,\omega)\\ &\overset{w.p.1}{\le} \rho_{\tilde{Y}}(t,\omega)\\ &\overset{\text{a.s.}}{\le} \rho_H(t) \wedge H(t). \qquad (4.32)\end{aligned}$$

Proof. We will first show that the process Y has a version $\tilde{Y} = \{\tilde{Y}(t)\}_{t\in\mathbb{R}}$ with continuous paths.

Observe that for all $a' < b'$, a', $b' \in \mathbb{R}$ and for all ρ,

$$0 < \rho < \inf_{t\in[a',b']} \rho_H^{\mathrm{unif}}(t),$$

we have

$$|H(t') - H(t'')| \le C|t' - t''|^\rho, \quad \text{for all } t',\ t'' \in [a',b'], \qquad (4.33)$$

where the constant C depends on a', b' and ρ. This follows from (4.5) by using the compactness of the interval $[a',b']$. Indeed, (4.5) implies that for all $t \in [a',b']$, we have

$$|H(t')-H(t'')| \le C(t)|t'-t''|^\rho, \quad t',t'' \in (t-\epsilon_t, t+\epsilon_t),$$

for some $C(t) > 0$ and $\epsilon_t > 0$. Since the interval $[a',b']$ is compact, there is a finite collection of open intervals $(t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$, $t_i \in [a',b']$, $i = 1,\ldots,n$, which covers $[a',b']$. Then, Relation (4.33) holds with $C := n(C(t_1) + \cdots + C(t_n))$.

Since $1/\alpha < \rho_H^{\mathrm{unif}}(t)$, $t \in \mathbb{R}$, the lower semi-continuity of the function $\rho_H^{\mathrm{unif}}(t)$ (Lemma 6.3) implies (see Lemma 6.4) that there exists ρ, such that

$$1/\alpha < \rho < \inf_{t\in[a',b']} \rho_H^{\mathrm{unif}}(t). \tag{4.34}$$

Relations (4.33) and (4.31) then imply that the conditions of Theorem 3.2 are fulfilled. Therefore, there exists a version Y^* of the process Y, with continuous paths on the interval $[a',b']$. Moreover, Y^* satisfies (3.5), which by (4.2), implies that for any $[a'',b''] \subset (a',b')$,

$$\rho \wedge \min_{t\in[a',b']} H(t) - 1/\alpha \overset{\text{a.s.}}{\le} \inf_{t\in[a'',b'']} \rho_{Y^*(\cdot,\omega)}^{\mathrm{unif}}(t). \tag{4.35}$$

Observe that the interval $[a',b']$ can be chosen arbitrarily large, thus in view of Lemma 6.1, one can construct a version $\tilde{Y}(t)$ of $Y(t)$, which has continuous paths, for all $t \in \mathbb{R}$.

We now focus on showing (4.32) using this version $\tilde{Y}$. The inequality

$$\rho_{\tilde{Y}}^{\mathrm{unif}}(t,\omega) \le^{w.p.1} \rho_{\tilde{Y}}(t,\omega)$$

in (4.32) follows directly from Relation (4.4). The inequality

$$\rho_{\tilde{Y}}(t,\omega) \overset{\text{a.s.}}{\le} \rho_H(t) \wedge H(t)$$

follows from Proposition 4.1, therefore it remains to show only the first inequality in (4.32).

Lemma 6.1 implies that if $Y^* = \{Y^*(t)\}_{t\in\mathbb{R}}$ is another version of Y, which has continuous paths, then

$$\rho_{\tilde{Y}(\cdot,\omega)}^{\mathrm{unif}}(t) \overset{w.p.1}{=} \rho_{Y^*(\cdot,\omega)}^{\mathrm{unif}}(t), \quad t \in \mathbb{R}.$$

Here $\overset{w.p.1}{=}$ means that the above equality is valid for all $t \in \mathbb{R}$ and for all $\omega \in \tilde{\Omega}$, where $\tilde{\Omega} \subset \Omega$ and $\mathbb{P}(\tilde{\Omega}) = 1$. This implies that the process $\tilde{Y}$ also satisfies (4.35) for all $[a'',b''] \subset (a',b')$, a', a'', b', $b'' \in \mathbb{R}$ and for any ρ such that the inequality (4.34) holds. Since ρ can be chosen arbitrarily close to $\inf_{t\in[a',b']} \rho_H^{\mathrm{unif}}(t)$, we obtain that

$$\inf_{t\in[a',b']} (\rho_H^{\mathrm{unif}}(t)\wedge H(t)) - 1/\alpha \overset{\text{a.s.}}{\le} \inf_{t\in[a'',b'']} \rho_{\tilde{Y}}^{\mathrm{unif}}(t,\omega), \tag{4.36}$$

holds for all $a' < a'' < b'' < b'$. Relation (4.36) holds for all rational $a'' < a' < b' < b''$, almost surely, and since the functions $\rho_H^{\mathrm{unif}}(t) \wedge H(t) - 1/\alpha$ and $\rho_{\tilde{Y}}^{\mathrm{unif}}(t,\omega)$ are lower semi-continuous (see Lemma 6.3), Lemma 6.5 implies the inequality

$$\rho_H^{\mathrm{unif}}(t) \wedge H(t) - 1/\alpha \le^{w.p.1} \rho_{\tilde{Y}(\cdot,\omega)}^{\mathrm{unif}}(t), \quad t \in \mathbb{R}$$

in (4.32). This completes the proof of the theorem. $\square$

The next result is a reformulation of Theorem 4.1 in terms of the uniform Hölder exponent $\rho_f^{\mathrm{unif}}((a,b))$ (see Lemma 4.1).

Corollary 4.1. *Assume the conditions of Theorem 4.1. Then, for all $a,b \in \mathbb{R}$, $a < b$, such that $\inf_{t\in[a,b]} \rho_H^{\mathrm{unif}}(t) = \inf_{t\in(a,b)} \rho_H^{\mathrm{unif}}(t)$, we have*

$$\begin{aligned} \rho_H^{\mathrm{unif}}([a,b]) \ \wedge \ \min_{t\in[a,b]} H(t) - 1/\alpha \\ \overset{\text{a.s.}}{\le} \rho_{\tilde{Y}}^{\mathrm{unif}}([a,b];\omega) \overset{\text{a.s.}}{\le} \rho_H^{\mathrm{unif}}([a,b]) \\ \wedge \ \min_{t\in[a,b]} H(t). \end{aligned} \tag{4.37}$$

Proof. By using that $\inf_{t\in[a,b]}(\rho_H^{\mathrm{unif}}(t) \wedge H(t)) = (\inf_{t\in[a,b]} \rho_H^{\mathrm{unif}}(t)) \wedge (\min_{t\in[a,b]} H(t))$, Lemma 4.1 and Relation (4.32), we obtain

$$\begin{aligned} \left(\inf_{t\in[a,b]} \rho_H^{\mathrm{unif}}(t)\right) \wedge \left(\min_{t\in[a,b]} H(t)\right) - 1/\alpha \\ \overset{\text{a.s.}}{\le} \inf_{t\in[a,b]} \rho_{\tilde{Y}}^{\mathrm{unif}}(t) \\ \overset{\text{a.s.}}{\le} \rho_{\tilde{Y}}^{\mathrm{unif}}([a,b]) \\ \overset{\text{a.s.}}{\le} \rho_H(t) \wedge H(t), \end{aligned}$$

where the last inequality is valid for all $t \in (a,b)\backslash\{0\}$. By assumption and in view of Lemma 4.1, we obtain that the left-hand-side of the last expression equals

$$\begin{aligned} \rho_H^{\mathrm{unif}}([a,b]) \wedge \min_{t\in[a,b]} H(t) - 1/\alpha \\ = \rho_H^{\mathrm{unif}}((a,b)) \wedge \min_{t\in[a,b]} H(t) - 1/\alpha. \end{aligned}$$

This implies the first inequality in (4.37).

The second inequality in (4.37), follows by using the fact that, for some sequence $t_n \in (a,b)\backslash\{0\}$, we have, as $n \to \infty$, $\rho_H(t_n) \wedge H(t_n) \to \inf_{t\in(a,b)}(\rho_H(t) \wedge H(t))$. Firstly, $\inf_{t\in(a,b)} H(t) = \min_{t\in[a,b]} H(t)$. Secondly, $\inf_{t\in(a,b)} \rho_H(t) = \inf_{t\in(a,b)} \rho_H^{\mathrm{unif}}(t)$, because by Proposition 4.3 in Seuret and Lévy-Vehel,[17] $\rho_H(t)$ and $\rho_H^{\mathrm{unif}}(t)$, namely the pointwise and uniform pointwise Hölder exponents of the Hölder-continuous function $H(t)$ coincide almost everywhere in (a,b). Since, by assumption, $\inf_{t\in(a,b)} \rho_H^{\mathrm{unif}}(t) = \inf_{t\in[a,b]} \rho_H^{\mathrm{unif}}(t)$, this implies the second inequality in (4.37). $\square$

5. THE GAUSSIAN CASE $\alpha = 2$

The Gaussian multifractional Brownian motion (MBM) process

$$Y(t) := \int_{\mathbb{R}} (t+s)_+^{H(t)-1/2} - (s)_+^{H(t)-1/2})M_2(ds), \tag{5.1}$$

where $M_2(ds)$ is a Gaussian measure, was introduced by Peltier and Lévy-Vehel.[2] When $\alpha = 2$ and $a^+ = 1$, $a^- = 0$, the LMSM process Y coincides with the MBM in (5.1).

We present here some further results on the Hölder regularity of the paths of the MBM process and, more generally, on those of the Gaussian LMSM $Y = \{Y(t)\}_{t\in\mathbb{R}}$. We start by reviewing some well-known results in the literature. Since the α-stable measure $M_{\alpha,\beta}(ds)$ is now Gaussian $\alpha = 2$, it is symmetric and the skewness intensity $\beta(s)$, $s \in \mathbb{R}$ does not play any role, in this section. $M_{\alpha,\beta}(ds)$ is then denoted $M_2(ds)$.

The results in the Gaussian case, involving Hölder exponents typically assume

$$\sup_{t\in J} H(t) < \rho_H^{\text{unif}}(J), \quad \text{for all } J \subset \mathbb{R}, \tag{5.2}$$

where $\rho_H^{\text{unif}}(J)$ is defined in (4.3) (see, for example, Proposition 10 and Sec. 3 in Peltier and Lévy-Vehel[2] and also Ayache and Taqqu[6]). The smoothness condition (5.2) expresses the fact that the Hölder regularity of $H(\cdot)$ is greater than $H(t)$, $t \in J$. In Theorem 5.1 below, we provide bounds on the pointwise and uniform pointwise Hölder exponents of the Gaussian LMSM process Y *without* the smoothness condition (5.2).

The next proposition, follows from Theorem 2.1 in Ayache and Taqqu[6] if $H \neq 1/2$ therein and $a^+ = a^-$. We present here a direct proof using the inequality (2.7).

Proposition 5.1. *Let $\alpha = 2$ and $X = \{X(u,v),\ u \in \mathbb{R},\ v \in (0,1)\}$ be the Gaussian field, defined by* (2.1). *The field X has a version $\tilde{X} = \{\tilde{X}(u,v),\ u \in \mathbb{R},\ v \in (0,1)\}$, which is continuous in $(u,v) \in \mathbb{R} \times (0,1)$.*

Proof. Let $[a,b] \subset \mathbb{R}$ and $[\epsilon, 1-\epsilon] \subset (0,1)$, $\epsilon > 0$. It is enough to show that X has a version $\tilde{X}$, which is continuous in $(u,v) \in [a,b]\times[\epsilon,1-\epsilon]$, since $\mathbb{R}\times(0,1)$ is a countable union of such compacts, Lemma 6.1 implies that there exists a version $\tilde{X}$ of the field X, which is continuous for all $(u,v) \in \mathbb{R}\times(0,1)$.

By Theorem 2.1, for all u', $u'' \in [a,b]$ and v', $v'' \in [\epsilon, 1-\epsilon]$, we have

$$\begin{aligned}
&\mathbb{E}(X(u',v') - X(u'',v''))^2 \\
&\quad \le C_K^2(|u'-u''|^{v'} + |v'-v''|)^2 \\
&\quad \le C_1(|u'-u''|^{\min\{v' : v' \in [\epsilon,1-\epsilon]\}} + |v'-v''|)^2 \\
&\quad \le C_2(|u'-u''|^2 + |v'-v''|^2)^{\epsilon/2},
\end{aligned} \tag{5.3}$$

where $K = [a,b]\times[\epsilon,1-\epsilon]$ and where C_1 and C_2 are some constants.

Observe that for any mean zero Gaussian variable Z, for all $r > 0$, we have

$$\mathbb{E}|Z|^r = C_{2,r}(\mathbb{E}Z^2)^{r/2}, \tag{5.4}$$

where $C_{2,r} > 0$. Thus, by (5.3), we have that for all $x' := (u',v')$, $x'' := (u'',v'') \in K$ and $r > 0$,

$$\begin{aligned}
\mathbb{E}|X(u',v') - X(u'',v'')|^r &= \mathbb{E}|X(x') - X(x'')|^r \\
&\le C_2^{r/2} C_{2,r}\|x' - x''\|^{r\epsilon/2},
\end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

The last relation shows that when $r\epsilon/2 > 2$, the field $X(u,v)$, $(u,v) \in K = [a,b]\times[\epsilon,1-\epsilon]$ satisfies the conditions of the multivariate version of the Kolmogorov's continuity criterion (see, for example, p. 55 in Karatzas and Shreve[21]). Therefore, X has a modification $\tilde{X}$ which is continuous for all $(u,v) \in [a,b]\times[\epsilon,1-\epsilon]$. This completes the proof of the proposition. □

Proposition 5.1 implies the following:

Corollary 5.1. *Let $\alpha = 2$ and let $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be the LMSM process, defined by* (1.7), *where the function H satisfies* (1.6). *If $H(t)$, $t \in \mathbb{R}$ is continuous, then the process Y has a version $\tilde{Y}$ with continuous paths.*

In view of Corollary 5.1, we may assume that $Y = \{Y(t)\}_{t\in(a,b)}$ has continuous paths. The following proposition shows that in the Gaussian case, the uniform pointwise Hölder exponent $\rho_Y^{\text{unif}}(t)$ at t of the process Y, which is random, is always greater than the smaller of $H(t)$ and its uniform pointwise Hölder exponent $\rho_H^{\text{unif}}(t)$ [compare with (4.32)].

Proposition 5.2. *Let $\alpha = 2$ and let $Y(t)$, $t \in (a,b) \subset \mathbb{R}$ $(a < b)$ be the LMSM process, defined by* (1.7), *where the function H satisfies* (1.6) *and is continuous. Assume that Y has continuous paths and that the uniform pointwise Hölder exponent $\rho_H^{\text{unif}}(t)$ of the function H is positive, for all $t \in (a,b)$.*

Then,

$$\mathbb{P}(\{\omega:\ \rho_H^{\rm unif}(t)\wedge H(t)\le\rho_Y^{\rm unif}(t,\omega),\quad \textit{for all } t\in(a,b)\})=1,\qquad(5.5)$$

where $\rho_Y^{\rm unif}(t,\omega)$ *denotes the uniform pointwise Hölder exponent of the continuous function* $Y(t,\omega)$, $t\in(a,b)$.

The proof of this proposition uses the following standard result:

Lemma 5.1. *Let* $Y=\{Y(t)\}_{t\in\mathbb{R}}$ *be a Gaussian process. Let* $[a,b], a<b$ *be a compact interval in* $\mathbb{R}$. *Suppose that there exist* $h>0$ *and* $C>0$, *such that for all* $t',t''\in[a,b]$, *we have*

$$\mathbb{E}(Y(t')-Y(t''))^2\le C|t'-t''|^{2h}.\qquad(5.6)$$

Then, there exists a modification $\tilde{Y}$ *of the process* Y, *such that, for all* $0<h'<h$, *almost surely,*

$$|\tilde{Y}(t',\omega)-\tilde{Y}(t'',\omega)|\le R(\omega)|t'-t''|^{h'},\ \ t',\,t''\in[a,b].\qquad(5.7)$$

That is, Relation (5.7) *holds for all* $\omega\in\tilde{\Omega}=\Omega_{h'}$, *where* $\mathbb{P}(\tilde{\Omega})=1$.

The proof of Lemma 5.1 follows by using (5.4) and the strong form of the Kolmogorov's continuity criterion (see, for example, Theorem 3.3.16 in Stroock[16]).

Proof of Proposition 5.2. Since $Y(t)=X(t,H(t))$, $t\in\mathbb{R}$, Theorem 2.1 and Relation (4.33) yield the inequality

$$\mathbb{E}(Y(t')-Y(t''))^2\le C|t'-t''|^{2h},\quad \text{for all } t',\ t''\in[a',b']\subset(a,b),$$

valid for any h, such that

$$0<h<\inf_{t\in[a',b']}(\rho_H^{\rm unif}(t)\wedge H(t)),\qquad(5.8)$$

where the constant C depends on a', b' and h. Relation (5.7), on the other hand, implies that $\mathbb{P}\{\omega\text{: } h\le\inf_{t\in[a',b']}\rho_{\tilde{Y}(\cdot,\omega)}^{\rm unif}(t)\}=1$, for any interval $[a',b']\subset(a,b)$. Since h can be chosen arbitrarily close to the right-hand-side of (5.8), we obtain that for any $[a'',b'']\subset(a',b')$, one has

$$\inf_{t\in[a',b']}(\rho_H^{\rm unif}(t)\wedge H(t))\overset{\text{a.s.}}{\le}\inf_{t\in[a'',b'']}\rho_{Y(\cdot,\omega)}^{\rm unif}(t).\qquad(5.9)$$

In Relation (5.9), we need not take a new version $\tilde{Y}$ of the process Y [see (5.7)], because here Y has continuous paths and any two continuous-path versions of Y are indistinguishable (see Lemma 6.1). The inequality (5.9) is valid, with probability one, for all rational $a'<a''<b''<b'$. Therefore, as in the proof of Theorem 4.1 above, by using the lower semi-continuity of the functions $\rho_H^{\rm unif}(t)\wedge H(t)$ and $\rho_{Y(\cdot,\omega)}^{\rm unif}(t)$ (Lemma 6.3) and Lemma 6.5, we get that

$$\rho_H^{\rm unif}(t)\wedge H(t)\le^{w.p.1}\rho_{Y(\cdot,\omega)}^{\rm unif}(t),\quad t\in(a,b).$$

This completes the proof of the proposition. □

Propositions 4.1 and 5.2 imply the following result.

Theorem 5.1. *Assume the conditions of Proposition* 5.2. *Suppose also that* $Y(t)\neq 0$, *a.s. for all* $t\in(a,b)\setminus\{0\}$, *Then, for all* $t\in(a,b)\setminus\{0\}$, *we have that*

$$\begin{aligned}\rho_H^{\rm unif}(t)\wedge H(t)&\overset{w.p.1}{\le}\rho_Y^{\rm unif}(t,\omega)\\&\overset{w.p.1}{\le}\rho_Y(t,\omega)\\&\overset{\text{a.s.}}{\le}\rho_H(t)\wedge H(t),\end{aligned}\qquad(5.10)$$

[*see* (4.7) *and* (4.8) *above*].

The next result relates the uniform Hölder exponent of the Gaussian LMSM process to the function H and its uniform Hölder exponent (compare with Corollary 4.1).

Corollary 5.2. *Assume the conditions of Theorem* 5.1. *Then, for all* $a',\,b'\in(a,b)$, $a'<b'$, *such that* $\inf_{t\in[a',b']}\rho_H^{\rm unif}(t)=\inf_{t\in(a',b')}\rho_H^{\rm unif}(t)$, *we have*

$$\begin{aligned}&\rho_H^{\rm unif}([a',b'])\wedge\min_{t\in[a',b']}H(t)\\&\quad\overset{\text{a.s.}}{\le}\rho_Y^{\rm unif}([a',b'];\omega)\\&\quad\overset{\text{a.s.}}{\le}\rho_H^{\rm unif}([a',b'])\wedge\min_{t\in[a',b']}H(t).\end{aligned}\qquad(5.11)$$

The proof of this result is similar to that of Corollary 4.1. It follows from (5.10) by using Lemma 4.1.

The following result shows an interesting property of the uniform pointwise Hölder exponent $\rho_Y^{\rm unif}(t)$ of the Gaussian LMSM, namely that the first inequality in (5.10) can be made into an equality if $\rho_H^{\rm unif}(t)\wedge H(t)$ is continuous. To show this, we use the fact that the pointwise and the uniform pointwise Hölder exponents of a Hölder continuous function H coincide on a uncountable everywhere dense set (see Proposition 4.3 in Seuret and Lévy-Vehel[17]). One needs to assume that the function $H(t)$ has some Hölder exponent $\rho_H^{\rm unif}((a,b))>0$, no matter how small.

Corollary 5.3. *Assume that* $\rho_H^{\rm unif}((a,b))>0$ *and that the conditions of Theorem* 5.1 *hold. If the set* *of discontinuity points, $\mathcal{D}$, of the function $\rho_H^{\rm unif}(t) \wedge H(t)$, $t \in (a,b)$ is countable and*

$$\rho_H(t) \wedge H(t) = \rho_H^{\rm unif}(t) \wedge H(t),$$

for all $t \in \mathcal{D}$, then

$$\rho_H^{\rm unif}(t) \wedge H(t) \stackrel{w.p.1}{=} \rho_Y^{\rm unif}(t,\omega), \quad t \in (a,b) \setminus \{0\}. \tag{5.12}$$

Proof. Consider the events

$$\mathcal{A}_t := \{\omega \colon \rho_H^{\rm unif}(t) \wedge H(t) < \rho_Y^{\rm unif}(t,\omega)\}.$$

By (5.10), we have that $\mathbb{P}(\mathcal{A}_t) = 0$, for all $t \in (a,b) \setminus \{0\}$ such that $\rho_H(t) \wedge H(t) = \rho_H^{\rm unif}(t) \wedge H(t)$. We will now show that $\mathbb{P}(\cup_{t\in(a,b)\setminus\{0\}}\mathcal{A}_t) = 0$. This will imply the equality $=^{w.p.1}$ in (5.12) and complete the proof of the statement.

Since $\mathcal{D} \subset (a,b)$ is countable and since $\rho_H(t) \wedge H(t) = \rho_H^{\rm unif}(t) \wedge H(t)$, for all $t \in \mathcal{D}$, we have that $\mathbb{P}(\cup_{t\in\mathcal{D}}\mathcal{A}_t) = 0$. Therefore, it suffices to show that the event $\mathcal{A} := \cup_{t\in(a,b)\setminus(\mathcal{D}\cup\{0\})}\mathcal{A}_t$ has probability zero.

Let $\mathcal{I}$ denote the set of all $t \in (a,b) \setminus \{0\}$, such that $\rho_H(t) = \rho_H^{\rm unif}(t)$. Since $\rho_H^{\rm unif}((a,b)) > 0$, Proposition 4.3 in Seuret and Lévy-Vehel[17] implies that the set $\mathcal{I}$ is *uncountable* and dense in every open sub-interval of (a,b). Hence, since $\mathcal{D}$ is *countable*, every sub-interval of (a,b) with rational end-points contains a point in the set $\mathcal{I} \setminus \mathcal{D}$. Therefore, there is a *countable* and *dense* subset, $\mathcal{J}$, of (a,b), which is also a subset of $\mathcal{I} \setminus \mathcal{D}$.

Observe that the event $\mathcal{B} := \bigcup_{t\in\mathcal{J}\setminus\{0\}}\mathcal{A}_t$ has probability zero, since $\mathcal{J}$ is countable and because $\rho_H(t) \wedge H(t) = \rho_H^{\rm unif}(t) \wedge H(t)$, for all $t \in \mathcal{J}$.

We will now show that $\mathcal{A} \subset \mathcal{B}$. If $\omega \in \mathcal{A}$, then $\rho_H^{\rm unif}(t) \wedge H(t) < \rho_Y^{\rm unif}(t,\omega)$, for some $t \in (a,b) \setminus \{0\}$ such that $t \notin \mathcal{D}$. Then by the lower semi-continuity of the function $\rho_Y^{\rm unif}(t)$ (Lemma 6.3), we have that the set $U(\omega) := \{s : c < \rho_Y^{\rm unif}(s,\omega)\}$ is open, where $c := (\rho_H^{\rm unif}(t) \wedge H(t) + \rho_Y^{\rm unif}(t,\omega))/2$. We have that $\rho_H^{\rm unif}(t) \wedge H(t) < c < \rho_Y^{\rm unif}(t,\omega)$ and hence $t \in U(\omega)$. Since $U(\omega)$ is open, for some $\epsilon = \epsilon(\omega) > 0$,

$$\rho_H^{\rm unif}(t) \wedge H(t) < c < \rho_Y^{\rm unif}(s,\omega), \quad \text{for all } s \in (t-\epsilon, t+\epsilon).$$

By using the continuity of the function $\rho_H^{\rm unif}(s) \wedge H(s)$ at $s = t \notin \mathcal{D}$ and the above inequality, we obtain that for some $\delta = \delta(\omega) > 0$, $\rho_H^{\rm unif}(s) \wedge H(s) \le c < \rho_Y^{\rm unif}(s,\omega)$, for all $s \in (t-\delta, t+\delta)$. Since $\mathcal{J}$ is dense in $(a,b)\setminus\{0\}$, there exists $s \in \mathcal{J} \cap (t-\delta, t+\delta)$, and hence $\omega \in \mathcal{B}$. Since $\omega \in \mathcal{A}$ was arbitrary, this shows that $\mathcal{A} \subset \mathcal{B}$ and that $\mathbb{P}(\mathcal{A}) \le \mathbb{P}(\mathcal{B}) = 0$. This completes the proof of the corollary. □

Remarks.

1. Let $a^+ \ne a^-$ and suppose that the condition (5.2) holds with $J = \mathbb{R}$. Then $H(t)$ is continuous and by Lemma 4.1, $\rho_H^{\rm unif}(t) \wedge H(t) = H(t)$, for all $t \in \mathbb{R}$. Corollary 5.3, Theorem 5.1 and Proposition 4.2(a) imply
$$H(t) \stackrel{w.p.1}{=} \rho_Y^{\rm unif}(t,\omega) \le \rho_Y(t,\omega) \stackrel{\rm a.s.}{=} H(t), \tag{5.13}$$
for all t in an increasing sequence of closed intervals and hence for all $t \in \mathbb{R}$.
2. The second equality ($=^{\rm a.s.}$) in Relation (5.13) was first established by Peltier and Lévy-Vehel,[2] under the assumption (5.2) and with $(a^+, a^-) = (1,0)$.
3. In Theorem 5.1, for simplicity, we focus only on the case when $Y(t) \ne 0$ a.s. Proposition 4.2 deals with the special cases where $Y(t) = 0$, that is, when $t = 0$ or $H(t) = 1/2$ and $a^+ = a^-$. In these cases, the upper bounds on $\rho_Y(t,\omega)$, given in (5.1), may not hold. Observe that in Theorem 4.1, we did not have to assume $Y(t) \ne 0$, a.s. for $t \ne 0$ because this is always the case when $1/\alpha < H(t)$.

Example of $\rho_Y(t,\omega) \stackrel{\rm a.s.}{<} H(t)$. If $H(t)$ does not satisfy the smoothness condition (5.2), then Relation (5.13) may not hold. Let, for example, $H(t)$ be a continuous function, such that in a neighborhood of t_0,

$$H(t) := 1/2 + |t - t_0|^{1/3},$$

for all $t \in (t_0 - 1/10, t_0 + 1/10)$, $t_0 \ne 0$, so that $H(t) \in [1/2, 1)$ in that neighborhood. By (4.1), we have that $\rho_H(t_0) = 1/3$ and hence $\rho_H^{\rm unif}(t_0) \le 1/3$. Note also that by using the inequalities $a^{1/3} - b^{1/3} \le |a-b|^{1/3}$, $a, b > 0$ and $|x - y| \ge ||x| - |y||$, we have that for all $t', t'' \in (t_0 - 1/10, t_0 + 1/10)$, $t' \ne t''$

$$\frac{|H(t') - H(t'')|}{|t' - t''|^\rho} = \frac{||t'-t_0|^{1/3} - |t''-t_0|^{1/3}|}{|t'-t''|^\rho} \le \frac{||t'-t_0| - |t''-t_0||^{1/3}}{||t'-t_0| - |t''-t_0||^\rho},$$

for any $\rho > 0$. The last expression converges to zero, as $t', t'' \to t_0$, for all $0 < \rho < 1/3$, which, in view of (4.2), implies $\rho_H^{\rm unif}(t_0) \ge 1/3$. Since $\rho_H^{\rm unif}(t_0) \le \rho_H(t_0) = 1/3$, we get $\rho_H^{\rm unif}(t_0) = 1/3$.

In this case,

$$\rho_H^{\text{unif}}(t_0) \wedge H(t_0) = \rho_H(t_0) \wedge H(t_0) = 1/3 < 1/2 = H(t_0)$$

and by Theorem 5.1, we have that, almost surely,

$$1/3 = \rho_Y^{\text{unif}}(t_0, \omega) = \rho_Y(t_0, \omega) < H(t_0) = 1/2.$$

This shows that Relation (5.13) does not always hold.

Ayache *et al.*[2] show, in particular, that given a function $H(\cdot)$ with values in $(0, b]$, $b < 1$, there is a continuous-path stochastic process Z, such that $\rho_Z(\cdot) =^{w.p.1} H(\cdot)$. This Z is a particular member of a family of processes called GMPREs. The MBM also belongs to that family, but in view of the above example, Z is, in general, not an MBM.

6. AUXILIARY RESULTS

The following elementary result is used in the proofs of Theorems 3.2, 4.1 and Proposition 5.1, above.

Lemma 6.1. *Let $Y = \{Y(t),\ t \in T\}$, be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the parameter t belongs to a separable metric space T. Assume that $\tilde{Y}$ and $\tilde{Y}^*$ are two versions of Y, which have continuous paths.*

Then, the processes $\tilde{Y}$ and $\tilde{Y}^*$ are indistinguishable, that is,

$$\mathbb{P}(\{\omega\colon \tilde{Y}(t,\omega) \neq \tilde{Y}^*(t,\omega), \text{ for some } t \in T\}) = 0. \tag{6.1}$$

Proof. The processes $\tilde{Y}$ and $\tilde{Y}^*$ are separable, since they have continuous paths. We will show that the event $\mathcal{N} := \{\omega\colon \tilde{Y}(t,\omega) \neq \tilde{Y}^*(t,\omega), \text{ for some } t \in T\}$ has probability zero. Since T is separable, then T has a countable and dense subset I. The events $\tilde{\mathcal{N}} := \{\omega\colon \tilde{Y}(t,\omega) \neq Y(t,\omega), \text{ for some } t \in I\}$ and $\tilde{\mathcal{N}}^* := \{\omega\colon \tilde{Y}^*(t,\omega) \neq Y(t,\omega), \text{ for some } t \in I\}$ have zero probabilities, since $\tilde{Y}$ and $\tilde{Y}^*$ are versions of Y and since the set I is countable. The continuity of the paths of $\tilde{Y}$ and $\tilde{Y}^*$ and the fact that I is dense in (a, b) imply that $\mathcal{N} \subset \tilde{\mathcal{N}} \cup \tilde{\mathcal{N}}^*$. This, by the completeness of the probability space, yields $\mathbb{P}(\mathcal{N}) = 0$. □

The next lemma is used in the proof of Theorem 3.2.

Lemma 6.2. *Let ξ be a strictly α-stable random variable, $\alpha \neq 1$, with characteristic function*

$$\mathbb{E}e^{i\theta\xi} = \exp\{-\|\xi\|_\alpha^\alpha |\theta|^\alpha (1 - i\, sign(\theta)\beta_\xi \times \tan(\pi\alpha/2))\}, \tag{6.2}$$

where $\beta_\xi \in [-1, 1]$ is the skewness coefficient of ξ. Then, for all $\gamma \in (0, \alpha)$,

$$\mathbb{E}|\xi|^\gamma \le C_{\alpha,\gamma} \|\xi\|_\alpha^\gamma,$$

where $C_{\alpha,\gamma}$ is a constant which does not depend on the skewness coefficient β_ξ.

Proof. Observe that by (6.2), $\xi =_d \|\xi\|_\alpha \xi_0$, where ξ_0 is a strictly α-stable random variable with unit scale coefficient. By Property 1.2.13 in Samorodnitsky and Taqqu,[11] we have that

$$\xi =_{\mathrm{d}} \|\xi\|_\alpha \left(\left(\frac{1+\beta_\xi}{2}\right)^{1/\alpha} \xi_1 - \left(\frac{1-\beta_\xi}{2}\right)^{1/\alpha} \xi_2 \right), \tag{6.3}$$

where ξ_1 and ξ_2 are independent, identically distributed strictly α-stable random variable with scale and skewness coefficients equal to 1 (that is, totally skewed to the right). By using the inequality $|x + y|^\gamma \le 2|x|^\gamma + 2|y|^\gamma$, valid for all x, $y \in \mathbb{R}$ and $\gamma \in (0, 2]$, and (6.3), we get

$$\mathbb{E}|\xi|^\gamma \le 4\|\xi\|_\alpha^\gamma \left(\frac{1+|\beta_\xi|}{2}\right)^{\gamma/\alpha} \mathbb{E}|\xi_1|^\gamma \le 4\mathbb{E}|\xi_1|^\gamma \|\xi\|_\alpha^\gamma =: C_{\alpha,\gamma}\|\xi\|_\alpha^\gamma,$$

since $|\beta_\xi| \le 1$. This completes the proof of the lemma. □

Proof of Lemma 4.1. The inequality $\rho_f^{\text{unif}}([a, b]) \le \rho_f^{\text{unif}}((a, b))$ follows directly from (4.3). We will now prove the first inequality in (4.6).

Let $h > \rho_f^{\text{unif}}([a, b])$ be arbitrary. By (4.3), there exist two sequences s_n', $s_n'' \in J$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \frac{|f(s_n') - f(s_n'')|}{|s_n' - s_n''|^h} = \infty.$$

Since $[a, b]$ is compact, f is bounded on $[a, b]$ and hence $s_n' - s_n'' \to 0$, as $n \to \infty$. Now, by the Bolzano-Weierstrass theorem, the compactness of $[a, b]$, implies that $s_{n_k}' \to s^*$, for some sub-sequence n_k, $k \in \mathbb{N}$, $n_k \to \infty$ and some $s^* \in [a, b]$. Therefore $s_{n_k}'' \to s^*$ and

$$\limsup_{s', s'' \to s^*} \frac{|f(s') - f(s'')|}{|s' - s''|^h} = \infty$$

and thus $h \ge \rho_f^{\text{unif}}(s^*) \ge \inf_{t\in[a,b]} \rho_f^{\text{unif}}(t)$. This shows that $\rho_f^{\text{unif}}([a, b]) \ge \inf_{t\in[a,b]} \rho_f^{\text{unif}}(t)$.

Let now $h > \inf_{t\in(a,b)} \rho_f^{\mathrm{unif}}(t)$. Then, there exists $t \in (a,b)$, such that $h > \rho_f^{\mathrm{unif}}(t)$ and thus, by (4.2),

$$\lim_{s',s''\to t,\ s',s''\in(a,b)} \frac{|f(s') - f(s'')|}{|s' - s''|^h} = \infty. \qquad (6.4)$$

Observe that it is important here that the interval (a,b) be open, because even if we had $h > \inf_{t\in[a,b]} \rho_f^{\mathrm{unif}}(t)$, we cannot extend (6.4) to hold for $s', s'' \in [a,b]$. The convergence (6.4), in view of (4.3), implies $h \geq \rho_f^{\mathrm{unif}}((a,b))$. Since $h > \inf_{t\in(a,b)} \rho_f^{\mathrm{unif}}(t)$ was arbitrary, this shows that $\inf_{t\in(a,b)} \rho_f^{\mathrm{unif}}(t) \geq \rho_f^{\mathrm{unif}}((a,b))$. This completes the proof of the lemma. □

Recall that a function $f(t)$ is lower semi-continuous if for any $\lambda \in \mathbb{R}$, the set $\{t\colon \lambda < f(t)\}$ is open. Equivalently, $f(t_0) \leq \liminf_{t\to t_0} f(t)$, for all $t_0 \in \mathbb{R}$ (see, for example, Chap. 2, Problem 50 in Royden[23]).

The following lemmas are used in the proofs of Theorem 4.1 and Proposition 5.2, above.

Lemma 6.3. *Let $H\colon \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. Then the functions $\rho_H^{\mathrm{unif}}(t)$ and $\rho_H^{\mathrm{unif}}(t) \wedge H(t)$, $t \in \mathbb{R}$ are lower semi-continuous.*

Proof. We will first show that the function ρ_H^{unif} is lower semi-continuous. Let $\lambda \in \mathbb{R}$ be arbitrary. If $\lambda < 0$, then since $0 \leq \rho_H^{\mathrm{unif}}(t)$, $t \in \mathbb{R}$, we get $\{t\colon \lambda < \rho_H^{\mathrm{unif}}(t)\} = \mathbb{R}$, which is open.

Let now $\lambda \geq 0$ and let $s \in U := \{t\colon \lambda < \rho_H^{\mathrm{unif}}(t)\}$. Then, $0 \leq \lambda < \mu < \rho_H^{\mathrm{unif}}(s)$ and hence [see (4.5)], $\lambda < \mu \leq \rho_H^{\mathrm{unif}}(t)$, for all $t \in (s-\epsilon, s+\epsilon)$, for some $\epsilon > 0$. This implies $(s-\epsilon, s+\epsilon) \subset U$, the set U is open and the function $\rho_H^{\mathrm{unif}}(t)$ is lower semi-continuous.

The function H is continuous and therefore lower semi-continuous. Since for all $\lambda \in \mathbb{R}$, the set

$$\{t\colon \lambda < \rho_H^{\mathrm{unif}}(t) \wedge H(t)\} = \{t\colon \lambda < \rho_H^{\mathrm{unif}}(t)\} \cap \{t\colon \lambda < H(t)\}$$

is open, it follows that the function $\rho_H^{\mathrm{unif}}(t) \wedge H(t)$ is also lower semi-continuous. □

Lemma 6.4. *Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be a lower semi-continuous function, such that $c < f(t)$, for all $t \in [a,b]$, $a < b$. Then, there exists $d \in \mathbb{R}$, such that*

$$c < d \leq \inf_{t\in[a,b]} f(t).$$

Proof. The proof follows from the fact that any lower semi-continuous function f, defined on a closed interval $[a,b]$ achieves its infimum. That is, there is $t_0 \in [a,b]$, such that $f(t_0) = \inf_{t\in[a,b]} f(t)$ (see, for example, Problem 50**i** on p. 52 in Royden[23]). □

Lemma 6.5. *Let $f\colon (a,b) \mapsto \mathbb{R}$ be a lower semi-continuous function and $g\colon (a,b) \mapsto \mathbb{R}$ be arbitrary. Suppose that,*

$$\inf_{t\in[a',b']} f(t) \leq \inf_{t\in[a'',b'']} g(t), \quad \text{for all } [a'',b''] \subset (a',b'), \qquad (6.5)$$

where $a', b', a'', b'' \in (a,b) \cap \mathbb{Q}$, $a' < b'$. Then,

$$f(t) \leq g(t), \quad \text{for all } t \in (a,b).$$

Proof. We will first show that

$$\inf_{t\in[a'',b'']} f(t) \leq \inf_{t\in[a'',b'']} g(t), \quad \text{for all } a'', b'' \in (a,b) \cap \mathbb{Q}. \qquad (6.6)$$

Let $[a'_n, b'_n] \downarrow [a'', b'']$, $n \to \infty$, where $a'_n, b'_n \in (a,b) \cap \mathbb{Q}$ and where $(a'_n, b'_n) \supset [a'', b'']$. Since the function f is lower semi-continuous it achieves its infimum on the closed interval $[a'_n, b'_n]$ (see, for example, p. 52 in Royden[23]). Let $f(t_n) = \inf_{t\in[a'_n,b'_n]} f(t)$ where $t_n \in [a'_n, b'_n]$, $n \in \mathbb{N}$. Now since $[a'_n, b'_n] \downarrow [a'', b'']$, $n \to \infty$, the Bolzano-Weierstrass theorem implies that $t_{n_k} \to t_0 \in [a'', b'']$, for some sub-sequence $n_k \to \infty$, $k \to \infty$. Thus, by the lower semi-continuity of f if follows that

$$f(t_0) \leq \liminf_{k\to\infty} f(t_{n_k}) = \lim_{n\to\infty} \inf_{t\in[a'_n,b'_n]} f(t).$$

Since $t_0 \in [a'', b''] \subset (a'_n, b'_n)$, the last inequality and (6.5) show that

$$\inf_{t\in[a'',b'']} f(t) \leq \lim_{n\to\infty} \inf_{t\in[a'_n,b'_n]} f(t) \leq \inf_{t\in[a'',b'']} g(t),$$

which implies (6.6).

We will now show that (6.6) implies $f(t) \leq g(t)$, $t \in (a,b)$. Suppose, *ad absurdum* that $f(t_0) > g(t_0)$ for some $t_0 \in (a,b)$. The lower semi-continuity of f implies that the set $U := \{t \in (a,b)\colon g(t_0) < f(t)\}$ is open. Note that $t_0 \in U$ and therefore, $(t_0 - \epsilon, t_0 + \epsilon) \subset U$, for some $\epsilon > 0$. Since $\mathbb{Q}$ is dense in (a,b), there exists an interval $[a'', b''] \subset (t_0 - \epsilon, t_0 + \epsilon]$, where $a'' < t_0 < b'' \in (a,b) \cap \mathbb{Q}$. By the definition of the set U it follows that $g(t_0) < f(t)$, for all $t \in [a'', b'']$ and hence

$$g(t_0) < \inf_{t\in[a'',b'']} f(t)$$

(see Lemma 6.4). The last inequality contradicts (6.6) because $\inf_{t\in[a'',b'']} g(t) \le g(t_0)$. This implies that $f(t) \le g(t),\ t \in (a,b)$ and completes the proof of the lemma. $\square$

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REFERENCES

1. S. Stoev and M. S. Taqqu, Stochastic properties of the linear multifractional stable motion, *Adv. Appl. Probab.* **36**(4) (2004) 1085–1115.
2. R. F. Peltier and J. L. Levy-Vehel, *Multifractional Brownian Motion: Definition and Preliminary Results*, Technical Report 2645 (Institut National de Recherche en Informatique et an Automatique, INRIA, Le Chesnay, France, 1995).
3. A. Benassi, S. Jaffard and D. Roux, Elliptic Gaussian random processes, *Rev. Math. Iberoam.* **13**(1) (1997) 19–90.
4. A. Ayache and J. Lévy-Véhel, The generalized multifractional Brownian motion, *Stat. Inf. Stoch. Process.* **3** (2000) 7–18.
5. A. Ayache, The generalized multifractional field: a nice tool for the study of the generalized multifractional Brownian motion, *J. Fourier Anal. Appl.* **8**(6) (2002) 581–601.
6. A. Ayache and M. S. Taqqu, Multifractional processes with random exponent, preprint (2003).
7. M. S. Taqqu Fractional Brownian motion and long-range dependence, in *Theory and Applications of Long-Range Dependence*, eds. P. Doukhan, G. Oppenheim and M. S. Taqqu, (Birkhäuser, 2003) pp. 5–38.
8. P. Embrechts and M. Maejima, *Self-Similar Processes* (Academic Press, 2003).
9. S. Cohen, From self-similarity to local self-similarity: the estimation problem, in *Fractals: Theory and Applications in Engineering*, eds. M. Dekking, J. L. Véhel, E. Lutton and C. Tricot, (Springer-Verlag, 1999).
10. S. Stoev and M. S. Taqqu, How rich is the class of multifractional Brownian motions?, Preprint (2004).
11. G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Processes*: *Stochastic Models with Infinite Variance* (Chapman and Hall, New York, London, 1994).
12. A. Benassi, S. Cohen and J. Istas, Identification and properties of real harmonizable fractional Lévy motions, *Bernoulli* **8** (2002) 97–115.
13. A. Benassi, S. Cohen and J. Istas, On roughness indices for fractional fields, *Bernoulli* **10**(2) (2004) 357–373.
14. J.-M. Bardet and P. Bertrand, Properties and identification of the multiscale fractional Brownian motion with biomedical applications, preprint (2003).
15. J. L. Doob, *Stochastic Processes* (Wiley, New York, 1953).
16. D. W. Stroock, *Probability Theory*: *An Analytic View* (Cambridge University Press, New York, 1993).
17. S. Seuret and J. Lévy-Vehel, The local Hölder function of a continuous function, *Appl. Comput. Harm. Anal.* **13**(3) (2002) 263–276.
18. B. Guiheneuf and J. Lévy-Vehel, Two-microlocal analysis and applications in signal processing, in *Wavelets and Multiscale Methods*, International Wavelets Conference (INRIA, Tangier, Morocco, 1998).
19. S. Jaffard and Y. Meyer, Wavelet methods for pointwise regularity and local oscillations of functions, in *Memoirs of the American Mathematical Society*, Vol. 587 (American Mathematical Society, Providence, RI, 1996).
20. B. Guiheneuf, S. Jaffard and J. Lévy-Vehel, Two results concerning chirps and 2-microlocal exponents prescription, *Appl. Comput. Harm. Anal.* **5**(4) (1998) 487–492.
21. I. Karatzas and S. E. Shreve, Brownian motion and stochastic calculus, in *Graduate Texts in Mathematics*, 2nd edn, Vol. 113 (Springer-Verlag, New York, 1991).
22. A. Ayache, S. Jaffard and M. S. Taqqu, Multifractional processes with most general Hölder exponents, preprint (2003).
23. H. L. Royden, *Real Analysis*, 3rd edn. (Macmillan, 1998).