# A Spanning Series Approach to Options*

Steven Heston†
University of Maryland

Alberto G. Rossi‡
University of Maryland

February 26, 2015

**Abstract**

This paper shows that Edgeworth expansions for option valuation are equivalent to approximating option payoffs using Hermite polynomials. Consequently, the value of an option is the value of an infinite series of replicating polynomials. The resulting formulas express option values in terms of skewness, kurtosis, and higher moments. Unfortunately, the Hermite series diverges for fat-tailed models, so we provide an alternative spanning series based on logistic polynomials. The new moment-based formulas are a computationally efficient alternative to Fourier transform valuation and can value options even when the characteristic function is not known. Applications include a series for Heston's (1993) stochastic volatility model, and the first convergent solution for the Hull and White (1987) model. (*JEL* G12)

*Keywords: stochastic volatility, orthogonal polynomials, options*

*The authors acknowledge helpful discussions with Gurdip Bakshi, Kathleen Hanley, Gerard Hoberg, Mark Loewenstein, Dilip Madan and Georgios Skoulakis. The authors also acknowledge the valuable comments received from the seminar participants of the University of Maryland and the conference participants at the Mont Tremblant Fifth Risk Management — in particular David Bates, Peter Christoffersen and Dacheng Xiu (the discussant). We welcome comments, including references to related papers we have inadvertently overlooked. Any remaining errors are our responsibility alone.

†R.H. Smith School of Business, University of Maryland, 4447 Van Munching Hall, College Park, MD 20742; Tel: (301) 405-9686; E-mail: sheston@rhsmith.umd.edu.

‡R.H. Smith School of Business, University of Maryland, 4457 Van Munching Hall, College Park, MD 20742; Tel: (301) 405-0703; E-mail: arossi@rhsmith.umd.edu.

# 1 Introduction

There is a long research program in finance aimed at characterizing risk and return in terms of moments of probability distributions. As indicated by Jarrow and Rudd (1982)'s title "The Valuation of Options for Arbitrary Return Distributions," this reflects a desire to generalize results beyond a restrictive parametric framework. Jarrow and Rudd (1982) pioneered the use of Edgeworth expansions for valuation of derivative securities, while Corrado and Su (1996) introduced Gram-Charlier expansions. These moment expansions influenced subsequent approximations by Madan and Milne (1994), Longstaff (1995), Abken, Madan, and Ramamurtie (1996a,b), Brenner and Eom (1997), Knight and Satchell (1997), Backus, Foresi, and Wu (2004), Feunou, Fontaine, and Tedongap (2009), and Xiu (2014).

These papers are part of a broader agenda that uses series approximations to valuation problems. Prominent examples include Cox, Ross, and Rubinstein (1979)'s binomial approximation to the Black-Scholes model and Hull and White (1987)'s stochastic volatility model. While these studies illustrate the practical appeal of series approximations, their methods are typically problem-specific and often lack economic intuition. In contrast, Corrado and Su (1996)'s fourth-order approximation became popular by intuitively relating option values to the skewness and kurtosis of the log-spot return distribution.[1] However, they did not provide any rigorous analysis of the approximation nor conditions under which it converged.

The present paper generalizes the Edgeworth probability expansion approach to an orthogonal polynomial theory of option valuation. It first approximates option payoffs using Hermite polynomial functions of the log-spot return. It then proves that, under suitable tail conditions, the value of these polynomial payoffs converges to the value of the option. This provides an exact infinite series

[1] Specific applications in finance include Rubinstein (1998) Ritchken, Sankarasubramanian, and Vijh (1993), and Collin-Dufresne and Goldstein (2002). In addition, a web search on "Edgeworth expansion option pricing" and "Hermite option pricing" reveals hundreds of related papers in econophysics, applied probability, computational finance, and the quantitative finance section of the mathematics working paper archive "arxiv.org".

formula that expresses option values in terms of variance, skewness, and higher moments of the log-spot returns. Our moment-based approach therefore formalizes the intuitive relation between option values and probability distributions.

Exact spanning with polynomials requires square-integrability conditions on the tails of the probability distribution. We show that conditions that support previous expansions are too restrictive for fat-tailed distributions common in finance, such as stochastic volatility or Levy jump models. Mathematically, this occurs because Hermite functions are too thin-tailed to span the probability densities of the log-spot return. Therefore, we derive a new set of logistic polynomial functions and prove that they successfully span fat-tailed distributions.

Our paper also shows how orthogonal polynomials are a natural analog of the transform methods. The difference is that orthogonal polynomials span payoffs with countable series, whereas the transform methods of Heston (1993), Bakshi and Madan (2000), and Chen and Joslin (2012) replicate payoffs with a continuum of trigonometric functions. Our spanning series approach is necessary when other solutions are unavailable, e.g., when the characteristic function in unknown. This is the case in the Hull and White (1987) model. Consequently, our propositions provide the first convergent formula for this model. But even in cases where Fourier methods are applicable, our new formulas can be more convenient and computationally effective. The Fourier method of Heston (1993) performs a numerical integral for every option value. In empirical work, this requires many costly evaluations of the characteristic function.[2] By contrast, our new approach requires calculation of the moments only once for all options in a dataset.

We apply the new formulas to the variance-gamma jump model of Madan and Seneta (1990) and the stochastic volatility models of Heston (1993) and Hull and White (1987). As our theory predicts, the Hermite formulas diverge for all these models. But our new logistic formulas converge in all cases.

The rest of the paper is organized as follows. Section 2 presents Gram-Charlier option expansions

[2]Other approaches such as the Fast Fourier Transform of Carr and Madan (1999) are efficient, but require more detailed programming.

and their relation to Hermite polynomials. Section 3 develops an analogous approach that uses orthogonal polynomials based on the standardized logistic density. It shows theoretically that popular Hermite polynomials diverge when the underlying spot returns process is fat-tailed due to jumps or stochastic volatility, while logistic polynomials do not. Section 4 compares our approach with transform methods. Section 5 compares Hermite and logistic polynomials in valuing options when the underlying returns follow the variance-gamma process of Madan and Seneta (1990). Section 6 repeats the exercise for the Heston (1993) and the Hull and White (1987) models. Section 7 concludes.

## 2 Gram-Charlier Option Expansions

Our analysis is inspired by the formulas of Corrado and Su (1996) and Backus, Foresi, and Wu (2004) that approximate option values in terms of risk-neutral moments of *log-spot* returns.[3] We depart from their analysis by analyzing results in terms of orthogonal polynomials instead of mere statistical expansions. The orthogonal polynomials provide an economic interpretation of spanning payoffs that approximate options according to an associated $L^2$ metric.

Given a risk-neutral density $p(x)$ that has mean zero and variance one, Corrado and Su (1996) perform an Edgeworth expansion around the standard normal density $n(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ to derive the Gram-Charlier A Series:

$$p(x) = n(x)\left(1 + \sum_{i=3}^{\infty} \frac{\kappa_i}{i!} He_i(x)\right), \tag{1}$$

[3] Using moments of log-returns is important because moments of the spot return may be infinite, whereas moments of the log-spot price are bounded above by the expected spot price. If the underlying spot price is a currency, then the risk-neutral expectation of the reciprocal also exists and all moments of the log-returns are finite. A separate theoretical issue is that moments may not determine a unique distribution; Stoyanov (1996) displays an infinite number of probability densities with the same moments as the lognormal distribution. In contrast, the normal distribution is uniquely identified by its moments. Our analysis in Sections 2 and 3 guarantees convergence to a unique probability distribution within a square-integrable class.

where $He_i(\cdot)$ is a Hermite polynomial:

$$He_i(x) = (-1)^i \frac{\frac{\partial^i}{\partial x^i} n(x)}{n(x)},$$

and where $\kappa_i$ is the $i$-th cumulant: it equals the expectation of the corresponding Hermite polynomial.[4] Hermite polynomials are easy to compute recursively using the relation:

$$\begin{aligned} He_0(x) &= 1, \\ He_1(x) &= x, \\ He_{i+1}(x) &= xHe_i(x) - iHe_{i-1}(x). \end{aligned}$$

Denote the future spot price by $S = \exp(\mu + \sigma x)$, and the strike price by $K$. The payoff on a call option is $Max(S - K, 0)$. The value of a call option is its risk-neutral expectation discounted at the compound interest rate $r$:

$$C(K) := e^{-r} \int_{-\infty}^{\infty} Max\left(e^{\mu+\sigma x} - K, 0\right) p(x) dx. \tag{2}$$

Integrating the Gram-Charlier series out to four terms yields a convenient call option approximation in terms of variance, skewness, and kurtosis:

$$C(K) \approx C_{He,0}(K) + \frac{skew}{3!}\, C_{He,3}(K) + \frac{kurtosis}{4!}\, C_{He,4}(K), \tag{3}$$

where $skew = E\left[x^3\right]$, $kurtosis = E\left[x^4 - 3\right]$, and $C_{He,0}(K)$ is is the Black-Scholes formula based on

[4]See http://en.wikipedia.org/wiki/Edgeworth_series#Gram.E2.80.93Charlier_A_series and http://en.wikipedia.org/wiki/Hermite_polynomials .

the normal distribution $N(\cdot)$:

$$C_{He,0}(K) = S_0 \; N\left(\frac{log(S_0/K) + r + \frac{1}{2}\sigma^2}{\sigma}\right) - e^{-r} \; K \; N\left(\frac{log(S_0/K) + r - \frac{1}{2}\sigma^2}{\sigma}\right),$$

where $S_0 = e^{\mu - r + \frac{1}{2}\sigma^2}$.

Appendix A expresses the higher terms using the normal distribution. Expression (3) is particularly handy because one could insert Bakshi, Kapadia, and Madan (2003)'s estimates of risk-neutral skewness and kurtosis to provide a handy generalization of the Black-Scholes formula that includes higher moments.

While simple and convenient, the Gram-Charlier approach fails to resolve some important questions. First, what is the *mathematical* nature of the approximation in Equation (3)? In particular, under what conditions does it become increasingly accurate and converge when adding more terms? Second, what is the *economic* interpretation of the approximation?

We pursue answers to these questions by exploiting the orthogonality and square-integrability properties of Hermite polynomials. Distinct Hermite polynomials $He_i(x)$ and $He_j(x)$, $i \neq j$, satisfy an orthogonality relation using the normal density as a weighting function.

**Condition 1** *The Hermite polynomials are orthogonal with respect to the normal density:*

$$\langle He_i, He_j \rangle = \int_{-\infty}^{\infty} He_i(x) He_j(x) n(x) dx = 0. \tag{4}$$

The norm or weighted average square of a Hermite polynomial is given by a factorial

$$\langle He_i, He_i \rangle = i!. \tag{5}$$

If a function $f(x)$ is weighted-square-integrable, i.e., $\langle f, f \rangle$ is finite, then we can approximate it with

Hermite polynomials by minimizing the weighted-square approximation error:

$$\min_{\{a_i\}_{i=0}^n} \int_{-\infty}^{\infty} \left\{ f(x) - \sum_{i=0}^{n} a_i He_i(x) \right\}^2 n(x)dx, \tag{6}$$

and the solution to this finite approximation problem is:

$$a_i = \frac{\langle f, He_i \rangle}{\langle He_i, He_i \rangle}. \tag{7}$$

Note that due to the orthogonality of the polynomials, the coefficient $a_i$ does not depend on the approximation level $n$. Since the Hermite polynomials form a complete orthogonal basis for the associated Hilbert space of square-integrable functions, a standard Hilbert space result is that the approximation in Equation (6) becomes exact and converges as $n$ approaches infinity.[5]

**Proposition 1** *If the weighted-average-square of a function is finite,* $\langle f, f \rangle < \infty$, *then*

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} a_i He_i(x). \tag{8}$$

This is an $L^2$ limit, which ensures convergence almost everywhere.

We can think of the Hermite polynomials approximating the payoff $f(x)$ of an option. But since polynomials explode in the tails, they do not provide suitable approximations for a probability density. For that purpose, we use Hermite functions, which are Hermite polynomials multiplied by the normal density.

**Definition 1** *A Hermite function is* $\Phi_i(x) := He_i(x)\ n(x)$.

Inserting this definition into the orthogonality condition (4) shows that the Hermite functions are orthogonal when the reciprocal of the normal density is used as a weighting function:

[5] See Proposition 27, Chapter 10, in Royden (1968)

**Condition 2** *The Hermite functions are orthogonal with respect to the reciprocal normal density:*

$$\int_{-\infty}^{\infty} \frac{\Phi_i(x)\Phi_j(x)}{n(x)} dx = 0. \tag{9}$$

Hence the Hermite functions form a basis for approximation in this new space. Consider a finite approximation to a probability density by way of Hermite functions

$$\min_{\{b_i\}_{i=0}^{n}} \int_{-\infty}^{\infty} \frac{\left(p(x) - \sum_{i=0}^{n} b_i \Phi_i(x)\right)^2}{n(x)} dx. \tag{10}$$

The solution is

$$b_i = \frac{1}{i!} \int_{-\infty}^{\infty} p(x) He_i(x) dx = \frac{1}{i!} E[He_i(x)].$$

This represents an important dual relationship to the earlier approximation reported in Equation (8). Since the $b_i$ coefficients are proportional to the expectations of Hermite polynomials, they are simply linear combinations of moments. In words, if we can approximate option payoffs with polynomials, then we can approximate the probability density in terms of moments. We formalize this as a proposition.[6]

**Proposition 2** *If $p(x)$ is weighted-square integrable:*

$$\int_{-\infty}^{\infty} \frac{p(x)^2}{n(x)} dx < \infty, \tag{11}$$

*then*

$$p(x) = \lim_{n \to \infty} \sum_{i=0}^{n} b_i \Phi_i(x). \tag{12}$$

Proposition 2 generalizes the Gram-Charlier A Series in Equation (1). Inserting the convergent

[6] Proposition 2 is also an application of Royden (1968). Note that in Proposition 2 the $L^2$ limit uses the reciprocal of the normal density as a weighting function.

series (12) into the call option valuation integral (2) produces an exact series for option values:

$$C(K) = \sum_{i=0}^{\infty} \frac{1}{i!} E\left[He_i\left(x\right)\right] C_{He,i}(K), \tag{13}$$

where $x = \frac{ln(S)-\mu}{\sigma}$ and

$$C_{He,i}(K) = e^{-r} \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \left(e^{\mu+\sigma x} - K\right)\ He_i(x)\ n(x)\ dx.$$

Appendix A provides analytical formulas for the $C_{He,i}(\cdot)$ terms.

There are two important contributions of this infinite series relative to previous approximation formulas. First, the infinite series (12) is an *exact* formula that achieves arbitrary accuracy according to an $L^2$ metric. Second, while it admits arbitrary parameters $\mu$ and $\sigma$, the coefficients of the first and second terms of the expansion (13) only disappear when $\mu$ and $\sigma$ equal the risk-neutral mean and standard deviation of the log-spot price. Corrado and Su (1996) and Backus, Foresi, and Wu (2004) inconsistently neglect those terms while setting the expected log-spot return equal to $r - \frac{\sigma^2}{2}$.[7]

# 3 Logistic Polynomials

Unfortunately, the weighted-square-integrability condition for Proposition 2 is quite restrictive; it requires the tails of the probability distribution to be thinner than a normal density. All fat-tailed models violate this condition, including models with stochastic volatility and large Poisson or Levy jumps. Due to the thin tails of the underlying Gaussian distribution, Hermite polynomials cannot span contingent claims on fat-tailed distributions that are ubiquitous in finance. To surmount these limitations, we

[7] Corrado (2007) acknowledged that this choice violated the martingale restriction.

introduce an analogous set of orthogonal polynomials based on the standardized logistic density

$$l(z) = \frac{\pi}{\sqrt{3}\left(\exp\left(\frac{\pi z}{\sqrt{3}}\right) + 2 + \exp\left(-\frac{\pi z}{\sqrt{3}}\right)\right)}, \tag{14}$$

where the logistic density takes its name from the sigmoid or "logistic" form of the distribution function

$$\int_{-\infty}^{x} l(z)dz = \frac{1}{1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)}.$$

It is possible to define orthogonal polynomials based on other densities. For example Feunou, Fontaine, and Tedongap (2009) use an Edgeworth expansion of the one-sided gamma density to derive Laguerre polynomials. The logistic density has the advantage of being a smooth distribution with fat tails in both positive and negative directions. Figure 1 shows that the logistic density has a symmetric, bell-shaped curve resembling the normal density. Its tails, however, are much fatter than those of the normal density. This feature will be critical to span option payoffs for fat-tailed distributions. The $n^{\text{th}}$ moment of the logistic density is

$$E[x^n] = (2^n - 2)3^{n/2}|B_n|,$$

where $B_n$ is the $n^{\text{th}}$ Bernoulli number. As shown in Table 1, the fourth moment of the standardized logistic distribution has only slightly more kurtosis than that of the normal distribution. But the higher moments grow rapidly; the tenth logistic moment is forty times the tenth Gaussian moment. This illustrates the explosive growth of higher moments of the logistic density.

We define the logistic polynomials, $Lo_i(x)$ to be identical to the Hermite polynomials for $i \leq 2$. By analogy with the Hermite Condition 1, logistic polynomials $Lo_i(x)$ and $Lo_j(x)$, for $i \neq j$, shall be orthogonal with respect to an inner product based on the standardized logistic probability density (14).

**Condition 3** *The logistic polynomials are orthogonal with respect to the logistic density:*

$$\langle Lo_i, Lo_j \rangle := \int_{-\infty}^{\infty} Lo_i(x) Lo_j(x) l(x) dx = 0. \tag{15}$$

Note that this section uses the bracket notation, $\langle . \rangle$, to denote the logistic inner product (15) instead of the Gaussian inner product (4) from the previous section. We can generate the logistic polynomials recursively via the Gram-Schmidt procedure:

$$Lo_n(x) = x^n - \sum_{i=0}^{n-1} \frac{\langle Lo_i, x^n \rangle}{\langle Lo_i, Lo_i \rangle} Lo_i(x), \tag{16}$$

but a computationally faster approach is to generate the polynomials using recursive relations for $i \geq 1$:

$$Lo_{i+1}(x) = x Lo_i(x) - \frac{3i^4}{(2i+1)(2i-1)} Lo_{i-1}(x). \tag{17}$$

The weighted-average-squares of the logistic polynomials have a slightly more complicated formula than the factorial Hermite case (5):

$$\langle Lo_i, Lo_i \rangle = \frac{3^i \ i!^4}{(2i-1)!!(2i+1)!!}. \tag{18}$$

Finally, we scale these polynomials by the logistic density to define logistic functions.

**Definition 2** *A logistic function is* $\Psi_i(x) := Lo_i(x) \ l(x)$.

Table 2 lists the first five Hermite and logistic polynomials along with their weighted-average squared values. Figure 2 compares squared Hermite functions with squared logistic functions. It shows that the $i^{th}$ function has $i$ "wiggles". The wiggles of the Hermite functions spread out within a parabola. In other words, the wiggles of the $i^{th}$ Hermite function spread out proportionally to the square-root of $i$. In contrast, the wiggles of the logistic function spread out much faster, proportional to $i$. This gives them greater ability to fit tail payoffs of options.

Appendix B proves that logistic polynomials span the associated space of square-integrable functions. Like the Hermite polynomials, the logistic polynomials form a complete orthogonal system. This is sufficient to establish the following analogue of Proposition 1.[8]

**Proposition 3** *If the weighted-average square of a function $f(x)$ is finite, $\langle f, f\rangle < \infty$, then*

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} a_i Lo_i(x), \tag{19}$$

*where*

$$a_i = \frac{\langle f, Lo_i\rangle}{\langle Lo_i, Lo_i\rangle}.$$

The orthogonality Condition 3 for logistic polynomials is equivalent to orthogonality Condition 4 for logistic functions.

**Condition 4** *The logistic functions are orthogonal with respect to the reciprocal logistic density:*

$$\int_{-\infty}^{\infty} \frac{\Psi_i(x)\Psi_j(x)}{l(x)} dx = 0. \tag{20}$$

By analogy with Proposition 2, one can approximate probability densities with logistic functions.[9]

**Proposition 4** *If the reciprocal–logistic-weighted-average square of a function $p(x)$ is finite,*

$$\int_{-\infty}^{\infty} \frac{p(x)^2}{l(x)} dx < \infty, \tag{21}$$

*then*

$$p(x) = \lim_{n\to\infty} \sum_{i=0}^{\infty} b_i \Psi_i(x), \tag{22}$$

[8] Note that Proposition 3 uses logistic weighting instead of normal weighting.

[9] Note that Proposition 4 uses reciprocal-logistic-weighting.

*where*

$$b_i = \frac{\int_{-\infty}^{\infty} p(x) Lo_i(x) dx}{\langle Lo_i, Lo_i \rangle}.$$

The reciprocal-logistic-weighted-average-square criterion (21) of Proposition 4 is a fairly lenient condition. It is tantamount to requiring exponentially bounded tails, i.e. $E\left[e^{\frac{\pi}{2\sqrt{3}}x}\right]$ and $E\left[e^{-\frac{\pi}{2\sqrt{3}}x}\right]$ must be finite. In contrast, the analogous condition (11) for Hermite polynomials requires the tails of the probability density to be no fatter than a normal density. The Hermite condition is violated for virtually all interesting fat-tailed distributions in finance.

In our application, $x$ is a rescaled logarithm of the spot price: $x = (log(S) - \mu)/\sigma$, so we need $E\left[S^{\frac{\pi}{2\sqrt{3}\sigma}}\right]$ and $E\left[S^{-\frac{\pi}{2\sqrt{3}\sigma}}\right]$ to be finite. One can verify these conditions for specific cases using the generating function of a model. In general, given that the expected spot price must exist, $E[S] < \infty$, the right tail condition $E\left[S^{\frac{\pi}{2\sqrt{3}\sigma}}\right] < \infty$ is satisfied for $\sigma > \frac{\pi}{2\sqrt{3}}$. In models used for currencies, the expectation of the reciprocal spot price must also exist, $E[S^{-1}] < \infty$. In such models the left tail condition $E\left[S^{-\frac{\pi}{2\sqrt{3}\sigma}}\right] < \infty$ is also satisfied when $\sigma > \frac{\pi}{2\sqrt{3}}$.

To illustrate the relation between propositions, we consider an Arrow-Debreu security that pays off when the state variable $x$ attains the value $k$, i.e., $f(x) = \delta(x - k)$. Using Proposition 3, Equation (19), we represent this delta function payoff in terms of logistic polynomials

$$\delta(x - k) = \sum_{i=0}^{\infty} \frac{Lo_i(k) l(k)}{\langle Lo_i, Lo_i \rangle} Lo_i(x). \tag{23}$$

The probability density $p(k)$ is equal to the expected value of the delta function (23)

$$p(k) = E[\delta(x - k)] = \sum_{i=0}^{\infty} \frac{E[Lo_i(x)]}{\langle Lo_i, Lo_i \rangle} \Psi_i(k). \tag{24}$$

But this is equivalent to the logistic function representation (22) in Proposition 4. This shows how approximating contingent claim payoffs gives the Arrow-Debreu state prices necessary to value options and other contingent claims.

We can apply this reasoning to derive a new formula for the value of a call option. Let the future spot price equal $e^{\mu+\sigma x}$. The payoff on a call option with spot price $S$ and strike price $K$ is $Max(0,\ S-K)$. Recall that the value of a call option $C(K)$ is equal to the discounted expected value of its payoff (2). Inserting the logistic representation (22) and integrating term-by-term shows

$$C(K)=\sum_{i=0}^{\infty}\frac{E\left[Lo_i\left(x\right)\right]}{\langle Lo_i, Lo_i\rangle}C_{Lo,i}(K), \tag{25}$$

where $x=\frac{ln(s)-\mu}{\sigma}$ and

$$C_{Lo,i}(K)=e^{-r}\int_{\frac{\log(K)-\mu}{\sigma}}^{\infty}\left(e^{\mu+\sigma x}-K\right)\ Lo_i(x)\ l(x)\ dx.$$

Appendix C provides closed-form expressions for the integrals $C_{Lo,i}$ in terms of special functions. The fourth-order logistic approximation resembles the Gram-Charlier approximation (3), but with different coefficients corresponding to the norms of the logistic polynomials listed in the last column of Table 2:

$$C_{Lo,0}(K)+E[x]\ C_{Lo,1}(K)+\frac{E[x^2-1]}{16/5}\ C_{Lo,2}(K)+\frac{E[x^3-\frac{21}{5}x]}{3888/175}\ C_{Lo,3}(K)+\frac{E[x^4-\frac{78}{7}x^2+\frac{243}{35}]}{331776/1225}\ C_{Lo,4}(K).$$

Again, the first- and second-order terms vanish by choosing $\mu$ and $\sigma$ to be the mean and standard deviation. The advantage of the logistic approximation is that it holds for fat-tailed models that violate the Hermite convergence condition of Proposition 2.

# 4 Comparison with Transform Methods

The previous sections showed how to span option payoffs with polynomials. Since the discounted expectation of a polynomial is a linear function of moments, this provides valuation formulas for options in terms of moments. This section compares that approach to traditional transform methods of Breeden and Litzenberger (1978), Heston (1993), Bakshi and Madan (2000) and Chen and Joslin (2012). Those methods effectively use a continuum of Arrow-Debreu or trigonometric functions to span options. Specifically, Breeden and Litzenberger (1978) use Dirac delta functions, while the Fourier approaches use trigonometric functions. This perspective unifies the approaches, showing that they differ mainly in the underlying choice of of fundamental basis securities.

Breeden and Litzenberger (1978) emphasize the relation of option values to Arrow-Debreu prices. We begin with a very simple illustration of payoff replication. Consider evaluating the normal density at three points using a trinomial implementation of the Breeden-Litzenberger model. Figure 3 shows the approximation where the log-spot return is 0 with probability $\frac{2}{3}$ and $\pm\sqrt{3}$ with probability $\frac{1}{6}$.

This is a terrible approximation to the Gaussian density at any specific point, in the sense that it is either zero or a Dirac delta function. But a call option payoff is an extremely smooth double integral of a Dirac delta function

$$Max(S-K,0)=\int_0^S\int_0^v \delta(u-K)dudv.$$

This allows the possibility of doing a terrible job at replication, but still getting a reasonable result for valuation. Note that the trinomial probability distribution in Figure 3 matches the first five moments of a normal distribution. It should therefore provide a good approximation for any contingent claim with a payoff that can be approximated by a quintic polynomial function of the compound spot return.

To continue this example, assume that the compound spot return is $\mu$ with probability $\frac{2}{3}$ and $\mu\pm\sigma$ with probability $\frac{1}{6}$.[10] If the interest rate is zero, then the trinomial call option value is

$$\hat{C}(S_0,K)=\frac{1}{6}Max(S_0e^{\mu-\sigma}-K,0)+\frac{2}{3}Max(S_0e^{\mu}-K,0)+\frac{1}{6}Max(S_0e^{\mu+\sigma}-K,0),$$

[10] To obey the martingale restriction, the parameter $\mu$ must satisfy: $\frac{1}{6}e^{\mu-\sigma}+\frac{2}{3}e^{\mu}+\frac{1}{6}e^{\mu+\sigma}=e^{r}$.

where $K$ denotes the exercise price and the current spot price is $S_0$. Note that the option value is piecewise linear in the spot price. Figure 4 shows that the trinomial option formula provides a surprisingly reasonable approximation to the Black-Scholes formula with $\sigma = \frac{1}{2}$.

Another popular approach to option valuation uses generalized Fourier transforms (Heston (1993), Bakshi and Madan (2000), Heston and Nandi (2000), and Chen and Joslin (2012)). This method proceeds from the formula

$$\begin{aligned} E[Max(S-K,0)] = E[S]&\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{K^{-i\phi}E[S^{i\phi+1}]}{i\phi E[S]}\right)d\phi\right) \\ &- K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{K^{-i\phi}E[S^{i\phi}]}{i\phi E[S]}\right)d\phi\right). \end{aligned} \tag{26}$$

The first term in Equation (26) is the expected proceeds from the spot price received at option exercise, while the second term is the expected value of payment of the exercise price. In the deterministic case we can eliminate the expectations and the Fourier formula simplifies to

$$Max(S-K,0) = \frac{S-K}{2} + \frac{S-K}{\pi}\int_0^\infty \frac{\sin(\phi\log(S/K))}{\phi}d\phi. \tag{27}$$

This means that we can span the call option payoff with a continuum of trigonometric functions involving sines of the log-spot return. Figure 5 plots the integrand of Equation (27) for an at-the-money option ($S = K$), along with a crude unit-step rectangular approximation using abscissas at $\phi = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

This is tantamount to approximating the option payoff by:

$$\begin{aligned} Max(S-K,0) \approx \frac{(S-K)}{2} + \frac{(S-K)}{\pi}\times& \\ \left(2\sin\left(\frac{1}{2}\log(S/K)\right) + \frac{2}{3}\sin\left(\frac{3}{2}\log(S/K)\right) + \frac{2}{5}\sin\left(\frac{5}{2}\log(S/K)\right)\right)&. \end{aligned} \tag{28}$$

While Figure 5 seems to be a very crude approximation in the frequency domain, Figure 6 shows that the three-point formula (28) provides a remarkably good approximation to a call payoff with unit strike price.

We would like to compare these trigonometric option implementations with comparable polynomial approximations. We observe that polynomials can be interpreted as a limiting case of trigonometric functions. As Figure 5 illustrates, one must evaluate the integrands of (26) at a finite set of abscissas. Compared to the semi-infinite region $[0, \infty]$, these abscissas are all near zero. This makes it natural to consider evaluating the integral in terms of local information about the integrand at the origin $\phi = 0$. The derivatives of the generating function are simply moments of the log-spot-price:

$$\left.\frac{\partial^n E[S^\phi]}{\partial \phi^n}\right|_{\phi=0} = E[\log(S)^n].$$

Hence, using polynomials is like using sine functions with very low frequencies.

Figure 7 shows the third-order approximations to the payoff of a call option with both Hermite and logistic polynomials. These polynomials provide similar approximations that are surprisingly accurate. Valuation will average out the overestimation and underestimation in the graphs, producing greater accuracy. But the exact results depend on the specific model at hand, so the next section will numerically evaluate the performance of Hermite and logistic polynomials using several prominent option models.

From a visual perspective, the polynomial approach compares favorably to the piece-wise linear Breeden-Litzenberger approximations in Figure 4 or the Fourier approximations in Figure 6. The polynomial graphs in Figure 7 are a straightforward application of Propositions 1 and 3. In contrast, we needed to choose the frequencies $\phi = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ in Equation (28) by trial-and-error to provide a visually appealing illustration in Figure 6. It is intuitive that evaluating Fourier integrals using any closely spaced abscissas will converge to correct option values. But from a practical perspective, the transform methods offer no prescription for efficient evaluation of the integrals. In addition, sine waves have no economic motivation and are weird in the context of valuing options. Finally, transform methods require the unnecessarily strong theoretical assumption that one can create a replicating portfolio with a continuum of assets. Therefore, the polynomial approach provides a practical and theoretically justified tool for spanning and valuing options.

# 5 Application to the Variance-Gamma Model

Recall the convergence condition for Hermite approximations in Proposition 2 is:

$$\int_{-\infty}^{\infty} \frac{p(x)^2}{n(x)} dx < \infty. \tag{29}$$

This condition shows that Hermite polynomials cannot generally approximate options for densities that are substantially fatter than a Gaussian distribution. Most modern dynamic models for derivatives valuation violate this condition because they generate fat tails using jumps or stochastic volatility.

For our first example, we use a model with a known solution because this allows us to rigorously assess the accuracy of our approximations. For this purpose, we choose the prominent Variance Gamma (V.G.) model of Madan and Seneta (1990). Empirical studies of the V.G. model include Madan, Carr, and Chang (1998), Carr, Geman, Madan, and Yor (2002) and Carr and Wu (2004). The model is also implemented in Bloomberg terminals. The V.G. model shows the misleading appeal of a Hermite approximation that is surprisingly accurate at the fourth-order, but subsequently diverges.

The V.G. model assumes that the log-spot return has independent increments generated by mixing the normal distribution over a distribution of variance. The mean of the log-spot-return is $\mu$, and the variance is $\sigma^2 V$, where $\mu$ and $\sigma^2$ are constant parameters. The distribution of V is gamma, with parameters $c$ and $\gamma$ and density $g(\nu)$:

$$g(\nu) = \frac{c^{\gamma} \nu^{\gamma-1} e^{-c\nu}}{\Gamma[\gamma]}, \tag{30}$$

where $\Gamma(\cdot)$ is the gamma function. To impose that the mean of V is unity, the restriction $\gamma = c$ is imposed.

Madan and Seneta (1990) show that the density of the log-spot-return can be written as:

$$f(x) = \frac{\sqrt{2/\upsilon}}{\sigma} \frac{(x\sqrt{2/\upsilon}/\sigma)^{(2/\upsilon-1)/2}}{2^{(2/\upsilon-1)/2}\Gamma(1/\upsilon)\sqrt{\pi}} \;\; K_{(2/\upsilon-1)/2}(x\sqrt{2/\upsilon}/\sigma), \tag{31}$$

where $\upsilon = \gamma/c^2$ and $K_w(x)$ is a Bessel function of the second kind of order $w$. Then the call option value is the expected discounted expectation over the risk-neutral density:

$$C(K) = e^{-r} \int_{-\infty}^{\infty} Max\left(e^{\mu+\sigma x} - K, 0\right) f(x)\; dx. \tag{32}$$

The V.G. is model is symmetric, so all odd central moments vanish. The central even moments are:

$$E\left[(R-\mu)^n\right] = 2^n\; \sigma^{2n}\; (2n-1)!!\; \frac{\Gamma[\gamma+n]}{\Gamma[\gamma]}. \tag{33}$$

The V.G. density has fat tails, with kurtosis of $3(1+\upsilon)$. We set the parameters of the model to $\gamma = 3/2$, $\sigma = \frac{\sqrt{1/(2\gamma)}}{10}$. This gives a variance of $0.1^2$, skewness equal to 0 and excess kurtosis of 2. We choose the $\gamma$ parameter small enough to make the Hermite method accurate at the fourth-order and large enough for the approximation to diverge slowly enough to fit on our graphs.[11]

Figure 8 shows Hermite and logistic approximations of the V.G. probability density for approximation orders 0, 4, and 8, and approximation orders 0, 4, and 20, respectively. Both approaches make a reasonable approximation of the V.G. density at the 4-th order of expansion, but the behavior of the two methods differs as we increase the level of approximation. The divergence of the Hermite method becomes apparent in the density of the 8-th order expansion. In particular, the 8-th order Hermite expansion generates a negative density in some regions of the spot-price support. This implies arbitrage opportunities in the form of negative values for butterfly spreads that pay off in this region.

[11] The martingale condition is $\mu = r + \gamma log(1-\sigma) + \gamma log(1+\sigma)$, where we use an interest rate $r = 0$.

The logistic method, on the other hand, is much better behaved and shows no sign of divergence even at the 20-th order of expansion. As we add moments, the approximation improves, as guaranteed by Proposition 4.[12]

Our interest, however, lies in valuing options where the underlying log-spot price is generated by a V.G. model. We report the results of this exercise in Table 3. The table reports Hermite and logistic approximations for call options that have an expiration of one year and a strike price of 100.[13] We allow the spot price to take the values 90 in Panel A, 100 in Panel B and 110 in Panel C to assess the performance of the methods for options that are out of the money, at the money and in the money, respectively. The table presents option values for orders of expansions ranging from 0 to 20, along with the exact option value obtained from the numerical integration of Equation (32). It also reports the absolute approximation errors, and the ratio of these errors with respect to the errors generated by the Black-Scholes option valuation.

Panel A of Table 3 shows the results when the spot price equals 90. In this case, the true option value equals 0.78. At the 0-th order of expansion, the Hermite method coincides with Black-Scholes and is 6 cents away from the true value. At the 4-th order of expansion, the Hermite method performs very well, as its approximation is almost equal to the exact value. At the 6-th order, however, the value is 12 cents away from the exact one and then quickly diverges. For example, at the 14-th order of expansion, the value implied by the Hermite method is negative at $-5.09$. The logistic method is much better behaved. The logistic value approximation at the $0-th$ moment is equal to 0.74 which has only 55% of the Black-Scholes error. The logistic value approximation is 0.76 at the 20-th order, which has only 26% of the Black-Scholes error.

Panel B reports the results when the spot price equals 100. In this case, the true option value equals 3.68. At the 0-th order of expansion, the Hermite method is 31 cents away from the true value. At the 4-th order of expansion, the Hermite method "gets lucky" again and its valuation is 2 cents

[12]Unreported computations show that the accuracy improves as we increase the expansion to 200 moments.

[13]As for the plots reported in Figure 8 we set $\gamma = 3/2$, $\sigma = \frac{\sqrt{1/(2\gamma)}}{10}$, and $\mu = r + \gamma log(1-\sigma) + \gamma log(1+\sigma)$.

away from the true one. At the 6-th order, however, the value is 20 cents away from the exact one and again quickly diverges. For example, at the 16-th order of expansion, the value implied by the Hermite method is negative at -49.6. The logistic method is, once again, much better behaved. The valuation at the 0-th moment is equal to 3.82 and its relative error with respect to Black-Scholes equals 55%. It then monotonically converges to 3.72 at the 20-th order, which is only 12% of the Black-Scholes error.

Panel C reports the results when the spot price equals 110. In this case, the true option value equals 10.93. The Hermite method does not "get lucky" this time, and the best valuation of the method is achieved at the 0-th order of expansion. The value then diverges quickly as more terms are added. The logistic method is, on the other hand, extremely lucky as it nails the correct value at the 0-th order. The subsequent logistic approximations are all more accurate than the Black-Scholes approximation and display nice convergence to the exact value.

Collectively, these results explain the appeal of the Hermite approach. In lucky cases, the fourth-order approximation can be strikingly accurate, and is never terrible. However, it ultimately diverges. The logistic approximation can also get lucky and give excellent values at low orders of approximation. More importantly, it is comparatively accurate at all orders, and converges nicely.

The Variance-Gamma model is useful to investigate how kurtosis affects option values. But due to its symmetry, it has no skewness. The next section examines stochastic volatility models that exhibit pronounced skewness.

## 6 Application to Stochastic Volatility Models

Another useful application of logistic polynomials is valuation in stochastic volatility models. Dragulescu and Yakovenko (2002) showed that stochastic volatility models have exponential tails. Therefore the Hermite convergence condition (29) is violated, and the resulting Hermite series diverge. But the logistic method still works.

We consider two influential models: the Heston (1993) model and the Hull and White (1987) model. The former is instructive because it has a closed-form characteristic function, which allows us to compare the polynomial moment methods with the Fourier method of valuation. The latter is particularly useful because the Hull-White model lacks a closed-form solution. By deriving the moments for the Hull-White process, our application provides the first convergent series for option values in the Hull-White model.

### 6.1 Heston (1993) Model

The Heston (1993) model assumes the following risk-neutral dynamics for a spot price $S(t)$:

$$\begin{aligned} dS &= r\ S\ dt + \sqrt{v}\ S\ dz_1 \\ dv &= (\alpha + \eta v)\ dt + \xi\ \sqrt{v}\ dz_2, \end{aligned} \tag{34}$$

where $r$ is the risk-free interest rate and $z_1(t)$ and $z_2(t)$ are Wiener processes with correlation $\rho$.

As shown by Heston (1993), the option values associated with the process (34) can be solved using the generating function of spot returns that takes the following form:

$$f(\phi, t, \tau) = E_t[S(t+\tau)^{\phi}] = e^{A(\tau,\phi)+B(\tau,\phi)v(t)},$$

where

$$\begin{aligned} A(\tau,\phi) &= r\ \phi\ \tau + \frac{\alpha}{\sigma^2}\left((\kappa - \rho\ \sigma\ \phi + d)\,\tau - 2\ Log\left[\frac{1-g\ e^{d\tau}}{1-g}\right]\right), \\ B(\tau,\phi) &= \frac{\kappa - \rho\ \sigma\ \phi + d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-g\ e^{d\tau}}\right], \end{aligned}$$

and

$$
\begin{aligned}
g &= \frac{\kappa + \rho\,\sigma\,\phi + d}{\kappa - \rho\,\sigma\,\phi - d}, \\
d &= \sqrt{(\rho\sigma\phi - \kappa)^2 + \sigma^2\,(\phi + \phi^2)}.
\end{aligned}
$$

We adopt the following parameterization of the model: $v = \sigma^2 = 0.01$, $\kappa = 0$, $\alpha = 0$, $r = 0$, $\tau = 180/365$, and set the correlation coefficient $\rho$ to -2/3 in order to assess how the Hermite and logistic methods perform in the presence of realistic values for the skewness and kurtosis of the underlying spot return density.

Adopting the same format of Table 3, Table 4 reports the results for the option values implied by the Heston model. Note that, in this case, the exact option values are computed using Heston (1993)'s Fourier inversion method.

Panel A of Table 4 shows the results when the spot price equals 90. In this case, the true option value equals 0.07. At the 0-th order of expansion, the Hermite method coincides with Black-Scholes and is 13 cents away from the true value. At the 4-th order of expansion, the Hermite method performs very well as the value it implies is 1 cent away from the exact one. At the 6-th order, however, the value is 3 cents away from the exact one and quickly diverges to implausibly large or even negative values. For example, at the 16-th order of expansion, the value implied by the Hermite method is negative at $-0.62$. The logistic method is much better behaved. The valuation at the 0-th moment is equal to 0.25 and its error relative to Black-Scholes equals 135%. The error then shrinks to 0.08 at the 20-th order, which is only 3% of the Black-Scholes error.

Panel B reports the results when the spot price equals 100. In this case, the true option value equals 2.74. At the 0-th order of expansion, the Hermite method is 9 cents away from the true value. At the 4-th order of expansion, the Hermite method improves to 6 cents away from the true value. At the 6-th order of expansion, the Hermite method improves even further as its valuation is 4 cents away from the correct one. At the 8-th order, however, the value is 8 cents away from the exact one and starts diverging. For example, at the 20-th order of expansion, the value implied by the Hermite method is negative at -21.3. The logistic method is, once again, much better behaved. The valuation at the 0-th moment is equal to 2.71 and its relative error with respect to Black-Scholes equals -32%. The value, however, does not display any divergent behavior as the expansion order increases and the its relative error with respect to Black-Scholes equals -23% at the 20-th order of expansion.

Panel C reports the results when the spot price equals 110. In this case, the true option value equals 10.47. The Hermite method improves its valuation up to the 14-th order of expansion and then starts diverging as more terms are added. The logistic method is, on the other hand, very well-behaved as the value is very close to the true one at low expansion orders, but the method does not display divergence as the expansion order increases.

In the same spirit as Figure 8, we plot the divergence of the Hermite method and the convergence of the logistic method in Figure 9. Note that the divergence is already apparent at the 8-th term for the Hermite method, while the logistic method approximates the true density increasingly well as the approximation order increases.

### 6.2 Hull and White (1987) Model

The Hull-White model assumes the following risk-neutral dynamics for the spot price $S(t)$:

$$\begin{aligned} dS &= r\ S\ dt + \sqrt{v}\ S\ dz_1 \\ dv &= \eta\ v\ dt + \xi\ v\ dz_2, \end{aligned} \tag{35}$$

where $r$ is the risk-free interest rate and $z_1(t)$ and $z_2(t)$ are Wiener processes with correlation $\rho$.[14]

The Hull-White model is important because it has been influential in derivatives valuation and has performed well in empirical tests by Christoffersen, Dorion, Jacobs, and Wang (2010) and Christoffersen, Jacobs, and Mimouni (2010). It is also important because it is the diffusion analogue of Bollerslev (1986)'s classic GARCH(1,1) model. This GARCH model has shown superior empirical

[14] We focus on $\rho \leq 0$ to preclude arbitrage (Heston, Loewenstein, and Willard (2007)).

results in studies by Duan (1995) and Hsieh and Ritchken (2005). Unfortunately though, the characteristic function for the Hull-White model is unknown. Hull and White (1987) were therefore forced to simulate discrete approximations for valuation.

Appendix D presents the solution for the moments of the Hull-White model. This enables the use of our series solutions. We adopt the following parameterization of the model: $\upsilon = 0.01$, $\eta = 0.001$, $r = 0$, $\xi = 1$, $\tau = 180/365$; and set the correlation coefficient $\rho$ to -2/3.

Table 5 presents option values for orders of expansion ranging from 0 to 20. For each method and expansion order, we report option values, the error with respect to the option value implied by logistic model with order of expansion equal to 20 — which we refer to as the "final value" henceforth — and the ratio of the error to the error generated by the Black-Scholes option valuation.

Panel A of Table 5 shows the results when the spot price equals 90. In this case, the final option value equals 0.08. At the 0-th order of expansion, the Hermite method coincides with Black-Scholes and is 12 cents away from the final value. At the 4-th order of expansion, the Hermite method performs very well as the value it implies is 1 cent away from the final one. At the 6-th order, however, the value is 10 cents away from the exact one and quickly diverges. The logistic method is much better behaved. The valuation at the 0-th moment is equal to 0.25 and its relative error to respect to Black-Scholes equals to 140%. It then converges to 0.08 at the 20-th order.

Panel B reports the results when the spot price equals 100. In this case, the final option value equals 2.69. At the 0-th order of expansion, the Hermite method is 13 cents away from the final value. At the 4-th order of expansion, the Hermite method improves as its valuation is 7 cents away from the final one. At the 6-th order, however, the value is 17 cents away from the final one and starts diverging. For example, at the 12-th order of expansion, the value implied by the Hermite method is negative at -15.9. The logistic method is, once again, much better behaved. The valuation at the 0-th moment is equal to 2.71 and its relative error with respect to Black-Scholes equals 12%. The value, however, does not display any divergent behavior as the expansion order increases.

Panel C reports the results when the spot price equals 110. In this case, the final option value equals 10.48. The Hermite method obtains the best valuation at the 4-th order of expansion and then starts diverging as more terms are added. The logistic method is, on the other hand, very well-behaved as the value is very close to the true one at low expansion orders, but the method does not display divergence as the expansion order increases.

Finally, in Figure 10 we plot the divergence of the Hermite method and the convergence of the logistic method. Note that the divergence is already apparent at the 4-th term for the Hermite method, while the density approximations of the logistic method are always well-behaved.

# 7 Conclusions

This paper uses orthogonal polynomials to span option payoffs. It shows that the value of an option is equal to the value of an infinite series of replicating payoffs. This provides an intuitive and computationally efficient alternative to Heston (1993)'s Fourier valuation.

The coefficients of our series are based on the moments of the risk-neutral distribution of log-spot returns. Hence, the new formulas express option values in terms of the volatility, skewness, kurtosis, and higher moments of the distribution.

Our analysis establishes that different tail conditions are required to support convergence of different series. Many interesting finance models violate some of these conditions due to fat tails caused by Levy jumps or stochastic volatility. In these cases, the strong assumptions required by previous formulas do not hold, but the weaker assumptions underlying our new logistic formulas do. This gives practical formulas for a variety of applications, including the first convergent solution for the Hull and White (1987) model.

By interpreting moment formulas in terms of spanning, our paper reconciles probability approximations to options with the more traditional arbitrage approach. This rigorously fulfills the research program of Jarrow and Rudd (1982) and Corrado and Su (1996) to value options under arbitrary return distributions.

## References

ABKEN, P. A., D. B. MADAN, AND S. RAMAMURTIE (1996a): "Estimation of risk-neutral and statistical densities by Hermite polynomial approximation: with an application to eurodollar futures options," *FRB Atlanta - Economic Review*, 96(5).

——— (1996b): "Pricing S & P 500 Index Options Using a Hilbert Space Basis," *FRB Atlanta - Economic Review*.

BACKUS, D., S. FORESI, AND L. WU (2004): "Accounting for biases in Black-Scholes," *Working Paper*.

BAKSHI, G., N. KAPADIA, AND D. MADAN (2003): "Stock return characteristics, skew laws, and the differential pricing of individual equity options," *Review of Financial Studies*, 16(1), 101–143.

BAKSHI, G., AND D. MADAN (2000): "Spanning and derivative-security valuation," *Journal of Financial Economics*, 55(2), 205–238.

BOLLERSLEV, T. (1986): "Generalized autoregressive conditional heteroskedasticity," *Journal of econometrics*, 31(3), 307–327.

BREEDEN, D. T., AND R. H. LITZENBERGER (1978): "Prices of state-contingent claims implicit in option prices," *Journal of business*, pp. 621–651.

BRENNER, M., AND Y. EOM (1997): "No-arbitrage option pricing: New evidence on the validity of the martingale property," *NYU - Working Paper*.

CARR, P., H. GEMAN, D. B. MADAN, AND M. YOR (2002): "The Fine Structure of Asset Returns: An Empirical Investigation*," *The Journal of Business*, 75(2), 305–333.

CARR, P., AND D. MADAN (1999): "Option valuation using the fast Fourier transform," *Journal of Computational Finance*, 2(4), 61–73.

CARR, P., AND L. WU (2004): "Time-changed Lévy processes and option pricing," *Journal of Financial Economics*, 71(1), 113–141.

CHEN, H., AND S. JOSLIN (2012): "Generalized transform analysis of affine processes and applications in finance," *Review of Financial Studies*, 25(7), 2225–2256.

CHRISTOFFERSEN, P., C. DORION, K. JACOBS, AND Y. WANG (2010): "Volatility components, affine restrictions, and nonnormal innovations," *Journal of Business & Economic Statistics*, 28(4), 483–502.

CHRISTOFFERSEN, P., K. JACOBS, AND K. MIMOUNI (2010): "Volatility dynamics for the S&P500: evidence from realized volatility, daily returns, and option prices," *Review of Financial Studies*, 23(8), 3141–3189.

COLLIN-DUFRESNE, P., AND R. S. GOLDSTEIN (2002): "Pricing swaptions within an affine framework," *The Journal of Derivatives*, 10(1), 9–26.

CORRADO, C. (2007): "The hidden martingale restriction in Gram-Charlier option prices," *Journal of futures markets*, 27(6), 517–534.

CORRADO, C. J., AND T. SU (1996): "Skewness and kurtosis in S&P 500 index returns implied by option prices," *Journal of Financial research*, 19, 175–192.

COX, J. C., S. A. ROSS, AND M. RUBINSTEIN (1979): "Option pricing: A simplified approach," *Journal of financial Economics*, 7(3), 229–263.

DRAGULESCU, A. A., AND V. M. YAKOVENKO (2002): "Probability distribution of returns in the Heston model with stochastic volatility*," *Quantitative Finance*, 2(6), 443–453.

DUAN, J.-C. (1995): "The GARCH option pricing model," *Mathematical finance*, 5(1), 13–32.

FEUNOU, B., J.-S. FONTAINE, AND R. TEDONGAP (2009): "The equity premium and the volatility spread: The role of risk-neutral skewness," *Bank of Canada Working Paper*.

Heston, S. L. (1993): "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of financial studies*, 6(2), 327–343.

Heston, S. L., M. Loewenstein, and G. A. Willard (2007): "Options and bubbles," *Review of Financial Studies*, 20(2), 359–390.

Heston, S. L., and S. Nandi (2000): "A closed-form GARCH option valuation model," *Review of Financial Studies*, 13(3), 585–625.

Hsieh, K., and P. Ritchken (2005): "An empirical comparison of GARCH option pricing models," *Review of Derivatives Research*, 8(3), 129–150.

Hull, J., and A. White (1987): "The pricing of options on assets with stochastic volatilities," *The Journal of Finance*, 42(2), 281–300.

Jarrow, R., and A. Rudd (1982): "Approximate option valuation for arbitrary stochastic processes," *Journal of financial Economics*, 10(3), 347–369.

Knight, J. L., and S. E. Satchell (1997): "The cumulant generating function estimation method," *Econometric Theory*, 13(02), 170–184.

Longstaff, F. A. (1995): "Option pricing and the martingale restriction," *Review of Financial Studies*, 8(4), 1091–1124.

Madan, D. B., P. P. Carr, and E. C. Chang (1998): "The variance gamma process and option pricing," *European finance review*, 2(1), 79–105.

Madan, D. B., and F. Milne (1994): "Contingent claims valued and hedged by pricing and investing in a basis," *Mathematical Finance*, 4(3), 223–245.

Madan, D. B., and E. Seneta (1990): "The variance gamma (VG) model for share market returns," *Journal of business*, pp. 511–524.

Ritchken, P., L. Sankarasubramanian, and A. M. Vijh (1993): "The valuation of path dependent contracts on the average," *Management Science*, 39(10), 1202–1213.

Royden, H. L. (1968): *Real analysis*. Macmillan New York.

Rubinstein, M. (1998): "Edgeworth binomial trees," *The Journal of Derivatives*, 5(3), 20–27.

Skabelin, A. (2005): "Pricing of options on assets with level dependent stochastic volatility," *SPIE Third International Symposium on Fluctuations and Noise*, pp. 110–124.

Stoyanov, J. M. (1996): *Counterexamples in probability*, vol. 10. Wiley.

Xiu, D. (2014): "Hermite polynomial based expansion of European option prices," *Journal of Econometrics*, 179, 158–177.

# Appendix: Analytical Formulas

## *A. Closed-Form Expressions for the Hermite Integrals*

Recall the definition of $C_{He,i}$ in Equation (13):

$$C_{He,i}(K) = e^{-r} \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \left(e^{\mu+\sigma x} - K\right) \; He_i(x) \; n(x) \; dx. \tag{A1}$$

For $i = 0$, we get the Black-Scholes formula:

$$C_{He,0}(K) = S_0 \; N\left(\frac{log(S_0/K) + r + \frac{1}{2}\sigma^2}{\sigma}\right) - e^{-r} \; K \; N\left(\frac{log(S_0/K) + r - \frac{1}{2}\sigma^2}{\sigma}\right),$$

where $S_0 = e^{\mu - r + \frac{1}{2}\sigma^2}$. For $i \geq 1$:

$$\begin{aligned} C_{He,i}(K) = {} & \sigma^i \; e^{\mu + \frac{\sigma^2}{2} - r} \; N\left(\frac{\sigma^2 + \mu - log(K)}{\sigma}\right) \\ & + e^{\mu + \frac{\sigma^2}{2} - r} \; n\left(\frac{log(K) - \mu}{\sigma} - \sigma\right) \sum_{j=0}^{i-1} \sigma^i He\left(i - 1 - j, \frac{log(K) - \mu}{\sigma}\right) \\ & - e^{-r} \; K \; n\left(\frac{log(K) - \mu}{\sigma}\right) \; He\left(i - 1, \frac{log(K) - \mu}{\sigma}\right) \end{aligned}$$

where $N(x)$ denotes the cumulative distribution function of a standard normal random variable and $n(x)$ is defined as $n(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$.

## B. *Completeness of Logistic Polynomials*

This appendix adapts a standard Fourier proof to show that polynomials are complete in the space of logistic-square-integrable functions.[15] Since the linear span of logistic polynomials is the space of all polynomials, and since logistic polynomials are orthogonal by construction, this establishes that logistic polynomials form a complete orthogonal system for this space.

To show that polynomials are complete, we must show that any function $f(x)$ satisfying

$$\int_{-\infty}^{\infty} f(x)l(x)x^n dx = 0$$

for all $n \geq 0$ must be identically zero. Define the Fourier transform of $f(x)l(x)$

$$F(\phi) := \int_{-\infty}^{\infty} \exp(\phi x) f(x)l(x)dx = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \int_{-\infty}^{\infty} f(x)l(x)x^n dx. \tag{B1}$$

where $|Re(\phi)| < \frac{\pi}{2\sqrt{3}}$. The function is analytic in this region and each term in the summation is zero. By the Identity Theorem, the Fourier transform must be identically zero, and the function $f(\cdot)$ must be zero almost everywhere. Completeness of logistic polynomials extends to completeness of logistic functions in the reciprocal-logistic-square-integrable space.

Standard Hilbert space results (Proposition 27 in Chapter 10, Royden (1968)) show that if $\{\Omega_i(x)\}_{i=0}^{\infty}$ is a complete orthogonal system and $f$ is an element of the weighted-square-integrable Hilbert space, then

$$f(x) = \sum_{i=0}^{\infty} \frac{\langle f, \Omega_i \rangle}{\langle \Omega_i, \Omega_i \rangle} \Omega_i(x). \tag{B2}$$

[15] http://en.wikipedia.org/wiki/Hermite_polynomials#Completeness

*C. Closed-Form Expressions for the Logistic Integrals*

Recall the definition of $C_{Lo,i}$ in Equation (25):

$$C_{Lo,i}(K) = e^{-r} \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \left(e^{\mu+\sigma x} - K\right) \; Lo_i(x) \; l(x) \; dx. \tag{C1}$$

For example, when $\mu = log(S_0) + r - \frac{1}{2}\sigma^2$, we have an analogue of Black-Scholes. Using put-call parity, we can write call values in terms of put values:

$$\begin{cases} C_{Lo,i}(K) = S_{Lo,i} + P_{Lo,i}(K) - Ke^{-r} & \text{for } i = 0, \\ C_{Lo,i}(K) = S_{Lo,i} + P_{Lo,i}(K) & \text{for } i \geq 1. \end{cases} \tag{C2}$$

In equation (C2), $S_i$ can be written as:

$$\begin{aligned} S_{Lo,i} &= e^{-r} \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} e^{\mu+\sigma x} \; Lo_i(x) \; l(x) \; dx \\ &= e^{\mu-r} \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} e^{\sigma x} \; Lo_i(x) \; l(x) \; dx \\ &= e^{\mu-r} \sum_{j=0}^{i} l_{ij} \; \frac{\partial^j}{\partial \sigma^j} \left(\sigma\sqrt{3}\csc(\sigma\sqrt{3})\right), \end{aligned} \tag{C3}$$

where $l_{ij}$ is the coefficient of $Lo_i(x)$ on $x^j$

$$l_{ij} = \frac{1}{j!} Lo_i^{(j)}(0) \tag{C4}$$

and $\frac{\partial^j}{\partial \sigma^j}(\cdot)$ denotes the $j-th$ partial derivative with respect to $\sigma$. The initial put option integral is

$$P_{Lo,0}(K) = e^{-r} \left[ \frac{K}{1 + e^{-\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}} - \frac{e^{\mu + (1+\frac{\sqrt{3}}{\pi}\sigma)\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}}{1 + e^{\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}} + \frac{\sqrt{3}}{\pi} e^{\mu} \; \sigma \; I\left(0, \frac{\sqrt{3}}{\pi}\sigma, \frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right) \right],$$

where

$$I(0,x,y) = {}_2F_1\left(1, 1+x, 2+x, -e^y\right) \frac{e^{y(1+x)}}{1+x},$$

and ${}_2F_1(\cdot)$ is the confluent hypergeometric function. It can be shown using recursive integration by parts, that for $i \geq 1$, $P_{Lo,i}$ is equal to:

$$\begin{aligned}
P_{Lo,i}(K) &= e^{-r} \int_{-\infty}^{\frac{log(K)-\mu}{\sigma}} \left(K - e^{\mu+\sigma x}\right)\ Lo_i(x)\ l(x)\ dx \\
&= e^{-r} \frac{K\ l_{i0}}{1 + e^{-\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}} \\
&+ e^{-r} K \sum_{j=1}^{i} l_{ij} \left(\frac{\sqrt{3}}{\pi}\right)^j \left( \frac{\left(\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right)^j}{1 + e^{-\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}} - j\ I\left(j-1, 0, \frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right)\right) \\
&+ e^{-r}\ l_{i0} \frac{\left(e^{\mu+\left(1+\frac{\sqrt{3}}{\pi}\sigma\right) \frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{t} \sigma}\right)}{1 + e^{\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}} - \frac{\sqrt{3}}{\pi} e^{\mu-r} \sigma\ I\left(0, \frac{\sqrt{3}}{\pi}\sigma, \frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right) \\
&- e^{\mu-r} \sum_{j=1}^{i}\ l_{ij}\ \left(\frac{\sqrt{3}}{\pi}\right)^i \left(\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right)^j \frac{e^{\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}(1+\frac{\sqrt{3}}{\pi}\sigma)}}{1 + e^{\frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}}} \\
&+ e^{\mu-r} \sum_{j=1}^{i}\ l_{ij}\ \left(\frac{\sqrt{3}}{\pi}\right)^i \frac{\sqrt{3}}{\pi}\ \sigma\ I\left(j, \frac{\sqrt{3}}{\pi}\sigma, \frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right) \\
&+ e^{\mu-r} \sum_{j=1}^{i}\ l_{ij}\ \left(\frac{\sqrt{3}}{\pi}\right)^i j\ I\left(j-1, \frac{\sqrt{3}}{\pi}\sigma, \frac{\pi}{\sqrt{3}} \frac{log(K)-\mu}{\sigma}\right),
\end{aligned}$$

where

$$I(j,x,y) = (-1)^j \left( e^{y(1+x)}\ \Gamma(j+1)\ \Phi\left(-e^y, j+1, 1+x\right) - (-1)^j \sum_{s=0}^{j-1} \binom{j}{s} (-y)^{j-s}\, I\left(s,x,y\right) \right),$$

$\Gamma(\cdot)$ is the gamma function, and $\Phi(\cdot)$ is the Lerch Transcendent:

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt.$$

*D. Moments of the Hull-White Model and Generalizations*

The basic Hull-White (1987) model uses the following risk-neutral dynamics for the spot price $S(t)$:

$$
\begin{aligned}
dS &= r\ S\ dt + \sqrt{v}\ S\ dz_1 \\
dv &= \mu(v)\ dt + \xi\ v\ dz_2,
\end{aligned}
$$

where $\mu(v) = \eta v$ and the Wiener processes $z_1(t)$ and $z_2(t)$ are uncorrelated. Hull and White present a series solution for this model, but unfortunately Skabelin (2005) shows that this series does not converge. In addition to this base model, we consider two generalizations. For the first generalization, we allow the Wiener processes to have constant correlation $\rho$. Heston, Loewenstein, and Willard (2007) show it is necessary for $\rho$ to be nonpositive in order to preclude arbitrage. For the second generalization, we allow mean-reversion in the form of a linear drift: $\mu(v) = \alpha + \eta v$.

The conditional moments $U(S,v,t) = E\left[U\left(S(T),v(T),T\right)|S(t)=S,v(t)=v\right]$ satisfy the backward equation:

$$
\frac{v\ S^2}{2}\frac{\partial^2 U}{\partial S^2} + \rho\ \xi\ v^{3/2}\ S\frac{\partial^2 U}{\partial S \partial v} + \frac{\xi^2 v^2}{2}\ \frac{\partial^2 U}{\partial v^2} + r\ S\frac{\partial U}{\partial S} + \mu(v)\frac{\partial U}{\partial v} + \frac{\partial U}{\partial t} = 0.
$$

The terminal condition is $U(S,v,T;n) = \log(S)^n$. For the case $n=1$, the solution is:

$$
U(S,v,t;1) = log(S) + (\alpha + r)(T-\tau) + \alpha\frac{1-e^{\eta(T-t)}}{2\eta^2} + v\frac{1-e^{\eta(T-\tau)}}{2\eta}.
$$

For $n > 1$, we can guess the following solution

$$
U(S,v,t;n) = \sum_{i=0}^{n}\binom{n}{i}\left(log(S) + r(T-t)\right)^{n-i} f(v,T-t;i).
$$

In the absence of mean-reversion ($\alpha$=0), the solution is

$$f(v,\tau;i)=\sum_{j=i}^{2i} v^{j/2}\sum_{k=1}^{j} g_{i,j,k}e^{h(k)\tau},$$

where

$$h(k)=-\frac{k}{2}\eta+k\frac{(k-2)}{8}\xi^2$$

and

$$\begin{array}{ll}
g_{1,2,1}=-\frac{1}{2\eta}, & g_{1,2,2}=\frac{1}{2\eta};\\
g_{2,2,1}=\frac{1}{\eta}; & g_{2,2,2}=-\frac{1}{\eta};\\
g_{2,3,1}=-\frac{8\rho\xi}{\eta(12\eta-3\xi^2)}; & g_{2,3,2}=\frac{8\rho\xi}{\eta(4\eta-3\xi^2)};\\
g_{2,3,3}=-\frac{8\rho\xi}{\eta(4\eta-3\xi^2)}+\frac{8\rho\xi}{\eta(12\eta-3\xi^2)}; & g_{2,4,1}=\frac{1}{4\eta^2-2\eta\xi^2};\\
g_{2,4,2}=-\frac{1}{2\eta(\eta-\xi^2)}; & g_{2,4,3}=0;\\
g_{2,4,4}=\frac{1}{2\eta(\eta-\xi^2)}-\frac{1}{(4\eta^2-2\eta\xi^2)}; & g_{3,3,1}=\frac{8\rho\xi}{\eta(4\eta-\xi^2)};\\
g_{3,3,2}=\frac{-24\rho\xi}{\eta(4\eta-3\xi^2)}; & g_{3,3,3}=-\frac{8\rho\xi}{\eta(4\eta-\xi^2)}+\frac{24\rho\xi}{\eta(4\eta-3\xi^2)}.
\end{array}$$

Additional $g_{i,j,k}$ coefficients satisfy the recursions:

$$g_{i,j,k}=\frac{\frac{i(i-2)}{2}\ g_{i-2,j-2,k}+\rho\ \xi\ \frac{i(j-1)}{2}\ g_{i-1,j-2,k}-\frac{i}{2}\ g_{i-1,j-2,k}}{h(k)-h(j)},\quad for\ k=\{1,...,j-2\}$$

$$g_{i,j,j-1}=\frac{\rho\ \xi\ i\frac{j-1}{2}\ g_{i-1,j-1,j-1}}{h(j-1)-h(j)},$$

$$g_{i,j,j}=-\sum_{k=1}^{j-1} g_{i,j,k}.$$

In the case with mean-reversion, the solution is

$$f(v,\tau;i)=\sum_{j=0}^{i} v^j g(\tau;i,j),$$

where $g(\tau; i, j)$ satisfies the more complicated differential equation

$$\frac{\partial g(\tau; i, j)}{\partial \tau} = -h(2j)\ g(\tau; i, j) + i\frac{(i-1)}{2}g(\tau; i-1, j-2) - \frac{i}{2}g(\tau, i-1, j-1) + \alpha(j+1)g(\tau; i, j+1),$$

which has the recursive solution:

$$g(\tau; i, j) = e^{-h(2j)\tau} \int_0^\tau e^{h(2j)s} \left( \frac{i(i-1)}{2} g(s; i-1, j-2) - \frac{i}{2} g(s, i-1, j-1) + \alpha(j+1) g(s; i, j+1) \right) ds.$$

# Tables and Figures

**Table 1. Comparison of Gaussian Distribution and Logistic Distribution Moments**

| $n$ | Gaussian $E[x^n]$ | Logistic $E[x^n]$ | Ratio |
|---|---|---|---|
| 2 | 1 | 1 | 1 |
| 4 | 3 | 4.2 | 1.4 |
| 6 | 15 | 39.9 | 2.7 |
| 8 | 105 | 686 | 6.5 |
| 10 | 945 | 18,814 | 40 |

This table reports the first ten even moments of a standard Gaussian distribution (second column) and standard logistic distribution (third column). The last column reports the ratio of the logistic and Gaussian moments.

**Table 2. Hermite and Logistic Polynomials along with their Weighted-Average Squared Values**

| $n$ | $He_n(x)$ | $\langle He_n, He_n \rangle$ | $Lo_n(x)$ | $\langle Lo_n, Lo_n \rangle$ |
|---|---|---|---|---|
| 0 | $1$ | 1 | $1$ | $1$ |
| 1 | $x$ | 1 | $x$ | $1$ |
| 2 | $x^2 - 1$ | 2 | $x^2 - 1$ | $\frac{16}{5}$ |
| 3 | $x^3 - 3x$ | 6 | $x^3 - \frac{21}{5}x$ | $\frac{3888}{175}$ |
| 4 | $x^4 - 6x^2 + 3$ | 24 | $x^4 - \frac{78}{7}x^2 + \frac{243}{35}$ | $\frac{331776}{1225}$ |
| 5 | $x^5 - 10x^3 + 15x$ | 120 | $x^5 - \frac{70}{3}x^3 + \frac{407}{7}x$ | $\frac{2764800}{539}$ |

This table reports Hermite polynomials in the second column along with their weighted-average squared values in the third column. The last two columns report logistic polynomials along with their weighted-average squared values.

## Table 3. Variance-Gamma Model Call Option Values

### Panel A. Stock Price = 90

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ |
| 0 | 0.71 | -0.06 | 1.00 | 0.74 | -0.03 | 0.55 |
| 4 | 0.77 | 0.00 | 0.03 | 0.74 | -0.04 | 0.61 |
| 6 | 0.65 | -0.12 | 1.95 | 0.74 | -0.04 | 0.56 |
| 8 | 1.00 | 0.23 | -3.57 | 0.74 | -0.03 | 0.50 |
| 10 | 0.13 | -0.64 | 10.2 | 0.75 | -0.03 | 0.44 |
| 12 | 2.66 | 1.88 | -29.9 | 0.75 | -0.02 | 0.39 |
| 14 | -5.09 | -5.86 | 93.0 | 0.75 | -0.02 | 0.35 |
| 16 | 18.4 | 17.6 | -280 | 0.76 | -0.02 | 0.32 |
| 18 | -37.1 | -37.9 | 601 | 0.76 | -0.02 | 0.29 |
| 20 | -108 | -109 | 1722 | 0.76 | -0.02 | 0.26 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Exact | 0.78 | 0.00 | 0.00 | 0.78 | 0.00 | 0.00 |

### Panel B. Stock Price = 100

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ |
| 0 | 3.99 | 0.31 | 1.00 | 3.82 | 0.14 | 0.46 |
| 4 | 3.66 | -0.02 | -0.06 | 3.78 | 0.10 | 0.32 |
| 6 | 3.88 | 0.20 | 0.65 | 3.76 | 0.08 | 0.25 |
| 8 | 3.40 | -0.28 | -0.90 | 3.75 | 0.06 | 0.21 |
| 10 | 4.55 | 0.87 | 2.82 | 3.74 | 0.06 | 0.18 |
| 12 | 0.86 | -2.82 | -9.18 | 3.73 | 0.05 | 0.16 |
| 14 | 15.1 | 11.4 | 37.2 | 3.73 | 0.05 | 0.15 |
| 16 | -49.6 | -53.3 | -174 | 3.72 | 0.04 | 0.14 |
| 18 | 288 | 284 | 928 | 3.72 | 0.04 | 0.13 |
| 20 | -1705 | -1709 | -5569 | 3.72 | 0.04 | 0.12 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Exact | 3.68 | 0.00 | 0.00 | 3.68 | 0.00 | 0.00 |

### Panel C. Stock Price = 110

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ |
| 0 | 10.95 | 0.03 | 1.00 | 10.93 | 0.00 | -0.06 |
| 4 | 10.90 | -0.03 | -1.22 | 10.91 | -0.02 | -0.77 |
| 6 | 10.84 | -0.08 | -3.16 | 10.90 | -0.02 | -0.95 |
| 8 | 11.09 | 0.17 | 6.38 | 10.90 | -0.03 | -0.98 |
| 10 | 10.29 | -0.64 | -24.6 | 10.90 | -0.02 | -0.95 |
| 12 | 13.25 | 2.32 | 89.1 | 10.90 | -0.02 | -0.92 |
| 14 | 1.39 | -9.54 | -365 | 10.90 | -0.02 | -0.87 |
| 16 | 53.65 | 42.7 | 1638 | 10.91 | -0.02 | -0.83 |
| 18 | -195 | -207 | -7928 | 10.91 | -0.02 | -0.79 |
| 20 | 1072 | 1061 | 40704 | 10.91 | -0.02 | -0.75 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Exact | 10.93 | 0.00 | 0.00 | 10.93 | 0.00 | 0.00 |

This table compares call option prices computed using the Hermite polynomial method and the logistic polynomial method. The spot returns follow Madan and Seneta (1990)'s Variance-Gamma process with parameters $\gamma = 3/2$, $\sigma = \sqrt{1/(2\gamma)}\frac{1}{10}$ and $\mu = r+\gamma log(1-\sigma)+\gamma log(1+\sigma)$. The options priced have an expiration of one year and a strike price of 100. The risk-free rate is set to zero and the price of the stock is 90 in Panel A, 100 in Panel B and 110 in Panel C. For the logistic and Hermite polynomials methods, we present option prices for orders of expansions ranging from 0 to 20. The exact solutions are obtained from the numerical integration of equation (28) in the paper. For each method and expansion order, we report option values, the error with respect to the exact option value and the ratio of the error to the error generated by the Black-Scholes option valuation.

## Table 4. Heston Model Call Option Values

### Panel A. Stock Price = 90

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | Error / B-S Error | Option Value | Error | Error / B-S Error |
| 0 | 0.20 | 0.13 | 1.00 | 0.25 | 0.18 | 1.35 |
| 4 | 0.08 | 0.01 | 0.10 | 0.14 | 0.07 | 0.52 |
| 6 | 0.04 | -0.03 | -0.23 | 0.11 | 0.03 | 0.26 |
| 8 | 0.11 | 0.04 | 0.31 | 0.09 | 0.02 | 0.16 |
| 10 | 0.01 | -0.06 | -0.46 | 0.09 | 0.01 | 0.11 |
| 12 | 0.12 | 0.05 | 0.38 | 0.08 | 0.01 | 0.08 |
| 14 | 0.14 | 0.07 | 0.54 | 0.08 | 0.01 | 0.06 |
| 16 | -0.62 | -0.69 | -5.27 | 0.08 | 0.01 | 0.05 |
| 18 | 3.33 | 3.26 | 24.9 | 0.08 | 0.00 | 0.04 |
| 20 | -13.1 | -13.2 | -100 | 0.08 | 0.00 | 0.03 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Exact | 0.07 | 0.00 | 0.00 | 0.07 | 0.00 | 0.00 |

### Panel B. Stock Price = 100

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | Error / B-S Error | Option Value | Error | Error / B-S Error |
| 0 | 2.83 | 0.09 | 1.00 | 2.71 | -0.03 | -0.32 |
| 4 | 2.68 | -0.06 | -0.65 | 2.69 | -0.04 | -0.49 |
| 6 | 2.78 | 0.04 | 0.43 | 2.70 | -0.04 | -0.46 |
| 8 | 2.66 | -0.08 | -0.90 | 2.70 | -0.04 | -0.42 |
| 10 | 2.87 | 0.13 | 1.48 | 2.70 | -0.03 | -0.38 |
| 12 | 2.44 | -0.30 | -3.38 | 2.71 | -0.03 | -0.34 |
| 14 | 3.50 | 0.76 | 8.52 | 2.71 | -0.03 | -0.31 |
| 16 | 0.56 | -2.17 | -24.5 | 2.71 | -0.02 | -0.28 |
| 18 | 9.65 | 6.91 | 77.7 | 2.71 | -0.02 | -0.25 |
| 20 | -21.3 | -24.1 | -270 | 2.72 | -0.02 | -0.23 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Exact | 2.74 | 0.00 | 0.00 | 2.74 | 0.00 | 0.00 |

### Panel C. Stock Price = 110

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | Error / B-S Error | Option Value | Error | Error / B-S Error |
| 0 | 10.31 | -0.16 | 1.00 | 10.34 | -0.13 | 0.83 |
| 4 | 10.51 | 0.04 | -0.27 | 10.43 | -0.04 | 0.26 |
| 6 | 10.43 | -0.04 | 0.23 | 10.46 | -0.01 | 0.04 |
| 8 | 10.51 | 0.04 | -0.26 | 10.48 | 0.01 | -0.04 |
| 10 | 10.44 | -0.03 | 0.17 | 10.48 | 0.01 | -0.07 |
| 12 | 10.44 | -0.03 | 0.21 | 10.48 | 0.01 | -0.08 |
| 14 | 10.74 | 0.27 | -1.72 | 10.48 | 0.01 | -0.07 |
| 16 | 9.27 | -1.20 | 7.55 | 10.48 | 0.01 | -0.07 |
| 18 | 15.11 | 4.64 | -29.2 | 10.48 | 0.01 | -0.06 |
| 20 | -7.02 | -17.49 | 110 | 10.48 | 0.01 | -0.05 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Exact | 10.47 | 00.00 | 0.00 | 10.47 | 0.00 | 0.00 |

This table compares call option prices computed using the Hermite polynomial method and the logistic polynomial method. The spot returns follow Heston (1993)'s square-root variance process with parameters $v = \sigma^2 = 0.01$, $\kappa = 0$, $\alpha = 0$, and $\rho = -2/3$. The options have an expiration of 180 days and a strike price of 100. The risk-free rate is set to zero and the price of the stock is 90 in Panel A, 100 in Panel B and 110 in Panel C. For the logistic and Hermite polynomials methods, we present option prices for orders of expansions ranging from 0 to 20. The exact solutions are computed using the Fourier method of Heston (1993). For each method and expansion order, we report option values, the error with respect to the exact option value and the ratio of the error to the error generated by the Black-Scholes option valuation.

## Table 5. Hull-White Model Call Option Values

### Panel A. Stock Price = 90

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ |
| 0 | 0.20 | 0.12 | 1.00 | 0.25 | 0.17 | 1.40 |
| 4 | 0.10 | 0.01 | 0.10 | 0.14 | 0.05 | 0.44 |
| 6 | -0.01 | -0.10 | -0.81 | 0.11 | 0.02 | 0.20 |
| 8 | 0.35 | 0.26 | 2.21 | 0.10 | 0.01 | 0.10 |
| 10 | -0.77 | -0.85 | -7.15 | 0.09 | 0.01 | 0.05 |
| 12 | 1.07 | 0.99 | 8.32 | 0.09 | 0.00 | 0.02 |
| 14 | 42.9 | 42.8 | 360 | 0.08 | 0.00 | 0.00 |
| 16 | -1311 | -1311 | -11033 | 0.08 | 0.00 | -0.01 |
| 18 | 38413 | 38413 | 323363 | 0.08 | 0.00 | -0.03 |
| 20 | -1362043 | -1362043 | -11465766 | 0.08 | 0.00 | 0.00 |

### Panel B. Stock Price = 100

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ |
| 0 | 2.83 | 0.13 | 1.00 | 2.71 | 0.02 | 0.12 |
| 4 | 2.63 | -0.07 | -0.50 | 2.68 | -0.02 | 0.26 |
| 6 | 2.87 | 0.17 | 1.29 | 2.68 | -0.01 | -0.08 |
| 8 | 2.20 | -0.49 | -3.67 | 2.68 | -0.01 | 0.02 |
| 10 | 5.24 | 2.55 | 19.0 | 2.69 | -0.01 | 0.00 |
| 12 | -15.9 | -18.6 | -139 | 2.69 | 0.00 | 0.00 |
| 14 | 203 | 200 | 1494 | 2.69 | 0.00 | 0.00 |
| 16 | -3142 | -3145 | -23429 | 2.69 | 0.00 | 0.00 |
| 18 | 72138 | 72135 | 537466 | 2.70 | 0.00 | 0.00 |
| 20 | -2443349 | -2443351 | -18204943 | 2.69 | 0.00 | 0.00 |

### Panel C. Stock Price = 110

| | Hermite | | | Logistic | | |
|---|---|---|---|---|---|---|
| | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ | Option Value | Error | $\frac{\text{Error}}{\text{B-S Error}}$ |
| 0 | 10.31 | -0.17 | 1.00 | 10.34 | -0.14 | 0.83 |
| 4 | 10.54 | 0.06 | -0.36 | 10.43 | -0.04 | -0.72 |
| 6 | 10.34 | -0.14 | 0.83 | 10.46 | -0.02 | 0.11 |
| 8 | 10.78 | 0.30 | -1.81 | 10.47 | 0.00 | -0.01 |
| 10 | 9.55 | -0.92 | 5.51 | 10.48 | 0.00 | 0.00 |
| 12 | 12.50 | 2.03 | -12.1 | 10.48 | 0.00 | 0.00 |
| 14 | 35.03 | 24.6 | -146 | 10.48 | 0.00 | 0.00 |
| 16 | -1000 | -1010 | 6030 | 10.48 | 0.00 | 0.00 |
| 18 | 34541 | 34531 | -206084 | 10.47 | 0.00 | 0.00 |
| 20 | -1442361 | -1442372 | 8608319 | 10.48 | 0.00 | 0.00 |

This table compares call option prices computed using the Hermite polynomial method and the logistic polynomial method. The spot returns follow Hull and White (1993)'s process with parameters $v = \sigma^2 = 0.01$, $\eta = 0$, $\xi = 0$, and $\rho = -2/3$. The options priced have an expiration of 180 days and a strike price of 100. The risk-free rate is set to zero and the price of the stock is 90 in Panel A, 100 in Panel B and 110 in Panel C. For the logistic and Hermite polynomials methods, we present option prices for orders of expansions ranging from 0 to 20. For each method and expansion order we report option values, the error with respect to the option value implied by logistic model with order of expansion equal to 20 and the ratio of the error to the error generated by the Black-Scholes option valuation.

**Logistic and Gaussian Densities**

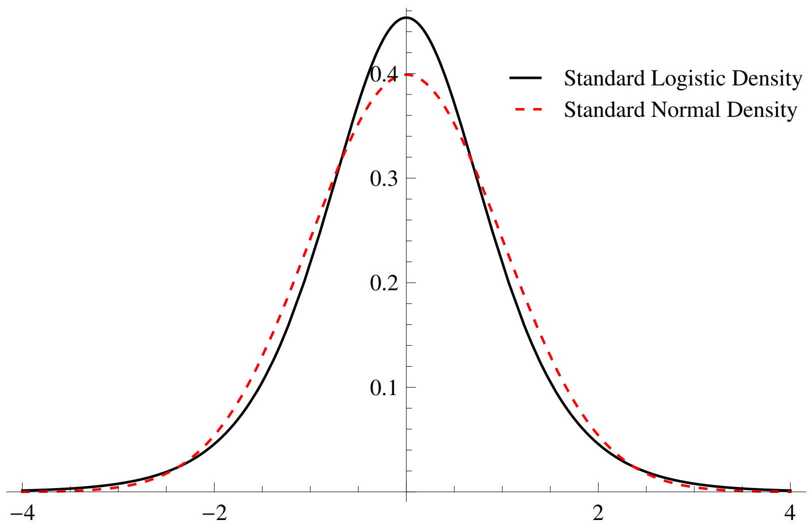
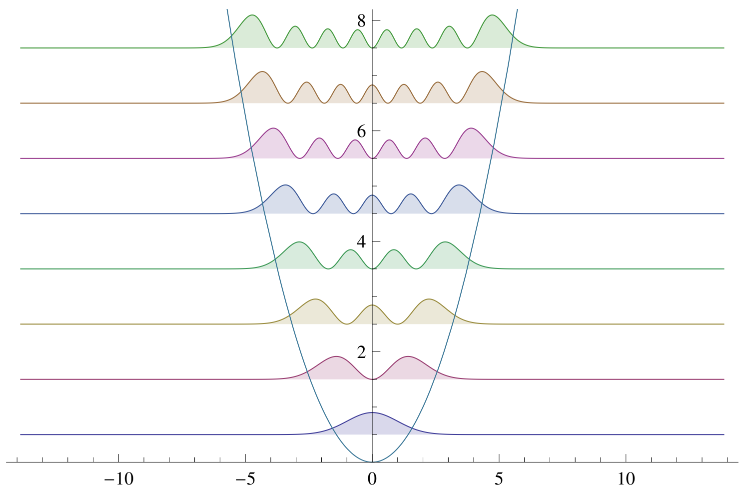
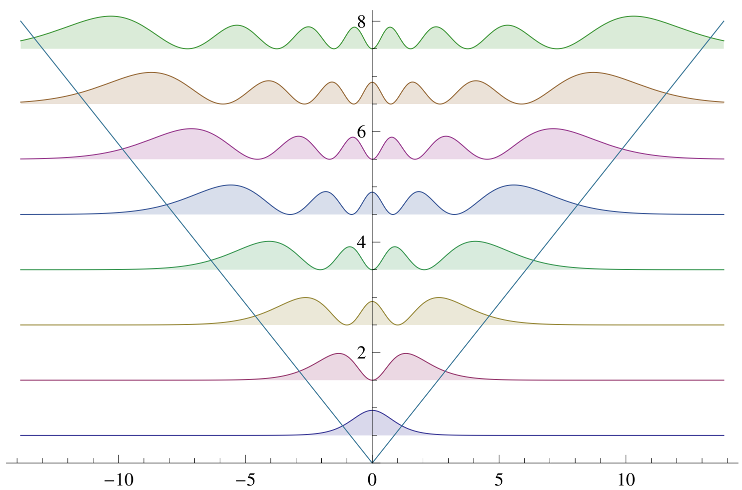



Figure 1: This figure plots a standard Logistic density and a standard Normal density.




(a) Hermite Functions




(b) Logistic Functions

Figure 2: This figure presents squared scaled and stacked Hermite (Top Panel) and logistic (Bottom Panel) functions up to order seven. The specific formulas of the functions plotted are $\sqrt{i+1}\ He_i(x)^2 \frac{n(x)}{\langle He_i, He_i \rangle} + i + \frac{1}{2}$ for the Hermite polynomials and $(i+1)\ Lo_i(x)^2\ \frac{l(x)}{\langle Lo_i, Lo_i \rangle} + i + \frac{1}{2}$ for the Logistic polynomials, for $i = 0, \ldots, 7$.

**Trinomial and Gaussian Densities**

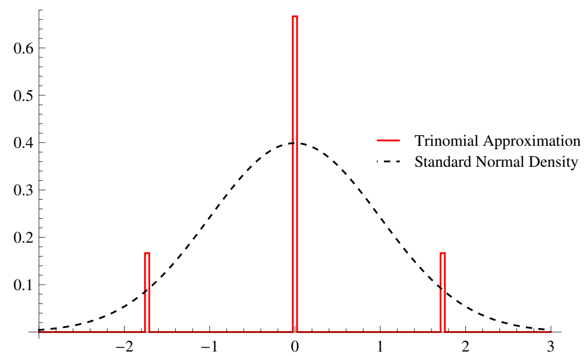



Figure 3: This figure plots a standard normal density as well as a trinomial approximation that takes the value 0 with probability $\frac{2}{3}$ and $\pm\sqrt{3}$ with probability $\frac{1}{6}$. Note that the first five moments of the two distributions coincide.

**Trinomial and Black-Scholes Call Values**

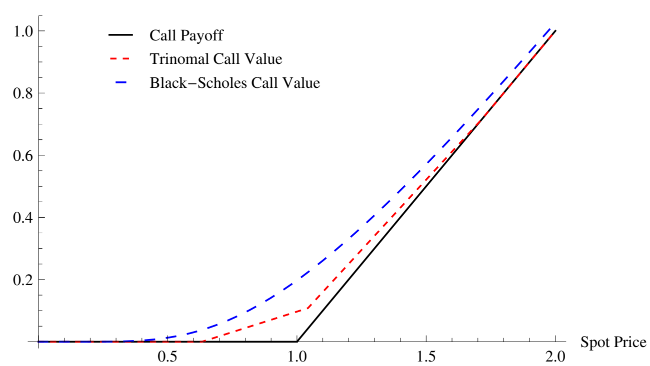



Figure 4: This figure plots call option payoffs, Black-Scholes call option values as well as call option values obtained using the trinomial implementation of the Breeden-Litzenberger model, where the log-spot return is $\mu$ with probability $\frac{2}{3}$, $\mu + \sigma$ with probability $\frac{1}{6}$ and $\mu - \sigma$ with probability $\frac{1}{6}$. To obey the martingale restriction the parameter $\mu$ must satisfy the relation $\frac{1}{6}e^{\mu-\sigma} + \frac{2}{3}e^{\mu} + \frac{1}{6}e^{\mu+\sigma} = e^{r}$. The strike price $K$ is set to 1, the spot price ranges from zero to two and the interest rate $r$ is set to zero.

**Rectangular Approximation to Fourier Integral**

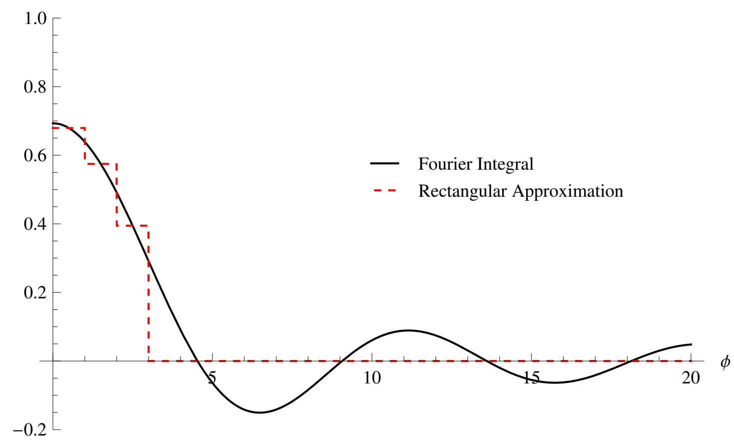



Figure 5: This figure plots the integrand $\int_0^\infty \frac{\sin(\phi \log(S/K))}{\phi} d\phi$ for an at-the-money option ($S = K$), along with a crude unit-step rectangular approximation using abscissas at $\phi = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

**Fourier Approximation to Call Payoff**

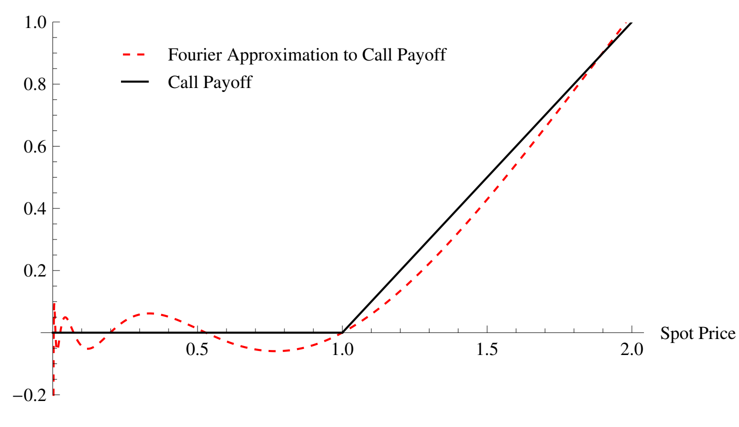



Figure 6: This figure plots the approximation to a Call Payoff obtained using the Fourier method at three abscissa points $\phi = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. For details, see expression (3) in the text.

## Call Payoff Approximation

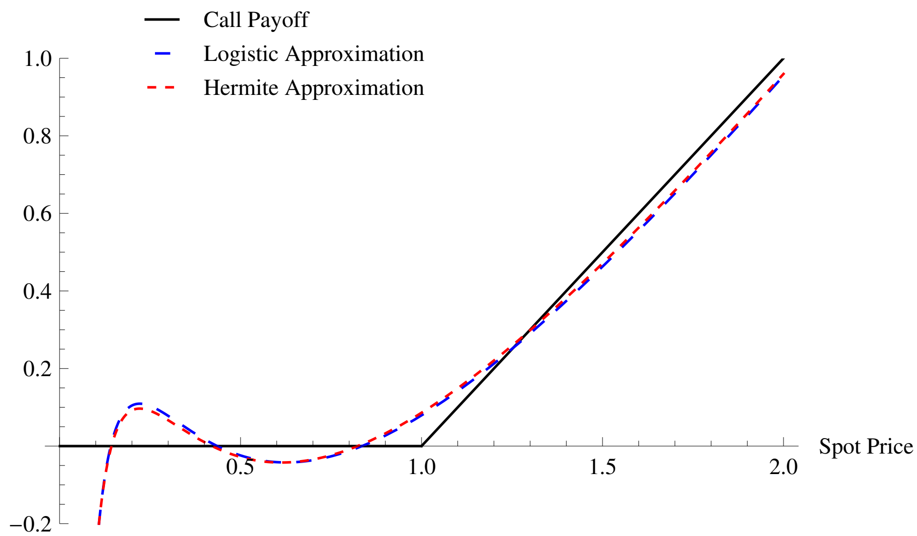
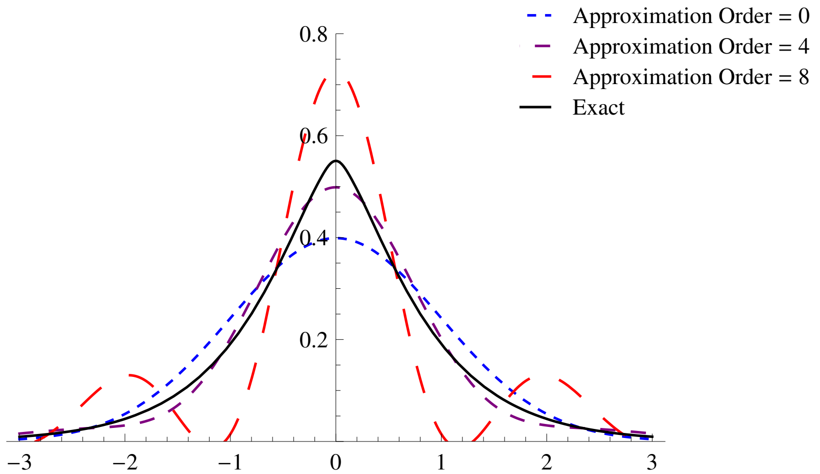
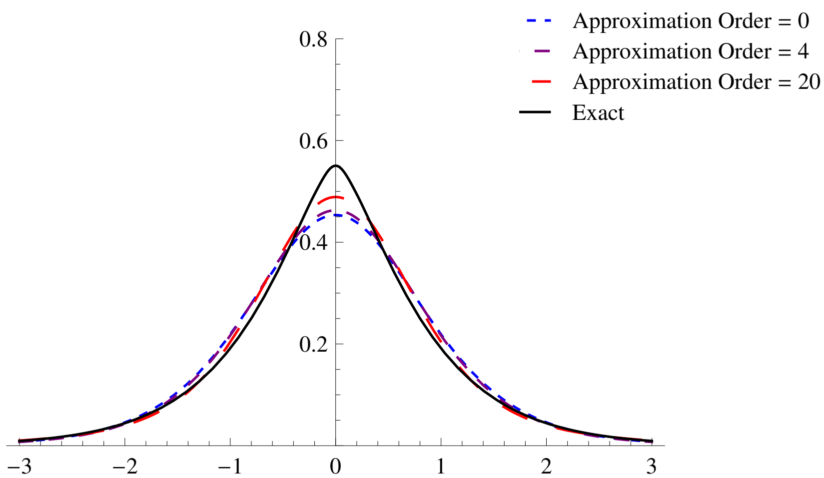



Figure 7: This figure plots the third-order approximation to a call payoff obtained using Logistic and Hermite polynomials. The underlying log-spot return has mean $\mu = 0$ and standard deviation $\sigma = 0.5$.

# Variance-Gamma Model Density Approximations



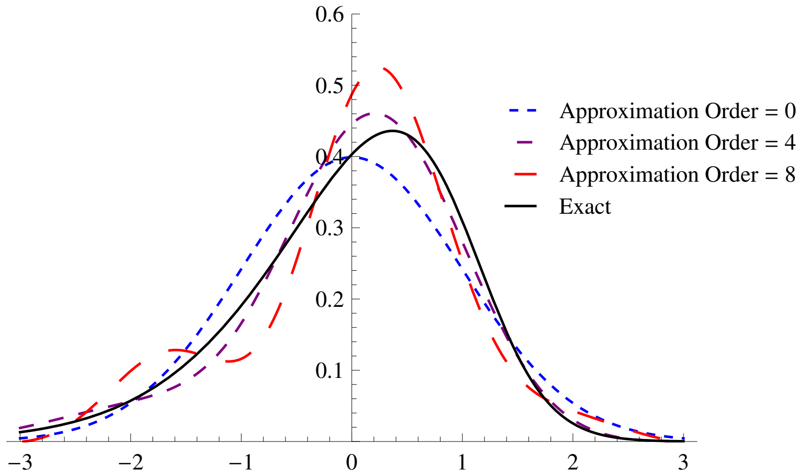

(a) Hermite Polynomials Approximations



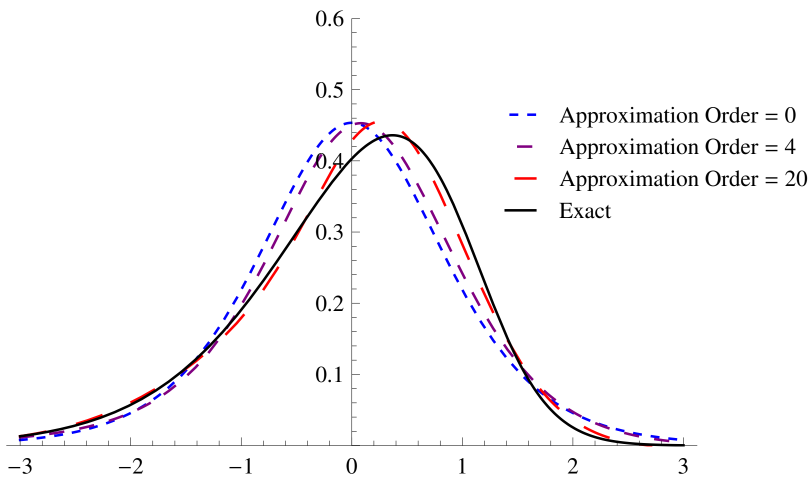

(b) Logistic Polynomials Approximations

Figure 8: This figure depicts density approximations of a Variance-Gamma model obtained using Hermite (top figure) and logistic (bottom figure) polynomials of various orders. The Variance-Gamma model is parameterized as follows: $\gamma = 3/2$, $\sigma = \sqrt{1/(2\gamma)}\frac{1}{10}$, and $\mu = r + \gamma log(1 - \sigma) + \gamma log(1 + \sigma)$. The exact density is obtained using numerical integration, following Madan and Seneta (1990).

## Heston Model Density Approximations



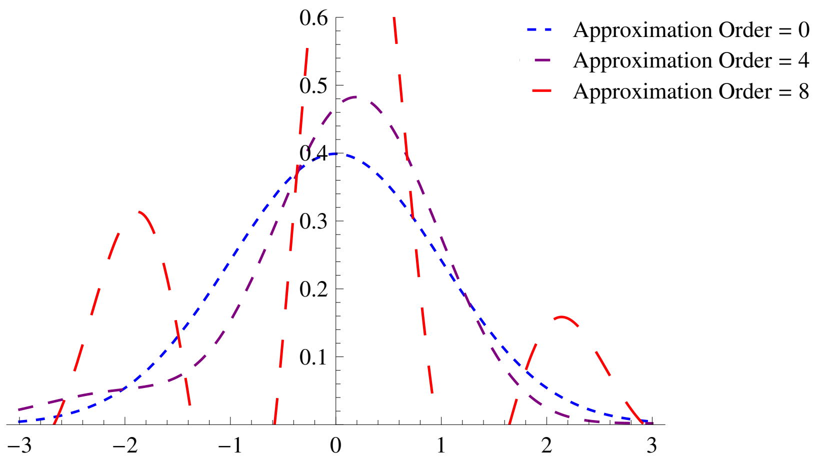

(a) Hermite Polynomials Approximations



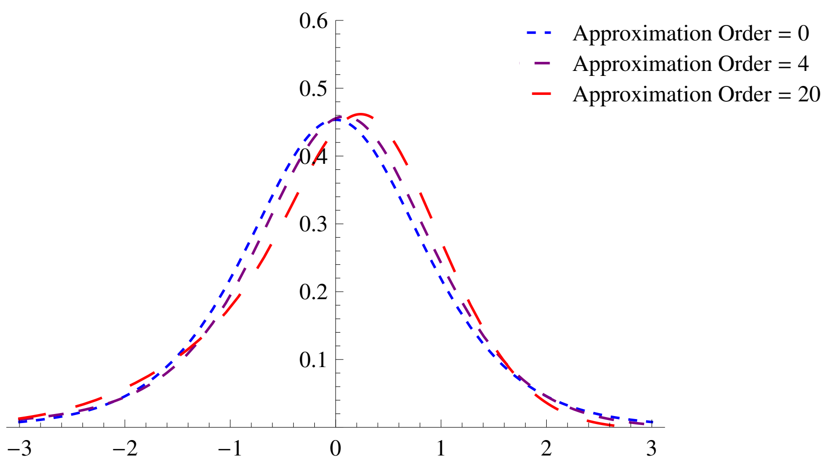

(b) Logistic Polynomials Approximations

Figure 9: This figure depicts density approximations of a Heston (1993) model obtained using Hermite (top figure) and logistic (bottom figure) polynomials of various orders. The Heston model is parameterized as follows: $v = \sigma^2 = 0.01$, $\kappa = 0$, $\alpha = 0$, and $\rho = -2/3$. The exact density is obtained using the Fourier method of Heston (1993).

## Hull-White Model Density Approximations




(a) Hermite Polynomials Approximations




(b) Logistic Polynomials Approximations

Figure 10: This figure depicts density approximations of a Hull-White model obtained using Hermite (top figure) and logistic (bottom figure) polynomials of various orders. The Hull-White model is parameterized as follows: $v = \sigma^2 = 0.01$, $\eta = 0$, $\xi = 0$, and $\rho = -2/3$.