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Interest Rate Caps "Smile" Too! But Can the LIBOR Market Models Capture the Smile?

ROBERT JARROW, HAITAO LI, and FENG ZHAO*

ABSTRACT

Using 3 years of interest rate caps price data, we provide a comprehensive documentation of volatility smiles in the caps market. To capture the volatility smiles, we develop a multifactor term structure model with stochastic volatility and jumps that yields a closed-form formula for cap prices. We show that although a three-factor stochastic volatility model can price at-the-money caps well, significant negative jumps in interest rates are needed to capture the smile. The volatility smile contains information that is not available using only at-the-money caps, and this information is important for understanding term structure models.

THE EXTENSIVE LITERATURE ON MULTIFACTOR DYNAMIC term structure models (hereafter, DTSMs) of the last decade mainly focuses on explaining bond yields and swap rates (see Dai and Singleton (2003) and Piazzesi (2003) for surveys of the literature). The pricing and hedging of over-the-counter interest rate derivatives such as caps and swaptions has attracted attention only recently. World-wide, caps and swaptions are among the most widely traded interest rate derivatives. According to the Bank for International Settlements, in recent years, their combined notional value exceeds 10 trillion dollars, which is many times larger than that of exchange-traded options. The accurate and efficient pricing and hedging of caps and swaptions is therefore of enormous practical importance. Moreover, because cap and swaption prices may contain information on term structure dynamics not contained in bond yields or swap rates (see Jagannathan, Kaplin, and Sun (2003) for a related discussion), Dai and Singleton (2003 p. 670) argue that there is an "enormous potential for new insights from using (interest rate) derivatives data in model estimations."

The extant literature on interest rate derivatives primarily focuses on two issues (see Section 5 of Dai and Singleton (2003)). The first issue is that of

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the so-called "unspanned stochastic volatility" (hereafter, USV) puzzle. Specifically, Collin-Dufresne and Goldstein (2002) and Heidari and Wu (2003) show that although caps and swaptions are derivatives written on LIBOR and swap rates, their prices appear to be driven by risk factors not spanned by the factors explaining LIBOR or swap rates. Though Fan, Gupta, and Ritchken (2003) argue that swaptions might be spanned by bonds, Li and Zhao (2006) show that multifactor DTSMs have serious difficulties in hedging caps and cap straddles. The second issue relates to the relative pricing between caps and swaptions. A number of recent papers, including Hull and White (2000), Longstaff, Santa-Clara, and Schwartz (2001) (hereafter, LSS), and Jagannathan et al. (2003), document a significant and systematic mispricing between caps and swaptions using various multifactor term structure models. As Dai and Singleton (2003 p. 668) point out, these two issues are closely related and the "ultimate resolution of this 'swaptions/caps puzzle' may require time-varying correlations and possibly factors affecting the volatility of yields that do not affect bond prices."

The evidence of USV suggests that, contrary to a fundamental assumption of most existing DTSMs, interest rate derivatives are not redundant securities and thus they contain unique information about term structure dynamics that is not available in bond yields and swap rates. USV also suggests that existing DTSMs need to be substantially extended to explicitly incorporate USV for pricing interest rate derivatives. However, as Collin-Dufresne and Goldstein (2002) show, it is rather difficult to introduce USV in traditional DTSMs: One must impose highly restrictive assumptions on model parameters to guarantee that certain factors that affect derivative prices do not affect bond prices. In contrast, it is relatively easy to introduce USV in the Heath, Jarrow, and Morton (1992) (hereafter, HJM) class of models, which include the LIBOR models of Brace, Gatarek, and Musiela (1997) and Miltersen, Sandmann, and Sondermann (1997), the random field models of Goldstein (2000), and the string models of Santa-Clara and Sornette (2001), indeed, any HJM model in which the forward rate curve has stochastic volatility and the volatility and yield shocks are not perfectly correlated exhibits USV. Therefore, in addition to the commonly known advantages of HJM models (such as perfectly fitting the initial yield curve), they offer the additional advantage of easily accommodating USV. Of course, the trade-off here is that in an HJM model, the yield curve is an input rather than a prediction of the model.

Recently, several HJM models with USV have been developed and applied to price caps and swaptions. Collin-Dufresne and Goldstein (2003) develop a random field model with stochastic volatility and correlation in forward rates. Applying the transform analysis of Duffie, Pan, and Singleton (2000), they obtain closed-form formulas for a wide variety of interest rate derivatives. However, they do not calibrate their models to market prices of caps and swaptions. Han (2002) extends the model of LSS (2001) by introducing stochastic volatility and correlation in forward rates. Han (2002) shows that stochastic volatility and correlation are important for reconciling the mispricing between caps and swaptions.

Our paper makes both theoretical and empirical contributions to the fast-growing literature on interest rate derivatives. Theoretically, we develop a multifactor HJM model with stochastic volatility and jumps in LIBOR forward rates. We allow LIBOR rates to follow the affine jump diffusions (hereafter, AJDs) of Duffie et al. (2000) and obtain a closed-form solution for cap prices. Given a small number of factors can explain most of the variation of bond yields, we consider low-dimensional model specifications based on the first few (up to three) principal components of historical forward rates. Though similar to Han (2002) in this respect, our models have several advantages. First, while Han's formulas, based on the approximation technique of Hull and White (1987), work well only for at-the-money (ATM) options, our formula, based on the affine technique, works well for all options. Second, we explicitly incorporate jumps in LIBOR rates, making it possible to differentiate between the importance of stochastic volatility versus jumps for pricing interest rate derivatives.

Our empirical investigation also substantially extends the existing literature by studying the relative pricing of caps with different strikes. Using a new data set that consists of 3 years of cap prices, we are among the first to provide comprehensive evidence of volatility smiles in the caps market. To our knowledge, we also conduct the first empirical analysis of term structure models with USV and jumps in capturing the smile. Because caps and swaptions are traded over-the-counter, the common data sources, such as Datastream, only supply ATM option prices. As a result, the majority of the existing literature uses only ATM caps and swaptions, with almost no documentation of the relative pricing of caps with different strike prices. In contrast, the attempt to capture the volatility smile in equity option markets has been the driving force behind the development of the equity option pricing literature for the past quarter of a century (for reviews of the equity option literature, see Duffie (2002), Campbell, Lo, and MacKinlay (1997), Bakshi, Cao, and Chen (1997), and references therein). Analogously, studying caps and swaptions with different strike prices could provide new insights about existing term structure models that are not available from using only ATM options.

Our analysis shows that a low-dimensional LIBOR rate model with three principal components, stochastic volatility for each component, and strong negative jumps is necessary to capture the volatility smile in the cap market reasonably well. The three yield factors capture the variation in the levels of LIBOR rates, and the stochastic volatility factors capture the time-varying volatilities of LIBOR rates. Though a three-factor stochastic volatility model can price ATM caps reasonably well, it fails to capture the volatility smile in the cap market; significant negative jumps in LIBOR rates are needed to do this. These results are consistent with the view that the volatility smile contains additional information, that is, the importance of negative jumps is revealed only through the pricing of caps across moneyness.

The rest of this paper is organized as follows. In Section I, we present the data and document the volatility smile in the caps market. In Section II, we develop our new market model with stochastic volatility and jumps, and we discuss the statistical methods for parameter estimation and model comparison. Section III reports the empirical findings and Section IV concludes.

I. A Volatility Smile in the Interest Rate Cap Markets

In this section, using 3 years of cap price data we provide a comprehensive documentation of volatility smiles in the cap market. The data come from SwapPX and include daily information on LIBOR forward rates (up to 10 years) and prices of caps with different strikes and maturities from August 1, 2000 to September 23, 2003. Jointly developed by GovPX and Garban-ICAP, SwapPX is the first widely distributed service delivering 24-hour real-time rates, data, and analytics for the world-wide interest rate swaps market. GovPX, established in the early 1990s by the major U.S. fixed-income dealers in a response to regulators' demands for increased transparency in the fixed-income markets, aggregates quotes from most of the largest fixed-income dealers in the world. Garban-ICAP is the world's leading swap broker specializing in trades between dealers and trades between dealers and large customers. The data are collected every day the market is open between 3:30 and 4 p.m. To reduce noise and computational burdens, we use weekly data (every Tuesday) in our empirical analysis. If Tuesday is not available, we first use Wednesday followed by Monday. After excluding missing data, we have a total of 164 weeks in our sample. To our knowledge, our data set is the most comprehensive available for caps written on dollar LIBOR rates (see Gupta and Subrahmanyam (2005) and Deuskar, Gupta, and Subrahmanyam (2003) for the only other studies that we are aware of in this area).

Interest rate caps are portfolios of call options on LIBOR rates. Specifically, a cap gives its holder a series of European call options, called caplets, on LIBOR forward rates. Each caplet has the same strike price as the others, but with different expiration dates. Suppose $L(t, T)$ is the 3-month LIBOR forward rate at $t \leq T$, for the interval from T to $T + \frac{1}{4}$. A caplet for the period $[T, T + \frac{1}{4}]$ struck at K pays $\frac{1}{4}(L(T, T) - K)^{+}$ at $T + \frac{1}{4}$.[1] Note that although the cash flow of this caplet is received at time $T + \frac{1}{4}$, the LIBOR rate is determined at time T. Hence, there is no uncertainty about the caplet's cash flow after the LIBOR rate is set at time T. In summary, a cap is just a portfolio of caplets whose maturities are 3 months apart. For example, a 5-year cap on 3-month LIBOR struck at 6% represents a portfolio of 19 separately exercisable caplets with quarterly maturities ranging from 6 months to 5 years, where each caplet has a strike price of 6%.

Given the caps in our data are written on 3-month LIBOR rates, our model and analysis focus on modeling the LIBOR forward rate curve. The data set provides 3-month LIBOR spot and forward rates at nine different maturities (3 and 6 months; 1, 2, 3, 4, 5, 7, and 10 years). As Figure 1 shows, the forward rate curve is relatively flat at the beginning of the sample period and it declines

[1] It can be shown that a caplet behaves like a put option on a zero-coupon bond.

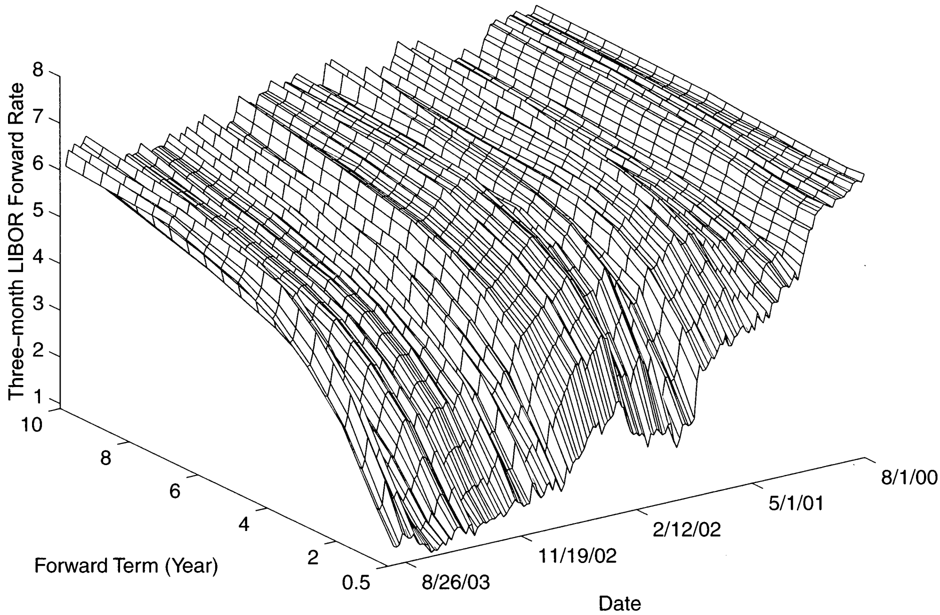


Figure 1. Term structure of 3-month LIBOR forward rates between August 1, 2000 and September 23, 2003.

over time, with the short end declining more than the long end. As a result, the forward rate curve becomes upward sloping in the later part of the sample.

The existing literature on interest rate derivatives mainly focuses on ATM contracts. One advantage of our data is that we observe prices of caps over a wide range of strikes and maturities. For example, every day for each maturity, there are 10 different strike prices: 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 8.0, 9.0, and 10.0% between August 1, 2000 and October 17, 2001; 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0 and 5.5% between October 18 and November 1, 2001; and 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0% between November 2, 2001 and July 15, 2002; 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, and 6.5% between July 16, 2002 and April 14, 2003; and 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, and 6.0% between April 15, 2003 and September 23, 2003. Moreover, caps have 15 different maturities throughout the whole sample period: 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0 years. This cross-sectional information on cap prices allows us to study the performance of existing term structure models in the pricing and hedging of caps for different maturity and moneyness.

Ideally, we would like to study caplet prices, which provide clear predictions of model performance across maturity. Unfortunately, we only observe cap prices. To simplify the empirical analysis, we consider the difference between the prices of caps with the same strike and adjacent maturities, which we refer to as *difference caps*. Thus, our analysis deals with the sum of the caplets between two neighboring maturities with the same strike. For example, 1.5-year difference caps with a specific strike represent the sum of the 1.25-year and 1.5-year caplets with the same strike.

Due to daily changes in LIBOR rates, difference caps realize different moneyness (defined as the ratio between the strike price and the average LIBOR forward rates underlying the caplets that form the difference cap) each day. Therefore, throughout our analysis, we focus on the prices of difference caps at given fixed moneyness. That is, each day we interpolate difference cap prices with respect to the strike price to obtain prices at fixed moneyness. Specifically, we use local cubic polynomials to preserve the shape of the original curves while smoothing over the grid points.[2] We refrain from extrapolation and interpolation over grid points without nearby observations, and we eliminate all observations that violate various arbitrage restrictions. We also eliminate observations with zero prices, and observations that violate either monotonicity or convexity with respect to the strikes.

Figure 2a plots the average Black (1976)-implied volatilities of difference caps across moneyness and maturity, while Figure 2b plots the average implied volatilities of ATM difference caps over the whole sample period. Consistent with the existing literature, the implied volatilities of difference caps with a moneyness between 0.8 to 1.2 have a humped shape with a peak at around a maturity of 2 years. However, the implied volatilities of all other difference caps decline with maturity. There is also a pronounced volatility skew for difference caps at all maturities, with the skew being stronger for short-term difference caps. The pattern is similar to that of equity options: In-the-money (ITM) difference caps have higher implied volatilities than do out-of-the-money (OTM) difference caps. The implied volatilities of the very short-term difference caps are more like a symmetric smile than a skew.

Figures 3a, 3b, and 3c, respectively, plot the time series of Black-implied volatilities for 2.5-, 5-, and 8-year difference caps across moneyness, while Figure 3d plots the time series of ATM implied volatilities of the three contracts. It is clear that the implied volatilities are time varying and they have increased dramatically (especially for 2.5-year difference caps) over our sample period. As a result of changing interest rates and strike prices, there are more ITM caps in the later part of our sample.

II. Market Models with Stochastic Volatility and Jumps: Theory and Estimation

In this section, we develop a multifactor HJM model with stochastic volatility and jumps in LIBOR forward rates to capture volatility smiles in the caps market. We also discuss model estimation and comparison using a wide cross-section of difference caps.

The volatility smile observed in the caps market suggests that the lognormal assumption of the standard LIBOR market models of Brace, Gatarek, and

[2] We also consider other interpolation schemes, such as linear interpolation, and obtain very similar results (the differences in interpolated prices and implied volatilities between local cubic polynomial and linear interpolation are less than 0.5%).

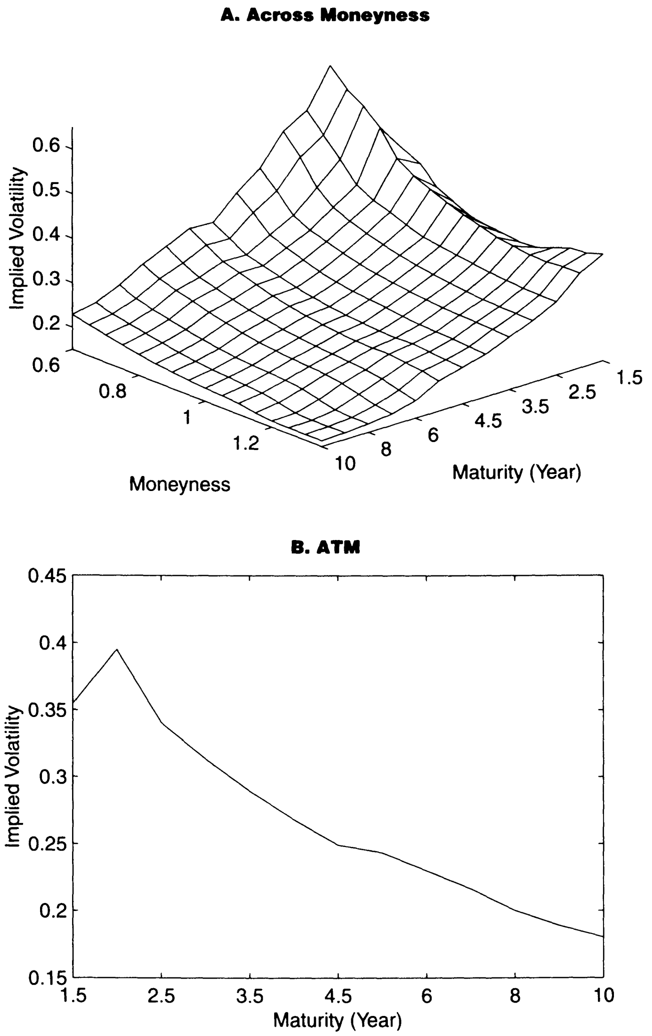


Figure 2. Average Black's implied volatilities of difference caps between August 1, 2000 and September 23, 2003.

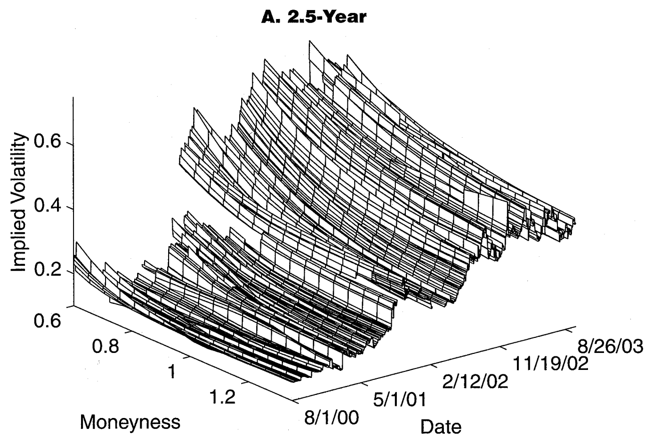
Musiela (1997) and Miltersen et al. (1997) is violated. Given the overwhelming evidence of stochastic volatility and jumps in interest rates,[3] we develop a multifactor HJM model of LIBOR rates with stochastic volatility and jumps to capture the smile. Instead of modeling the unobservable instantaneous spot rate or forward rate, we focus on the LIBOR forward rates, which are observable and widely used in the market.

[3] Andersen and Lund (1997) and Brenner, Harjes, and Kroner (1996) show that stochastic volatility or GARCH significantly improve the performance of pure diffusion models for spot interest rates. Das (2002), Johannes (2004), and Piazzesi (2005) show that jumps are important for capturing interest rate dynamics.

Throughout our analysis, we restrict the cap maturity T to a finite set of dates $0 = T_0 < T_1 < \cdots < T_K < T_{K+1}$, and we assume that the intervals $T_{k+1} - T_k$ are equally spaced by δ, a quarter of a year. Let $L_k(t) = L(t, T_k)$ be the LIBOR forward rate for the actual period $[T_k, T_{k+1}]$, and let $D_k(t) = D(t, T_k)$ be the price of a zero-coupon bond maturing at T_k. We then have

$$L\left(t, T_k\right) = \frac{1}{\delta}\left(\frac{D\left(t, T_k\right)}{D\left(t, T_{k+1}\right)} - 1\right), \text{ for } k = 1, 2, \ldots K. \tag{1}$$

For LIBOR-based instruments such as caps, floors, and swaptions, it is convenient to consider pricing under the forward measure. We will therefore focus on the dynamics of the LIBOR forward rates $L_k(t)$ under the forward measure $\mathbb{Q}^{k+1}$, which is essential for pricing caplets maturing at T_{k+1}. Under this measure, the discounted price of any security using $D_{k+1}(t)$ as the numeraire is a



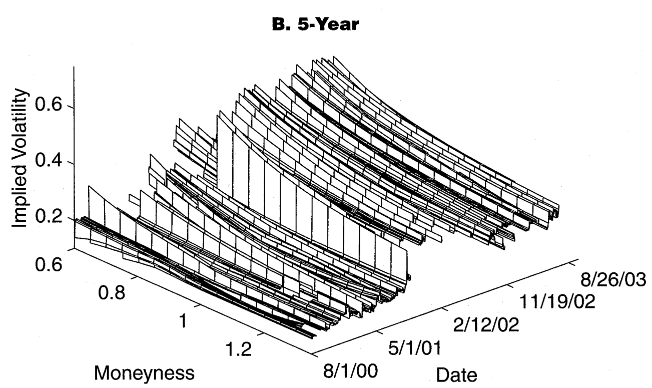
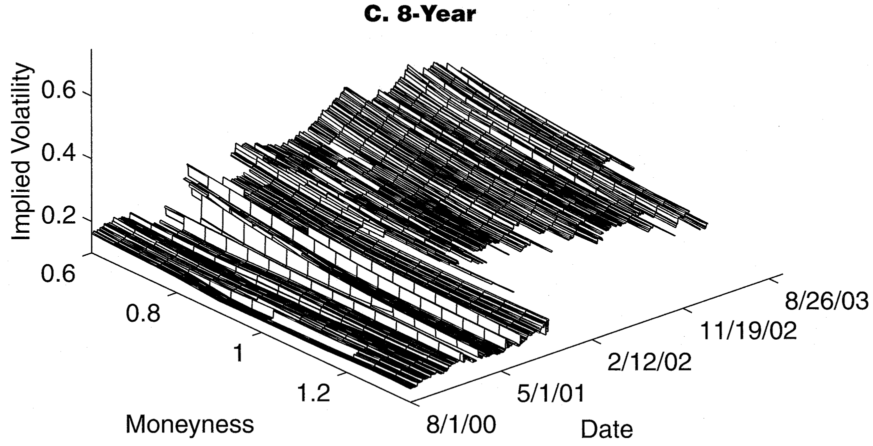


Figure 3. Black's implied volatilities of 2.5-, 5-, and 8-year difference caps between August 1, 2000 and September 23, 2003.



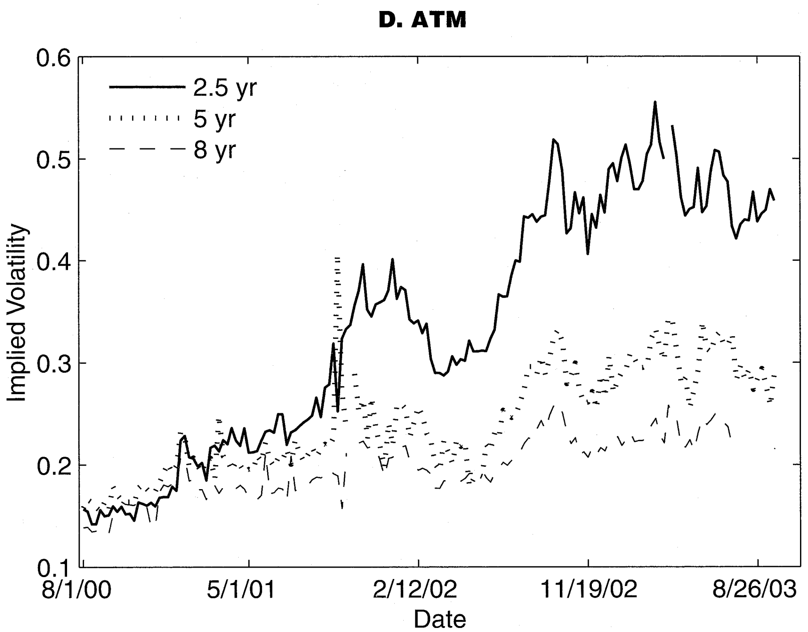


Figure 3.—*Continued*

martingale. Thus, the time-t price of a caplet maturing at T_{k+1} with a strike price of X is

$$Caplet(t, T_{k+1}, X) = \delta D_{k+1}(t) E_t^{\mathbb{Q}^{k+1}}\left[(L_k(T_k) - X)^+\right], \tag{2}$$

where $E_t^{\mathbb{Q}^{k+1}}$ is taken with respect to $\mathbb{Q}^{k+1}$ given the information set at t. The key to valuation is modeling the evolution of $L_k(t)$ under $\mathbb{Q}^{k+1}$ realistically and yet parsimoniously to yield closed-form pricing formula. To achieve this goal, we rely on the flexible affine jump diffusions (AJDs) of Duffie et al. (2000) to model the evolution of LIBOR rates.

We assume that under the physical measure $\mathbb{P}$, the dynamics of LIBOR rates are given by the following system of SDEs, for $t \in [0, T_k)$ and $k = 1, \ldots, K$:

$$\frac{dL_k(t)}{L_k(t)} = \alpha_k(t)\,dt + \sigma_k(t)\,dZ_k(t) + dJ_k(t), \tag{3}$$

where $\alpha_k(t)$ is an unspecified drift term, $Z_k(t)$ is the k^{th} element of a K-dimensional correlated Brownian motion with covariance matrix $\Psi(t)$, and $J_k(t)$ is the k^{th} element of a K-dimensional independent pure jump process that is assumed to be independent of $Z_k(t)$ for all k. To introduce stochastic volatility and correlation, we could allow the volatility of each LIBOR rate $\sigma_k(t)$ and each individual element of $\Psi(t)$ to follow a stochastic process. However, such a model is unnecessarily complicated and difficult to implement. Instead, we consider a low-dimensional model based on the first few principal components of historical LIBOR forward rates. We assume that the entire LIBOR forward curve is driven by a small number of factors $N < K$ ($N \leq 3$ in our empirical analysis). By focusing on the first N principal components of historical LIBOR rates, we can reduce the dimension of the model from K to N.

Following LSS (2001) and Han (2002), we assume that the instantaneous covariance matrix of changes in LIBOR rates shares the same eigenvectors as the historical covariance matrix. Suppose that the historical covariance matrix can be approximated as $H = U\Lambda_0 U'$, where Λ_0 is a diagonal matrix whose diagonal elements are the first N-largest eigenvalues in descending order, and the N columns of U are the corresponding eigenvectors. Of course, with jumps in LIBOR rates, both the historical and instantaneous covariance matrix of LIBOR rates may contain a component that is due to the jumps. Therefore, our approach implicitly assumes that the first three principal components from the historical covariance matrix only capture the variation in LIBOR rates due to continuous shocks, with the impact of jumps contained in the residuals. This assumption means that the instantaneous covariance matrix of changes in LIBOR rates with fixed time-to-maturity, Ω_t, shares the same eigenvectors as H. That is,

$$\Omega_t = U\Lambda_t U', \tag{4}$$

where Λ_t is a diagonal matrix whose i^{th} diagonal element, denoted by $V_i(t)$, can be interpreted as the instantaneous variance of the i^{th} common factor driving the yield curve evolution at t. We assume that $V(t)$ evolves according to the square-root process, which has been widely used in the literature for modeling stochastic volatility (see, e.g., Heston (1993)):

$$dV_i(t) = \kappa_i(\bar{v}_i - V_i(t))\,dt + \xi_i\sqrt{V_i(t)}\,d\tilde{W}_i(t), \tag{5}$$

where $\tilde{W}_i(t)$ is the i^{th} element of an N-dimensional independent Brownian motion that we assume is independent of $Z_k(t)$ and $J_k(t)$ for all k.[4]

Though (4) and (5) specify the instantaneous covariance matrix of LIBOR rates with fixed time-to-maturity, in our analysis we need the instantaneous covariance matrix of LIBOR rates with fixed maturities Σ_t. At $t = 0$, Σ_t coincides with Ω_t; for $t > 0$, we obtain Σ_t from Ω_t through interpolation. Specifically, we assume that $U_{s,j}$ is piecewise constant, that is, for time-to-maturity $s \in (T_k, T_{k+1})$,

$$U_s^2 = \frac{1}{2}\left(U_k^2 + U_{k+1}^2\right). \tag{6}$$

We further assume that $U_{s,j}$ is constant for all caplets belonging to the same difference cap. For the family of the LIBOR rates with maturities $T = T_1, T_2, \ldots T_K$, we denote by U_{T-t} the time-t matrix that consists of the rows of U_{T_k-t}, and obtain the time-t covariance matrix of the LIBOR rates with fixed maturities,

$$\Sigma_t = U_{T-t}\Lambda_t U'_{T-t}. \tag{7}$$

To stay within the family of AJDs, we assume that the random jump times arrive with a constant intensity λ_J, and conditional on the arrival of a jump, the jump size follows a normal distribution, $N(\mu_J, \sigma_J^2)$. Intuitively, the conditional probability at time t of another jump within the next small time interval Δt is $\lambda_J \Delta t$ and, conditional on a jump event, the mean relative jump size is $\mu = \exp(\mu_J + \frac{1}{2}\sigma_J^2) - 1$. For simplicity, we assume that different forward rates follow the same jump process with a constant jump intensity. It is not difficult, however, to introduce different jump processes for individual LIBOR rates and to let the jump intensity depend on the state of the economy within the AJD framework. We do not pursue this extension herein. We also assume that the shocks driving LIBOR rates, volatility, and jumps (both jump time and size) are mutually independent from each other.

Given the above assumptions, we have the following dynamics of LIBOR rates under the physical measure $\mathbb{P}$:

$$\frac{dL_k(t)}{L_k(t)} = \alpha_k(t)\,dt + \sum_{j=1}^{N} U_{T_k-t,j}\sqrt{V_j(t)}\,dW_j(t) + dJ_k(t), \quad k = 1, 2, \ldots, K. \tag{8}$$

To price caps, we need the dynamics of LIBOR rates under the appropriate forward measure. The existence of stochastic volatility and jumps results in an incomplete market and hence the nonuniqueness of forward martingale measures. Our approach for eliminating this nonuniqueness is to specify the market

[4] Many empirical studies on interest rate dynamics (see, e.g., Andersen and Lund (1997), Ball and Torous (1999), and Chen and Scott (2001)) show that correlation between stochastic volatility and interest rates is close to zero. That is, there is not a strong "leverage" effect for interest rates as there is for stock prices. The independence assumption between stochastic volatility and LIBOR rates in our model captures this stylized fact.

prices of both the volatility and jump risks to change from the physical measure $\mathbb{P}$ to the forward measure $\mathbb{Q}^{k+1}$. Following the existing literature, we model the volatility risk premium as $\eta_j^{k+1}\sqrt{V_j(t)}$, for $j = 1, \ldots, N$. For the jump risk premium, we assume that under the forward measure $\mathbb{Q}^{k+1}$, the jump process has the same distribution as that under $\mathbb{P}$, except that the jump size follows a normal distribution with mean μ_J^{k+1} and variance σ_J^2. Thus, the mean relative jump size under $\mathbb{Q}^{k+1}$ is $\mu^{k+1} = \exp(\mu_J^{k+1} + \frac{1}{2}\sigma_J^2) - 1$. Our specification of the market prices of jump risks allows the mean relative jump size under $\mathbb{Q}^{k+1}$ to be different from that under $\mathbb{P}$, accommodating a premium for jump size uncertainty. This approach, which is also adopted by Pan (2002), artificially absorbs the risk premium associated with the timing of the jump by the jump size risk premium. In our empirical analysis, we make the simplifying assumption that the volatility and jump risk premiums are linear functions of time-to-maturity, that is, $\eta_j^{k+1} = c_{jv}(T_k - 1)$ and $\mu_J^{k+1} = \mu_J + c_J(T_k - 1)$.

In order to estimate the volatility and jump risk premiums, one needs to investigate the joint dynamics of LIBOR rates under both the physical and forward measure, as in Chernov and Ghysels (2000), Pan (2002), and Eraker (2004). In our empirical analysis, however, we only study the dynamics under the forward measures. Therefore, we can only identify the differences in risk premiums between the forward measures with different maturities. Our specifications of both risk premiums implicitly use the 1-year LIBOR rate as a reference point. Due to the no-arbitrage restriction, the risk premiums of shocks to LIBOR rates for different forward measures are intimately related to each other. If shocks to volatility and jumps also are correlated with shocks to LIBOR rates, then volatility and jump risk premiums for different forward measures should be closely related to each other. However, in our model, shocks to LIBOR rates are independent of those to volatility and jumps, and, as a result, the change of measure of LIBOR shocks does not affect that of volatility and jump shocks. Due to stochastic volatility and jumps, the underlying LIBOR market is no longer complete and there is no unique forward measure. This gives us the freedom to choose the functional forms of η_j^{k+1} and μ_J^{k+1}. See Andersen and Brotherton-Ratcliffe (2001) for a similar discussion.

Given the above market prices of risks, we can write the dynamics of $\log(L_k(t))$ under forward measure $\mathbb{Q}^{k+1}$ as

$$d\log(L_k(t)) = -\left(\lambda_J\mu^{k+1} + \frac{1}{2}\sum_{j=1}^{N} U_{T_k-t,j}^2 V_j(t)\right)dt + \sum_{j=1}^{N} U_{T_k-t,j}\sqrt{V_j(t)}\,dW_j^{\mathbb{Q}^{k+1}}(t) + dJ_k^{\mathbb{Q}^{k+1}}(t). \tag{9}$$

For pricing purposes, the above process can be further simplified to

$$d\log(L_k(t)) = -\left(\lambda_J \mu^{k+1} + \frac{1}{2}\sum_{j=1}^{N} U_{T_k-t,j}^2 V_j(t)\right) dt + \sqrt{\sum_{j=1}^{N} U_{T_k-t,j}^2 V_j(t)}\, dZ_k^{\mathbb{Q}^{k+1}}(t) + dJ_k^{\mathbb{Q}^{k+1}}(t), \tag{10}$$

where $Z_k^{\mathbb{Q}^{k+1}}(t)$ is a standard Brownian motion under $\mathbb{Q}^{k+1}$. Note that (10) has the same distribution as (9). The dynamics of $V_i(t)$ under $\mathbb{Q}^{k+1}$ then become

$$dV_i(t) = \kappa_i^{k+1}\left(\bar{v}_i^{k+1} - V_i(t)\right) dt + \xi_i \sqrt{V_i(t)}\, d\tilde{W}_i^{\mathbb{Q}^{k+1}}(t), \tag{11}$$

where $\tilde{W}^{\mathbb{Q}^{k+1}}$ is independent of $Z^{\mathbb{Q}^{k+1}}$, $\kappa_j^{k+1} = \kappa_j - \xi_j \eta_j^{k+1}$, and $\bar{v}_j^{k+1} = \frac{\kappa_j \bar{v}_j}{\kappa_j - \xi_j \eta_j^{k+1}}$, $j = 1, \ldots, N$. The dynamics of $L_k(t)$ under the forward measure $\mathbb{Q}^{k+1}$ are completely captured by (10) and (11).

Given that LIBOR rates follow AJDs under both the physical measure and the forward measure, we can directly apply the transform analysis of Duffie et al. (2000) to derive a closed-form formula for cap price. Denote the state variables at t as $Y_t = (\log(L_k(t)), V_t)'$ and the time-t expectation of $e^{u \cdot Y_{T_k}}$ under the forward measure $\mathbb{Q}^{k+1}$ as $\psi(u, Y_t, t, T_k) \triangleq E_t^{\mathbb{Q}^{k+1}}[e^{u \cdot Y_{T_k}}]$. Let $u = (u_0, 0_{1\times N})'$. Then the time-$t$ expectation of LIBOR rate at T_k equals

$$\begin{aligned} E_t^{\mathbb{Q}^{k+1}}\{\exp[u_0 \log(L_k(T_k))]\} &= \psi(u_0, Y_t, t, T_k) \\ &= \exp[a(s) + u_0 \log(L_k(t)) + B(s)' V_t], \end{aligned} \tag{12}$$

where $s = T_k - t$ and closed-form solutions of $a(s)$ and $B(s)$ (an N-by-1 vector) are obtained by solving a system of Ricatti equations (see the Appendix).

Following Duffie et al. (2000), we define

$$G_{a,b}\left(y; Y_t, T_k, \mathbb{Q}^{k+1}\right) = E_t^{\mathbb{Q}^{k+1}}\left[e^{a \cdot \log(L_k(T_k))} 1_{\{b \cdot \log(L_k(T_k)) \le y\}}\right], \tag{13}$$

and its Fourier transform

$$\begin{aligned} \mathcal{G}_{a,b}(v; Y_t, T_k, \mathbb{Q}^{k+1}) &= \int_R e^{ivy}\, dG_{a,b}(y) \\ &= E_t^{\mathbb{Q}^{k+1}}\left[e^{(a+ivb)\cdot \log(L_k(T_k))}\right] \\ &= \psi(a + ivb, Y_t, t, T_k). \end{aligned} \tag{14}$$

Levy's inversion formula gives

$$G_{a,b}(y; Y_t, T_k, \mathbb{Q}^{k+1}) = \frac{\psi(a+ivb, Y_t, t, T_k)}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\left[\psi(a+ivb, Y_t, t, T_k)e^{-ivy}\right]}{v}\,dv. \tag{15}$$

The time-0 price of a caplet that matures at T_{k+1} with a strike price of X equals

$$Caplet(0, T_{k+1}, X) = \delta D_{k+1}(0) E_0^{\mathbb{Q}^{k+1}}\left[(L_k(T_k) - X)^+\right], \tag{16}$$

where the expectation is given by the inversion formula

$$E_0^{\mathbb{Q}^{k+1}}[L_k(T_k) - X]^+ = G_{1,-1}(-\ln X; Y_0, T_k, \mathbb{Q}^{k+1}) - XG_{0,-1}(-\ln X; Y_0, T_k, \mathbb{Q}^{k+1}). \tag{17}$$

The new model developed in this section nests some of the most important models in the literature, such as LSS (2001) (with constant volatility and no jumps) and Han (2002) (with stochastic volatility and no jumps). The closed-form formula for cap prices makes empirical implementation of our model very convenient and provides some advantages over existing methods. For example, while Han (2002) develops approximations of ATM cap and swaption prices using the techniques of Hull and White (1987), such an approach might not work well for away-from-the-money options. In contrast, our method works well for all options, which is important for explaining the volatility smile.

In addition to introducing stochastic volatility and jumps, our multifactor HJM model also has advantages over the standard LIBOR market models of Brace et al. (1997) and Miltersen et al. (1997), or their extensions that are often applied to caps in practice. In particular, Andersen and Brotherton-Ratcliffe (2001) and Glasserman and Kou (2003) develop LIBOR models with stochastic volatility and jumps, respectively. While our model provides a unified multifactor framework to characterize the evolution of the whole yield curve, the LIBOR market models typically make separate specifications of the dynamics of LIBOR rates with different maturities. As LSS (2001) suggest, the standard LIBOR models are "more appropriately viewed as a collection of different univariate models, where the relationship between the underlying factors is left unspecified." In contrast, the dynamics of LIBOR rates with different maturities under their related forward measures are internally consistent with each other given their dynamics under the physical measure and the market prices of risks. Once our model is estimated using one set of prices, it can be used to price and hedge other fixed-income securities.

We estimate our new market model using prices from a wide cross-section of difference caps with different strikes and maturities. Every week we observe prices of difference caps with 10 moneyness and 13 maturities. However, due to changing interest rates, we do not have enough observations in all moneyness/maturity categories throughout the sample. Thus, we focus on the 53 moneyness/maturity categories that have less than 10% of missing values over the sample estimation period. The moneyness (in %) and maturity (in years) of all difference caps belong to the sets {0.7, 0.8, 0.9, 1.0, 1.1} and {1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0}, respectively. The difference caps with time-to-maturity less than or equal to 5 years represent portfolios of two caplets, while those with time-to-maturity longer than 5 years represent portfolios of four caplets.

We estimate the model parameters by minimizing the sum of squared percentage pricing errors (SSE) of all relevant difference caps. Due to the wide range of moneyness and maturities of the difference caps we investigate, there could be significant differences in the difference cap prices. Using percentage pricing errors helps to mitigate this problem. Consider the time-series observations $t = 1, \ldots, T$, of the prices of 53 difference caps with moneyness m_i and time-to-maturities τ_i, $i = 1, \ldots, M = 53$. Let θ represent the model parameters, which remain constant over the sample period. Let $C(t, m_i, \tau_i)$ be the observed price of a difference cap with moneyness m_i and time-to-maturity τ_i and let $\hat{C}(t, \tau_i, m_i, V_t(\theta), \theta)$ denote the corresponding theoretical price under a given model, where $V_t(\theta)$ is the model-implied instantaneous volatility at t given model parameters θ. For each i and t, denote the percentage pricing error as

$$u_{i,t}(\theta) = \frac{C(t, m_i, \tau_i) - \hat{C}\left(t, m_i, \tau_i, V_t(\theta), \theta\right)}{C(t, m_i, \tau_i)}, \tag{18}$$

where $V_t(\theta)$ is defined as

$$V_t(\theta) = \arg\min_{\{V_t\}} \sum_{i=1}^{M} \left[\frac{C(t, m_i, \tau_i) - \hat{C}(t, m_i, \tau_i, V_t, \theta)}{C(t, m_i, \tau_i)}\right]^2. \tag{19}$$

That is, we estimate the unobservable volatility variables V_t for a specific parameter θ by minimizing the SSEs of all difference caps at t. We assume that under the correct model specification and the true model parameters, $\theta^0, V_t(\theta^0) = V_t^0$, the true instantaneous stochastic volatility. This assumption is similar to that of Pan (2002), who first backs out $V_t(\theta)$ from short-term ATM options given θ, however, our approach has the advantage of fully utilizing the information in the prices of all difference caps to estimate the volatility variables at t. We then estimate the model parameters θ by minimizing the SSE over the entire sample. That is,

$$\hat{\theta} = \arg\min_{\{\theta\}} \sum_{t=1}^{T} u_t'(\theta) u_t(\theta), \tag{20}$$

where $u_t(\theta)$ is a vector of length M such that the ith element equals $u_{i,t}(\theta)$.

The above approach is essentially the nonlinear least squares method discussed in Gallant (1987). As Gallant (1987) (p. 153) points out, the least squares estimator can be cast into the form of a method of moments estimator by using the first derivatives of the SSE with respect to the model parameters as moment conditions. As a result, our estimation method can be regarded as a special case of the implied-state generalized method of moments (IS-GMM) of Pan (2002) with a special set of moment conditions. Note that since the number of moments always equals the number of model parameters in our method, our model is exactly identified. Therefore, the consistency of our parameter estimators can be established by invoking Pan's arguments. Unlike the generalized least squares method or GMM, however, we use an identity weighting matrix in our method. The main reason for this choice is that optimal weighting matrices are generally different across different models, which makes the comparison of results across models more difficult.

One implicit assumption we make to obtain consistent estimators is that the conditional mean of the pricing errors, given all the independent variables such as LIBOR rates and volatilities, equals zero. This assumption, which is used previously in econometrics when the regressors are stochastic (see, for example, White (2001) and Wooldridge (2002)), is slightly weaker than the assumption that the pricing errors are independent of all the explanatory variables. The latter is more likely to be violated in our data. Due to potential dependences among the pricing errors and between the pricing errors and the independent variables, we estimate standard errors using a robust covariance matrix estimator adapted from (7.26) of Wooldridge (2002) to the nonlinear model considered above. Wooldridge (2002) shows that this covariance matrix estimator is valid without any second-moment assumptions on the pricing errors except that the second moments are well defined. This covariance matrix estimator also allows for general dependence of the conditional variances of the pricing errors with the independent variables. Nonetheless, as is often the case in empirical investigations, even these weak assumptions may be violated, in which case our estimates would be inconsistent.

First, we compare model performance using the likelihood ratio test following LSS (2001). That is, the total number of observations (both cross-sectional and time series) times the difference between the logarithms of the SSEs between two models follows an asymptotic χ^2 distribution. Similar to LSS (2001), we treat the implied instantaneous volatility variables as parameters. Thus, the degree of freedom of the χ^2 distribution equals the difference in the number of model parameters and the total number of implied volatility variables across the two models.

In addition, to test whether one model has statistically smaller pricing errors, we also adopt an approach developed by Diebold and Mariano (1995) in the time-series forecast literature. While the overall SSE equals $\sum_{t=1}^{T} u_t' u_t$, we define SSE at t as $\varepsilon(t) = u_t' u_t$. Consider two models with weekly SSEs $\{\varepsilon_1(t)\}_{t=1}^{T}$ and $\{\varepsilon_2(t)\}_{t=1}^{T}$, respectively.[5] The null hypothesis that the two models have the same pricing errors is $E[\varepsilon_1(t)] = E[\varepsilon_2(t)]$, or $E[d(t)] = 0$, where $d(t) = \varepsilon_1(t) - \varepsilon_2(t)$.

[5] Diebold and Mariano (1995) compare model performance based on out-of-sample forecast errors. Applying the test to our setting, we compare model performance based on in-sample SSEs.

Diebold and Mariano (1995) show that if $\{d(t)\}_{t=1}^{\mathcal{T}}$ is covariance stationary with short memory, then

$$\sqrt{\mathcal{T}}(\bar{d}-\mu_d) \sim N(0, 2\pi f_d(0)), \tag{21}$$

where $\bar{d}=\frac{1}{\mathcal{T}}\sum_{t=1}^{\mathcal{T}}[\varepsilon_1(t)-\varepsilon_2(t)]$, $f_d(0)=\frac{1}{2\pi}\sum_{q=-\infty}^{\infty}\gamma_d(q)$, and $\gamma_d(q)=E[(d_t-\mu_d)(d_{t-q}-\mu_d)]$. In large samples, $\bar{d}$ is approximately normally distributed with mean μ_d and variance $2\pi f_d(0)/T$. Thus, under the null hypothesis of equal pricing errors, the statistic

$$S=\frac{\bar{d}}{\sqrt{2\pi\hat{f}_d(0)/\mathcal{T}}} \tag{22}$$

is distributed asymptotically as $N(0,1)$, where $\hat{f}_d(0)$ is a consistent estimator of $f_d(0)$. We estimate the variance of the test statistic using the Bartlett estimate of Newey and West (1987). To compare the overall performance of the two models, we use the above statistic to measure whether one model has significantly smaller SSEs. We also can use the above statistic to measure whether one model has smaller squared percentage pricing errors for difference caps in a specific moneyness/maturity group.

III. Empirical Results

In this section, we provide empirical evidence on the performance of six different models in capturing the cap volatility smile. The first three models, SV1, SV2, and SV3, allow one, two, and three principal components respectively to drive the forward rate curve, each with its own stochastic volatility. The next three models, SVJ1, SVJ2, and SVJ3, introduce jumps in LIBOR rates in each of the previous SV models. Note that SVJ3 is the most comprehensive model, nesting all the others as special cases. We begin by examining the separate performance of each of the SV and SVJ models. We then compare performance across the two classes of models.

The estimation of all models is based on the principal components extracted from historical LIBOR forward rates between June 1997 and July 2000. Following the bootstrapping procedure of LSS (2001), we construct the LIBOR forward curve using weekly LIBOR and swap rates from Datastream. Figure 4 shows that the three principal components can be interpreted as in Litterman and Scheinkman (1991). The first principal component, the "level" factor, represents a parallel shift of the forward rate curve. The second principal component, the "slope" factor, twists the forward rate curve by moving the short and long ends of the curve in opposite directions. The third principal component, the "curvature" factor, increases the curvature of the curve by moving the short and long ends of the curve in one direction and the middle range of the curve in the other direction. These three factors explain 77.78%, 14.35%, and 7.85% of the variation of LIBOR rates up to 10 years, respectively.

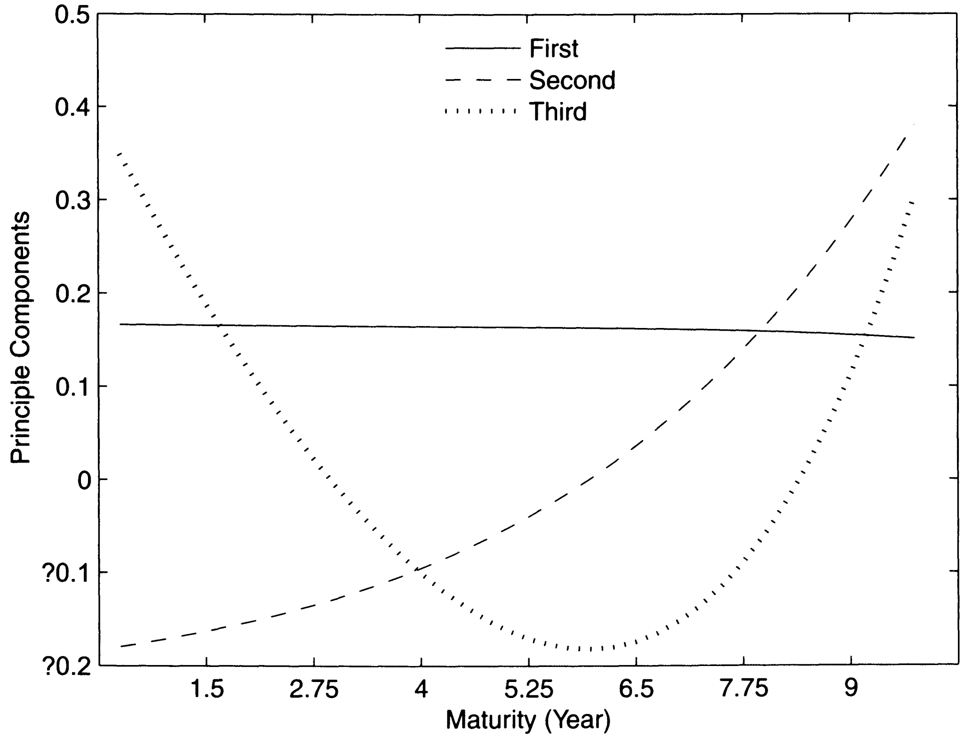


Figure 4. The first three principal components of weekly percentage changes of 3-month LIBOR rates between June 2, 1997 and July 31, 2000.

A. Performance of Stochastic Volatility Models

The SV models contribute to cap pricing in four important ways. First, the three principal components capture variation in the levels of LIBOR rates caused by innovations in the level, slope, and curvature factors. Second, the stochastic volatility factors capture the fluctuations in the volatilities of LIBOR rates reflected in the Black-implied volatilities of ATM caps.[6] Third, the stochastic volatility factors also introduce fatter tails in LIBOR rate distributions than implied by the lognormal model, which helps capture the volatility smile. Finally, given our model structure, innovations of stochastic volatility factors also affect the covariances between LIBOR rates with different maturities. The first three factors are more important for our applications, however, because difference caps are much less sensitive to time-varying correlations than are swaptions (see Han (2002)). Our discussion of the performance of the SV models focuses on the estimates of the model parameters and the latent volatility variables, and the time-series and cross-sectional pricing errors of difference caps.

[6] Throughout our discussion, volatilities of LIBOR rates refer to market implied volatilities from cap prices and are different from volatilities estimated from historical data.

Table I

Parameter Estimates of Stochastic Volatility Models

This table reports parameter estimates and standard errors of the one-, two-, and three-factor stochastic volatility models (SV1, SV2, and SV3, respectively). We obtain the estimates by minimizing the sum of squared percentage pricing errors (SSE) of difference caps in 53 moneyness and maturity categories observed on a weekly frequency from August 1, 2000 to September 23, 2003. The objective functions reported in the table are rescaled SSEs over the entire sample at the estimated model parameters and are equal to the RMSE of difference caps. The volatility risk premium of the ith stochastic volatility factor for forward measure Q^{k+1} is defined as $\eta_i^{k+1} = c_{iv}(T_k - 1)$.

	SV1		SV2		SV3	
Parameter	Estimate	Std. Err	Estimate	Std. Err	Estimate	Std. Err
κ_1	0.0179	0.0144	0.0091	0.0111	0.0067	0.0148
κ_2			0.1387	0.0050	0.0052	0.0022
κ_3					0.0072	0.0104
$\bar{v}_1$	1.3727	1.1077	1.7100	2.0704	2.1448	4.7567
$\bar{v}_2$			0.0097	0.0006	0.0344	0.0142
$\bar{v}_3$					0.1305	0.1895
ξ_1	1.0803	0.0105	0.8992	0.0068	0.8489	0.0098
ξ_2			0.0285	0.0050	0.0117	0.0065
ξ_3					0.1365	0.0059
c_{1v}	−0.0022	0.0000	−0.0031	0.0000	−0.0015	0.0000
c_{2v}			−0.0057	0.0010	−0.0007	0.0001
c_{3v}					−0.0095	0.0003
Objective function	0.0834		0.0758		0.0692	

A comparison of the parameter estimates of the three SV models in Table I shows that the level factor has the most volatile stochastic volatility, followed, in decreasing order, by the curvature and slope factors. The long-run mean ($\bar{v}_1$) and volatility of volatility (ξ_1) of the first volatility factor are much greater than those of the other two factors. This suggests that the fluctuations in the volatility of LIBOR rates are mainly due to the time-varying volatility of the level factor. The estimates of the volatility risk premium of the three models are significantly negative, suggesting that the stochastic volatility factors of longer-maturity LIBOR rates under the forward measure are less volatile with lower long-run mean and faster speed of mean reversion. This is consistent with the fact that the Black-implied volatilities of longer-maturity difference caps are less volatile than those of short-term difference caps.

Our parameter estimates are consistent with the volatility variables inferred from the prices of difference caps in Figure 5. The volatility of the level factor is the highest among the three (although at lower absolute levels in the more sophisticated models), starting at a low level and steadily increasing and stabilizing at a high level in the later part of the sample period. The volatility of the slope factor is much lower and is relatively stable throughout the entire sample period. The volatility of the curvature factor is generally between that of the

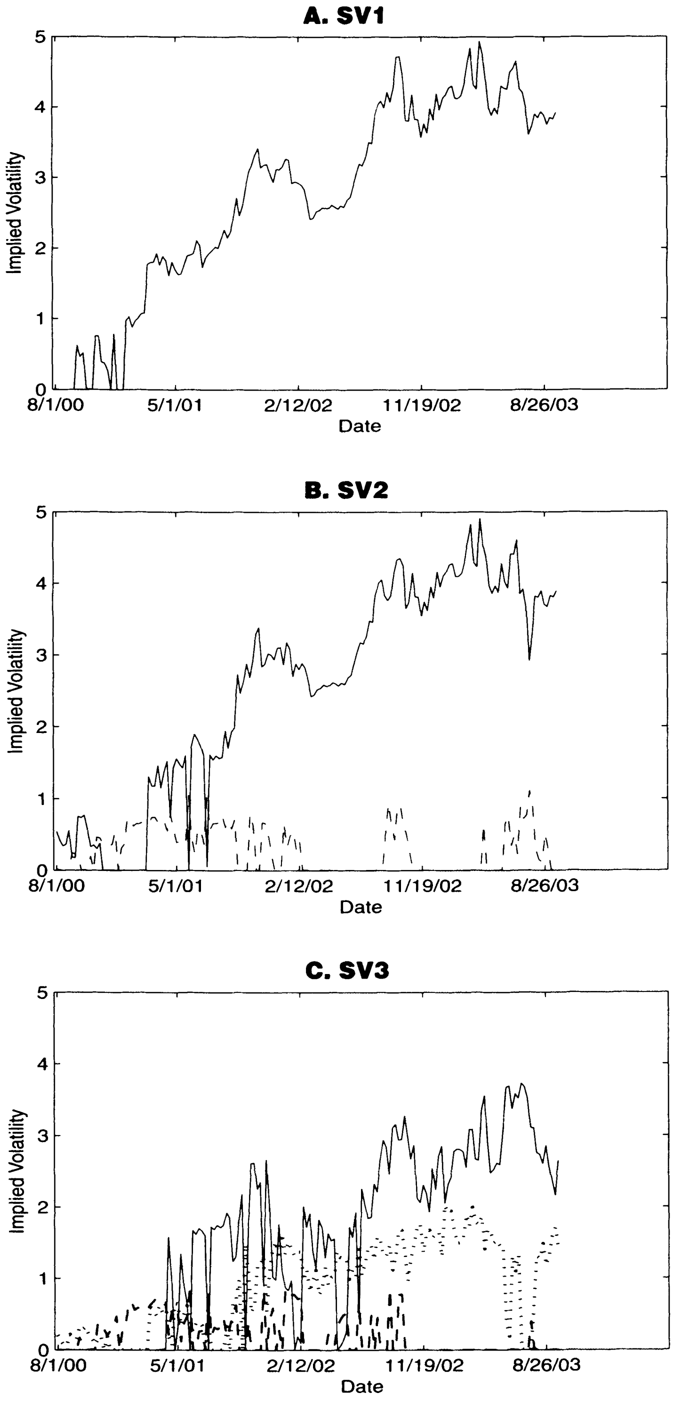


Figure 5. The implied volatilities from SV models between August 1, 2000 and September 23, 2003 (solid: first factor; dashed: second factor; dotted: third factor).

first and second factors. The steady increase of the volatility of the level factor is consistent with the increase of Black-implied volatilities of ATM difference caps throughout our sample period. In fact, the correlation between the Black-implied volatilities of most difference caps and the implied volatility of the level factor is higher than 0.8. The correlations between the Black-implied volatilities and the other two volatility factors are much weaker. The importance of stochastic volatility is obvious: The fluctuations in Black-implied volatilities show that a model with constant volatility simply would not be able to capture even the general level of cap prices.

The other aspects of model performance are the time-series and cross-sectional pricing errors of difference caps. The likelihood ratio tests in Panel A of Table II overwhelmingly reject SV1 and SV2 in favor of SV2 and SV3, respectively. The Diebold–Mariano statistics in Panel A of Table II also show that SV2 and SV3 have significantly smaller SSEs than do SV1 and SV2, respectively, suggesting that the more sophisticated SV models improve the pricing of all caps.

Figure 6 plots the time series of the root mean square errors (RMSEs) of the three SV models over our sample period. The RMSE at t is calculated as $\sqrt{u_t'(\hat{\theta})u_t(\hat{\theta})/M}$. We plot RMSEs instead of SSEs because the former provides a more direct measure of average percentage pricing errors of difference caps. Except for two special periods in which all models have extremely large pricing errors, the RMSEs of all models are rather uniform over the entire sample period, with the best model (SV3) having RMSEs slightly above 5%. The two special periods with high pricing errors, namely the period between the second half of December 2000 and the first half of January 2001, and the first half of October 2001, coincide with high prepayments in mortgage-backed securities (MBS). Indeed, the MBAA refinancing index and prepayment speed (see Figure 3 of Duarte (2004)) show that after a long period of low prepayments between the middle of 1999 and late 2000, prepayments dramatically increased at the end of 2000 and the beginning of 2001, with an additional dramatic increase of prepayments at the beginning of October 2001. As widely recognized in the fixed-income market,[7] excessive hedging demands for prepayment risk using interest rate derivatives may push derivative prices away from their equilibrium values, which could explain the failure of our models during these two special periods.[8]

In addition to overall model performance as measured by SSEs, we also examine the cross-sectional pricing errors of difference caps with different moneyness and maturities. We first look at the squared percentage pricing errors, which measure both the bias and variability of the pricing errors. Then we look at the average percentage pricing errors (the difference between market and

[7] We would like to thank Pierre Grellet Aumont from Deutsche Bank for his helpful discussions on the influence of MBS markets on OTC interest rate derivatives.

[8] While the prepayments rates were also high in the later part of 2002 and for most of 2003, they might not have come as surprises to participants in the MBS markets given the two previous special periods.

Table II

Comparison of the Performance of Stochastic Volatility Models

This table reports model comparison based on likelihood ratio and Diebold–Mariano statistics. The total number of observations (both cross-sectional and time series), which equals 8,545 over the entire sample, times the difference between the logarithms of the SSEs between two models follows a χ^2 distribution asymptotically. We treat implied volatility variables as parameters. Thus, the degree of freedom of the χ^2 distribution is 168 for the SV2-SV1 and SV3-SV2 pairs, because SV2 (SV3) has four more parameters and 164 additional implied volatility variables than SV1 (SV2). The 1% critical value of $\chi^2(168)$ is 214. The Diebold–Mariano statistics are calculated according to equation (22) with a lag order q of 40, and they follow an asymptotic standard Normal distribution under the null hypothesis of equal pricing errors. A negative statistic means that the more sophisticated model has smaller pricing errors. Bold entries mean that the statistics are significant at the 5% level.

Model Pairs	D–M Stats	Likelihood Ratio Stats $\chi^2(168)$
Panel A: Likelihood Ratio and Diebold–Mariano Statistics for Overall Model Performance Based on SSEs		
SV2-SV1	−1.931	**1624**
SV3-SV2	**−6.351**	**1557**

Moneyness	1.5 Yr	2 Yr	2.5 Yr	3 Yr	3.5 Yr	4 Yr	4.5 Yr	5 Yr	6 Yr	7 Yr	8 Yr	9 Yr	10 Yr
Panel B: Diebold–Mariano Statistics between SV2 and SV1 for Individual Difference Caps Based on Squared Percentage Pricing Errors													
0.7	–	–	–	–	**2.433**	**2.895**	**2.385**	**5.414**	**4.107**	**5.701**	**2.665**	−1.159	−1.299
0.8	–	–	−0.061	0.928	1.838	1.840	**2.169**	**6.676**	**3.036**	**2.274**	−0.135	−1.796	−1.590
0.9	–	−1.553	**−1.988**	**−2.218**	−1.064	−1.222	**−3.410**	−1.497	0.354	−0.555	−1.320	−1.439	−1.581
1.0	−0.295	**−5.068**	**−2.693**	−1.427	−1.350	−1.676	**−3.498**	**−3.479**	**−2.120**	−1.734	−1.523	−0.133	**−2.016**
1.1	−1.260	**−4.347**	−1.522	0.086	−1.492	**−3.134**	**−3.439**	**−3.966**	–	–	–	–	–
Panel C: Diebold–Mariano Statistics between SV3 and SV2 for Individual Difference Caps Based on Squared Percentage Pricing Errors													
0.7	–	–	–	–	1.493	1.379	0.229	−0.840	**−3.284**	**−5.867**	**−4.280**	−0.057	**−2.236**
0.8	–	–	**−3.135**	−1.212	1.599	1.682	−0.052	−0.592	**−3.204**	**−6.948**	**−4.703**	1.437	−1.079
0.9	–	**−2.897**	**−3.771**	**−3.211**	1.417	1.196	**−2.237**	−1.570	−1.932	**−6.920**	−1.230	**−2.036**	−1.020
1.0	−0.849	**−3.020**	**−3.115**	−0.122	0.328	**−3.288**	**−3.342**	**−3.103**	1.351	1.338	0.139	**−4.170**	−0.193
1.1	0.847	**−2.861**	0.675	0.315	**−3.650**	**−3.523**	**−2.923**	**−2.853**	–	–	–	–	–

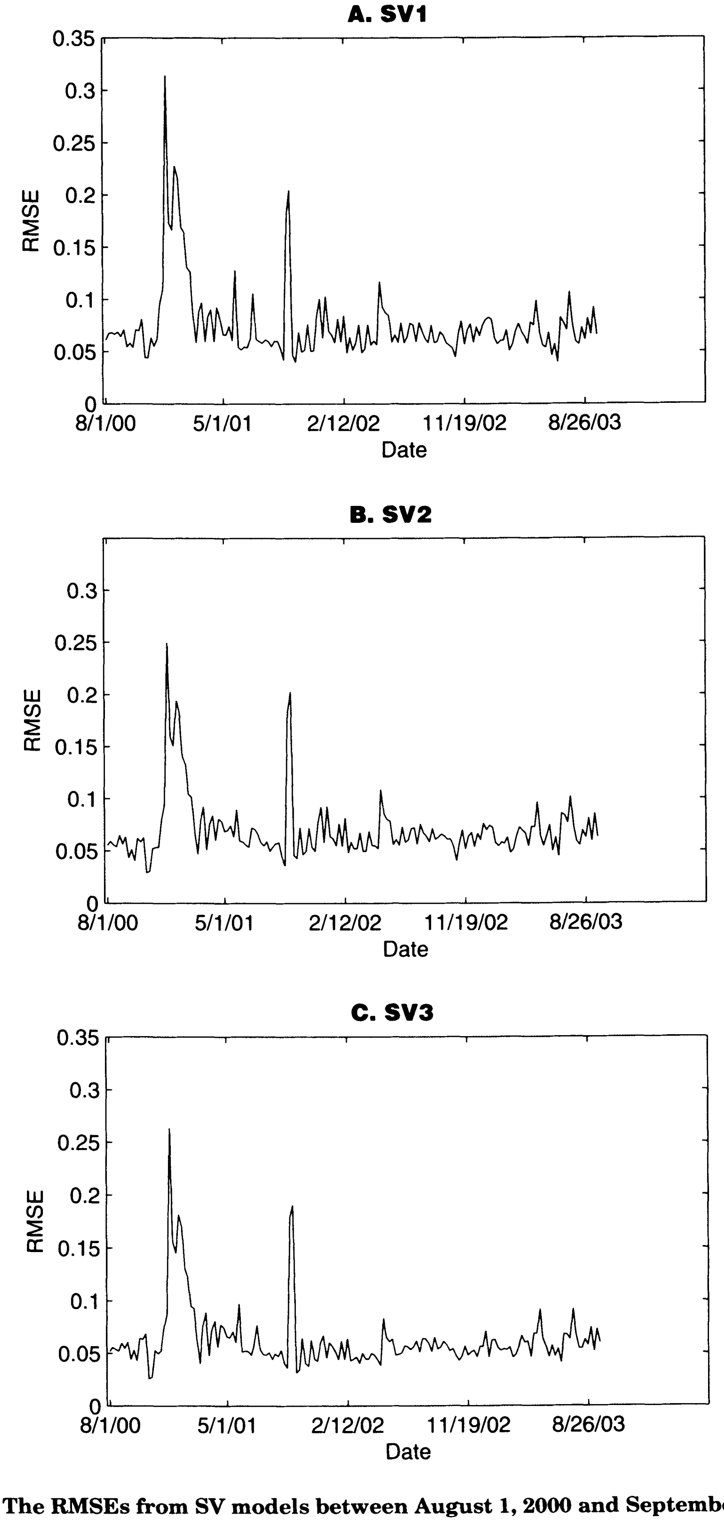


Figure 6. The RMSEs from SV models between August 1, 2000 and September 23, 2003.

model prices divided by the market price) to determine, on average, whether SV models can capture the volatility smile in the cap market.

The Diebold-Mariano statistics of squared percentage pricing errors of individual difference caps between SV2 and SV1 in Panel B of Table II show that SV2 reduces the pricing errors of SV1 for some but not all difference caps. SV2 has the most significant reductions in pricing errors of SV1 for mid- and short-term around-the-money difference caps. On the other hand, SV2 has larger pricing errors for deep ITM difference caps. The Diebold–Mariano statistics between SV3 and SV2 in Panel C of Table II show that SV3 significantly reduces the pricing errors of many short- (2- to 3-year) and mid-term (3.5- to 5-year) around-the-money, and long-term (6- to 10-year) ITM difference caps.

Table III reports the average percentage pricing errors of all difference caps under the three SV models. Panel A of Table III shows that, on average, SV1 underprices short-term and overprices mid- and long-term ATM difference caps, and underprices ITM and overprices OTM difference caps. This suggests that SV1 cannot generate sufficient skewness in the implied volatilities to be consistent with the data. Panel B shows that SV2 achieves some improvements over SV1, mainly for some short-term (less than 3-yr) ATM, and mid-term (3.5- to 5-year) slightly OTM difference caps. However, SV2 has worse performance for most deep ITM ($m = 0.7$ and 0.8) difference caps, actually worsening the underpricing of ITM caps. Panel C of Table III shows that relative to SV1 and SV2, SV3 has smaller average percentage pricing errors for most long-term (7- to 10-year) ITM, mid-term (3.5- to 5-year) OTM, and short-term (2- to 2.5-year) ATM difference caps, and larger average percentage pricing errors for mid-term (3.5- to 5-year) ITM difference caps. There is still significant underpricing of ITM and overpricing of OTM difference caps under SV3.

Overall, the results show that stochastic volatility factors are essential for capturing the time-varying volatilities of LIBOR rates. The Diebold–Mariano statistics in Table II show that, in general, more sophisticated SV models have smaller pricing errors than simpler models, although the improvements are more important for close-to-the-money difference caps. The average percentage pricing errors in Table III show, however, that even the most sophisticated SV model cannot generate enough volatility skew to be consistent with the data. Though previous studies, such as Han (2002) show that a three-factor stochastic volatility model similar to ours performs well in pricing ATM caps and swaptions, our analysis shows that the model fails to completely capture the volatility smile in the cap markets. Our findings highlight the importance of studying the relative pricing of caps with different moneyness to reveal the inadequacies of existing term structure models; that is, these inadequacies cannot be obtained from studying only ATM options.

B. Performance of Stochastic Volatility and Jump Models

One important reason for the failure of SV models is that the stochastic volatility factors are independent of LIBOR rates. As a result, the SV models can only generate a symmetric volatility smile, but not the asymmetric smile or

Table III

Average Percentage Pricing Errors of Stochastic Volatility Models

This table reports average percentage pricing errors of difference caps with different moneyness and maturities of the three stochastic volatility models (SV1, SV2, and SV3, respectively). Average percentage pricing errors are defined as the difference between market price and model price divided by the market price.

Moneyness	1.5 Yr	2 Yr	2.5 Yr	3 Yr	3.5 Yr	4 Yr	4.5 Yr	5 Yr	6 Yr	7 Yr	8 Yr	9 Yr	10 Yr
Panel A: Average Percentage Pricing Errors of SV1													
0.7	–	–	–	–	0.034	0.0258	0.0122	0.0339	0.0361	0.0503	0.0344	0.0297	0.0402
0.8	–	–	0.0434	0.0412	0.0323	0.018	0.0106	0.0332	0.0322	0.0468	0.0299	0.0244	0.0325
0.9	–	0.1092	0.0534	0.0433	0.0315	0.01	0.0003	0.0208	0.0186	0.0348	0.0101	0.0062	0.0158
1.0	0.0293	0.1217	0.0575	0.0378	0.0227	−0.0081	−0.0259	−0.0073	−0.0079	0.0088	−0.0114	−0.0192	−0.0062
1.1	−0.1187	0.0604	−0.0029	−0.0229	−0.034	−0.0712	−0.0815	−0.0562	–	–	–	–	–
Panel B: Average Percentage Pricing Errors of SV2													
0.7	–	–	–	–	0.0482	0.0425	0.0304	0.0524	0.0544	0.0663	0.0456	0.0304	0.0378
0.8	–	–	0.0509	0.051	0.0443	0.032	0.0258	0.0486	0.0472	0.0586	0.0344	0.0138	0.0202
0.9	–	0.1059	0.0498	0.0421	0.0333	0.0145	0.0069	0.0284	0.0265	0.0392	0.0054	−0.0184	−0.008
1.0	−0.0002	0.0985	0.0369	0.0231	0.0134	−0.0123	−0.0261	−0.005	−0.0042	0.008	−0.024	−0.0572	−0.0403
1.1	−0.1056	0.0584	−0.0085	−0.026	−0.0326	−0.0653	−0.0721	−0.0454	–	–	–	–	–
Panel C: Average Percentage Pricing Errors of SV3													
0.7	–	–	–	–	0.0489	0.0437	0.0308	0.0494	0.0431	0.0466	0.031	0.03	0.028
0.8	–	–	0.044	0.0476	0.0462	0.0378	0.0322	0.0506	0.0367	0.0365	0.0226	0.0249	0.0139
0.9	–	0.0917	0.0367	0.0379	0.0398	0.0288	0.0226	0.0377	0.0178	0.0145	−0.0026	0.0068	−0.0109
1.0	−0.0126	0.0782	0.0198	0.0194	0.0252	0.0105	−0.0012	0.011	−0.0129	−0.0221	−0.0299	−0.0192	−0.0432
1.1	−0.1184	0.0314	−0.0323	−0.0336	−0.0212	−0.0397	−0.0438	−0.0292	–	–	–	–	–

Table IV

Parameter Estimates of Stochastic Volatility and Jump Models

This table reports parameter estimates and standard errors of the one-, two-, and three-factor stochastic volatility and jump models (SVJ1, SVJ2, and SVJ3, respectively). We obtain the estimates by minimizing the sum of squared percentage pricing errors (SSE) of difference caps in 53 moneyness and maturity categories observed on a weekly frequency from August 1, 2000 to September 23, 2003. The objective functions reported in the table are rescaled SSEs over the entire sample at the estimated model parameters and are equal to the RMSE of difference caps. The volatility risk premium of the ith stochastic volatility factor and the jump risk premium for forward measure Q^{k+1} are defined as $\eta_i^{k+1} = c_{iv}(T_k - 1)$ and $\mu_J^{k+1} = \mu_J + c_J(T_k - 1)$, respectively.

	SVJ1		SVJ2		SVJ3	
Parameter	Estimate	Std. Err	Estimate	Std. Err	Estimate	Std. Err
κ_1	0.1377	0.0085	0.0062	0.0057	0.0069	0.0079
κ_2			0.0050	0.0001	0.0032	0.0000
κ_3					0.0049	0.0073
$\bar{v}_1$	0.1312	0.0084	0.7929	0.7369	0.9626	1.1126
$\bar{v}_2$			0.3410	0.0030	0.2051	0.0021
$\bar{v}_3$					0.2628	0.3973
ξ_1	0.8233	0.0057	0.7772	0.0036	0.6967	0.0049
ξ_2			0.0061	0.0104	0.0091	0.0042
ξ_3					0.1517	0.0035
c_{1v}	−0.0041	0.0000	−0.0049	0.0000	−0.0024	0.0000
c_{2v}			−0.0270	0.0464	−0.0007	0.0006
c_{3v}					−0.0103	0.0002
λ	0.0134	0.0001	0.0159	0.0001	0.0132	0.0001
μ_J	−3.8736	0.0038	−3.8517	0.0036	−3.8433	0.0063
c_J	0.2632	0.0012	0.3253	0.0010	0.2473	0.0017
σ_J	0.0001	3.2862	0.0003	0.8723	0.0032	0.1621
Objective function	0.0748		0.0670		0.0622	

skew observed in the data. The pattern of the smile in the cap market is rather similar to that of index options: ITM calls (and OTM puts) are overpriced, and OTM calls (and ITM puts) are underpriced relative to the Black model. Similarly, the smile in the cap market could be due to a market expectation of dramatically declining LIBOR rates. In this section, we examine the contribution of jumps in LIBOR rates in capturing the volatility smile. Our discussion of the performance of the SVJ models parallels that of the SV models.

Parameter estimates in Table IV show that the three stochastic volatility factors of the SVJ models resemble those of the SV models closely. The level factor still has the most volatile stochastic volatility, followed by the curvature and the slope factors. With the inclusion of jumps, the stochastic volatility factors in the SVJ models, especially those of the level factor, tend to be less volatile than those of the SV models (lower long-run mean and volatility of volatility). Negative estimates of the volatility risk premium show that the volatility of the longer-maturity LIBOR rates under the forward measure have lower long-run mean and faster speed of mean-reversion. Figure 7 shows that the volatility

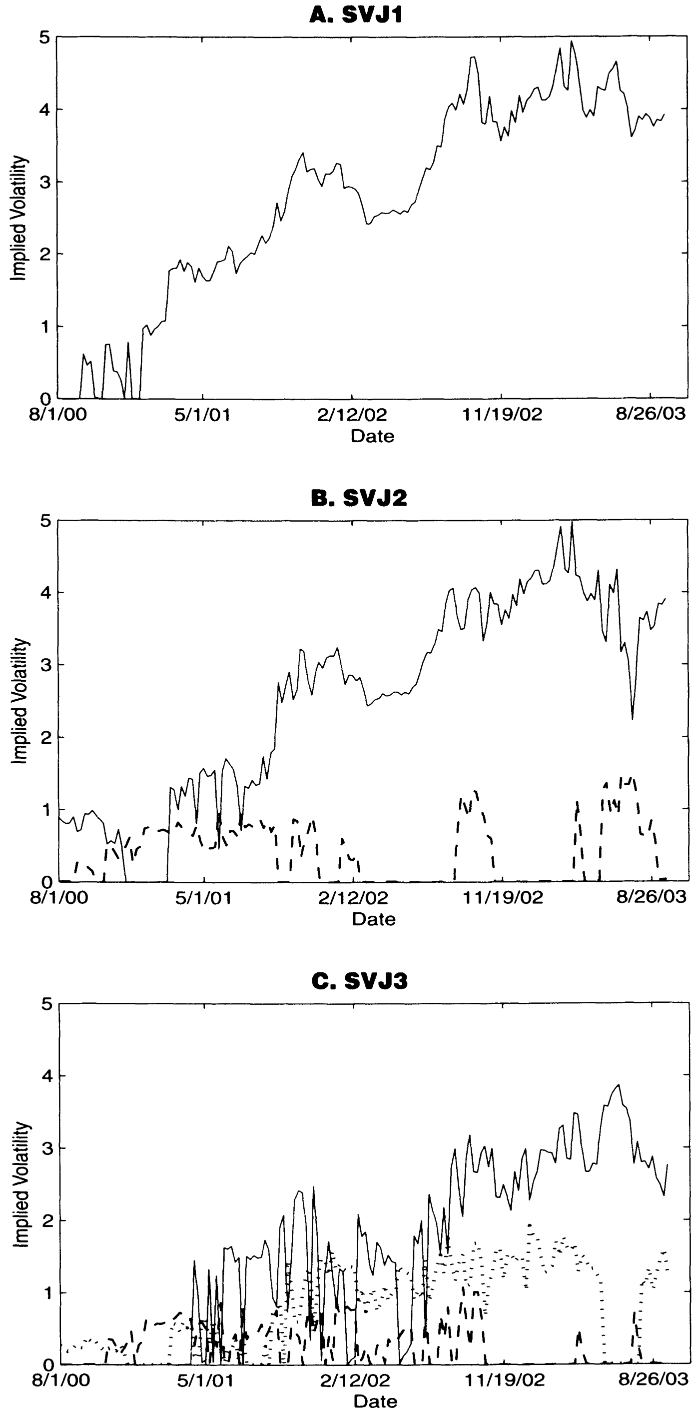


Figure 7. The implied volatilities from SVJ models between August 1, 2000 and September 23, 2003 (solid: first factor; dashed: second factor; dotted: third factor).

of the level factor experiences a steady increase over the entire sample period, while the volatility of the other two factors is relatively stable over time.

Most important, we find overwhelming evidence of strong negative jumps in LIBOR rates under the forward measure. To the extent that cap prices reflect market expectations of the future evolution of LIBOR rates, the evidence suggests that the market expects a dramatic decrease in LIBOR rates over our sample period. Such an expectation might be justifiable given that the economy has been in recession during a major part of our sample period. This is similar to the volatility skew in the index equity option market, which reflects investor fears of a stock market crash such as that of 1987. Compared to the estimates from index options (see, e.g., Pan (2002)), we observe lower estimates of jump intensity (about 1.5% per annual), but much higher estimates of jump size. The positive estimates of a jump risk premium suggest that the jump magnitude of longer-maturity forward rates tends to be smaller. Under SVJ3, the mean relative jump size, $\exp(\mu_J + c_J(T_k - 1) + \sigma_J^2/2) - 1$, for 1-, 5-, and 10-year LIBOR rates are -97%, -94%, and -80%, respectively. However, we do not find any incidents of negative moves in LIBOR rates under the physical measure with a size close to that under the forward measure. This large discrepancy between jump sizes under the physical and forward measures resembles that between the physical and risk-neutral measure for index options (see, e.g., Pan (2002)). This could be a result of a huge jump risk premium.

The likelihood ratio tests in Panel A of Table V again overwhelmingly reject SVJ1 and SVJ2 in favor of SVJ2 and SVJ3, respectively. The Diebold–Mariano statistics in Panel A of Table V show that SVJ2 and SVJ3 have significantly smaller SSEs than do SVJ1 and SVJ2, respectively, suggesting that the more sophisticated SVJ models significantly improve the pricing of all difference caps. Figure 8 plots the time series of RMSEs of the three SVJ models over our sample period. In addition to the two special periods in which the SVJ models have large pricing errors, the SVJ models have larger RMSEs than do the SV models during the first 20 weeks of the sample. This should not be surprising given the relatively stable forward rate curve and a less pronounced volatility smile in the first 20 weeks. The RMSEs of all the SVJ models are rather uniform over the rest of the sample period.

The Diebold–Mariano statistics of squared percentage pricing errors of individual difference caps in Panel B of Table V show that SVJ2 significantly improves the performance of SVJ1 for long-, mid-, and short-term around-the-money difference caps. The Diebold–Mariano statistics in Panel C of Table V show that SVJ3 significantly reduces the pricing errors of SVJ2 for long-term ITM, and some mid- and short-term around-the-money difference caps.

The average percentage pricing errors in Table VI show that the SVJ models capture the volatility smile much better than the SV models do. Panels A, B, and C of Table VI show that the three SVJ models have smaller average percentage pricing errors than the corresponding SV models for most difference caps. In particular, the degree of mispricing of ITM and OTM difference caps is greatly weakened. For example, Panel C of Table III shows that for many

Table V

Comparison of the Performance of Stochastic Volatility and Jump Models

This table reports model comparison based on likelihood ratio and Diebold–Mariano statistics. The total number of observations (both cross-sectional and time series), which equals 8,545 over the entire sample, times the difference between the logarithms of the SSEs between two models follows a χ^2 distribution asymptotically. We treat implied volatility variables as parameters. Thus, the degree of freedom of the χ^2 distribution is 168 for the SVJ2-SVJ1 and SVJ3-SVJ2 pairs, because SVJ2 (SVJ3) has four more parameters and 164 additional implied volatility variables than SVJ1 (SVJ2). The 1% critical value of $\chi^2(168)$ is 214. The Diebold–Mariano statistics are calculated according to equation (22) with a lag order q of 40 and thus follow an asymptotic standard Normal distribution under the null hypothesis of equal pricing errors. A negative statistic means that the more sophisticated model has smaller pricing errors. Bold entries mean that the statistics are significant at the 5% level.

Panel A: Likelihood Ratio and Diebold–Mariano Statistics for Overall Model Performance Based on SSEs

Model Pairs	D–M Stats	Likelihood Ratio Stats $\chi^2(168)$
SVJ2-SVJ1	**−2.240**	**1886**
SVJ3-SVJ2	**−7.149**	**1256**

Moneyness	1.5 Yr	2 Yr	2.5 Yr	3 Yr	3.5 Yr	4 Yr	4.5 Yr	5 Yr	6 Yr	7 Yr	8 Yr	9 Yr	10 Yr
Panel B: Diebold–Mariano Statistics between SVJ2 and SVJ1 for Individual Difference Caps Based on Squared Percentage Pricing Errors													
0.7	–	–	–	–	−0.234	−0.308	−1.198	−0.467	−0.188	0.675	−0.240	−0.774	−0.180
0.8	–	–	−1.346	−0.537	−0.684	−1.031	−1.892	−1.372	−0.684	−0.365	−0.749	−1.837	−1.169
0.9	–	−1.530	−1.914	−0.934	−1.036	−1.463	**−3.882**	**−3.253**	−0.920	−1.588	**−2.395**	**−3.287**	−0.686
1.0	−0.094	**−3.300**	**−2.472**	−1.265	−1.358	−1.647	**−2.186**	**−2.020**	−0.573	−1.674	−1.396	**−2.540**	−0.799
1.1	−1.395	**−5.341**	−0.775	0.156	−0.931	**−3.141**	**−3.000**	**−2.107**	–	–	–	–	–
Panel C: Diebold–Mariano Statistics between SVJ3 and SVJ2 for Individual Difference Caps Based on Squared Percentage Pricing Errors													
0.7	–	–	–	–	0.551	0.690	−1.023	−1.023	−1.133	**−2.550**	−1.469	−0.605	−1.920
0.8	–	–	−1.036	−0.159	1.650	1.609	−1.714	−1.898	−0.778	**−3.191**	**−3.992**	**−2.951**	**−3.778**
0.9	–	−1.235	**−2.057**	−0.328	2.108	1.183	**−2.011**	−1.361	−0.249	**−2.784**	−1.408	**−3.411**	**−2.994**
1.0	−1.594	−1.245	**−2.047**	−0.553	−0.289	−0.463	**−2.488**	−1.317	2.780	0.182	−0.551	−1.542	−1.207
1.1	−0.877	−1.583	−0.365	−0.334	−1.088	−2.040	**−3.302**	−1.259	–	–	–	–	–

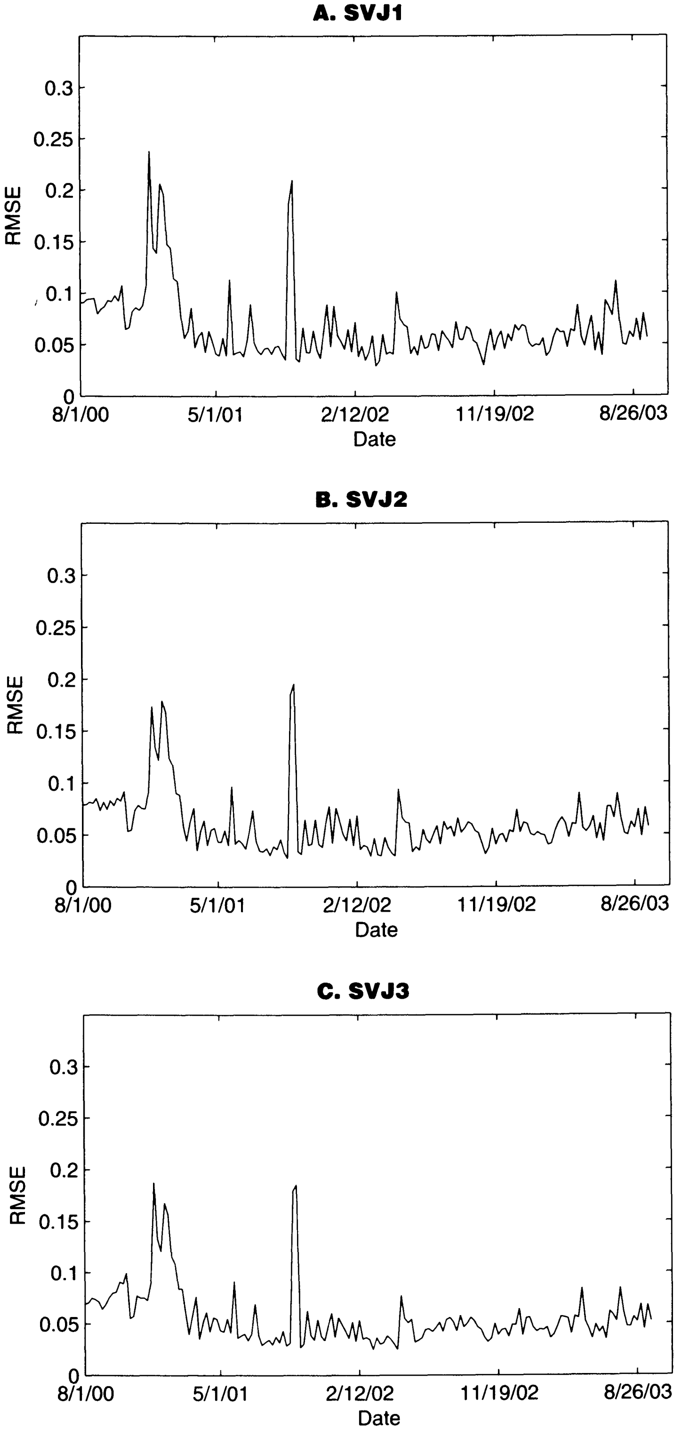


Figure 8. The RMSEs from SVJ models between August 1, 2000 and September 23, 2003.

Table VI

Average Percentage Pricing Errors of Stochastic Volatility and Jump Models

This table reports average percentage pricing errors of difference caps with different moneyness and maturities of the three stochastic volatility and jump models. Average percentage pricing errors are defined as the difference between market price and model price divided by market price.

Moneyness	1.5 Yr	2 Yr	2.5 Yr	3 Yr	3.5 Yr	4 Yr	4.5 Yr	5 Yr	6 Yr	7 Yr	8 Yr	9 Yr	10 Yr
Panel A: Average Percentage Pricing Errors of SVJ1													
0.7	–	–	–	–	0.0164	0.0073	−0.0092	0.01	0.0102	0.0209	−0.0001	−0.0061	0.0077
0.8	–	–	0.014	0.0167	0.0116	−0.0014	−0.0091	0.0111	0.007	0.0207	−0.0009	−0.0076	0.0053
0.9	–	0.0682	0.0146	0.0132	0.0112	−0.0035	−0.0103	0.0104	0.0038	0.0204	−0.0062	−0.0114	0.0042
1.0	−0.009	0.0839	0.0233	0.016	0.0158	−0.0004	−0.0105	0.0105	0.0062	0.0194	0.0013	−0.0083	0.0094
1.1	−0.098	0.0625	−0.0038	−0.0144	−0.0086	−0.0255	−0.0199	0.0094	–	–	–	–	–
Panel B: Average Percentage Pricing Errors of SVJ2													
0.7	–	–	–	–	0.0243	0.0148	−0.0008	0.0188	0.0175	0.0279	0.0116	0.0106	0.0256
0.8	–	–	0.0232	0.0271	0.0211	0.0062	−0.0035	0.0172	0.0137	0.0255	0.0081	0.0061	0.0139
0.9	–	0.0698	0.019	0.0205	0.0172	−0.0012	−0.0119	0.0068	0.0039	0.0198	−0.0041	−0.0047	−0.002
1.0	−0.0375	0.0668	0.013	0.0131	0.015	−0.0058	−0.0214	−0.0047	−0.0054	0.0127	−0.0058	−0.0112	−0.0128
1.1	−0.089	0.0612	−0.0048	−0.0094	0.0003	−0.0215	−0.0273	−0.0076	–	–	–	–	–
Panel C: Average Percentage Pricing Errors of SVJ3													
0.7	–	–	–	–	0.0261	0.0176	0.0008	0.017	0.0085	0.0167	0.0008	−0.0049	−0.0021
0.8	–	–	0.0222	0.0249	0.0223	0.0115	0.0027	0.0185	0.0016	0.0131	0.004	−0.0008	−0.0063
0.9	–	0.0713	0.014	0.0155	0.0182	0.0073	−0.0002	0.0129	−0.0108	0.0072	0.0044	0.0048	−0.0092
1.0	−0.0204	0.0657	0.005	0.0054	0.0142	0.0033	−0.0068	0.0047	−0.0232	−0.001	0.019	0.0206	−0.0058
1.1	−0.0688	0.0528	−0.02	−0.0242	−0.0085	−0.0199	−0.0182	−0.0028	–	–	–	–	–

difference caps, the average percentage pricing errors under SVJ3 are less than 1%, demonstrating that the model can successfully capture the smile.

Table VII compares the performance of the SVJ and SV models. As we showed before, during the first 20 weeks of our sample the SVJ models have much higher RMSEs than do the SV models. As a result, the likelihood ratio and Diebold–Mariano statistics between the three SVJ-SV model pairs over the entire sample are somewhat smaller than those of the sample period without the first 20 weeks. Nonetheless, all the SV models are overwhelmingly rejected in favor of their corresponding SVJ models by both tests. The Diebold–Mariano statistics of individual difference caps in Panels B, C, and D show that the SVJ models significantly improve the performance of the SV models for most difference caps across moneyness and maturity. The most interesting results are in Panel D, which shows that SVJ3 significantly reduces the pricing errors of most ITM difference caps of SV3, strongly suggesting that the negative jumps are essential for capturing the asymmetric smile in the cap market.

Our analysis demonstrates that a low-dimensional model with (1) three principal components driving the forward rate curve, (2) stochastic volatility of each component, and (3) strong negative jumps captures the volatility smile in the cap markets reasonably well. The three yield factors capture the variation of the levels of LIBOR rates, while the stochastic volatility factors are essential to capture the time-varying volatilities of LIBOR rates. Even though the SV models can price ATM caps reasonably well, they fail to capture the volatility smile in the cap market. Instead, significant negative jumps in LIBOR rates are needed to capture the smile. These results highlight the importance of studying the pricing of caps across moneyness: The importance of negative jumps is revealed only through the pricing of away-from-the-money caps. Excluding the first 20 weeks and the two special periods, SVJ3 has an average RMSE of 4.5%. Given that the bid–ask spread is about 2% to 5% in our sample for ATM caps, and because ITM and OTM caps tend to have even higher percentage spreads (see, e.g., Deuskar et al. (2003)), this can be interpreted as good pricing performance.

Despite this good performance, there are strong indications that SVJ3 is misspecified and that the inadequacies of the model seem to be related to MBS markets. For example, though SVJ3 works reasonably well for most of the sample period, it has large pricing errors during several periods that coincide with high prepayment activities in the MBS markets.

Moreover, even though we assume that the stochastic volatility factors are independent of LIBOR rates, Table VIII reports strong negative correlations between the implied volatility variables of the first factor and the LIBOR rates. Given that existing empirical studies do not find a strong "leverage" effect for interest rates under the physical measure, the negative correlations in Table VIII are evidence of model misspecification. This result suggests that when interest rates are low, cap prices become too high for the model to capture and the implied volatilities would have to become abnormally high to fit the observed cap prices. One possible explanation of the leverage effect is that higher demand for caps to hedge prepayments from MBS markets in low

Table VII

Comparison of the Performance of SV and SVJ Models

This table reports model comparison based on likelihood ratio and Diebold–Mariano statistics. The total number of observations (both cross-sectional and time series), which equals 8,545 over the entire sample and 7,485 excluding the first 20 weeks, times the difference between the logarithms of the SSEs between two models follows a χ^2 distribution asymptotically. We treat implied volatility variables as parameters. Thus, the degree of freedom of the χ^2 distribution is four for the SVJ-SV1, SVJ2-SV2, and SVJ3-SV3 pairs, because SVJ models have four more parameters than and an equal number of additional implied volatility variables as the corresponding SV models. The 1% critical value of $\chi^2(4)$ is 13. The Diebold–Mariano statistics are calculated according to equation (22) with a lag order q of 40 and follow an asymptotic standard Normal distribution under the null hypothesis of equal pricing errors. A negative statistic means that the more sophisticated model has smaller pricing errors. Bold entries mean that the statistics are significant at the 5% level.

Panel A: Likelihood Ratio and Diebold–Mariano Statistics for Overall Model Performance Based on SSEs

Model Pairs	D–M Stats (Whole Sample)	D–M Stats (Without First 20 Weeks)	Likelihood Ratio Stats $\chi^2(4)$ (Whole Sample)	Likelihood Ratio Stats $\chi^2(4)$ (Without First 20 Weeks)
SVJ1-SV1	**−2.972**	**−3.006**	**1854**	**2437**
SVJ2-SV2	**−3.580**	**−4.017**	**2115**	**2688**
SVJ3-SV3	**−3.078**	**−3.165**	**1814**	**2497**

Panel B: Diebold–Mariano Statistics between SVJ1 and SV1 for Individual Difference Caps Based on Squared Percentage Pricing Errors (without First 20 Weeks)

Moneyness	1.5 Yr	2 Yr	2.5 Yr	3 Yr	3.5 Yr	4 Yr	4.5 Yr	5 Yr	6 Yr	7 Yr	8 Yr	9 Yr	10 Yr
0.7	–	–	–	–	**−3.050**	**−3.504**	−0.504	−1.904	**−4.950**	**−3.506**	**−3.827**	**−2.068**	**−2.182**
0.8	–	–	**−3.243**	**−12.68**	**−9.171**	−1.520	−0.692	−1.195	**−2.245**	**−1.986**	−1.920	−1.353	−1.406
0.9	–	**−7.162**	**−9.488**	**−2.773**	−1.948	−1.030	−1.087	−0.923	−0.820	−0.934	−1.176	−1.109	−1.166
1.0	0.670	**−5.478**	**−2.844**	−1.294	−1.036	**−4.001**	**−3.204**	−1.812	**−2.331**	−1.099	−1.699	**−2.151**	**−2.237**
1.1	−0.927	−0.435	−0.111	−0.261	−1.870	**−2.350**	−1.710	−0.892	–	–	–	–	–

(continued)

Table VII—*Continued*

Moneyness	1.5 Yr	2 Yr	2.5 Yr	3 Yr	3.5 Yr	4 Yr	4.5 Yr	5 Yr	6 Yr	7 Yr	8 Yr	9 Yr	10 Yr
Panel C: Diebold–Mariano Statistics between SVJ2 and SV2 for Individual Difference Caps Based on Squared Percentage Pricing Errors (without First 20 Weeks)													
0.7	–	–	–	–	**−9.696**	**−7.714**	**−2.879**	**−3.914**	**−14.01**	**−7.387**	**−7.865**	**−3.248**	−1.637
0.8	–	–	**−7.353**	**−5.908**	**−5.612**	**−2.917**	−1.669	**−2.277**	**−7.591**	**−4.610**	**−4.182**	0.397	2.377
0.9	–	**−7.013**	**−6.271**	**−2.446**	**−2.145**	−1.246	−1.047	−1.309	**−2.856**	−1.867	−0.183	0.239	3.098
1.0	1.057	**−6.025**	**−2.736**	−1.159	−0.688	**−4.478**	**−4.410**	**−3.754**	−0.404	−0.416	−0.881	**−2.504**	0.023
1.1	−0.969	−0.308	0.441	−1.284	**−2.179**	**−3.148**	**−2.874**	**−2.267**	–	–	–	–	–
Panel D: Diebold–Mariano Statistics between SVJ3 and SV3 for Individual Difference Caps Based on Squared Percentage Pricing Errors (without First 20 Weeks)													
0.7	–	–	–	–	**−7.040**	**−8.358**	**−7.687**	**−10.49**	**−7.750**	**−5.817**	**−5.140**	**−3.433**	**−3.073**
0.8	–	–	**−4.732**	**−7.373**	**21.74**	**−7.655**	**−5.145**	**−4.774**	**−6.711**	**−3.030**	**−2.650**	**−2.614**	−1.239
0.9	–	**−1.980**	**−2.571**	**−2.501**	**−6.715**	**−3.622**	**−1.985**	**−2.384**	−1.938	−1.114	−0.768	**−4.119**	−1.305
1.0	−0.530	−1.124	−1.305	−1.353	−1.909	−0.880	1.023	0.052	0.543	**−2.110**	−0.359	−0.492	**−2.417**
1.1	−1.178	1.395	−1.424	**−2.218**	−1.834	**−2.151**	−1.537	−0.337	–	–	–	–	–

Table VIII

Correlations between LIBOR Rates and Implied Volatility Variables

This table reports the correlations between LIBOR rates and implied volatility variables from SVJ3. Given the parameter estimates of SVJ3 in Table IV, the implied volatility variables are estimated at t by minimizing the SSEs of all difference caps at t.

	L(t,1)	L(t,3)	L(t,5)	L(t,7)	L(t,9)	$V_1(t)$	$V_2(t)$	$V_3(t)$
$V_1(t)$	−0.8883	−0.8772	−0.8361	−0.7964	−0.7470	1	−0.4163	0.3842
$V_2(t)$	0.1759	0.235	0.2071	0.1545	0.08278	−0.4163	1	−0.0372
$V_3(t)$	−0.5951	−0.485	−0.4139	−0.3541	−0.3262	0.3842	−0.0372	1

interest rate environments could artificially push cap prices and implied volatilities up. Therefore, extending our models to incorporate factors from MBS markets seems to be a promising direction for future research.

IV. Conclusion

In this paper, we make significant theoretical and empirical contributions to the fast-growing literature on LIBOR and swap-based interest rate derivatives. Theoretically, we develop a multifactor HJM model that explicitly takes into account the new empirical features of term structure data, namely, unspanned stochastic volatility and jumps. Our model provides a closed-form formula for cap prices, which greatly simplifies empirical implementation of the model. Empirically, we provide one of the first comprehensive analyses of the relative pricing of caps with different moneyness. Using a comprehensive data set that consists of 3 years of cap prices with different strike and maturity, we document a volatility smile in the cap market. Although previous studies show that multifactor stochastic volatility models can price ATM caps and swaptions well, we find that they fail to capture the volatility smile in the cap market, and that a three-factor model with stochastic volatility and significant negative jumps is needed to capture the smile. Our results show, indeed, that the volatility smile contains new information that is not available in ATM caps.

Note that our paper is only one of the first attempts to explain the volatility smile in OTC interest rate derivatives markets. Thus, although our model exhibits reasonably good performance, there are several aspects of the model that are not completely satisfactory. Given that the volatility smile has guided the development of a rich equity option pricing literature since Black and Scholes (1973) and Merton (1973), we hope that the volatility smile documented here will help further similar development of term structure models in the years to come.

Appendix

The solution to the characteristic function of $\log(L_k(T_k))$,

$$\psi(u_0, Y_t, t, T_k) = \exp[a(s) + u_0 \log(L_k(t)) + B(s)'V_t], \tag{A1}$$

$a(s)$, and $B(s)$, $0 \leq s \leq T_k$, satisfies the following system of Ricatti equations:

$$\frac{dB_j(s)}{ds} = -\kappa_j^{k+1} B_j(s) + \frac{1}{2} B_j^2(s)\xi_j^2 + \frac{1}{2}[u_0^2 - u_0] U_{s,j}^2, \quad 1 \leq j \leq N, \tag{A2}$$

$$\frac{da(s)}{ds} = \sum_{j=1}^{N} \kappa_j^{k+1} \theta_j^{k+1} B_j(s) + \lambda_J [\Gamma(u_0) - 1 - u_0(\Gamma(1) - 1)], \tag{A3}$$

where the function Γ is

$$\Gamma(x) = \exp\left(\mu_J^{k+1} x + \frac{1}{2}\sigma_J^2 x^2\right). \tag{A4}$$

The initial conditions are $B(0) = 0_{N\times 1}$ and $a(0) = 0$, and κ_j^{k+1} and θ_j^{k+1} are the parameters of the $V_j(t)$ process under $\mathbb{Q}^{k+1}$.

For any $l < k$, given that $B(T_l) = B_0$ and $a(T_l) = a_0$, we have the closed-form solutions for $B(T_{l+1})$ and $a(T_{l+1})$. Define constants $p = [u_0 - u_0^2] U_{s,j}^2$, $q = \sqrt{(\kappa_j^{k+1})^2 + p\xi_j^2}$, $c = \frac{p}{q - \kappa_j^{k+1}}$, and $d = \frac{p}{q + \kappa_j^{k+1}}$. Then we have

$$B_j(T_{l+1}) = c - \frac{(c+d)(c - B_{j0})}{(d + B_{j0})\exp(-q\delta) + (c - B_{j0})}, \quad 1 \leq j \leq N, \tag{A5}$$

$$a(T_{l+1}) = a_0 - \sum_{j=1}^{N} \left[\kappa_j^{k+1}\theta_j^{k+1}\left(d\delta + \frac{2}{\xi_j^2}\ln\left(\frac{(d + B_{j0})\exp(-q\delta) + (c - B_{j0})}{c+d}\right)\right)\right]$$
$$+ \lambda_J \delta[\Gamma(u_0) - 1 - u_0(\Gamma(1) - 1)]. \tag{A6}$$

Finally, $B(T_k)$ and $a(T_k)$ can be computed via iteration.

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