Applications of Malliavin Calculus to Monte-Carlo Methods in Finance

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Outline

Motivation

In finance, the demand of hedging financial instruments, especially (exotic) options is based on the calculation of Greeks

- explicit formula (very simple examples, e.g. European Call);
- Monte Carlo Simulation the finite difference approximation of the differentials
  - perform very poorly when pay-off function is not smooth enough
  - hard to handle complex pay-off function case with accuracy
  - inadequacy for the treatment of American-type options (part II)

Some Mathematical Notations...

The underlying assets are assumed to be given by $\{X(t); 0 \leq t \leq T\}$, whose dynamic are described by stochastic differential equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t)$$

where $\{W(t), 0 \leq t \leq T\}$ is a Brownian motion with values in $\mathbb{R}^n$.

Given $0 \leq t_1 \leq \ldots \leq t_m = T$, we consider the function

$$u(x) = \mathbb{E}[\phi(X(t_1), \ldots, X(t_m))|X(0) = x]$$

where $u(x)$ describes the prices of a contingent claim defined by the pay-off function ϕ involving the times $(t_1, \ldots, t_m)$.

Main findings and contributions of this paper

In this paper, using Malliavin calculus we will show that all the differentials of interest can be expressed as

$$\mathbb{E}[\pi\phi(X(t_1),\ldots,X(t_m))|X(0)=x]$$

where the weight π does not depend on the pay-off function ϕ.

Some simplified theoretical aspects

The asset prices can be written

$$price = \mathbb{E}_{Q_0}[\text{pay-offs}]$$

where $\mathbb{E}_{Q_0}$ is the expected value under the risk neutral probability Q_0.
The marginal changes of Q will lead to new prices according to

$$\begin{aligned} variation\ of\ prices &= new\ prices - old\ prices \\ &= \mathbb{E}_{Q}[\text{pay-offs}] - \mathbb{E}_{Q_0}[\text{pay-offs}] \\ &= \mathbb{E}_{Q_0}[\text{pay-offs} \times \pi] \end{aligned}$$

where

$$\pi = \frac{dQ - dQ_0}{dQ_0}$$

Some simplified theoretical aspects, con't

Suppose that the probability Q lies within a parametrized family (Q_λ). $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the marginal moves of the market can be assessed through the derivatives

$$\frac{\partial}{\partial \lambda_i}(price) = \mathbb{E}_{Q_0}[\text{pay-offs} \times \pi_i]$$

where $G = \frac{dQ}{dQ_0}$ and $\pi_i = \frac{\partial G}{\partial \lambda_i}$, i.e. π_i is the logarithmic derivative of Q at Q_0 in the λ_i direction.

Notations

Let $\{W(t), 0 \leq t \leq T\}$ be a n-dimensional Brownian motion defined on a complete probability space $(\omega, \mathcal{F}, P)$. Let $\mathcal{L}$ be the set of r.v. F of the form

$$F = f\left(\int_0^\infty h_1(t)dW(t), \ldots, \int_0^\infty h_n(t)dW(t)\right), \quad f \in \mathcal{P}(\mathbb{R}^n)$$

where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of infinitely differentiable and rapidly decreasing function on $\mathbb{R}^n$ and $h_1, \ldots, h_n \in L^2(\omega \times \mathbb{R}_+)$. The Malliavin derivative DF of F is defined as the process $\{D_t F, t \geq 0\}$ of $L^2(\omega \times \mathbb{R}_+)$ with values in $L^2(\mathbb{R}_+)$ which we denote by H

$$D_t F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\int_0^\infty h_1(t)dW(t), \ldots, \int_0^\infty h_n(t)dW(t)\right) h_t(t), \quad t \geq 0 \, a.s.$$

Properties of Malliavin Calculus

Property 1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with partial derivatives and $F = (F_1, \dots, F_n)$ a random vector whose components belong to $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and

$$D_t \phi(F) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F) D_t F_i, \qquad t \geq 0 \ a.s.$$

Properties of Malliavin Calculus,con't

Property 2. Let $\{Y(t), t \geq 0\}$ be the associated first variation process of $X(t)$ and

$$dY(t) = b'(X(t))Y(t)dt + \sum_{i=1}^{n} \sigma_i'(X(t))Y(t)dW^i(t), \quad Y(0) = I_n$$

where I_n is the identity matrix of $\mathbb{R}^n$, primes denote derivatives and σ_i is the i-th column vector of σ. Then the process $\{X(t), t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by

$$D_s X(t) = Y(t)Y^{-1}(s)\sigma(X(s))1_{\{s \leq t\}}, \quad s \geq 0 \; a.s.$$

Hence, if $\psi \in C_b^1(\mathbb{R}^n)$ then we have

$$\begin{aligned} D_s \psi(X_T) &= \nabla\psi(X_T)Y(T)Y^{-1}(s)\sigma(X(s))1_{\{s \leq T\}}, \quad s \geq 0 \; a.s. \\ D_s \int_0^\infty \psi(X_t)dt &= \int_s^T \nabla\psi(X_T)Y(T)Y^{-1}(s)\sigma(X(s))dt \quad a.s. \end{aligned}$$

Properties of Malliavin Calculus,con't

Property 3. Let u be a stochastic process. Then $u \in Dom(\delta)$ if for any $\phi \in \mathbb{D}^{1,2}$, we have

$$\mathbb{E}(< D\phi, u >_H) = \mathbb{E}\left(\int_0^\infty D_t\phi u(t)dt\right) \leq C(u)||\phi||_{1,2}$$

if $u \in Dom(\delta)$, we define $\delta(u)$ by

$$\mathbb{E}(\phi\delta(u)) = \mathbb{E}(< D\phi, u >_H) \quad for\ any\ \phi \in \mathbb{D}^{1,2}$$

Properties of Malliavin Calculus,con't

Property 4. Let u be an adapted stochastic process in $L^2(\omega \times \mathbb{R}_+)$. Then we have

$$\delta(u) = \int_0^\infty [u(t)]^* dW(t)$$

Property 5. Let F be an $\mathcal{F}_T-$adapted random variable which belongs to $\mathbb{D}^{1,2}$ then for any u in $dom(\delta)$ we have

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u(t) dt$$

Property 6. Let F be an a random variable which belongs to $\mathbb{D}^{1,2}$. Then we have

$$F = \mathbb{F} + \int_0^T \mathbb{E}(D_t F | \mathcal{F}_t) dW(t)$$

Greeks calculation: assumption

We denote by $\{Y(t), 0 \leq t \leq T\}$ the first variation process associated to $\{X(t), 0 \leq t \leq T\}$ defined by the stochastic differential equation

$$\begin{aligned} Y(0) &= I_n \\ dY(t) &= b'(X(t))Y(t)dt + \sum_{i=1}^{n} \sigma_i'(X(t))Y(t)dW_i(t) \end{aligned}$$

Assumption 3.1 The diffusion matrix satisfies the uniform ellipticity condition

$$\exists \epsilon > 0, \xi^* \sigma^*(x)\sigma(x)\xi \geq \epsilon |\xi|^2 \quad for\ any\ \xi, x \in \mathbb{R}^n$$

Variations in the drift coefficient and initial condition

Property 3.1. The function $\epsilon \mapsto u^\epsilon(x)$ is differentiable in $\epsilon = 0$, for any $x \in \mathbb{R}^n$, and we have

$$\frac{\partial}{\partial \epsilon} u^\epsilon(x)\bigg|_{\epsilon=0} = \mathbb{E}\left[\phi(X(.)) \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) > \bigg| X(0) = x\right]$$

Property 3.2. Under Assumption 3.1, for any $x \in \mathbb{R}^n$ and for any $a \in \Gamma_m$, we have

$$\nabla u(x) = \mathbb{E}^x\left[\phi(X(t_1), \ldots, X(t_m)) \int_0^T a(t)[\sigma^{-1}(X(t))(Y(t))]^* dW(t)\right]$$

Assumption 3.2. The diffusion matrix $\sigma + \epsilon\tilde{\sigma}$ satisfies the uniform ellipticity condition for any ϵ

$$\exists \eta > 0, \xi^*(\sigma + \epsilon\tilde{\sigma})^*(x)(\sigma + \epsilon\tilde{\sigma})(x)\xi \geq \eta|\xi|^2 \quad for\ any\ \xi, x \in \mathbb{R}^n$$

Property 3.3. Under Assumption 3.2, for any a in $\tilde{\Gamma}_m$, we have

$$\left.\frac{\partial}{\partial \epsilon} u^\epsilon(x)\right|_{\epsilon=0} = \mathbb{E}^x\left[\phi(X(t_1), \ldots, X(t_m))\delta(\sigma^{-1}(X)Y\tilde{\beta}_a(T))\right]$$

where

$$\tilde{\beta}_a(t) = \sum_{i=1}^{m} a(t)(\beta(t_i) - \beta(t_{i-1}))_-\{t_{i-1} \leq t \leq t_i\}$$

and where $\delta(\sigma^{-1}(X)Y\tilde{\beta}_a)(T)$ is the Skorohod integral of the anticipating process

$$\{\sigma^{-1}(X(t))Y(t)\tilde{\beta}_a(T); 0 \leq t \leq T\}$$

Greeks in European case

Consider the European type payoff function under Black-Scholes framework

$$\begin{aligned} rho_{\tilde{r}(t)} = \; & \mathbb{E}\left[e^{-\int_0^T r(t)dt}\phi(S_T)\int_0^T \frac{\tilde{r}(t)}{\sigma S_t}dW_t\right] \\ & -\mathbb{E}\left[\int_0^T \tilde{r}(t)dt e^{-\int_0^T r(t)dt}\phi(S_T)\right] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial x}(0,x) &= \mathbb{E}\left[e^{-\int_0^T r(t)dt}\phi(S_T)\frac{W_T}{x\sigma T}\right] \\ \frac{\partial^2 u}{\partial x^2}(0,x) &= \mathbb{E}\left[e^{-\int_0^T r(t)dt}\phi(S_T)\frac{1}{x^2\sigma T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right] \\ \frac{\partial u}{\partial \sigma}(0,x) &= \mathbb{E}\left[e^{-\int_0^T r(t)dt}\phi(S_T)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right] \end{aligned}$$

Greeks in Asian case

Consider the payoff function of the form $\phi(\int_0^T S_t dt)$ (Asian option)

$$\frac{\partial u}{\partial x}(0,x) = \mathbb{E}\left[e^{-\int_0^T r(t)dt}\phi(\int_0^T S_t dt)\left(\frac{2}{x\sigma}\frac{\int_0^T Y_t dW_t}{\int_0^T Y_t dt} + \frac{1}{x}\right)\right]$$

Consider the payoff function of the form $\phi(S_T, \int_0^T S_t dt)$ (Asian barrier in option)

$$\frac{\partial u}{\partial x}(0,x) = \mathbb{E}\left[e^{-\int_0^T r(t)dt}\phi\left(S_T, \int_0^T S_t dt\right)\delta(G)\right]$$

where G is the random process

$$G(s) = (a + \alpha s)\frac{Y_s}{\sigma S_s} + (b + \beta s)\frac{2Y_s^2}{\sigma S_s \int_0^T S_u du}$$

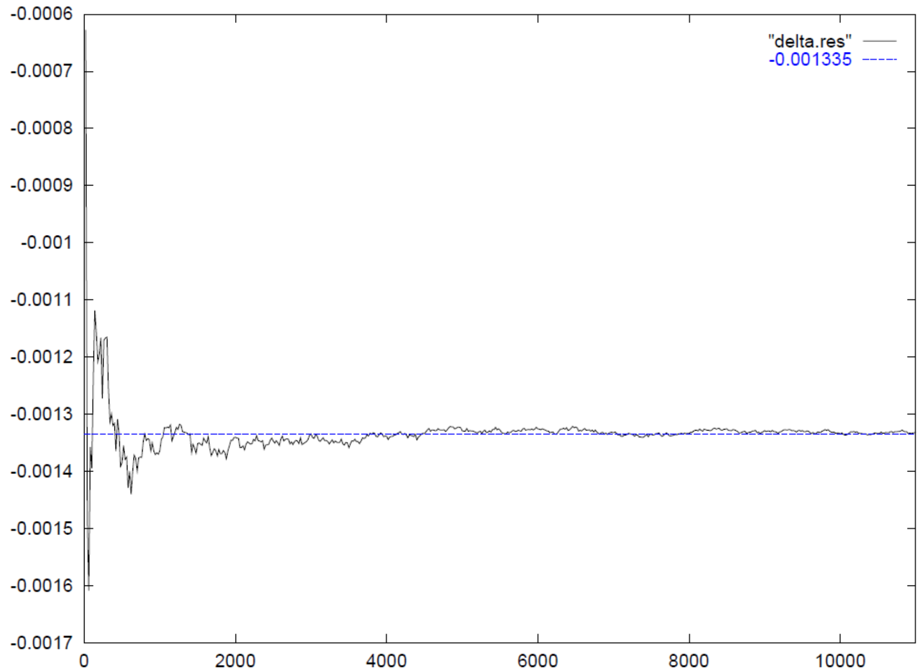

Fig. 1. Delta for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

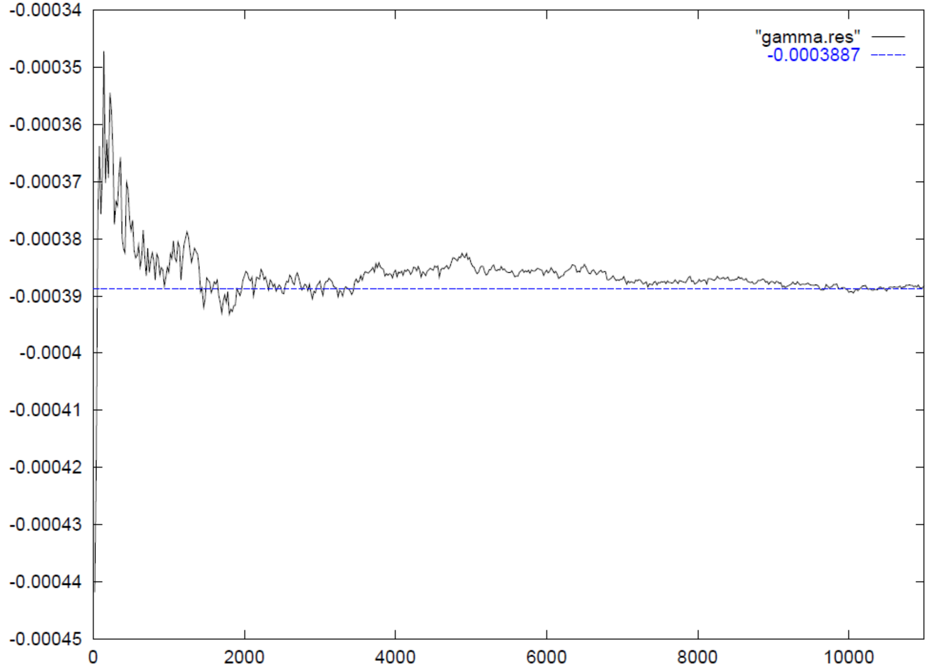

Fig. 2. Gamma for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

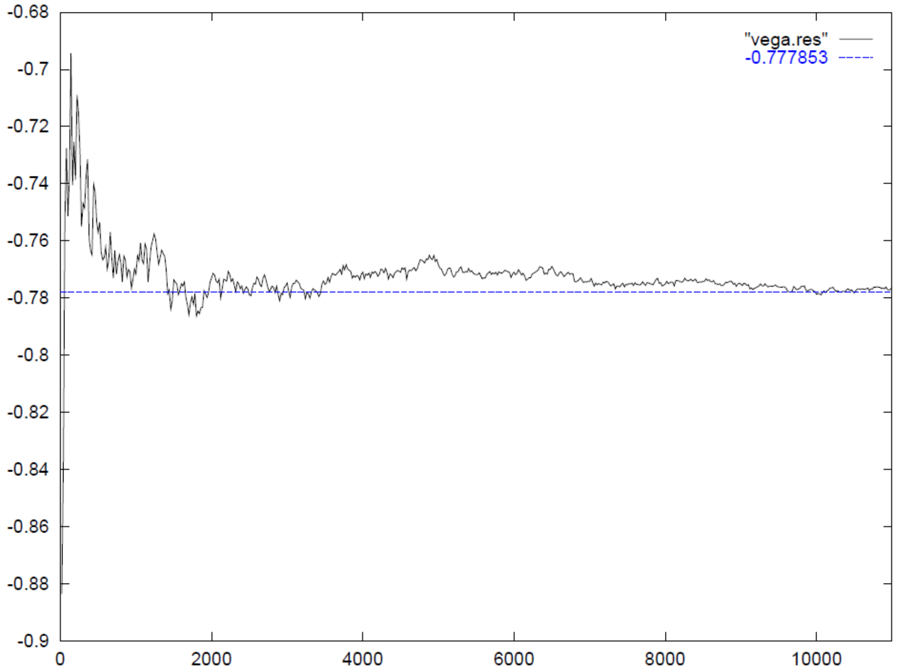

Fig. 3. Vega for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

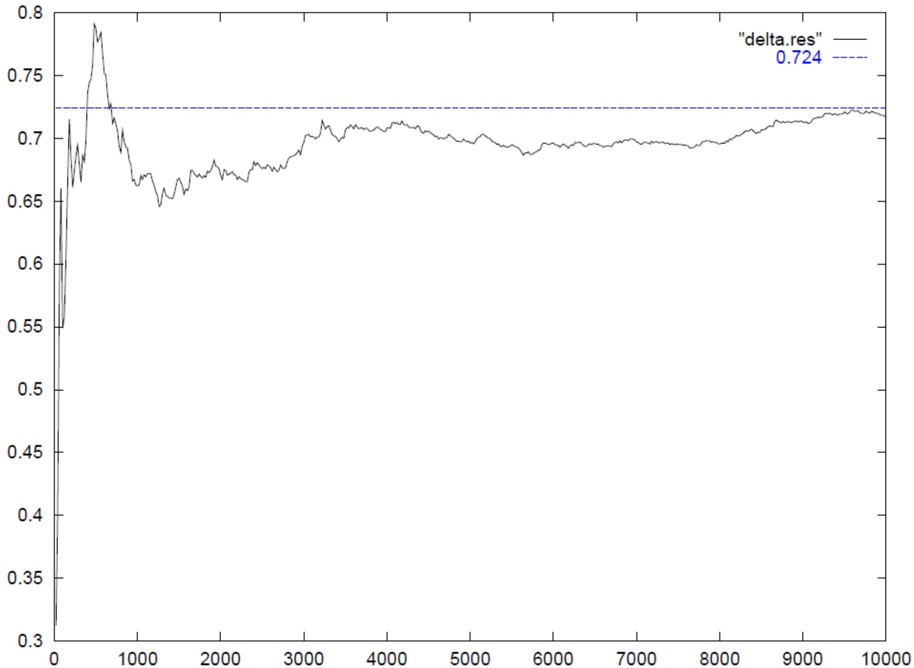

Fig. 4. Delta for a asian option with pay-off $(\int_0^T S_s\, ds - K)_+$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year, $K = 100$. We use standard Monte Carlo generation.

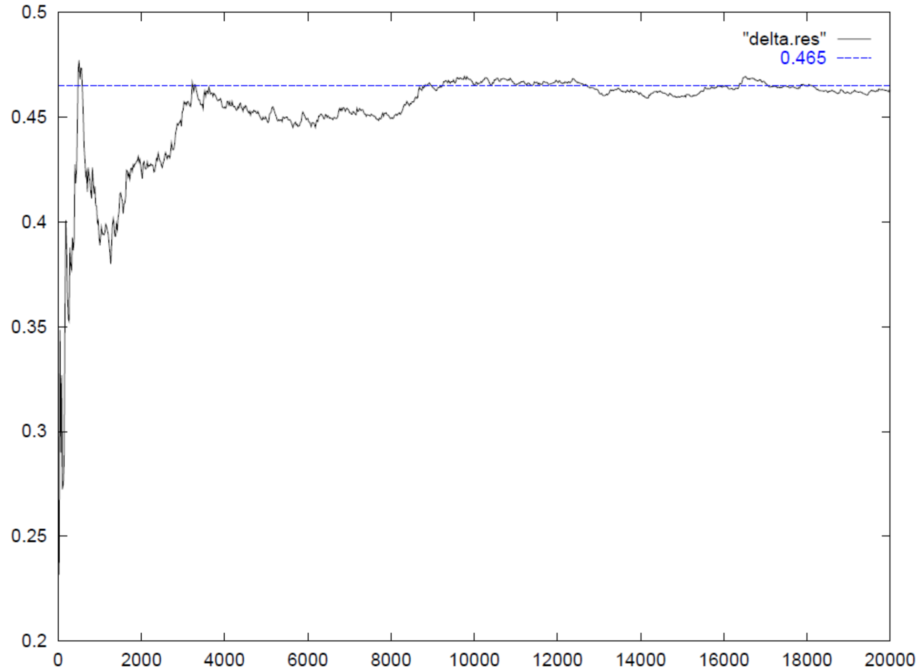

Fig. 5. Delta for a complex option with pay-off $1_{\{\int_0^1 W_s\,ds \geq B\}}(W_1 - K)_+$. We use standard Monte Carlo generation.

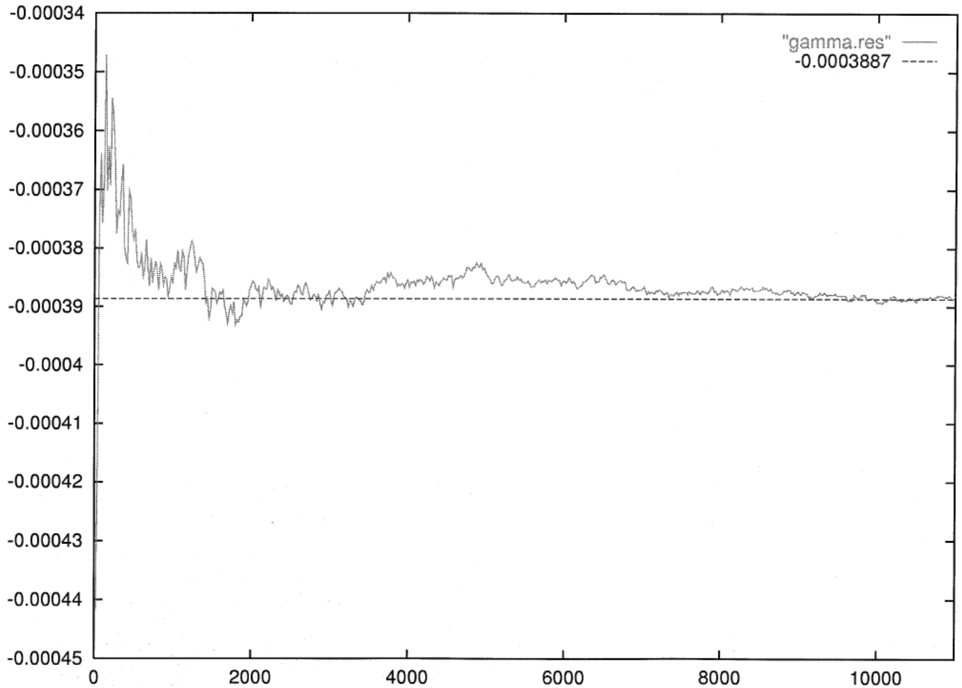

Fig. 6. Gamma of a call option computed by global and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ (localization parameter). We use low discrepency sequences.

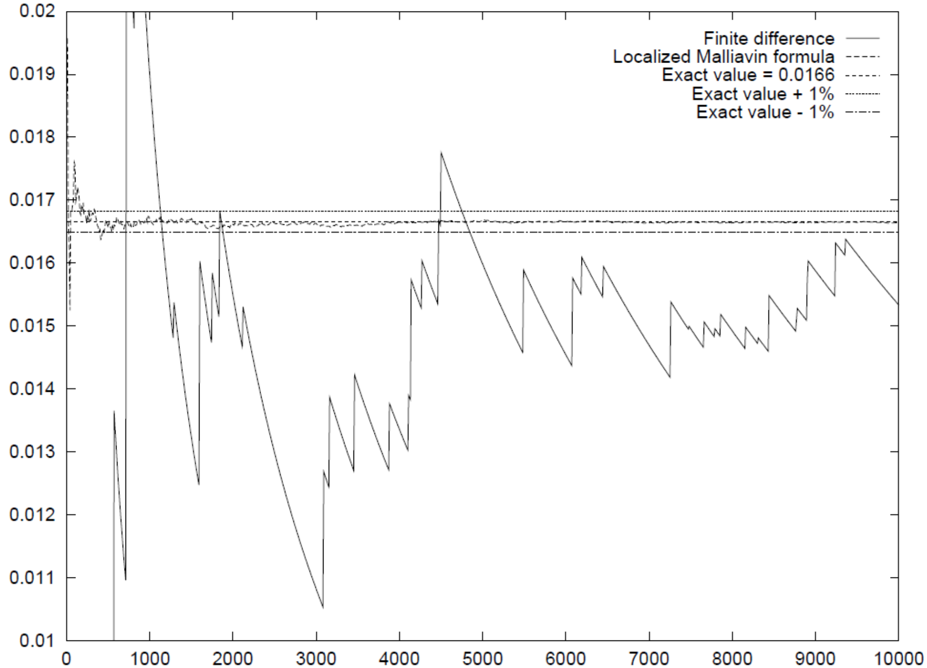

Fig. 7. Gamma of a call option computed by finite difference and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ (localization parameter). We use low discrepency sequences.

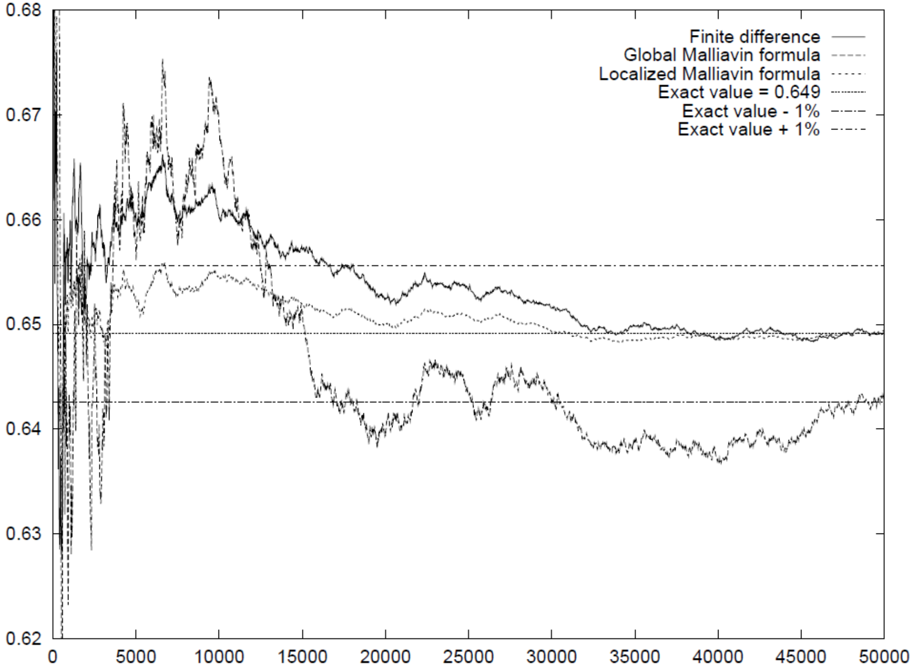

Fig. 8. Delta of an average call option computed by finite difference, global and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ for the localization parameter. We use pseudo random sequences.

Reference

1. Broadie, M., Glasserman, P.: Estimating security price derivatives using simulation, Manag. Sci. **42**, 269–285 (1996)
2. Friedman, A.: Stochastic Differential Equations and Applications, Vol. 1. New York: Academic Press 1975
3. Glasserman, P., Yao D.D.: Some guidelines and guarantees for common random numbers, Manag. Sci. **38**, 884–908 (1992)
4. Glynn, P.W.: Optimization of stochastic systems via simulation. In: Proceedings of the 1989 Winter simulation Conference. San Diego: Society for Computer Simulation 1989, pp. 90–105
5. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, Berlin Heidelberg New York: Springer 1988
6. Kloeden, P.E., Platen, E.: Numerical Solution of Stochastic Differential Equations, Berlin Heidelberg New York: Springer 1992
7. L'Ecuyer, P., Perron, G.: On the convergence rates of IPA and FDC derivative estimators, Oper. Res. **42**, 643–656 (1994)
8. Malliavin, P.: Stochastic Analysis. (Grundlehren der Mathematischen Wissenschaften, Bd. 313) Berlin Heidelberg New York: Springer 1997
9. Nualart, D.: Malliavin Calculus and Related Topics. (Probability and its Applications) Berlin Heidelberg New York: Springer 1995
10. Protter, P.: Stochastic Integration and Differential Equations. Berlin Heidelberg New York: Springer 1990
11. Üstünel, A.S.: An introduction to Analysis on Wiener Space. Berlin Heidelberg New York: Springer 1992
12. Watanabe, S.: Stochastic Differential Equations and Malliavin Calculus. Tata Institute of Fundamental Research, Bombay. Berlin Heidelberg New York: Springer 1984