Applications of Malliavin Calculus to Monte-Carlo Methods in Finance

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Outline

Motivation

In finance, the demand of hedging financial instruments, especially (exotic) options is based on the calculation of Greeks

- explicit formula (very simple examples, e.g. European Call);
- Monte Carlo Simulation the finite difference approximation of the differentials
  - perform very poorly when pay-off function is not smooth enough
  - hard to handle complex pay-off function case with accuracy
  - inadequacy for the treatment of American-type options (part II)

Some Mathematical Notations...

The underlying assets are assumed to be given by $\{X(t); 0 \leq t \leq T\}$, whose dynamic are described by stochastic differential equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t)$$

where $\{W(t), 0 \leq t \leq T\}$ is a Brownian motion with values in $\mathbb{R}^n$.

Given $0 \leq t_1 \leq \ldots \leq t_m = T$, we consider the function

$$u(x) = \mathbb{E}[\phi(X(t_1), \ldots, X(t_m))|X(0) = x]$$

where $u(x)$ describes the prices of a contingent claim defined by the pay-off function ϕ involving the times $(t_1, \ldots, t_m)$.

Main findings and contributions of this paper

In this paper, using Malliavin calculus we will show that all the differentials of interest can be expressed as

$$\mathbb{E}[\pi\phi(X(t_1),\ldots,X(t_m))|X(0)=x]$$

where the weight π does not depend on the pay-off function ϕ.

Some simplified theoretical aspects

The asset prices can be written

$$price = \mathbb{E}_{Q_0}[\text{pay-offs}]$$

where $\mathbb{E}_{Q_0}$ is the expected value under the risk neutral probability Q_0. The marginal changes of Q will lead to new prices according to

$$\begin{aligned} variation\ of\ prices &= new\ prices - old\ prices \\ &= \mathbb{E}_{Q}[\text{pay-offs}] - \mathbb{E}_{Q_0}[\text{pay-offs}] \\ &= \mathbb{E}_{Q_0}[\text{pay-offs} \times \pi] \end{aligned}$$

where

$$\pi = \frac{dQ - dQ_0}{dQ_0}$$

Some simplified theoretical aspects, con't

Suppose that the probability Q lies within a parametrized family (Q_λ). $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the marginal moves of the market can be assessed through the derivatives

$$\frac{\partial}{\partial \lambda_i}(price) = \mathbb{E}_{Q_0}[\text{pay-offs} \times \pi_i]$$

where $G = \frac{dQ}{dQ_0}$ and $\pi_i = \frac{\partial G}{\partial \lambda_i}$, i.e. π_i is the logarithmic derivative of Q at Q_0 in the λ_i direction.

Notations

Let $\{W(t), 0 \leq t \leq T\}$ be a n-dimensional Brownian motion defined on a complete probability space $(\omega, \mathcal{F}, P)$. Let $\mathcal{L}$ be the set of r.v. F of the form

$$F = f\left(\int_0^\infty h_1(t)dW(t), \ldots, \int_0^\infty h_n(t)dW(t)\right), \quad f \in \mathcal{P}(\mathbb{R}^n)$$

where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of infinitely differentiable and rapidly decreasing function on $\mathbb{R}^n$ and $h_1, \ldots, h_n \in L^2(\omega \times \mathbb{R}_+)$. The Malliavin derivative DF of F is defined as the process $\{D_t F, t \geq 0\}$ of $L^2(\omega \times \mathbb{R}_+)$ with values in $L^2(\mathbb{R}_+)$ which we denote by H

$$D_t F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\int_0^\infty h_1(t)dW(t), \ldots, \int_0^\infty h_n(t)dW(t)\right) h_t(t), \quad t \geq 0\, a.s.$$

Property 1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with partial derivatives and $F = (F_1, \dots, F_n)$ a random vector whose components belong to $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and

$$D_t \phi(F) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F) D_t F_i, \qquad t \geq 0 \; a.s.$$

Properties of Malliavin Calculus,con't

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Property 3. Let u be a stochastic process. Then $u \in Dom(\delta)$ if for any $\phi \in \mathbb{D}^{1,2}$, we have

$$\mathbb{E}(< D\phi, u >_H) = \mathbb{E}\left(\int_0^{\infty} D_t\phi u(t)dt\right) \leq C(u)||\phi||_{1,2}$$

if $u \in Dom(\delta)$, we define $\delta(u)$ by

$$\mathbb{E}(\phi\delta(u)) = \mathbb{E}(< D\phi, u >_H) \quad for\ any\ \phi \in \mathbb{D}^{1,2}$$

Properties of Malliavin Calculus,con't

Property 4. Let u be an adapted stochastic process in $L^2(\omega \times \mathbb{R}_+)$. Then we have

$$\delta(u) = \int_0^\infty [u(t)]^* dW(t)$$

Property 5. Let F be an $\mathcal{F}_T-$adapted random variable which belongs to $\mathbb{D}^{1,2}$ then for any u in $dom(\delta)$ we have

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u(t) dt$$

Property 6. Let F be an a random variable which belongs to $\mathbb{D}^{1,2}$. Then we have

$$F = \mathbb{F} + \int_0^T \mathbb{E}(D_t F | \mathcal{F}_t) dW(t)$$

Greeks calculation: assumption

We denote by $\{Y(t), 0 \leq t \leq T\}$ the first variation process associated to $\{X(t), 0 \leq t \leq T\}$ defined by the stochastic differential equation

$$\begin{aligned} Y(0) &= I_n \\ dY(t) &= b'(X(t))Y(t)dt + \sum_{i=1}^{n} \sigma_i'(X(t))Y(t)dW_i(t) \end{aligned}$$

Assumption 3.1 The diffusion matrix satisfies the uniform ellipticity condition

$$\exists \epsilon > 0, \xi^* \sigma^*(x) \sigma^(x) \xi \geq \epsilon |\xi|^2 \quad for\ any\ \xi, x \in \mathbb{R}^n$$

Variations in the drift coefficient and initial condition

Property 3.1. The function $\epsilon \mapsto u^\epsilon(x)$ is differentiable in $\epsilon = 0$, for any $x \in \mathbb{R}^n$, and we have

$$\frac{\partial}{\partial \epsilon} u^\epsilon(x)\bigg|_{\epsilon=0} = \mathbb{E}\left[\phi(X(.)) \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) > \bigg| X(0) = x\right]$$

Property 3.2. Under Assumption 3.1, for any $x \in \mathbb{R}^n$ and for any $a \in \Gamma_m$, we have

$$\nabla u(x) = \mathbb{E}^x\left[\phi(X(t_1), \ldots, X(t_m)) \int_0^T a(t)[\sigma^{-1}(X(t))(Y(t))]^* dW(t)\right]$$

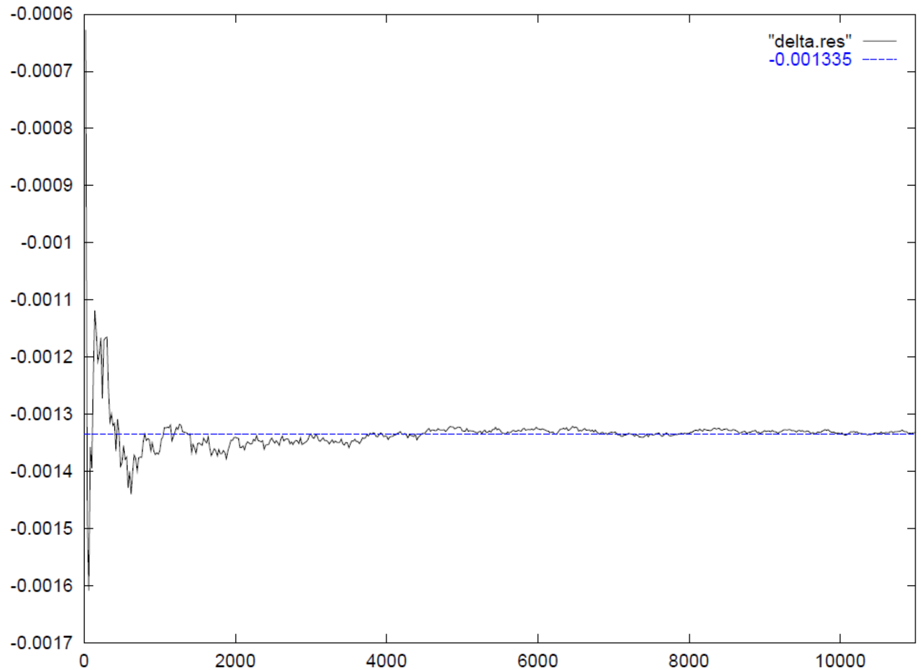

Fig. 1. Delta for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

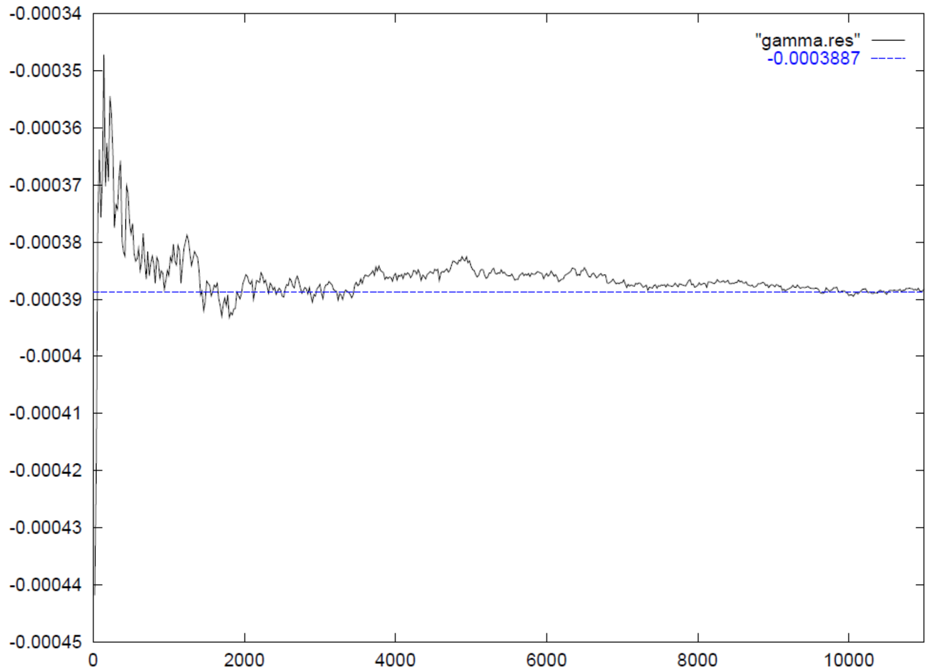

Fig. 2. Gamma for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

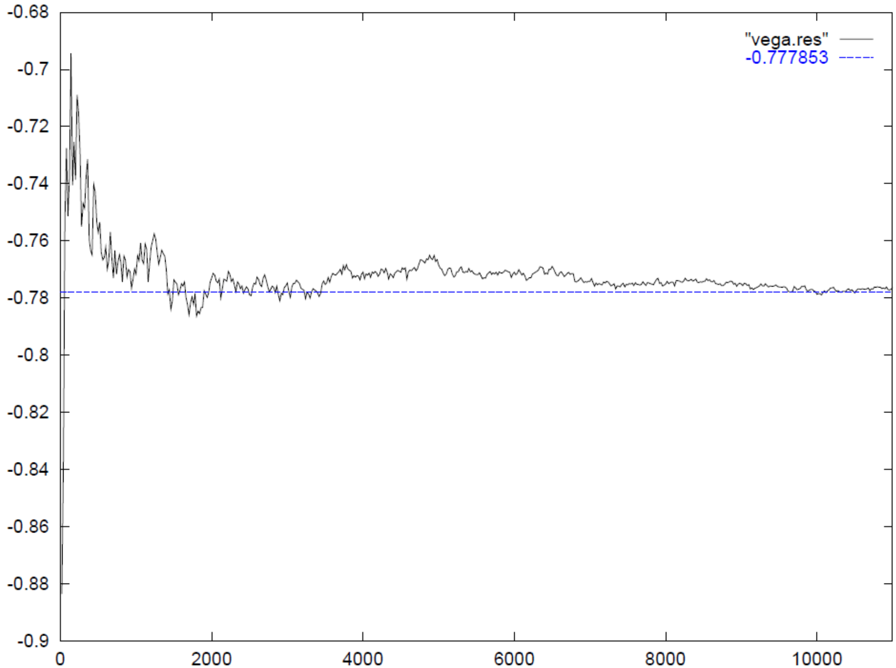

Fig. 3. Vega for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

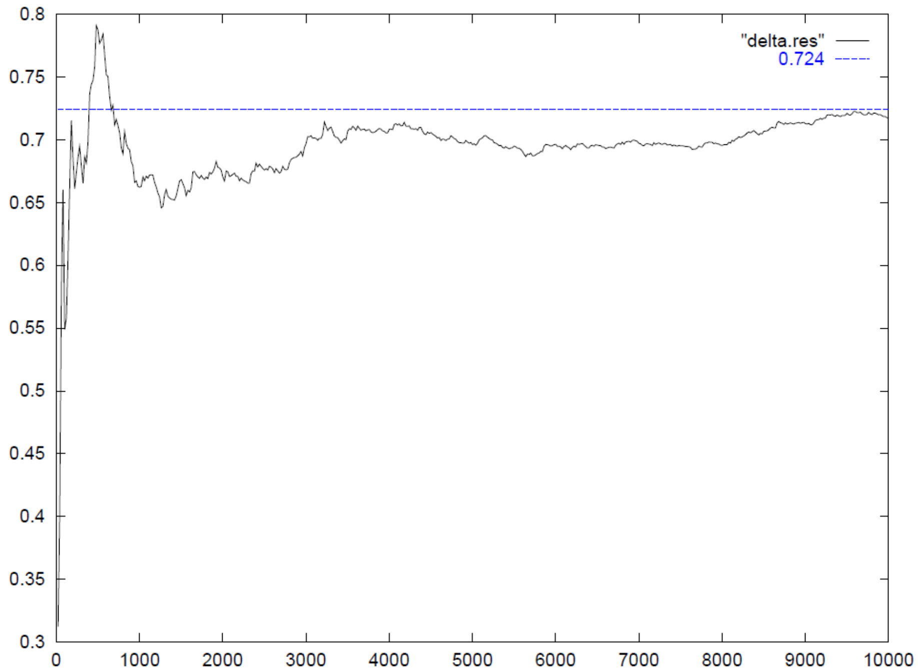

Fig. 4. Delta for a asian option with pay-off $(\int_0^T S_s\, ds - K)_+$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year, $K = 100$. We use standard Monte Carlo generation.

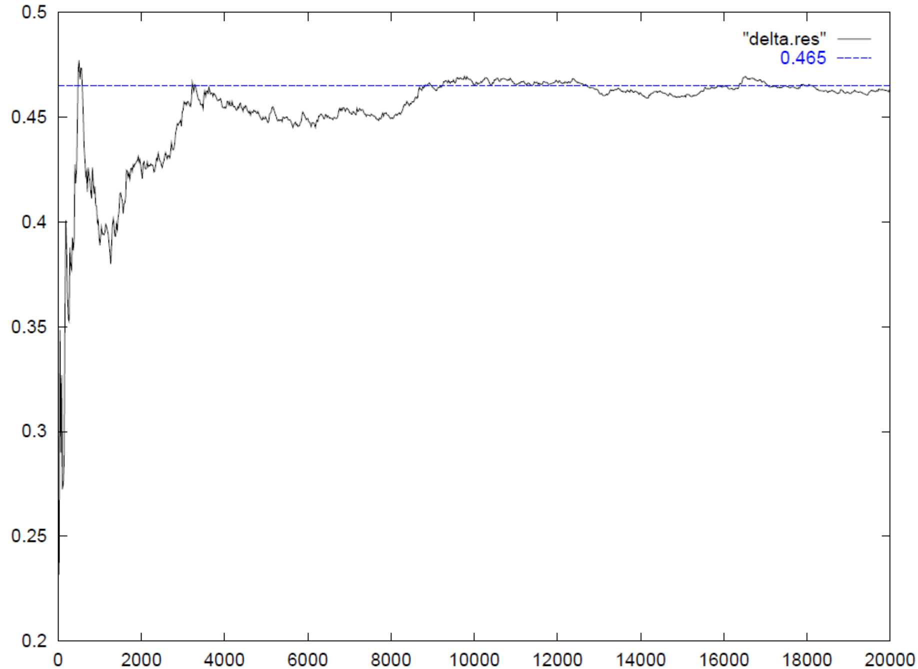

Fig. 5. Delta for a complex option with pay-off $1_{\{\int_0^1 W_s\, ds \geq B\}}(W_1 - K)_+$. We use standard Monte Carlo generation.

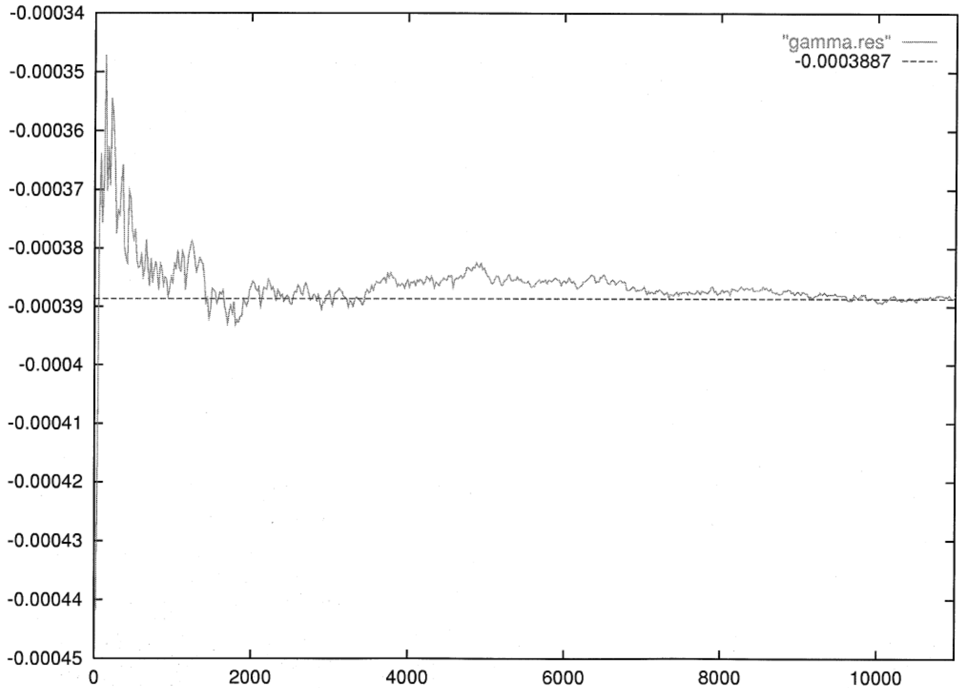

Fig. 6. Gamma of a call option computed by global and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ (localization parameter). We use low discrepency sequences.

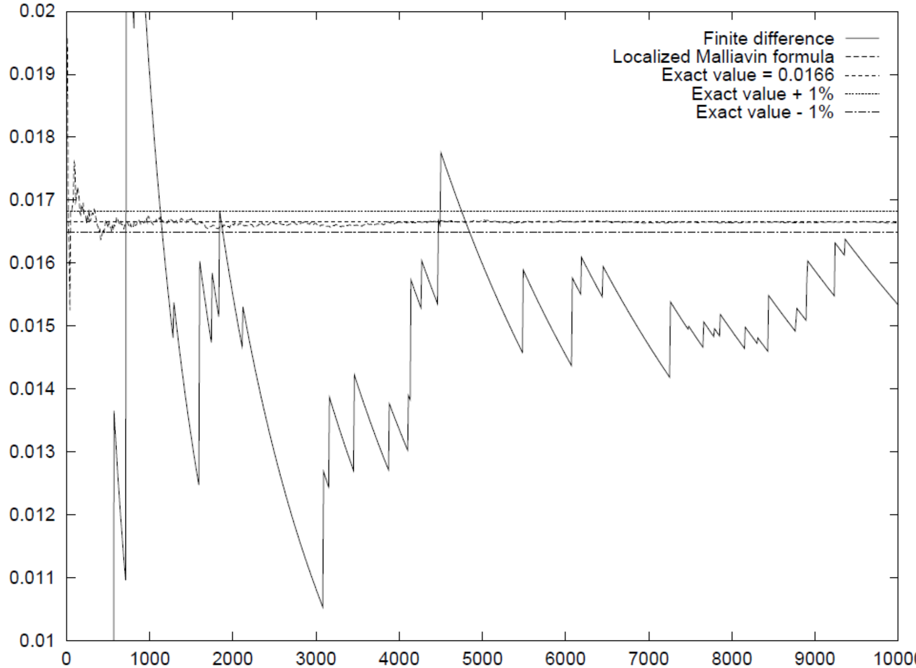

Fig. 7. Gamma of a call option computed by finite difference and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ (localization parameter). We use low discrepency sequences.

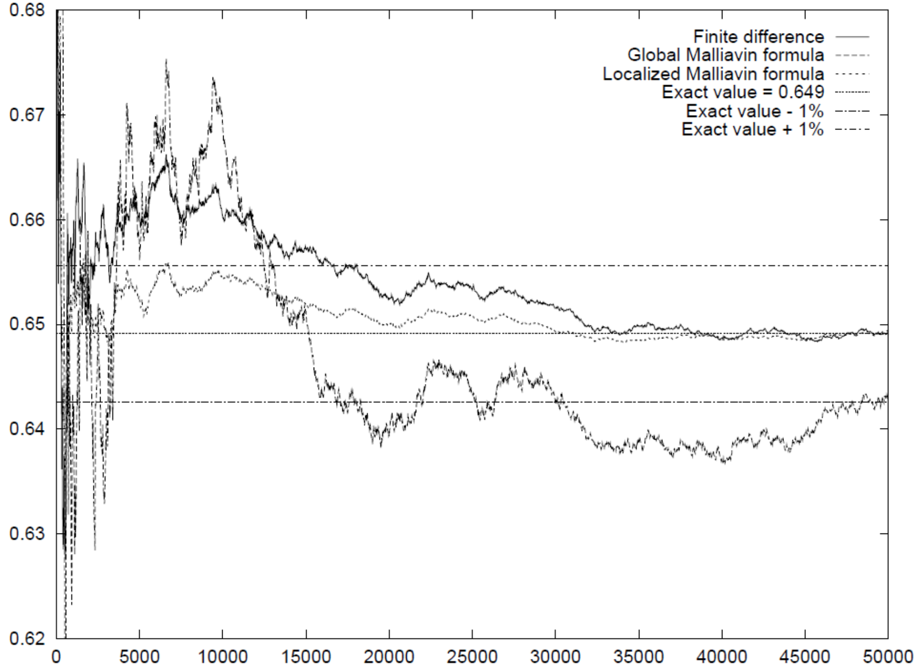

Fig. 8. Delta of an average call option computed by finite difference, global and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ for the localization parameter. We use pseudo random sequences.

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