Stochastic Processes for Interest Rates and Equilibrium Bond Prices

Author(s): Terry A. Marsh and Eric R. Rosenfeld
Source: *The Journal of Finance*, Vol. 38, No. 2, Papers and Proceedings Forty-First Annual Meeting American Finance Association New York, N.Y. December 28-30, 1982 (May, 1983), pp. 635-646
Published by: Wiley for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328002
Accessed: 07-02-2017 22:11 UTC

THE JOURNAL OF FINANCE • VOL. XXXVIII, NO. 2 • MAY 1983

Stochastic Processes for Interest Rates and Equilibrium Bond Prices

TERRY A. MARSH and ERIC R. ROSENFELD*

1. Introduction

DEFAULT FREE BONDS REPRESENT, in essence, money sold forward: the investor gives dollars (say) to the Government now in exchange for a contract which promises to pay, with probability one, a given number of dollars at maturity, and possibly a stream of intermediate coupon payments prior to maturity. Only recently have financial economists begun to examine the pricing of those bonds in a portfolio context. In that framework, nominal "liquidity premiums" or risk premiums on long-term default-free bonds are attributable to uncertainty about taste, technological, and price level changes which may occur prior to maturity of the bonds (and prior to each coupon payment if any). To the extent that the investor is unable to diversify away that uncertainty, there is a systematic risk attached to the marginal utility which he or she can derive from the use in consumption or investment of the dollar payoffs on the bonds.

In some term structure models, the instantaneous or "short-term" rate of interest, denoted here as a function of time t by $R(t)$ in nominal terms and $r(t)$ in real terms, can be taken as a sufficient statistic[1] for the source of systematic uncertainty about the marginal utility of the proceeds to be derived from default-free bonds maturing beyond the next instant (e.g., Merton [1974], Vasicek [1977], Cox, Ingersoll, and Ross [1977]). Specific assumptions about tastes, technology, and utility functions lead to specific equilibrium stochastic processes for the instantaneous (short-term) riskless rate, and longer term bond prices which are modelled as functionally dependent upon maturity and that short term riskless rate.

If a closed form solution exists for the long-term bond price as a function of maturity and the short term interest rate, it will be conditional, inter alia, on the parameters of the stochastic process for that short term rate. Marsh [1981] proposed that the closed form solutions could be tested in terms of the restrictions implied cross-sectionally on the parameters of the time series process for the returns (or prices) of longer term bonds, each with a given term to maturity. For an analogy outside the term structure literature, think of the closed form solution given by Black and Scholes [1972] for the price of an option on a nondividend paying stock with lognormal price dynamics. Given the model, the cross-section of time series for the changes in prices for each of the several options written on the same stock are not unrestricted: their dispersion is functionally linked to the same volatility parameter for the stock price. Thus, the option model, including

* Massachusetts Institute of Technology and Harvard University, respectively.

[1] This use of the term is similar, but not identical, to its statistical use.

the underlying dynamics for the stock price, could be tested without even observing the stock price.[2] Indeed, in certain cases, the pricing model may provide an economically sensible method for filtering the underlying variable's stochastic process.[3] Alternatively, the implied restriction could be tested aross the bivariate time series for the stock price and any one option.

In either the term structure models or option models, the assumed stochastic process for the independent variable(s) is critical. Tests of the cross-sectional restrictions just mentioned are one way of testing the goodness of fit of those assumed processes. This paper, however, reports only the portion of our work concerned with direct tests of stochastic process models for short term rates of interest, something which is roughly analogous to the univariate "identify" step in Box-Jenkins analysis.

The models we test are comprehensive when considered relative to the pertinent literature. We focus on a mean-reverting constant elasticity of variance diffusion model (particular versions of which are studied in Feller [1951], Black [1976], and Cox [1975]), which is nested within the typical diffusion-Poisson jump model. We allow expansions of the states in the diffusion component in two alternative ways, the most interesting of which involves subordinating the mean-reverting drift to a directing process. The single state constant elasticity of diffusion component itself nests the Ornstein-Uhlenbeck normal model, the lognormal, and the "square root" process for both interest rate changes and percentage interest rate changes. These are the three models typically encountered in the term structure literature. In this paper, we examine *only* these diffusion models for nominal interest rate changes.

2. Stochastic Process Models

We consider the time homogeneous class of diffusions which can be represented as:[4]

$$dR = f(R)R\,dt + \sigma(R)R\,dZ \tag{1}$$

The hypothesis tests in which we are interested involve continuous-time stochastic processes which are nested within (1). That is, we focus on hypothetical densities from the same family of distributions, so that discriminating between

[2] There is no mystery in using multiple options to estimate simultaneously volatility parameters and implied stock prices (e.g., Manaster and Rendleman [1982]), or in using observed prices to substitute out parameters for unobserved state variables (e.g., Garman [1979], or in the term structure context, Merton [1974] and Brennan and Schwartz [1982]).

[3] For example, we might be able to achieve a better signal-to-noise ratio in investigating stock price jumps by studying out-of-the-money options than would be possible by studying the discrete-time stock price series itself.

[4] As an example of a model which fits within (1) and would contain the models used in the relevant literature, consider:

$$dR = (a_0 + a_1R + a_2R^2)\,dt + \sigma R^{\beta/2}\,dz$$

$$0 \le \beta < 2$$

$$R > 0.$$

them involves only parameter estimation.[5] We need only ensure, then, that the family of distributions is not itself too parsimonious to adequately fit the interest rate series.

Three checks on this "identify" step seem possible: (i) the diffusion model can be itself embedded within a diffusion-Poisson model. As Rosenfeld [1982] shows, the resulting distribution may be analyzed in precisely the same manner as the assumed diffusion process is here; (ii) we could extend the first order differential equation (1) by expanding states and adding higher order differentials. This step is non-trivial, and fortunately is probably not warranted. It is not trivial because anything beyond a first order stochastic differential equation flirts with the "aliasing" problem in estimation (if indeed it remained tractable). Expansion seems unwarranted because p'th order continuous time autoregressions usually aggregate to ARMA (p, p) processes in discrete time (e.g., Phadke and Wu [1974]), and there is much evidence to suggest that nothing more than an AR(1) model is needed to fit first differences in interest rates; (iii) it is much more interesting to overfit the diffusion model (1) by subordinating its drift and volatility terms than by proceeding as in (ii), because this bears more directly on how we conjecture the continuous time process works for interest rates—innovations arise partly through a process directing $f(R)$ and/or $\mathfrak{s}(R)$ in (1), which *then*, together with the dZ shock, impact dR, rather than as in (ii), where a shock simultaneously perturbs R and the p differential terms.[6]

In this paper, we will illustrate our methodology with a generalized case of the constant elasticity of variance diffusion processes:[7]

$$dR = (AR^{-(1-\beta)} + BR)\, dt + \mathfrak{s}_R R^{\beta/2}\, dZ \tag{2}$$

(2) nests the constant elasticity of variance process studied in the stock price context by Cox [1975] and Black [1976]. The constant elasticity of variance process includes, in turn, the "square root" and normal processes, and as a limiting case, the lognormal model.

If $\beta = 1$, (2) becomes the square root process with mean reverting drift considered by Cox, Ingersoll, and Ross [1977]. If $\beta = 0$, it becomes,

$$dR = (A/R + BR)\, dt + \mathfrak{s}_R\, dZ. \tag{3}$$

If $a_2 = 0$, $\beta = 0$, we obtain the Ornstein-Uhlenbeck elastic random walk model in Vasicek [1977]; if $a_2 = 0$, $\beta = 1$, "the square root" model of (Cox, Ingersoll and Ross [1977]; and if $a_0 = 0$, $\beta = 2$, the model studied in Merton [1975a] for which $1/R$ is a weighted integral of lognormals. The variations of (*) can be interpreted in different ways, e.g., if $a_0 = 0$, $\beta = 2$, the percentage interest rate changes are normal. In line with both the form of the drift term in (*) and the discussion of Section 2, a barrier needs to be imposed on the normal model at zero to make it a sensible alternative for nominal interest rates. Fortunately, the density function for that truncated variable is a linear combination of two solutions to (1) (with $\beta = 2$), and hence, its density is still nested by (1).

[5] Nested hypothesis tests avoid a lot of problems concerning model selection criteria (e.g., Leamer [1979]), or in the distribution context, the need to adjust likelihood ratios for tests between different families of distributions (e.g., Cox [1961] [1962], Jackson [1968]).

[6] We are effectively using prior knowledge to identify (in a Box-Jenkins sense), the continuous time model (and, incidentally, overcome any aliasing problem).

[7] It may be verified that (2) is a fairly mechanical extension of Cox's [1975] transformation. Even for the process (1), we have to "pick" $f(R)$ so that, given $\beta(R)$, it can be transformed into a process which is manageable.

When R becomes small in (3), and $A > 0$, the first term dominates, and large positive changes are expected. If R is very large, the second term dominates, and the process behaves much like an Ornstein-Uhlenbeck process (with proportional drift). Thus, besides the need to bound $R \geq 0$ as discussed earlier, version (3) of the normal model seems unsatisfactory at high levels of R. As $\beta \rightarrow 2$, (2) approaches a limiting lognormal model with proportional drift $(A + B)$, where A and B are not separately identified. Model (3) again has the undesirable feature that mean reversion is not built into the drift—a limitation that arises from the special case (2) rather than from our methodology. Finally, note that variants of (2) can be interpreted in a different way: e.g., if $\beta \rightarrow 2$, percentage changes become normal with constant drift.

Equation (2) can be transformed to:

$$dy = (c + by)\,dt + \sqrt{a'y}\,dZ \tag{4}$$

where:

$$\begin{aligned} y &= R^{2-\beta} \\ a' &= .5\sigma_R(2-\beta)^2 \\ c &= (2-\beta)(A + .5\sigma_R(1-\beta)) \\ b &= B(2-\beta). \end{aligned}$$

Defining $a \equiv 2a'$, Feller [1951, 6.2] gives the following closed form solution for the conditional density of y generated as in (4):

$$\begin{aligned} f(R_i;\, \Delta t,\, R_{i-1}) = \frac{b}{a(e^{b\Delta t}-1)} &\left\{\exp \frac{-b(R_i + R_{i-1}e^{b\Delta t})}{a(e^{b\Delta t}-1)}\right] \\ &\times (4b^2 e^{-b\Delta t} R_i/R_{i-1})^{(c-a)/2a} \\ &\times I_{1-c/a}\left[\frac{2b}{a(1-e^{-b\Delta t})}\,(e^{-b\Delta t}R_iR_{i-1})^{1/2}\right] \end{aligned} \tag{5}$$

where $I_k(x)$ is the modified Bessel function

$$I_k(x) = \textstyle\sum_{s=0}^{\infty} \dfrac{(x/2)^{2s+k}}{S!\,\Gamma(S+1+k)}.$$

One comment about (5) is in order. Feller emphasizes that the character of the process (4) and its solution change dramatically depending upon whether $c \leq 0$, $0 < c < a$, or, in terms of the original diffusion, when $(2-\beta)(A + .5\sigma_R(1-\beta)) \leq \underline{0}$, $0 < (2-\beta)(A + .5\sigma_R(1-\beta)) < .5\sigma_R(2-\beta)^2$, respectively. The solution given in Feller [1951] is for the case $c \leq 0$ (where no boundary conditions can be imposed at zero), or the case $0 < c < a$ (where an "absorbing" barrier at zero boundary condition is satisfied). When $c > a(c > 0)$, Feller [1951, Lemma 4] shows that a non-negative solution for the density function exists where zero is both an absorbing and reflecting barrier. We have not obtained or estimated the solution in this case.[8]

[8] By definition, the flux $f(t)$ in Feller's [1951, (2.2)] is zero at $x = 0$ and it seems that the solution would be obtained by setting $f(t) = 0$ instead of substituting (his) (5.2) in (3.4).

3. Continuous Time Estimation

If a continuous time process is being sampled at discrete intervals, a discrete time model usually exists for the discrete time observations which will correspond exactly to the underlying continuous time process in the sense that it generates exactly the same data at discrete points as does the continuous model. Two questions naturally arise: (i) if an exact discrete time model always exists, why not develop financial models for (say) long term bond prices or option prices in discrete time, and not bother with the (possibly complicated) isomorphism in continuous time; and (ii) how "wrong" will parameter estimates be if they are obtained from estimation of the continuous time model with discrete observations where differentials are replaced by finite differences?

Discussion of the first question often revolves around whether time "really is" continuous, but this seems misdirected. As Merton [1975b] emphasized, the length of the time interval is either an implicit or explicit argument in all multiperiod models—solving models in continuous time makes explicit the assumption for one particular case. Casual observation suggests that there is discreteness in trading, but transacting should be included endogenously in the models—it makes little sense to arbitrarily designate a "representative" trading interval.

If continuity of time is assumed in building financial models, then it seems logical to estimate their exact discrete time equivalents. Alternatively, if a given discrete interval is assumed in a model construction, then that is the appropriate differencing interval (if, for example, the observation interval is longer than the assumed interval, further temporal aggregation may be required).

This logic seems straightforward. The more interesting question is the second: how much injustice does it do to the continuous time model if the researcher, instead of working out the exact discrete model, tests an approximate discrete-time version thereof by substituting differences for differentials? Clearly, the answer depends upon the nature of the stochastic process—e.g., if the "variable" were almost deterministic, ($\sigma = 0$), there would be almost no bias. Marsh and Rosenfeld [1983] investigate the asymptotic bias for the discrete time approximations of the continuous time models used here when applied to stock prices and interest rate data and show that the bias can be important (in a typical metric or loss function) even for quite short lengths of the differencing interval.

There is at least one additional and important consideration. It is not really "fair" to examine the fidelity of the approximation as the length of the differencing interval is reduced. As in most markets, data is not itself generated in real time. Consequently, observation error (in a percentage sense) increases as the interval shrinks.[9] Suppose that we reduced our monthly and weekly observation intervals for the nominal riskless rate to a "day." Suppose further, that we wanted to use bank overnight rates to construct our daily interest rate series. Those rates are quoted throughout the day, so that if we simply took the "closest" possible bids, a percentage error of up to half a day's rate would be possible. The federal funds rate is available daily, but may measure the "true" interest rate with error on Wednesdays when reserve requirements have to be met. Thus, if

[9] A familiar example is the increased relative importance of non-trading in daily versus monthly stock returns.

Table 1

Maximum Likelihood Estimation of Stochastic Process Models in Their Continuous-Time Formulation

Model:

$$/dR = (AR^{-(1-\beta)} + BR)\,dt + \sigma_R^{\beta/2}\,dz$$

A MONTHLY

$\beta = 0$

	A	B	σ_R^2	Log Likelihood
3/53–6/81	0.0047	0.0481	0.0026	1623.83
	(0.0047)	(0.4063)	(0.0002)	
3/53–12/59	0.0005	−1.4708	0.0007	449.73
	(0.0003)	(1.2195)	(0.0001)	
1/60–12/70	0.0014	−0.9557	0.0008	710.44
	(0.0008)	(0.6307)	(0.0001)	
1/71–6/81	0.0041	−0.3646	0.0059	547.35
	(0.0041)	(0.9688)	(0.0007)	

$\beta = 1$

	A	B	σ_R^2	Log Likelihood
3/53–6/81	0.0088	2.9224	0.003	1960.97
	(0.0733)	(4.3806)	(0.0016)	
3/53–12/59	0.0202	4.1756	0.0019	458.91
	(0.0272)	(6.0023)	(0.0027)	
1/60–12/70	0.0848	4.3605	0.0001	878.22
	(0.0935)	(8.0672)	(0.0116)	
1/71–6/81	0.0783	4.5117	0.015	728.93
	(0.0948)	(12.9952)	(0.0567)	

$\beta = 2$

	$A + B$	σ_R^2	Log Likelihood
3/53–6/81	0.3396	0.0438	1994.81
	(1.6667)	(0.0034)	
3/53–12/59	0.4575	0.2693	473.84
	(5.3073)	(0.0418)	
1/60–12/70	0.0735	0.0116	897.26
	(1.9271)	(0.0014)	
1/71–6/81	0.5740	0.0127	739.14
	(2.0208)	(0.0016)	

B. WEEKLY

$\beta = 0$

	A	B	σ_R^2	Log Likelihood
Jan. 78–May 82	0.0208	−1.4583	0.0326	1144.92
	(0.0052)	(0.6771)	(0.0031)	
Jan. 78–Mar. 82	0.0021	0.2604	0.0033	664.42
	(0.0051)	(0.7292)	(0.0004)	
Mar. 80–May 82	0.0365	−2.6042	0.0208	541.47
	(0.0163)	(1.3021)	(0.0028)	

$\beta = 1$

	A	B	σ_2^R	Log Likelihood
Jan. 78–May 82	0.2301	2.6186	0.015	1259.21
	(0.1905)	(18.7314)	(0.0455)	
Jan. 78–Mar. 80	0.2499	4.5908	0.0078	652.56
	(0.190)	(15.3511)	(0.0326)	
Mar. 80–May 82	0.1896	1.7456	0.0215	615.44
	(0.3415)	(19.4157)	(0.0691)	

Table 1—*continued*

	$\beta = 2$		
	$A + B$	σ_R^2	Log Likelihood
Jan. 78–May 82	0.4155	0.5265	1326.22
	(0.3476)	(0.0494)	
Jan. 78–Mar. 80	0.5180	0.1458	821.49
	(0.2592)	(0.0194)	
Mar. 80–May 82	0.7672	0.9052	589.90
	(0.6431)	(0.1199)	

Standard Errors are in parenthesis below the point estimates.
Coefficient estimates are annualized.

we can estimate continuous time models in terms of their exact discrete equivalents, we have a good deal more flexibility in reducing measurement errors in our data.

Finally, note that we need to assume that the term premium is zero over the discrete interval (i.e., the weekly and monthly T-bill rates do constitute integrals of the instantaneous rate), and presumably, this assumption becomes less tenable as the length of the interval increases.

4. Data and Preliminary Analysis

The estimates reported here are obtained for two data series: rates of return on T-bills with one month to maturity over the period March 1953 to June 1981, and rates of return on T-bills with one week to maturity over the period January 1978 to May 1982. Both series are constructed as in Fama [1975].[10] Figures 1(a) and 1(b) contain plots of the monthly series of rates and changes in rates, respectively. Figures 2(a) and 2(b) are the weekly analogs.

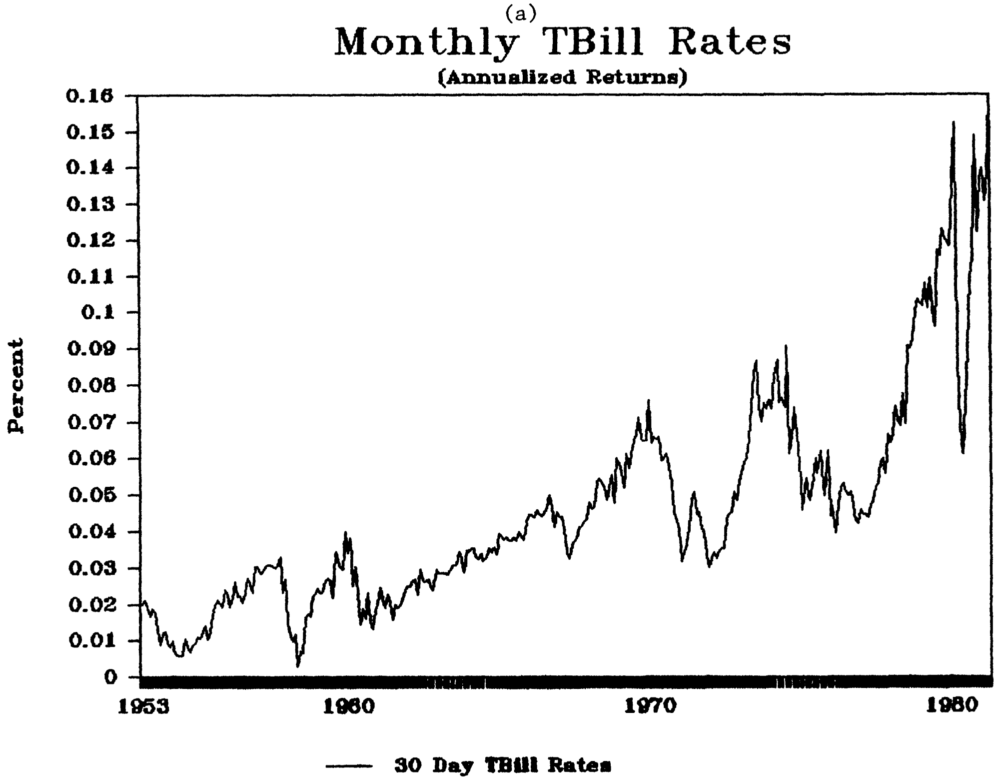
Histograms (not given) of the discrete time data tend to have "fat tails," consistent with earlier work by Roll [1970]. We interpret this as "first glance" evidence that a directing process such as that discussed earlier may be at work on the drift and volatility parameters.

We can see immediately from the figures that there appears to be (roughly) some positive relation between volatility of the changes in both the weekly and monthly rates and the level of those rates. It must be remembered that October 6, 1979 marked the beginning of the Federal reserve policy of de-emphasizing interest rate stability and emphasizing instead target growth rates for monetary aggregates, along with lower target levels for those growth rates.[11] Since that time, monetary growth rates over short periods, Treasury Bill rates, and changes in industrial production have all displayed greater volatility than historically. We do not need to infer causality from the evidence in order to conclude that

[10] For the weekly data, bid and ask prices were collected each week from the *Wall Street Journal* on Thursday (Wednesday if Thursday was a Holiday). Months were standardized to 30.4 days and weeks to 7.02 days. An interesting question concerns the degree to which extra data on (say) weekly highs and lows could be used to improve the power of our tests (or efficiency of our estimators) of our continuous time models (the question has been addressed for discrete-time estimation by Garman and Klass [1980] and for the geometric Brownian motion model by Parkinson [1980]).

[11] Ex post growth of $M1$ from October 1979 to June 1981 averaged about 6% per annum.

(a)


Monthly TBill Rates
(Annualized Returns)
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Percent
1953
1960
1970
1980
30 Day TBill Rates

(b)

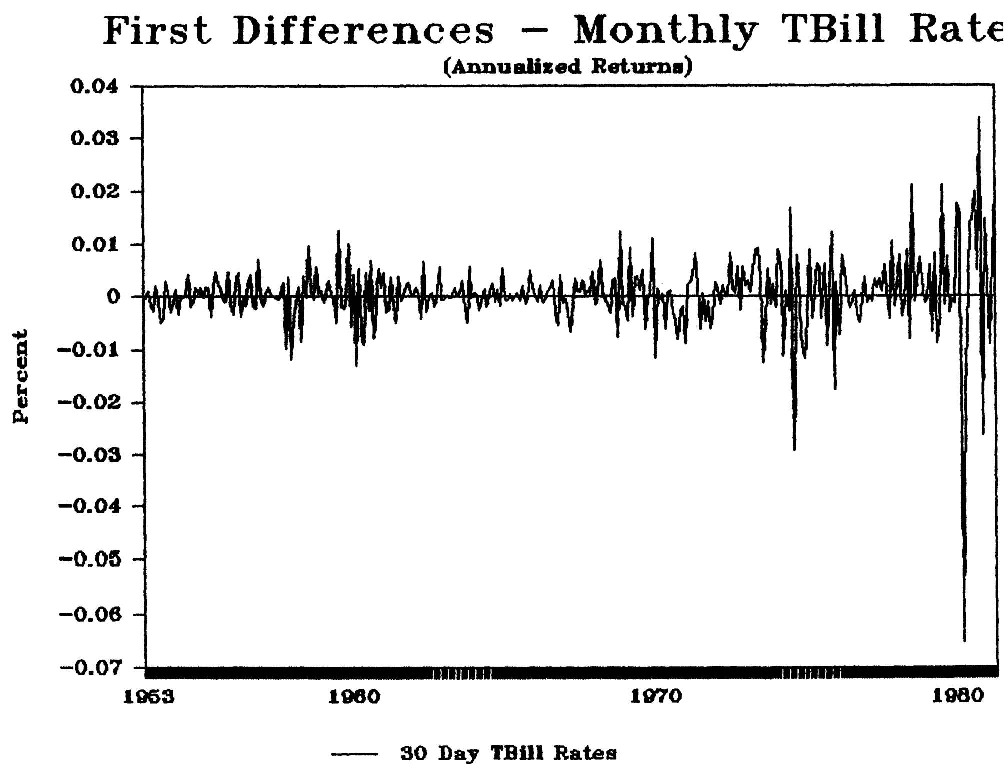
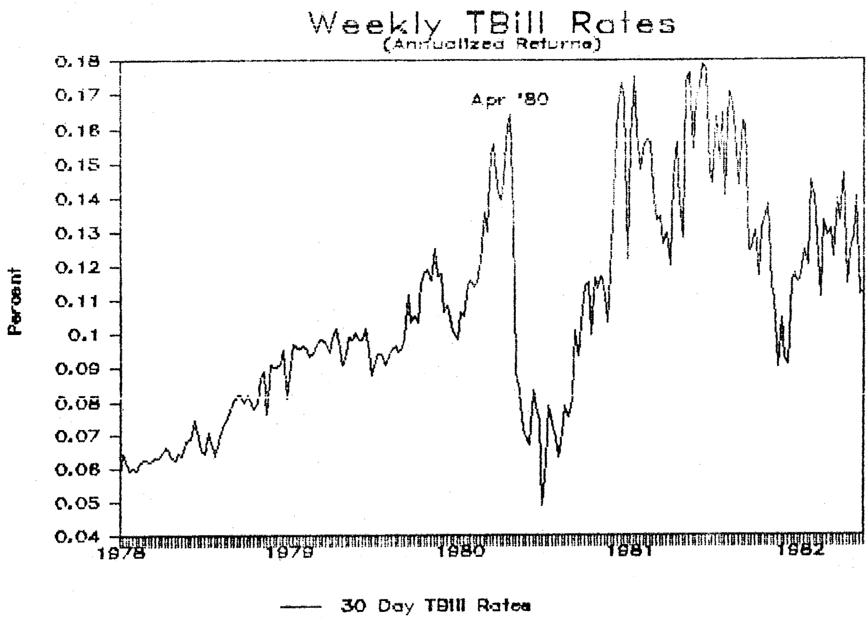

First Differences – Monthly TBill Rate
(Annualized Returns)
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
−0.05
−0.06
−0.07
Percent
1953
1960
1970
1980
30 Day TBill Rates

Figure 1

(a)


Weekly TBill Rates
(Annualized Returns)
Apr '80
Percent
0.18
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
1978
1979
1980
1981
1982
30 Day TBill Rates

(b)

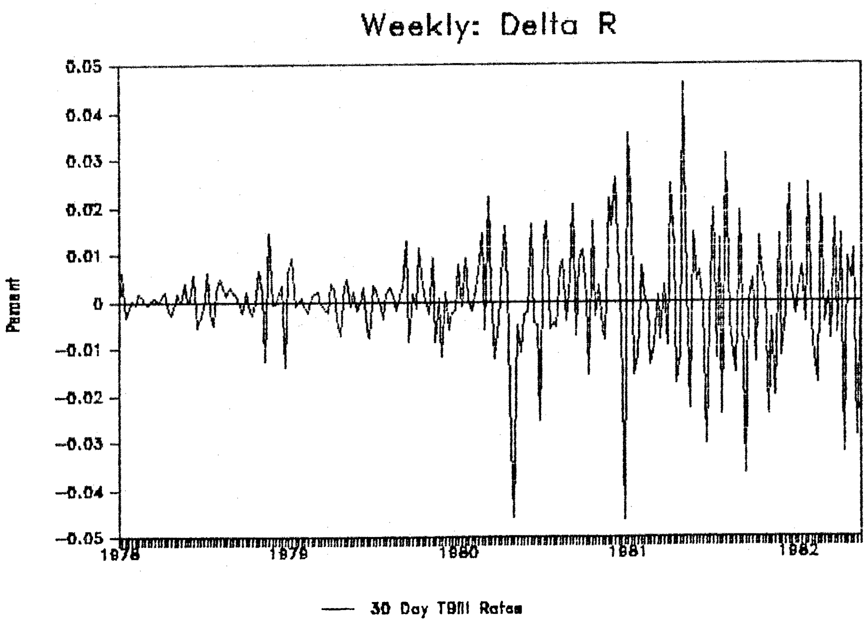

Weekly: Delta R
Percent
0.05
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
−0.05
1978
1979
1980
1981
1982
30 Day TBill Rates

Figure 2

the models fitted to data covering this recent period (particulary our weekly data) might tend to be "period-specific." The higher volatility might also result in higher observed bid-ask spreads, but it is not clear that it has any effect on the bid-ask average.

It is well to emphasize that we deal here with only a univariate model for the nominal riskless rate of interest and (not described here), a model for the bivariate series of expected rate of interest and expected rate of inflation. In dealing with the reduced form or final form equations for these variables, rather than with any underlying structural model, we do not have an explicit[12] "theory" guiding model identification (beyond obvious boundary conditions such as the nominal interest rate must be non-negative).

To examine the continuous-time formulation of the stochastic process for interest rates given in equation (1), a maximum likelihood estimation procedure is used. That is the actual density function corresponding to (1), $f(R_i{:}dt, R_{i-1} \mid A, B, \sigma^2_R, \beta)$ is used to maximize, over the parameter values, the likelihood of generating an interest rate series.

The results of the maximum likelihood estimation procedure are presented in Table 1. From the standard errors of the point estimates, it appears that it is very difficult to estimate the mean related parameters (A and B). For the monthly data, we cannot reject the hypothesis that either A or B is equal to zero in any of the models. The variance related parameter, σ^2_R, is significant in virtually all of the models.

A comparison of the likelihood values for the monthly data shows that over the full time period, the lognormal model ($\beta = 2$) is the most likely *of the three models* fitted in Table 1. However, we have not yet attempted to find that value of β which maximizes the likelihood function for (2).

We conclude by making two brief points: (i) our paper is intended to illustrate the methodology in estimating continuous time interest rate models, and to do this, we have space to consider only three special cases of the specialized class (2) of the specialized diffusion class. Since little theory is available to guide us, much more checking of models is warranted; (ii) the models fitted here are picking up the positive relation, obvious from Figures 1 and 2, between interest rate changes and their levels. This relation is consistent with the loose stylized fact that the variance of the inflation rate increases with its level. If differentials are replaced by differences in (2), it becomes a linear difference equation in which we can test for heteroscedasticity using procedures given by Breusch and Pagan [1979], White [1980], Koenker [1980] and Engle [1982]. These tests (not reported here) confirm the continuous time results, including the lack of mean reversion in the weekly sample period.[13]

REFERENCES

Beckers, S., 1980, "The constant elasticity of variance model and its implications for option pricing," *The Journal of Finance*, 35 (3), June, 661–673.

[12] Model building obviously involves priors, usually in the form of implicit or explicit assumptions. For a discussion of the role of a model prior see Box [1980].

[13] Few would argue for a political context in which inflation *rates* would wander off like a mean nonstationary series—the more serious debate undoubtedly centers on the likely half-life of the mean-reverting series relative to the length of our sample.

Betancourt, R. R., 1977, "On the consequences of planing interval specification error for the estimation of dynamic models," *Journal of Econometrics*, 6, 237–242.

Black, F., 1976, "Studies of stock price volatility changes," Proceedings of the 1976 Meeting of the American Statistical Association, Business and Economic Statistics Section.

Box, G. E. P., 1980, "Sampling and Bayes' inference in scientific modelling and robustness," *Journal of the Royal Statistical Society*, Series A, 143 (4), 383–430.

Brennan, M. J., and E. S. Schwartz, 1982, "An equilibrium model of bond pricing and a test of market efficiency," *Journal of Financial and Quantitative Analysis*, 17 (3), September, 301–330.

Breusch, T. S., and A. R. Pagan, "A simple test for heteroscedasticity and random coefficient variation," *Econometrica*, 47 (5), September, 1287–1294.

Cox, D. R., 1961, "Tests of separate families of hypotheses," In: J. Neyman, ed., Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics, Vol. 2, (University of California Press, Berkeley).

Cox, D. R., 1962, "Further results on tests of separate families of hypotheses," *Journal of the Royal Statistical Society*, Series B (Methodological), 24 (2), 406–424.

Cox, J. C., 1975, "Notes on option pricing I: Constant elasticity of variance diffusions," Unpublished Note.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross, 1977, "A theory of the term structure of interest rates and the valuation of interest-dependent claims," Research Paper (Revised), Graduate School of Business, University of Chicago.

Engle, R. F., 1982, "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation," *Econometrica*, 50 (4), July, 987–1007.

Fama, E. F., 1975, "Short-term interest rates as predictors of inflation," *The American Economic Review*, 65 (3), 269–282.

Feller, W., 1951, "Two singular diffusion problems," *Annals of Mathematics*, 54 (2), July, 173–182.

Garman, M. B., 1979, "A synthesis of the equilibrium theory of arbitrage," Research Paper (Revised), University of California at Berkeley, January.

Jackson, O. A. Y., 1968, "Some results on tests of separate families of hypotheses, *Biometrika*, 55 (2), 355–363.

Jackson, O. A. Y., and M. J. Klass, 1980, "On the estimation of security price volatilities from historical data," *Journal of Business*, 53 (1), 68–78.

Kendall, M., and A. Stuart, 1979, *The Advanced Theory of Statistics*, Vol. 2, MacMillan Publishing Co., Inc., New York.

Koenker, R., 1981, "A note on studentizing a test for heteroscedasticity," *Journal of Econometrics*, 17, 107–112.

Leamer, E. E., 1979, "Model choice and specification analysis," *The Handbook of Econometrics*, ed., Z. Griliches and M. D. Intrilligator.

Manaster, S., and R. J. Rendleman, Jr., 1982, "Option prices as predictors of equilibrium stock prices," *The Journal of of Finance*, 37 (4), September, 1043–1057.

Marsh, T., 1980, "Equilibrium term structure models: Test methodology," *The Journal of Finance*, 35 (5), May, 421–435.

Marsh, T., and E. R. Rosenfeld, 1983, "Discrete time approximations to continuous time models in finance," Unpublished Paper, Sloan School of Management, MIT, 50 Memorial Drive, Cambridge, MA, 02139.

Merton, R. C., 1974, "On the term structure of interest rates and other things," Unpublished Note, MIT.

Merton, R. C., 1975a, "An asymptotic theory of growth under uncertainty," *The Review of Economic Studies*, 62 (3), July, 375–393.

Merton, R. C., 1975b, "Theory of finance from the perspective of continuous time," *Journal of Financial and Quantitative Analysis*, November, 659–674.

Parkinson, M., 1980, "The extreme value method for estimating the variance of the rate of return," *Journal of Business*, 53 (1), 61–65.

Phadke, M. S., and S. M. Wu, 1974, "Modeling of continuous stochastic processes from discrete observations with application to sunspots data," *Journal of the American Statistical Association*, 69 (346), Applications Section, 325–329.

Roll, R., 1970, *The Behavior of Interest Rates: An Application of the Efficient Market Model to U.S. Treasury Bills*, Basic Books, Inc., New York.

Rosenfeld, R., 1982, "Stochastic processes of common stock returns: An empirical examination," Harvard Business School Working Paper 82-34, January.
Vasicek, O., 1977, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, 5 (2), November, 177–188.
White, H., 1980, "A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity, *Econometrica*, 48 (4), May, 817–838.

DISCUSSION

S. M. TURNBULL*: Dunn and Singleton argue that previous work on the pricing of GNMA pass through securities such as Dunn and McConnell (1981), make various restrictive assumptions about the stochastic processes describing the term structure of interest rates and preferences. The objective of this paper is to test a consumption based asset pricing model and to identify the parameters of the representative agent's utility function, without imposing restrictive assumptions on the probability distributions describing the relevant economic variables.

Consider the Dunn and McConnell (1981) paper. The key assumptions of that paper are: the instantaneous rate of interest can be described by a mean reverting process, and individuals have logarithmic utility functions. The assumption that individuals have logarthmic utility is very strong and is unlikely to hold. Given the work of Brennan and Schwartz (1980), the assumption that interest rates follow a mean reverting process can certainly be questioned. Does this imply that the Dunn and McConnell paper is worthless? Not necessarily. Whether the model is of any use, depends upon the accuracy of its predictions.

The authors argue that by using a discrete time consumption model, it is possible to price securities without imposing restrictive assumptions on preference and the stochastic processes describing asset prices. The first order conditions hold for any asset. Hence, in testing this model, one can pick any type of security, as the results should be the same if the model is well specified.

There are a number of difficulties with this paper.

In common with all asset pricing models is the problem of aggregation. The conditions for aggregation are quite restrictive, as the authors note. This does impose some restrictions on the family of utility functions that can be used, given that it is necessary to estimate the parameters characterizing the utility function. In a multigood economy, what is the appropriate price index? This issue, and the type of econometric biases introduced by using an inappropriate index, are not addressed by the authors. Given that individuals consume from after tax income, the effects of taxation on the pricing of assets and the econometric biases introduced by neglecting taxation need to be considered. The basic form of the model requires aggregate consumption at a particular point in time. Empirically, there are always lags in measuring consumption and this introduces biases in the estimation. These effects may be particularly important when the utility function depends upon past levels of consumption. The magnitude of the biases introduced by this errors in variables problem needs to be considered. Given that the model holds for any security, the empirical estimation of the parameters of the utility

* University of Toronto