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Squared Bessel Processes and Their Applications to the Square Root Interest Rate Model

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Abstract. We study the Bessel processes with time-varying dimension and their applications to the extended Cox-Ingersoll-Ross model with time-varying parameters. It is known that the classical CIR model is a modified Bessel process with deterministic time and scale change. We show that this relation can be generalized for the extended CIR model with time-varying parameters, if we consider Bessel process with time-varying dimension. This enables us to evaluate the arbitrage free prices of discounted bonds and their contingent claims applying the basic properties of Bessel processes. Furthermore we study a special class of extended CIR models which not only enables us to fit every arbitrage free initial term structure, but also to give the extended CIR call option pricing formula.

Key words: extended CIR model, Bessel process, time-varying dimension, option pricing.

1. Introduction

In this paper, we study the extended CIR term structure model. Let $(\Omega, (\mathcal{F}_t)_{0\leqslant t}, P)$ be a filtered probability space satisfying the usual conditions and $\{W_t; 0 \leqslant t\}$ be a Wiener process under P. Also we assume that the filtration $\mathcal{F}$ is generated by W and the null sets of $\mathcal{F}_\infty$ (remark that it is then automatically right continuous). Let r_t, the spot rate at time t, follow the equation

$$dr_t = (\alpha_t - \beta_t r_t)dt + \sigma_t\sqrt{r_t}dW_t,$$

where α_t, β_t and σ_t are time-varying deterministic functions. This term structure model was first considered by Cox-Ingersoll-Ross [3] for the constant parameters and for some modifications on the mean reverting level. On the other hand, Pitman-Yor [14] have studied the functional of Bessel processes with constant dimension which have the deep relationship between the classical CIR model as shown by Deelstra [4]. That is, the arbitrage free discount bond prices for the classical CIR

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model is equivalent to a special functional's value of Bessel processes with constant dimension. Thus we can derive the classical CIR results through the purely probabilistic approach shown by Pitman-Yor [14].

The extensions of the classical CIR model with time-varying parameters were studied by Beaglehole-Tenny [1], Hull-White [10], Jamshidian [11], Maghsoodi [13] and Rogers [17]. Maghsoodi [13] and Rogers [17] considered the relationship between the extended CIR model and integer dimensional Ornstein-Uhlenbeck processes. They showed that the separation of variable method gives the exact solution of the extended CIR model and derived the extended CIR call option formula for the constant integer dimension case. Although these researches enable us to study the special case of extended CIR model, they do not permit the study of the general case. Hull-White [10] treated the extended CIR model numerical approach. They derived a partial differential equation for the arbitrage free pure discount bond prices and evaluated them numerically. Furthermore, Beaglehole-Tenny [1] and Jamshidian [11] have studied the general extended CIR model by the separation of variables method. Beaglehole-Tenny [1] derived the arbitrage free discount bond prices for general time-varying parameters. Also Jamshidian derived the arbitrage free call option prices for time-varying parameters with constant dimension. However his argument is still valid under time-varying dimension and can easily be covered by the methods of this paper. Nevertheless it is important to study the probabilistic aspect of the extended CIR model which enables us to explain why such a separation of variables method works well. Recently, Carmona [2] have studied the functional of Bessel processes with time-varying dimension. Then it is possible to study the extended CIR model through the generalized relationship between the extended CIR model and the time-varying Bessel processes. The object of this paper is to apply the basic properties of the squared Bessel processes with time-varying dimension to study the extended CIR model in a purely probabilistic approach.

This paper is organized as follows. In Section 2, we summarize the basic results of squared Bessel processes with time-varying dimension as obtained by Carmona [2] and derive the supplementary results which play key role to study the extended CIR model. Then in Sections 3 and 4, we study the extended CIR model by using the results shown in Section 2. First we show that there exists an equivalent martingale measure for the square root risk premium process. Then we derive the arbitrage free prices of pure discount bonds and their call options under this measure change. Finally in Section 5, we study a special class of extended CIR model which enables us to show the results in Sections 4 and 5, explicitly.

2. Squared Bessel Processes with Time-Varying Dimension

First we summarize the main results for the squared Bessel processes with time-varying dimension derived by Carmona [2]. A process $X_t^{(\delta)}$ that follows

$$X_0^{(\delta)} = 0, \tag{2.1}$$

where $\delta : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a time-varying dimension function, is called a squared Bessel process with time-varying dimension function δ. For this class of diffusion processes, the following relationship holds, which is well known for constant δ [18].

THEOREM 2.1. (Carmona [2, Proposition 3.3, p. 24]). *Let $X_t^{(\delta)}$ and $X_t^{(\delta')}$ be independent squared Bessel processes with time-varying dimension functions δ, $\delta' \geqslant 0$ and initial states x, $x' \geqslant 0$, respectively. Then*

$$\{X_t^{(\delta)} + X_t^{(\delta')}; 0 \leqslant t\} \overset{law}{=} \{X_t^{(\delta+\delta')}; 0 \leqslant t\}, \tag{2.2}$$

where $X_0^{(\delta+\delta')} = x + x'$. □

For the notational convenience, we denote $E_x[\,\cdot\,] = E[\,\cdot\,|X_0^{(\delta)} = x]$. In general, it is difficult to derive the probability distribution of the squared Bessel processes with time-varying dimension explicitly. However we can derive the Laplace transform as follows.

THEOREM 2.2 (Carmona [2, Proposition 3.4, p. 27], Revuz-Yor [19, p. 411]). *For $\lambda \geqslant 0$,*

$$E_x\left[\exp\left\{-\lambda X_t^{(\delta)}\right\}\right] = \exp\left\{-\lambda \frac{x}{1+2\lambda t} - \int_0^t \frac{\lambda \delta_u}{1+2\lambda(t-u)} du\right\}. \tag{2.3}$$

Especially when δ_u is constant,

$$E_x\left[\exp\left\{-\lambda X_t^{(\delta)}\right\}\right] = (1+2\lambda t)^{-\frac{\delta}{2}} \exp\left\{-\lambda \frac{x}{1+2\lambda t}\right\}, \tag{2.4}$$

and its Laplace inversion is given by

$$f(y; t, x) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x+y}{2t}\right\} I_\nu\left(\frac{\sqrt{xy}}{t}\right), \quad t > 0,\ x > 0, \tag{2.5}$$

where $\nu = \frac{\delta}{2} - 1$ and

$$I_z(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(z+1+k)} \left(\frac{x}{2}\right)^{2k+z}. \tag{2.6}$$

□

Next we shall consider the evaluation of the functional:

$$E_{t,x}\left[\exp\left\{-\frac{1}{2}\int_t^T X_u^{(\delta)} m_u du\right\} f\left(X_T^{(\delta)}\right)\right], \tag{2.7}$$

where $E_{t,x}[\,\cdot\,] = E[\,\cdot\,|X_t^{(\delta)} = x]$, f is a measurable function and $m_u : [t, T] \to \boldsymbol{R}_+$ is a time-varying nonnegative function. First we shall consider the case $f = 1$.

Let $\varphi(\cdot, T) : [t, T] \to \boldsymbol{R}_+$ be a solution of the following second order differential equation:

$$\varphi_{uu}(u, T) = m_u \varphi(u, T), \quad \varphi(t, T) = k, \quad \varphi_u(T, T) = 0, \quad t \leqslant u \leqslant T, \tag{2.8}$$

where $\varphi_u(u, T) = \frac{\partial \varphi(u,T)}{\partial u}$, $\varphi_{uu}(u, T) = \frac{\partial^2 \varphi(u,T)}{\partial u^2}$ and $k > 0$ is some positive value. We can easily check that $\varphi(u, T)$ is strictly positive, convex and continuous with respect to $u \in [t, T]$ for given T. Also the solution is separable for the initial state, i.e., $\varphi|_{\varphi(t,T)=k} = k\varphi|_{\varphi(t,T)=1}$ for all $k > 0$.

LEMMA 2.3 (Carmona [2, Proposition 3.4, p. 24]). *If $\varphi(u, T)$ is the solution of the differential Equation (2.8), then for any initial value $\varphi(t, T) = k > 0$,*

$$\begin{aligned} & E_{t,x}\left[\exp\left\{-\frac{1}{2}\int_t^T X_u^{(\delta)} m_u du\right\}\right] \\ &\quad = \exp\left\{\frac{\varphi_u(t, T)}{2\varphi(t, T)}x\right\}\exp\left\{\frac{1}{2}\int_t^T \frac{\varphi_u(u, T)}{\varphi(u, T)}\delta_u du\right\}. \end{aligned} \tag{2.9}$$

□

This lemma enables us to evaluate (2.7) with constant f through the second order differential equation (2.8). Now we show the relationship between (2.1) and the stochastic differential equation:

$$dY_t = (a_t + b_t Y_t)dt + c_t\sqrt{Y_t}dW_t\,. \tag{2.10}$$

The weak solution for the stochastic differential Equation (2.10) is represented by a squared Bessel process with time and state changes as follows [14, 19].

LEMMA 2.4. *For any initial state $y \in \boldsymbol{R}_+$ at time $t \geqslant 0$,*

$$\{Y_u; t \leqslant u, Y_t = y\} \overset{law}{=} \left\{s_u X_{\tau_u}^{(\delta)}; t \leqslant u, X_{\tau_t}^{(\delta)} = \frac{y}{s_t}\right\}, \tag{2.11}$$

where

$$\left\{\begin{array}{lcl} s_u & = & \exp\{\int_0^u b_v dv\}, \\ \tau_u & = & \frac{1}{4}\int_0^u \frac{c_v^2}{s_v} dv, \\ \delta_u & = & \frac{4(a \circ \tau^{-1})_u}{(c^2 \circ \tau^{-1})_u}. \end{array}\right. \tag{2.12}$$

Proof. Let $\tilde{Y}_u = s_u X_{\tau_u}^{(\delta)}$. By Itô's lemma,

$$\begin{aligned} d\tilde{Y}_u &= ds_u X_{\tau_u}^{(\delta)} + s_u dX_{\tau_u}^{(\delta)} \\ &= b_u \tilde{Y}_u du + s_u\left(2\sqrt{X_{\tau_u}^{(\delta)}}dW_{\tau_u} + \delta_{\tau_u} d\tau_u\right) \\ &= (a_u + b_u \tilde{Y}_u)du + c_u\sqrt{\tilde{Y}_u}dW'_u, \end{aligned}$$

where $\{W'_u; 0 \leqslant u\}$ is another Wiener process defined by

$$W'_u = 2\int_0^{\tau_u} \frac{\sqrt{S_{\tau_v^{-1}}}}{c_{\tau_v^{-1}}} dW_v .$$

Then $\{Y_u; t \leqslant u\}$ and $\{\tilde{Y}_u; t \leqslant u\}$ are diffusion processes with identical differential operator and initial state. Hence their weak solutions have the same distribution law. □

From Lemmas 2.3 and 2.4, we can evaluate (2.7) in separable form for arbitrary f, which plays key role to derive the arbitrage free call option pricing formula.

LEMMA 2.5. *Let $\varphi(u, T)$ is a solution of the differential Equation (2.8). Then for any initial value $\varphi(t, T) = k > 0$,*

$$\begin{aligned} &E_{t,x}\left[\exp\left\{-\frac{1}{2}\int_t^T X_u^{(\delta)} m_u du\right\} f\left(X_T^{(\delta)}\right)\right] \\ &\quad = E_{t,x}\left[\exp\left\{-\frac{1}{2}\int_t^T X_u^{(\delta)} m_u du\right\}\right] E_{\tau_t, \frac{x}{s_t}}\left[f\left(s_T X_{\tau_T}^{(\delta\circ\tau^{-1})}\right)\right], \end{aligned} \tag{2.13}$$

where

$$\begin{cases} s_u &= \exp\{2\int_0^u \frac{\varphi_v(v,T)}{\varphi(v,T)} dv\}, \\ \tau_u &= \int_0^u \frac{1}{s_v} dv. \end{cases} \tag{2.14}$$

Proof. Let $F_t = \frac{\varphi_u(u,T)}{\varphi(u,T)}$ and let M_u be the martingale part of $X_u^{(\delta)}$, that is $M_u = X_u^{(\delta)} - \int_0^u \delta_v dv$. Since F_u is a continuous function of finite variation, we have

$$F_T X_T^{(\delta)} = F_t x + \int_t^T F_u dX_u^{(\delta)} + \int_t^T X_u^{(\delta)} dF_u .$$

Furthermore

$$\begin{aligned} \int_t^T X_u^{(\delta)} dF_u &= \int_t^T X_u^{(\delta)} \frac{\varphi_{uu}(u, T)}{\varphi_u(u, T)} du - \int_t^T X_u^{(\delta)} \left(\frac{\varphi_u(u, T)}{\varphi(u, T)}\right)^2 du \\ &= \int_t^T X_u^{(\delta)} m_u du - \int_t^T X_u^{(\delta)} F_u^2 du . \end{aligned}$$

Then

$$\begin{aligned} Z_{t,T} &= \mathcal{E}\left(\frac{1}{2}\int_t^{\cdot} F_v dM_v\right)_T \\ &= \exp\left\{-\frac{1}{2}\int_t^T F_u \delta_u du - \frac{1}{2} F_t x\right\} \exp\left\{-\frac{1}{2}\int_t^T X_u^{(\delta)} m_u du\right\}. \end{aligned}$$

Define the conditional probability measure $\hat{P}[\,\cdot\,|\mathcal{F}_t]$ by $\hat{P}[A|\mathcal{F}_t] = E[1_A Z_{t,T}|\mathcal{F}_t]$ for all $A \in \mathcal{F}_T$. By the Girsanov theorem [12, p. 191], $\tilde{W}_u = W_u - \int_t^u F_v \sqrt{X_v^{(\delta)}}dv$ is a Wiener process under $\hat{P}[\cdot|\mathcal{F}_t]$. From Lemma 2.4 for $c = 2$ and

$$\begin{aligned} dX_u^{(\delta)} &= \delta_u du + 2\sqrt{X_u^{(\delta)}}dW_u \\ &= (\delta_u + 2F_u X_u^{(\delta)})du + 2\sqrt{X_u^{(\delta)}}d\tilde{W}_u\,, \end{aligned}$$

we get

$$\begin{aligned} &\{X_u^{(\delta)};\, t \leqslant u,\, X_t^{(\delta)} = x\} \text{ under } \hat{P} \overset{law}{=} \{s_u X_{\tau_u}^{(\delta\circ\tau^{-1})}; \\ &t \leqslant u,\, s_t X_{\tau_t}^{(\delta\circ\tau^{-1})} = x\} \text{ under } P. \end{aligned}$$

This means that

$$E_{t,x}\left[Z_{t,T} f\left(X_T^{(\delta)}\right)\right] = E_{\tau_t, \frac{x}{s_t}}\left[f\left(s_T X_{\tau_T}^{(\delta\circ\tau^{-1})}\right)\right]. \tag{2.15}$$

On the other hand, we can easily check that

$$\begin{aligned} &E_{t,x}\left[Z_{t,T} f\left(X_T^{(\delta)}\right)\right] \\ &= \exp\left\{-\frac{1}{2}\int_t^T F_u \delta_u du - \frac{1}{2}F_t x\right\} E_{t,x}\left[\exp\left\{-\frac{1}{2}\int_t^T X_u^{(\delta)} m_u du\right\} f\left(X_T^{(\delta)}\right)\right]. \end{aligned} \tag{2.16}$$

Substituting (2.9) and (2.15) into (2.16), we obtain (2.13) after arrangement. □

3. Arbitrage Free Pure Discount Bond Price

Hereafter we study the extended Cox-Ingersoll-Ross term structure model:

$$dr_t = (\alpha_t - \beta_t r_t)dt + \sigma_t \sqrt{r_t}dW_t\,, \tag{3.1}$$

where α_t, β_t and σ_t are time-varying functions. We assume that $\alpha_t, \beta_t \geqslant 0$, $\inf_{0\leqslant t} \sigma_t > 0$ and σ_t is continuously differentiable with respect to t. From Lemma 2.4, we have the following property.

COROLLARY 3.1.

$$\{r_u;\, t \leqslant u,\, r_t = r\} \overset{law}{=} \left\{\theta_u X_{\nu_u}^{(\gamma)};\, t \leqslant u,\, X_{\nu_t}^{(\gamma)} = \frac{r}{\theta_t}\right\}, \tag{3.2}$$

where

$$\begin{cases} \theta_u = \exp\{-\int_0^u \beta_v dv\}, \\ \nu_u = \frac{1}{4}\int_0^u \frac{\sigma_v^2}{\theta_v} dv, \\ \gamma_u = \frac{4(\alpha\circ\nu^{-1})_u}{(\sigma^2\circ\nu^{-1})_u}. \end{cases} \tag{3.3}$$

Our object is to evaluate the arbitrage free pure discount bond price which pays \$1 at the maturity T. Let $\tilde{P}$ be an equivalent measure. Then from the general asset pricing theory [8, 9], the arbitrage free pure discount bond price $B(t, r; T)$ at time t is given by

$$B(t, r; T) = \tilde{E}_{t,r}\left[\exp\left\{-\int_t^T r_u du\right\}\right], \tag{3.4}$$

where $\tilde{E}_{t,r}[\,\cdot\,]$ denotes the conditional expectation of $\tilde{P}$ under $r_t = r$.

First we show the existence of an equivalent martingale measure when the measure change is taken for the time-varying square root risk premium process $\lambda_t\sqrt{r_t}$. We assume that λ_t is a time-varying bounded function. Define the process $\tilde{W}_t$ by

$$\tilde{W}_t = W_t + \int_0^t \lambda_u\sqrt{r_u}du. \tag{3.5}$$

Then the interest rate r_t follows

$$dr_t = (\alpha_t - \tilde{\beta}_t r_t)dt + \sigma_t\sqrt{r_t}d\tilde{W}_t\,, \tag{3.6}$$

where $\tilde{\beta}_t = \beta_t + \lambda_t\sigma_t$. By the Girsanov's theorem [12, p. 191], $\{\tilde{W}_t; 0 \leqslant t \leqslant T\}$ becomes a Wiener process under the equivalent measure change for $P|_{\mathcal{F}_T}$ defined by the Radon-Nikodym derivative:

$$\rho_T = \exp\left\{-\int_0^T \lambda_t\sqrt{r_t}dW_t - \frac{1}{2}\int_0^T \lambda_t^2 r_t dt\right\}, \tag{3.7}$$

if $E[\rho_T] = 1$. In this class of measure changes, both the original process (3.1) and the risk neutralized process (3.6) belong to the extended CIR class. Furthermore we can guarantee the existence of the measure changes as follows.

THEOREM 3.2. *For any $T < \infty$, $E[\rho_T] = 1$ and hence there exists an equivalent martingale measure $\tilde{P}$ to $P|_{\mathcal{F}_T}$ defined by*

$$\tilde{P}[A] = E[1_A\rho_T] \quad \text{for } A \in \mathcal{F}_T\,. \tag{3.8}$$

Proof. First we show that $E[\rho_T|\mathcal{F}_t] = 1$, through Novikov's criterion [12, p. 198] for small $\Delta = T - t > 0$. From Corollary 3.1,

$$\begin{aligned}
\int_t^T \lambda_u^2 r_u du &\overset{law}{=} \int_t^T \lambda_u^2\theta_u X_{\nu_u}^{(\gamma)}du \\
&= \int_{\nu_t}^{\nu_T} \lambda_{\nu_v^{-1}}^2\theta_{\nu_v^{-1}}X_v^{(\gamma)}\frac{d\nu_v^{-1}}{dv}dv \\
&= \int_{\nu_t}^{\nu_T} X_v^{(\gamma)}\left(\frac{2\lambda_{\nu_v^{-1}}\theta_{\nu_v^{-1}}}{\sigma_{\nu_v^{-1}}}\right)^2 dv.
\end{aligned}$$

Then Novikov's criterion associated with $\rho_T|_{\mathcal{F}_t}$ is given by

$$E\left[\exp\left\{\frac{1}{2}\int_t^T \lambda_u^2 r_u du\right\}\middle|\ \mathcal{F}_t\right] = E\left[\exp\left\{\frac{1}{2}\int_{\nu_t}^{\nu_T}\left(\frac{2\lambda_{\nu_v^{-1}}\theta_{\nu_v^{-1}}}{\sigma_{\nu_v^{-1}}}\right)^2 X_v^{(\gamma)}dv\right\}\middle|\ \mathcal{F}_{\nu_t}\right] < \infty. \tag{3.9}$$

From the comparison theorem [16, p. 364],

$$0 \leqslant X_{\nu_t}^{(\gamma)} \leqslant X_{\nu_t}^{(\bar{\gamma})}, \quad 0 \leqslant t, \quad P\text{-a.s.},$$

where $\bar{\gamma} = \sup_{0\leqslant t}\gamma_t < \infty$. This together with the Markov property of $\{r_t; 0 \leqslant t\}$ and $\{X_t^{(\bar{\gamma})}; 0 \leqslant t\}$, we get

$$\begin{aligned} E\left[\exp\left\{\frac{1}{2}\int_t^T \lambda_u^2 r_u du\right\}\middle|\ \mathcal{F}_t\right] &= E\left[\exp\left\{\frac{1}{2}\int_t^T \lambda_u^2 r_u du\right\}\middle|\ r_t = r\right] \\ &\leqslant E\left[\exp\left\{\frac{q^2}{2}\int_{\nu_t}^{\nu_T} X_{\nu_u}^{(\bar{\gamma})}du\right\}\middle|\ X_{\nu_t}^{(\bar{\gamma})} = x\right], \end{aligned} \tag{3.10}$$

where $q = \sup_{0\leqslant t}\left|\frac{2\lambda_t\theta_t}{\sigma_t}\right| < \infty$ and $x = \frac{r}{\theta_t}$. Let $\zeta(u, \nu_T)$ be the solution for the second order differential equation:

$$\begin{cases} \zeta_{uu}(u, \nu_T) = -q^2\zeta(u, \nu_T), & \nu_t \leqslant u \leqslant \nu_T, \\ \zeta_u(\nu_T, \nu_T) = 0. \end{cases} \tag{3.11}$$

It is easy to check that the solution of Equation (3.11) is given by

$$\zeta(u, \nu_T) = k\sin\left(q(u - \nu_T) + \frac{\pi}{2}\right), \text{ for some } k \in \boldsymbol{R}. \tag{3.12}$$

We can choose $k \in \boldsymbol{R}$ so that $\zeta(u, \nu_T) > 0$ for $\nu_t \leqslant u \leqslant \nu_T$ whenever $\Delta' = \nu_T - \nu_t < \frac{\pi}{2q}$. Let $F_u = -\frac{\zeta_u(u,\nu_T)}{\zeta(u,\nu_T)}$. From Itô's lemma,

$$\begin{aligned} F_{\nu_T}X_{\nu_T}^{(\bar{\gamma})} &= F_{\nu_t}x + \int_{\nu_t}^{\nu_T} F_u dX_u^{(\bar{\gamma})} + \int_{\nu_t}^{\nu_T} X_u^{(\bar{\gamma})}dF_u \\ &= F_{\nu_t}x + \int_{\nu_t}^{\nu_T} F_u dX_u^{(\bar{\gamma})} + q^2\int_{\nu_t}^{\nu_T} X_u^{(\bar{\gamma})}du + \int_{\nu_t}^{\nu_T} F_u^2 X_u^{(\bar{\gamma})}du \\ &= 0. \end{aligned}$$

Also let M_u be the martingale part of squared Bessel process $X_t^{(\bar{\gamma})}$ and us consider the exponential martingale for the local martingale $\int_t^T F_u dM_u$. Then

$$\begin{aligned}
Z_{\nu_t,\nu_T} &= \mathcal{E}\left(-\frac{1}{2}\int_{\nu_t}^{\cdot} F_u dM_u\right)_{\nu_T} \\
&= \exp\left\{-\frac{1}{2}\int_{\nu_t}^{\nu_T} F_u dX_u^{(\bar{\gamma})} + \frac{\bar{\gamma}}{2}\int_{\nu_t}^{\nu_T} F_u du - \frac{1}{2}\int_{\nu_t}^{\nu_T} F_u^2 X_u^{(\bar{\gamma})} du\right\} \\
&= \zeta(\nu_t,\nu_T)^{-\frac{\bar{\gamma}}{2}} \exp\left\{\frac{q^2}{2}\int_{\nu_t}^{\nu_T} X_u^{(\bar{\gamma})} du\right\} \exp\left\{-\frac{\zeta_u(\nu_t,\nu_T)}{2\zeta(\nu_t,\nu_T)} x\right\}.
\end{aligned}$$

Since Z_{ν_t,ν_T} is a positive local martingale, it is supermartingale. Hence we have

$$\begin{aligned}
&E[Z_{\nu_t,\nu_T}|X_{\nu_t}^{(\bar{\gamma})} = x] \\
&= \zeta(\nu_t,\nu_T)^{-\frac{\bar{\gamma}}{2}} E\left[\exp\left\{\frac{q^2}{2}\int_{\nu_t}^{\nu_T} X_u^{(\bar{\gamma})} du\right\}\middle|\mathcal{F}_{\nu_t}\right] \exp\left\{-\frac{\zeta_u(\nu_t,\nu_T)}{2\zeta(\nu_t,\nu_T)} x\right\} \qquad (3.13)\\
&\leqslant E[Z_{\nu_t,\nu_t}|X_{\nu_t}^{(\bar{\gamma})} = x] = 1.
\end{aligned}$$

The last inequality in (3.13) means

$$E\left[\exp\left\{\frac{q^2}{2}\int_{\nu_t}^{\nu_T} X_u^{(\bar{\gamma})} du\right\}\middle| X_{\nu_t}^{(\bar{\gamma})} = x\right] \leqslant \zeta(\nu_t,\nu_T)^{\frac{\bar{\gamma}}{2}} \exp\left\{\frac{\zeta_u(\nu_t,\nu_T)}{2\zeta(\nu_t,\nu_T)} x\right\} < \infty,$$
$$x \geqslant 0.$$

Thus (3.10) becomes finite and Novikov's criterion (3.9) is satisfied. Next suppose that $T < \infty$ and let $\{T_i; 0 \leqslant i \leqslant N\}$ be an increasing sequence such that $0 = T_0 < T_1 < \cdots < T_N = T$ and $\nu_{T_{i+1}} - \nu_{T_i} < \frac{q}{2\pi}$. By the result shown above, we have

$$\begin{aligned}
&E\left[\frac{\rho_{T_{i+1}}}{\rho_{T_i}}\middle| r_{T_i} = r\right] \\
&= E\left[\exp\left\{-\int_{T_i}^{T_{i+1}} \lambda_t\sqrt{r_t}dW_t - \frac{1}{2}\int_{T_i}^{T_{i+1}} \lambda_t^2 r_t dt\right\}\middle| r_{T_i} = r\right] \qquad (3.14)\\
&= 1, \quad \text{for all} \quad r \geqslant 0,\ 0 \leqslant i \leqslant N-1.
\end{aligned}$$

Furthermore from the definition of ρ_t and the Markov property of $\{r_t; 0 \leqslant t\}$,

$$E\left[\frac{\rho_{T_{i+1}}}{\rho_{T_i}}\middle|\mathcal{F}_{T_i}\right] = E\left[\frac{\rho_{T_{i+1}}}{\rho_{T_i}}\middle| r_{T_i}\right] = 1. \qquad (3.15)$$

Then from (3.14) and (3.15),

$$\begin{aligned}
E[\rho_T] &= E\left[\prod_{0\leqslant i\leqslant N-1}\left(\frac{\rho_{T_{i+1}}}{\rho_{T_i}}\right)\right] \\
&= E\left[\frac{\rho_{T_1}}{\rho_{T_0}} \cdots E\left[\frac{\rho_{T_N}}{\rho_{T_{N-1}}}\middle|\mathcal{F}_{T_{N-1}}\right]\cdots\right] \\
&= 1.
\end{aligned}$$

Thus we have $E[\rho_T] = 1$ for any $T < \infty$. □

Next we shall consider the arbitrage free pure discount bond price (3.4) under the equivalent martingale measure defined by (3.8).

THEOREM 3.3.

$$B(t, r; T) = \exp\left\{\int_t^T \frac{2\alpha_u \eta_u(u, T)}{\sigma_u^2 \eta(u, T)} du\right\} \exp\left\{\frac{2\eta_u(t, T)}{\sigma_t^2 \eta(t, T)} r\right\}, \tag{3.16}$$

where $\eta(u, T)$ is a solution for the differential equation:

$$\begin{cases} \eta_{uu}(u, T) - \left(\tilde{\beta}_u + 2\frac{\sigma_u'}{\sigma_u}\right) \eta_u(u, T) - \frac{1}{2}\sigma_u^2 \eta(u, T) = 0, \ t \leqslant u \leqslant T, \\ \eta_u(T, T) = 0, \ \eta(t, T) = k > 0, \end{cases} \tag{3.17}$$

for some $k > 0$. Especially when $\delta_u = \frac{4\alpha_u}{\sigma_u^2}$ is constant,

$$B(t, r; T) = \left(\frac{\eta(T, T)}{\eta(t, T)}\right)^{\frac{\delta}{2}} \exp\left\{\frac{2\eta_u(t, T)}{\sigma_t^2 \eta(t, T)} r\right\}. \tag{3.18}$$

Proof. From Corollary 3.1 and Girsanov's theorem,

$$\begin{aligned} \int_t^T r_u du\Big|_{r_t = r} \text{ under } \tilde{P} \ &\overset{law}{=} \int_t^T \tilde{\theta}_u X_{\tilde{\nu}_u}^{(\tilde{\gamma})} du\Big|_{\tilde{\theta}_t X_{\tilde{\nu}_t}^{(\tilde{\gamma})} = r} \text{ under } P \\ &= \int_{\tilde{\nu}_t}^{\tilde{\nu}_T} \tilde{\theta}_{\tilde{\nu}_v^{-1}} X_v^{(\tilde{\gamma})} \frac{d\tilde{\nu}_v^{-1}}{dv} dv\Big|_{\tilde{\theta}_t X_{\tilde{\nu}_t}^{(\tilde{\gamma})} = r} \\ &= \frac{1}{2}\int_{\tilde{\nu}_t}^{\tilde{\nu}_T} X_v^{(\tilde{\gamma})} \tilde{m}_v dv\Big|_{X_{\tilde{\nu}_t}^{(\tilde{\gamma})} = \frac{r}{\tilde{\theta}_t}}, \end{aligned}$$

where

$$\begin{cases} \tilde{\theta}_u &= \exp\{-\int_0^u \tilde{\beta}_v dv\}, \\ \tilde{\nu}_u &= \frac{1}{4}\int_0^u \frac{\sigma_v^2}{\tilde{\theta}_v} dv, \\ \tilde{\gamma}_u &= \frac{4(\alpha \circ \tilde{\nu}^{-1})_u}{(\sigma^2 \circ \tilde{\nu}^{-1})_u}, \\ \tilde{m}_u &= \frac{8(\tilde{\theta}^2 \circ \tilde{\nu}^{-1})_u}{(\sigma^2 \circ \tilde{\nu}^{-1})_u}. \end{cases} \tag{3.19}$$

Let $\varphi(v, \tilde{\nu}_T)$ be a solution of the following differential equation:

$$\begin{aligned} &\varphi_{vv}(v, \tilde{\nu}_T) = \tilde{m}_v \varphi(v, \tilde{\nu}_T), \ \varphi(\tilde{\nu}_t, \tilde{\nu}_T) = k > 0, \ \varphi_v(\tilde{\nu}_T, \tilde{\nu}_T) = 0, \\ &\tilde{\nu}_t \leqslant v \leq \tilde{\nu}_T. \end{aligned} \tag{3.20}$$

Then from Corollary 3.1, (3.19) and Lemma 2.3,

$$\begin{aligned}
&\tilde{E}_{t,r}\left[\exp\left\{-\int_t^T r_u du\right\}\right] \\
&= E_{\tilde{\nu}_t,\frac{r}{\tilde{\theta}_t}}\left[\exp\left\{-\frac{1}{2}\int_{\tilde{\nu}_t}^{\tilde{\nu}_T} X_v^{(\tilde{\gamma})}\tilde{m}_v dv\right\}\right] \qquad (3.21)\\
&= \exp\left\{\frac{\varphi_v(\tilde{\nu}_t,\tilde{\nu}_T)}{2\varphi(\tilde{\nu}_t,\tilde{\nu}_T)}\frac{r}{\tilde{\theta}_t}\right\}\exp\left\{\frac{1}{2}\int_{\tilde{\nu}_t}^{\tilde{\nu}_T}\frac{\varphi_v(v,\tilde{\nu}_T)}{\varphi(v,\tilde{\nu}_T)}\tilde{\gamma}_v dv\right\}.
\end{aligned}$$

Now let $\eta(u,T)=\varphi(\tilde{\nu}_u,\tilde{\nu}_T),\ t\leqslant u\leqslant T$. Then we can easily check that

$$\begin{cases}
\eta_u(u,T) &= \varphi_v(\tilde{\nu}_u,\tilde{\nu}_T)\tilde{\nu}_u',\\
\eta_{uu}(u,T) &= \varphi_{vv}(\tilde{\nu}_u,\tilde{\nu}_T)\tilde{\nu}_u'^2+\varphi_v(\tilde{\nu}_u,\tilde{\nu}_T)\tilde{\nu}_u'',
\end{cases}$$

and hence

$$\frac{\varphi_v(\tilde{\nu}_t,\tilde{\nu}_T)}{\varphi(\tilde{\nu}_t,\tilde{\nu}_T)}=\frac{4\tilde{\theta}_t\eta_u(t,T)}{\sigma_t^2\eta(t,T)}, \qquad (3.22)$$

$$\int_{\tilde{\nu}_t}^{\tilde{\nu}_T}\frac{\varphi_v(v,\tilde{\nu}_T)}{\varphi(v,\tilde{\nu}_T)}\tilde{\gamma}_v dv=\int_t^T\frac{4\alpha_u\eta_u(u,T)}{\sigma_u^2\eta(u,T)}du. \qquad (3.23)$$

Furthermore,

$$\begin{aligned}
(3.20)\ \Leftrightarrow\ &\begin{cases}
\left(\dfrac{\sigma^2_{\tilde{\nu}_v^{-1}}}{4\tilde{\theta}_{\tilde{\nu}_v^{-1}}}\right)^2\varphi_{vv}(v,\tilde{\nu}_T)-\frac{1}{2}\sigma^2_{\tilde{\nu}_v^{-1}}\varphi(v,\tilde{\nu}_T)=0,\ \tilde{\nu}_t\leqslant v\leqslant\tilde{\nu}_T\\
\varphi(\tilde{\nu}_t,\tilde{\nu}_T)=k,\ \varphi_v(\tilde{\nu}_T,\tilde{\nu}_T)=0
\end{cases}\\
\Leftrightarrow\ &(3.17).
\end{aligned}$$

Substituting (3.22) and (3.23) into (3.21), we arrive at (3.16). □

We shall now show the relationship between the separation of variable approach [11] and our approach. The pure discount bond price function $B(t,r;T)$ which is consistent with the arbitrage free property satisfies the fundamental differential equation:

$$\begin{cases}
B_u(u,r;T)+\frac{1}{2}\sigma_u^2 rB_{rr}(u,r;T)+(\alpha_u-\tilde{\beta}_u r)B_r(u,r;T)-rB(u,r;T)=0,\\
t\leqslant u\leqslant T, \qquad (3.24)\\
B(T,r;T)=1,\ r\geqslant 0.
\end{cases}$$

Suppose that $B(t,r;T)$ is separable in each variable such that $B(t,r;T)=\exp\{-a(t,T)-b(t,T)r\}$. Then we can easily check (3.24) is satisfied whenever $a(t,T)$ and $b(t,T)$ satisfy the following differential equations:

$$a_u(u,T)+\alpha_u b(u,T)=0,\ a(T,T)=0,\ t\leqslant u\leq T, \qquad (3.25)$$

$$b_u(u,T) - \tilde{\beta}_u b(u,T) - \tfrac{1}{2}\sigma_u^2 b^2(u,T) + 1 = 0,\ b(T,T) = 0,\quad t \leqslant u \leqslant T. \tag{3.26}$$

From (3.25), $a(t,T) = \int_t^T \alpha_u b(u,T)du$ and hence

$$B(t,r;T) = \exp\left\{-\int_t^T \alpha_u b(u,T)du - b(t,T)r\right\},\ 0 \leqslant t \leqslant T, \tag{3.27}$$

where $b(u,T)$ is the solution of the differential Equation (3.26). Substitute

$$b(u,T) = -\frac{2\eta_u(u,T)}{\sigma_u^2 \eta(u,T)}, \tag{3.28}$$

then we can confirm (3.16) $\Leftrightarrow$ (3.27) and (3.17) $\Leftrightarrow$ (3.26). Here notice that to derive the arbitrage free price by separation of variable approach, we must presume the special class solution for $B(t,r;T)$. However this assumption is not needed to derive the exact arbitrage free price by Lemma 2.3.

4. Arbitrage Free Call Option Price

Next we study the arbitrage free call option price. Let us consider the call option on a T_b-maturity pure discount bond with exercise price $K \in [0,1)$, which expires at T_o ($T_o \leqslant T_b$). Then its arbitrage free price $C(t,r;T_o,T_b,K)$ at time t is given by [8, 9]:

$$C(t,r;T_o,T_b,K) = \tilde{E}_{t,r}\left[\exp\left\{-\int_t^{T_o} r_u du\right\}\left(B(T_o,r_{T_o};T_b) - K\right)_+\right], \tag{4.1}$$

where $\tilde{E}_{t,r}[\,\cdot\,]$ denotes the conditional expectation of $\tilde{P}$ under $r_t = r$. We assume that the measure change is taken for the time-varying square root risk premium process $\lambda_t\sqrt{r_t}$ and the equivalent martingale measure is given by (3.8).

THEOREM 4.1.

$$\begin{aligned} &C(t,r;T_o,T_b,K) \\ &\quad = B(t,r;T_b)P_r[\check{\theta}_{t,T_o}X^{(\check{\gamma}^t)}_{\check{\nu}_{t,T_o}} \leqslant r^*] - KB(t,r;T_o)P_r[\hat{\theta}_{t,T_o}X^{(\hat{\gamma}^t)}_{\hat{\nu}_{t,T_o}} \leqslant r^*], \end{aligned} \tag{4.2}$$

where $r^* \in \boldsymbol{R}_+$ is the unique solution for $B(T_o, r^*; T_b) = K$, ($r^* = \infty$ when $K = 0$) and

$$\begin{cases} \hat{\theta}_{t,u} & = \left(\frac{\eta(u,T_o)}{\eta(t,T_o)}\right)^2 \exp\{-\int_t^u \tilde{\beta}_v dv\}, \\ \hat{\nu}_{t,u} & = \frac{1}{4}\int_t^u \frac{\sigma_v^2}{\hat{\theta}_{t,v}} dv, \\ \hat{\gamma}_u^t & = \frac{4(\alpha \circ \hat{\nu}_{t,\cdot}^{-1})_u}{(\sigma^2 \circ \hat{\nu}_{t,\cdot}^{-1})_u}, \\ \mu_t & = -\frac{2\eta_u(T_o,T_b)}{\sigma_{T_o}^2 \eta(T_o,T_b)} \hat{\theta}_{t,T_o}, \\ \check{\theta}_{t,u} & = \left(\frac{1+2\mu_t(\hat{\nu}_{t,T_o}-\hat{\nu}_{t,u})}{1+2\mu_t\hat{\nu}_{t,T_o}}\right)^2 \hat{\theta}_{t,u}, \\ \check{\nu}_{t,u} & = \frac{1}{4}\int_t^u \frac{\sigma_v^2}{\check{\theta}_{t,v}} dv = \frac{1+2\mu_t\hat{\nu}_{t,T_o}}{1+2\mu_t(\hat{\nu}_{t,T_o}-\hat{\nu}_{t,u})} \hat{\nu}_{t,u}, \\ \check{\gamma}_u^t & = \frac{4(\alpha \circ \check{\nu}_{t,\cdot}^{-1})_u}{(\sigma^2 \circ \check{\nu}_{t,\cdot}^{-1})_u}, \\ \eta(u,T) & : \quad \textit{a solution for the differential Equation (3.17).} \end{cases} \tag{4.3}$$

The Laplace transforms of $\check{\theta}_{t,T_o} X_{\check{\nu}_{t,T_o}}^{(\check{\gamma}^t)}|_{X_0^{(\check{\gamma}^t)}=x}$ and $\hat{\theta}_{t,T_o} X_{\hat{\nu}_{t,T_o}}^{(\hat{\gamma}^t)}|_{X_0^{(\hat{\gamma}^t)}=x}$ are given by:

$$\begin{aligned} &E_x\left[\exp\left\{-\lambda\bar{\theta}_{t,T_o} X_{\bar{\nu}_{t,T_o}}^{(\bar{\gamma}^t)}\right\}\right] \\ &= \exp\left\{-\frac{\lambda\bar{\theta}_{t,T_o}x}{1+2\lambda\bar{\theta}_{t,T_o}\bar{\nu}_{t,T_o}} - \int_0^{\bar{\nu}_{t,T_o}} \frac{\lambda\bar{\theta}_{t,T_o}\bar{\gamma}_u^t}{1+2\lambda\bar{\theta}_{t,T_o}(\bar{\nu}_{t,T_o}-u)} du\right\}, \end{aligned} \tag{4.4}$$

where $(\bar{\theta}_{t,T_o}, \bar{\nu}_{t,T_o}, \bar{\gamma}_u^t)$ denotes $(\check{\theta}_{t,T_o}, \check{\nu}_{t,T_o}, \check{\gamma}_u^t)$ or $(\hat{\theta}_{t,T_o}, \hat{\nu}_{t,T_o}, \hat{\gamma}_u^t)$. Especially when $\delta_u = \frac{4\alpha_u}{\sigma_u^2}$ is constant,

$$P_r[\bar{\theta}_{t,T_o} X_{\bar{\nu}_{t,T_o}}^{(\bar{\gamma}^t)} \leqslant r^*] = F_{\chi^2}\left(\frac{r^*}{\bar{\theta}_{t,T_o}\bar{\nu}_{t,T_o}}; \delta, \frac{r}{\bar{\nu}_{t,T_o}}\right), \tag{4.5}$$

where $F_{\chi^2}(\cdot; \delta, c^2)$ denotes the cumulated distribution function of the noncentral χ^2-distribution of δ degrees of freedom and non-centrality parameter c^2.

Proof. From Corollary 3.1 and Girsanov's theorem,

$$\left(\int_t^T r_u du, r_T\right)\Bigg|_{r_t=r} \text{ under } \tilde{P} \overset{law}{=} \left(\frac{1}{2}\int_{\tilde{\nu}_t}^{\tilde{\nu}_T} X_v^{(\tilde{\gamma})}\tilde{m}_v dv, \tilde{\theta}_T X_{\tilde{\nu}_T}^{(\tilde{\gamma})}\right)\Bigg|_{\tilde{\theta}_t X_{\tilde{\nu}_t}^{(\tilde{\gamma})}=r} \text{ under } P,$$

where $\tilde{\theta}_u$, $\tilde{v}_u$, $\tilde{\gamma}_u$ and $\tilde{m}_u$ are defined by (3.19). This together with Lemma 2.5 allows us to deduce

$$
\begin{aligned}
&C(t, r; T_o, T_b, K) \\
&= \tilde{E}_{t,r}\left[\exp\left\{-\int_t^T r_u du\right\}\left(B(T_o, r_{T_o}; T_b) - K\right)_+\right] \\
&= \tilde{E}_{t,r}\left[\exp\left\{-\int_t^T r_u du\right\}\left(B(T_o, r_{T_o}; T_b) - K\right) 1_{\{r_{T_o} \leqslant r^*\}}\right] \\
&= E_{\tilde{v}_t, \frac{r}{\tilde{\theta}_t}}\left[\exp\left\{-\frac{1}{2}\int_{\tilde{v}_t}^{\tilde{v}_{T_o}} X_v^{(\tilde{\gamma})}\tilde{m}_v dv\right\}\left(B(T_o, \tilde{\theta}_{T_o} X_{\tilde{v}_{T_o}}^{(\tilde{\gamma})}; T_b) - K\right) 1_{\{\tilde{\theta}_{T_o} X_{\tilde{v}_{T_o}}^{(\tilde{\gamma})} \leqslant r^*\}}\right] \qquad (4.6) \\
&= E_{\tilde{v}_t, \frac{r}{\tilde{\theta}_t}}\left[\exp\left\{-\frac{1}{2}\int_{\tilde{v}_t}^{\tilde{v}_{T_o}} X_v^{(\tilde{\gamma})}\tilde{m}_v dv\right\}\right] E_{\hat{v}_t, \frac{r}{\hat{\theta}_t}}\left[\left(B(T_o, \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}; T_b) - K\right) 1_{\{\hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})} \leqslant r^*\}}\right] \\
&= B(t, r; T_o)\left(E_{\hat{v}_t, \frac{r}{\hat{\theta}_t}}\left[B(T_o, \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}; T_b) 1_{\{\hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})} \leqslant r^*\}}\right] - K P_{\hat{v}_t, \frac{r}{\hat{\theta}_t}}\left[\hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})} \leqslant r^*\right]\right),
\end{aligned}
$$

where

$$
\begin{cases}
\hat{\theta}_u = \left(\frac{\eta(u, T_o)}{\eta(0, T_o)}\right)^2 \exp\{-\int_0^u \tilde{\beta}_v dv\}, \\
\hat{v}_u = \frac{1}{4}\int_0^u \frac{\sigma_v^2}{\hat{\theta}_v} dv, \\
\hat{\gamma}_u = \frac{4(\alpha \circ \hat{v}^{-1})_u}{(\sigma^2 \circ \hat{v}^{-1})_u}.
\end{cases} \qquad (4.7)
$$

By the same way,

$$
\begin{aligned}
B(t, r; T_b) &= \tilde{E}_{t,r}\left[\exp\left\{-\int_t^T r_u du\right\} B(T_o, r_{T_o}; T_b)\right] \\
&= B(t, r; T_o) E_{\hat{v}_t, \frac{r}{\hat{\theta}_t}}\left[B(T_o, \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}; T_b)\right].
\end{aligned} \qquad (4.8)
$$

Let us define the Radon-Nikodym derivative:

$$
\begin{aligned}
Z_{\hat{v}_t, \hat{v}_{T_o}} &= \left.\frac{d\hat{P}}{dP}\right|_{\hat{\theta}_t X_{\hat{v}_t} = r} \\
&= \frac{B(T_o, \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}; T_b)}{E_{\hat{v}_t, \frac{r}{\hat{\theta}_t}}\left[B(T_o, \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}; T_b)\right]} \\
&= \frac{\exp\left\{\frac{2\eta_u(T_o, T_b)}{\sigma_{T_o}^2 \eta(T_o, T_b)} \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}\right\}}{E_{\hat{v}_t, \frac{r}{\hat{\theta}_t}}\left[\exp\left\{\frac{2\eta_u(T_o, T_b)}{\sigma_{T_o}^2 \eta(T_o, T_b)} \hat{\theta}_{T_o} X_{\hat{v}_{T_o}}^{(\hat{\gamma})}\right\}\right]}.
\end{aligned}
$$

The last equality follows from (3.16), where due to the definition, $Z_{\nu_t,\nu_{T_o}} > 0$ and $E_{\hat{\nu}_t,\frac{r}{\hat{\theta}_t}}[Z_{\nu_t,\nu_{T_o}}] = 1$. Let $\hat{Y}_u = \hat{\theta}_u X^{(\hat{\gamma})}_{\hat{\nu}_u}$. By Itô's lemma,

$$d\hat{Y}_u = \left(\alpha_u + \left(\frac{2\eta_u(u,T_o)}{\eta(u,T_o)} - \tilde{\beta}_u\right)\hat{Y}_u\right)du + \sigma_u\sqrt{\hat{Y}_u}\,d\hat{W}_u, \tag{4.9}$$

where $\{\hat{W}_u; 0 \leqslant u\}$ is another Wiener process defined by

$$\hat{W}_u = \frac{1}{2}\int_0^{\hat{\nu}_u} \frac{\sqrt{\hat{\theta}_{\hat{\nu}_v^{-1}}}}{\sigma_{\hat{\nu}_v^{-1}}}\,dW_v.$$

Now we shall consider the martingale representation of $Z_{\hat{\nu}_t,\hat{\nu}_{T_o}}$ with respect to $\{\hat{W}_u; t \leqslant u \leqslant T_o\}$. Let us define the conditional expectation:

$$\begin{aligned} f(u,y) &= E\left[\exp\left\{\frac{2\eta_u(T_o,T_b)}{\sigma_{T_o}^2\eta(T_o,T_b)}\hat{Y}_{T_o}\right\}\middle|\, \hat{Y}_u = y\right] \\ &= E\left[\exp\left\{-\mu X^{(\hat{\gamma})}_{\hat{\nu}_{T_o}}\right\}\middle|\, X^{(\hat{\gamma})}_{\hat{\nu}_u} = \frac{y}{\hat{\theta}_u}\right], \end{aligned}$$

where

$$\mu = -\frac{2\eta_u(T_o,T_b)}{\sigma_{T_o}^2\eta(T_o,T_b)}\hat{\theta}_{T_o} > 0.$$

From Theorem 2.2,

$$f(u,y) = \exp\left\{-\frac{\mu y}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u))\hat{\theta}_u} - \int_u^{T_o}\frac{\mu\alpha_v}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_v))\hat{\theta}_v}\,dv\right\},$$

and hence

$$f_y(u,y) = -\frac{\mu}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u))\hat{\theta}_u}f(u,y).$$

Now let $\hat{Z}_{t,u} = \frac{f(u,\hat{Y}_u)}{f(t,\hat{Y}_t)}$. Then $\{\hat{Z}_{t,u}; t \leqslant u \leqslant T_o\}$ is a martingale under P such that $\hat{Z}_{t,t} = 1$ and $\hat{Z}_{t,T_o} = Z_{\hat{\nu}_t,\hat{\nu}_{T_o}}$. This together with (4.9) yields

$$\frac{d\hat{Z}_{t,u}}{\hat{Z}_{t,u}} = -\frac{\mu}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u))\hat{\theta}_u}\sigma_u\sqrt{\hat{Y}_u}\,d\hat{W}_u,$$

which means

$$\begin{aligned} \hat{Z}_{t,T_o} = Z_{\hat{\nu}_t,\hat{\nu}_{T_o}} &= \exp\left\{-\frac{1}{2}\int_t^{T_o}\left(\frac{\mu\sigma_u\sqrt{\hat{Y}_u}}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u))\hat{\theta}_u}\right)^2 du\right. \\ &\qquad \left. - \int_t^{T_o}\frac{\mu\sigma_u\sqrt{\hat{Y}_u}}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u))\hat{\theta}_u}\,d\hat{W}_u\right\}. \end{aligned}$$

Let

$$\check{W}_u = W_u + \int_0^u \frac{\mu\sigma_v\sqrt{\hat{Y}_v}}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_v))\hat{\theta}_v}dv,\ t \leqslant u \leqslant T_o.$$

By Girsanov's theorem, $\{\check{W}_u; t \leqslant u \leqslant T_o\}$ becomes a Wiener process under $\hat{P}|_{\hat{Y}_t=r}$. Using $\check{W}_u$, (4.9) is equivalent to:

$$\begin{aligned} d\hat{Y}_u &= \left(\alpha_u + \left(\frac{2\eta_u(u,T_o)}{\eta(u,T_o)} - \tilde{\beta}_u - \frac{\mu\sigma_u^2}{(1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u))\hat{\theta}_u}\right)\hat{Y}_u\right)du \\ &\quad + \sigma_u\sqrt{\hat{Y}_u}d\check{W}_u. \end{aligned} \tag{4.10}$$

Hence from Lemma 2.4 and (4.10), we get

$$\begin{aligned} &B(t,r;T_o)E_{\hat{\nu}_t,\frac{r}{\hat{\theta}_t}}\left[B(T_o,\hat{\theta}_{T_o}X^{(\hat{\gamma})}_{\hat{\nu}_{T_o}};T_b)1_{\{\hat{\theta}_{T_o}X^{(\hat{\gamma})}_{\hat{\nu}_{T_o}}\leqslant r^*\}}\right] \\ &= B(t,r;T_b)E_{\hat{\nu}_t,\frac{r}{\hat{\theta}_t}}\left[1_{\{\hat{\theta}_{T_o}X^{(\hat{\gamma})}_{\hat{\nu}_{T_o}}\leqslant r^*\}}Z_{\hat{\nu}_t,\hat{\nu}_{T_o}}\right] \\ &= B(t,r;T_b)\hat{P}\left[\hat{Y}_{T_o}\leqslant r^*|\hat{Y}_t=r\right] \\ &= B(t,r;T_b)P_{\check{\nu}_t,\frac{r}{\check{\theta}_t}}\left[\check{\theta}_{T_o}X^{(\check{\gamma})}_{\check{\nu}_{T_o}}\leqslant r^*\right], \end{aligned} \tag{4.11}$$

where

$$\begin{cases} \check{\theta}_u &= \left(\frac{1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u)}{1+2\mu\hat{\nu}_{T_o}}\right)^2\hat{\theta}_u, \\ \check{\nu}_u &= \frac{(1+2\mu\hat{\nu}_{T_o})\hat{\nu}_u}{1+2\mu(\hat{\nu}_{T_o}-\hat{\nu}_u)}, \\ \check{\gamma}_u &= \frac{4(\alpha\circ\check{\nu}^{-1})_u}{(\sigma^2\circ\check{\nu}^{-1})_u}. \end{cases}$$

Let us use $(\bar{\theta}_t, \bar{\theta}_{t,u}, \bar{\nu}_u, \bar{\nu}_{t,u}, \bar{\gamma}_u, \bar{\gamma}_u^t)$ as $(\hat{\theta}_t, \hat{\theta}_{t,u}, \hat{\nu}_u, \hat{\nu}_{t,u}, \hat{\gamma}_u, \hat{\gamma}_u^t)$ or $(\check{\theta}_t, \check{\theta}_{t,u}, \check{\nu}_u, \check{\nu}_{t,u}, \check{\gamma}_u, \check{\gamma}_u^t)$. Then from Theorem 2.2,

$$\begin{aligned} &E\left[\exp\left\{-\lambda\bar{\theta}_{T_o}X^{(\bar{\gamma})}_{\bar{\nu}_{T_o}}\right\}\Big|\,\bar{\theta}_tX^{(\bar{\gamma})}_{\bar{\nu}_t}=x\right] \\ &= \exp\left\{-\frac{\lambda\bar{\theta}_{T_o}\frac{x}{\bar{\theta}_t}}{1+2\lambda\bar{\theta}_{T_o}(\bar{\nu}_{T_o}-\bar{\nu}_t)} - \int_{\bar{\nu}_t}^{\bar{\nu}_{T_o}}\frac{\lambda\bar{\theta}_{T_o}\bar{\gamma}_u}{1+2\lambda\bar{\theta}_{T_o}(\bar{\nu}_{T_o}-u)}du\right\} \\ &= \exp\left\{-\frac{\lambda\bar{\theta}_{t,T_o}x}{1+2\lambda\bar{\theta}_{t,T_o}\bar{\nu}_{t,T_o}} - \int_0^{\bar{\nu}_{t,T_o}}\frac{\lambda\bar{\theta}_{t,T_o}\bar{\gamma}_u^t}{1+2\lambda\bar{\theta}_{t,T_o}(\bar{\nu}_{t,T_o}-u)}du\right\} \\ &= E\left[\exp\left\{-\lambda\bar{\theta}_{t,T_o}X^{(\bar{\gamma}^t)}_{\bar{\nu}_{t,T_o}}\right\}\Big|\,X^{(\bar{\gamma}^t)}_0=x\right]. \end{aligned}$$

Here we use the relation $\bar{\theta}_{T_o} = \bar{\theta}_t \bar{\theta}_{t,T_o}, \bar{\theta}_t(\bar{\nu}_{T_o} - \bar{\nu}_t) = \bar{\nu}_{t,T_o}$ to get the second equality. Thus the Laplace transforms of $\bar{\theta}_{T_o} X^{(\bar{\gamma})}_{\bar{\nu}_{T_o}}|_{\bar{\theta}_t X^{(\bar{\gamma})}_{\bar{\nu}_t}=x}$ and $\bar{\theta}_{t,T_o} X^{(\bar{\gamma}')}_{\bar{\nu}_{t,T_o}}|_{X^{(\bar{\gamma}')}_0=x}$ coincide and

$$P_{\bar{\nu}_t, \frac{r}{\bar{\theta}_t}}[\bar{\theta}_{T_o} X^{(\bar{\gamma})}_{\bar{\nu}_{T_o}} \leqslant r^*] = P_r[\bar{\theta}_{t,T_o} X^{(\bar{\gamma}')}_{\bar{\nu}_{t,T_o}} \leqslant r^*].$$

Hence together with (4.6) and (4.11), we arrive at (4.2). Next suppose that δ_u is constant. Let $Y(\delta, c^2)$ be a random variable which follows the noncentral χ^2-distribution with δ degrees of freedom and noncentral parameter c^2. The Laplace transform of $Y(\delta, c^2)$ is given by [15]:

$$E_x[\exp\{-\lambda Y(\delta, c^2)\}] = (1 + 2\lambda)^{-\frac{\delta}{2}} \exp\left\{-\frac{\lambda}{1+2\lambda} c^2\right\}.$$

Hence from (2.4), $X^{(\delta)}_t|_{X^{(\delta)}_0=x} \overset{law}{=} tY(\delta, \frac{x}{t})$, which yields (4.5). □

Using the evaluation formula (4.2), we shall show how to duplicate the T_o-maturity payoff of a call option on a T_b-maturity pure discount bond with exercise price K. Let us consider the portfolio value process $\{V_u; t \leqslant u \leqslant T_o\}$ which consists of the two bonds with maturity T_b and T_o. That is:

$$V_u = w_{u,T_b} B(u, r; T_b) + w_{u,T_o} B(u, r; T_o), \ t \leqslant u \leqslant T_o,$$

where $w_{u,T}$ denotes the number of T-maturity pure discount bonds included in the portfolio at time u. Applying the formula (4.2), let

$$\begin{cases} w_{u,T_b}|_{r_u=r} &= P_r[\check{\theta}_{u,T_o} X^{(\check{\gamma}^u)}_{\check{\nu}_{u,T_o}} \leqslant r^*], \\ w_{u,T_o}|_{r_u=r} &= -K P_r[\hat{\theta}_{u,T_o} X^{(\hat{\gamma}^u)}_{\hat{\nu}_{u,T_o}} \leqslant r^*]. \end{cases}$$

Then from (4.1) and Theorem 4.1,

$$\begin{aligned} &\exp\left\{-\textstyle\int_t^u r_v dv\right\} V_u \\ &\quad = \tilde{E}_{u,r}\left[\exp\left\{-\textstyle\int_t^{T_o} r_v dv\right\} \left(B(T_o, r_{T_o}; T_b) - K\right)_+\right], \end{aligned} \tag{4.12}$$

$$V_{T_o} = \left(B(T_o, r_{T_o}; T_b) - K\right)_+. \tag{4.13}$$

(4.12) means that $\{\exp\{-\int_t^u r_v dv\} V_u; t \leqslant u \leqslant T_o\}$ is a martingale under $\tilde{P}$ and hence the portfolio value process keeps budget constraints during $t \leqslant u \leqslant T_o$. This together with (4.13) guarantees that we can duplicate the T_o-maturity value of call option by the portfolio value process $\{V_u; t \leqslant u \leqslant T_o\}$ starting from the initial budget $V_t = C(t, r; T_o, T_b, K)$.

5. Special Case Analysis

In this section, we consider the special class of term structures which enables us to deduce the explicit form of function $\eta(u, T)$. Let us fix the current time t and assume that:

$$\tilde{\beta}_u = \bar{\beta}_t \frac{\sigma_u}{\sigma_t} - \frac{\sigma_u'}{\sigma_u}, \quad t \leqslant u. \tag{5.1}$$

We can easily check that (5.1) is equivalent to

$$\sigma_u = \frac{\sigma_t}{\exp\{-\int_t^u \tilde{\beta}_v dv\} - \bar{\beta}_t \int_t^u \exp\{\int_v^u \tilde{\beta}_w dw\} dv}, \quad t \leqslant u.$$

Let $\bar{\eta}$ be the function which satisfies the differential equation:

$$\begin{cases} \bar{\eta}''(v) + \bar{\beta}_t \bar{\eta}'(v) - \frac{1}{2}\sigma_t^2 \bar{\eta}(v) = 0, \; 0 \leqslant v, \\ \bar{\eta}(0) = 1, \; \bar{\eta}'(0) = 0. \end{cases} \tag{5.2}$$

The solution of (5.2) is given by

$$\bar{\eta}(v) = \frac{1}{\xi_t} e^{-\frac{\bar{\beta}_t}{2} v} \left(\xi_t \cosh \xi_t v + \frac{\bar{\beta}_t}{2} \sinh \xi_t v \right), \tag{5.3}$$

where $\xi_t = \frac{1}{2}\sqrt{\bar{\beta}_t^2 + 2\sigma_t^2}$. Direct computation yields

$$\begin{cases} \frac{\bar{\eta}(w)}{\bar{\eta}(v)} = e^{-\frac{\bar{\beta}_t}{2}(w-v)} \frac{\xi_t \cosh \xi_t w + \frac{\bar{\beta}_t}{2} \sinh \xi_t w}{\xi_t \cosh \xi_t v + \frac{\bar{\beta}_t}{2} \sinh \xi_t v}, \\ \frac{\bar{\eta}'(v)}{\bar{\eta}(v)} = \frac{\sigma_t^2}{2} \frac{\sinh \xi_t v}{\xi_t \cosh \xi_t v + \frac{\bar{\beta}_t}{2} \sinh \xi_t v}, \\ \left(\frac{\bar{\eta}'(v)}{\bar{\eta}(v)} \right)' = \frac{\sigma_t^2 \xi_t^2}{2(\xi_t \cosh \xi_t v + \frac{\bar{\beta}_t}{2} \sinh \xi_t v)^2}. \end{cases} \tag{5.4}$$

If we define the function $\eta(u, T)$ by

$$\eta(u, T) = \bar{\eta}(\omega_{u,T}), \quad \omega_{u,T} = \int_u^T \frac{\sigma_v}{\sigma_t} dv, \quad t \leqslant u \leqslant T, \tag{5.5}$$

then

$$\begin{cases} \eta_u(u, T) = -\bar{\eta}'(\omega_{u,T}) \frac{\sigma_u}{\sigma_t}, \\ \eta_{uu}(u, T) = \bar{\eta}''(\omega_{u,T}) \left(\frac{\sigma_u}{\sigma_t} \right)^2 - \bar{\eta}'(\omega_{u,T}) \frac{\sigma_u'}{\sigma_t}. \end{cases} \tag{5.6}$$

Substituting (5.6) into (3.17), we see that (5.2) implies (3.17). Hence (5.5) is an solution of (3.17) for all $T \geqslant t$. This together with Theorem 3.3, (5.4) and (5.6) yields

$$B(u, r; T) = \exp \left\{ \begin{array}{l} -\int_u^T \frac{\sigma_t \alpha_v \sinh \xi_t \omega_{v,T}}{\sigma_v(\xi_t \cosh \xi_t \omega_{v,T} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{v,T})} dv \\ -\frac{\sigma_t \sinh \xi_t \omega_{u,T}}{\sigma_u(\xi_t \cosh \xi_t \omega_{u,T} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T})} r \end{array} \right\}, \quad t \leqslant u \leqslant T. \tag{5.7}$$

Next we shall consider the call option evaluation. From (4.3), (5.1), (5.4) and (5.6),

$$\hat{\theta}_{u,v} = \frac{\sigma_v(\xi_t \cosh \xi_t \omega_{v,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{v,T_o})^2}{\sigma_u(\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o})^2}, \quad t \leqslant u \leqslant v \leqslant T_o\,. \tag{5.8}$$

Substituting this into (4.3), we deduce

$$\begin{aligned}
&\hat{\nu}_{u,T_o}\\
&= \frac{1}{4}\int_u^{T_o} \frac{\sigma_v^2}{\hat{\theta}_{u,v}} dv\\
&= \sigma_u \left(\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o}\right)^2 \int_u^{T_o} \sigma_v (\xi_t \cosh \xi_t \omega_{v,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{v,T_o})^{-2} dv\\
&= -\frac{\sigma_u}{2\xi_t \sigma_t}\left(\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o}\right)^2 \left(\frac{\bar{\beta}_t}{2\xi_t} - \frac{\xi_t \sinh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \cosh \xi_t \omega_{u,T_o}}{\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o}}\right)\\
&= \frac{\sigma_t \sigma_u}{4\xi_t^2} \sinh \xi_t \omega_{u,T_o} \left(\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o}\right).
\end{aligned} \tag{5.9}$$

Hence we get

$$\hat{\theta}_{u,T_o} \hat{\nu}_{u,T_o} = \frac{\sigma_t \sigma_{T_o} \sinh \xi_t \omega_{u,T_o}}{4(\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o})}. \tag{5.10}$$

By the same method, we can show that:

$$\begin{cases}
\mu_u = \frac{2\xi_t^2}{\sigma_t \sigma_u} \frac{\bar{\eta}'(\omega_{T_o,T_b})}{\bar{\eta}(\omega_{T_o,T_b})} \left(\xi_t \cosh \xi_t \omega_{u,T_o} + \frac{\bar{\beta}_t}{2} \sinh \xi_t \omega_{u,T_o}\right)^{-2},\\
\check{\theta}_{u,T_o} = \dfrac{\sigma_{T_o} \xi_t^2}{\sigma_u \left(\xi_t \cosh \xi_t \omega_{u,T_o} + \left(\frac{\bar{\beta}_t}{2} + \frac{\bar{\eta}'(\omega_{T_o,T_b})}{\bar{\eta}(\omega_{T_o,T_b})}\right) \sinh \xi_t \omega_{u,T_o}\right)^2},\\
\check{\nu}_{u,T_o} = \frac{\sigma_t \sigma_u}{4\xi_t^2} \sinh \xi_t \omega_{u,T_o} \left(\xi_t \cosh \xi_t \omega_{u,T_o} + \left(\frac{\bar{\beta}_t}{2} + \frac{\bar{\eta}'(\omega_{T_o,T_b})}{\bar{\eta}(\omega_{T_o,T_b})}\right) \sinh \xi_t \omega_{u,T_o}\right),\\
\check{\theta}_{u,T_o} \check{\nu}_{u,T_o} = \dfrac{\sigma_t \sigma_{T_o} \sinh \xi_t \omega_{u,T_o}}{4\left(\xi_t \cosh \xi_t \omega_{u,T_o} + \left(\frac{\bar{\beta}_t}{2} + \frac{\bar{\eta}'(\omega_{T_o,T_b})}{\bar{\eta}(\omega_{T_o,T_b})}\right) \sinh \xi_t \omega_{u,T_o}\right)}.
\end{cases} \tag{5.11}$$

To compute the probability $P_r[\bar{\theta}_{t,T_o} X^{(\bar{\gamma}^t)}_{\bar{\nu}_{t,T_o}} \leqslant r^*]$, we can use the relationship between $X^{(\bar{\gamma}^t)}_{\bar{\nu}_{t,T_o}}$ and $\hat{Y}_{T_o}$:

$$\begin{aligned}
P_r[\hat{\theta}_{t,T_o} X^{(\hat{\gamma}^t)}_{\hat{\nu}_{t,T_o}} \leqslant r^*] &= P_{t,r}[\hat{Y}_{T_o} \leqslant r^*],\\
P_r[\check{\theta}_{t,T_o} X^{(\check{\gamma}^t)}_{\check{\nu}_{t,T_o}} \leqslant r^*] &= \hat{P}_{t,r}[\hat{Y}_{T_o} \leqslant r^*].
\end{aligned} \tag{5.12}$$

From (4.9) and (4.10), $\{\hat{Y}_u; t \leqslant u \leqslant T_o\}$ follows the stochastic differential equation:

$$\begin{aligned}
d\hat{Y}_u &= \left(\alpha_u - \left(\frac{2\sigma_u \bar{\eta}'(\omega_{u,T_o})}{\sigma_t \bar{\eta}(\omega_{u,T_o})} + \tilde{\beta}_u\right) \hat{Y}_u\right) dt + \sigma_u \sqrt{\hat{Y}_u} d\hat{W}_u\\
&= \left(\alpha_u - \left(\frac{2\sigma_u \bar{\eta}'(\omega_{u,T_o})}{\sigma_t \bar{\eta}(\omega_{u,T_o})} + \tilde{\beta}_u + \frac{\mu_u \sigma_u^2}{(1 + 2\mu_u \hat{\nu}_{u,T_o})\hat{\theta}_{u,T_o}}\right) \hat{Y}_u\right) dt + \sigma_u \sqrt{\hat{Y}_u} d\check{W}_u,
\end{aligned}$$

where $\{\hat{W}_u; t \leqslant u \leqslant T_o\}$ ($\{\check{W}_u; t \leqslant u \leqslant T_o\}$, respectively) are the Wiener processes under P ($\hat{P}$). Then we can numerically compute the probabilities (5.12) using (5.4) through (5.11) and the call option price is given by

$$\begin{aligned} C(t, r; T_o, T_b, K) &= B(t, r; T_b)\hat{P}_{t,r}[\hat{Y}_{T_o} \leqslant r^*] \\ &\quad - KB(t, r; T_o)P_{t,r}[\hat{Y}_{T_o} \leqslant r^*]. \end{aligned} \tag{5.13}$$

Furthermore, we shall consider the additional assumption:

$$\delta_u = \frac{4\alpha_u}{\sigma_u^2} \text{ is constant for all } u \geqslant t. \tag{5.14}$$

Let $\bar{B}(u, r; T)$ ($\bar{C}(t, r; T_o, T_b, K)$, respectively) be the price of pure discount bond (call option on T_b-maturity pure discount bond) when the parameters α_u, $\tilde{\beta}_u$ and σ_u are constant for $u \geqslant t$. That is

$$\begin{aligned} \bar{B}(u, r; T) &= \bar{\eta}(T-u)^{-\frac{\delta}{2}} \exp\left\{-\frac{2\bar{\eta}'(T-u)}{\sigma_t^2\bar{\eta}(T-u)}r\right\}, \quad t \leqslant u \leqslant T, \\ \bar{C}(t, r; T_o, T_b, K) &= \bar{B}(t, r; T_b)F_{\chi^2}\left(\frac{\bar{r}^*}{\check{\theta}_{t,T_o}\check{\nu}_{t,T_o}}; \delta, \frac{r}{\check{\nu}_{t,T_o}}\right) \\ &\quad - K\bar{B}(t, r; T_o)F_{\chi^2}\left(\frac{\bar{r}^*}{\tilde{\theta}_{t,T_o}\tilde{\nu}_{t,T_o}}; \delta, \frac{r}{\tilde{\nu}_{t,T_o}}\right), \end{aligned}$$

where $\bar{r}^*$ is the unique solution for $\bar{B}(T_o, \bar{r}^*; T_b) = K$ and

$$\begin{cases} \tilde{\nu}_{t,T_o} = \frac{\sigma_t^2}{4\xi_t^2}\sinh\xi_t(T_o-t)\left(\xi_t\cosh\xi_t(T_o-t) + \frac{\bar{\beta}_t}{2}\sinh\xi_t(T_o-t)\right), \\ \tilde{\theta}_{t,T_o}\tilde{\nu}_{t,T_o} = \frac{\sigma_t^2\sinh\xi_t(T_o-t)}{4(\xi_t\cosh\xi_t(T_o-t)+\frac{\bar{\beta}_t}{2}\sinh\xi_t(T_o-t))}, \\ \check{\nu}_{t,T_o} = \frac{\sigma_t^2}{4\xi_t^2}\sinh\xi_t(T_o-t)\left(\xi_t\cosh\xi_t(T_o-t) + \left(\frac{\bar{\beta}_t}{2} + \frac{\bar{\eta}'(T_b-T_o)}{\bar{\eta}(T_b-T_o)}\right)\sinh\xi_t(T_o-t)\right), \\ \check{\theta}_{t,T_o}\check{\nu}_{t,T_o} = \frac{\sigma_t^2\sinh\xi_t(T_o-t)}{4\left(\xi_t\cosh\xi_t(T_o-t)+\left(\frac{\bar{\beta}_t}{2}+\frac{\bar{\eta}'(T_b-T_o)}{\bar{\eta}(T_b-T_o)}\right)\sinh\xi_t(T_o-t)\right)}. \end{cases}$$

Then from (5.7) and (5.13), we arrive at

$$\begin{aligned} &B(u, r; T) = \bar{B}\left(u, \frac{\sigma_t}{\sigma_u}r; u + \omega_{u,T}\right), \quad t \leqslant u \leqslant T, \\ &C(t, r; T_o, T_b, K) = \bar{C}(t, r; t + \omega_{t,T_o}, t + \omega_{t,T_b}, K). \end{aligned}$$

These relations mean that under the assumptions (5.1) and (5.14), we can evaluate the pure discount bonds and their call options only by changing the time: $\omega_{u,T} = \int_u^T \frac{\sigma_u}{\sigma_t}du$, $t \leqslant u \leqslant T$ for the classical CIR formulas. Here note that at the money interest rate r^* for $B(T_o, r; T_b)$ is given by $\frac{\sigma_{T_o}}{\sigma_t}\bar{r}^*$ when $\bar{B}(T_o, \bar{r}^*; T_o + \omega_{T_o,T_b}) = K$. The possibility of these relations was first pointed out by Rogers [17] for the pure discount bond prices. Finally we derive the volatility structure $\{\sigma_u; t \leqslant u\}$ which is consistent with the given arbitrage free initial forward rate curve $\{f(t,u); t \leqslant u\}$. From the definition of forward rate,

$$\begin{aligned} f(t,u) &= -\frac{\partial}{\partial u}\log B(t,r;u) \\ &= \frac{\delta\sigma_u\bar{\eta}'(\omega_{t,u})}{2\sigma_t\bar{\eta}(\omega_{t,u})} + \frac{2\sigma_u}{\sigma_t^3}\left(\frac{\bar{\eta}'(\omega_{t,u})}{\bar{\eta}(\omega_{t,u})}\right)' r, \end{aligned} \tag{5.15}$$

and

$$\int_t^u f(t,v)dv = \frac{\delta}{2}\log\bar{\eta}(\omega_{t,u}) + \frac{2}{\sigma_t^2}\frac{\bar{\eta}'(\omega_{t,u})}{\bar{\eta}(\omega_{t,u})}r.$$

For some given $t, r \geqslant 0$, let us define the function:

$$F_{t,r}(x) = \frac{\delta}{2}\log\bar{\eta}(x) + \frac{2}{\sigma_t^2}\frac{\bar{\eta}'(x)}{\bar{\eta}(x)}r.$$

Since

$$F'_{t,r}(x) = \frac{\delta}{2}\frac{\bar{\eta}'(x)}{\bar{\eta}(x)} + \frac{2}{\sigma_t^2}\left(\frac{\bar{\eta}'(x)}{\bar{\eta}(x)}\right)' r > 0,$$

the inverse function $F_{t,r}^{-1}(y)$ is well defined. Then (5.15) is equivalent to

$$\sigma_u = \frac{f(t,u)}{F'_{t,r}\left(F_{t,r}^{-1}(\int_t^u f(t,v)dv)\right)}\sigma_t,\ t \leqslant u.$$

This enables us to define the volatility structure $\{\sigma_u; t \leqslant u\}$ which is consistent with the given arbitrage free (i.e., positive) initial forward rate curve $\{f(t,u); t \leqslant u\}$.

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