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ABSTRACT

This paper expresses the constant elasticity of variance option pricing formula in terms of the noncentral chi-square distribution. This allows the application of well-known approximation formulas and the derivation of a whole class of closed-form solutions. In addition, a simple and efficient algorithm for computing this distribution is presented.

THIS PAPER SHOWS THAT the constant elasticity of variance (CEV) formula can be expressed as a function of the noncentral chi-square distribution. A simple and efficient algorithm for computing this distribution is presented. Approximations to this distribution can be used to estimate accurately the CEV formula when the computation of the exact solution is problematic.

Section I discusses theories and some evidence on the association between volatility and price level. Section II presents the particular kind of relationship assumed by the CEV model and reviews empirical evidence on the model. Section III shows that the CEV formula can be expressed in terms of the noncentral chi-square distribution. A simple algorithm for computing this distribution is derived in Section IV. Section V presents some special "closed-form" solutions to this distribution. Finally, Section VI shows that an approximation to the CEV formula can be used when the exact solution converges slowly.

I. Relationship between Volatility and Price Level

Several theoretical arguments imply an association between stock price and volatility. Geske [11], Black [3], and Christie [5] consider the effects of financial leverage on the variance of the stock. An increase in the stock price reduces the debt-equity ratio of the firm and therefore reduces the variance of the stock's returns. Black hypothesizes that price changes may also affect volatility through their impact on operating leverage. In addition, he proposes a reverse causal relationship whereby an increase in the volatility of stocks causes prices to fall. In Rubinstein's [17] displaced diffusion model, the relation between price and volatility of returns depends on the firm's asset mix and debt-equity ratio.

Empirical evidence supports the hypothesis that volatility changes with stock price. Schmalensee and Trippi [19], examining just over a year of weekly data on six stocks, find a strong negative relationship between stock price changes and changes in implied volatility. Black [3], using over ten years of data on thirty stocks, finds that a given proportional increase (decrease) in stock price is

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generally associated with an even larger proportional decrease (increase) in standard deviation.

II. Empirical Evidence Supporting the CEV Model

The CEV model proposes the following deterministic relationship between stock price, S, and volatility:

$$\sigma(S, t) = \delta \cdot S^{(\beta-2)/2}.$$

The elasticity of return variance with respect to price equals $\beta - 2$, and, if $\beta < 2$, volatility and price are inversely related. The stock price is assumed to be governed by the diffusion process

$$dS = \mu S dt + \delta S^{\beta/2} dZ,$$

where dZ is a Weiner process. If $\beta = 2$ (i.e., the elasticity is zero), prices are lognormally distributed and the variance of returns is constant, as is assumed in the Black-Scholes model.

Beckers [2] estimates β for forty-seven stocks using a year of daily data. He finds thirty-seven of the estimates to be significantly less than two and concludes that the CEV diffusion process "could be a better descriptor of the actual stock price behavior than the traditionally used lognormal model." Christie [5] estimates elasticities for 379 firms using over sixteen years of quarterly data. The elasticities are found to be generally negative and, as expected, strongly influenced by the firms' debt-equity ratios. Choi and Longstaff [4] find evidence that seasonal volatility of soybean futures is described well by a CEV process.

Empirical evidence on the CEV option pricing model is somewhat limited. MacBeth and Merville [14] compare deviations of market from model prices for both the Black-Scholes and CEV call pricing models. Based on a daily sample of options on six stocks over one year, they conclude that the CEV model better explains market prices than the Black-Scholes model. (See Manaster [15] for criticisms of their methodology.)

Emanuel and MacBeth [9] expand on the sample used in the MacBeth and Merville study (by including an additional year of data) and test the predictive power of the two models. They find that, when the prediction period is less than one month (i.e., model prices are generated using parameters estimated no more than a month earlier), the CEV model predicts market prices better than the Black-Scholes model. In addition, the elasticities of each stock are found to vary considerably over time but are generally negative.

Hauser and Bagley [12] compare the predictive power of the CEV and Garman and Kohlhagen [10] models in pricing call options on five currencies over a six-month sample period. The CEV model is found generally to perform better.

III. The CEV Formula

As discussed in Cox and Ross [7], when a perfect hedge can be formed between a European option and its underlying stock, the option can be valued by determining the discounted expected value of its payoff assuming risk neutrality.

Cox [6] shows that, for a CEV process with β less than two, the density function of S_T, the stock price at T, conditional on S_t, the current stock price, in a risk-neutral world is

$$f(S_T, T; S_t, t) = (2-\beta)k^{1/(2-\beta)}(xw^{1-2\beta})^{1/(4-2\beta)}e^{-x-w}I_{1/(2-\beta)}(2\sqrt{xw}), \tag{1}$$

where

$$k = \frac{2(r-a)}{\delta^2(2-\beta)[e^{(r-a)(2-\beta)\tau}-1]},$$

$$x = kS_t^{2-\beta}e^{(r-a)(2-\beta)\tau},$$

$$w = kS_T^{2-\beta},$$

and $\tau = T - t$; $I_q(\cdot)$ is the modified Bessel function of the first kind of order q; r denotes the riskless interest rate; and a denotes the continuous proportional dividend rate.

The European call price with exercise price E is then

$$C = e^{-r\tau}\int_E^\infty f(S_T, T; S_t, t)(S_T - E)\, dS_T.$$

Making the change of variable $S_T = (w/k)^{1/(2-\beta)}$ and simplifying gives

$$C = S_t e^{-a\tau}\int_y^\infty e^{-w-x}(w/x)^{1/(4-2\beta)}I_{1/(2-\beta)}(2\sqrt{xw})\, dz$$

$$+ Ee^{-r\tau}\int_y^\infty e^{-w-x}(x/w)^{1/(4-2\beta)}I_{1/(2-\beta)}(2\sqrt{xw})\, dz,$$

where

$$y = kE^{2-\beta}.$$

The integrands are both noncentral chi-square density functions.[1] The first is $p[2w; 2 + 2/(2-\beta), 2x]$, with $2 + 2/(2-\beta)$ degrees of freedom and noncentral parameter $2x$. Integrating with respect to w from y to infinity gives the corresponding complementary distribution function $Q[2y; 2 + 2/(2-\beta), 2x]$.

The second integrand is equal to $p[2x; 2 + 2/(2-\beta), 2w]$, where $2w$ is now the noncentral parameter. Noting that $I_n(z) = I_{-n}(z)$ if n is an integer, this can be written instead as $p[2w; 2 - 2/(2-\beta), 2x]$ when $1/(2-\beta)$ is an integer. This yields the formula

$$C = S_t e^{-a\tau}Q[2y; 2 + 2/(2-\beta), 2x]$$
$$- Ee^{-r\tau}Q[2y; 2 - 2/(2-\beta), 2x]. \tag{2}$$

In the Appendix, it is shown that

$$\int_y^\infty p[2x; 2 + 2/(2-\beta), 2w]\, dw = 1 - Q[2x; 2/(2-\beta), 2y],$$

[1] See Johnson and Kotz [13, p. 133] for the noncentral chi-square density function.

which provides a call price formula applicable to all $\beta < 2$:[2]

$$C = S_t e^{-a\tau} Q[2y;\ 2 + 2/(2-\beta),\ 2x] - Ee^{-r\tau}(1 - Q[2x;\ 2/(2-\beta),\ 2y]). \tag{3}$$

Comparing (2) and (3) gives the identity

$$Q(z;\ 2n,\ \kappa) + Q(\kappa;\ 2-2n,\ z) = 1 \tag{4}$$

for integral n and $z, \kappa > 0$.

IV. Computing the Noncentral Chi-Square Distribution

The complementary noncentral chi-square distribution function, $Q(z;\ v,\ \kappa)$, evaluated at z, with v degrees of freedom and noncentral parameter κ, may be represented as a weighted average of complementary central probability functions:[3]

$$Q(z;\ v,\ \kappa) = \sum_{n=0}^{\infty} e^{-\kappa/2} \frac{(\kappa/2)^n}{\Gamma(n+1)} Q(z;\ v+2n,\ 0), \tag{5}$$

for $z, \kappa > 0$. The function $Q(z;\ v + 2n,\ 0)$, in turn, is related to the complementary gamma distribution function:

$$Q(z;\ v+2n,\ 0) = G[n + (v/2),\ z/2],$$

where

$$G(m,\ t) = \int_t^{\infty} g(m,\ x)\, dx,$$

$$g(m,\ x) = \frac{e^{-x} x^{m-1}}{\Gamma(m)}.$$

Substituting into (5) and employing the notation for the gamma density function, $g(\cdot,\cdot)$, gives

$$Q(z;\ v,\ \kappa) = \sum_{n=1}^{\infty} g(n,\ \kappa/2) G[n + (v-2)/2,\ z/2]$$

$$\text{or } Q(2z;\ 2v,\ 2\kappa) = \sum_{n=1}^{\infty} g(n,\ \kappa) G(n + v - 1,\ z). \tag{6}$$

The evaluation of the infinite sum can be computationally expensive, especially when z and κ are large (e.g., when β is close to two, volatility is low, or τ is small in the CEV formula).

[2] Emanuel and MacBeth [9] derive a formula for $\beta > 2$, which can be shown to be equivalent to

$$C = S_t e^{-a\tau} Q[2x;\ 2/(\beta-2),\ 2y] - Ee^{-r\tau}(1 - Q[2y;\ 2 + 2/(\beta-2),\ 2x]).$$

[3] See Abramowitz and Stegun [1, equations 26.4.25 (p. 942) and 26.4.19 (p. 941)]. It is simple to verify this formula by integrating the density function, as is done in the Appendix.

Integrating by parts yields a recurrence formula for the complementary gamma function:[4]

$$G(m+1, t) = \int_t^\infty \frac{e^{-x}x^m}{\Gamma(m+1)}\,dx$$

$$= \frac{-e^{-x}x^m}{\Gamma(m+1)}\Bigg|_t^\infty + \int_t^\infty \frac{e^{-x}x^{m-1}}{\Gamma(m)}\,dx$$

$$= g(m+1, t) + G(m, t). \tag{7}$$

Applying (7) to (6),

$$\begin{aligned} Q(2z;\, v,\, 2\kappa) = {} & g(1, \kappa)[G(v-1, z) + g(v, z)] \\ & + g(2, \kappa)[G(v-1, z) + g(v, z) + g(1+v, z)] \\ & + g(3, \kappa)[G(v-1, z) + g(v, z) \\ & + g(1+v, z) + g(2+v, z)] \\ & + \cdots. \end{aligned}$$

Because the double sum is absolutely convergent (all the terms, of course, are positive), the order of the terms can be rearranged. Noting that

$$\textstyle\sum_{n=1}^{\infty} g(n, \kappa) = e^{-\kappa} \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} = 1$$

and summing down columns yields[5]

$$\begin{aligned} Q(2z;\, 2v,\, 2\kappa) = {} & G(v-1, z) + g(v, z) \\ & + g(1+v, z)[1 - g(1, \kappa)] \\ & + g(2+v, z)[1 - g(1, \kappa) - g(2, \kappa)] \\ & + \cdots. \end{aligned} \tag{8}$$

Using (7) it is clear that

$$G(v-1, z) + \textstyle\sum_{n=0}^{\infty} g(n+v, z) = \lim\limits_{n\to\infty} G(n+v, z) = 1.$$

Applying this to (8) finally gives

$$\begin{aligned} Q(2z;\, 2v,\, 2\kappa) = {} & 1 - g(1+v, z)[g(1, \kappa)] \\ & - g(2+v, z)[g(1, \kappa) + g(2, \kappa)] \\ & - g(3+v, z)[g(1, \kappa) + g(2, \kappa) + g(3, \kappa)] \\ & - \cdots. \end{aligned}$$

[4] This recurrence formula is well known and is reported in Abramowitz and Stegun [1, p. 262].

[5] Equation (8) is similar to an algorithm for the noncentral chi-square distribution function derived by Robertson [16].

That is,

$$Q(2z;\ 2v,\ 2\kappa) = 1 - \textstyle\sum_{n=1}^{\infty} g(n + v,\ z) \sum_{i=1}^{n} g(i,\ \kappa). \tag{9}$$

The distribution function can be expressed as an infinite double sum of gamma functions and requires the computation of no incomplete gamma functions. The computation cost of the double sum is only a small fraction of the cost of the original series.

Equation (9) allows the following simple iterative algorithm to be used to compute the infinite sum (when z and κ are not too large). First initialize the following four variables:

$$\begin{aligned} gA &= \frac{e^{-z}z^{v}}{\Gamma(1+v)} && \{= g(1+v,\ z)\}, \\ gB &= e^{-\kappa} && \{= g(1,\ \kappa)\}, \\ Sg &= gB, && \\ R &= 1 - gA \cdot Sg. && \end{aligned}$$

Then repeat the following loop beginning with $n = 2$ and incrementing n by one after each iteration. The loop is terminated when the contributions to the sum, R, are declining and very small:

$$\begin{aligned} gA &= gA \cdot \frac{z}{n+v-1} && \{= g(n+v,\ z)\}, \\ gB &= gB \cdot \frac{\kappa}{n-1} && \{= g(n,\ \kappa)\}, \\ Sg &= Sg + gB && \{= g(1,\ \kappa) + \cdots + g(n,\ \kappa)\}, \\ R &= R - gA \cdot Sg && \{= \text{the } n\text{th partial sum}\}. \end{aligned}$$

At each iteration, gA equals $g(n + v,\ z)$, gB equals $g(n,\ \kappa)$, and Sg equals $g(1,\ \kappa) + \cdots + g(n,\ \kappa)$. The only nonelementary function necessary to compute is a single gamma function. (When v is a multiple of ½, this too reduces to an elementary function.)

It is simpler than the algorithm derived by Robertson [16] because, as mentioned above, no incomplete gamma functions need to be computed. (Robertson's algorithm requires that one such function be computed.)

V. Special Cases

For odd degrees of freedom, $Q(z;\ v,\ \kappa)$ can be represented by the sum of normal distributions and elementary functions.[6] Letting $N'(\cdot)$ denote the standard

[6] The formula for $Q(z;\ 3,\ k)$ is in Johnson and Kotz [13]. The formulas for all other odd-valued degrees of freedom can be found by using the formula for the modified Bessel function of the first kind of order $n + ½$. This can be found in Johnson and Kotz or Abramowitz and Stegun [1].

normal density function and $N(\cdot)$ the normal distribution, the formulas for degrees of freedom of 1, 3, and 5 are

$$Q(z;\,1,\,\kappa) = N(\sqrt{\kappa} - \sqrt{z}) + N(-\sqrt{\kappa} - \sqrt{z})$$

$$Q(z;\,3,\,\kappa) = Q(z;\,1,\,\kappa) + [N'(\sqrt{\kappa} - \sqrt{z}) - N'(\sqrt{\kappa} + \sqrt{z})]/\sqrt{\kappa}$$

$$\begin{aligned} Q(z;\,5,\,\kappa) = Q(z;\,1,\,\kappa) + \kappa^{-3/2}[(\kappa - 1 + \sqrt{\kappa z}) \cdot N'(\sqrt{\kappa} - \sqrt{z}) \\ - (\kappa - 1 - \sqrt{\kappa z}) \cdot N'(\sqrt{\kappa} + \sqrt{z})]. \end{aligned}$$

The case of $\beta = 0$ (i.e., volatility inversely related to S) corresponds to 3 and 1 degree(s) of freedom in the two distribution functions in (3). It is a straightforward exercise to verify the simple formula for this case derived by Cox and Ross [7].

Simple formulas for a whole family of values of β ($4/3$, $8/5$, $12/7$, $\cdots$) can also be derived. Though not particularly interesting in themselves, the formula for $\beta = 4/3$ (corresponding to 5 and 3 degrees of freedom) can be combined with the formulas for $\beta = 0$ and $\beta = 2$ (the Black-Scholes formula) to interpolate CEV prices for β's between zero and two. The resulting estimates are very accurate over a very wide range of parameters.[7]

VI. Approximating the Noncentral Chi-Square Distribution

The algorithm suggested for computing $Q(2z;\,2v,\,2\kappa)$ may converge slowly when z and κ are large. In addition, overflow and underflow errors may be encountered. Fortunately, approximations exist that are accurate for this range of parameters.

A number of approximations to the noncentral chi-square distribution have been developed. (See Johnson and Kotz [13, Chapter 28] for a review.) One particularly good approximation (presented below slightly simplified) is derived by Sankaran [18]:

$$Q(z;\,v,\,\kappa) \sim \frac{1 - hp[1 - h + 0.5(2 - h)mp] - [z/(v + \kappa)]^h}{h\sqrt{2p(1 + mp)}}, \tag{10}$$

where

$$h = 1 - (2/3)(v + \kappa)(v + 3\kappa)(v + 2\kappa)^{-2},$$

$$p = \frac{v + 2\kappa}{(v + \kappa)^2},$$

$$m = (h - 1)(1 - 3h).$$

[7] Newton's interpolation formula can be applied to fit a second-degree polynomial to three CEV values for which "closed-form" solutions exist. If $f(\beta)$ denotes the call price, then the interpolated price for any β between 0 and 2 is

$$\begin{aligned} f(\beta) = f(0) + (3/4)\beta[f(4/3) - f(0)] \\ + (1/4)\beta(\beta - 4/3)[3f(0) - 9f(4/3) + 6f(2)]. \end{aligned}$$

The accuracy of Cox's square-root approximation declines as volatility and time to expiration increase. The interpolated estimate is much more robust.

Cox [6] applies this formula to derive the approximation to the square-root process ($\beta = 1$) reported in Beckers [2]. Because $1/(2 - \beta)$ is an integer for this case, either formula (2) or (3) may be used to compute the call price. Cox's approximation can be derived using (2).

The CEV formula expressed as (3), however, shows that the approximation formula (10) is applicable for any value of β. The approximation is generally extremely accurate when x and y are large. These are precisely the circumstances under which the exact solution may be difficult to compute. For small x and y the approximation deteriorates, but the exact solution converges quickly.

VII. Conclusion

This paper presents a simple and efficient algorithm for computing the noncentral chi-square distribution function and applies this to the CEV option pricing formula. For large x or y even this algorithm is slow, but an approximation for this parameter range is extremely accurate.

The noncentral chi-square distribution has many applications, including estimating the power of the chi-square test and coverage problems in ballistics. In addition, in Cox, Ingersoll, and Ross [8], it is the distribution of the interest rate following a mean-reverting process. The function appears explicitly in their formula for a call option on a discount bond.

Appendix

$$\begin{aligned}\int_y^\infty p(2z;\,2v,\,2\kappa)\,d\kappa &= \int_y^\infty e^{-z-\kappa}(z/\kappa)^{(v-1)/2} I_{v-1}(2\sqrt{\kappa z})\,d\kappa \\ &= \int_y^\infty e^{-z-\kappa}(z/\kappa)^{(v-1)/2}(\sqrt{z\kappa})^{v-1}\sum_{n=0}^{\infty}\frac{(z\kappa)^n}{\Gamma(n+1)\Gamma(n+v)}\,d\kappa \\ &= \sum_{n=1}^{\infty} g(n+v-1,\,z)G(n,\,y) \\ &= \sum_{n=1}^{\infty} g(n+v-1,\,z)\sum_{i=1}^{n} g(i,\,y).\end{aligned}$$

where $\lambda = (v-2)/2$. Comparing this expression with (9) gives

$$\int_y^\infty p(2z;\,2v,\,2\kappa)\,d\kappa = 1 - Q(2z;\,2v-2,\,2y).$$

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