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SOLUTION OF THE EXTENDED CIR TERM STRUCTURE AND BOND OPTION VALUATION

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The extended Cox-Ingersoll-Ross (ECIR) models of interest rates allow for time-dependent parameters in the CIR square-root model. This article presents closed-form pathwise unique solutions of these unsolved stochastic differential equations (s.d.e.s) in terms of functionals of their driving Brownian motion and parameters. It is shown that quadratics in solution of linear s.d.e.s solve the ECIR model if and only if the *dimension* of the model is a positive integer and that this solution can be achieved by construction of a pathwise unique *generalized* Ornstein-Uhlenbeck process from the ECIR Brownian motion. For real valued dimensions an extension of the time-change theorem of Dubins and Schwarz (1965) is presented and applied to show that a lognormal process solves the model through a stochastic time change. Pathwise equivalence to a rescaled time-changed Bessel square process is also established. These novel results are applied to characterize zero-hitting time and to produce transition density and zero-hitting conditions for the ECIR spot rate. The CIR term structure is then extended to ECIR under no arbitrage, and its solutions and the transition density are represented under a new ECIR martingale measure. The findings are employed to derive a closed-form ECIR bond option valuation formula which generalizes that obtained by CIR (1985).

KEY WORDS: interest rates, square-root models, solution of stochastic differential equations, representation of martingales, time and measure transformations, term structure, bond option pricing

1. INTRODUCTION

The extended Cox-Ingersoll-Ross (ECIR) model of the term structure of interest rates which allows for time varying parameters was first considered by Hull and White (1990), where numerical valuation of the instruments was suggested. This paper derives the pathwise unique solutions of the unsolved stochastic dynamics of the ECIR term structure in terms of functionals of the Brownian motion driving the spot rate and its parameter functions. In addition to providing interesting mathematical and practical insight on the behavior of the modeled interest rates, these solutions are shown to produce other useful novel results such as characterization of the spot rate zero-hitting time, generalization of the zero-hitting conditions and the derivation of the extended spot rate transition densities under new martingale measures which are of particular significance in valuation of interest rate derivatives. This will be illustrated through the derivation of an ECIR bond option valuation formula. The solutions presented here can also be used in pathwise applications such as forecasting and efficient pathwise simulations and model validation. This is also important since due to the singularity of the square-root models numerical solutions via standard discretization methods are generally highly unstable (see Maghsoodi and Harris 1987).

The CIR model is among those which satisfy most of the desirable properties expected from an attractive term structure model, such as consistency with a term structure theory and

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methodology, nonnegativity of the interest rates, stochastic volatility, empirical evidence, analytical tractability and availability of exact pricing formulae and uncomplicated fitting of parameters to the initial term structure (Cox, Ingersoll, and Ross 1985, Brown and Dybvig 1986, Hull and White 1990, Longstaff and Schwartz 1992, Chen and Scott 1992, Brown and Schaefer 1994).

Cox, Ingersoll, and Ross (1985) modeled the evolution of the spot rate, $r(t)$, by the s.d.e.

$$\text{(1.1)(CIR)} \qquad dr(t) = k(\theta - r(t))\,dt + \sigma\sqrt{r(t)}\,dw(t), \qquad r(0) > 0,$$

where $w(t)$ is a standard Brownian motion, k, θ, and σ, are positive constants, representing reversion rate, asymptotic interest rate and volatility parameters respectively. The quantity $\delta \triangleq 4\theta k/\sigma^2$ plays a critical role in the behavior of the process. We assume that it is positive and finite and will refer to it as the *dimension* of the process $r(t)$. By the CIR(δ) process we shall mean the pathwise unique strong solution (see, e.g., Revus and Yor 1991) of the s.d.e. (1.1). CIR(d) will be used when δ is a positive integer d. Feller (1951) studied the Fokker-Plank-Kolmogorov equation for the transition density of CIR(δ) and showed that if $\delta \geq 2$, $r(t)$ always remains positive, and if $\delta < 2$, it can reach zero but will never become negative. The probability distribution function of $r(t)$ was shown to be a rescaled noncentral chi-square with δ degrees of freedom. Shiga and Watanabe (1973) showed that the law of CIR(d) is that of the squared norm of the Ornstein-Uhlenbeck process. Maghsoodi (1992, 1993) gave a pathwise relation between the driving Brownian motions of these processes and proved that CIR(d) is in fact a.s.P. equivalent to the squared norm of a pathwise unique Ornstein-Uhlenbeck process (see Lemma 2.1 and Corollary 2.2). Furthermore Maghsoodi (1993) showed that CIR(δ) is a.s.P. equivalent to a time changed lognormal process (see Corollary 2.3).

An important generalization of the model (1.1)(CIR) is to allow for time-dependent parameters to represent possible time-varying expected trends and volatilities of the market and the economy. Thus let $\theta(t)$, $k(t)$, and $\sigma(t)$ be positive real-valued bounded and continuous functions on $[0, \infty)$ such that the dimension (function) $\delta(t) \triangleq 4\theta(t)k(t)/\sigma^2(t)$ is bounded. By the ECIR($\delta(t)$) process we shall mean the pathwise unique strong solution of the s.d.e.:

$$\text{(1.2)(ECIR)} \qquad dr(t) = k(t)(\theta(t) - r(t))\,dt + \sigma(t)\sqrt{r(t)}\,dw(t), \qquad r(0) > 0.$$

ECIR(d) will be used when $\delta(t) \equiv d \in \mathbf{N}$, for all $0 \leq t < \infty$. Lemma 2.1 shows that given the parameters and the driving Brownian motion of ECIR(d), there exists a pathwise unique *generalized* d-dimensional Ornstein-Uhlenbeck process from which this Brownian motion can be uniquely constructed. It is then proved in Theorem 2.1 that the ECIR(d) process is equivalent a.s.P. to the squared norm of this process. Detailed proofs of Lemma 2.1 and Theorem 2.1 and other implications of the proofs have been given in Maghsoodi (1992, 1993). Investigating the scope of the methods of Lemma 2.1 and Theorem 2.1, it is proved in Theorem 2.2 that in general ECIR($\delta(t)$) admits a quadratic representation in the solution of d-dimensional linear (in the narrow sense) s.d.e.s, if and only if its dimension is a positive integer. In Theorem 2.3 the result of Theorem 2.1 is applied to derive the novel transition density of the ECIR(d) process.

Solutions of the more general cases of real-valued dimensions are presented in Section 2.2. Volkonskii (1958) showed the equivalence in law of a stopped Bessel process to a lognormal process. Williams (1974) showed that the exponential of Brownian motion with drift is equivalent to a time-changed Bessel process (see also Geman and Yor 1993). Pitman and Yor (1982) showed that a class of CIR(δ) processes are equivalent in law to a time-changed BESQ(δ) process (Bessel square with dimension δ). Lemma 2.3 shows that the ECIR(δ) (thus the CIR(δ)) process is in fact a.s.P. equivalent to a pathwise unique rescaled time-changed BESQ(δ) process. This result is used to derive transition density of the ECIR(δ) process which generalizes that obtained by Feller (1951) (see also CIR 1985). The general solutions are presented in Theorems 2.4 and 2.5 where it is shown that the ECIR($\delta(t)$) process is a.s.P. equivalent to a lognormal process through a stochastic time change. This result is proved by presenting an extension (Lemma 2.2) of the time-change theorem of Dubins and Schwarz, proved in Maghsoodi (1993). Theorems 2.4 and 2.5 also characterize the zero-hitting time of the spot rate and generalize the conditions for inaccessibility of zero interest rates to the ECIR models.

In Section 3 the ECIR($\delta(t)$) term structure is presented. In Section 3.1 the dynamics of the entire ECIR($\delta(t)$) term structure is derived under no arbirage. Dybvig and Ross (1987), Heston (see Duffie 1992) and recently Rogers (1995) have pointed out the equivalence of no arbitrage and equilibrium pricing. This was directly shown for the ECIR($\delta(t)$) model in Maghsoodi (1992, 1993). In particular it was shown to satisfy the "*forward rate drift restriction*" of Heath, Jarrow, and Morton (1992) (see (3.16)). In Section 3.2 a novel ECIR($\delta(t)$) equivalent martingale measure is presented. The methods developed in the preceding sections are applied to solve the ECIR(d) and ECIR(δ) spot rates and obtain their transition density under this measure. This density is used in the derivation of ECIR(δ) bond option valuation formula (Maghsoodi 1992) which generalizes that of CIR(1985). The analyses and derivations of Section 3 are entirely based on the stochastic dynamics of the term structure, with a martingale measure methodology akin to the fundamental works of Harrison and Pliska (1981) and Heath, Jarrow, and Morton (1992).

2. REPRESENTATIONS OF THE SPOT RATES

Consider the probability space $(\Omega, \mathbf{F}, \mathbf{P})$, with filtration $\mathrm{F} = \{F_t; 0 \le t < \infty\}$, satisfying the usual conditions. In what follows, unless otherwise specified the random processes are adapted, the Brownian motions are standard, and the relationships hold for $0 \le t \le T < \infty$ (a.s.P.). The symbol $\triangleq$ will be used when introducing notations or definitions.

2.1. Integer-Valued Dimensions

Suppose the state variables follow a generalized d-dimensional Ornstein-Uhlenbeck process $Y(t) = (y_1(t), \ldots, y_d(t))$, with mutually independent components each satisfying the s.d.e.

$$dy_i(t) = -\frac{1}{2}k(t)y_i(t)\,dt + \frac{1}{2}\sigma(t)\,dw_i(t), \qquad i = 1, 2, \ldots, d. \tag{2.1}$$

Let $\mathcal{K}(t, s) \triangleq \exp(\frac{1}{2}\int_t^s k(u)\,du)$. Then

$$y_i(t) = \mathcal{K}^{-1}(0, t)\left[\frac{1}{2}\int_0^t \sigma(u)\mathcal{K}(0, u)\,dw_i(u) + y_i(0)\right] \quad a.s.P. \tag{2.2}$$

LEMMA 2.1. *Given a one-dimensional (P, F_t)-Brownian motion $w(t)$, there exists a d-dimensional (P, F_t)-Brownian motion $\bar{W}(t) = (\bar{w}_1(t), \dots, \bar{w}_d(t))$, such that*

$$w(t) = \sum_{i=1}^{d} \int_0^t \frac{y_i(u)}{\|Y(u)\|} \, d\bar{w}_i(u) \quad a.s.P. \tag{2.3}$$

Furthermore, it is possible to choose $\bar{W} = W$ a.s.P.

THEOREM 2.1. *Let $\theta(\cdot)$, $k(\cdot)$, and $\sigma(\cdot)$ be positive real-valued, bounded and continuous functions on $0 \le t \le T$. Then the pathwise unique strong solution* ECIR(d) *is given by*

$$r(t) = \|Y(t)\|^2 \quad a.s.P. \tag{2.4}$$

Proof. By Lemma 2.1 we can construct the Brownian motion $W(t)$, and thus the process $Y(t)$ from $w(t)$. Application of Itô's lemma to $\|Y(t)\|^2$ gives

$$dr(t) = \left(-k(t)\|Y(t)\|^2 + \frac{1}{4}\sigma^2(t)d\right) dt + \sigma(t) \sum_{i=1}^{d} y_i(t) \, dw_i(t).$$

Using (2.4) and $d \equiv 4\theta(t)k(t)/\sigma^2(t)$ we obtain

$$dr(t) = k(t)\,(\theta(t) - r(t))\, dt + \sigma(t)\sqrt{r(t)} \sum_{i=1}^{d} \frac{y_i(t)}{\|Y(t)\|} \, dw_i(t).$$

The proof of the theorem now follows from (2.3). □

Thus Lemma 2.1 shows the existence and pathwise uniqueness of the process $Y(t)$ which can be suitably constructed from the ECIR(d) Brownian motion. Theorem 2.1 proves that with this construction (2.4) is the ECIR(d) process. The proof of Lemma 2.1 uses representation of local martingales as stochastic integrals (RLMSI) (see, e.g., Ikeda and Watanabe 1989). A direct consequence of (2.4) is the derivation of zero-hitting conditions for ECIR(d). The proof of this and Lemma 2.1 and implications of the proof of Theorem 2.1 have been given in Maghsoodi (1992, 1993). A comment (without proof) on obtaining integer-dimensioned ECIR models from norm square of "Gauss Markov" processes has recently been remarked in Rogers (1995).

REMARK 2.1 (Additivity). An interesting generalization of the result of Shiga and Watanabe (1973) on the additivity of Bessel square processes follows from the proof of Lemma 2.1. That is, if ECIR$(\delta_1(t))$ and ECIR$(\delta_2(t))$ are independent and have the same $k(t)$ and $\sigma(t)$, then

$$\text{ECIR}(\delta_1(t)) + \text{ECIR}(\delta_2(t)) = \text{ECIR}(\delta_1(t) + \delta_2(t)) \quad \text{a.s.P.},$$

and this process has the same $k(t)$ and $\sigma(t)$. This follows since

$$\int_0^t \frac{\sqrt{r_1}\,dw_1 + \sqrt{r_2}\,dw_2}{\sqrt{r_1 + r_2}}$$

is a Brownian motion.

COROLLARY 2.1. *For* $0 \leq t \leq s \leq T$, *the* ECIR(d) *process* $r(s)$ *is given by*

$$r(s) = \mathcal{K}^{-2}(t, s) \sum_{i=1}^{d} \left[\frac{1}{2} \int_t^s \sigma(u)\mathcal{K}(t, u)\, dw_i(u) + y_i(t) \right]^2 \quad a.s.P. \tag{2.5}$$

where $\|Y(t)\|^2 = r(t)$ *a.s.P.*

Proof. See Maghsoodi (1993). □

It follows from (2.5) that

$$\lim_{s\to\infty} E(r(s)) = \lim_{s\to\infty} \frac{\sigma^2(s)d}{4k(s)} = \lim_{s\to\infty} \theta(s). \tag{2.6}$$

That is, the asymptotic central value of ECIR(d) model. The solution of the unextended CIR(d) model (1.1) now follows as a special case.

COROLLARY 2.2. *For all* $0 \leq t \leq s \leq T$, *the* CIR(d) *process is given by*

$$r(s) = \exp(-k(s-t)) \sum_{i=1}^{d} \left[\frac{1}{2} \int_t^s \sigma \exp\left(\frac{1}{2}k(u-t)\right) dw_i(u) + x_i(t) \right]^2 \quad a.s.P., \tag{2.7}$$

$$x_i(t) = \frac{\sigma}{2} e^{-kt/2} \left[\int_0^t e^{ku/2}\, dw_i(u) + \frac{2}{\sigma} x_i(0) \right], \qquad i = 1, \ldots, d, \tag{2.8}$$

where $x_i(t)$ *are independent Ornstein-Uhlenbeck processes, and* $\|X(t)\|^2 = \sum_{i=1}^{d} x_i^2(t) = r(t)$ *a.s.P.*

Detailed proof of Corollary 2.2 has been given in Maghsoodi (1992, 1993). Another form of the relation (2.7) has since been reported in Geman and Yor (1993).

Naturally the above representations pose the general question as to how wide a class of ECIR models are representable by the above methods. Theorem 2.2 shows that in general the ECIR$(\delta(t))$ interest rate has a quadratic representation in the solution of linear s.d.e.s via RLMSI, if and only if the dimension is a fixed positive integer. Suppose the state follows a

d-dimensional adapted Gaussian process, $X(t) = (x_1(t), \ldots, x_d(t))$, which is the pathwise unique strong solution of the system of linear (in the narrow sense) s.d.e.s:

$$(2.9) \qquad dX(t) = (\mathbf{a}(t)X(t) + b(t))\,dt + B(t)\,d\bar{W}(t),$$

where $\bar{W}(t)$ is a (P, F_t) q-dimensional Brownian motion, $\mathbf{a}(t)$, $b(t)$, and $B(t)$, are $d \times d$, $d \times 1$, and $d \times q$, bounded and continuously differentiable functions on $[0, T]$.

THEOREM 2.2. *Let $A(t)$, $C(t)$, and $\eta(t)$ be $d \times d$, $d \times 1$, and 1×1 respectively, and continuously differentiable. Then the* ECIR$(\delta(t))$ *process admits the representation:*

$$(2.10) \qquad r(t) = \tilde{Q}(t, X(t)) \triangleq X^T(t)A(t)X(t) + C^T(t)X(t) + \eta(t) \quad a.s.P.$$

via RLMSI *if and only if* $\delta(t) \equiv d$.

Proof. See Appendix A. □

Various cases of the model (2.10) have been considered in the literature. For example, Beaglehole and Tenney (1991) took the parameters in (2.9) and (2.10) except $b(t)$ constant, and within a more general framework Jamshidian (1993a) considers diagonal matrices. Constantinides (1992) and Duffie and Kan (1993) consider constant coefficients within other general frameworks. In general, the proof of Theorem 2.2 implies that a large class of multifactor (quadratic Gaussian) models may be reduced to one-factor (CIR or ECIR) models (see also Remark 2.1). Examples and further implications of the proof of Theorem 2.2 have been given in Maghsoodi (1993). Jamshidian (1993a) reports general conditions for the bond price to have a quadratic in $X(t)$ as exponent.

THEOREM 2.3 (Transition Density of ECIR(d)). *Let $\{r(s); 0 \le t < s < \infty\}$ denote* ECIR(d) *process and set $G(t,s) \triangleq 4[\int_t^s \sigma^2(s)\mathcal{K}^{-2}(u,s)\,du]^{-1}$. Then conditional on $r(t)$, $G(t,s)r(s)$ has noncentral chi-square distribution with d degrees of freedom and noncentrality parameter $\lambda(t,s) = 4r(t)V^{-2}(t,s)$, with $V^2(t,s) \triangleq [\int_t^s \sigma^2(u)\mathcal{K}^2(t,u)\,du]$.*

Proof. By (2.5), we can write

$$(2.11) \qquad 4\mathcal{K}^2(t,s)V^{-2}(t,s)r(s) = \sum_{i=1}^{d} \left[Z_i + 2y_i(t)V^{-1}(t,s)\right]^2,$$

where Z_i are independent standard normal random variables. The left-hand side of (2.11) is $G(t,s)r(s)$ and by definition its right-hand side is a noncentral chi-square random variable, with d degrees of freedom and noncentrality parameter $\lambda(t,s)$. □

Thus the conditional distribution of the ECIR process $r(s)$, is a rescaled noncentral chi-square distribution $\chi^2(G(t,s)r(s); d, \lambda(t,s))$. Hence, from the formula of the density (see, e.g., Johnson and Kotz 1970), we obtain the transition density

$$\text{(2.12)} \quad p(t, r(t); s, r(s)) = \frac{1}{2} G \left(\frac{Gr(s)}{\lambda} \right)^{(d-2)/4} \times \exp\left(-\frac{1}{2}(\lambda + Gr(s)) \right) I_{(d-2)/2}(\sqrt{\lambda Gr(s)}),$$

where $G \triangleq G(t, s)$ and $I_\alpha(\cdot)$ is the modified Bessel function of the first kind of order α.

2.2. Real-Valued Dimensions

The following extension of Dubins and Schwarz's (1965) theorem will be used. It allows time change in terms of strictly increasing homeomorphisms $\phi(t)$ rather than just t. The proof has been given in Maghsoodi (1993).

LEMMA 2.2. *Let $\{M(t), 0 \leq t < \infty\}$ be a (P, F_t) continuous local martingale such that $\zeta \triangleq \lim_{t\to\infty}\langle M\rangle(t) = \infty$ a.s.P. Let $\phi(t) = \int_0^t h^2(v)\,dv$ be a homeomorphism on $0 \leq t < \infty$ with inverse $\xi(\cdot)$, such that $\lim_{t\to\infty}\phi(t) = \infty$. For each $0 \leq t < \infty$, define the stopping time*

$$\tau(t) = \inf\{s \geq 0; \langle M\rangle(s) > \phi(t)\}.$$

Then the time-changed process, $N(t) \triangleq M(\tau(t))$, $0 \leq t < \infty$, is a $(P, \mathcal{G}_t)$ square-integrable martingale, with $\mathcal{G}_t \triangleq F_{\tau(t)}$, and can be represented as

$$N(t) = \int_0^t h(v)\,dw(v) \quad a.s.P.,$$

where $w(t)$ is a $(P, \mathcal{G}_t)$-Brownian motion.

REMARK 2.2. If $P\{\zeta < \infty\} > 0$, then in analogy to the corresponding modification of Dubins and Schwarz's theorem (see, e.g., Rogers and Williams 1987) Lemma 2.2 will still hold up to the process lifetime ζ by defining

$$\tau(t) = \begin{cases} \inf\{s \geq 0; \langle M\rangle(s) > \phi(t)\}, & 0 \leq t < \xi(\zeta), \\ \infty, & t \geq \xi(\zeta). \end{cases}$$

Consider now the linear scalar s.d.e.

$$\text{(2.13)} \quad dy(t) = \frac{1}{2}(D(t, \omega) - 1)y(t)\,dt + y(t)\,d\bar{w}(t),$$

where $\bar{w}(t)$ is a (P, F_t)-Brownian motion, $D(t, \omega)$ is, in general, a path-history-dependent adapted process, and $y^2(0) = r(0)$ a.s.P. Define the (P, F_t)-martingale $M(t)$ by

$$M(t) = \int_0^t y(u)\, d\bar{w}(u). \tag{2.14}$$

Let

$$\zeta \triangleq \sup_{t \geq 0} \int_0^t y^2(u)\, du, \qquad \phi(t) = \frac{1}{4}\int_0^t \sigma^2(v)\mathcal{K}^2(0, v)\, dv. \tag{2.15}$$

For each $0 \leq t < \infty$, define the stopping time

$$\tau(t) = \begin{cases} \inf\{s \geq 0; \int_0^s y^2(u)\, du > \phi(t)\} & 0 \leq t < \xi(\zeta), \\ \infty, & t \geq \xi(\zeta). \end{cases} \tag{2.16}$$

THEOREM 2.4 (Solution of ECIR(δ)). *Set $D(t, \omega) \equiv \delta$ a.s.P. Then the following hold:*

(i) *$\langle M \rangle_t$ is a (P, F_t)-submartingale.*
(ii) *$y^2(t)$ is a (P, F_t)-submartingale.*
(iii) *The process $N(t) \triangleq M(\tau(t))$ is a $(P, \mathcal{G}_t)$ square-integrable martingale and has the representation*

$$N(t) = \frac{1}{2}\int_0^t \sigma(v)\mathcal{K}(0, v)\, dw(v), \qquad 0 \leq t < \infty, \tag{2.17}$$

where $w(t)$ is a $(P, \mathcal{G}_t)$-Brownian motion.
(iv) *The* ECIR(δ) *process is given by*

$$\begin{aligned} r(t) &= \mathcal{K}^{-2}(0, t) y^2(\tau) \\ &= r(0)\exp[(\delta - 2)\tau - \int_0^t k(v)\, dv + 2\bar{w}(\tau)] \quad a.s.P., \qquad 0 \leq t \leq \xi(\zeta). \end{aligned} \tag{2.18}$$

(v) *$\xi(\zeta)$ is the first zero-hitting time of $r(t)$.*
(vi) *In finite time, $r(t)$ avoids zero a.s.P. if $\delta \geq 2$, and it hits zero a.s.P if $0 < \delta < 2$.*

Proof. See Appendix B. □

When $0 < \delta < 2$, the lifetime of the representation (2.18) is $\xi(\zeta) < \infty$ a.s.P., i.e. the first zero-hitting time of $r(t)$, which is finite a.s.P. However, for the reflected case, the representation may be used afresh, away from the singularity, at $\xi(\zeta)^+$ with new $r(0) = 0^+$. The solution of the unextended CIR(δ) model (1.1) now follows as a special case of Theorem 2.4.

COROLLARY 2.3 (Solution of CIR(δ)). *The conclusions of Theorem 2.4 hold for the* CIR(δ) *model with $D(t,\omega) \equiv \delta$ a.s.P. and $\phi(t) = \frac{1}{4}\int_0^t \sigma^2\gamma^2(0,v)\,dv$, where $\gamma(t,s) \triangleq \exp(\frac{1}{2}k(t-s))$. In particular this spot rate is given by*

$$\begin{aligned}(2.19)\quad r(t) &= \gamma^{-2}(0,t)y^2(\tau(t)) \\ &= r(0)\exp((\delta-2)\tau - kt + 2\bar{w}(\tau)) \quad a.s.P., \qquad 0 \le t < \xi(\zeta).\end{aligned}$$

LEMMA 2.3 (Relations to BESQ(δ) Process). *The* ECIR(δ) *spot rate $\{r(s), 0 \le t \le s < \infty\}$ is given by*

$$r(s) = \mathcal{K}^{-2}(t,s)\bar{r}\left(\frac{1}{4}\int_t^s \sigma^2(u)\mathcal{K}^2(t,u)\,du\right) \quad a.s.P. \tag{2.20}$$

where $\bar{r}$ is a BESQ(δ) *process with $\bar{r}(0) = r(0)$. In addition, the transition density of $r(s)$ is given by (2.12) with d replaced by δ.*

Proof. This is a special case of the proof of Lemma 3.2 given in Appendix C. □

THEOREM 2.5 (Solution of ECIR($\delta(t)$)). *Let $\phi(\cdot)$ be defined by (2.15) and set*

$$D(t,\omega) = \delta\left(\xi\left(\left(\int_0^t y^2(v)\,dv\right)\right)\right) \quad a.s.P., \qquad 0 \le t < \infty. \tag{2.21}$$

Then the conclusions of Theorem 2.4 hold for the ECIR($\delta(t)$) *process. Furthermore there exists an equivalent probability measure P^* on* **F**, *and a (P^*, F_t)-Brownian motion $w^*(t)$, such that $r(t)$ has the representation*

$$r(t) = r(0)\exp 2\left(w^*(\tau) - \tau - \int_0^t k(v)\,dv\right) \quad a.s.P^*.\ \&\ P., \qquad 0 \le t < \xi(\zeta). \tag{2.22}$$

Proof. See Appendix B. □

3. THE ECIR($\delta(t)$) TERM STRUCTURE

3.1. The Term Structure Dynamics

In this section an entire term structure of interest rates is developed on the basis of ECIR($\delta(t)$) spot rates and no arbitrage. Let $P(t,T)$ denote the price at time t, of the T-maturity discount bond. Following CIR (1985) we set

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \quad \text{a.s.P.}, \qquad 0 \le t \le T, \tag{3.1}$$

with $A(T, T) = 1$ and $B(T, T) = 0$. Unless otherwise specified, the subscripts t and T will denote partial derivatives with respect to the first time variable (current time) and the second time variable (maturity time) respectively. Application of Itô's lemma to (3.1) and using the spot rate dynamics gives the bond dynamics:

$$dP(t, T) = \mu_P(t, T, \omega)P(t, T)\,dt + \sigma_P(t, T, \omega)P(t, T)\,d\omega(t), \qquad 0 \le t \le T, \tag{3.2}$$

$$\mu_P(\omega) = \frac{A_t}{A} - B\theta(t)k(t) - \left(B_t - Bk(t) - \frac{1}{2}\sigma^2(t)B^2\right)r(t), \tag{3.3}$$

$$\sigma_P(\omega) = -B\sigma(t)\sqrt{r(t)}, \tag{3.4}$$

where the variables t and T have been omitted. Following Hull and White (1990), the CIR (1985) market price of risk is extended to

$$\Lambda(t) = -\frac{\lambda(t)\sqrt{r(t)}}{\sigma(t)}, \tag{3.5}$$

where $\lambda(t)$ is bounded and continuously differentiable. The no-arbitrage condition (see, e.g., Heath, Jarrow, and Morton 1992) requires that

$$\frac{\mu_P(\omega) - r(t)}{\sigma_P(\omega)} = \frac{\lambda(t)\sqrt{r(t)}}{\sigma(t)} \quad \text{a.s.P.} \tag{3.6}$$

Substituting (3.3) and (3.4) in (3.6) we obtain

$$B_t(t, T) - \nu(t)B(t, T) - \frac{1}{2}\sigma^2(t)B^2(t, T) + 1 = 0, \tag{3.7}$$

$$A(t, T) = \exp\left(-\int_t^T \theta(u)k(u)B(u, T)\,du\right), \tag{3.8}$$

where $\nu(t) \triangleq \lambda(t) + k(t)$. Equations (3.7) and (3.8) and similar have been derived (under equilibrium), e.g., in Hull and White (1990), Longstaff and Schwartz (1992), and Jamshidian (1993a,b) and in the context of "affine models," e.g., in Duffie and Kan (1993).

Differentiation of (3.7) with respect to T and elimination of $\sigma(t)$ gives

$$BB_{tT} + \nu(t)BB_T - 2B_tB_T - 2B_T = 0, \quad B(T, T) = 0, \quad B_T(T, T) = 1, \tag{3.9}$$

the solution of which in terms of initial boundary is[2]

$$B(t,T) = 2\left[\nu(t) + \frac{2B_T(0,t)}{B(0,T) - B(0,t)} + \frac{B_{TT}(0,t)}{B_T(0,t)}\right]^{-1}. \tag{3.10}$$

The initial boundary curves $B(0, s)$ and $A(0, s)$, $s \in [t, T]$, are uniquely determined by the initial term structure and the initial term structure volatilities (see Remark 3.1). When ν and σ are fixed (3.10) gives

$$B(t,T) = 2\left[\nu + \gamma \coth\left(\frac{1}{2}\gamma(T-t)\right)\right]^{-1}, \tag{3.11}$$

where $\gamma \triangleq (\nu^2 + 2\sigma^2)^{1/2}$, which is the solution for the unextended model obtained by CIR (1985). Substituting from (3.4) and (3.6) into (3.2) we obtain the ECIR bond price dynamics under no arbitrage:

$$dP(t,T) = r(t)(1 - \lambda(t)B(t,T))P(t,T)\,dt - B(t,T)\sigma(t)\sqrt{r(t)}P(t,T)\,dw(t). \tag{3.12}$$

The instantaneous forward rate at time t for maturity date $T \geq t$, $f(t, T)$, is defined by

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T} \quad \text{a.s.P.} \tag{3.13}$$

Hence differentiating the logarithm of (3.1) with respect to T we obtain

$$\begin{aligned} f(t,T) &= B_T(t,T)r(t) + c(t,T) \quad \text{a.s.P.}, \\ c(t,T) &\triangleq -\frac{A_T(t,T)}{A(t,T)} = \int_t^T \theta(u)k(u)B_T(u,T)\,du. \end{aligned} \tag{3.14}$$

Therefore using the methods of Section 2 the ECIR($\delta(t)$) bond prices and forward rates can also be solved via (3.1) and (3.14) respectively, in terms of the parameters and functionals of the spot rate Brownian motion. Applying Itô's lemma to the right-hand side of (3.14) and using the spot rate dynamics, the ECIR forward rates' dynamics is obtained under no arbitrage

$$df(t,T) = [B_{Tt} - B_T k(t)](f(t,T) - c)B_T^{-1}\,dt + \sqrt{B_T}\sigma(t)\sqrt{f(t,T) - c}\,dw(t). \tag{3.15}$$

This completes modeling (and solution) of the ECIR($\delta(t)$) term structure dynamics under no arbitrage. The bond price (3.1) together with (3.7) and (3.8) staisfies CIR's (1985)

[2]Substitute $B(t,T) = 2(\nu(t) + D(t,T))^{-1}$ in (3.7) and (3.9), solve for $D_T(t,T)$ by using two values of T, and use the solution in $D(t,T) = D_{Tt}(t,T)/D_T(t,T)$.

fundamental partial differential equation. Using (3.2) and (3.6) it can easily be checked that the ECIR($\delta(t)$) term structure presented also satisfies all the conditions of the new methodology of Heath, Jarrow, and Morton (1992). In particular, by using (3.7) together with the drift $\mu_f = [B_{Tt} - B_T k(t)]r(t)$ and the diffusion $\sigma_f = B_T \sigma(t)\sqrt{r(t)}$ of the forward rates, it is directly verified that they satisfy the *forward rate drift restriction*

$$(3.16)\ \mu_f(t, T, \omega) = -\sigma_f(t, T, \omega)\left[\int_t^T \sigma_f(t, v, \omega)\, dv + \Lambda(t)\right], \qquad 0 \le t \le T \quad \text{a.s.P.}$$

3.2. A Martingale Measure and Bond Option Valuation

By the fundamental results of Harrison and Pliska (1981) and Heath, Jarrow, and Morton (1992), (3.16) implies that there exists a unique equivalent "risk-neutral" martingale measure $\tilde{P}$ on $(\Omega, \mathbf{F})$ with respect to which the ECIR($\delta(t)$) discounted bond price process is a martingale and

$$(3.17) \qquad \tilde{w}(t) = w(t) - \int_0^t \Lambda(u)\, du$$

is a $(\tilde{P}, F_t)$-Brownian motion. Furthermore, for $0 \le t \le s \le T$, the time t price, $C(t)$, of a European call option on the T maturity bond $P(t, T)$, expiring at s with exercise price K, is given by

$$(3.18) \qquad C(t) = \tilde{E}\left(\exp\left(-\int_t^s r(u)\, du\right) C(s)/F_t\right),$$

where $\tilde{E}(\cdot)$ denotes the expectation with respect to $\tilde{P}$. A further change of measure will be useful. From (3.12) and (3.17)

$$(3.19)\ P(t, s) = P(0, s)\exp\left[\int_0^t \left(r(u) - \frac{1}{2}\sigma_P^2(u, s)\right) du + \int_0^t \sigma_P(u, s)\, d\tilde{w}(u)\right].$$

Hence, for each $s \in [t, T]$, the process

$$\begin{aligned}(3.20) M^s(t) &\triangleq \frac{P(t, s)}{P(0, s)}\exp\left[-\int_0^t r(u)\, du\right]\\ &= \exp\left[-\frac{1}{2}\int_0^t \sigma_P^2(u, s)\, du + \int_0^t \sigma_P(u, s)\, d\tilde{w}(u)\right]\\ &= 1 + \int_0^t \sigma_P(u, s)\exp\left[-\frac{1}{2}\int_0^u \sigma_P^2(x, s)\, dx + \int_0^u \sigma_P(x, s)\, d\tilde{w}(x)\right] d\tilde{w}(u)\end{aligned}$$

is a $(\tilde{P}, F_t)$ martingale. Thus by Girsanov's theorem (see, e.g., Karatzas and Shreve 1988) for each $s \in [t, T]$ there exists an equivalent probability measure P^s on $(\Omega, \mathbf{F})$, with density

M^s with respect to $\tilde{P}$ such that

$$(3.21) \qquad w^s(t) = \tilde{w}(t) - \int_0^t \sigma_P(u,s)\,du = w(t) - \int_0^t (\Lambda(u) + \sigma_P(u,s))\,du$$

is a (P^s, F_t)-Brownian motion. Let $E^s(\cdot)$ denote the expectation with respect to P^s. Then by Girsanov's formula

$$(3.22) \qquad E^s\left(C(s)/F_t\right) = [M^s(t)]^{-1}\tilde{E}\left(M^s(s)C(s)/F_t\right).$$

But from (3.20)

$$(3.23) \qquad \frac{M^s(s)}{M^s(t)} = \frac{1}{P(t,s)}\exp\left[-\int_t^s r(u)\,du\right].$$

Substituting (3.23) in (3.22) and using (3.18) we obtain

$$(3.24) \qquad C(t) = P(t,s)E^s\left(C(s)/F_t\right).$$

This formula is simpler to compute provided the transition density of $C(s)$ under P^s is known. Lemmas 3.1 and 3.2 will lead the way. By substituting for $dw(t)$ from (3.21) in (3.15), and using (3.14) we find the forward rates dynamics under P^s:

$$(3.25) \quad df(t,s) = B_s(t,s)^{1/2}\sigma(t)\sqrt{f(t,s) - c(t,s)}\,dw^s(t), \qquad 0 \le t \le s \le T,$$

Therefore the s-maturity forward rates are local martingales with respect to P^s. Now, by applying Itô's lemma to the right-hand side of (3.14) and using (3.25), it follows that under P^s the spot rate preserves its ECIR structure with new reversion rate $k^s(t) = B_{st}(t,s)B_s^{-1}(t,s)$. More specifically,

$$(3.26) \qquad dr(t) = \theta(t)k(t)\,dt - B_{st}(t,s)B_s^{-1}(t,s)r(t)\,dt + \sigma(t)\sqrt{r(t)}\,dw^s(t).$$

Probability measures similar to P^s have been applied to other models under various names (see Merton 1973, El Karoui et al. 1992, and Jamshidian 1993a). We shall refer to P^s as the s-*forward* measure. Thus the representation methods of Section 2 can be applied to solve the entire ECIR term structure under P^s too. Let $W^s(t) = (w_1^s(t), \ldots, w_d^s(t))$ be a (P^s, F_t) d-dimensional Brownian motion, and define the process $Z^s(t) = (z_1^s(t), \ldots, z_d^s(t))$ by

$$(3.27) \qquad dz_i^s(t) = \frac{1}{2}\sigma(t)B_s^{1/2}(t,s)\,dw_i^s(t).$$

LEMMA 3.1. *Under the s-forward measure the* ECIR(d) *process $r(s)$ is given by*

$$(3.28) \quad r(s) = \|Z^s(s)\|^2 = \sum_{i=1}^d \left[\frac{1}{2}\int_t^s \sigma(u)B_s^{1/2}(u,s)\,dw_i^s(u) + z_i^s(t)\right]^2 \quad a.s. P^s.$$

Furthermore, under P^s, for $s > t$ and conditional on the value of $r(t)$, $r(s)d/c(t,s)$ has noncentral chi-square distribution with d degrees of freedom and noncentrality parameter $B_s(t,s)r(t)d/c(t,s)$.

Proof. This is analogous to the proofs of Corollary 2.1 and Theorem 2.3 and has been given in Maghsoodi (1992, 1993). □

LEMMA 3.2. *Under the s-forward measure the* ECIR(δ) *process is given by*

$$r(u) = \mathcal{K}^{s^{-2}}(t,u)\bar{r}\left(\frac{1}{4}\int_t^u \sigma^2(v)\mathcal{K}^{s^2}(t,v)\,dv\right) \quad a.s.P^s., \qquad 0 \le t \le u \le s \le T, \tag{3.29}$$

where $\mathcal{K}^{s^2}(t,u) \triangleq \exp\int_t^u k^s(v)\,dv = B_s(u,s)B_s^{-1}(t,s)$ and $\bar{r}$ is a BESQ(δ) *process with $\bar{r}(0) = r(t)$. In addition, the conditional distribution of Lemma 3.1 is valid with d replaced by δ.*

Proof. See Appendix C. □

Other implications of the proof of (3.29) can be found in Maghsoodi (1992). Application of the ECIR(δ) model seems plausible as the volatility parameter may be expected to depend on the levels of the reversion rate and the central value.

THEOREM 3.1 (Bond Option Price, Maghsoodi 1992). *Consider the* ECIR(δ) *term structure model. For $0 \le t \le s \le T$, the time t price $C(t)$, of a European call option on the T-maturity discount bond $P(t,T)$, expiring at s with exercise price K, is given by*

$$C(t) = P(t,T)\chi^2(\rho_2;\delta,\gamma_2) - KP(t,s)\chi^2(\rho_1;\delta,\gamma_1), \tag{3.30}$$

where

$$\begin{aligned}
&\rho_1 \triangleq Jr^*,\\
&J \triangleq \tfrac{\delta}{c(t,s)},\\
&r^* \triangleq B(s,T)^{-1}\log\left(\tfrac{A(s,T)}{K}\right),\\
&\gamma_1 \triangleq JB_s(t,s)r(t),\\
&\rho_2 \triangleq \rho_1 + 2B(s,T)r^*,\\
&\gamma_2 \triangleq \gamma_1\left(\tfrac{2B(s,T)}{J} + 1\right)^{-1},\\
&A(s,T) \triangleq \exp\left[-\tfrac{\delta}{4}\int_s^T \sigma^2(u)B(u,T)\,du\right],\\
&c(t,s) \triangleq -\tfrac{A_s(t,s)}{A(t,s)} = \tfrac{\delta}{4}\int_t^s \sigma(u)^2 B_s(u,s)\,du,
\end{aligned}$$

and $B(s,T)$ is given by (3.10).

Proof. By (3.24) the price of the European call option on the bond is

$$C(t) = P(t,s)E^s\left([P(s,T) - K]^+/F_t\right). \tag{3.31}$$

Hence, by (3.1) and Lemma 3.2,

$$(3.32)\quad \mathcal{C}(t) = P(t,s)E^s\left(\left[A(s,T)\exp\left(-\frac{1}{\delta}B(s,T)c(t,s)\bar{Q}\right) - K\right]^+ / F_t\right),$$

where $\bar{Q}$ has the noncentral chi-square distribution of Lemma 3.1 with δ degrees of freedom. Hence,

$$(3.33)\quad \mathcal{C}(t) = P(t,s)\int_{-\infty}^{\rho_1} A(s,T)\exp\left(-\frac{1}{\delta}B(s,T)c(t,s)x\right)d\chi^2(x;\delta,\gamma_1) - KP(t,s)\int_{-\infty}^{\rho_1} d\chi^2(x;\delta,\gamma_1).$$

The second term in (3.33) is that of (3.30). Change of the variable of integration in the first integral to $y = (\frac{2}{\delta}c(t,s)B(s,T)+1)x \triangleq (S+1)x$, and manipulation of the noncentral χ^2 density gives

$$(3.34)\quad \mathcal{C}(t) = P(t,s)A(s,T)(S+1)^{-\delta/2}\exp\left[\frac{1}{2}\gamma_1((S+1)^{-1}-1)\right] \int_{-\infty}^{\rho_2} d\chi^2(y;\delta,\gamma_2) - KP(t,s)\chi^2(\rho_1;\delta,\gamma_1).$$

Now, since $\lim_{K\downarrow 0}\mathcal{C}(t) = P(t,T)$ and $\lim_{K\downarrow 0}\rho_2 = \infty$, the term in front of the integral in (3.34) is $P(t,T)$. Hence (3.30) follows. □

For arbitrary shape of the initial term structure and its volatilities, the functions $(A\cdot,\cdot)$, $B(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are arbitrary, and thus the comparative statics of the ECIR bond option value (3.30) are indeterminate. For example, to study $\partial\mathcal{C}(t)/\partial s$ for fixed T, the sign of $\partial\gamma_1/\partial s$ is required, which in turn requires the sign of $B_{ss}(t,s)$, which in general is indeterminate. When $\theta(t)$, $k(t)$ and $\sigma(t)$ are all constant for all $0 \le t \le T$, the discount bond option price (3.30) reduces to that obtained by CIR (1985) for the unextended model.

REMARK 3.1. To implement the bond option valuation formula (3.30) we may estimate $B(0,t)$ from the initial term structure and its volatilities using the formula (Hull and White 1990)

$$(3.35)\quad B(0,t) = \frac{t\sigma_R(0,t)R(0,t)}{\sigma(0)\sqrt{r(0)}},$$

where $R(t,T) = 1/(T-t)\int_t^T f(t,v)\,dv$ denotes the yield with volatility $\sigma_R(t,T)$. $A(0,t)$ can now be uniquely determined from (3.1) and (3.13). The spot rate parameters may similarly be estimated from the initial term structure (Maghsoodi 1992).

EXAMPLE 3.1. If the initial forward rates curve is given by $f(0,t) = r(0)e^{-ht}$, then $R(0,t) = (r(0)/ht)(1-e^{-ht})$. Hence if the initial yield volatility curve is given by

$\sigma_R(0,t) = ve^{gt}$, (3.35) gives

$$B(0,t) = \frac{1}{h} e^{gt}(1 - e^{-ht}), \tag{3.36}$$

and from (3.1) and (3.13) we obtain

$$A(0,t) = \exp\left[\frac{r(0)}{h}(1 - e^{gt})(e^{-ht} - 1)\right]. \tag{3.37}$$

4. SUMMARY

We have seen that the CIR and the ECIR spot rate stochastic differential equations can be solved pathwise uniquely in terms of model parameters and the driving Brownian motion. For ECIR(d) a unique generalized Ornstein-Uhlenbeck process was constructed from this Brownian motion such that its norm squared gave the pathwise unique spot rate. It was then shown that in general the role of quadratics in Gaussian processes as solution is limited to integer dimensions only.

For real-valued dimensions an extension of the time change theorem of Dubins and Schwarz was presented and applied to show that stopped lognormal processes solve these cases and the solutions characterize the first zero-hitting time of the spot rates. A relation to a BESQ(δ) process was also established. The solutions were applied to produce the ECIR transition density, an additivity property and to generalize zero-hitting conditions.

We have seen that the ECIR($\delta(t)$) term structure can be developed under no arbitrage and can be analytically tractable. The term structure dynamics was solved and the transition density was derived under a new ECIR martingale measure. These were applied to derive a closed-form ECIR(δ) bond option valuation formula. The findings can also be applicable in model validation, forecasting and in valuation of other interest rate derivatives. Such applications, together with empirical tests are also the subject of future research.

APPENDIX A

Proof of Theorem 2.2. The proof of the "if" part, is that of Theorem 2.1. To prove the "only if" part, without loss of generality, we can assume that $A(t)$ is symmetric. Applying Itô's lemma to $\tilde{Q}(t, X(t))$ and using Lemma 2.1, we obtain

$$S(t) \triangleq B(t)B^T(t) \equiv \frac{1}{4}\sigma^2(t)A^{-1}(t) \tag{A.1}$$

as a necessary condition for a.s.P. equivalence of the diffusion terms. Likewise, for the drift terms we obtain

$$\dot{A}(t) = -(k(t)I + \mathbf{a}^T(t))A(t) - A(t)\mathbf{a}(t), \tag{A.2}$$

$$C^T(t)S(t)C(t) = \sigma^2(t)\eta(t), \tag{A.3}$$

$$\dot{C}(t) = -(k(t)I + \mathbf{a}^T(t))C(t) - 2A(t)b(t), \tag{A.4}$$

$$\dot{\eta}(t) = -k(t)\eta(t) - C^T(t)b(t) + \rho(t), \tag{A.5}$$

$$\rho(t) \triangleq \theta(t)k(t) - \frac{1}{4}\sigma^2(t)d.$$

Solving (A.2) we obtain

$$A(t) = \mathcal{K}^{-2}(0,t)\Phi(t)A(0)\Phi^T(t), \tag{A.6}$$

where $A(0)$ is positive definite and symmetric, and $\Phi(t)$ is the fundamental solution of the deterministic linear system $\dot{\Phi}(t) = -\mathbf{a}^T(t)\Phi(t)$. Solving (A.4) and (A.5) we obtain

$$C(t) = \mathcal{K}^{-2}(0,t)\Phi(t)\left(C(0) - 2A(0)\int_0^t \Phi^T(u)b(u)\,du\right), \tag{A.7}$$

$$\eta(t) = \mathcal{K}^{-2}(0,t)\left[\eta(0 + \int_0^t \mathcal{K}^2(0,u)(\rho(u) - C^T(u)b(u))\,du\right]. \tag{A.8}$$

Given $\mathbf{a}(t)$ and $b(t)$, all the parameter functions of the representation (2.10) are uniquely and independently determined. In addition, (A.3) must hold. Using (A.1) and (A.6)–(A.8) in (A.3) we obtain

$$\frac{1}{4}\bar{C}^T(t)A^{-1}(0)\bar{C}(t) = \eta(0) + \int_0^t \mathcal{K}^2(0,u)\left(\rho(u) - C^T(u)b(u)\right)du, \tag{A.9}$$

where $\bar{C}(t) \triangleq C(0) - 2A(0)\int_0^t \Phi^T(u)b(u)\,du$. Differentiating (A.9) we obtain $\mathcal{K}^2(0,t)\rho(t) \equiv 0$, which holds if and only if $\delta(t) \equiv d$. □

APPENDIX B

Proof of Theorem 2.4. For $0 \le s \le t$, (2.13) and (2.14) imply that

$$E\{\langle M\rangle(t)/F_s\rangle\} = \langle M\rangle(s) + \int_s^t E\{y^2(u)/F_s\}\,du, \tag{B.1}$$

$$E\{y^2(t)/F_s\} = y^2(s)\exp(\delta(t-s)). \tag{B.2}$$

Parts (i) and (ii) of the theorem now follow from (B.1) and (B.2) respectively. Part (iii) follows from Lemma 2.2 and Remark 2.2. To prove (iv) apply Itô's lemma to (2.18) and use (2.13) to obtain

$$dr(t) = -k(t)r(t)\,dt + \mathcal{K}^{-2}(0,t)\left(\delta y^2(\tau)\,d\tau + 2y^2(\tau)\,d\bar{w}(\tau)\right), \tag{B.3}$$

which after time substitution gives the ECIR model (1.2). Let T_0 denote the first zero-hitting time of $r(t)$ and note that from (2.16) and part (iv):

$$\tau(t) = \frac{1}{4}\int_0^t \frac{\sigma^2(v)}{r(v)}\,dv, \qquad 0 \le t < \xi(\zeta). \tag{B.4}$$

Hence, $\tau(T_0) = \infty$ a.s.P. Thus $T_0 = \xi(\zeta)$ a.s.P. which proves part (v). Since the time change in (2.20) is a strictly increasing homeomorphism the zero-hitting conditions follow from those of BESQ(δ) process which proves part (vi). □

Proof of Theorem 2.5. By boundedness and continuity of $\delta(\cdot)$, the s.d.e. (2.13) has a nonexploding pathwise unique solution. Thus conclusion (i) follows as in the proof of Theorem 2.4. From (2.13),

$$\begin{aligned} y^2(t) &= y^2(s)\exp 2\left[\bar{w}(t) - \bar{w}(s) - (t-s) + \frac{1}{2}\int_s^t D(v,\omega)\,dv\right], \\ &\quad 0 \le s \le t < \infty \quad \text{a.s.P.} \end{aligned} \tag{B.5}$$

$$E\{y^2(t)/F_s\} = y^2(s)E\left\{\exp\left[\int_s^t D(v,\omega)\,dv\right]/F_s\right\} \quad \text{a.s.P.} \tag{B.6}$$

which verifies (ii). The proofs of parts (iii)–(vi) are analogous to those of Theorem 2.4. By boundedness and continuity of $\delta(\cdot)$, Novikov's condition (see Karatzas and Shreve 1988) is satisfied and

$$\mathcal{M}(t) = \exp -\frac{1}{2}\left[\int_0^t D(u,\omega)\,d\bar{w}(u) + \frac{1}{2}\int_0^t D^2(u,\omega)\,du\right] \tag{B.7}$$

is a (P, F_t)-martingale. Hence, by Girsanov's theorem, there exists an equivalent probability measure P^*, with density $\mathcal{M}(t)$ with respect to P, such that

$$w^*(t) = \bar{w}(t) + \frac{1}{2}\int_0^t D(v,\omega)\,dv \tag{B.8}$$

is a (P^*, F_t)-Brownian motion. Hence, from (B.5),

$$y(t) = y(0)\exp(w^*(t) - t) \quad \text{a.s.}\tilde{\text{P}}\ \&\ \text{P}. \tag{B.9}$$

The time change conditions can be equivalently written under P^* in terms of $w^*(t)$. The representation (2.22) now follows from (2.18) and (B.9). □

APPENDIX C

Proof of Lemma 3.2. Define the time change

$$\tau^s(u) = \frac{1}{4}\int_t^u \sigma^2(v)\mathcal{K}^{s^2}(t, v)\,dv. \tag{C.1}$$

Hence

$$\bar{w}(\tau^s(u)) \triangleq \frac{1}{2}\int_t^u \sigma(v)\mathcal{K}^s(t, v)\,dw^s(v) \quad \text{a.s.P}^s \tag{C.2}$$

is a (P^s, F_{τ^s})-Brownian motion. Application of Itô's lemma to (3.29) gives

$$dr(u) = -k(u)\mathcal{K}^{s^{-2}}(t, u)\bar{r}(\tau^s(u))\,du + \mathcal{K}^{s^{-2}}(t, u)\,d\bar{r}(\tau^s(u))$$

from which we obtain the ECIR dynamics (3.26) under P^s, upon using (C.1), (C.2), and

$$d\bar{r}(\tau^s) = \delta\,d\tau^s + 2\sqrt{\bar{r}(\tau^s)}\,d\bar{w}(\tau^s). \tag{C.3}$$

For the proof of the second part, note that by (3.29) if $\bar{p}(t, x, y)$ is the transition density of the BESQ(δ) process then the transition density $p(t, x; s, y)$ of the ECIR process with fixed dimension $\delta > 0$ satisfies

$$p(t, x; s, y) = \mathcal{K}^{s^2}(t, s)\bar{p}(\tau^s(s), x, \mathcal{K}^{s^2}(t, s)y). \tag{C.4}$$

But we note that

$$\frac{\mathcal{K}^{s^2}(t, s)}{\tau^s(s)} = \frac{\delta}{c(t, s)}, \tag{C.5}$$

since the left-hand side is

$$\begin{aligned} 4\left[\int_t^s \sigma^2(v)\mathcal{K}^{s^{-2}}(t, v)\mathcal{K}^{s^{-2}}(t, s)\,dv\right]^{-1} &= 4\left[\int_t^s \sigma^2(v)\mathcal{K}^{s^{-2}}(v, s)\,dv\right]^{-1} \\ &= 4\left[\int_t^s \sigma^2(v)B_s(v, s)\,dv\right]^{-1} \\ &= \frac{\delta}{c(t, s)}, \end{aligned} \tag{C.6}$$

where the last equality follows from (3.14). Now substituting the new variables in the transition density,

$$\bar{p}(t, x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{\delta-2}{4}} \exp\left[-\frac{x+y}{2t}\right] I_{\frac{\delta-2}{2}}\left(\frac{\sqrt{xy}}{t}\right) \tag{C.7}$$

of BESQ(δ) and using (C.5) the following transition density is obtained for $r(t)$ under P^s:

$$\text{(C.8)}\quad p(t, x; s, y) = \frac{\delta}{2c(t,s)}\left(\frac{y}{B_s(t,s)x}\right)^{(\delta-2)/4} \times \exp\left[-\frac{B_s(t,s)\delta x + \delta y}{2c(t,s)}\right] I_{(\delta-2)/2}\left(\frac{\delta\sqrt{B_s(t,s)xy}}{c(t,s)}\right),$$

which by (2.12) is the required rescaled noncentral chi-square density. □

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