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A jump to default extended CEV model: an application of Bessel processes

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Abstract We develop a flexible and analytically tractable framework which unifies the valuation of corporate liabilities, credit derivatives, and equity derivatives. We assume that the stock price follows a diffusion, punctuated by a possible jump to zero (default). To capture the positive link between default and equity volatility, we assume that the hazard rate of default is an increasing affine function of the instantaneous variance of returns on the underlying stock. To capture the negative link between volatility and stock price, we assume a constant elasticity of variance (CEV) specification for the instantaneous stock volatility prior to default. We show that deterministic changes of time and scale reduce our stock price process to a standard Bessel process with killing. This reduction permits the development of completely explicit closed form solutions for risk-neutral survival probabilities, CDS spreads, corporate bond values, and European-style equity options. Furthermore, our valuation model is sufficiently flexible so that it can be calibrated to exactly match arbitrarily given term structures of CDS spreads, interest rates, dividend yields, and at-the-money implied volatilities.

Keywords Default · Credit spread · Corporate bonds · Equity derivatives · Credit derivatives · Implied volatility skew · CEV model · Bessel processes

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1 Introduction

In this paper, we propose a simple framework which unifies the valuation of corporate liabilities, credit derivatives, and equity derivatives. We have designed our model to parsimoniously capture several fundamental empirical observations. The first observation is that credit default swap (henceforth CDS) spreads and corporate bond yields are both positively related to implied volatilities of equity options. The second observation is that the realized volatility of a stock is negatively related to its price level. Similarly, equity implied volatilities tend to be decreasing convex functions of the option's strike price. The model that we propose to capture all of these empirical regularities is both flexible and analytically tractable. The flexibility of the model allows it to be rendered perfectly consistent with arbitrarily given term structures of CDS spreads, interest rates, dividend yields, and at-the-money implied volatilities. The analytical tractability of the model stems from the close connection of our stock price process to Bessel processes. This connection leads us to explicit closed form formulas for survival probabilities and for arbitrage-free values of corporate bonds, credit derivatives, and equity options.

Before introducing our model specification, we first review some empirical evidence in support of the positive relationship between default probabilities and equity volatility. In particular, several studies find a link between corporate bond yields and equity volatility. For example, Campbell and Taksler [13] find that equity volatility can explain as much cross-sectional variation in bond yields as can credit ratings. They and Kassam [43] further observe that recent increases in corporate bond yields can be explained by the recent upward trend in idiosyncratic equity volatility. Similarly, Cremers et al. [21] show that the implied volatility of individual stock options contains important information for credit spreads. In particular, individual option prices contain information on the likelihood of rating migrations. Conversely, Hilscher [39] develops a measure of equity volatility implied by the yield spread on US investment grade corporate bonds, and finds that this measure is highly significant when forecasting volatility.

Since bond yields also proxy for other variables besides default, several other empirical studies have focused on the link between credit default swap (CDS) spreads and equity volatilities. For example, Cremers et al. [21] show that CDS rates are positively correlated with both stock option implied volatility levels and the slope of the implied volatility plot against moneyness. Consigli [19] analyzes the statistical relationship between CDS spreads and equity volatility for six stocks over 2002–2003. He finds that implied volatility movements drive significant spread movements. Similarly, Carr and Wu [16] regress CDS spreads on stock option implied volatilities for four companies and find R^2's ranging from 36 to 82%. Finally, Zhu et al. [56] also examine the relationship between stock volatility and credit spreads. After controlling for ratings, macro-financial variables, and firm's accounting information, they find that equity volatility and a jump risk measure together explain 75% of the variation in credit spreads.

There is also a great deal of empirical evidence that realized stock volatility is negatively related to stock price. This so-called leverage effect was first discussed in [8] and has enjoyed empirical support from many studies, e.g., [18]. The term "leverage effect" has become generic in describing the negative correlation between stock returns and their volatilities. However, various other explanations to this relationship between volatility and returns have also been proposed in the financial economics literature, e.g., [5,11,12,37].

It has similarly been observed that equity implied volatility is decreasing in the option's strike price. As this relationship has been consistently observed for both single name and index options since 1987, it is commonly referred to as the "implied volatility skew". The skew in individual stock options was thoroughly examined in [26]. Dennis et al. [27] also show that changes in implied volatility of individual stocks are negatively related to returns, particularly when systematic risk is high.

A broad class of structural models provides theoretical support for both of these fundamental empirical regularities. A large class of models, which finds inspiration from Merton [49], assumes that default occurs the first time that the firm's asset value crosses a lower barrier. For these models, a reduction in the firm's asset value causes a decrease in the value of existing debt, but an increase in the debt-equity ratio, unless debt is retired. The bond price decline raises the promised yield and CDS spreads, while the increase in leverage causes an increase in equity volatility. These responses to the decline in firm value lead to the observed positive relationship between standard measures of default likelihood and realized stock volatility. Furthermore, the reduction in firm value leads to a reduction in stock price leading to the observed negative relationship between the stock's volatility and the stock price. This negative relationship also produces the observed negative relationship between implied volatility and strike price.

Our reduced-form model assumes that the stock price follows a diffusion, punctuated by a possible jump to zero. Consistent with the above empirical evidence showing that default indicators are positively related to equity volatility, we assume that the instantaneous risk-neutral hazard rate of default is an increasing affine function of the instantaneous variance of the underlying stock. To be consistent with the leverage effect and the implied volatility skew, we further assume a constant elasticity of variance (CEV) process for the stock price prior to default. Together, these assumptions imply that the risk-neutral hazard rate is affine in a declining power function of the stock price. We call the resulting stock price process the *jump to default extended CEV* process, which we abbreviate to *JDCEV*.

An overview of this paper is as follows. In Sect. 2, we develop some general results that arise when the instantaneous volatility and the risk-neutral hazard rate of default are both functions of just the stock price and time. In Sect. 3, we develop our framework for unifying the valuation of corporate liabilities, credit derivatives and equity derivatives, by regarding them all as contingent claims written on the defaultable stock. In Sect. 4, we specify our jump to default extended CEV model for the risk-neutral dynamics of the defaultable stock. In Sect. 5, we solve our model in closed form by appealing to the theory of Bessel processes. By combining time changes, scale changes, and measure changes, we reduce the contingent claim valuation problem to the problem of computing expectations of some function of the standard Bessel process. In particular, we obtain completely explicit closed form formulas for risk-neutral survival probabilities, CDS spreads, corporate bond values under alternative recovery assumptions, and stock option values. Section 6 presents an asymptotic analysis of the model dynamics. In Sect. 7, we numerically illustrate the implications of our JDCEV stock price process by calculating examples of risk-neutral survival probabilities, term structures of credit spreads, and implied volatility skews. We also discuss the extent to which put options on defaultable stocks can be regarded as credit derivatives. Section 8 concludes the paper.

2 Jump to default extended diffusion

In this section and the next, we develop relatively general results which will be used in the sequel when we specialize to the jump to default extended CEV process. We start with a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a standard Brownian motion $\{B_t, t \geq 0\}$ and an exponential random variable with unit parameter $e \sim \mathrm{Exp}(1)$ independent of B. We assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{Q}$ as given. We model the *pre-default* stock dynamics under the EMM as a time-inhomogeneous diffusion process $\{S_t, t \geq 0\}$ solving a stochastic differential equation (SDE)

$$\mathrm{d}S_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\,\mathrm{d}t + \sigma(S_t, t)S_t\,\mathrm{d}B_t, \quad S_0 = S > 0, \tag{2.1}$$

where $r(t) \geq 0, q(t) \geq 0, \sigma(S,t) > 0$ and $\lambda(S,t) \geq 0$ are the time-dependent risk-free interest rate, time-dependent dividend yield, time- and state-dependent instantaneous stock volatility, and time- and state-dependent default intensity, respectively. The function $r(t)$ can be obtained from a given yield curve (calibrated to the term structure of default-free interest rates) and the function $q(t)$ can be adapted from dividend forecasts or from a given term structure of forward or futures prices (by letting $q(t)$ be a generalized function, discrete proportional dividends can be handled). Carr and Javaheri [15] show that the functions $\lambda(S,t)$ and $\sigma(S,t)$ can in principle be inferred from a complete term and strike structure of implied equity volatilities observed on two or more days. For simplicity we assume that $r(t)$ and $q(t)$ are continuously differentiable in time on $[0,\infty)$ and that $\sigma(S,t)$ and $\lambda(S,t)$ are continuously differentiable in the stock price and time on $(0,\infty)\times[0,\infty)$ (these assumptions are not necessary, but will simplify the subsequent discussion). We also assume that $\sigma(S,t)$ and $\lambda(S,t)$ remain uniformly bounded as $S \to \infty$ (we do not assume they remain bounded as $S \to 0$). In Sect. 4 we will further specify σ and λ to the form (4.1) and (4.2).

Since we assumed the functions $\sigma(S,t)$ and $\lambda(S,t)$ remain bounded as $S \to \infty$, the process does not explode to infinity. Since we do not assume that $\sigma(S,t)$ and $\lambda(S,t)$ remain bounded as $S \to 0$, the process may hit zero, depending on the behavior of $\sigma(S,t)$ and $\lambda(S,t)$ as $S \to 0$. We assume the process is killed at the first hitting time of zero, $T_0 = \inf\{t \geq 0 : S_t = 0\}$, and is sent to a *cemetery state* Δ, where it remains forever (Δ is an absorbing state). If the process does not hit zero, we set $T_0 = \infty$ by convention. In general, the extended state space for the process S is $E^\Delta = (0,\infty) \cup \{\Delta\}$. In what follows, we denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the filtration generated by the process S.

To model jump to default, we introduce a jump-to-default *hazard process* $\{\Lambda_t, t \geq 0\}$. If $\lambda(S,t)$ remains bounded as $S \to 0$, $\Lambda_t = \int_0^t \lambda(S_u,u)\mathrm{d}u$. If $\lambda(S,t) \to \infty$ as $S \to 0$ (the intensity process $\lambda_t = \lambda(S_t,t)$ explodes at T_0), the hazard process is $[0,\infty]$-valued:

$$\Lambda_t = \begin{cases} \int_0^t \lambda(S_u,u)\mathrm{d}u, & t < T_0 \\ \infty, & t \geq T_0 \end{cases}.$$

We model the random time $\tilde{\zeta}$ of jump to default as the first time when the hazard process Λ is greater or equal to the random level $e \sim \mathrm{Exp}(1)$, i.e.,

$$\tilde{\zeta} = \inf\{t \geq 0 : \Lambda_t \geq \mathrm{e}\}.$$

At time $\tilde{\zeta}$, the stock jumps to the cemetery (bankruptcy) state Δ, where it remains forever (the cemetery (default) state Δ can be identified with zero by setting $\Delta = 0$). We assume equity holders do not receive any recovery in the event of bankruptcy and their equity position becomes worthless. We denote the defaultable stock process $S^\Delta = \{S_t^\Delta, t \geq 0\}$. We note that, in general, in this model default can happen either at time T_0 via diffusion to zero or at time $\tilde{\zeta}$ via a jump to default, whichever comes first. The time of default ζ (*lifetime* of the process S^Δ in the terminology of Markov processes) is then decomposed into a predictable and a totally inaccessible part (see also [17] for a model where the default time has both predictable and totally inaccessible parts); we have

$$\zeta = T_0 \wedge \tilde{\zeta}.$$

If $\lambda(S,t) \to \infty$ as $S \to 0$, it is possible that the diffusion process S^Δ with killing at rate $\lambda(S,t)$ never hits zero, while the process S in (2.1) without killing can hit zero. Intuitively, as the killing rate (intensity of jump to default) increases towards infinity as the process diffuses towards zero, the process will be almost surely killed through a jump to default from a positive value (will be sent to the cemetery state Δ from a positive value) before it has the opportunity to diffuse down to zero. In such case, $\tilde{\zeta} < T_0$ a.s., $\zeta = \tilde{\zeta}$ a.s., and $S_{\zeta-}^\Delta > 0$ a.s.

To keep track of how information is revealed over time, we follow Elliott et al. [33] by introducing a default indicator process $\{D_t, t \geq 0\}, D_t = \mathbf{1}_{\{t \geq \zeta\}}$, the filtration $\mathbb{D} = \{\mathcal{D}_t, t \geq 0\}$ generated by D, and an enlarged filtration $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ (recall that $\mathbb{F}$ is the filtration generated by the pre-default process S).

To summarize the results of this section, we model the price of the defaultable stock as a time-inhomogeneous diffusion process $\{S_t^\Delta, t \geq 0\}$ with state space $E^\Delta = (0,\infty) \cup \{\Delta\}$, initial value $S_0 = x > 0$, diffusion coefficient $\sigma(x,t)x$, drift $[r(t) - q(t) + \lambda(x,t)]x$, and killing rate $\lambda(x,t)$. In our notation, $\{S_t, t \geq 0\}$ is the pre-default stock price process (2.1), while $\{S_t^\Delta, t \geq 0\}$ is the defaultable stock price process, so that $S_t^\Delta = S_t$ for $t < \zeta$ and $S_t^\Delta = \Delta$ for all $t \geq \zeta$. The addition of the hazard rate in the drift rate in the pre-default dynamics (2.1) compensates for default to insure that the total expected rate of return to the stockholders is equal to the risk-free interest rate in the risk-neutral economy, and the discounted gain process to the stockholders (accounting for continuous price changes, dividends, and possible default with zero recovery to the stockholders) is a martingale under the EMM (this addition of jump intensity in the drift rate was already discussed in Merton [50]). We call the resulting stock price process a *jump to default extended diffusion*.

3 Unified valuation of corporate liabilities, credit derivatives, and equity derivatives

Conditioning on the information available at time $t \geq 0$ and using the Markov property of S, the risk-neutral probability of surviving beyond some fixed time $T > t$ is given by

$$\begin{aligned}
\mathbb{Q}(\zeta > T|\mathcal{G}_t) &= \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\big[\mathrm{e}^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\mathbf{1}_{\{T_0>T\}}\big|\mathcal{F}_t\big] \\
&= \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\big[\mathrm{e}^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\mathbf{1}_{\{m_{0,T}^S>0\}}\big|\mathcal{F}_t\big] \\
&= \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\big[\mathrm{e}^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\mathbf{1}_{\{m_{0,t}^S>0\}}\mathbf{1}_{\{m_{t,T}^S>0\}}\big|\mathcal{F}_t\big] \\
&= \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\big[\mathrm{e}^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\mathbf{1}_{\{m_{t,T}^S>0\}}\big|S_t\big] = \mathbf{1}_{\{\zeta>t\}}Q(S_t,t;T),
\end{aligned}$$

where $m_{t,T}^S = \min_{s\in[t,T]} S_s$ is the minimum of the process S on the time interval $[t,T]$, we use the fact that $\{T_0 > T\} = \{m_{0,T}^S > 0\}$, and we introduce the notation

$$Q(S,t;T) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\mathbf{1}_{\{m_{t,T}^S>0\}}\big|S_t = S\right]. \tag{3.1}$$

We first consider three relatively simple "building block claims", which form a basis for more complex securities:

(i) A European-style contingent claim with maturity (expiration) at time $T > 0$ and payoff $\Psi(S_T)$ at T, given no default by T, and no recovery if default happens by T;
(ii) a recovery payment of one dollar paid at the maturity date T if default occurs by T;
(iii) a recovery payment of one dollar paid at the default time ζ if default occurs by T.

The valuation of the three building blocks at some time $t \in [0, T)$ is standard in the reduced-form credit risk modeling framework (see [29,30,33,40,41,48], as well as the recent monographs of Bielecki and Rutkowski [7], Duffie and Singleton [31], Jeanblanc et al. [42], Lando [44], and Schönbucher [53]):

(i) European claim with no recovery:

$$\begin{aligned} &e^{-\int_t^T r(u)\mathrm{d}u}\mathbb{E}\left[\Psi(S_T)\mathbf{1}_{\{\zeta>T\}}\middle|\,\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\zeta>t\}}e^{-\int_t^T r(u)\mathrm{d}u}\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\Psi(S_T)\mathbf{1}_{\{m_{t,T}^S>0\}}\middle|\,S_t\right]; \end{aligned} \tag{3.2}$$

(ii) Fixed recovery paid at the maturity date T:

$$e^{-\int_t^T r(u)\mathrm{d}u}\mathbb{E}\left[\mathbf{1}_{\{\zeta\le T\}}\middle|\,\mathcal{G}_t\right]=e^{-\int_t^T r(u)\mathrm{d}u}[1-\mathbf{1}_{\{\zeta>t\}}Q(S_t,t;T)]; \tag{3.3}$$

(iii) Fixed recovery paid at the default time ζ:

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_t^\zeta r(u)\mathrm{d}u}\mathbf{1}_{\{\zeta\le T\}}\middle|\,\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\left[\int_t^T e^{-\int_t^u[r(v)+\lambda(S_v,v)]\mathrm{d}v}\lambda(S_u,u)\mathbf{1}_{\{T_0>u\}}\mathrm{d}u\middle|\mathcal{F}_t\right] \\ &\quad+\mathbf{1}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^{T_0}[r(u)+\lambda(S_u,u)]\mathrm{d}u}\mathbf{1}_{\{t<T_0\le T\}}\middle|\,\mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\zeta>t\}}\int_t^T e^{-\int_t^u r(v)\mathrm{d}v}\mathbb{E}\left[e^{-\int_t^u \lambda(S_v,v)\mathrm{d}v}\lambda(S_u,u)\mathbf{1}_{\{m_{t,u}^S>0\}}\middle|\,S_t\right]\mathrm{d}u \qquad (3.4) \\ &\quad+\mathbf{1}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^{T_0}[r(u)+\lambda(S_u,u)]\mathrm{d}u}\mathbf{1}_{\{t<T_0\le T\}}\middle|\,S_t\right]. \qquad (3.5) \end{aligned}$$

These valuations reduce to computing risk-neutral expectations of the form

$$\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)\mathrm{d}u}\Psi(S_T)\mathbf{1}_{\{m_{t,T}^S>0\}}\middle|\,S_t=S\right] \tag{3.6}$$

for some $\Psi(S_T)$. In addition, if the process can hit zero via diffusion before it is killed by a jump, then to value the recovery payment at the time of default one also needs to compute the second term (3.5) involving the hitting time of zero (the first term (3.4) corresponds to the recovery paid at the time of jump to default, assuming the process is killed by a jump before it hits zero via diffusion). In the jump to default extended CEV model studied in this paper the process is almost surely killed by a jump before it can diffuse to zero and, hence, this term vanishes and all calculations in the JDCEV model reduce to computing expectations of the form (3.6).

These three building blocks can be used to value corporate liabilities, credit derivatives, and equity derivatives. In particular, for fixed $T > 0$, a *defaultable zero-coupon bond* with unit face value and no recovery can be represented as the European claim with $\Psi(S_T) = 1$ and valued at time $t < T$, given no default by time t and stock price $S_t = S > 0$ at time t, as the discounted risk-neutral survival probability

$$B(S, t; T) = \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} Q(S, t; T). \tag{3.7}$$

The building blocks (ii) and (iii) correspond to defaultable zero-coupon bonds under *fractional recovery of treasury* and *fractional recovery of face value*, respectively (see, e.g., [44], p. 120). Defaultable bonds with coupons can be valued as portfolios of defaultable zeros.

Assuming a constant recovery rate on the reference bond, the floating leg of a CDS pays a fixed amount at the time of default ζ if $\zeta \leq T$. It can be valued using our building block (iii). The fixed leg of a CDS is an annuity of fixed cash payments made at discrete payment dates up until the earlier of the default time and expiration. It can be valued as a portfolio of defaultable zero-coupon bonds with no recovery.

A *European call option* with strike $K > 0$ with the payoff $(S_T - K)^+$ at expiration T has no recovery if the firm defaults. A *European put option* with strike $K > 0$ with the payoff $(K - S_T)^+$ can be decomposed into two parts: the put payoff $(K - S_T)^+ \mathbf{1}_{\{\zeta > T\}}$, given no default by time T, and a recovery payment equal to the strike K at expiration in the event of default $\zeta \leq T$. Assuming no default by time $t \in [0, T)$ and stock price $S_t = S > 0$ at time t, the pricing formulas for European-style call and put options at time t take the form

$$C(S, t; K, T) = \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} \mathbb{E}\left[\mathrm{e}^{-\int_t^T \lambda(S_u, u)\mathrm{d}u} (S_T - K)^+ \mathbf{1}_{\{m_{t,T}^S > 0\}} \middle| S_t = S \right], \tag{3.8}$$

$$P(S, t; K, T) = P_0(S, t; K, T) + P_D(S, t; K, T) \tag{3.9}$$

$$= \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} \mathbb{E}\left[\mathrm{e}^{-\int_t^T \lambda(S_u, u)\mathrm{d}u} (K - S_T)^+ \mathbf{1}_{\{m_{t,T}^S > 0\}} \middle| S_t = S \right] \tag{3.10}$$

$$+ K\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} [1 - Q(S, t; T)], \tag{3.11}$$

respectively. One notes that the put pricing formula (3.9) consists of two parts: the present value $P_0(S, t; K, T)$ of the put payoff conditional on no default given by (3.10) (this can be interpreted as the down-and-out put with the down-and-out barrier at zero), as well as the present value $P_D(S, t; K, T)$ of the cash payment K in the event of default given by (3.11). This "recovery" part of the put is a *European-style default claim*, a credit derivative that pays a fixed cash amount K at maturity T if the underlying firm has defaulted by time T. We emphasize that in our model, corporate liabilities, credit derivatives, and equity options are all valued in a unified framework as contingent claims written on the defaultable stock.

While we will now focus on deriving explicit closed-form expressions for European-style securities by probabilistic methods, the framework of this section can be straightforwardly extended to the valuation of American-style options and more complicated securities with American features, such as convertible bonds. The standard results imply that the value function solves the appropriate partial differential equation (PDE) on the appropriate domain and subject to appropriate terminal and boundary conditions. The solution can be derived via finite differences (e.g., see [1]) or by a lattice (e.g., see [22]).

4 A jump to default extended CEV stock price process

To be consistent with the leverage effect and the implied volatility skew, we specify the instantaneous volatility as that of a constant elasticity of variance (CEV) process (see [20,23–25,42,46,54]) for background on the CEV process; we use notation consistent with Davydov and Linetsky [23,24] and Linetsky [46]):

$$\sigma(S,t) = a(t)S^{\beta}, \tag{4.1}$$

where $\beta < 0$ is the volatility elasticity parameter and $a(t) > 0$ is the time-dependent volatility scale parameter. To be consistent with the empirical evidence linking corporate bond yields and CDS spreads to equity volatility, we specify the default intensity as an affine function of the instantaneous variance of the underlying stock, i.e.,

$$\lambda(S,t) = b(t) + c\,\sigma^2(S,t) = b(t) + c\,a^2(t)S^{2\beta}, \tag{4.2}$$

where $b(t) \geq 0$ is a deterministic non-negative function of time and $c > 0$ is a positive constant parameter governing the sensitivity of λ to σ^2. The functions $b(t)$ and $a(t)$ can be determined by reference to given term structures of credit spreads and at-the-money implied volatilities.

We shall see in Sect. 5 that for $c \geq 1/2$, the SDE (2.1) with σ and λ specified by (4.1) and (4.2) has a unique non-exploding solution. This solution is a diffusion process on $(0,\infty)$ where zero and infinity are both unattainable boundaries. When $c \in (0,1/2)$, the situation is more complicated. For $c \in (0,1/2)$, infinity is an unattainable (natural) boundary for all $\beta < 0$, while zero is an exit boundary for $\beta \in [c-1/2,0)$. For $c \in (0,1/2)$ and $\beta < c-1/2$, zero is a regular boundary, and we specify it as a killing boundary by adjoining a killing boundary condition. Thus, for $c \in (0,1/2)$ the process S can hit zero and is sent to the cemetery state Δ at the first hitting time of zero, T_0. Note that for $c \geq 1/2$, $T_0 = \infty$ since zero is an unattainable boundary. As we shall see in Sect. 5, even though for $c \in (0,1/2)$ the pre-default process S can hit zero, since the intensity $\lambda(S,t)$ goes to infinity as S goes to zero, zero is an unattainable boundary for the process S^{Δ} killed at the rate $\lambda(S,t)$ ($\tilde{\zeta} < T_0$ a.s. and $\zeta = \tilde{\zeta}$ a.s.). Intuitively, on all sample paths in which the pre-default stock price S diffuses down toward zero, the inverse relationship between λ and S causes the process to be killed from a positive value before it can reach zero via diffusion. Consequently, the second term (3.5) in the expression for the price of the recovery at default vanishes identically in the JDCEV model.

To summarize our specification, under the EMM, we model the price of the defaultable stock as a time-inhomogeneous diffusion process $\{S_t^{\Delta}, t \geq 0\}$ with state space $E^{\Delta} = (0, \infty) \cup \{\Delta\}$, initial value $S_0 = x > 0$, diffusion coefficient $a(t)x^{\beta+1}$, drift $[r(t) - q(t) + b(t) + c\,a^2(t)x^{2\beta}]x$, and killing rate $b(t) + c\,a^2(t)x^{2\beta}$. We refer to the stock price process as the *jump to default extended CEV* process, or *JDCEV* for short.

Our JDCEV model has a number of advantages relative to the defaultable stock models already in the literature. The special case of constant arrival rate λ and constant volatility σ is mentioned in [50] and elaborated upon in [41]. This model is very tractable and produces downward sloping implied volatility skews consistent with observation. It is also straightforward to extend this model to deterministically time varying default arrival rates and instantaneous volatilities, so as to accomodate observed term structures of credit spreads and at-the-money implied volatilities. Despite all of these benefits, credit spreads will not behave randomly, and in particular will be independent of stock price movements. Furthermore, the determinacy of volatility is inconsistent with both stochastic volatility and the leverage effect. Madan and Unal [48] (see also the correction in [36]) and Linetsky [47] address the former issue by suggesting analytically tractable specifications for the default intensity function $\lambda(S, t)$. Unfortunately, the instantaneous stock volatility remains constant in both models, $\sigma(S, t) = \sigma$, so that stochastic volatility and the leverage effect remain unaddressed. Furthermore, the risk of default is the sole determinant of the implied volatility skew in all of these models. In practice, the skew tends to flatten out as maturity increases and these time homogeneous models are unable to capture the magnitude of the skew at both long and short maturities. To address the observation that realized volatilities fluctuate and are negatively correlated to returns, Campi et al. [14] and Atlan and Leblanc [4] both suggest letting the instantaneous variance have increments which are stochastic and negatively correlated with returns. Atlan and Leblanc [4] treat the stock price as a Bessel process running on an absolutely continuous stochastic clock. As the resulting process is continuous, short term credit spreads will vanish. Campi et al. [14] suggest a CEV specification for variance, which differs from ours only in that they impose time-homogeneity. However, they assume constant arrival rate of default, which leads to the implication that short term credit spreads have low (but not zero) sensitivity to stock price levels and implied volatilities. By letting the hazard rate and the instantaneous variance both depend on the stock price, our JDCEV model accomodates large negative correlations between default indicators and stock prices, and between realized volatilities and stock prices. Furthermore, by forcing the hazard rate and the instantaneous variance to depend on the stock price in the same manner, we induce the large positive correlation between default indicators and volatilities that have been observed in the market. The parameters β and c both play a role in determining the slope of the volatility skew, which gives more flexibility in accommodating slopes which vary with term.

Typically, the models which let λ and/or σ depend on the stock price S do not allow time-dependent parameters if one insists on retaining analytical tractability. As a result, given term structures of credit spreads and at-the-money options will not be repriced by these models. A deterministic time change of these models would accommodate the same dependence on time for λ and σ, but this *deux ex machina* is patently unrealistic. The desire to exactly match given term structures of CDS spreads and/or implied volatility is particularly strong for practitioners, who make markets in these liquid securities. This twin desire for tractability and consistency sometimes forces researchers to choose parametric forms for $\sigma(S,t)$ and $\lambda(S,t)$ which introduce the possibility of negative default intensity λ. For example, Gaussian models are sometimes used for the credit spread (see, e.g., [9]).

Our JDCEV model addresses some of the deficiencies of these earlier models. The parametric forms for $\sigma(S,t)$ and $\lambda(S,t)$ are such that they generate positive processes for the stock price and default intensity prior to default. Our specification leads to closed form formulas for risk-neutral survival probabilities and credit risky derivative security values. However, we can include the arbitrary functions of time $a(t)$ and $b(t)$ in our specification, which can be determined by reference to given term structures of credit spreads and at-the-money implied volatilities.

We also remark that our pre-default diffusion process described by the SDE (2.1) with σ and λ specified by (4.1) and (4.2) is a CEV process with the additional term $ca^2(t)S^{2\beta+1}$ in the drift. In the special case with $c = 1$ and constant parameters, this process appears in [38] in their work on growth optimal portfolios, where this additional drift term with $c = 1$ serves as the market price of risk (these authors call this process *modified CEV*). The process with $c = 1$ and constant parameters is also considered in [25] who show that it is identical in law to the standard CEV process conditioned to stay strictly positive. The diffusion with arbitrary $c \in \mathbb{R}$ and constant parameters was extensively studied in the monograph by Lehnigk [45] using PDE methods. In this monograph, the process is called the *generalized Feller process* as it nests Feller's (1951) square-root diffusion as a special case with $\beta = -1/2$. The default extended CEV process proposed in this paper nests all of these processes previously considered in the literature. Our generalizations are financially relevant as they include killing (default), as well as time-dependent parameters, while retaining analytical tractability due to the remarkable properties of Bessel processes, discussed in the next section.

5 Solution of the jump to default extended CEV model via the theory of Bessel processes

Fortunately, our jump to default extended CEV model is fully analytically tractable. In particular, the three building block claims can be valued in closed form, leading to closed form valuation formulas for corporate bonds, credit default swaps, stock options, and other credit and equity derivatives. The analytical solution is due to the theory of Bessel processes (see [10,35,42, Chapter 6, 51, Chapter XI] for background on Bessel processes).

Let $\{R_t^{(\nu)}, t \geq 0\}$ be a Bessel process of index ν and started at $x, BES^{(\nu)}(x)$. Recall that for $\nu \geq 0$, zero is an unattainable entrance boundary. For $\nu \leq -1$, zero is an exit boundary. For $\nu \in (-1, 0)$, zero is a regular boundary. In our application, we specify zero as a killing boundary and send the Bessel process with $\nu < 0$ to the cemetery state Δ at the first hitting time of zero, i.e., at $T_0^R = \inf\{t \geq 0 : R_t^{(\nu)} = 0\}$ (see, e.g., [10] pp. 133–134 for a summary).

Proposition 5.1 *Let* $\{S_t, t \geq 0\}$ *be the process evolving according to* (2.1) *with* σ *and* λ *specified in* (4.1) *and* (4.2). *For* $c \geq 1/2$, *the process* (2.1) *can be represented as a re-scaled and time-changed power of a Bessel process, namely*

$$\left\{S_t = \mathrm{e}^{\int_0^t \alpha(u)\mathrm{d}u}(|\beta| R_{\tau(t)}^{(\nu)})^{1/|\beta|}, t \geq 0\right\}, \tag{5.1}$$

where the deterministic time change is

$$\tau(t) = \int_0^t a^2(u)\mathrm{e}^{-2|\beta|\int_0^u \alpha(s)\mathrm{d}s}\mathrm{d}u, \tag{5.2}$$

and

$$\nu = \frac{c - 1/2}{|\beta|} \in \mathbb{R}, \;\; R_0^{(\nu)} = x = \frac{1}{|\beta|}S^{|\beta|} > 0, \;\; \alpha(t) = r(t) - q(t) + b(t). \tag{5.3}$$

For $c \in (0, 1/2)$ $(\nu < 0)$, *the same representation holds before the first hitting time of zero, i.e.,*

$$\left\{S_t = \mathrm{e}^{\int_0^t \alpha(u)\mathrm{d}u}\left(|\beta|\, R_{\tau(t)}^{(\nu)}\right)^{1/|\beta|}, 0 \leq t < T_0^S = \tau^{-1}\left(T_0^R\right)\right\}, \tag{5.4}$$

where $T_0^S = \tau^{-1}\left(T_0^R\right)$ $\left(T_0^R = \tau\left(T_0^S\right)\right)$ *is the first hitting time of zero for the process* S_t $(R_t^{(\nu)})$ *and* τ^{-1} *is the inverse function of the deterministic time change function* τ (*both processes are killed and sent to the cemetery state* Δ *at the first hitting time of zero*).

Proof First assume $c \geq 1/2$. This corresponds to $\nu \geq 0$. Recall that for $\nu \geq 0$, the $BES^{(\nu)}(x)$ can be represented as the unique strong solution of the SDE

$$R_t^{(\nu)} = x + (\nu + 1/2)\int_0^t \frac{\mathrm{d}u}{R_u^{(\nu)}} + B_t, \;\; t \geq 0.$$

Consider the process $\left\{X_t = \mathrm{e}^{|\beta| \int_0^t \alpha(u)\mathrm{d}u} R^{(\nu)}_{\tau(t)}, t \geq 0\right\}$. From Itô's formula,

$$X_t = x + \int_0^t \left(\alpha(u)|\beta| X_u + \frac{(\nu + 1/2)a^2(u)}{X_u}\right) \mathrm{d}u + \int_0^t \mathrm{e}^{|\beta| \int_0^u \alpha(s)\mathrm{d}s} \mathrm{d}B_{\tau(u)}, \quad t \geq 0.$$

The process $\left\{\int_0^t \mathrm{e}^{|\beta| \int_0^u \alpha(s)\mathrm{d}s} \mathrm{d}B_{\tau(u)}, t \geq 0\right\}$ is a continuous martingale with quadratic variation

$$\int_0^t \mathrm{e}^{2|\beta| \int_0^u \alpha(s)\mathrm{d}s} \mathrm{d}\tau(u) = \int_0^t a^2(u)\mathrm{d}u,$$

and hence can be represented as $\int_0^t \mathrm{e}^{|\beta| \int_0^u \alpha(s)\mathrm{d}s} B_{\tau(u)} = \int_0^t a(u)\mathrm{d}\tilde{B}_u$, where $\tilde{B}$ is a standard Brownian motion; explicitly, $\tilde{B}_t = \int_0^t a^{-1}(u)\mathrm{e}^{|\beta| \int_0^u \alpha(s)\mathrm{d}s} \mathrm{d}B_{\tau(u)}$. Finally, define the process $\left\{S_t = (|\beta| X_t)^{1/|\beta|}, t \geq 0\right\}$. From Itô's formula,

$$S_t = S + \int_0^t [\alpha(u) + c\, a^2(u) S_u^{2\beta}] S_u \,\mathrm{d}u + \int_0^t a(u) S_u^{\beta+1} \,\mathrm{d}\tilde{B}_u, \quad t \geq 0.$$

Now assume $c \in (0, 1/2)$. This corresponds to $\nu < 0$. The process $R^{(\nu)}$ with $\nu < 0$ is killed at the first hitting time of zero T_0^R. The proof is the same as for $c \geq 1/2$, with all relations valid before T_0^R for the Bessel process, correspondingly, before $T_0^S = \tau^{-1}(T_0^R)$ for the process S. □

The following result is a corollary of Proposition 5.1. It can be used to compute expectations of the form (3.6).

Proposition 5.2 *Let $S > 0$. Then for any $0 \leq t < T$ we have*

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{-c \int_t^T a^2(u) S_u^{2\beta} \mathrm{d}u\right\} \Psi(S_T) \mathbf{1}_{\{m_{t,T}^S > 0\}} \Big| S_t = S\right] \\ &= E_x^{(\nu)}\left[\exp\left\{-\frac{c}{\beta^2} \int_0^\tau \frac{\mathrm{d}u}{R_u^2}\right\} \Psi\left(\mathrm{e}^{\int_t^T \alpha(s)\mathrm{d}s} (|\beta| R_\tau)^{1/|\beta|}\right) \mathbf{1}_{\{T_0^R > \tau\}}\right], \end{aligned} \tag{5.5}$$

where $E_x^{(\nu)}$ denotes expectation with respect to the law of $BES^{(\nu)}(x)$. Here the index ν and the starting value x of the Bessel process R are as in Proposition 5.1, *and the time τ is*

$$\tau = \tau(t, T) = \int_t^T a^2(u) \mathrm{e}^{-2|\beta| \int_t^u \alpha(s)\mathrm{d}s} \mathrm{d}u. \tag{5.6}$$

Proof From Proposition 5.1, we have

$$\int_t^T a^2(u)S_u^{2\beta}\mathrm{d}u = \frac{1}{\beta^2}\int_t^T a^2(u)\mathrm{e}^{-2|\beta|\int_0^u \alpha(v)\mathrm{d}v}\frac{\mathrm{d}u}{R^2_{\tau(u)}} = \frac{1}{\beta^2}\int_{\tau(t)}^{\tau(T)}\frac{\mathrm{d}s}{R_s^2}.$$

Substituting this into (3.6), we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-c\int_t^T a^2(u)S_u^{2\beta}\mathrm{d}u\right\}\Psi(S_T)\mathbf{1}_{\{m^S_{t,T}>0\}}\Big|S_t=S\right]\\
&= E^{(\nu)}\left[\exp\left\{-\frac{c}{\beta^2}\int_{\tau(t)}^{\tau(T)}\frac{\mathrm{d}s}{R_s^2}\right\}\Psi\left(\mathrm{e}^{\int_0^T\alpha(s)\mathrm{d}s}(|\beta|R_{\tau(T)})^{1/|\beta|}\right)\mathbf{1}_{\{m^R_{\tau(t),\tau(T)}>0\}}\Big|R_{\tau(t)}=x\gamma(t)\right],
\end{aligned}$$

where $m^R_{u,v} = \min_{s\in[u,v]} R_s$ is the minimum of the Bessel process R on $[u,v]$ and $\gamma(t) = \mathrm{e}^{-|\beta|\int_0^t\alpha(v)\mathrm{d}v}$. Observe that for $u \geq t \geq 0$ we can write $\tau(u) = \tau(t)+\tau(t,u)\gamma^2(t)$, where $\tau(t,u) = \int_t^u a^2(v)\mathrm{e}^{-2|\beta|\int_t^v\alpha(s)\mathrm{d}s}\mathrm{d}v$. In particular, when $u = T$, set $\tau = \tau(t,T)$ to simplify notation.

Recall the scaling property of Bessel processes ([52], Proposition 1.6, p. 443 and Proposition 1.10, p. 446): if $\{R_t, t\geq 0\}$ is a $\mathrm{BES}^{(\nu)}(x)$, then for any $c > 0$ the process $\{c^{-1}R_{c^2t}, t\geq 0\}$ is a $\mathrm{BES}^{(\nu)}(c^{-1}x)$ (note that the scaling affects the initial state).

Since the Bessel process is a time-homogeneous Markov process that satisfies the scaling property, we have

$$\begin{aligned}
&E^{(\nu)}\left[\exp\left\{-\frac{c}{\beta^2}\int_{\tau(t)}^{\tau(T)}\frac{\mathrm{d}s}{R_s^2}\right\}\Psi\left(\mathrm{e}^{\int_0^T\alpha(s)\mathrm{d}s}(|\beta|R_{\tau(T)})^{1/|\beta|}\right)\mathbf{1}_{\left\{m^R_{\tau(t),\tau(T)}>0\right\}}\Big|R_{\tau(t)}=x\gamma(t)\right]\\
&= E^{(\nu)}\Bigg[\exp\left\{-\frac{c}{\beta^2}\int_0^\tau\frac{\gamma^2(t)\mathrm{d}u}{R^2_{\tau(t)+u\gamma^2(t)}}\right\}\\
&\qquad\times\Psi\left(\mathrm{e}^{\int_0^T\alpha(s)\mathrm{d}s}(|\beta|R_{\tau(t)+\tau\gamma^2(t)})^{1/|\beta|}\right)\mathbf{1}_{\left\{m^R_{\tau(t),\tau(t)+\tau\gamma^2(t)}>0\right\}}\Big|R_{\tau(t)}=x\gamma(t)\Bigg]\\
&= E^{(\nu)}\left[\exp\left\{-\frac{c}{\beta^2}\int_0^\tau\frac{\gamma^2(t)\mathrm{d}u}{R^2_{u\gamma^2(t)}}\right\}\Psi\left(\mathrm{e}^{\int_0^T\alpha(s)\mathrm{d}s}(|\beta|R_{\tau\gamma^2(t)})^{1/|\beta|}\right)\mathbf{1}_{\{m^R_{0,\tau\gamma^2(t)}>0\}}\Big|R_0=x\gamma(t)\right]\\
&= E^{(\nu)}\left[\exp\left\{-\frac{c}{\beta^2}\int_0^\tau\frac{\mathrm{d}u}{R_u^2}\right\}\Psi\left(\mathrm{e}^{\int_t^T\alpha(s)\mathrm{d}s}(|\beta|R_\tau)^{1/|\beta|}\right)\mathbf{1}_{\left\{T_0^R>\tau\right\}}\Big|R_0=x\right].
\end{aligned}$$

In the last equality we used the scaling property with the constant $\gamma(t)$ and the equality of events $\{m^R_{0,\tau} > 0\} = \{T^R_0 > \tau\}$. This completes the proof. □

Proposition 5.2 reduces the JDCEV process killed at the rate $c\,a^2(t)S^{2\beta}$ to the Bessel process with index ν killed at the rate $c\beta^{-2}R^{-2}$. Recall that when $\nu < 0$ the Bessel process without killing can hit zero. Applying Feller's boundary classification criteria (e.g., [10], pp. 14–15), we verify that, in contrast, the Bessel process with $\nu < 0$ with killing at the rate αR^{-2} never hits zero for any $\alpha > 0$ (zero is an unattainable boundary, since the process is killed and sent to the cemetery state before it can diffuse to zero, as the killing rate rapidly increases towards infinity like R^{-2} as the process R falls towards zero).

The next step will remove the factor $\exp\left\{(-c/\beta^2)\int_0^\tau \mathrm{d}u/R^2_u\right\}$ in (5.5). We need the following result due to Yor [55] and Pitman and Yor [51] (see also [52, p. 450] and [42, Section 6.1.5]).

Proposition 5.3 *Let $P^{(\nu)}_x$ be the law of the Bessel process $R^{(\nu)}$ started at $x > 0$ and let $\mathcal{R}_t$ be its canonical filtration. For $\nu \geq 0$ and $\mu \geq 0$ the following absolute continuity relation holds:*

$$P^{(\nu)}_x\Big|_{\mathcal{R}_t} = \left(\frac{R_t}{x}\right)^{\nu-\mu} \exp\left(-\frac{\nu^2-\mu^2}{2}\int_0^t \frac{\mathrm{d}u}{R^2_u}\right) P^{(\mu)}_x\Big|_{\mathcal{R}_t}.$$

For $\nu < 0$ and $\mu \geq 0$ the following absolute continuity relation holds before T^R_0, the first hitting time of zero (since the Bessel process with $\mu > 0$ does not hit zero, we can omit $\{t < T_0\}$ on the right-hand side):

$$P^{(\nu)}_x\Big|_{\mathcal{R}_t\cap\{t<T^R_0\}} = \left(\frac{R_t}{x}\right)^{\nu-\mu} \exp\left(-\frac{\nu^2-\mu^2}{2}\int_0^t \frac{\mathrm{d}u}{R^2_u}\right) P^{(\mu)}_x\Big|_{\mathcal{R}_t}.$$

Proof For $\mu \geq 0$ and $\nu \geq 0$ this result is due to Yor [55] and Pitman and Yor [51] (see also [52, p. 450]). The proof is via the application of Girsanov's theorem and their demonstration that the relevant Radon-Nikodým derivative is a martingale with unit mean. The absolute continuity result before T_0 for $\nu < 0$ and $\mu \geq 0$ can be found in [42, Section 6.1.5]). □

Applying the absolute continuity relationships in Proposition 5.3, we further reduce the calculation of (5.5) as follows.

Proposition 5.4 *Let $S > 0$ and $0 \leq t < T$. Then we have*

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{-c\int_t^T a^2(u)S^{2\beta}_u \mathrm{d}u\right\}\Psi(S_T)\mathbf{1}_{\left\{m^S_{t,T}>0\right\}}\Bigg|S_t = S\right]\\ &= E^{(\nu_+)}_x\left[\left(\frac{R_\tau}{x}\right)^{-1/|\beta|}\Psi\left(\mathrm{e}^{\int_t^T \alpha(s)\mathrm{d}s}(|\beta|R_\tau)^{1/|\beta|}\right)\right],\end{aligned} \tag{5.7}$$

where the new Bessel process index is

$$\nu_+ = \nu + \frac{1}{|\beta|} = \frac{c+1/2}{|\beta|} > 0. \tag{5.8}$$

Proof For $\nu = (c-1/2)/|\beta| \in \mathbb{R}$ and $\nu_+ = \nu + 1/|\beta| = (c+1/2)/|\beta| > 0$, Proposition 5.3 implies that we have the absolute continuity relationship

$$\exp\left(-\frac{c}{\beta^2}\int_0^\tau \frac{\mathrm{d}u}{R_u^2}\right) P_x^{(\nu)}\Big|_{\mathcal{R}_\tau \cap \{\tau < T_0^R\}} = \left(\frac{R_\tau}{x}\right)^{-1/|\beta|} P_x^{(\nu_+)}\Big|_{\mathcal{R}_\tau}$$

(for $c \geq 1/2$ (correspondingly, $\nu \geq 0$), $\{t < T_0^R\}$ can be dropped since the process never hits zero). Applying this to (5.5), we arrive at the result (5.7). □

We have thus reduced the valuation problem to one of computing an expected value of a known function of the standard Bessel process $\mathrm{BES}^{(\nu_+)}(x)$ with index $\nu_+ = (c+1/2)/|\beta| > 0$, started at $x = (1/|\beta|)S^{|\beta|} > 0$, and evaluated at the fixed time $\tau = \tau(t,T)$ defined in (5.6). Recall that for $\nu \geq 0$ the density of $R_\tau > 0$ started at $R_0 = x > 0$ is given by (see, e.g., [52, p. 446, 10, p. 373, Eq.(4.1.0.6)]. Note that these references use different notations for densities of transition semigroups. Indeed, densities are given with respect to the Lebesgue measure in the former, and with respect to the speed measure in the latter. Here we give the density with respect to the Lebesgue measure):

$$P_x^{(\nu)}(R_\tau \in \mathrm{d}y) = p^{(\nu)}(\tau;x,y)\mathrm{d}y = \frac{y}{\tau}\left(\frac{y}{x}\right)^\nu \exp\left(-\frac{x^2+y^2}{2\tau}\right) I_\nu\left(\frac{xy}{\tau}\right)\mathrm{d}y,$$

where $I_\nu(z)$ is the Bessel function of the third kind of index ν,

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}.$$

Recall that the non-central chi-square distribution $\chi^2(\delta,\alpha)$ with δ degrees of freedom and non-centrality parameter $\alpha > 0$ has the density

$$f_{\chi^2}(x;\delta,\alpha) = \frac{1}{2}\mathrm{e}^{-(\alpha+x)/2}\left(\frac{x}{\alpha}\right)^{\frac{\nu}{2}} I_\nu(\sqrt{x\alpha})\mathbf{1}_{\{x>0\}}, \tag{5.9}$$

where $\nu = \delta/2 - 1$. The Bessel process density is expressed in terms of the non-central chi-square density as

$$p^{(\nu)}(\tau;x,y) = \left(\frac{2y}{\tau}\right) f_{\chi^2}\left(\frac{y^2}{\tau};\delta,\frac{x^2}{\tau}\right).$$

This reduces the calculation of the expectation (5.7) to the problem of evaluating integrals whose kernel is the non-central chi-square density. We will need the following lemma.

Lemma 5.1 *Let X be a $\chi^2(\delta,\alpha)$ random variable, $\nu = \delta/2 - 1$, $p > -(\nu+1)$, and $k > 0$. The p-th moments and truncated p-th moments are given by*

$$\begin{aligned}\mathcal{M}(p;\delta,\alpha) &= E^{\chi^2(\delta,\alpha)}[X^p] \\ &= 2^p \mathrm{e}^{-\alpha/2} \frac{\Gamma(p+\nu+1)}{\Gamma(\nu+1)} {}_1F_1(p+\nu+1,\nu+1,\alpha/2), \end{aligned} \tag{5.10}$$

$$\begin{aligned}\Phi^+(p,k;\delta,\alpha) &= E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X>k\}}] \\ &= 2^p \sum_{n=0}^{\infty} \mathrm{e}^{-\alpha/2} \left(\frac{\alpha}{2}\right)^n \frac{\Gamma(\nu+p+n+1,k/2)}{n!\,\Gamma(\nu+n+1)}, \end{aligned} \tag{5.11}$$

$$\begin{aligned}\Phi^-(p,k;\delta,\alpha) &= E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X\le k\}}] \\ &= 2^p \sum_{n=0}^{\infty} \mathrm{e}^{-\alpha/2} \left(\frac{\alpha}{2}\right)^n \frac{\gamma(\nu+p+n+1,k/2)}{n!\,\Gamma(\nu+n+1)}, \end{aligned} \tag{5.12}$$

where $\Gamma(a)$ is the standard Gamma function, $\gamma(a,x) = \int_0^x y^{a-1}\mathrm{e}^{-y}\mathrm{d}y$ is the incomplete Gamma function, $\Gamma(a,x) = \Gamma(x) - \gamma(a,x)$ is the complementary incomplete Gamma function, and

$${}_1F_1(a,b,x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \quad \text{where } (a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1),\ n>0,$$

is the Kummer confluent hypergeometric function.

Proof Use the series representation of the Bessel function $I_\nu(z)$ and compute the relevant integrals term by term. □

By definition, the three functions defined in Lemma 5.1 satisfy the identity

$$\Phi^+(p,k;\delta,\alpha) + \Phi^-(p,k;\delta,\alpha) = \mathcal{M}(p;\delta,\alpha). \tag{5.13}$$

We can now express the risk-neutral survival probability and the value functions for credit and equity derivatives in terms of these three special functions.

Proposition 5.5 *Let x, ν_+, and $\tau = \tau(t,T)$ be defined as in* (5.3), (5.8), *and* (5.6), *respectively, and let $\delta_+ = 2(\nu_+ + 1)$. Assume that default has not happened by time $t \ge 0$, i.e., $\zeta > t$, and $S_t = S > 0$.*

(i) *The risk-neutral survival probability* (3.1) *is given by*

$$Q(S,t;T) = \mathrm{e}^{-\int_t^T b(u)\mathrm{d}u} \left(\frac{x^2}{\tau}\right)^{1/(2|\beta|)} \mathcal{M}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{\tau}\right). \tag{5.14}$$

(ii) *The value* (3.4) *of a claim that pays one dollar at the time* ζ *of jump to default is given by*

$$\begin{aligned}\int_t^T \mathrm{e}^{-\int_t^u [r(s)+b(s)]\mathrm{d}s} \Bigg\{ & b(u)\left(\frac{x^2}{\tau(t,u)}\right)^{1/(2|\beta|)} \mathcal{M}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{\tau(t,u)}\right) \qquad (5.15)\\ & + ca^2(u)S^{2\beta}\mathrm{e}^{-2|\beta|\int_t^u \alpha(s)\mathrm{d}s}\left(\frac{x^2}{\tau(t,u)}\right)^{1/(2|\beta|)+1}\mathcal{M}\left(-\frac{1}{2|\beta|}-1;\delta_+,\frac{x^2}{\tau(t,u)}\right)\Bigg\}\mathrm{d}u\end{aligned}$$

(the second part (3.5) *vanishes identically since the JDCEV process is killed by a jump to default before it has the opportunity to diffuse to zero).*

(iii) *The call option price* (3.8) *is given by*

$$\begin{aligned}C(S,t;K,T) = {} & \mathrm{e}^{-\int_t^T q(u)\mathrm{d}u} S\Phi^+\left(0,\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right) \qquad (5.16)\\ & -\mathrm{e}^{-\int_t^T [r(u)+b(u)]\mathrm{d}u} K\left(\frac{x^2}{\tau}\right)^{1/(2|\beta|)} \Phi^+\left(-\frac{1}{2|\beta|},\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right),\end{aligned}$$

where

$$k = k(t,T) = \tfrac{1}{|\beta|}K^{|\beta|}\mathrm{e}^{-|\beta|\int_t^T \alpha(u)\mathrm{d}u}. \tag{5.17}$$

(iv) *The price of the put payoff conditional on no default by time* T (3.10) *is given by*

$$\begin{aligned}P_0(S,t;K,T) = {} & \mathrm{e}^{-\int_t^T [r(u)+b(u)]\mathrm{d}u} K\left(\frac{x^2}{\tau}\right)^{1/(2|\beta|)} \Phi^-\left(-\frac{1}{2|\beta|},\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right)\\ & -\mathrm{e}^{-\int_t^T q(u)\mathrm{d}u} S\Phi^-\left(0,\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right), \qquad (5.18)\end{aligned}$$

and the recovery part $P_D(S,t;K,T)$ *of the put option is given by* (3.11) *with the survival probability* (5.14).

Proof The proof is by straightforward calculations using the results in Proposition 5.4 and Lemma 5.1 and is omitted. □

Note that the put-call parity is verified from the identity (5.13):

$$C(S,t;K,T) - P(S,t;K,T) = \mathrm{e}^{-\int_t^T q(u)\mathrm{d}u} S - \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} K.$$

Remark 5.1 **Time-homogeneous model with constant parameters.** To obtain survival probabilities and contingent claim values under the time-homogeneous model with constant parameters $r \geq 0, q \geq 0$, $b \geq 0$, and $a > 0$, observe that the expressions for τ and k reduce to

$$\tau = \tau(t,T) = \begin{cases} \frac{a^2}{2|\beta|(r-q+b)}(1 - \mathrm{e}^{-2|\beta|(r-q+b)(T-t)}), & r-q+b \neq 0 \\ a^2(T-t), & r-q+b = 0 \end{cases}, \tag{5.19}$$

$$k = k(t,T) = \frac{1}{|\beta|} K^{|\beta|} \mathrm{e}^{-|\beta|(r-q+b)(T-t)}. \tag{5.20}$$

Remark 5.2 **Standard time-homogeneous and inhomogeneous CEV models.** The standard CEV model of Cox [20] is nested within our general specification. In fact, our model nests a more general time-inhomogeneous version of Cox's model with time-dependent interest rate, dividend yield, and volatility scale parameters $r(t), q(t)$, and $a(t)$, respectively. To obtain this special case, set $b = 0$ and $c = 0$, so that default can only occur when the stock price diffuses into zero (recall that the CEV process with $\beta < 0$ hits zero with positive probability). In contrast, it is well known that for $\beta = 0$, the limiting process of geometric Brownian motion never hits zero. Intuitively, the reason that the standard CEV process can hit zero while geometric Brownian motion cannot is due to the increased volatility of the former process at low stock prices. While the standard CEV model does allow for default, it should be noted that short term credit spreads still vanish as default can only happen via hitting the boundary at zero (no surprise). Moreover, the default probability is unrealistically small for empirically reasonable parameter values for β and a of the CEV process volatility. Thus, if one wishes to realistically model default, it is reasonable to extend the model by introducing stock price-dependent default intensity, as is done in our jump to default extended CEV model.

To facilitate comparisons of the standard and jump to default extended CEV models, we observe that when $b = c = 0$, the (risk-neutral) survival probability reduces to the well-known survival probability in the standard CEV model (assume no default before time $t \geq 0$ and $S_t = S > 0$):

$$Q(S,t;T) = \mathbb{Q}(T_0 > T | S_t = S) = \frac{\gamma(|\nu|, x^2/(2\tau))}{\Gamma(|\nu|)},$$

where $|\nu| = 1/(2|\beta|)$ in this case. The (risk-neutral) default probability is given by

$$\mathbb{Q}(t < T_0 \leq T | S_t = S) = \frac{\Gamma(|\nu|, x^2/(2\tau))}{\Gamma(|\nu|)}.$$

In the time-homogeneous case, this probability was first obtained by Cox. Similarly, other results in our Proposition 5.5 reduce to the corresponding results for the time-inhomogeneous CEV model when $b = c = 0$ (and in particular, to Cox's results in the constant parameter case). When $b > 0$ is a positive constant and $c = 0$, our model reduces to the CEV model with killing at a constant rate recently considered by Campi et al. [14]. In this model the default intensity is independent of the stock price and return volatility. Note that when the intensity of jump to default is a constant, it is possible that default happens by either a jump to default from a positive value or by hitting zero via diffusion. This is in contrast to our JDCEV model with jump to default intensity exploding to infinity as the stock price falls towards zero. While Proposition 5.5 is formulated assuming $c > 0$, all results remain valid in the limiting case $c = 0$ with one modification. The claim that pays one dollar at the time of default also acquires a non-vanishing second contribution (3.5) stemming from the possibility of default via diffusion to zero when $c = 0$. For the case of constant parameters r, q, a, and b, this contribution can be computed using results in [23,24,46] on hitting times of the CEV process.

Remark 5.3 **Computation.** For $p = 0$, $\Phi^-(0, k; \delta, \alpha)$ and $\Phi^+(0, k; \delta, \alpha)$ reduce to the non-central chi-square distribution function and the complementary distribution function, respectively. There exists an extensive literature devoted to the efficient computation of these functions, with several alternative representations available (e.g., [6,28,32,54] and references therein). For certain ranges of parameter values, some of these representations are more computationally efficient than the series of incomplete Gamma functions in Lemma 5.1 for $p = 0$. These representations can also be applied to the computation of the more general truncated moment functions for $p \neq 0$. We do not pursue these numerical issues here. For all numerical computations of option prices in this paper, we used the series of incomplete Gamma functions in Lemma 5.1 implemented in the *Mathematica* software package on a PC. The incomplete Gamma function, as well as the confluent hypergeometric function entering into the moment formula (5.10) (see also Eq. (5.6) in [3] for the computation of negative moments of the squared Bessel process similar to our Eq. (5.10)), are available in *Mathematica* as built-in functions. To compute call and put option values, the series of Gamma functions in Lemma 5.1 were truncated to insure six decimal place accuracy.

Remark 5.4 **Instantaneous credit spread process.** Our expression (5.14) for the risk-neutral survival probability is very similar to the survival probability obtained in Andreasen (2003) in his interesting *credit explosives* model. Andreasen directly models the instantaneous credit spread as a diffusion process with quadratic drift and cubic instantaneous variance. Under a certain parameter restriction, he shows that this process can explode to infinity in finite time, thereby triggering a default. In our JDCEV model, the instantaneous credit spread is equal to the default intensity $\lambda(S,t) = b(t) + c\,a^2(t)S^{2\beta}$. Itô's formula implies that, when $b = 0$ and $c > 0$, the process $\lambda_t = \lambda(S_t,t) = c\,a^2(t)S^{2\beta}$ is a diffusion process with quadratic drift and cubic instantaneous variance. This process can explode to infinity for $c \in (0, 1/2)$. Recall that when $c \in (0, 1/2)$, our *pre-default* stock price process can hit zero. For $\beta < 0$, S_t hitting zero is the same event as λ_t exploding to infinity. Thus, in fact, the JDCEV model nests the credit explosives model of Andreasen [2] when $b = 0$ and $c \in (0, 1/2)$. The focus of Andreasen [2] is exclusively on credit derivatives and he develops tractable valuation models for CDS and first to default baskets. In contrast, we endogenize the hazard rate by assuming that it is affine in a negative power of the defaultable stock price. This link allows us to value equity derivatives as well as credit derivatives and provides a direct relationship between them. It also permits direct observation of the default intensity via the underlying stock price or volatility, obviating the need to imply it out from market prices of credit derivatives.

6 Asymptotic analysis

In this section, we investigate the short and long maturity behavior of credit spreads in our model. Consider the model with time-independent parameters (see Remark 5.1). Assuming that the unit face value T-maturity zero-coupon bond recovers nothing in the event of bankruptcy, the T-maturity credit spread (observed at time zero $t = 0$) is defined as usual as:

$$\mathcal{S}(S, T) = -\frac{1}{T} \ln B(S, 0; T) - r = -\frac{1}{T} \ln Q(S, 0; T),$$

where $Q(S, 0; T)$ is the survival probability. As the time to maturity T approaches zero, the jump to default extended CEV model implies that the short maturity credit spread is

$$\mathcal{S}(S, T) \sim b + c\, a^2 S^{2\beta} \text{ as } T \to 0.$$

To investigate the long maturity asymptotics of the term structure of credit spreads, we define the asymptotic credit spread $\mathcal{S}_\infty := \lim_{T\to\infty} \mathcal{S}(S, T)$. We are also interested in the probability of ultimate survival $\mathbb{Q}(\zeta = \infty | S_0 = S) = \lim_{T\to\infty} Q(S, 0; T)$.

Proposition 6.1 *The probability of ultimate survival is given by*

$$\mathbb{Q}(\zeta = \infty | S_0 = S) = \begin{cases} \left(\frac{x^2}{\tau_\infty}\right)^{1/(2|\beta|)} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau_\infty}\right) > 0, & b = 0 \text{ and } r - q > 0 \\ 0, & \text{otherwise} \end{cases},$$

where, for $b = 0$ and $r - q > 0$,

$$\tau_\infty = \lim_{T\to\infty} \tau(0, T) = \frac{a^2}{2|\beta|(r - q)}.$$

The asymptotic credit spread is given by

$$\mathcal{S}_\infty = \begin{cases} b, & r - q + b \geq 0 \\ q - r, & r - q + b < 0 \end{cases}.$$

Proof The proof is by straightforward calculation of the limits with the survival probability (5.14) specialized to the constant parameter case; see (5.19). □

When $b = 0$ and $r > q$, as the stock price increases towards infinity, the volatility and the default intensity both decrease towards zero, while the drift rate is positive. This allows for some positive probability for the stock process to ultimately escape default. When $b > 0$ and $r - q + b > 0$, as the stock price increases towards infinity under the influence of the positive drift, the volatility decreases towards zero and the default intensity decreases towards its constant term b. The process cannot avoid ultimate default, and, for large T, the term structure of credit spreads will asymptotically flatten out towards the asymptotic credit spread b. When the dividend yield q is sufficiently large so that $q > r + b$, the drift rate is negative, the process is pushed towards zero, and the intensity of default increases as the stock price decreases, thus precipitating inevitable default. For large T, the term structure of credit spreads flattens out towards the asymptotic spread $\mathcal{S}_\infty = q - r$.

7 Numerical examples: default probabilities, credit spreads, and implied volatility skews in the JDCEV model

We now give numerical examples of risk-neutral default probabilities, term structures of credit spreads, and implied volatility skews in the default extended CEV model. In this section, we consider the time-homogeneous model with constant parameters for simplicity. It is convenient to parametrize the local volatility function as $\sigma(S) = \sigma_*(S/S^*)^\beta$, where $S^* > 0$ is some reference stock price level and $\sigma_* > 0$ is the volatility at that reference level, $\sigma(S^*) = \sigma_*$, so that σ_* serves as the volatility scale parameter. To calibrate the model in applications, one typically selects $S^* = S_0$, the initial stock price at the time of calibration. With this parametrization, $a = \sigma_*(S^*)^{-\beta}$.

In the following set of examples, we assume that the initial stock price $S_0 = 50$, volatility scale parameter $\sigma_* = 0.2$ (20%), $r = 0.05$ (risk-free rate of 5%), $q = 0$ (no dividends), and that the elasticity parameter $\beta = -1$ (corresponding to the absolute diffusion model). Our initial interest is in the dependence of various observables such as term credit spreads and implied volatilities on the two parameters b and c governing the default intensity $\lambda(S) = b + c\sigma_*^2(S/S^*)^{2\beta}$. We consider the cases with $b = 0$ and $b = 0.02$ (the latter choice adds 2% per annum to default intensity) and $c = 1/2$ and $c = 1$. At the initial stock price level $S_0 = 50$ and the volatility scale parameter $\sigma_* = 0.2$, the contribution to default intensity due to the variance term $c\sigma^2(S)$ at this stock price level is 0.04. As the stock price falls (increases), the volatility increases (decreases) and the

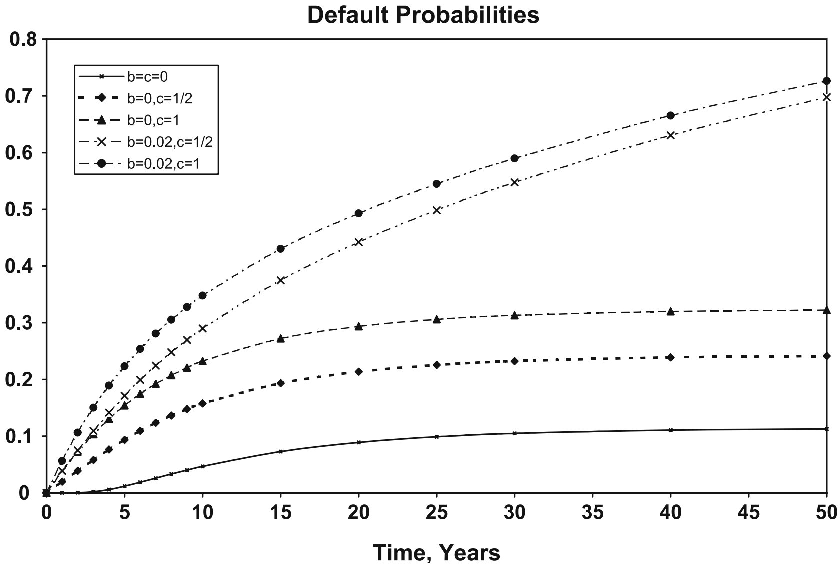

Fig. 1 Risk-neutral default probabilities. Parameter values: $S = S^* = 50, \sigma_* = 0.2, \beta = -1$, $r = 0.05, q = 0, b = 0, 0.02, c = 0, 1/2, 1$

default intensity also increases (decreases). For comparison, we also consider the standard CEV model with $b = c = 0$.

Figure 1 plots the (risk-neutral) default probability by time T as a function of T. We observe that in the standard CEV model, the default probability (due to the possibility of hitting zero through diffusion) is very small but positive (it does not become meaningfully greater than zero until about five years). In contrast, this default probability is substantially increased in our default extended CEV model, where the default event can also arrive as a surprise. As expected, increasing b and c increases the default probability. Figure 2 illustrates the corresponding shapes of the term structure of zero-coupon credit spreads, assuming zero recovery. The credit spread curves start at the instantaneous credit spread equal to the default intensity $b + c\,\sigma_*^2$ and asymptotically go towards the asymptotic yield $\mathcal{S}_\infty = b$. We stress that in this example we chose constant b for simplicity. By proper choice of the time-dependent function $b(t)$, any given finite term structure of term credit spreads can be matched, thereby allowing calibration to the term structure of CDS spreads or corporate bond yields.

Figure 3 plots the option implied volatility against the strike price in the JDCEV model with $b = 0.02$ and $c = 1$ for expirations of 3 months, 6 months, 1 year, and 5 years (these implied volatility curves correspond to the term structure of credit spreads in Fig. 2 with $b = 0.02$ and $c = 1$). The implied volatilities are obtained by first computing the call pricing formula of Proposition 5.5 for a given strike and expiration and then implying out the Black-Scholes implied volatility. The current stock price is $S = 50$. We observe the characteristic

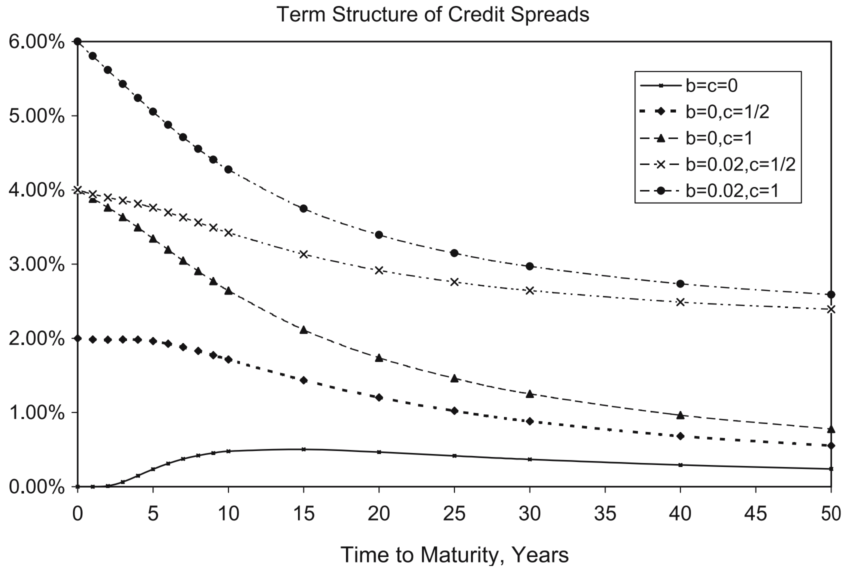

Fig. 2 Term structures of credit spreads. Parameter values: $S = S^* = 50, \sigma_* = 0.2, \beta = -1, r = 0.05, q = 0, b = 0, 0.02, c = 0, 1/2, 1$

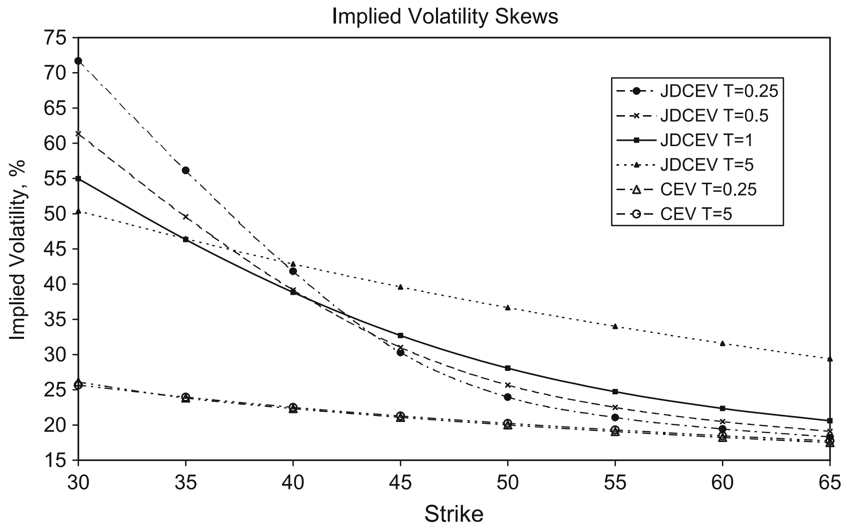

Fig. 3 Implied volatility skews. Parameter values: $S = S^* = 50, \sigma_* = 0.2, \beta = -1, r = 0.05, q = 0$. For CEV model: $b = c = 0$. For JDCEV model: $b = 0.02, c = 1$. For JDCEV times to expiration are $T = 0.25, 0.5, 1, 5$ years. Implied volatilities are plotted against strike

decreasing and convex implied volatility skew with implied volatilities increasing for lower strikes, as the local volatility and the default intensity both increase as the stock price declines. Moreover, shorter expirations exhibit steeper skews, and the skews gradually flatten out for longer maturities.

For comparison, implied volatility skews for the standard CEV model with $b = c = 0$ and 6 months and 5 years to maturity are also plotted. Two observations are immediately apparent. First, the JDCEV model skews for all expirations are above the corresponding standard CEV curves and are also steeper. This is no surprise as the JDCEV model also includes the default jump. The second observation is that for the standard CEV model, the skews are essentially the same for all expirations (the 6 month and 5 year skews are so close to each other that the difference is hardly visible on the plot). In contrast, the JDCEV skews exhibit characteristic patterns of steeper skews for shorter maturities and gradually declining skews for longer maturities, which is in accordance with empirical observations. Proportionally, the default jump increases the volatilities of the shorter expirations more than it does the longer expirations. Thus, the JDCEV model is capable of not only capturing the skew for one given expiration, but also reproducing the pattern of decreasing skew steepness as expiration increases.

We stress that in this numerical example, the volatility scale parameter a (or σ_*) was chosen to be constant for simplicity. However, our time-inhomogeneous JDCEV model allows one to choose a time-dependent function $a(t)$ to match a given arbitrary term structure of at-the-money implied volatilities. After the function $b(t)$ has been selected to match the credit spread curve and the function $a(t)$ has been selected to match the at-the-money implied volatility term structure, the model still has two remaining free parameters, β and c. Just as the parameter $\beta < 0$ dictates the sensitivity of the log of instantaneous volatility to returns in the standard CEV model, the parameter c specifies the sensitivity of the default arrival rate λ to the instantaneous variance in our JDCEV model. These parameters can be selected to best fit some measure of the implied volatility skew at two maturities (in contrast with the standard CEV, which can only fit some skewness measure at one maturity). Since the steepness of the skew typically monotonically declines with expiration, fitting the model to two skews at two expirations, e.g., a short expiration and a long expiration, will usually provide a satisfactory fit to the entire implied volatility surface.

Finally, we investigate the default claim embedded in the put option. Table 1 presents prices of one-year put options in the JDCEV model with $\beta = -1$, $\sigma_* = 0.2, b = 0.02$, and $c = 1$. The current stock price is $S = S^* = 50$. For a range of strikes from \$5 to \$75, the table presents the price of the put payoff conditional on no default (down-and-out put with the knock-out barrier at zero) computed from (5.18), the price of the default claim embedded in the put given by (3.11) and (5.14), and the total price of the put (3.9). We observe that for strikes below \$40, the values of these deep out-of-the-money puts are comprised almost entirely of the value of the default claim. We conclude that in our model, deep out-of-the-money put options are essentially credit derivatives that give their purchaser protection against default of the underlying firm. Indeed, when a company is hit with bad news, the observed volume for deep out-of-the-money equity puts is typically of the same magnitude as the volume for at-the-money puts, while volume at other strikes is lower. Hence, this implication of our model appears to be well known to market participants.

Table 1 Put prices in JDCEV model ($P = P_0 + P_D$)

K	$P_0(S,t;K,T)$	$P_D(S,t;K,T)$	$P(S,t;K,T)$
5	3.3×10^{-8}	0.26819	0.26819
10	2.0×10^{-6}	0.53638	0.53638
20	0.00036	1.07277	1.07313
30	0.01499	1.60915	1.62414
40	0.23407	2.14553	2.37960
45	0.67715	2.41372	3.09087
50	1.62988	2.68192	4.31180
55	3.32780	2.95011	6.27791
60	5.88779	3.21830	9.10609
65	9.23827	3.48649	12.7248

Parameter values: $S = S^* = 50, \sigma_* = 0.2, \beta = -1, r = 0.05, q = 0, b = 0.02, c = 1, T = 1$ year

8 Summary and future research

In this paper, we treat corporate bonds, credit derivatives, and stock options in a unified framework. To illustrate the framework, we propose and solve in closed form a parsimonious extension of the CEV model to incorporate a jump to default. By combining time changes, scale changes, and measure changes, we reduce the problem of determining survival probabilities, corporate bond prices, and stock option prices in our model to one of calculating moments and truncated moments of Bessel processes evaluated at a fixed time. We find a closed form solution to the latter problem in terms of the non-central chi-square distribution function. In contrast to most tractable models with nonnegative stock prices, volatilities, and hazard rates, our model can be calibrated to exactly match arbitrarily given term structures of credit spreads, interest rates, dividend yields, and at-the-money implied volatilities. We can also match the slopes of the implied volatility skew for two expiration dates, e.g., the nearest term and the most distant term. The results of our analysis provide further insights into the linkages between corporate credit spreads, CDS spreads, and volatility skews. We hope that this development of a flexible and analytically tractable model will spur further research by academics and practitioners into the development of a unified framework for pricing, trading, and risk managing corporate liabilities, credit derivatives, and equity derivatives.

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