Chapter 13
Forward Rate Modeling

This chapter is concerned with interest rate modeling, in which the mean reversion property plays an important role. We consider the main short rate models (Vasicek, CIR, CEV, affine models) and the computation of bond prices in such models. Next we consider the modeling of forward rates in the HJM and BGM models, as well as in two-factor models.

13.1 Short Term Models and Mean Reversion

Vasicek Model

The first model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion, is the Vasicek [111] model, which is based on the Ornstein-Uhlenbeck process. Here, the short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ solves the equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \tag{13.1}$$

where $a, b, \sigma \in \mathbb{R}$ and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with solution

$$r_t = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma \int_0^t e^{-b(t-s)} dB_s, \quad t \in \mathbb{R}_+. \tag{13.2}$$

The probability distribution of r_t is Gaussian at all times t, with mean

$$r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}),$$

and variance

$$\sigma^2 \int_0^t (e^{-b(t-s)})^2 ds = \sigma^2 \int_0^t e^{-2bs} ds = \frac{\sigma^2}{2b}(1 - e^{-2bt}), \qquad t \in \mathbb{R}_+.$$

In large time t, this distribution converges to the Gaussian $\mathcal{N}(a/b, \sigma^2/(2b))$ distribution when $b > 0$, which is also the *invariant* (or stationary) distribution of $(r_t)_{t\in\mathbb{R}_+}$. However, a drawback of the Vasicek model is to allow for negative values of r_t.

Figure 13.1 presents a random simulation of $t \longmapsto r_t$ in the Vasicek model with $r_0 = a/b = 5\%$, *i.e.* the reverting property of the process is with respect to its initial value $r_0 = 5\%$. Note that the interest rate in Figure 13.1 may become negative, which can be unusual for interest rates but may nevertheless happen.

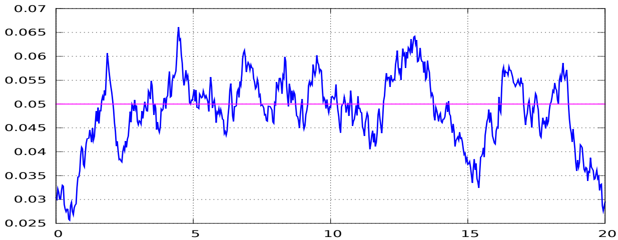

Fig. 13.1: Graph of the short rate $t \longmapsto r_t$ in the Vasicek model.

Cox-Ingersoll-Ross (CIR) Model

The Cox-Ingersoll-Ross (CIR) [18] model brings a solution to the positivity problem encountered with the Vasicek model, by the use the nonlinear stochastic differential equation

$$dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}dB_t, \qquad \alpha \geq 0, \quad \beta > 0.$$

The probability distribution of r_t at time $t > 0$ admits the noncentral Chi square probability density function given by

$$f_{r_t}(x) \tag{13.3}$$
$$= \frac{2\beta}{\sigma^2(1-e^{-\beta t})} \exp\left(-\frac{2\beta(x + r_0 e^{-\beta t})}{\sigma^2(1-e^{-\beta t})}\right)\left(\frac{x}{r_0 e^{-\beta t}}\right)^{\alpha\beta/\sigma^2 - 1/2} I_{2\alpha\beta/\sigma^2-1}\left(\frac{4\beta\sqrt{r_0 x e^{-\beta t}}}{\sigma^2(1-e^{-\beta t})}\right),$$

$x > 0$, where

$$I_\lambda(z) := \left(\frac{z}{2}\right)^\lambda \sum_{k=0}^\infty \frac{(z^2/4)^k}{k!\Gamma(\lambda+k+1)}, \qquad z \in \mathbb{R},$$

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is the modified Bessel function of the first kind, cf. Corollary 24 in [2]. Note that $f_{r_t}(x)$ is not defined at $x = 0$ if $\alpha\beta/\sigma^2 - 1/2 < 0$, *i.e.* $2\alpha\beta < \sigma^2$, in which case the probability distribution of r_t admits a point mass at $x = 0$. On the other hand, r_t remains almost surely strictly positive under the Feller condition $2\alpha\beta \geq \sigma^2$, cf. the study of the associated probability density in Lemma 4 of [36].

In large time t, using the asymptotics

$$I_\lambda(z) \simeq_{z\to 0} \frac{1}{\Gamma(\lambda+1)}\left(\frac{z}{2}\right)^\lambda,$$

the density (13.3) becomes the Gamma density

$$f(x) = \frac{1}{\Gamma(2\alpha\beta/\sigma^2)}\left(\frac{2\beta}{\sigma^2}\right)^{2\alpha\beta/\sigma^2} x^{-1+2\alpha\beta/\sigma^2} e^{-2\beta x/\sigma^2}, \qquad x > 0. \tag{13.4}$$

with shape parameter $2\alpha\beta/\sigma^2$ and scale parameter $\sigma^2/(2\beta)$, which is also the *invariant distribution* of r_t.

Other classical mean reverting models include the Courtadon (1982) model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t,$$

where α, β, σ are nonnegative, and the exponential Vasicek model

$$dr_t = r_t(\eta - a\log r_t)dt + \sigma r_t dB_t,$$

where a, η, σ are nonnegative constant coefficients, cf. Exercises 4.11 and 4.12.

Constant Elasticity of Variance (CEV)

Constant Elasticity of Variance models are designed to take into account nonconstant volatilities that can vary as a power of the underlying asset. The Marsh-Rosenfeld (1983) model

$$dr_t = (\beta r_t^{-(1-\gamma)} + \alpha r_t)dt + \sigma r_t^{\gamma/2} dB_t$$

where α, β, σ, γ are nonnegative constants and β is the variance (or diffusion) elasticity coefficient, covers most of the CEV models. Here, constant elasticity refers to the constant ratio

$$\frac{dv(r)/v(r)}{dr/r} = \frac{rv'(r)}{v(r)} = \frac{d\log v(r)}{d\log r} = \frac{d\log r^{\gamma/2}}{d\log r} = \frac{\gamma}{2}$$

between the relative change $dv(r)/v(r)$ in the variance $v(r)$ and the relative change dr/r in r.

For $\gamma = 1$ this is the CIR model, and for $\beta = 0$ we get the standard CEV model

$$dr_t = \alpha r_t dt + \sigma r_t^{\gamma/2} dB_t.$$

If $\gamma = 2$ this yields the Dothan [30] model

$$dr_t = \alpha r_t dt + \sigma r_t dB_t.$$

Affine Models

The class of short rate interest rate models admits a number of generalizations that can be found in the references quoted in the introduction of this chapter, among which is the class of affine models of the form

$$dr_t = (\eta(t) + \lambda(t) r_t) dt + \sqrt{\delta(t) + \gamma(t) r_t} dB_t. \tag{13.5}$$

Such models are called affine because the associated zero-coupon bonds can be priced using an *affine* PDE of the type (13.13) below, as will be seen after Proposition 13.2.

They also include the Ho-Lee model

$$dr_t = \theta(t) dt + \sigma dB_t,$$

where $\theta(t)$ is a deterministic function of time, as an extension of the Merton model $dr_t = \theta dt + \sigma dB_t$, and the Hull-White model [56], cf. Section 13.1

$$dr_t = (\theta(t) - \alpha(t) r_t) dt + \sigma(t) dB_t$$

which is a time-dependent extension of the Vasicek model.

Calibration of the Vasicek model

The Vasicek equation

$$dr_t = (a - b r_t) dt + \sigma dB_t$$

can be discretized according to a discrete time sequence $(t_k)_{k=0,1,\ldots,n}$ as

$$r_{t_{k+1}} - r_{t_k} = (a - b r_{t_k}) \Delta t + \sigma Z_k, \qquad k \in \mathbb{N},$$

where $\Delta t := t_{k+1} - t_k$ and $(Z_k)_{k \geq 0}$ is a Gaussian white noise with variance Δt, *i.e.* a sequence of independent, centered and identically distributed $\mathcal{N}(0, \Delta t)$

Gaussian random variables.

We find

$$r_{t_{k+1}} = r_{t_k} + (a - br_{t_k})\Delta t + \sigma Z_k = a\Delta t + (1 - b\Delta t)r_{t_k} + \sigma Z_k, \qquad k \in \mathbb{N}.$$

By minimization of the residual

$$\sum_{k=0}^{n-1} \left(r_{t_{k+1}} - a\Delta t - (1 - b\Delta t) r_{t_k} \right)^2$$

over a and b using Ordinary Least Square (OLS) regression, we can estimate the parameters a and b respectively as the empirical mean and covariance of t $(r_{t_k})_{k=0,1,\ldots,n}$, *i.e.*

$$\begin{cases} \hat{a}\Delta t = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} r_{t_{k+1}} - \dfrac{1}{n}(1 - \hat{b}\Delta t) \sum_{k=0}^{n-1} r_{t_k} \\ \text{and} \\ 1 - \hat{b}\Delta t = \dfrac{\displaystyle\sum_{k=0}^{n-1} r_{t_k} r_{t_{k+1}} - \dfrac{1}{n} \sum_{k=0}^{n-1} r_{t_k} \sum_{l=0}^{n-1} r_{t_{l+1}}}{\displaystyle\sum_{k=0}^{n-1} r_{t_k} r_{t_k} - \dfrac{1}{n} \sum_{k=0}^{n-1} r_{t_k} \sum_{l=0}^{n-1} r_{t_l}} \\ \qquad = \dfrac{\displaystyle\sum_{k=0}^{n-1} \left(r_{t_k} - \dfrac{1}{n} \sum_{l=0}^{n-1} r_{t_l} \right) \left(r_{t_{k+1}} - \dfrac{1}{n} \sum_{l=0}^{n-1} r_{t_{l+1}} \right)}{\displaystyle\sum_{k=0}^{n-1} \left(r_{t_k} - \dfrac{1}{n} \sum_{k=0}^{n-1} r_{t_k} \right)^2}. \end{cases}$$

This also yields

$$\sigma^2 \Delta t = \mathrm{Var}[\sigma Z_k] = \mathrm{Var}\left[r_{t_{k+1}} - (1 - b\Delta t) r_{t_k} - a\Delta t \right], \qquad k \in \mathbb{N},$$

hence σ can be estimated as

$$\hat{\sigma}^2 \Delta t = \frac{1}{n} \sum_{k=0}^{n-1} \left(r_{t_{k+1}} - r_{t_k}(1 - \hat{b}\Delta t) - \hat{a}\Delta t \right)^2.$$

Defining $\tilde{r}_{t_k} := r_{t_k} - a/b$, $k \in \mathbb{N}$, we have

$$\begin{aligned} \tilde{r}_{t_{k+1}} &= r_{t_{k+1}} - a/b \\ &= r_{t_k} - a/b + (a - br_{t_k})\Delta t + \sigma Z_k \end{aligned}$$

$$
\begin{aligned}
&= r_{t_k} - a/b - b(r_{t_k} - a/b)\Delta t + \sigma Z_k \\
&= \tilde{r}_{t_k} - b\tilde{r}_{t_k}\Delta t + \sigma Z_k \\
&= (1 - b\Delta t)\tilde{r}_{t_k} + \sigma Z_k, \qquad k \in \mathbb{N}.
\end{aligned}
$$

In other words, the sequence $(\tilde{r}_{t_k})_{k\in\mathbb{N}}$ is modeled according to an autoregressive AR(1) time series in which the current state X_n of the system is expressed as the linear combination

$$
X_n := \sigma Z_n + \alpha_1 X_{n-1}, \qquad n \geq 1,
$$

which can be solved recursively as the series

$$
X_n = \sigma Z_n + \alpha_1(\sigma Z_{n-1} + \alpha_1 X_{n-2}) = \cdots = \sigma \sum_{k=0}^{\infty} \alpha_1^k Z_{n-k},
$$

which converges when $|\alpha_1| < 1$, *i.e.* $|1 - b\Delta t| < 1$.

Note that the variance of X_n is given by

$$
\begin{aligned}
\mathrm{Var}[X_n] &= \sigma^2 \,\mathrm{Var}\left[\sum_{k=0}^{\infty} \alpha_1^k Z_{n-k}\right] \\
&= \sigma^2 \Delta t \sum_{k=0}^{\infty} \alpha_1^{2k} \\
&= \sigma^2 \Delta t \sum_{k=0}^{\infty} (1 - b\Delta t)^{2k} \\
&= \frac{\sigma^2 \Delta t}{1 - (1 - b\Delta t)^2} \\
&= \frac{\sigma^2 \Delta t}{2b\Delta t - b^2(\Delta t)^2} \\
&\simeq \frac{\sigma^2}{2b},
\end{aligned}
$$

which is the expected variance of the Vasicek process in the stationary regime.

```
library(quantmod)
getSymbols("^TNX",from="2012-01-01",to="2016-01-01",src="yahoo")
rate=Ad(`TNX`)
chartSeries(rate,up.col="blue",theme="white")
n = sum(!is.na(rate))
```

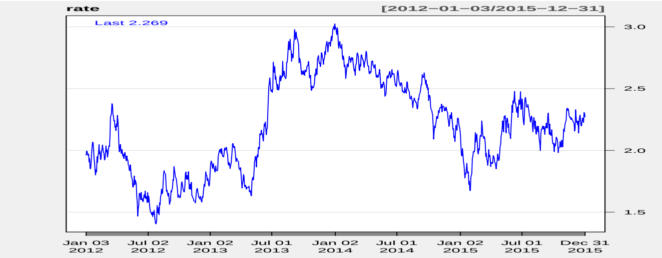

Fig. 13.2: CBOE 10 Year Treasury Note yield.

```
ratek=as.vector(rate)
ratekplus1 <- c(ratek[-1],0)
b <- (sum(ratek*ratekplus1) - sum(ratek)*sum(ratekplus1)/n)/(sum(ratek*ratek) - sum(ratek)*sum(
      ratek)/n)
a <- sum(ratekplus1)/n-b*sum(ratek)/n
sigma <- sqrt(sum((ratekplus1-b*ratek-a)^2)/n)
```

```
for (i in 1:100) {
ar.sim<-arima.sim(model=list(ar=c(b)),n.start=100,n)
y=ratek[1]+a/b+sigma*ar.sim
time <- as.POSIXct(time(TNX), format = "%Y-%m-%d")
yield <- xts(x = y, order.by = time)
chartSeries(yield,up.col="blue",theme="white")
Sys.sleep(0.5)
}
```

Fig. 13.3: Calibrated Vasicek *samples*.

13.2 Zero-Coupon Bonds and Coupon Bonds

A zero-coupon bond is a contract priced $P(t,T)$ at time $t < T$ to deliver $P(T,T) = 1\$$ at time T. In addition to its value at maturity, a bond may yield a periodic *coupon* payment at regular time intervals until the maturity date.

Fig. 13.4: Five dollar Louisiana bond of 1875 with 7.5% biannual coupons.

The computation of the arbitrage price $P_0(t,T)$ of a zero-coupon bond based on an underlying short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is a basic and important issue in interest rate modeling.

Deterministic short rates

In case the short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is a deterministic function of time, a standard arbitrage argument shows that the price $P(t,T)$ of the bond is given by

$$P(t,T) = e^{-\int_t^T r_s ds}, \qquad 0 \le t \le T. \tag{13.6}$$

Stochastic short rates

In case $(r_t)_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t$-adapted random process the formula (13.6) is no longer valid as it relies on future information, and we replace it with

$$P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t \right], \qquad 0 \le t \le T, \tag{13.7}$$

under a risk-neutral measure $\mathbb{P}^*$. It is natural to write $P(t,T)$ as a conditional expectation under a martingale measure, as the use of conditional expectation helps to "filter out" the future information past time t contained in $\displaystyle\int_t^T r_s ds$. The expression (13.7) makes sense as the "best possible estimate" of the future quantity $e^{-\int_t^T r_s ds}$ in mean square sense, given information known up to time t.

Coupon bonds

Pricing bonds with non-zero coupon is not difficult since in general the amount and periodicity of coupons are deterministic.* In the case of a constant, continuous-time coupon yield at rate $c > 0$ the price $P_c(t,T)$ of the coupon bond is given by

$$P_c(t,T) = e^{c(T-t)} P_0(t,T), \qquad 0 \leq t \leq T.$$

In the sequel we will only consider zero-coupon bonds priced as $P(t,T) = P_0(t,T)$, $0 \leq t \leq T$.

Martingale property of discounted bond prices

The following proposition shows that Assumption **(A)** of Chapter 12 is satisfied, in other words, the bond price process $t \longmapsto P(t,T)$ can be used as a numéraire.

Proposition 13.1. *The discounted bond price process*

$$t \longmapsto e^{-\int_0^t r_s ds} P(t,T)$$

is a martingale under $\mathbb{P}^*$.

Proof. We have

$$\begin{aligned} e^{-\int_0^t r_s ds} P(t,T) &= e^{-\int_0^t r_s ds} \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t \right] \\ &= \mathbb{E}^*\left[e^{-\int_0^t r_s ds} e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t \right] \\ &= \mathbb{E}^*\left[e^{-\int_0^T r_s ds} \,\Big|\, \mathcal{F}_t \right], \end{aligned}$$

and this suffices to conclude since by the "tower property" (17.37) of conditional expectations, any process of the form $t \longmapsto \mathbb{E}^*[F \mid \mathcal{F}_t]$, $F \in L^1(\Omega)$, is a martingale, cf. Relation (6.2). □

Bond pricing PDE

We assume from now on that the underlying short rate process is solution to the stochastic differential equation

* However, coupon default cannot be excluded.

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dB_t \tag{13.8}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

Since all solutions of stochastic differential equations such as (13.8) have the Markov property, cf *e.g.* Theorem V-32 of [96], the arbitrage price $P(t,T)$ can be rewritten as a function $F(t, r_t)$ of r_t, *i.e.*

$$P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t\right] = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, r_t\right] = F(t, r_t),$$

and depends on r_t only instead of depending on all information available in $\mathcal{F}_t$ up to time t, meaning that the pricing problem can now be formulated as a search for the function $F(t,x)$.

Proposition 13.2. *(Bond pricing PDE) The bond pricing PDE for $P(t,T) = F(t, r_t)$ is written as*

$$\boxed{xF(t,x) = \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x), \quad (13.9)}$$

$t \in \mathbb{R}_+$, $x \in \mathbb{R}$, subject to the terminal condition

$$F(T,x) = 1, \qquad x \in \mathbb{R}. \tag{13.10}$$

Proof. From Itô's formula we have

$$\begin{aligned}
&d\left(e^{-\int_0^t r_s ds}P(t,T)\right) = -r_t e^{-\int_0^t r_s ds}P(t,T)dt + e^{-\int_0^t r_s ds}dP(t,T) \\
&= -r_t e^{-\int_0^t r_s ds}F(t,r_t)dt + e^{-\int_0^t r_s ds}dF(t,r_t) \\
&= -r_t e^{-\int_0^t r_s ds}F(t,r_t)dt + e^{-\int_0^t r_s ds}\frac{\partial F}{\partial x}(t,r_t)(\mu(t,r_t)dt + \sigma(t,r_t)dB_t) \\
&+ e^{-\int_0^t r_s ds}\left(\frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t)dt + \frac{\partial F}{\partial t}(t,r_t)dt\right) \\
&= e^{-\int_0^t r_s ds}\sigma(t,r_t)\frac{\partial F}{\partial x}(t,r_t)dB_t \\
&+ e^{-\int_0^t r_s ds}\left(-r_tF(t,r_t) + \mu(t,r_t)\frac{\partial F}{\partial x}(t,r_t) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t) + \frac{\partial F}{\partial t}(t,r_t)\right)dt.
\end{aligned} \tag{13.11}$$

Given that $t \longmapsto e^{-\int_0^t r_s ds}P(t,T)$ is a martingale, the above expression (13.11) should only contain terms in dB_t (cf. Corollary II-1, page 72 of [96]), and all terms in dt should vanish inside (13.11). This leads to the identities

$$\begin{cases} -r_tF(t,r_t)+\mu(t,r_t)\dfrac{\partial F}{\partial x}(t,r_t)+\dfrac{1}{2}\sigma^2(t,r_t)\dfrac{\partial^2 F}{\partial x^2}(t,r_t)+\dfrac{\partial F}{\partial t}(t,r_t)=0 \\ \\ d\left(e^{-\int_0^t r_s ds}P(t,T)\right)=e^{-\int_0^t r_s ds}\sigma(t,r_t)\dfrac{\partial F}{\partial x}(t,r_t)dB_t. \end{cases} \tag{13.12a}$$

Condition (13.10) is due to the fact that $P(T,T)=\$1$. □

In the case of an interest rate process modeled by (13.5) we have (13.12b)

$$\mu(t,x)=\eta(t)+\lambda(t)x \qquad \text{and} \qquad \sigma(t,x)=\sqrt{\delta(t)+\gamma(t)x},$$

hence (13.9) yields the *affine* PDE

$$xF(t,x)=\frac{\partial F}{\partial t}(t,x)+(\eta(t)+\lambda(t)x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}(\delta(t)+\gamma(t)x)\frac{\partial^2 F}{\partial x^2}(t,x), \tag{13.13}$$

$t\in\mathbb{R}_+$, $x\in\mathbb{R}$. By (13.12b), the above proposition also shows that

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= \frac{1}{P(t,T)}d\left(e^{\int_0^t r_s ds}e^{-\int_0^t r_s ds}P(t,T)\right) \\ &= \frac{1}{P(t,T)}\left(r_tP(t,T)dt+e^{\int_0^t r_s ds}d\left(e^{-\int_0^t r_s ds}P(t,T)\right)\right) \\ &= r_tdt+\frac{1}{P(t,T)}e^{\int_0^t r_s ds}d\left(e^{-\int_0^t r_s ds}P(t,T)\right) \\ &= r_tdt+\frac{1}{F(t,r_t)}\frac{\partial F}{\partial x}(t,r_t)\sigma(t,r_t)dB_t \\ &= r_tdt+\sigma(t,r_t)\frac{\partial \log F}{\partial x}(t,r_t)dB_t. \end{aligned} \tag{13.14}$$

In the Vasicek case

$$dr_t=(a-br_t)dt+\sigma dW_t,$$

the bond price takes the form

$$F(t,r_t)=P(t,T)=e^{A(T-t)+r_tC(T-t)},$$

where $A(\cdot)$ and $C(\cdot)$ are functions of time, cf. (13.17) below, and (13.14) yields

$$\frac{dP(t,T)}{P(t,T)}=r_tdt-\frac{\sigma}{b}(1-e^{-b(T-t)})dW_t, \tag{13.15}$$

since $F(t,x)=e^{A(T-t)+xC(T-t)}$.

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Note that more generally, all affine short rate models as defined in Relation (13.5), including the Vasicek model, will yield a bond pricing formula of the form

$$P(t,T) = e^{A(T-t)+r_t C(T-t)},$$

cf. *e.g.* § 3.2.4. of [11].

Probabilistic PDE Solution

Next we solve the PDE (13.9) by a direct computation of the conditional expectation

$$P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \;\Big|\; \mathcal{F}_t \right], \tag{13.16}$$

in the Vasicek [111] model

$$dr_t = (a - br_t)dt + \sigma dB_t,$$

i.e. when the short rate $(r_t)_{t\in\mathbb{R}_+}$ has the expression

$$r_t = g(t) + \int_0^t h(t,s)dB_s = r_0 e^{-bt} + \frac{a}{b}(1-e^{-bt}) + \sigma \int_0^t e^{-b(t-s)} dB_s,$$

where

$$g(t) := r_0 e^{-bt} + \frac{a}{b}(1-e^{-bt}), \qquad t \in \mathbb{R}_+,$$

and

$$h(t,s) := \sigma e^{-b(t-s)}, \qquad 0 \le s \le t,$$

are deterministic functions.

Letting $u \vee t = \max(u,t)$, using the fact that Wiener integrals are Gaussian random variables and the Gaussian moment generating function, we have

$$\begin{aligned}
P(t,T) &= \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \;\Big|\; \mathcal{F}_t \right] \\
&= \mathbb{E}^*\left[e^{-\int_t^T (g(s)+\int_0^s h(s,u)dB_u)ds} \;\Big|\; \mathcal{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds\right) \mathbb{E}^*\left[e^{-\int_t^T \int_0^s h(s,u)dB_u ds} \;\Big|\; \mathcal{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds\right) \mathbb{E}^*\left[e^{-\int_0^T \int_{u\vee t}^T h(s,u)dsdB_u} \;\Big|\; \mathcal{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_{u\vee t}^T h(s,u)dsdB_u\right) \mathbb{E}^*\left[e^{-\int_t^T \int_{u\vee t}^T h(s,u)dsdB_u} \;\Big|\; \mathcal{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_t^T h(s,u)dsdB_u\right) \mathbb{E}^*\left[e^{-\int_t^T \int_u^T h(s,u)dsdB_u} \;\Big|\; \mathcal{F}_t \right]
\end{aligned}$$

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$$
\begin{aligned}
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_t^T h(s,u)dsdB_u\right) \mathbb{E}^*\left[e^{-\int_t^T \int_u^T h(s,u)dsdB_u}\right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_t^T h(s,u)dsdB_u + \frac{1}{2}\int_t^T \left(\int_u^T h(s,u)ds\right)^2 du\right) \\
&= \exp\left(-\int_t^T (r_0 e^{-bs} + \frac{a}{b}(1-e^{-bs}))ds - \sigma \int_0^t \int_t^T e^{-b(s-u)}dsdB_u\right) \\
&\times \exp\left(\frac{\sigma^2}{2}\int_t^T \left(\int_u^T e^{-b(s-u)}ds\right)^2 du\right) \\
&= \exp\left(-\int_t^T (r_0 e^{-bs} + \frac{a}{b}(1-e^{-bs}))ds - \frac{\sigma}{b}(1-e^{-b(T-t)})\int_0^t e^{-b(t-u)}dB_u\right) \\
&\times \exp\left(\frac{\sigma^2}{2}\int_t^T e^{2bu}\left(\frac{e^{-bu}-e^{-bT}}{b}\right)^2 du\right) \\
&= \exp\left(-\frac{r_t}{b}(1-e^{-b(T-t)}) + \frac{1}{b}(1-e^{-b(T-t)})\left(r_0 e^{-bt} + \frac{a}{b}(1-e^{-bt})\right)\right) \\
&\times \exp\left(-\int_t^T \left(r_0 e^{-bs} + \frac{a}{b}(1-e^{-bs})\right) ds + \frac{\sigma^2}{2}\int_t^T e^{2bu}\left(\frac{e^{-bu}-e^{-bT}}{b}\right)^2 du\right) \\
&= e^{A(T-t)+r_t C(T-t)},
\end{aligned}
\tag{13.17}
$$

where

$$
C(T-t) := -\frac{1}{b}(1-e^{-b(T-t)}),
$$

and

$$
A(T-t) := \frac{4ab-3\sigma^2}{4b^3} + \frac{\sigma^2-2ab}{2b^2}(T-t) + \frac{\sigma^2-ab}{b^3}e^{-b(T-t)} - \frac{\sigma^2}{4b^3}e^{-2b(T-t)}.
\tag{13.18}
$$

Analytical PDE Solution

In order to solve the PDE (13.9) analytically we may look for a solution of the form

$$
F(t,x) = e^{A(T-t)+xC(T-t)},
\tag{13.19}
$$

where $A(\cdot)$ and $C(\cdot)$ are functions to be determined under the conditions $A(0)=0$ and $C(0)=0$. Substituting (13.19) into the PDE (13.9) with the Vasicek coefficients $\mu(t,x) = (a-bx)$ and $\sigma(t,x) = \sigma$ shows that

$$
\begin{aligned}
xe^{A(T-t)+xC(T-t)} &= -(A'(T-t) - xC'(T-t))e^{A(T-t)+xC(T-t)} \\
&\quad +(a-bx)C(T-t)e^{A(T-t)+xC(T-t)} \\
&\quad +\frac{1}{2}\sigma^2 C^2(T-t)e^{A(T-t)+xC(T-t)}.
\end{aligned}
$$

By identification of terms for $x = 0$ and $x \neq 0$, this yields the system of differential equations

$$\begin{cases} A'(s) = aC(s) + \dfrac{\sigma^2}{2} C^2(s) \\ \\ C'(s) = 1 - bC(s), \end{cases}$$

which can be solved to recover the above value of $P(t,T) = F(t, r_t)$.

Some Bond Price Simulations

In this section we consider again the Vasicek model, in which the short rate $(r_t)_{t\in\mathbb{R}_+}$ is solution to (13.1). Figure 13.5 presents a random simulation of $t \longmapsto P(t,T)$ in the same Vasicek model. The graph of the corresponding deterministic bond price obtained for $a = b = \sigma = 0$ is also shown on the Figure 13.5.

Fig. 13.5: Graphs of $t \longmapsto P(t,T)$ and $t \longmapsto e^{-r_0(T-t)}$.

Figure 13.6 presents a random simulation of $t \longmapsto P(t,T)$ for a (non-zero) coupon bond with price $P_c(t,T) = e^{c(T-t)}P(t,T)$, and coupon rate $c > 0$, $0 \leq t \leq T$.

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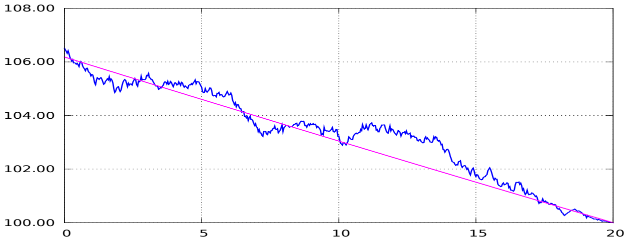

Fig. 13.6: Graph of $t \longmapsto P(t,T)$ for a bond with a 2.3% coupon.

The above simulation can be compared to the actual market data of a coupon bond in Figure 13.7 below.

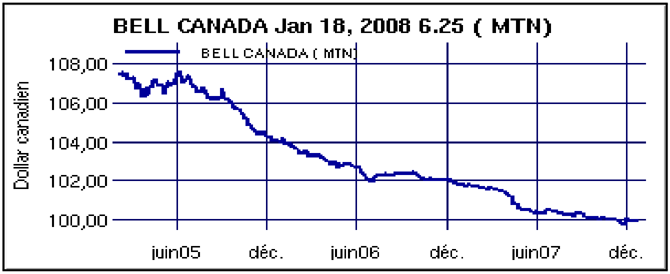

Fig. 13.7: Bond price graph with maturity 01/18/08 and coupon rate 6.25%.

See Exercise 13.3 for a bond pricing formula in the CIR model.

Bond pricing in the Dothan model

In the Dothan [30] model, the short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is modeled according to a geometric Brownian motion

$$dr_t = \lambda r_t dt + \sigma r_t dB_t, \tag{13.20}$$

where the volatility $\sigma > 0$ and the drift $\lambda \in \mathbb{R}$ are constant parameters and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. In this model the short term interest rate r_t remains always positive, while the proportional volatility term σr_t accounts for the sensitivity of the volatility of interest rate changes to the level of the rate r_t.

On the other hand, the Dothan model is the only lognormal short rate model that allows for an analytical formula for the zero coupon bond price

$$P(t,T)=\mathbb{E}\left[e^{-\int_t^T r_s ds}\ \Big|\ \mathcal{F}_t\right],\qquad 0\le t\le T.$$

For convenience of notation we let $p=1-2\lambda/\sigma^2$ and rewrite (13.20) as

$$dr_t=(1-p)\frac{1}{2}\sigma^2 r_t dt+\sigma r_t dB_t,$$

with solution

$$r_t=r_0\exp\left(\sigma B_t-p\sigma^2 t/2\right),\qquad t\in\mathbb{R}_+,\tag{13.21}$$

where $p\sigma/2$ identifies to the market price of risk. By the Markov property of $(r_t)_{t\in\mathbb{R}_+}$, the bond price $P(t,T)$ is a function $F(\tau,r_t)$ of r_t and of the time to maturity $\tau=T-t$:

$$P(t,T)=F(\tau,r_t)=\mathbb{E}\left[e^{-\int_t^T r_s ds}\ \Big|\ r_t\right],\qquad 0\le t\le T.\tag{13.22}$$

By computation of the conditional expectation (13.22) using (10.6) we easily obtain the following result, cf. [83], where the function $\theta(v,t)$ is defined in (10.4).

Proposition 13.3. *The zero-coupon bond price $P(t,T)=F(T-t,r_t)$ is given for all $p\in\mathbb{R}$ by*

$$F(\tau,x)=e^{-\sigma^2p^2\tau/8}\int_0^\infty\int_0^\infty e^{-ux}\exp\left(-2\frac{(1+z^2)}{\sigma^2 u}\right)\theta\left(\frac{4z}{\sigma^2 u},\frac{\sigma^2\tau}{4}\right)\frac{du}{u}\frac{dz}{z^{p+1}},\tag{13.23}$$

$x>0$.

Proof. By Proposition 10.1 the probability distribution of the time integral $\int_0^{T-t}e^{\sigma B_s-p\sigma^2 s/2}ds$ is given by

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T-t}e^{\sigma B_s-p\sigma^2 s/2}ds\in dy\right)\\
&=\int_{-\infty}^\infty\mathbb{P}\left(\int_0^t e^{\sigma B_s-p\sigma^2 s/2}ds\in dy,\ B_t-p\sigma t/2\in dz\right)dz\\
&=\frac{\sigma}{2}\int_{-\infty}^\infty e^{-p\sigma z/2-p^2\sigma^2 t/8}\exp\left(-2\frac{1+e^{\sigma z}}{\sigma^2 y}\right)\theta\left(\frac{4e^{\sigma z/2}}{\sigma^2 y},\frac{\sigma^2 t}{4}\right)\frac{dy}{y}dz\\
&=e^{-p^2\sigma^2(T-t)/8}\int_0^\infty\exp\left(-2\frac{1+z^2}{\sigma^2 y}\right)\theta\left(\frac{4z}{\sigma^2 y},\frac{\sigma^2(T-t)}{4}\right)\frac{dz}{z^{p+1}}\frac{dy}{y},\quad y>0,
\end{aligned}$$

where the exchange of integrals is justified by the Fubini theorem and the nonnegativity of integrands. Hence by (10.6) and (13.21) we find

$$\begin{aligned}
F(T-t, r_t) &= P(t,T) \\
&= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Bigm| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(-r_t \int_t^T e^{\sigma(B_s-B_t)-\sigma^2 p(s-t)/2} ds\right) \Bigm| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(-x \int_t^T e^{\sigma(B_s-B_t)-\sigma^2 p(s-t)/2} ds\right)\right]_{x=r_t} \\
&= \mathbb{E}\left[\exp\left(-x \int_0^{T-t} e^{\sigma B_s-\sigma^2 ps/2} ds\right)\right]_{x=r_t} \\
&= \int_0^\infty e^{-r_t y} \mathbb{P}\left(\int_0^{T-t} e^{\sigma B_s - p\sigma^2 s/2} ds \in dy\right) \\
&= e^{-p^2\sigma^2(T-t)/8} \int_0^\infty e^{-r_t y} \int_0^\infty \exp\left(-2\frac{1+z^2}{\sigma^2 y}\right) \theta\left(\frac{4z}{\sigma^2 y}, \frac{\sigma^2(T-t)}{4}\right) \frac{dz}{z^{p+1}} \frac{dy}{y}.
\end{aligned}$$

□

See [83] and [82] for more results on bond pricing in the Dothan model, and [92] for numerical computations.

Zero coupon bond price and yield data

The following zero coupon bond price data was downloaded at EMMA from the Municipal Securities Rulemaking Board.

ORANGE CNTY CALIF PENSION OBLIG CAP APPREC-TAXABLE-REF-SER A (CA)
CUSIP: 68428LBB9
Dated Date: 06/12/1996 (June 12, 1996)
Maturity Date: 09/01/2016 (September 1st, 2016)
Interest Rate: 0.0 %
Principal Amount at Issuance: $26,056,000
Initial Offering Price: 19.465

```
library(quantmod)
bondprice <- read.table("bond_data_R.txt",col.names = c("Date","HighPrice","LowPrice","
      HighYield","LowYield","Count","Amount"))
head(bondprice)
time <- as.POSIXct(bondprice$Date, format = "%Y-%m-%d")
price <- xts(x = bondprice$HighPrice, order.by = time)
yield <- xts(x = bondprice$HighYield, order.by = time)
chartSeries(price,up.col="blue",theme="white")
chartSeries(yield,up.col="blue",theme="white")
```

	Date	HighPrice	LowPrice	HighYield	LowYield	Count	Amount
1	2016-01-13	99.082	98.982	1.666	1.501	2	20000
2	2015-12-29	99.183	99.183	1.250	1.250	1	10000
3	2015-12-21	97.952	97.952	3.014	3.014	1	10000
4	2015-12-17	99.141	98.550	2.123	1.251	5	610000
5	2015-12-07	98.770	98.770	1.714	1.714	2	10000
6	2015-12-04	98.363	98.118	2.628	2.280	2	10000

Fig. 13.8: Orange Cnty Calif prices.

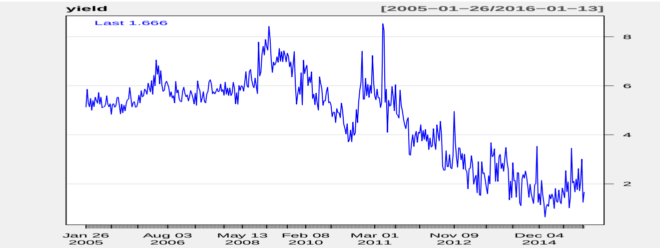

Fig. 13.9: Orange Cnty Calif yields.

13.3 Forward Rates

A forward interest rate contract (or forward rate agreement, FRA) gives its holder a loan decided at present time t and to be delivered over a future period of time $[T, S]$ at a rate denoted by $f(t, T, S)$, $t \leq T \leq S$, and called a forward rate. When $T = t$, the *spot* forward rate $f(t, t, T)$ is also called the *yield*.

Figure 13.10 presents a typical yield curve on the LIBOR (London Interbank Offered Rate) market with t =07 May 2003.

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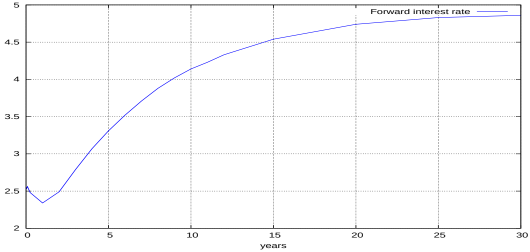

TimeSerieNb	505
AsOfDate	7-mai-03
2D	2,55
1W	2,53
1M	2,56
2M	2,52
3M	2,48
1Y	2,34
2Y	2,49
3Y	2,79
4Y	3,07
5Y	3,31
6Y	3,52
7Y	3,71
8Y	3,88
9Y	4,02
10Y	4,14
11Y	4,23
12Y	4,33
13Y	4,4
14Y	4,47
15Y	4,54
20Y	4,74
25Y	4,83
30Y	4,86

Fig. 13.10: Forward rate graph $T \longmapsto f(t, t, T)$.

Maturity transformation, *i.e.*, the ability to transform short term borrowing (debt with short maturities, such as deposits) into long term lending (credits with very long maturities, such as loans), is among the roles of banks. Profitability is then dependent on the difference between long rates and short rates.

Forward rates from bond prices

Let us determine the arbitrage or "fair" value of this rate using the instruments available in a bond market, which are bonds priced at $P(t, T)$ for various maturity dates $T > t$.

The loan can be realized using the instruments (here, bonds) available on the market by proceeding in two steps:

1) At time t, borrow the amount $P(t, S)$ by issuing (or short selling) one bond with maturity S, which means refunding \$1 at time S.

2) Since the money is only needed at time T, the rational investor will invest the amount $P(t, S)$ over the period $[t, T]$ by buying a (possibly fractional) quantity $P(t, S)/P(t, T)$ of a bond with maturity T priced $P(t, T)$ at time t. This will yield the amount

$$\$1 \times \frac{P(t, S)}{P(t, T)}$$

at time T.

As a consequence, the investor will actually receive $P(t, S)/P(t, T)$ at time T, to refund \$1 at time S.

The corresponding forward rate $f(t, T, S)$ is then given by the relation

$$\frac{P(t,S)}{P(t,T)}\exp\left((S-T)f(t,T,S)\right)=\$1, \qquad 0\le t\le T\le S, \tag{13.24}$$

where we used exponential compounding, which leads to the following definition (13.25).

Definition 13.4. *The forward rate $f(t,T,S)$ at time t for a loan on $[T,S]$ is given by*

$$\boxed{f(t,T,S)=\frac{\log P(t,T)-\log P(t,S)}{S-T}. \tag{13.25}}$$

The *spot* forward rate (or yield) $f(t,t,T)$ is given by

$$f(t,t,T)=-\frac{\log P(t,T)}{T-t}, \quad \text{or} \quad P(t,T)=e^{-(T-t)f(t,t,T)}, \qquad 0\le t\le T. \tag{13.26}$$

The instantaneous forward rate $f(t,T)$ is defined by taking the limit of $f(t,T,S)$ as $S\searrow T$, *i.e.*

$$\begin{aligned} f(t,T) :&= \lim_{S\searrow T} f(t,T,S)\\ &= -\lim_{S\searrow T}\frac{\log P(t,S)-\log P(t,T)}{S-T}\\ &= -\lim_{\varepsilon\searrow 0}\frac{\log P(t,T+\varepsilon)-\log P(t,T)}{\varepsilon}\\ &= -\frac{\partial \log P(t,T)}{\partial T}\\ &= -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}. \end{aligned} \tag{13.27}$$

The above equation (13.27) can be viewed as a differential equation to be solved for $\log P(t,T)$ under the initial condition $P(T,T)=1$, which yields the following proposition.

Proposition 13.5. *We have*

$$P(t,T)=\exp\left(-\int_t^T f(t,s)ds\right), \qquad 0\le t\le T. \tag{13.28}$$

Proof. We check that

$$\log P(t,T)=\log P(t,T)-\log P(t,t)=\int_t^T\frac{\partial \log P(t,s)}{\partial s}ds=-\int_t^T f(t,s)ds.$$

$\square$

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Proposition 13.5 also shows that

$$
\begin{aligned}
f(t,t) &= \frac{\partial}{\partial T} \int_t^T f(t,s) ds_{|T=t} \\
&= -\frac{\partial}{\partial T} \log P(t,T)_{|T=t} \\
&= -\frac{1}{P(t,T)}_{|T=t} \frac{\partial}{\partial T} P(t,T)_{|T=t} \\
&= -\frac{\partial}{\partial T} \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t \right]_{|T=t} \\
&= \mathbb{E}^* \left[r_T e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t \right]_{|T=t} \\
&= \mathbb{E}^*[r_t \mid \mathcal{F}_t] \\
&= r_t,
\end{aligned}
$$

i.e. the short rate r_t can be recovered from the instantaneous forward rate as

$$
r_t = f(t,t) = \lim_{T \searrow t} f(t,T).
$$

As a consequence of (13.24) and (13.28) the forward rate $f(t,T,S)$, $0 \leq t \leq T \leq S$, can be recovered from (13.25) and the instantaneous forward rate $f(t,s)$, as:

$$
\begin{aligned}
f(t,T,S) &= \frac{\log P(t,T) - \log P(t,S)}{S-T} \\
&= -\frac{1}{S-T} \left(\int_t^T f(t,s)ds - \int_t^S f(t,s)ds \right) \\
&= \frac{1}{S-T} \int_T^S f(t,s)ds, \qquad 0 \leq t \leq T < S.
\end{aligned}
\tag{13.29}
$$

In particular, the *spot* forward rate or *yield* $f(t,t,T)$ can be written as

$$
f(t,t,T) = -\frac{\log P(t,T)}{T-t} = \frac{1}{T-t} \int_t^T f(t,s)ds, \qquad 0 \leq t < T.
$$

Differentiation with respect to T of the above relation shows that

$$
\frac{\partial}{\partial T} f(t,t,T) = -\frac{1}{(T-t)^2} \int_t^T f(t,s)ds + \frac{1}{T-t} f(t,T), \qquad 0 \leq t < T,
$$

and

$$
\begin{aligned}
f(t,T) &= \frac{1}{T-t} \int_t^T f(t,s)ds + (T-t) \frac{\partial}{\partial T} f(t,t,T) \\
&= f(t,t,T) + (T-t) \frac{\partial}{\partial T} f(t,t,T), \qquad 0 \leq t < T.
\end{aligned}
$$

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Forward Swap Rates

The first interest rate swap occured in 1981 between IBM and the World Bank. In particular, an interest rate swap makes it possible to exchange a sequence of variable forward rates $f(t,T_k,T_{k+1})$, $k=1,\ldots,n-1$, against a fixed rate κ over a time period $[T_1,T_n]$. Over the succession of time intervals $[T_1,T_2],\ldots,[T_{n-1},T_n]$ defining a *tenor structure*, see Section 14.1 for details, the combination of such exchanges will generate a cumulative discounted cash flow

$$\left(\sum_{k=1}^{n-1}(T_{k+1}-T_k)e^{-\int_t^{T_{k+1}} r_s ds} f(t,T_k,T_{k+1})\right)-\left(\sum_{k=1}^{n-1}\kappa(T_{k+1}-T_k)e^{-\int_t^{T_{k+1}} r_s ds}\right)$$
$$=\sum_{k=1}^{n-1}(T_{k+1}-T_k)e^{-\int_t^{T_{k+1}} r_s ds}(f(t,T_k,T_{k+1})-\kappa),$$

at time t, in which we used simple (or linear) interest rate compounding. This cash flow is used to make the contract fair, and it can be priced *at time* t as

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{n-1}(T_{k+1}-T_k)e^{-\int_t^{T_{k+1}} r_s ds}(f(t,T_k,T_{k+1})-\kappa)\;\Big|\;\mathcal{F}_t\right]\\
&\quad=\sum_{k=1}^{n-1}(T_{k+1}-T_k)(f(t,T_k,T_{k+1})-\kappa)\,\mathbb{E}\left[e^{-\int_t^{T_{k+1}} r_s ds}\;\Big|\;\mathcal{F}_t\right]\\
&\quad=\sum_{k=1}^{n-1}(T_{k+1}-T_k)P(t,T_{k+1})\big(f(t,T_k,T_{k+1})-\kappa\big).
\end{aligned}$$

The swap rate $S(t,T_1,T_n)$ is by definition the value of the rate κ that makes the contract fair by making this cash flow vanish. In other words, $S(t,T_1,T_n)$ is the fixed rate over $[T_1,T_n]$ that will be agreed in exchange for the family of forward rates $f(t,T_k,T_{k+1})$, $k=1,\ldots,n-1$, and it solves

$$\sum_{k=1}^{n-1}(T_{k+1}-T_k)P(t,T_{k+1})\big(f(t,T_k,T_{k+1})-S(t,T_1,T_n)\big)=0, \tag{13.30}$$

i.e.

$$\begin{aligned}
0&=\sum_{k=1}^{n-1}(T_{k+1}-T_k)P(t,T_{k+1})f(t,T_k,T_{k+1})-S(t,T_1,T_n)\sum_{k=1}^{n-1}(T_{k+1}-T_k)P(t,T_{k+1})\\
&=\sum_{k=1}^{n-1}(T_{k+1}-T_k)P(t,T_{k+1})f(t,T_k,T_{k+1})-P(t,T_1,T_n)S(t,T_1,T_n),
\end{aligned}$$

which shows that

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$$S(t, T_1, T_n) = \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t, T_{k+1}) f(t, T_k, T_{k+1}), \quad (13.31)$$

where

$$P(t, T_1, T_n) := \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t, T_{k+1}), \qquad 0 \leq t \leq T_1,$$

is the annuity numéraire, or present value at time t of future \$1 receipts at times $T_2, T_3, \ldots, T_n$, weighted by the time intervals $T_{k+1} - T_k$, $k = 1, 2 \ldots, n-1$.

LIBOR Rates

Recall that the forward rate $f(t, T, S)$, $0 \leq t \leq T \leq S$, is defined using exponential compounding, from the relation

$$f(t, T, S) = -\frac{\log P(t, S) - \log P(t, T)}{S - T}. \quad (13.32)$$

In order to compute swaption prices one prefers to use forward rates as defined on the London InterBank Offered Rates (LIBOR) market instead of the standard forward rates given by (13.32).

The forward LIBOR $L(t, T, S)$ for a loan on $[T, S]$ is defined using linear compounding, *i.e.* by replacing (13.32) with the relation

$$1 + (S - T) L(t, T, S) = \frac{P(t, T)}{P(t, S)},$$

which yields the following definition.

Definition 13.6. *The forward LIBOR rate $L(t, T, S)$ at time t for a loan on $[T, S]$ is given by*

$$L(t, T, S) = \frac{1}{S - T} \left(\frac{P(t, T)}{P(t, S)} - 1 \right), \qquad 0 \leq t \leq T < S. \quad (13.33)$$

Note that (13.33) above yields the same formula for the (LIBOR) instantaneous forward rate

$$\begin{aligned} f(t, T) :&= \lim_{S \searrow T} L(t, T, S) \\ &= \lim_{S \searrow T} \frac{P(t, S) - P(t, T)}{(S - T) P(t, S)} \end{aligned}$$

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$$
\begin{aligned}
&= \lim_{\varepsilon \searrow 0} \frac{P(t,T+\varepsilon) - P(t,T)}{\varepsilon P(t,T+\varepsilon)} \\
&= \frac{1}{P(t,T)} \lim_{\varepsilon \searrow 0} \frac{P(t,T+\varepsilon) - P(t,T)}{\varepsilon} \\
&= -\frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T} \\
&= -\frac{\partial \log P(t,T)}{\partial T},
\end{aligned}
$$

as (13.27).

In addition, Relation (13.33) shows that the LIBOR rate can be viewed as a forward price $\hat{X}_t = X_t/N_t$ with numéraire $N_t = (S-T)P(t,S)$ and $X_t = P(t,T) - P(t,S)$, according to Relation (12.7) of Chapter 12. As a consequence, from Proposition 12.3, the LIBOR rate $(L(t,T,S))_{t\in[T,S]}$ is a martingale under the forward measure $\hat{\mathbb{P}}$ defined by

$$
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = \frac{1}{P(0,S)} e^{-\int_0^S r_t dt}.
$$

LIBOR Swap Rates

The LIBOR swap rate $S(t,T_1,T_n)$ satisfies the same relation as (13.30) with the forward rate $f(t,T_k,T_{k+1})$ replaced with the LIBOR rate $L(t,T_k,T_{k+1})$, *i.e.*

$$
\sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t,T_{k+1}) (L(t,T_k,T_{k+1}) - S(t,T_1,T_n)) = 0.
$$

Proposition 13.7. *We have*

$$
S(t,T_1,T_n) = \frac{P(t,T_1) - P(t,T_n)}{P(t,T_1,T_n)}, \qquad 0 \leq t \leq T_1. \tag{13.34}
$$

Proof. By (13.31) and (13.33) we have

$$
\begin{aligned}
S(t,T_1,T_n) &= \frac{1}{P(t,T_1,T_n)} \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t,T_{k+1}) L(t,T_k,T_{k+1}) \\
&= \frac{1}{P(t,T_1,T_n)} \sum_{k=1}^{n-1} P(t,T_{k+1}) \left(\frac{P(t,T_k)}{P(t,T_{k+1})} - 1 \right) \\
&= \frac{1}{P(t,T_1,T_n)} \sum_{k=1}^{n-1} (P(t,T_k) - P(t,T_{k+1})) \\
&= \frac{P(t,T_1) - P(t,T_n)}{P(t,T_1,T_n)} \qquad (13.35)
\end{aligned}
$$

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by a telescoping sum. □

Clearly, a simple expression for the swap rate such as that of Proposition 13.7 cannot be obtained using the standard (*i.e.* non-LIBOR) rates defined in (13.32).

When $n = 2$, the swap rate $S(t, T_1, T_2)$ coincides with the forward rate $L(t, T_1, T_2)$:

$$S(t, T_1, T_2) = L(t, T_1, T_2), \tag{13.36}$$

and the bond prices $P(t, T_1)$ can be recovered from the forward swap rates $S(t, T_1, T_n)$.

Similarly to the case of LIBOR rates, Relation (13.34) shows that the LIBOR swap rate can be viewed as a forward price with (annuity) numéraire $N_t = P(t, T_1, T_n)$ and $X_t = P(t, T_1) - P(t, T_n)$. Consequently the LIBOR swap rate $(S(t, T_1, T_n)_{t\in[T,S]}$ is a martingale under the forward measure $\hat{\mathbb{P}}$ defined from (12.1) by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = \frac{P(T_1, T_1, T_n)}{P(0, T_1, T_n)} e^{-\int_0^{T_1} r_t dt}.$$

Yield curve data

We refer to Chapter III-12 of [14] on the R package "YieldCurve" [48] for the following code and further details on yield curve and interest rate modeling using R.

```
install.packages("YieldCurve")
require(YieldCurve)
data(FedYieldCurve)
first(FedYieldCurve,'3 month')
last(FedYieldCurve,'3 month')
mat.Fed=c(0.25,0.5,1,2,3,5,7,10)
n=50
plot(mat.Fed, FedYieldCurve[n,], type="o",xlab="Maturities structure in years", ylab="Interest rates
      values")
title(main=paste("Federal Reserve yield curve observed at",time(FedYieldCurve[n], sep=" ") ))
grid()
```

The next Figure 13.11 is plotted using this code* which is adapted from http://www.quantmod.com/examples/chartSeries3d/chartSeries3d.alpha.R

* Click to open or download.

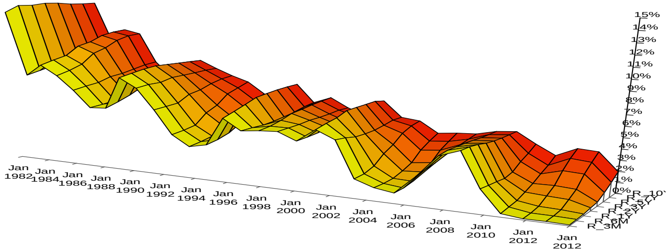

Fig. 13.11: Federal Reserve yield curves from 1982 to 2012.

European Central Bank (ECB) data can be similarly obtained.

```
data(ECBYieldCurve)
first(ECBYieldCurve,'3 month')
last(ECBYieldCurve,'3 month')
mat.ECB<-c(3/12, 0.5, 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30)
for (n in 200:400) {
plot(mat.ECB, ECBYieldCurve[n,], type="o",xlab="Maturities structure in years", ylab="Interest
      rates values",ylim=c(3.1,5.1))
title(main=paste("European Central Bank yield curve observed at",time(ECBYieldCurve[n], sep=" ")
      ))
grid()
}
```

The next Figure 13.12 represents the output of the above script.

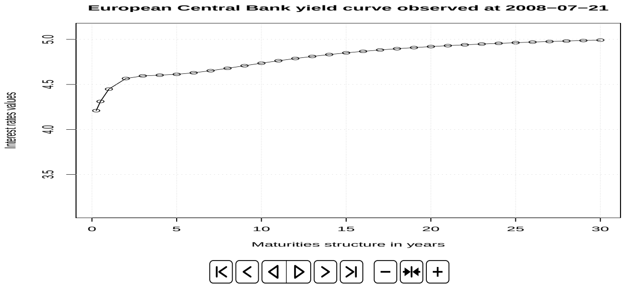

Fig. 13.12: European Central Bank yield curves.*

* The animation works in Acrobat Reader on the entire pdf file.

13.4 The HJM Model

In the previous chapter we have focused on the modeling of the short rate $(r_t)_{t\in\mathbb{R}_+}$ and on its consequences on the pricing of bonds $P(t,T)$, from which the forward rates $f(t,T,S)$ and $L(t,T,S)$ have been defined.

In this section we choose a different starting point and consider the problem of directly modeling the instantaneous forward rate $f(t,T)$. The graph given in Figure 13.13 presents a possible random evolution of a forward interest rate curve using the Musiela convention, *i.e.* we will write

$$g(x) = f(t,t+x) = f(t,T),$$

under the substitution $x = T - t$, $x \geq 0$, and represent a sample of the instantaneous forward curve $x \longmapsto f(t,t+x)$ for each $t \in \mathbb{R}_+$.

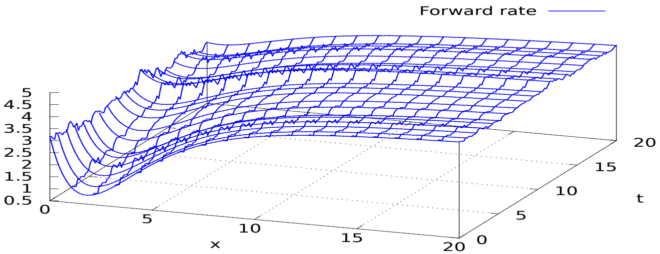

Fig. 13.13: Stochastic process of forward curves.

In the Heath-Jarrow-Morton (HJM) model, the instantaneous forward rate $f(t,T)$ is modeled under $\mathbb{P}$ by a stochastic differential equation of the form

$$d_t f(t,T) = \alpha(t,T)dt + \sigma(t,T)dB_t, \tag{13.37}$$

where $t \longmapsto \alpha(t,T)$ and $t \longmapsto \sigma(t,T)$, $0 \leq t \leq T$, are allowed to be random (adapted) processes. In the above equation, the date T is fixed and the differential d_t is with respect to t.

Under basic Markovianity assumptions, a HJM model with deterministic coefficients $\alpha(t,T)$ and $\sigma(t,T)$ will yield a short rate process $(r_t)_{t\in\mathbb{R}_+}$ of the form

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dB_t,$$

cf. § 6.6 of [87], which is the Hull-White model [56], with explicit solution

$$r_t = r_s e^{-\int_s^t b(\tau)d\tau} + \int_s^t e^{-\int_u^t b(\tau)d\tau} a(u)du + \int_s^t \sigma(u)e^{-\int_u^t b(\tau)d\tau} dB_u,$$

$0 \leq s \leq t$.

The HJM Condition

How to "encode" absence of arbitrage in the defining equation (13.37) is an important question. Recall that under absence of arbitrage, the bond price $P(t, T)$ has been constructed as

$$P(t,T) = \mathbb{E}^* \left[\exp\left(- \int_t^T r_s ds \right) \Big| \mathcal{F}_t \right] = \exp\left(- \int_t^T f(t,s)ds \right), \quad (13.38)$$

cf. Proposition 13.5, hence the discounted bond price process is given by

$$t \longmapsto \exp\left(- \int_0^t r_s ds \right) P(t,T) = \exp\left(- \int_0^t r_s ds - \int_t^T f(t,s)ds \right) \quad (13.39)$$

is a martingale by Proposition 13.1 and Relation (13.28). This latter property will be used to characterize absence of arbitrage in the HJM model.

Proposition 13.8. *(HJM Condition [53]). Under the condition*

$$\boxed{\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)ds, \qquad t \in [0,T], \qquad (13.40)}$$

which is known as the HJM absence of arbitrage condition, *the discounted bond price process* (13.39) *becomes a martingale.*

Proof. Consider the spot forward rate or yield

$$f(t,t,T) = \frac{1}{T-t} \int_t^T f(t,s)ds,$$

and let

$$X_t = \int_t^T f(t,s)ds = -\log P(t,T), \qquad 0 \leq t \leq T,$$

with the relation

$$f(t,t,T) = \frac{1}{T-t} \int_t^T f(t,s)ds = \frac{X_t}{T-t}, \qquad 0 \leq t \leq T, \qquad (13.41)$$

where the dynamics of $t \longmapsto f(t,s)$ is given by (13.37). We note that when $f(t,s) = g(t)h(s)$ is a smooth function which satisfies the separation of variables property we have the relation

$$d_t \int_t^T g(t)h(s)ds = -g(t)h(t)dt + g'(t) \int_t^T h(s)dsdt,$$

which extends to $f(t,s)$ as

$$d_t \int_t^T f(t,s)ds = -f(t,t)dt + \int_t^T d_t f(t,s)ds,$$

which can be seen as a form of the Leibniz integral rule. Therefore we have

$$\begin{aligned} d_t X_t &= -f(t,t)dt + \int_t^T d_t f(t,s)ds \\ &= -f(t,t)dt + \int_t^T \alpha(t,s)dsdt + \int_t^T \sigma(t,s)dsdB_t \\ &= -r_t dt + \left(\int_t^T \alpha(t,s)ds\right) dt + \left(\int_t^T \sigma(t,s)ds\right) dB_t, \end{aligned}$$

hence

$$|d_t X_t|^2 = \left(\int_t^T \sigma(t,s)ds\right)^2 dt.$$

Hence by Itô's calculus we have

$$\begin{aligned} d_t P(t,T) &= d_t e^{-X_t} \\ &= -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} (d_t X_t)^2 \\ &= -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} \left(\int_t^T \sigma(t,s)ds\right)^2 dt \\ &= -e^{-X_t} \left(-r_t dt + \int_t^T \alpha(t,s)dsdt + \int_t^T \sigma(t,s)dsdB_t\right) \\ &\quad + \frac{1}{2} e^{-X_t} \left(\int_t^T \sigma(t,s)ds\right)^2 dt, \end{aligned}$$

and the discounted bond price satisfies

$$\begin{aligned} & d_t \left(\exp\left(-\int_0^t r_s ds\right) P(t,T)\right) \\ &= -r_t \exp\left(-\int_0^t r_s ds - X_t\right) dt + \exp\left(-\int_0^t r_s ds\right) d_t P(t,T) \\ &= -r_t \exp\left(-\int_0^t r_s ds - X_t\right) dt - \exp\left(-\int_0^t r_s ds - X_t\right) d_t X_t \\ &\quad + \frac{1}{2} \exp\left(-\int_0^t r_s ds - X_t\right) \left(\int_t^T \sigma(t,s)ds\right)^2 dt \\ &= -r_t \exp\left(-\int_0^t r_s ds - X_t\right) dt \\ &\quad - \exp\left(-\int_0^t r_s ds - X_t\right) \left(-r_t dt + \int_t^T \alpha(t,s)dsdt + \int_t^T \sigma(t,s)dsdB_t\right) \\ &\quad + \frac{1}{2} \exp\left(-\int_0^t r_s ds - X_t\right) \left(\int_t^T \sigma(t,s)ds\right)^2 dt \end{aligned}$$

$$= -\exp\left(-\int_0^t r_s ds - X_t\right)\int_t^T \sigma(t,s)dsdB_t$$
$$-\exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \alpha(t,s)dsdt - \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2\right)dt.$$

Thus, the discounted bond price process

$$t \longmapsto \exp\left(-\int_0^t r_s ds\right)P(t,T)$$

will be a martingale provided that

$$\int_t^T \alpha(t,s)ds - \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 = 0, \qquad 0 \le t \le T. \tag{13.42}$$

Differentiating the above relation with respect to T, we get

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds,$$

which is in fact equivalent to (13.42). □

13.5 Forward Vasicek Rates

In this section we consider the Vasicek model, in which the short rate process is the solution (13.2) of (13.1) as illustrated in Figure 13.1.

In the Vasicek model, the forward rate is given by

$$f(t,T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}$$
$$= -\frac{r_t(C(S-t) - C(T-t)) + A(S-t) - A(T-t))}{S-T}$$
$$= -\frac{\sigma^2 - 2ab}{2b^2}$$
$$-\frac{1}{S-T}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)(e^{-b(S-t)} - e^{-b(T-t)}) - \frac{\sigma^2}{4b^3}(e^{-2b(S-t)} - e^{-2b(T-t)})\right),$$

and spot forward rate, or yield, satisfies

$$f(t,t,T) = -\frac{\log P(t,T)}{T-t} = -\frac{r_t C(T-t) + A(T-t)}{T-t}$$
$$= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)(1 - e^{-b(T-t)}) - \frac{\sigma^2}{4b^3}(1 - e^{-2b(T-t)})\right).$$

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In this model, the forward rate $t \longmapsto f(t,T,S)$ can be represented as in Figure 13.14, with here $b/a > r_0$.

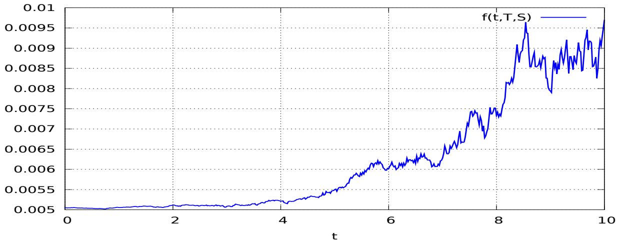

Fig. 13.14: Forward rate process $t \longmapsto f(t,T,S)$.

Note that the forward rate cure $t \longmapsto f(t,T,S)$ appears flat for small values of t, *i.e.* longer rates are more stable, while shorter rates show higher volatility or risk. Similar features can be observed in Figure 13.15 for the instantaneous short rate given by

$$f(t,T) := -\frac{\partial \log P(t,T)}{\partial T} \tag{13.43}$$
$$= r_t e^{-b(T-t)} + \frac{a}{b}(1 - e^{-b(T-t)}) - \frac{\sigma^2}{2b^2}(1 - e^{-b(T-t)})^2,$$

from which the relation $\lim_{T \searrow t} f(t,T) = r_t$ can be easily recovered.

The instantaneous forward rate $t \longmapsto f(t,T)$ can be represented as in Figure 13.15, with here $t = 0$ and $b/a > r_0$:

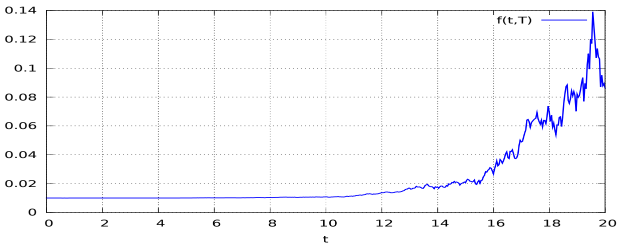

Fig. 13.15: Instantaneous forward rate process $t \longmapsto f(t,T)$.

The HJM coefficients in the Vasicek model are in fact deterministic and taking $a = 0$ we have

$$d_t f(t,T) = \sigma^2 e^{-b(T-t)} \int_t^T e^{b(t-s)} ds dt + \sigma e^{-b(T-t)} dB_t,$$

i.e.

$$\alpha(t,T) = \sigma^2 e^{-b(T-t)} \int_t^T e^{b(t-s)} ds = \sigma^2 e^{-b(T-t)} \frac{1-e^{-b(T-t)}}{b},$$

and $\sigma(t,T) = \sigma e^{-b(T-t)}$, and the HJM condition reads

$$\alpha(t,T) = \sigma^2 e^{-b(T-t)} \int_t^T e^{b(t-s)} ds = \sigma(t,T) \int_t^T \sigma(t,s) ds. \tag{13.44}$$

Random simulations of the Vasicek instantaneous forward rates are provided in Figures 13.16 and 13.17.

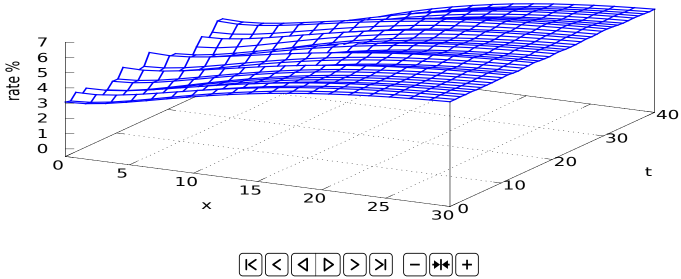

Fig. 13.16: Forward instantaneous curve $(t,x) \longmapsto f(t,t+x)$ in the Vasicek model.[*]

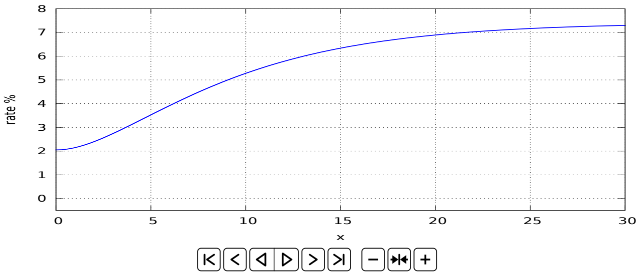

Fig. 13.17: Forward instantaneous curve $x \longmapsto f(0,x)$ in the Vasicek model.[†]

[†] The animation works in Acrobat Reader on the entire pdf file.

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For $x = 0$ the first "slice" of this surface is actually the short rate Vasicek process $r_t = f(t,t) = f(t,t+0)$ which is represented in Figure 13.18 using another discretization.

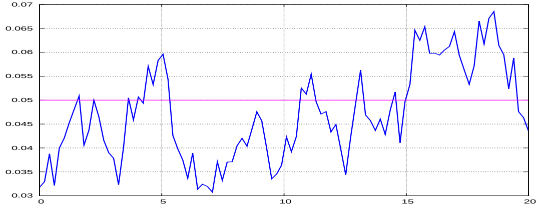

Fig. 13.18: Short term interest rate curve $t \longmapsto r_t$ in the Vasicek model.

Another example of market data is given in the next Figure 13.19, in which the red and blue curves refer respectively to July 21 and 22 of year 2011.

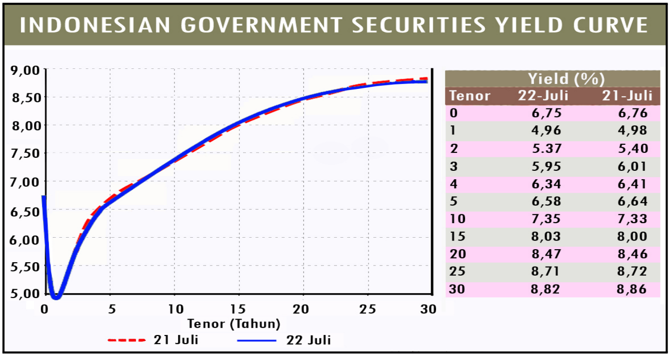

Fig. 13.19: Market example of yield curves (13.26).

13.6 Modeling Issues

Parametrization of Forward Rates

In the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by 4 coefficients z_1, z_2, z_3, z_4, as

$$g(x) = z_1 + (z_2 + z_3 x)e^{-xz_4}, \qquad x \geq 0.$$

An example of a graph obtained by the Nelson-Siegel parametrization is given in Figure 13.20, for $z_1 = 1$, $z_2 = -10$, $z_3 = 100$, $z_4 = 10$.

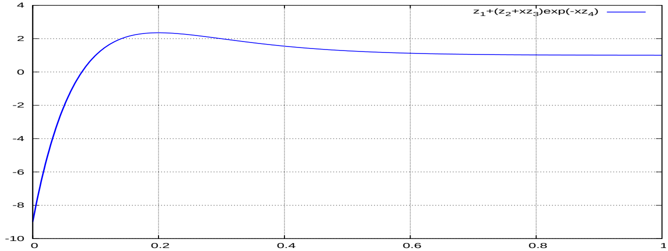

Fig. 13.20: Graph of $x \longmapsto g(x)$ in the Nelson-Siegel model.

The Svensson parametrization has the advantage to reproduce two humps instead of one, the location and height of which can be chosen *via* 6 parameters z_1, z_2, z_3, z_4, z_5, z_6 as

$$g(x) = z_1 + (z_2 + z_3 x)e^{-xz_4} + z_5 x e^{-xz_6}, \qquad x \geq 0.$$

A typical graph of a Svensson parametrization is given in Figure 13.21, for $z_1 = 7$, $z_2 = -5$, $z_3 = -100$, $z_4 = 10$, $z_5 = -1/2$, $z_6 = -1$.

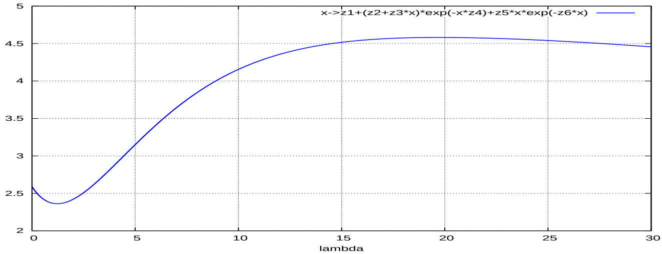

Fig. 13.21: Graph of $x \longmapsto g(x)$ in the Svensson model.

Figure 13.22 presents a fit of the market data of Figure 13.10 using a Svensson curve.

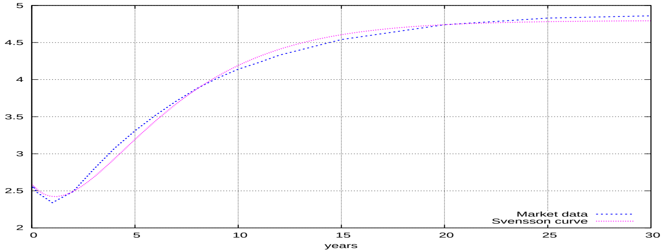

Fig. 13.22: Comparison of market data *vs* a Svensson curve.

It can be shown, cf. § 3.5 of [7], that the forward yield curves of the Vasicek model are included neither in the Nelson-Siegel space, nor in the Svensson space.

In addition, such curves do not appear to correctly model the market forward curves considered above, cf. *e.g.* Figure 13.10.
In the Vasicek model we have

$$\frac{\partial f}{\partial T}(t,T) = \left(-br_t + a - \frac{\sigma^2}{b} + \frac{\sigma^2}{b}e^{-b(T-t)}\right)e^{-b(T-t)},$$

and one can check that the sign of the derivatives of f can only change once at most. As a consequence, the possible forward curves in the Vasicek model are limited to one change of "regime" per curve, as illustrated in Figure 13.23 for various values of r_t, and in Figure 13.24.

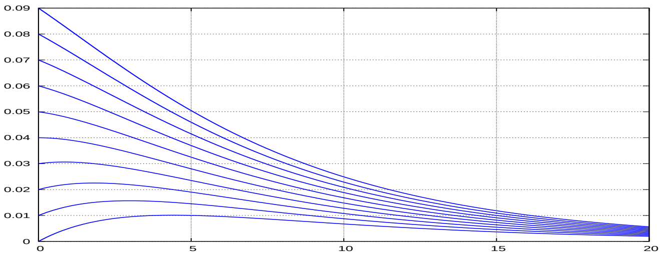

Fig. 13.23: Graphs of forward rates.

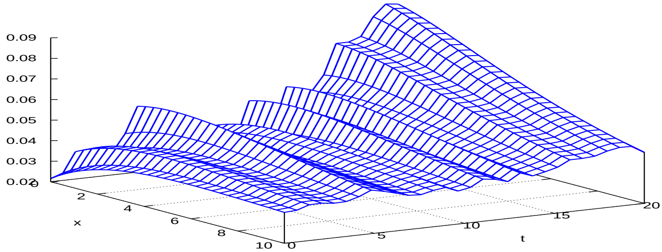

Fig. 13.24: Forward instantaneous curve $(t, x) \longmapsto f(t, t + x)$ in the Vasicek model.

One may think of constructing an instantaneous rate process taking values in the Svensson space, however this type of modelization is not consistent with absence of arbitrage, and it can be proved that the HJM curves cannot live in the Nelson-Siegel or Svensson spaces, cf. §3.5 of [7].

Another way to deal with the curve fitting problem is to use deterministic shifts for the fitting of one forward curve, such as the initial curve at $t = 0$, cf. *e.g.* § 8.2 of [87].

The Correlation Problem and a Two-Factor Model

The correlation problem is another issue of concern when using the affine models considered so far. Let us compare three bond price simulations with maturity $T_1 = 10$, $T_2 = 20$, and $T_3 = 30$ based on the same Brownian path, as given in Figure 13.25. Clearly, the bond prices $P(t, T_1)$ and $P(t, T_2)$ with maturities T_1 and T_2 are linked by the relation

$$P(t, T_2) = P(t, T_1) \exp(A(t, T_2) - A(t, T_1) + r_t(C(t, T_2) - C(t, T_1))), \quad (13.45)$$

meaning that bond prices with different maturities could be deduced from each other, which is unrealistic.

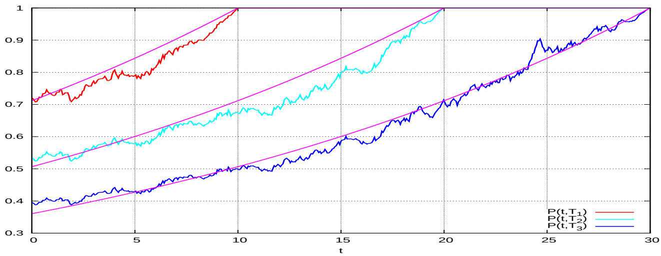

Fig. 13.25: Graph of $t \longmapsto P(t, T_1)$.

In affine short rates models, by (13.45), $\log P(t,T_1)$ and $\log P(t,T_2)$ are linked by the linear relationship

$$\begin{aligned}
\log P(t,T_2) &= \log P(t,T_1) + A(t,T_2) - A(t,T_1) + r_t(C(t,T_2) - C(t,T_1)) \\
&= \log P(t,T_1) + A(t,T_2) - A(t,T_1) + (C(t,T_2) - C(t,T_1))\frac{\log P(t,T_1) - C(t,T_1)}{A(t,T_1)} \\
&= \left(1 + \frac{C(t,T_2) - C(t,T_1)}{A(t,T_1)}\right)\log P(t,T_1) \\
&\quad + A(t,T_2) - A(t,T_1) - (C(t,T_2) - C(t,T_1))\frac{C(t,T_1)}{A(t,T_1)}
\end{aligned}$$

with constant coefficients, which yields the perfect (anti)correlation

$$\mathrm{Cor}(\log P(t,T_1), \log P(t,T_2)) = \pm 1,$$

depending on the sign of the coefficient $1 + (C(t,T_2) - C(t,T_1))/A(t,T_1)$, cf. § 8.3 of [87], A solution to the correlation problem is to consider two control processes $(X_t)_{t\in\mathbb{R}_+}$, $(Y_t)_{t\in\mathbb{R}_+}$ which are solution of

$$\begin{cases} dX_t = \mu_1(t,X_t)dt + \sigma_1(t,X_t)dB_t^{(1)}, \\ \\ dY_t = \mu_2(t,Y_t)dt + \sigma_2(t,Y_t)dB_t^{(2)}, \end{cases} \tag{13.46}$$

where $(B_t^{(1)})_{t\in\mathbb{R}_+}$, $(B_t^{(2)})_{t\in\mathbb{R}_+}$ have correlated Brownian motion with

$$\mathrm{Cov}(B_s^{(1)}, B_t^{(2)}) = \rho \min(s,t), \qquad s,t \in \mathbb{R}_+, \tag{13.47}$$

and

$$dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt, \tag{13.48}$$

for some correlation parameter $\rho \in [-1,1]$. In practice, $(B^{(1)})_{t\in\mathbb{R}_+}$ and $(B^{(2)})_{t\in\mathbb{R}_+}$ can be constructed from two independent Brownian motions

$(W^{(1)})_{t\in\mathbb{R}_+}$ and $(W^{(2)})_{t\in\mathbb{R}_+}$, by letting

$$\begin{cases} B_t^{(1)} = W_t^{(1)}, \\ B_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}, \qquad t \in \mathbb{R}_+, \end{cases}$$

and Relations (13.47) and (13.48) are easily satisfied from this construction.

In two-factor models one chooses to build the short term interest rate r_t *via*

$$r_t = X_t + Y_t, \qquad t \in \mathbb{R}_+.$$

By the previous standard arbitrage arguments we define the price of a bond with maturity T as

$$\begin{aligned} P(t,T) &:= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| X_t,\ Y_t\right] \\ &= F(t, X_t, Y_t), \end{aligned} \tag{13.49}$$

since the couple $(X_t, Y_t)_{t\in\mathbb{R}_+}$ is Markovian. Applying the Itô formula with two variables to

$$t \longmapsto F(t, X_t, Y_t) = P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right],$$

and using the fact that the discounted process

$$t \longmapsto e^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right) \Big| \mathcal{F}_t\right]$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, we can derive a PDE

$$\begin{aligned} &-(x+y)F(t,x,y) + \mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y) + \mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y) \\ &+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y) \\ &+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x \partial y}(t,x,y) + \frac{\partial F}{\partial t}(t, X_t, Y_t) = 0, \end{aligned} \tag{13.50}$$

on $\mathbb{R}^2$ for the bond price $P(t,T)$. In the Vasicek model

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$$\begin{cases} dX_t = -aX_t dt + \sigma dB_t^{(1)}, \\ \\ dY_t = -bY_t dt + \eta dB_t^{(2)}, \end{cases}$$

this yields the solution $F(t,x,y)$ of (13.50) as

$$P(t,T) = F(t,X_t,Y_t) = F_1(t,X_t)F_2(t,Y_t)\exp\left(U(t,T)\right), \tag{13.51}$$

where $F_1(t,X_t)$ and $F_2(t,Y_t)$ are the bond prices associated to X_t and Y_t in the Vasicek model, and

$$U(t,T) = \rho\frac{\sigma\eta}{ab}\left(T - t + \frac{e^{-a(T-t)}-1}{a} + \frac{e^{-b(T-t)}-1}{b} - \frac{e^{-(a+b)(T-t)}-1}{a+b}\right)$$

is a correlation term which vanishes when $(B_t^{(1)})_{t\in\mathbb{R}_+}$ and $(B_t^{(2)})_{t\in\mathbb{R}_+}$ are independent, *i.e.* when $\rho = 0$, cf [11], Chapter 4, Appendix A, and § 8.4 of [87].

Partial differentiation of $\log P(t,T)$ with respect to T leads to the instantaneous forward rate

$$f(t,T) = f_1(t,T) + f_2(t,T) - \rho\frac{\sigma\eta}{ab}(1 - e^{-a(T-t)})(1 - e^{-b(T-t)}), \tag{13.52}$$

where $f_1(t,T)$, $f_2(t,T)$ are the instantaneous forward rates corresponding to X_t and Y_t respectively, cf. § 8.4 of [87].

An example of a forward rate curve obtained in this way is given in Figure 13.26.

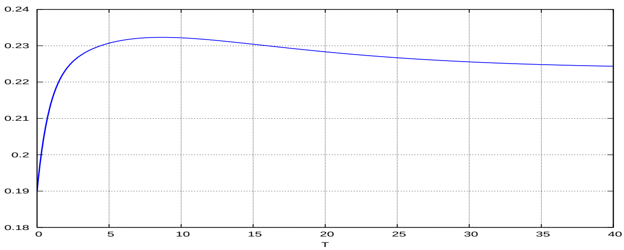


Fig. 13.26: Graph of forward rates in a two-factor model.

Next in Figure 13.27 we present a graph of the evolution of forward curves in a two factor model.

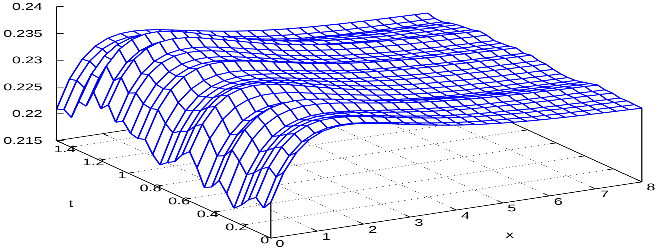


Fig. 13.27: Random evolution of forward rates in a two-factor model.

Fitting the Nelson-Siegel and Svensson models to yield curve data

Recall that in the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by four coefficients z_1, z_2, z_3, z_4, as

$$f(t, t+y) = z_1 + (z_2 + z_3 y)e^{-yz_4}, \qquad y \geq 0. \tag{13.53}$$

Recall taking $x = T - t$, the yield $f(t, t, T)$ is given as

$$\begin{aligned}
f(t, t, T) &= \frac{1}{T-t}\int_t^T f(t, s)ds \\
&= \frac{1}{x}\int_0^x f(t, t+y)dy \\
&= z_1 + \frac{z_2}{x}\int_0^x e^{-yz_4}dy + \frac{z_3}{x}\int_0^x ye^{-yz_4}dy \\
&= z_1 + z_2\frac{1-e^{-xz_4}}{xz_4} + z_3\frac{1-e^{-xz_4}+xe^{-xz_4}}{xz_4}.
\end{aligned}$$

The expression (13.53) can be represented in the parametrization

$$f(t, t+x) = z_1 + (z_2 + z_3 x)e^{-xz_4} = \beta_0 + \beta_1 e^{-x/\lambda} + \frac{\beta_2}{\lambda}xe^{-x/\lambda}, \qquad x \geq 0,$$

cf. [14], with $\beta_0 = z_1$, $\beta_1 = z_2$, $\beta_2 = z_3/z_4$, $\lambda = 1/z_4$.

```
require(YieldCurve)
data(ECBYieldCurve)
mat.ECB<-c(3/12, 0.5, 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30)
first(ECBYieldCurve, '1 month')
Nelson.Siegel(first(ECBYieldCurve, '1 month'), mat.ECB)
```

```
for (n in seq(from=70, to=290, by=10)) {
ECB.NS <- Nelson.Siegel(ECBYieldCurve[n,], mat.ECB)
ECB.S <- Svensson(ECBYieldCurve[n,], mat.ECB)
ECB.NS.yield.curve <- NSrates(ECB.NS, mat.ECB)
ECB.S.yield.curve <- Srates(ECB.S, mat.ECB,"Spot")
plot(mat.ECB, as.numeric(ECBYieldCurve[n,]), type="o", lty=1, col=1,ylab="Interest rates", xlab=
      "Maturity in years", ylim=c(3.2,4.8))
lines(mat.ECB, as.numeric(ECB.NS.yield.curve), type="l", lty=3,col=2,lwd=2)
lines(mat.ECB, as.numeric(ECB.S.yield.curve), type="l", lty=2,col=6,lwd=2)
title(main=paste("ECB yield curve observed at",time(ECBYieldCurve[n], sep=" "),"vs fitted yield
      curve"))
legend('bottomright', legend=c("ECB data","Nelson-Siegel","Svensson"),col=c(1,2,6), lty=1, bg='
      gray90')
grid()
}
```

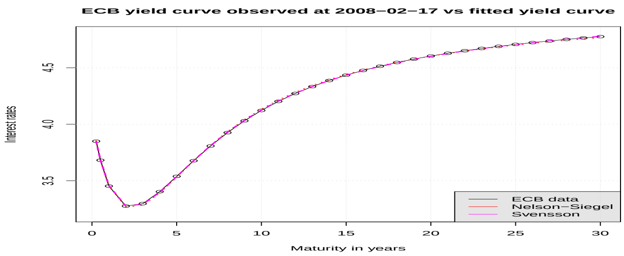


Fig. 13.28: ECB data *vs* fitted yield curve.*

13.7 The BGM Model

The models (HJM, affine, etc.) considered in the previous chapter suffer from various drawbacks such as nonpositivity of interest rates in Vasicek model, and lack of closed form solutions in more complex models. The BGM [9] model has the advantage of yielding positive interest rates, and to permit to derive explicit formulas for the computation of prices for interest rate derivatives such as caps and swaptions on the LIBOR market.

In the BGM model we consider two bond prices $P(t, T_1)$, $P(t, T_2)$ with maturities T_1, T_2 and the forward measure

$$\frac{d\mathbb{P}_2}{d\mathbb{P}_2^*} = \frac{e^{-\int_0^{T_2} r_s ds}}{P(0, T_2)},$$

* The animation works in Acrobat Reader on the entire pdf file.

with numeraire $P(t, T_2)$, cf. (12.6). The forward LIBOR rate $L(t, T_1, T_2)$ is modeled as a geometric Brownian motion under $\mathbb{P}_2$, *i.e.*

$$\frac{dL(t, T_1, T_2)}{L(t, T_1, T_2)} = \gamma_1(t) dB_t^{(2)}, \tag{13.54}$$

$0 \leq t \leq T_1$, $i = 1, \ldots, n-1$, for some deterministic function $\gamma_1(t)$, with solution

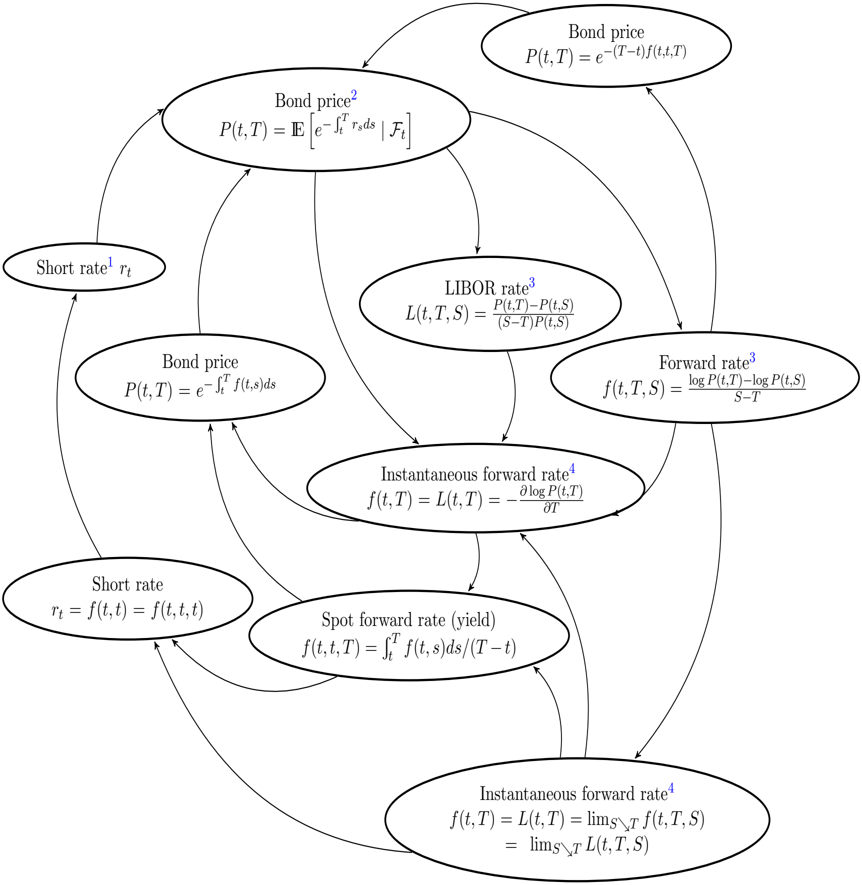
$$L(u, T_1, T_2) = L(t, T_1, T_2) \exp\left(\int_t^u \gamma_1(s) dB_s^{(2)} - \frac{1}{2} \int_t^u |\gamma_1|^2(s) ds \right),$$

i.e. for $u = T_1$,

$$L(T_1, T_1, T_2) = L(t, T_1, T_2) \exp\left(\int_t^{T_1} \gamma_1(s) dB_s^{(2)} - \frac{1}{2} \int_t^{T_1} |\gamma_1|^2(s) ds \right).$$

Since $L(t, T_1, T_2)$ is a geometric Brownian motion under $\mathbb{P}_2$, standard caplets can be priced at time $t \in [0, T_1]$ from the Black-Scholes formula.

The following graph 13.29 summarizes the notions introduced in this chapter.



[1] Can be modeled by Vasiçek and other short rate models
[2] Can be modeled from $dP(t,T)/P(t,T)$.
[3] Can be modeled in the BGM model
[4] Can be modeled in the HJM model

Fig. 13.29: Graph of stochastic interest rate modeling.

Exercises

Exercise 13.1 Consider a tenor structure $\{T_1, T_2\}$ and a bond with maturity T_2 and price given at time $t \in [0, T_2]$ by

$$P(t, T_2) = \exp\left(-\int_t^{T_2} f(t, s)ds\right), \qquad t \in [0, T_2],$$

where the instantaneous yield curve $f(t, s)$ is parametrized as

$$f(t, s) = r_1 \mathbb{1}_{[0,T_1]}(s) + r_2 \mathbb{1}_{[T_1,T_2]}(s), \qquad s \in [t, T_2].$$

Find a formula to estimate the values of r_1 and r_2 from the data of $P(0, T_2)$ and $P(T_1, T_2)$.

Same question for when $f(t, s)$ is parametrized as

$$f(t, s) = r_1 s \mathbf{1}_{[0,T_1]}(s) + (r_1 T_1 + r_2(s - T_1))\mathbf{1}_{[T_1,T_2]}(s), \qquad s \in [t, T_2].$$

Exercise 13.2 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion started at 0 under the risk-neutral measure $\mathbb{P}^*$. We consider a short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ in a Ho-Lee model with constant deterministic volatility, defined by

$$dr_t = adt + \sigma dB_t,$$

where $a > 0$ and $\sigma > 0$. Let $P(t, T)$ will denote the arbitrage price of a zero-coupon bond in this model:

$$P(t, T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \;\Big|\; \mathcal{F}_t\right], \qquad 0 \le t \le T. \tag{13.55}$$

a) State the bond pricing PDE satisfied by the function $F(t, x)$ defined *via*

$$F(t, x) := \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \;\Big|\; r_t = x\right], \qquad 0 \le t \le T.$$

b) Compute the arbitrage price $F(t, r_t) = P(t, T)$ from its expression (13.55) as a conditional expectation.

   Hint. One may use the *integration by parts* relation

$$\begin{aligned}\int_t^T B_s ds &= TB_T - tB_t - \int_t^T s dB_s \\ &= (T - t)B_t + T(B_T - B_t) - \int_t^T s dB_s\end{aligned}$$

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$$= (T-t)B_t + \int_t^T (T-s)dB_s,$$

and the Laplace transform identity $\mathbb{E}[e^{\lambda X}] = e^{\lambda^2\eta^2/2}$ for $X \simeq \mathcal{N}(0,\eta^2)$.

c) Check that the function $F(t,x)$ computed in question (b) does satisfy the PDE derived in question (a).
d) Compute the forward rate $f(t,T,S)$ in this model.

From now on we let $a = 0$.

e) Compute the instantaneous forward rate $f(t,T)$ in this model.
f) Derive the stochastic equation satisfied by the instantaneous forward rate $f(t,T)$.
g) Check that the HJM absence of arbitrage condition is satisfied in this equation.

Exercise 13.3 Consider the CIR process $(r_t)_{t\in\mathbb{R}_+}$ solution of

$$dr_t = -ar_t dt + \sigma\sqrt{r_t}dB_t,$$

where $a,\sigma > 0$ are constants $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion started at 0.

a) Write down the bond pricing PDE for the function $F(t,x)$ given by

$$F(t,x) := \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big| r_t = x\right], \qquad 0 \leq t \leq T.$$

Hint: Use Itô calculus and the fact that the discounted bond price is a martingale.

b) Show that the PDE of Question (a) admits a solution of the form $F(t,x) = e^{A(T-t)+xC(T-t)}$ where the functions $A(s)$ and $C(s)$ satisfy ordinary differential equations to be also written down together with the values of $A(0)$ and $C(0)$.

Exercise 13.4 Given $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion, consider a HJM model given by

$$d_t f(t,T) = \frac{\sigma^2}{2}T(T^2-t^2)dt + \sigma T dB_t. \qquad (13.56)$$

a) Show that the HJM condition is satisfied by (13.56).
b) Compute $f(t,T)$ by solving (13.56).

Hint: We have $f(t,T) = f(0,T) + \int_0^t d_s f(s,T) = \cdots$

c) Compute the short rate $r_t = f(t,t)$ from the result of Question (b).

d) Show that the short rate r_t satisfies a stochastic differential equation of the form

$$dr_t = \eta(t)dt + (r_t - f(0,t))\psi(t)dt + \xi(t)dB_t,$$

where $\eta(t)$, $\psi(t)$, $\xi(t)$ are deterministic functions to be determined.

Exercise 13.5 Let $(r_t)_{t\in\mathbb{R}_+}$ denote a short term interest rate process. For any $T > 0$, let $P(t,T)$ denote the price at time $t \in [0,T]$ of a zero coupon bond defined by the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma_t^T dB_t, \qquad 0 \le t \le T, \tag{13.57}$$

under the terminal condition $P(T,T) = 1$, where $(\sigma_t^T)_{t\in[0,T]}$ is an adapted process. Let the forward measure $\mathbb{P}_T$ be defined by

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}} \,\Big|\, \mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T.$$

Recall that

$$B_t^T := B_t - \int_0^t \sigma_s^T ds, \qquad 0 \le t \le T,$$

is a standard Brownian motion under $\mathbb{P}_T$.

a) Solve the stochastic differential equation (13.57).
b) Derive the stochastic differential equation satisfied by the discounted bond price process

$$t \longmapsto e^{-\int_0^t r_s ds} P(t,T), \qquad 0 \le t \le T,$$

and show that it is a martingale.
c) Show that

$$\mathbb{E}\left[e^{-\int_0^T r_s ds} \,\Big|\, \mathcal{F}_t\right] = e^{-\int_0^t r_s ds} P(t,T), \qquad 0 \le t \le T.$$

d) Show that

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathcal{F}_t\right], \qquad 0 \le t \le T.$$

e) Compute $P(t,S)/P(t,T)$, $0 \le t \le T$, show that it is a martingale under $\mathbb{P}_T$ and that

$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2}\int_t^T (\sigma_s^S - \sigma_s^T)^2 ds\right).$$

f) Assuming that $(\sigma_t^T)_{t\in[0,T]}$ and $(\sigma_t^S)_{t\in[0,S]}$ are deterministic functions, compute the price

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \;\Big|\; \mathcal{F}_t\right] = P(t,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+ \;\Big|\; \mathcal{F}_t\right]$$

of a bond option with strike price κ.

Recall that if X is a centered Gaussian random variable with mean m_t and variance v_t^2 given $\mathcal{F}_t$, we have

$$\mathbb{E}[(e^X-K)^+ \mid \mathcal{F}_t] = e^{m_t+v_t^2/2}\Phi\left(\frac{v_t}{2}+\frac{1}{v_t}(m_t+v_t^2/2-\log K)\right)$$
$$-K\Phi\left(-\frac{v_t}{2}+\frac{1}{v_t}(m_t+v_t^2/2-\log K)\right)$$

where $\Phi(x)$, $x\in\mathbb{R}$, denotes the Gaussian cumulative distribution function.

Exercise 13.6 (Exercise 4.10 continued). Bridge model. Assume that the price $P(t,T)$ of a zero coupon bond is modeled as

$$P(t,T) = e^{-\mu(T-t)+X_t^T}, \qquad t\in[0,T],$$

where $\mu>0$.

a) Show that the terminal condition $P(T,T)=1$ is satisfied.
b) Compute the forward rate

$$f(t,T,S) = -\frac{1}{S-T}(\log P(t,S)-\log P(t,T)).$$

c) Compute the instantaneous forward rate

$$f(t,T) = -\lim_{S\searrow T}\frac{1}{S-T}(\log P(t,S)-\log P(t,T)).$$

d) Show that the limit $\lim_{T\searrow t} f(t,T)$ does not exist in $L^2(\Omega)$.
e) Show that $P(t,T)$ satisfies the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + \frac{1}{2}\sigma^2 dt - \frac{\log P(t,T)}{T-t}dt, \qquad t\in[0,T].$$

f) Show, using the results of Exercise 13.5-(d), that

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s^{\mathrm{T}} ds} \;\Big|\; \mathcal{F}_t\right],$$

where $(r_t^{\mathrm{T}})_{t\in[0,T]}$ is a process to be determined.
g) Compute the conditional density

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}} \,\Big|\, \mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s^{\mathrm{T}} ds}$$

of the forward measure $\mathbb{P}_T$ with respect to $\mathbb{P}$.

h) Show that the process

$$\tilde{B}_t := B_t - \sigma t, \qquad 0 \leq t \leq T,$$

is a standard Brownian motion under $\mathbb{P}_T$.

i) Compute the dynamics of X_t^S and $P(t,S)$ under $\mathbb{P}_T$.
Hint: Show that

$$-\mu(S-T) + \sigma(S-T)\int_0^t \frac{1}{S-s} dB_s = \frac{S-T}{S-t} \log P(t,S).$$

j) Compute the bond option price

$$\mathbb{E}\left[e^{-\int_t^T r_s^{\mathrm{T}} ds}(P(T,S)-K)^+ \,\Big|\, \mathcal{F}_t\right] = P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+ \,\Big|\, \mathcal{F}_t\right],$$

$0 \leq t < T < S$.

Exercise 13.7 (Exercise 4.13 continued). Write down the bond pricing PDE for the function

$$F(t,x) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \,\Big|\, r_t = x\right]$$

and show that in case $\alpha = 0$ the corresponding bond price $P(t,T)$ equals

$$P(t,T) = e^{-B(T-t)r_t}, \qquad 0 \leq t \leq T,$$

where

$$B(x) = \frac{2(e^{\gamma x}-1)}{2\gamma + (\beta+\gamma)(e^{\gamma x}-1)},$$

with $\gamma = \sqrt{\beta^2 + 2\sigma^2}$.

Exercise 13.8 Consider a short rate process $(r_t)_{t\in\mathbb{R}_+}$ of the form $r_t = h(t) + X_t$, where $h(t)$ is a deterministic function and $(X_t)_{\mathbb{R}_+}$ is a Vasicek process started at $X_0 = 0$.

a) Compute the price $P(0,T)$ at time $t = 0$ of a bond with maturity T, using $h(t)$ and the function $A(T)$ defined in (13.18) for the pricing of Vasicek bonds.
b) Show how the function $h(t)$ can be estimated from the market data of the initial instantaneous forward rate curve $f(0,t)$.

Exercise 13.9

a) Given two LIBOR spot rates $L(t,t,T)$ and $L(t,t,S)$, compute the corresponding LIBOR forward rate $L(t,T,S)$.
b) Assuming that $L(t,t,T) = 2\%$, $L(t,t,S) = 2.5\%$ and $t = 0$, $T = 1$, $S = 2T = 2$, would you buy a LIBOR forward contract over $[T,2T]$ with rate $L(0,T,2T)$ if $L(T,T,2T)$ remained at $L(T,T,2T) = L(0,0,T) = 2\%$?

Exercise 13.10 Stochastic string model [103]. Consider an instantaneous forward rate $f(t,x)$ solution of

$$d_t f(t,x) = \alpha x^2 dt + \sigma d_t B(t,x), \tag{13.58}$$

with a flat initial curve $f(0,x) = r$, where x represents the time to maturity, and $(B(t,x))_{(t,x)\in\mathbb{R}_+^2}$ is a standard *Brownian sheet* with covariance

$$\mathbb{E}[B(s,x)B(t,y)] = (\min(s,t))(\min(x,y)), \qquad s,t,x,y \in \mathbb{R}_+,$$

and initial conditions $B(t,0) = B(0,x) = 0$ for all $t,x \in \mathbb{R}_+$.

a) Solve the equation (13.58) for $f(t,x)$.
b) Compute the short term interest rate $r_t = f(t,0)$.
c) Compute the value at time $t \in [0,T]$ of the bond price

$$P(t,T) = \exp\left(-\int_0^{T-t} f(t,x)dx\right)$$

with maturity T.
d) Compute the variance $\mathbb{E}\left[\left(\int_0^{T-t} B(t,x)dx\right)^2\right]$ of the centered Gaussian random variable $\int_0^{T-t} B(t,x)dx$.
e) Compute the expected value $\mathbb{E}[P(t,T)]$.
f) Find the value of α such that the discounted bond price

$$e^{-rt}P(t,T) = \exp\left(-rT - \frac{\alpha}{3}t(T-t)^3 - \sigma\int_0^{T-t} B(t,x)dx\right), \quad t \in [0,T].$$

satisfies $e^{-rt}\,\mathbb{E}[P(t,T)] = e^{-rT}$.
g) Compute the bond option price $\mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right)(P(T,S)-K)^+\right]$ by the Black-Scholes formula, knowing that

$$\mathbb{E}[(xe^{m+X}-K)^+] = xe^{m+\frac{v^2}{2}}\Phi(v+(m+\log(x/K))/v) - K\Phi((m+\log(x/K))/v),$$

when X is a centered Gaussian random variable with mean $m = r\tau - v^2/2$ and variance v^2.

Exercise 13.11 (Exercise 13.6 continued).

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a) Compute the forward rate

$$f(t,T,S) = -\frac{1}{S-T}(\log P(t,S) - \log P(t,T)).$$

b) Compute the instantaneous forward rate

$$f(t,T) = -\lim_{S \searrow T} \frac{1}{S-T}(\log P(t,S) - \log P(t,T)).$$

c) Show that the limit $\lim_{T \searrow t} f(t,T)$ does not exist in $L^2(\Omega)$.

d) Show that $P(t,T)$ satisfies the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + \frac{1}{2}\sigma^2 dt - \frac{\log P(t,T)}{T-t} dt, \qquad t \in [0,T].$$

e) Show, using the results of Exercise 13.5-(c), that

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s^T ds} \Big| \mathcal{F}_t \right],$$

where $(r_t^T)_{t\in[0,T]}$ is a process to be determined.

f) Compute the conditional density

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}} \Big| \mathcal{F}_t \right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s^T ds}$$

of the forward measure $\mathbb{P}_T$ with respect to $\mathbb{P}$.

g) Show that the process

$$\tilde{B}_t := B_t - \sigma t, \qquad 0 \le t \le T,$$

is a standard Brownian motion under $\mathbb{P}_T$.

h) Compute the dynamics of X_t^S and $P(t,S)$ under $\mathbb{P}_T$.

Hint: Show that

$$-\mu(S-T) + \sigma(S-T)\int_0^t \frac{1}{S-s} dB_s = \frac{S-T}{S-t}\log P(t,S).$$

i) Compute the bond option price

$$\mathbb{E}\left[e^{-\int_t^T r_s^T ds}(P(T,S)-K)^+ \Big| \mathcal{F}_t \right] = P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+ \Big| \mathcal{F}_t \right],$$

$0 \le t < T < S$.