On Some Exponential Functionals of Brownian Motion
Author(s): Marc Yor
Source: *Advances in Applied Probability*, Vol. 24, No. 3 (Sep., 1992), pp. 509-531
Published by: Applied Probability Trust
Stable URL: http://www.jstor.org/stable/1427477
Accessed: 03-02-2017 23:40 UTC

REFERENCES
Linked references are available on JSTOR for this article:
http://www.jstor.org/stable/1427477?seq=1&cid=pdf-reference#references_tab_contents
You may need to log in to JSTOR to access the linked references.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at http://about.jstor.org/terms

Applied Probability Trust is collaborating with JSTOR to digitize, preserve and extend access to *Advances in Applied Probability*

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

Adv. Appl. Prob. **24**, 509–531 (1992)
Printed in N. Ireland
© *Applied Probability Trust* 1992

ON SOME EXPONENTIAL FUNCTIONALS OF BROWNIAN MOTION

MARC YOR,* *Université Paris VI*

Abstract

In this paper, distributional questions which arise in certain mathematical finance models are studied: the distribution of the integral over a fixed time interval $[0, T]$ of the exponential of Brownian motion with drift is computed explicitly, with the help of computations previously made by the author for Bessel processes. The moments of this integral are obtained independently and take a particularly simple form. A subordination result involving this integral and previously obtained by Bougerol is recovered and related to an important identity for Bessel functions. When the fixed time T is replaced by an independent exponential time, the distribution of the integral is shown to be related to last-exit-time distributions and the fixed time case is recovered by inverting Laplace transforms.

BROWNIAN MOTION WITH DRIFT; BESSEL FUNCTIONS; BESSEL PROCESSES; LAST-EXIT TIMES

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60J65
SECONDARY 90A09

1. Introduction

1.1. For the last four or five years, a number of applied probabilists, working in the domain of mathematical finance, and more precisely on path-dependent options, the so-called 'Asian options', a particular case of which are average-value options, have been interested in the distribution of

$$\int_0^t ds \exp(aB_s + bs), \tag{1.a}$$

for a fixed time t, and reals a and b, where $(B_s, s \geqq 0)$ denotes a real-valued Brownian motion starting from 0.

Thanks to the scaling properties of Brownian motion, it suffices to find this distribution for $a = 2$ for instance. We shall now write

$$A_t^{(\nu)} = \int_0^t ds \exp 2(B_s + \nu s), \tag{1.b}$$

and discuss the distribution of $A_t^{(\nu)}$. When $\nu = 0$, we simply write A_t.

At this point, it may be worth emphasizing that some quantities of interest in mathematical finance are the moments of $A_t^{(\nu)}$, and, perhaps more importantly, the

Received 10 May 1991; revision received 17 July 1991.
* Postal address: Laboratoire de Probabilités, Université Paris VI, 4 Place Jussieu, Tour 56-3$^{\text{eme}}$ Etage, 75252 Paris Cedex 05, France.

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

function

$$(*)\qquad C(t, k) \equiv \boldsymbol{E}((A_t^{(\nu)} - k)^+).$$

We come back to this point in detail in Section 1.7 of this introduction.

Hence, a fairly thorough knowledge of the law of A_t is needed in order to get, as much as possible, an explicit expression of $C(t, k)$.

1.2. We now remark that, thanks to Girsanov's theorem, we have, for any Borel function $f:\mathbb{R}_+ \to \mathbb{R}_+$:

$$\text{(1.c)}\qquad \boldsymbol{E}[f(A_t^{(\nu)})] = \boldsymbol{E}[f(A_t)\exp(\nu B_t - \tfrac{1}{2}\nu^2 t)]$$

so that the distribution of $A_t^{(\nu)}$ may be obtained once the joint distribution of $(A_t, \exp(B_t))$ is known. This may be done in terms of the semigroup of the hyperbolic Brownian motion on Poincaré's half-plane (see Bougerol [2]).

In order to be complete and, at the same time, not to confuse the reader with different approaches, the connections with hyperbolic Brownian motion are presented concisely in Section 7 of the present paper.

However, the main approach chosen in this paper, for the study of the law of A_t, and more generally $A_t^{(\nu)}$, is by relating $(\exp(B_t + \nu t), t \geqq 0)$ and the Bessel process $(\rho_u^{(\nu)}, u \geqq 0)$ with index ν. Some important facts concerning Bessel processes and related computations are collected in Section 2.

1.3. An interesting by-product of this computation is the following striking identity in law obtained by Bougerol [2]:

$$\text{(1.d)}\qquad \text{for any fixed } t > 0, \qquad \sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t},$$

where $(\gamma_u, u \geqq 0)$ is a one-dimensional Brownian motion starting from 0, and independent of B.

Routine computations show that (1.d) is equivalent to the relation

$$\text{(1.e)}\quad \text{for } u \in \mathbb{R}, \qquad \boldsymbol{E}\left[\frac{1}{\sqrt{A_t}}\exp\left(-\frac{u^2}{2A_t}\right)\right] = \frac{1}{\sqrt{(1+u^2)t}}\exp -\frac{1}{2t}(\operatorname{Arg\,sh} u)^2$$

an identity which characterizes the law of A_t.

In Section 3, we show that both sides of (1.e) admit the same Laplace transform (in t) thereby proving (1.e), hence (1.d). The key argument which we use to prove this Laplace transform identity is the identity in law:

$$\text{(1.f)}\qquad \exp(B_t) = \rho_{A_t} \qquad (t \geqq 0)$$

where ρ is a two-dimensional Bessel process starting from 1.

It finally turns out that Bougerol's identity in law may be understood as a probabilistic interpretation of the following special case of the Lipschitz–Hankel

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

formulae (see Watson [13], p. 386, for example): for $a \geqq 1$, and $\nu > 0$, we have

$$\nu \int_0^\infty \frac{dt}{t} \exp(-at) I_\nu(t) = \frac{1}{(a + \sqrt{a^2 - 1})^\nu}.$$

1.4. In Section 4, we show, using elementary arguments independent of the previous computations, that there exists an explicit sequence of polynomials $(P_n, n \in \mathbb{N})$ such that:

$$E\left[\left(\int_0^t ds \exp(2B_s)\right)^n\right] = \frac{1}{4^n} E[P_n(\exp 2B_t)]. \tag{1.g}$$

In particular, this formula allows us to compute the moments of A_t, and it is easily shown that formula (1.g) agrees with Bougerol's formula (1.d).

However, we also show, in the same paragraph, that the knowledge of the moments of A_t does not *a priori* determine the law of A_t, since Carleman's criterion does not apply. Hence, our derivation of formula (1.g) in Section 4, is, *a priori*, strictly weaker than Bougerol's result (1.d).

1.5. In Section 5, we compute the law of $A_{S_\theta}^{(\nu)}$, where S_θ is an exponential random variable, with parameter $(\frac{1}{2}\theta^2)$, which is independent of B. For simplicity, we now discuss the case $\nu = 0$.

We first recall that Williams [14] showed that, given $B_{S_\theta} = x > 0$, the process $(B_u, u \leqq S_\theta)$ is distributed as $(B_u + \theta u; u \leqq L_x^\theta)$, where $L_x^\theta \equiv \sup\{u : B_u + \theta u = x\}$; changing B into $-B$, it immediately follows that, given $B_{S_\theta} = x < 0$, the process $(B_u, u \leqq S_\theta)$ is distributed as $(B_u - \theta u; u \leqq L_x^{-\theta})$, where $L_x^{-\theta} \equiv \sup\{u : B_u - \theta u = x\}$. Now, the time-change identity (1.f) may also be extended to

$$\exp(B_t + \theta t) = \rho^{(\theta)}_{A_t^{(\theta)}}, \tag{1.f$_\theta$}$$

where $(\rho_u^{(\theta)}; u \geqq 0)$ denotes a Bessel process with index θ (for dimension $d_\theta \equiv 2(\theta + 1)$) starting from 1.

As a consequence of both (1.f) and (1.f)$_\theta$, we obtain that, given $B_{S_\theta} = x > 0$, we have:

$$A_{S_\theta} \overset{\text{(law)}}{=} \Lambda_{e^x}^{(\theta)}, \tag{1.h}$$

where $\Lambda_a^{(\theta)} \equiv \sup\{u > 0 : \rho_u^{(\theta)} = a\}$.

Moreover, as is well known in the general context of transient diffusions (see Pitman and Yor [9], for example), the law of $\Lambda_a^{(\theta)}$ is intimately linked with the semigroup of $\rho^{(\theta)}$, which is known explicitly. Hence, an explicit formula for the law of A_{S_θ} is easily deduced from formula (1.h).

1.6. In Section 6, we invert the Laplace transforms obtained in Section 5 to derive the law of A_t, given B_t, from computations done in [15], relative to the two-dimensional Bessel process $(\rho_u, u \geqq 0)$.

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

1.7. We now give some detailed explanations as to why the knowledge of the distribution of $A(t)$ is most interesting when dealing with average-value (financial) options (in what follows, we shall write AV-options for short).

The following presentation of such options is taken from Kemna and Vorst [7]: consider a perfect security market which is open continuously, offers a constant riskless interest rate r to all borrowers and lenders, and in which no transaction costs and/or takes are incurred. It is further assumed that the underlying asset on which the option is based is equal to a stock with price $S(t)$, which satisfies the linear stochastic differential equation

$$dS_t = \alpha S_t\, dt + \sigma S_t\, dB_t, \qquad S_0 = 1, \tag{1.i}$$

where α and σ are non-negative constants. Hence, we deduce from (1.i) that

$$S_t = \exp(\sigma B_t + \alpha t - \tfrac{1}{2}\sigma^2 t). \tag{1.j}$$

We then introduce the process

$$\mathscr{A}(t) = \frac{1}{T - T_0}\int_{T_0}^{t} du\, S_u, \qquad T_0 \leqq t \leqq T, \tag{1.k}$$

where T is the maturity date, and $[T_0, T]$ is the final time interval over which the average value of the stock is calculated. The payoff on the AV-option can be expressed as $(\mathscr{A}(T) - K)^+$, where K is the exercise price of the option.

Kemna and Vorst ([7], p. 121) then show that the value of the AV-option at time $t \in [T_0, T]$ is given by

$$\tilde{C}(S(t), \mathscr{A}(t), t) = \exp(-r(T-t))\boldsymbol{E}[(\mathscr{A}(T) - K)^+ \mid \mathscr{F}_t] \tag{1.l}$$

where $\mathscr{F}_t$ is the σ-field generated by the past of S, up to time t.

Thanks to the independence of the increments of B, it is immediate that the expression (1.l) may be expressed in terms of the function C defined above in $(*)$, but now k is an explicit expression depending on $\mathscr{A}(t)$.

Because of the current lack of knowledge about the law of A_t, and more generally of $A_t^{(\nu)}$, different authors working on this subject have proceeded to various approximations (see, for example, Bouaziz et al. [1], Vorst [12]) and Monte Carlo simulations (Kemna and Vorst [7], who even made the very strange claim that 'it is impossible to derive an explicit analytic expression for an AV-option'). In a future publication, we hope to develop fully the computation of $C(t, k)$, defined in $(*)$, with the help of our knowledge of the law of $A_t^{(\nu)}$.

The main advantage of the AV-options is that they depend on the entire past history of the market, and hence, at least heuristically, reduce the risk of price manipulations of the underlying asset at the maturity date. This may explain why, more generally, path-dependent options seem to be widely used on some (Asian, but also Western) marketplaces, in particular to protect some corporations against

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

potentially hostile takeovers (see Bouaziz et al. [1] and Kemna and Vorst [7] for actual examples).

A second advantage of AV-options is that the price of an AV-option is always strictly less than the price of the corresponding standard European option; in mathematical terms, this is expressed by the inequality

$$E[(\mathscr{A}(T)-K)^+] \leqq E[(S_T-K)^+] \tag{1.m}$$

which we now show to be a simple consequence of Jensen's inequality. Indeed, we have, trivially,

$$(\mathscr{A}(T)-K)^+ \leqq \frac{1}{T-T_0}\int_{T_0}^{T} du\,(S_u-K)^+$$

and, since $(S_u,\ u\geqq 0)$ is a submartingale, the inequality

$$(S_u-K)^+ \leqq E[(S_T-K)^+ \mid \mathscr{F}_u] \qquad (u\leqq T) \tag{1.n}$$

holds. The inequality (1.m) follows by taking expectations of both sides of (1.n), and then averaging over $[T_0, T]$.

1.8. To conclude this introduction, we should like to sum up the contents of this paper as follows: the law of A, and more generally $A^{(\nu)}$, taken at a fixed time or at an independent exponential time, is obtained, and characterized in several ways, whilst the actual computation of $C(t, k)$ (see $(*)$) is being postponed.

Moreover, the moments of $A_t^{(\nu)}$ are obtained explicitly (see Corollary 2 of Theorem 1).

2. Some preliminaries on Bessel processes

The information about Bessel processes which is presented in this section may be found in Yor [15] and Pitman and Yor [9].

2.1. The key fact in the present paper is the relationship

$$\exp(B_t+\nu t) = \rho^{(\nu)}_{A_t^{(\nu)}}, \qquad t\geqq 0, \tag{2.a}$$

where $(\rho_u^{(\nu)},\ u\geqq 0)$ is a Bessel process with index ν, that is, an $\mathbb{R}_+$-valued diffusion with infinitesimal generator $\mathscr{L}^\nu$ given by

$$\mathscr{L}^\nu f(x) = \tfrac{1}{2}f''(x) + \frac{2\nu+1}{2x}f'(x), \qquad f\in C_b^2(\mathbb{R}_+^*).$$

2.2. For any $\nu\in\mathbb{R}$, and $a\geqq 0$, we denote by P_a^ν the law on $C(\mathbb{R}_+,\mathbb{R}_+)$ of $\rho^{(\nu)}$, when starting from a. We write $(R_t,\ t\geqq 0)$ for the canonical process on $C(\mathbb{R}_+,\mathbb{R}_+)$, and we denote by $\{\mathscr{R}_t=\sigma(R_s, s\leqq t),\ t\geqq 0\}$ the canonical filtration.

Using Girsanov's theorem, it is easily shown that, for $\nu\geqq 0$, the mutual absolute

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

continuity relation holds, for $a>0$:

$$P^{\nu}_{a|\mathcal{R}_t}=\left(\frac{R_t}{a}\right)^{\nu}\exp\left(-\frac{\nu^2}{2}\int_0^t\frac{ds}{R_s^2}\right)P^0_{a|\mathcal{R}_t}. \tag{2.b}$$

In the case $\nu<0$, the Bessel process $(\rho_u^{(\nu)}, u\geqq 0)$ reaches 0, hence the law $P^{\nu}_{a|\mathcal{R}_t}$ cannot be equivalent to $P^0_{a|\mathcal{R}_t}$; nonetheless, we have, for $\nu<0$, and $a>0$,

$$P^{\nu}_{a|\mathcal{R}_t\cap(t<T_0)}=\left(\frac{R_t}{a}\right)^{\nu}\exp\left(-\frac{\nu^2}{2}\int_0^t\frac{ds}{R_s^2}\right)P^0_{a|\mathcal{R}_t}, \tag{2.b$'$}$$

with $T_0\equiv\inf\{u>0:R_u=0\}$. (Note that the formula (2.b$'$) is also valid for $\nu\geqq 0$, since then $P^{\nu}_a(T_0=\infty)=1$.) Comparing formulae (2.b) and (2.b$'$), we also obtain: for $\nu\geqq 0$, and $a>0$,

$$P^{-\nu}_{a|\mathcal{R}_t\cap(t<T_0)}=\left(\frac{a}{R_t}\right)^{2\nu}P^{\nu}_{a|\mathcal{R}_t}. \tag{2.c}$$

2.3. In what follows, we shall also need the explicit form of the density of the Bessel semigroup $P^{\nu}_t(a, dr)=p^{\nu}_t(a, r)\,dr$, for $\nu\geqq 0$; these densities are known to be

$$p^{\nu}_t(a, r)=\left(\frac{r}{a}\right)^{\nu}\frac{r}{t}\exp\left(-\frac{1}{2t}(a^2+r^2)\right)I_{\nu}\left(\frac{ar}{t}\right), \tag{2.d}$$

for $a>0$, $r\geqq 0$, and $t>0$ (see, for example, Revuz and Yor [10], p. 415).

Comparing formulae (2.b) and (2.d), we now obtain the important relationship

$$E^0_a\left[\exp\left(-\frac{\nu^2}{2}\int_0^t\frac{ds}{R_s^2}\right)\,\middle|\, R_t=r\right]=\left(\frac{I_{|\nu|}}{I_0}\right)\left(\frac{ar}{t}\right). \tag{2.e}$$

This formula (2.e) gives a probabilistic interpretation for the Hartman–Watson probability measure on $\mathbb{R}_+$, which we denote as $\eta_r(du)$, for $r>0$, and which is defined and characterized by

$$\left(\frac{I_{|\nu|}}{I_0}\right)(r)=\int_0^{\infty}\exp\left(-\frac{\nu^2 u}{2}\right)\eta_r(du),\quad \text{for } \nu\in\mathbb{R}. \tag{2.e$'$}$$

For more probabilistic interpretations of the (originally) purely analytic work of Hartman and Watson [4], we refer the reader to Yor [15] and Pitman and Yor [9].

2.4. Finally, we shall make some use in what follows of the formulae (2.f) and (2.g) below involving Bessel functions:

(i) formula (2.f) below is a particular case of the more general Lipschitz–Hankel formulae (see, for example, Watson [13], p. 386, formula (7)): for $\nu\geqq 0$, and $a\geqq 1$,

$$\nu\int_0^{\infty}\frac{dt}{t}\exp(-at)I_{\nu}(t)=\frac{1}{(a+\sqrt{a^2-1})^{\nu}}. \tag{2.f}$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

This formula is also found in the literature in the slightly different, but equivalent form: for $\nu \geqq 0$, and $\theta \geqq 0$,

$$\text{(2.f}'\text{)} \qquad \nu \int_0^\infty \frac{dt}{t} \exp\left(-(\theta+1)t\right) I_\nu(t) = \frac{2^\nu}{(\sqrt{\theta+2}+\sqrt{\theta})^{2\nu}}.$$

(ii) We also need to recall the classical integral representation for the Bessel function K_ν: for any $\nu \in \mathbb{R}$,

$$\text{(2.g)} \qquad K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} \exp -\left(t + \frac{z^2}{2t}\right)$$

(see, for example, Watson [13], p. 183, formula (15)).
We shall also use the well-known formulae

$$\text{(2.g}'\text{)} \qquad K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}.$$

3. A proof of Bougerol's identity in law

As we remarked in the Introduction, Bougerol's identity (1.d) is equivalent to (1.e). In order to prove formula (1.e), it is sufficient to show that both sides admit the same Laplace transform in t; hence, all we need to prove is the following identity: for any $\theta \geqq 0$,

$$\text{(3.a)} \qquad \begin{aligned} &\int_0^\infty \frac{dt}{\sqrt{(1+u^2)t}} \exp\left(-\frac{\theta^2 t}{2}\right) \exp\left(-\frac{1}{2t}(\operatorname{Arg}\operatorname{sh} u)^2\right) \\ &= \int_0^\infty dt \exp\left(-\frac{\theta^2 t}{2}\right) \boldsymbol{E}\left[\frac{1}{\sqrt{A_t}} \exp\left(-\frac{u^2}{2A_t}\right)\right] \end{aligned}$$

holds. We denote by $l(\theta, u)$, the left-hand side and by $r(\theta, u)$, the right-hand side of (3.a).

We first transform $r(\theta, u)$ using the expression of $(\exp(B_t), t \geqq 0)$ given in (2.a) in terms of $(\rho_u, u \geqq 0)$, a two-dimensional Bessel process starting from 1. We then obtain, with the notation

$$H_s = \int_0^s \frac{du}{\rho_u^2},$$

$$\text{(3.b)} \qquad \begin{aligned} r(\theta, u) &= \boldsymbol{E}\left[\int_0^\infty dH_s \exp\left(-\frac{\theta^2 H_s}{2}\right) \frac{1}{\sqrt{s}} \exp\left(-\frac{u^2}{2s}\right)\right] \\ &= \boldsymbol{E}\left[\int_0^\infty \frac{ds}{\sqrt{s}} \frac{1}{\rho_s^2} \exp\left(-\frac{\theta^2 H_s}{2}\right) \exp\left(-\frac{u^2}{2s}\right)\right]. \end{aligned}$$

We now replace $\exp(-\theta^2 H_s/2)$ by its conditional expectation, given ρ_s, so that,

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

using the formulae (2.e), and then (2.d), we obtain

$$r(\theta, u)=\int_0^\infty \frac{ds}{\sqrt{s}}\left(\frac{1}{s}\right)\int_0^\infty \frac{d\rho\,\rho}{\rho^2}\exp\left(-\frac{1+\rho^2}{2s}\right)I_\theta\left(\frac{\rho}{s}\right)\exp\left(-\frac{u^2}{2s}\right),$$

$$=\int_0^\infty \frac{d\rho}{s^{\frac{3}{2}}}\int_0^\infty \frac{d\rho}{\rho}\exp\left(-\frac{1+\rho^2+u^2}{2s}\right)I_\theta\left(\frac{\rho}{s}\right),$$

$$=\int_0^\infty \frac{d\rho}{\rho}\int_0^\infty \frac{ds}{s^{\frac{3}{2}}}\exp\left(-\frac{1+\rho^2+u^2}{2s}\right)I_\theta\left(\frac{\rho}{s}\right). \tag{3.c}$$

We now define

$$M(\rho, u)=\int_0^\infty \frac{ds}{s^{\frac{3}{2}}}\exp\left(-\frac{1+\rho^2+u^2}{2s}\right)I_\theta\left(\frac{\rho}{s}\right).$$

Making the change of variables $s=\rho/\xi$, we obtain

$$M(\rho, u)=\frac{1}{\sqrt{\rho}}\int_0^\infty \frac{d\xi}{\sqrt{\xi}}\exp\left(-\left(\frac{1+\rho^2+u^2}{2\rho}\right)\xi\right)I_\theta(\xi),$$

so that, using Fubini's theorem in (3.c), we obtain

$$r(\theta, u)=\int_0^\infty \frac{d\xi}{\sqrt{\xi}}N(u, \xi)I_\theta(\xi), \tag{3.d}$$

where

$$N(u, \xi)=\int_0^\infty \frac{d\rho}{\rho^{\frac{3}{2}}}\exp\left(-\frac{1}{2}\left(\frac{1+u^2}{\rho}+\rho\right)\xi\right).$$

Making some elementary changes of variables, we obtain, using (2.g) and (2.g′) for $\nu=\frac{1}{2}$:

$$N(u, \xi)=\frac{\sqrt{2\pi}}{\xi}\frac{1}{\sqrt{1+u^2}}\exp\left(-\xi\sqrt{1+u^2}\right).$$

Consequently, we deduce from (3.d) that

$$r(\theta, u)=\frac{\sqrt{2\pi}}{\sqrt{1+u^2}}\frac{1}{\theta}\frac{1}{(\sqrt{1+u^2}+u)^\theta}.$$

On the other hand, the left-hand side $l(\theta, u)$ of (3.a) is easily seen, thanks to the formulae (2.g) and (2.g′), for $\nu=-\frac{1}{2}$, to be equal to

$$l(\theta, u)=\frac{\sqrt{2\pi}}{\sqrt{1+u^2}}\frac{1}{\theta}\exp-\theta(\operatorname{Arg\,sh} u),$$

which is equal to $r(\theta, u)$, since $\operatorname{Arg\,sh} u \equiv \log\left(u+\sqrt{1+u^2}\right)$.

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

4. The moments of A_t

4.1. Elementary arguments, which do not bear upon the previous section, and which involve essentially the independence of the increments of B, allow us to obtain an explicit formula for

$$E\left[\left(\int_0^t ds \exp(B_s)\right)^n \exp(\mu B_t)\right],$$

for any $n \in \mathbb{N}$, and $\mu \geqq 0$.

In order to simplify the presentation, and to be able to extend easily some of the computations for the Brownian case to the case of some other processes with independent increments, we shall write, for $\lambda \in \mathbb{R}$,

$$E[\exp(\lambda B_t)] = \exp t\varphi(\lambda), \quad \text{where, here, } \varphi(\lambda) = \frac{\lambda^2}{2}. \tag{4.a}$$

We then have the following result.

Theorem 1. 1. *Let $\mu \geqq 0$, $n \in \mathbb{N}$, and $\alpha > \varphi(\mu + n) \equiv \frac{1}{2}(\mu + n)^2$. Then, the formula*

$$\int_0^\infty dt \exp(-\alpha t) E\left[\left(\int_0^t ds \exp B_s\right)^n \exp(\mu B_t)\right] = \frac{n!}{\prod_{j=0}^{n} (\alpha - \varphi(\mu + j))} \tag{4.b}$$

holds.

2. *Let $\mu \geqq 0$, $n \in \mathbb{N}$, and $t \geqq 0$. Then, we have*

$$E\left[\left(\int_0^t ds \exp(B_s)\right)^n \exp(\mu B_t)\right] = E[P_n^{(\mu)}(\exp B_t) \exp(\mu B_t)] \tag{4.c}$$

where $(P_n^{(\mu)}, n \in \mathbb{N})$ is the following sequence of polynomials:

$$P_n^{(\mu)}(z) = n! \left(\sum_{j=0}^{n} c_j^{(\mu)} z^j\right), \quad \textit{with} \quad c_j^{(\mu)} = \left(\prod_{\substack{k \neq j \\ 0 \leqq k \leqq n}} (\varphi(\mu + j) - \varphi(\mu + k))\right)^{-1}.$$

Proof. 1. We define

$$\begin{aligned}\Phi_{n,t}(\mu) &= E\left[\left(\int_0^t ds \exp(B_s)\right)^n \exp(\mu B_t)\right] \\ &= n!\, E\left[\int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \exp(B_{s_1} + \cdots + B_{s_n} + \mu B_t)\right].\end{aligned}$$

We then remark that

$$\begin{aligned}&E[\exp(\mu B_t + B_{s_1} + \cdots + B_{s_n})] \\ &\quad = E[\exp\{\mu(B_t - B_{s_1}) + (\mu + 1)(B_{s_1} - B_{s_2}) + \cdots + (\mu + n)B_{s_n}\}] \\ &\quad = \exp\{\varphi(\mu)(t - s_1) + \varphi(\mu + 1)(s_1 - s_2) + \cdots + \varphi(\mu + n)s_n\}.\end{aligned}$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

Therefore, we have

$$\int_0^\infty dt \exp(-\alpha t)\Phi_{n,t}(\mu)$$

$$= n! \int_0^\infty dt \exp(-\alpha t) \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \exp\{\varphi(\mu)(t-s_1) + \cdots + \varphi(\mu+n)s_n\}$$

$$= n! \int_0^\infty ds_n \exp -(\alpha - \varphi(\mu+n))s_n \int_{s_n}^\infty ds_{n-1} \exp -(\alpha - \varphi(\mu+n-1))(s_{n-1}-s_n)$$

$$\cdots \int_{s_1}^\infty dt \exp -(\alpha - \varphi(\mu))(t-s_1),$$

so that, in the case $\alpha > \varphi(\mu+n)$, we obtain formula (4.b) by integrating successively the $(n+1)$ exponential functions.

2. Next, we use the additive decomposition formula

$$\frac{1}{\prod_{j=0}^n (\alpha - \varphi(\mu+j))} = \sum_{j=0}^n c_j^{(\mu)} \frac{1}{(\alpha - \varphi(\mu+j))}$$

where $c_j^{(\mu)}$ is given as stated in the theorem, and we obtain, for $\alpha > \varphi(\mu+n)$:

$$\int_0^\infty dt \exp(-\alpha t)\Phi_{n,t}(\mu) = n! \sum_{j=0}^n c_j^{(\mu)} \int_0^\infty dt \exp(-\alpha t) \exp(\varphi(\mu+j)t),$$

a formula from which we deduce

$$\Phi_{n,t}(\mu) = n! \sum_{j=0}^n c_j^{(\mu)} \exp(\varphi(\mu+j)t) = n! \sum_{j=0}^n c_j^{(\mu)} \boldsymbol{E}[\exp(jB_t)\exp(\mu B_t)]$$

$$= \boldsymbol{E}[P_n^{(\mu)}(\exp B_t)\exp(\mu B_t)].$$

Hence, we have proved formula (4.c).

As a consequence of Theorem 1, we have the following.

Corollary 1. *For any $\lambda \in \mathbb{R}$, and any $n \in \mathbb{N}$, we have*

$$\lambda^{2n} \boldsymbol{E}\left[\left(\int_0^t du \exp(\lambda B_u)\right)^n\right] = \boldsymbol{E}[P_n(\exp \lambda B_t)] \tag{4.d}$$

where

$$P_n(z) = 2^n(-1)^n\left\{\frac{1}{n!} + 2\sum_{j=1}^n \frac{n!\,(-z)^j}{(n-j)!\,(n+j)!}\right\}. \tag{4.e}$$

Proof. Thanks to the scaling property of Brownian motion, it suffices to prove formula (4.d) for $\mu = 1$, and any $t \geqq 0$. In this case, we remark that formula (4.d) is then precisely formula (4.c) taken with $\mu = 0$, once the coefficients $c_j^{(0)}$ have been

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

identified as

$$c_0^{(0)} = (-1)^n \frac{2^n}{(n!)^2}; \qquad c_j^{(0)} = \frac{2^n(-1)^{n-j}\,2}{(n-j)!\,(n+j)!} \qquad (1 \leqq j \leqq n);$$

therefore, it now appears that the polynomial P_n is precisely $P_n^{(0)}$, and this ends the proof of (4.d).

It may also be helpful to write down explicitly the moments of $A_t^{(\nu)}$.

Corollary 2. *For any* $\lambda \in \mathbb{R}^*$, $\mu \in \mathbb{R}$, *and* $n \in \mathbb{N}$, *we have*

$$\text{(4.d}'\text{)} \qquad \lambda^{2n} \boldsymbol{E}\left[\left(\int_0^t du \exp \lambda(B_u + \mu u)\right)^n\right] = n! \sum_{j=0}^{n} c_j^{(\mu/\lambda)} \exp\left(\left(\frac{\lambda^2 j^2}{2} + \lambda j \mu\right)t\right).$$

In particular, we have, for $\mu = 0$,

$$\text{(4.d}''\text{)} \quad \lambda^{2n} \boldsymbol{E}\left[\left(\int_0^t du \exp \lambda B_u\right)^n\right] = n! \left\{\frac{(-1)^n}{(n!)^2} + 2 \sum_{j=1}^{n} \frac{(-1)^{n-j}}{(n-j)!\,(n+j)!} \exp \frac{\lambda^2 j^2 t}{2}\right\}.$$

Taking $\lambda = 2$ in formula (4.d), we obtain

$$\text{(4.f)} \qquad \boldsymbol{E}[A_t^n] = \frac{1}{4^n} \boldsymbol{E}[P_n(\exp 2B_t)].$$

A different computation of $\boldsymbol{E}[A_t^n]$ follows from Bougerol's identity:

$$\text{(1.d)} \qquad \text{for any } t \geqq 0, \qquad \gamma_{A_t} \stackrel{\text{(law)}}{=} \sinh(B_t),$$

from which we deduce

$$\text{(4.g)} \qquad \boldsymbol{E}[\gamma_1^{2n}] \boldsymbol{E}[A_t^n] = \boldsymbol{E}[(\sinh(B_t))^{2n}].$$

We now show that formulae (4.f) and (4.g) give the same value for $\boldsymbol{E}[A_t^n]$, thereby providing a partial checking of Theorem 1 and its corollary.

Lemma 1. *The identity*

$$\text{(4.h)} \qquad \frac{1}{4^n} \boldsymbol{E}[P_n(\exp 2B_t)] = \frac{\boldsymbol{E}[(\sinh B_t)^{2n}]}{\boldsymbol{E}(B_1^{2n})}$$

holds for all $n \in \mathbb{N}$, *and* $t \geqq 0$.

Proof. We write g_{2n} for $\boldsymbol{E}(B_1^{2n})$. Developing both sides of (4.h) with the help of the distribution of B_t, we see that (4.h) is equivalent to

$$\frac{1}{4^n} \int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2}{2t}\right) P_n(\exp 2x) = \frac{1}{g_{2n}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2}{2t}\right) (\sinh x)^{2n},$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

which is itself equivalent to

$$\text{(4.h}')\quad \begin{aligned}&\int_0^\infty dx \exp(-x^2/2t)(P_n(\exp(2x)) + P_n(\exp(-2x)))\\ &\quad = \frac{4^n}{g_{2n}} 2\int_0^\infty dx \exp(-x^2/2t)(\sinh x)^{2n}.\end{aligned}$$

From the injectivity of the Laplace transform, we deduce that (4.h′) is satisfied for all $t > 0$ if, and only if

$$\tfrac{1}{2}(P_n(\exp(2x)) + P_n(\exp(-2x))) = \frac{4^n}{g_{2n}}(\sinh x)^{2n},$$

or, equivalently, if we use the variable $y = \exp(2x)$,

$$\text{(4.i)}\quad \frac{1}{2}\left(P_n(y) + P_n\left(\frac{1}{y}\right)\right) = \frac{1}{g_{2n}}\left(y + \frac{1}{y} - 2\right)^n.$$

We now remark that

$$\left(y + \frac{1}{y} - 2\right)^n \equiv \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^{2n} \stackrel{\text{def}}{=} Q_n(y)$$

is given by the formula

$$\text{(4.j)}\quad Q_n(y) = \sum_{k=0}^{2n} C_{2n}^k y^{n-k}(-1)^k.$$

We now write

$$\text{(4.j}')\quad \begin{aligned} Q_n(y) &= \sum_{k=0}^{n-1} C_{2n}^k y^{n-k}(-1)^k + (-1)^n C_{2n}^n + \sum_{k=n+1}^{2n} C_{2n}^{n-(k-n)} \frac{(-1)^{n-(k-n)}}{y^{k-n}} \\ &= (-1)^n\left\{C_{2n}^n + \sum_{k=1}^{n} C_{2n}^{n-k}(-y)^k + \sum_{k=1}^{n} C_{2n}^{n-k}\left(-\frac{1}{y}\right)^k\right\}.\end{aligned}$$

On the other hand, the formula $g_{2n} = (2n)!/2^n n!$, which follows from $\boldsymbol{E}[\exp(\lambda B_1)] = \exp(\frac{1}{2}\lambda^2)$ by differentiation, is well known. Hence, the formulae (4.j′) and (4.e) imply that

$$\frac{1}{g_n} Q_n(y) = \frac{1}{2}\left(P_n(y) + P_n\left(\frac{1}{y}\right)\right),$$

which is precisely (4.i).

4.2. Although we do not want to sidetrack the reader with questions of secondary importance, it may be worthwhile to mention that the results of Theorem 1 can be extended to a large class of processes $(X_t, t \geqq 0)$ with independent increments, instead of $(B_t, t \geqq 0)$. We hope to develop such a general study in a

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

further paper, but we may remark here that, for $X_t = B_t^{(\nu)} \equiv B_t + \nu t$, with $\nu \geqq 0$, formulae (4.b) and (4.c) may be extended with the function $\varphi^{(\nu)}$ now defined to be $\varphi^{(\nu)}(\lambda) = \frac{1}{2}\lambda^2 + \nu\lambda$. In particular, for μ, $\nu \geqq 0$, there exist polynomials ${}^{(\nu)}P_n^{(\mu)}$ such that

$$(4.c)_\nu \quad \boldsymbol{E}\left[\left(\int_0^t ds \exp(B_s^{(\nu)})\right)^n \exp(\mu B_t^{(\nu)})\right] = \boldsymbol{E}[{}^{(\nu)}P_n^{(\mu)}(\exp B_t^{(\nu)}) \exp(\mu B_t^{(\nu)})].$$

Now, thanks to Girsanov's theorem, the above identity is equivalent to

$$(4.c)'_\nu \quad \boldsymbol{E}\left[\left(\int_0^t ds \exp(B_s)\right)^n \exp((\mu+\nu)B_t)\right] = \boldsymbol{E}[{}^{(\nu)}P_n^{(\mu)}(\exp B_t) \exp((\mu+\nu)B_t)]$$

and, comparing this with formula (4.c), we easily obtain the following.

Lemma 2. *For any $n \in \mathbb{N}$, μ, $\nu \geqq 0$, we have ${}^{(\nu)}P_n^{(\mu)} = P_n^{(\mu+\nu)}$.*

Proof. Comparing formulae $(4.c)'_\nu$ and (4.c) with the parameter μ replaced there by $(\mu+\nu)$, we obtain

$$\boldsymbol{E}[{}^{(\nu)}P_n^{(\mu)}(\exp B_t)] = \boldsymbol{E}[P_n^{(\mu+\nu)}(\exp B_t) \exp(\mu+\nu)B_t].$$

Denote $\lambda = \mu + \nu$. Now, a Laplace transform argument (in t) shows that the preceding equality is true if and only if

$$P(\rho)\rho^\lambda + P\left(\frac{1}{\rho}\right)\left(\frac{1}{\rho}\right)^\lambda = Q(\rho)\rho^\lambda + Q\left(\frac{1}{\rho}\right)\left(\frac{1}{\rho}\right)^\lambda, \qquad \rho > 0$$

where we have written P for ${}^{(\nu)}P_n^{(\mu)}$ and Q for $P_n^{(\mu+\nu)}$.

Now, letting $\rho \to \infty$, we obtain that the coefficients of highest degree (i.e. n) of P and Q are equal. Iterating this procedure, we show that the coefficients of any given degree k ($\leqq n$) of P and Q are equal, hence $P = Q$.

Remark. The identity shown in Lemma 2 could also be proved directly upon looking at the explicit formulae for ${}^{(\nu)}P_n^{(\mu)}$ and $P_n^{(\mu+\nu)}$ (see just below (4.c)). Indeed, it suffices to remark that

$$\varphi^{(\nu)}(\mu+x) - \varphi^{(\nu)}(\mu+y) \equiv (\mu+\nu)(x-y) = \varphi^{(\mu+\nu)}(x) - \varphi^{(\mu+\nu)}(y).$$

However, it also seemed interesting to develop the arguments of the above proof, which are not computational.

4.3. We now remark that the polynomials $P_n \equiv P_n^{(0)}$ are closely linked with certain hypergeometric polynomials. We first recall the definition of the hypergeometric function F with parameters α, β, γ:

$$F(\alpha, \beta, \gamma; z) = \sum_{z=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} z^k,$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

where

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} \equiv \lambda(\lambda + 1) \cdots (\lambda + k - 1)$$

(see Lebedev [8], p. 238). In particular, we have

$$F(-n, 1, n+1; z) = \sum_{j=0}^{n} \frac{(-n)_j (1)_j}{(n+1)_j} \frac{z^j}{j!} = \sum_{j=0}^{n} \frac{(n!)^2 (-z)^j}{(n+j)!\,(n-j)!}.$$

As a consequence, we deduce from formula (4.e) the following relation:

$$\text{(4.k)} \qquad P_n(z) = \frac{(-2)^n}{n!} \{2F(-n, 1, n+1; z) - 1\}.$$

4.4. Although in Section 4.1 above, we proved the identity

$$\text{(4.g)} \qquad \boldsymbol{E}[\gamma_1^{2n}]\boldsymbol{E}[A_t^n] = \boldsymbol{E}[(\sinh B_t)^{2n}], \quad \text{for any } n \in \mathbb{N},$$

it does not seem possible to deduce Bougerol's identity in law (1.d) directly from (4.g). Indeed, the law of $\sinh(B_t)$ is not determined by its moments, and, in fact, it is easy to exhibit some explicit distributions on $\mathbb{R}$ which have the same moments as $\sinh(B_t)$.

For simplicity, we take $t = 1$, and we recall (see Stoyanov [11], p. 89) that if N is a standard $\mathrm{N}(0, 1)$ random variable, then, for any ε with $0 < |\varepsilon| \leqq 1$, and any $p \in \mathbb{Z}^*$, $\exp(N)$ and $\exp(N_{(\varepsilon,p)})$ have the same positive and negative moments, i.e.

$$\boldsymbol{E}[\exp(kN)] = \boldsymbol{E}[\exp(kN_{(\varepsilon,p)})], \quad \text{for any } k \in \mathbb{Z},$$

if $\boldsymbol{P}(N_{(\varepsilon,p)} \in dx) \stackrel{\text{def}}{=} \boldsymbol{P}(N \in dx)(1 + \varepsilon \sin(\pi p x))$. Consequently, $\sinh(N)$ and $\sinh(N_{(\varepsilon,p)})$ also have the same positive and negative moments, although they have different distributions. However, we have not been able to show that $\sinh(N_{(\varepsilon,p)})$ can be represented in law as $\gamma_{A_{(\varepsilon,p)}}$, for some non-negative random variable $A_{(\varepsilon,p)}$ which is independent of the Brownian motion $(\gamma_t, t \geqq 0)$. Hence, the possibility of deducing (1.d) directly from (4.g) is not entirely ruled out, although it seems very unlikely.

4.5. Coming back to (4.c) and $(4.\mathrm{c})_\nu$, we give yet another example of such formulae, this time for the standard Cauchy process $(C_t, t \geqq 0)$, the law of which, as a process with homogeneous independent increments, is determined by

$$\boldsymbol{E}[\exp(i\lambda C_t)] = \exp(-t|\lambda|) \qquad (t \geqq 0, \lambda \in \mathbb{R}).$$

Computations similar to those developed in the proof of Theorem 1 yield the following very simple formula: for any $t \geqq 0$, and any $\mu \geqq 0$,

$$\text{(4.l)} \qquad \boldsymbol{E}\left[\left(\int_0^t ds \exp(iC_s)\right)^n \exp(i\mu C_t)\right] = \boldsymbol{E}[(1 - \exp(iC_t))^n \exp(i\mu C_t)].$$

5. The law of $A^{(\nu)}$ taken at an independent exponential time

5.1. We now consider S_θ, an exponentially distributed random variable with parameter $\frac{1}{2}\theta^2$, that is, $\boldsymbol{P}(S_\theta \in dt) = \frac{1}{2}\theta^2 \exp(-\frac{1}{2}\theta^2 t)\,dt$, which is assumed moreover to be independent of B.

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

A simple variant of the arguments used in Section 3 above will yield the following result.

Theorem 2. *We recall that, for* $\mu \geqq 0$, $p_t^\mu(a, \rho)\,d\rho$ *denotes the semigroup of the Bessel process of index* μ, *which is given by formula* (2.d).

1. *The joint law of* $(\exp(B_{S_\theta}), A_{S_\theta})$ *is given by*

$$P(\exp(B_{S_\theta}) \in d\rho; A_{S_\theta} \in du) = \frac{\theta^2}{2\rho^{2+\theta}} p_u^\theta(1, \rho)\,d\rho\,du. \tag{5.a}$$

2. *More generally, if* $\nu \in \mathbb{R}$, *we set*: $\lambda = (\theta^2 + \nu^2)^{1/2}$, *and we have*

$$P(\exp(B_{S_\theta}^{(\nu)}) \in d\rho; A_{S_\theta}^{(\nu)} \in du) = \frac{\theta^2}{2\rho^{2+\lambda-\nu}} p_u^\lambda(1, \rho)\,d\rho\,du. \tag{5.b}$$

Proof. 1. Consider $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ two Borel functions. We have

$$\begin{aligned}\Phi_\theta(f, g) &\stackrel{\text{def}}{=} E[f(\exp B_{S_\theta}) g(A_{S_\theta})] \\ &= \frac{\theta^2}{2} E\left[\int_0^\infty dt \exp\left(-\frac{\theta^2 t}{2}\right) f(\exp B_t) g(A_t)\right].\end{aligned}$$

Now, using the formulae (2.a), (2.e) and (2.d), we obtain

$$\begin{aligned}\Phi_\theta(f, g) &= \frac{\theta^2}{2} E\left[\int_0^\infty dH_u \exp\left(-\frac{\theta^2 H_u}{2}\right) f(\exp B_{H_u}) g(u)\right] \\ &= \frac{\theta^2}{2}\left[\int_0^\infty \frac{du}{\rho_u^2} \exp\left(-\frac{\theta^2 H_u}{2}\right) f(\rho_u) g(u)\right] \\ &= \frac{\theta^2}{2}\int_0^\infty du\, g(u) \int_0^\infty \frac{d\rho}{\rho^{2+\theta}} f(\rho) p_u^\theta(1, \rho),\end{aligned}$$

which implies formula (5.a).

2. More generally, we have

$$\begin{aligned}\Phi_\theta^{(\nu)}(f, g) &\stackrel{\text{def}}{=} E[f(\exp(B_{S_\theta}^{(\nu)})) g(A_{S_\theta}^{(\nu)})] \\ &= \frac{\theta^2}{2}\int_0^\infty dt \exp\left(-\frac{\theta^2 t}{2}\right) E[f(\exp B_t^{(\nu)}) g(A_t^{(\nu)})] \\ &= \frac{\theta^2}{2}\int_0^\infty dt \exp\left(-\frac{\theta^2 t}{2}\right) E\left[f(\exp B_t) g(A_t) \exp\left(\nu B_t - \frac{\nu^2 t}{2}\right)\right] \\ &= \frac{\theta^2}{(\theta^2+\nu^2)}\left(\frac{\theta^2+\nu^2}{2}\right)\int_0^\infty dt \exp\left(-\frac{\lambda^2 t}{2}\right) E[f(\exp B_t) g(A_t) \exp(\nu B_t)] \qquad (5.c) \\ &= \frac{\theta^2}{(\theta^2+\nu^2)} E[f(\exp B_{S_\lambda}) g(A_{S_\lambda}) \exp(\nu B_{S_\lambda})] \\ &= \frac{\theta^2}{2}\int_0^\infty du\, g(u) \int_0^\infty \frac{d\rho}{\rho^{\lambda+2}} \rho^\nu f(\rho) p_u^\lambda(1, \rho), \quad \text{from (5.a)},\end{aligned}$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

which implies formula (5.b). Note that in (5.c), we have used Girsanov's theorem to relate the law of $(B_u^{(\nu)}, u \leqq t)$ to that of $(B_u, u \leqq t)$.

From formula (5.b), we immediately obtain the conditional law of $A_{S_\theta}^{(\nu)}$ given $B_{S_\theta}^{(\nu)}$.

Corollary. We keep the notation used in Theorem 2. We have, for any $\nu \in \mathbb{R}$,

$$\text{(5.d)} \qquad \boldsymbol{P}(B_{S_\theta}^{(\nu)} \in dx) = \frac{\theta^2}{2\lambda} \exp(-\lambda\,|x| + \nu x)\, dx,$$

and

$$\text{(5.d}'\text{)} \qquad \boldsymbol{P}(A_{S_\theta}^{(\nu)} \in du \mid B_{S_\theta}^{(\nu)} = x) = \lambda \exp(-x + 2\lambda x^-) p_u^\lambda(1, e^x)\, du$$

where $x^- = (-x)1_{(x \leqq 0)}$.

Proof. The formula (5.d) may be obtained using Girsanov's theorem to reduce it to the case $\nu = 0$, in which case we find the well-known result

$$\boldsymbol{P}(B_{S_\lambda} \in dx) = \frac{\lambda}{2} \exp(-\lambda\,|x|)\, dx.$$

Once formula (5.d) has been obtained, formula (5.d′) follows from (5.b).

5.2. We now discuss the statements of Theorem 2 and its corollary in relation to the explicit description of the laws of last-passage times for certain transient diffusions on $\mathbb{R}_+$, and, in particular, Bessel processes with dimension $d > 2$.

The following general discussion may be found in Pitman and Yor [9], Section 6, Williams [14], Revuz and Yor [10] (Exercises (4.16), p. 298, and (1.16), p. 378).

We consider the canonical realization on $C(\mathbb{R}_+, \mathbb{R}_+)$ of a regular diffusion $(R_t, t \geqq 0; P_x, x \in \mathbb{R}_+)$ with infinite lifetime, and we suppose for simplicity that

$$\text{(5.e)} \qquad P_x(T_0 < \infty) = 0, \qquad x > 0,$$

$$\text{(5.f)} \qquad P_x\Big(\lim_{t\to\infty} R_t = \infty\Big) = 1, \qquad x > 0.$$

As a consequence of (5.e) and (5.f), a scale function s for this diffusion satisfies $s(0+) = -\infty$ and $s(\infty) < \infty$. We can therefore suppose that $s(\infty) = 0$. Let Γ be the infinitesimal generator of the diffusion, and take the speed measure m to be such that $\Gamma = \frac{1}{2}(d/dm)(d/ds)$. According to Itô and McKean ([5], p. 149), there exists a continuous function

$$p^{\cdot}: \begin{array}{l} (\mathbb{R}_+^*)^3 \to \mathbb{R}_+^* \\ (t, x, y) \to p_t(x, y) \end{array}$$

which is symmetric in x and y, and such that the semigroup P_t of the diffusion is given by

$$P_t(x, dy) = p_t(x, y) m(dy).$$

We can now state the following result.

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

Theorem 3 (Pitman and Yor [9], Theorem 6.1). *Let* (R_t, P_x) *b a regular diffusion on* $\mathbb{R}_+$, *which satisfies the hypotheses* (5.e) *and* (5.f). *Then*:

1. *for all* $a, b > 0$,

$$P_a(L_b \in dt) = \frac{-1}{2s(b)} p_t'(a, b)\, dt \qquad (t > 0), \tag{5.g}$$

2. *for* $a \leqq b$, *the formula* (5.g) *defines an infinitely divisible probability distribution on* $\mathbb{R}_+$,
3. *for* $a < b$, *we have*

$$P_b(T_a < \infty) \equiv P_b(L_a > 0) = \frac{s(b)}{s(a)} \quad \textit{and} \quad P_b(L_a \in dt; L_a > 0) = \frac{-1}{2s(a)} p_t'(b, a)\, dt,$$

so that

$$P_b(L_a \in dt \mid L_a > 0) = \frac{-1}{2s(b)} p_t'(a, b)\, dt = P_a(L_b \in dt).$$

As a consequence of Theorem 3, we may now interpret, after making some elementary computations concerning the speed measures and scale functions of Bessel processes, the corollary of Theorem 2 as follows.

Proposition 1. *We keep the notation used in Theorem* 2. *Moreover, we denote by* $L^{(\lambda)}(a, b)$ *the last-passage time in* b *of the Bessel process* $\rho^{(\lambda)}$ *starting from* a. *Then, for any* $\nu \in \mathbb{R}$, *the quantity*

$$\boldsymbol{P}(A_{S_\theta}^{(\nu)} \in du \mid B_{S_\theta}^{(\nu)} = x)$$

is equal to

$$\boldsymbol{P}(L^{(\lambda)}(1, e^x) \in du) \quad \textit{if} \quad x \geqq 0,$$

and to

$$\boldsymbol{P}(L^{(\lambda)}(1, e^x) \in du \mid L^{(\lambda)}(1, e^x) > 0) = \boldsymbol{P}(L^{(\lambda)}(e^x, 1) \in du) \quad \text{if} \quad x < 0.$$

5.3. We now show how the results in Theorem 2 and its corollary, together with Proposition 1, lead to the following result:

$$\text{for } \nu > 0, \qquad A_\infty^{(-\nu)} \underset{\text{(a)}}{\overset{\text{(law)}}{=}} \frac{1}{2Z_\nu} \underset{\text{(b)}}{\overset{\text{(law)}}{=}} L^{(\nu)}(0, 1), \tag{5.h}$$

where Z_ν denotes a standard gamma variable with parameter ν, i.e.

$$\boldsymbol{P}(Z_\nu \in dt) = \frac{t^{\nu-1}}{\Gamma(\nu)} \exp(-t)\, dt \qquad (t > 0)$$

(for a more direct derivation of (5.h) and related references, see [16]). Since $S_\theta \xrightarrow[\theta \to 0]{(P)} \infty$, it will be sufficient, in order to prove (5.h), to pass to the limit as $\theta \to 0$,

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

in formulae (5.d) and (5.d′). In fact, writing $\mu = 1/2\nu$, we shall show

$$(5.\mathrm{h}') \qquad (-\theta^2 B^{(-\nu)}_{S_\theta}; A^{(-\nu)}_{S_\theta}) \xrightarrow[\theta\to 0]{\text{(law)}} \left(e_\mu; \frac{1}{2Z_\nu}\right)$$

where e_μ and Z_ν are independent, and $\boldsymbol{P}(e_\mu \in dy) = \frac{1}{2}\nu \exp(-(y/2\nu))\,dy \ (y>0)$. The convergence

$$-\theta^2 B^{(-\nu)}_{S_\theta} \xrightarrow[\theta\to 0]{\text{(law)}} e_\mu$$

is immediate, since

$$-\theta^2 B^{(-\nu)}_{S_\theta} \overset{\text{(law)}}{=} (-\theta^2)(\sqrt{S_\theta}\, B_1 - \nu S_\theta), \quad \text{and} \quad \frac{\theta^2}{2} S_\theta \overset{\text{(law)}}{=} e_1.$$

Then, in order to prove ((5.h), (a)), it suffices to show that, for any function $f: \mathbb{R}_+ \to \mathbb{R}_+$, continuous, with compact support, and any $y>0$, the quantity

$$(5.\mathrm{d}'') \quad \boldsymbol{E}\left[f(A^{(-\nu)}_{S_\theta}) \mid B^{(-\nu)}_{S_\theta} = -\frac{y}{\theta^2}\right] \equiv \lambda \exp\left\{\frac{y}{\theta^2}(1+2\lambda)\right\} \int_0^\infty du\, p^\lambda_u\left(1, \exp\left(-\frac{y}{\theta^2}\right)\right) f(u)$$

converges, as $\theta \to 0$, towards

$$\int_0^\infty \frac{du}{\Gamma(\nu) 2^\nu u^{\nu+1}} \exp\left(-\frac{1}{2u}\right) f(u).$$

This is a consequence of the explicit formula (2.d) for $p^\lambda_u(1, \xi)$ and of the well-known equivalence result

$$I_\lambda(\xi) \sim \left(\frac{\xi}{2}\right)^\lambda \frac{1}{\Gamma(\lambda+1)}, \quad \text{as} \quad \xi \to 0,$$

uniformly as λ varies in a compact subset of $\mathbb{R}^*_+$. On the other hand, we shall prove ((5.h, (b)), or rather the identity

$$(5.\mathrm{i}) \qquad A^{(-\nu)}_\infty \overset{\text{(law)}}{=} L^{(\nu)}(0, 1)$$

with the help of Proposition 1. Indeed, from this proposition, we have, for any $y>0$, and any bounded continuous function $f: \mathbb{R}_+ \to \mathbb{R}_+$,

$$\begin{aligned} \boldsymbol{E}\left[f(A^{(-\nu)}_{S_\theta}) \mid B^{(-\nu)}_{S_\theta} = -\frac{y}{\theta^2}\right] &= \boldsymbol{E}\left[f\left(L^{(\lambda)}\left(1, \exp\left(-\frac{y}{\theta^2}\right)\right)\right) \,\middle|\, L^{(\lambda)}\left(1, \exp\left(-\frac{y}{\theta^2}\right)\right) > 0\right] \\ &= \boldsymbol{E}\left[f\left(L^{(\lambda)}\left(\exp\left(-\frac{y}{\theta^2}\right), 1\right)\right)\right] \end{aligned}$$

and, finally, letting $\theta \to 0$, the last-written quantity converges to $\boldsymbol{E}[f(L^{(\nu)}(0, 1))]$, thereby proving (5.i).

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

6. The law of A taken at a fixed time

6.1. Our aim in this section is to give a formula, which we would like to be as explicit as possible, for

$$P(A(t) \in du \mid B_t = x) \overset{\text{def}}{=} a_t(x, u)\, du. \tag{6.a}$$

This computation will be closely linked with the following form of the density of the Hartman–Watson distribution $\eta_r(du)$, which we defined above in (2.e′). We take the following formula from Yor ([15], p. 85):

$$I_0(r)\eta_r(du) = \theta_r(u)\, du, \quad \text{with} \quad \theta_r(u) = \frac{r}{(2\pi^3 u)^{\frac{1}{2}}} \exp\left(\frac{\pi^2}{2u}\right)\psi_r(u),$$

and

$$\psi_r(u) = \int_0^\infty dy \exp(-y^2/2u) \exp(-r(\cosh y))(\sinh y) \sin\left(\frac{\pi y}{u}\right). \tag{6.b}$$

Before getting any further into the computation of $a_t(x, u)$, we remark that, thanks to Girsanov's theorem relating the laws of $(B_s + \nu s; s \leqq t)$ and $(B_s; s \leqq t)$, we have

$$P(A_t^{(\nu)} \in du \mid B_t + \nu t = x) = a_t(x, u)\, du \tag{6.a′}$$

so that it is really sufficient to consider only the case $\nu = 0$.

6.2. The main result in this section is the following.

Proposition 2. *If we denote* $P(A_t \in du \mid B_t = x) = a_t(x, u)\, du$*, then we have*

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) a_t(x, u) = \frac{1}{u} \exp\left(-\frac{1}{2u}(1 + e^{2x})\right) \theta_{e^x/u}(t). \tag{6.c}$$

Proof. Consider $f, g: \mathbb{R}_+ \to \mathbb{R}_+$ two Borel functions; then, we have, on the one hand, for $\mu \geqq 0$:

$$E\left[\int_0^\infty dt \exp\left(-\frac{\mu^2 t}{2}\right) f(\exp B_t) g(A_t)\right]$$
$$= \int_0^\infty dt \exp\left(-\frac{\mu^2 t}{2}\right) \int_{-\infty}^\infty \frac{dx}{\sqrt{2\pi t}} f(e^x) \exp\left(-\frac{x^2}{2t}\right) \int_0^\infty du\, g(u) a_t(x, u),$$

by definition of $a_t(x, u)$; on the other hand, the same quantity is, by formula (5.a), equal to

$$\int_0^\infty du\, g(u) \int_0^\infty \frac{d\rho}{\rho^{\mu+2}} f(\rho) p_u^\mu(1, \rho)$$
$$= \int_{-\infty}^\infty dx \exp(-(\mu + 1)x) f(e^x) \int_0^\infty du\, g(u) p_u^\mu(1, e^x).$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

Comparing the two expressions we have just obtained, we see that

$$\int_0^\infty dt \exp -\frac{1}{2}\left(\mu^2 t + \frac{x^2}{t}\right) a_t(x, u) = \exp(-(\mu+1)x) p_u^\mu(1, e^x). \tag{6.d}$$

Now, using the explicit form of p^μ given by formula (2.d), and the definition of the function θ given above (6.b), the right-hand side of (6.d) may be written as

$$\frac{1}{u}\exp\left(-\frac{1}{2u}(1+e^{2x})\right)\int_0^\infty dt\, \theta_{e^x/u}(t) \exp\left(-\frac{\mu^2 t}{2}\right),$$

and, finally, formula (6.c) follows from (6.d) using the injectivity of the Laplace transform.

From formulae (6.b) and (6.c)), we now deduce an interesting expression for $\boldsymbol{E}[f(\exp B_t)g(A_t)]$, where $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ are two Borel functions.

Corollary. We have

$$\boldsymbol{E}[f(\exp B_t)g(A_t)] = c_t \int_0^\infty dy \int_0^\infty dv\, f(y) g\left(\frac{1}{v}\right) \exp\left(-\frac{v}{2}(1+y^2)\right) \psi_{yv}(t), \tag{6.e}$$

where

$$c_t = \frac{1}{(2\pi^2 t)^{\frac{1}{2}}} \exp(\pi^2/2t).$$

Proof. By definition of $a_t(x, u)$, we have

$$\begin{aligned}
\boldsymbol{E}[f(\exp B_t)g(A_t)] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^\infty dx \exp\left(-\frac{x^2}{2t}\right) \int_0^\infty du\, g(u) a_t(x, u) f(e^x) \\
&= \int_{-\infty}^\infty dx \int_0^\infty \frac{du}{u} \exp\left(-\frac{1}{2u}(1+e^{2x})\right) \theta_{e^x/u}(t) f(e^x) g(u) \quad \text{(from (6.c))} \\
&= c_t \int_{-\infty}^\infty dx \int_0^\infty \frac{du}{u} \exp\left(-\frac{1}{2u}(1+e^{2x})\right) \frac{e^x}{u} \psi_{e^x/u}(t) f(e^x) g(u) \\
&= c_t \int_0^\infty dy \int_0^\infty dv \exp\left(-\frac{v}{2}(1+y^2)\right) \psi_{yv}(t) f(y) g\left(\frac{1}{v}\right).
\end{aligned}$$

6.3. From the formulae (6.b) and (6.e), we obtain an 'explicit' expression for the density $\alpha_t(v)$ of A_t, i.e. $\boldsymbol{P}(A_t \in dv) = \alpha_t(v)\, dv$, which is nonetheless complicated.

Indeed, we deduce from (6.e) that, for any Borel function $g : \mathbb{R}_+ \to \mathbb{R}_+$, we have:

$$\boldsymbol{E}[g(A_t)] = c_t \int_0^\infty dv\, g\left(\frac{1}{v}\right) \exp\left(-\frac{v}{2}\right) \alpha_t(v), \tag{6.f}$$

where

$$\alpha_t(v) = \int_0^\infty dy\, \psi_{yv}(t) \exp\left(-\frac{vy^2}{2}\right) = \int_0^\infty dx \frac{1}{\sqrt{v}} \psi_{x\sqrt{v}}(t) \exp\left(-\frac{x^2}{2}\right).$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

We then found it interesting to check that, from (6.f), we are able (at least!) to recover the fact that $E[(A_t)^n] < \infty$, for any $n \in \mathbb{N}$. In order to prove this, it is sufficient to show that $\alpha_t(v) = O(v^k)$, as $v \to 0$, for any $k \in \mathbb{N}$, which, in turn, is implied by the property

$$\psi_r(t) = O(r^k), \quad \text{as} \quad r \to 0, \text{ for any } k \in \mathbb{N}. \tag{6.g}$$

Proof of (6.g). The key argument in the proof is the probabilistic representation of $\psi_r(t)$ as

$$\psi_r(t) = \frac{\sqrt{2\pi t}}{2i} E[\exp(-r\cosh(\sqrt{t}\,G))\{\exp\sqrt{t}(1+i\lambda)G - \exp\sqrt{t}(1-i\lambda)G\}]$$

where $\lambda = \pi/t$, and G is a standard N(0, 1) gaussian variable. Now, (6.g) follows from the fact that all derivatives of $\psi_r(t)$, with respect to r, when taken at $r = 0$, are equal to 0, which is a consequence of

$$E[(\cosh(\sqrt{t}\,G))^k\{\exp\sqrt{t}(1+i\lambda)G - \exp\sqrt{t}(1-i\lambda)G\}] = 0, \quad \text{for any } k \in \mathbb{N}. \tag{6.h}$$

In order to prove (6.h), we shall show that, for any $\theta = k\sqrt{t}$, with $k \in \mathbb{Z}$, the quantity

$$h(\theta) \stackrel{\text{def}}{=} E[\exp(\theta G)\{\exp\sqrt{t}(1+i\lambda)G - \exp\sqrt{t}(1-i\lambda)G\}]$$

is equal to 0. Indeed, for any $\theta \in \mathbb{R}$, we have

$$h(\theta) = \exp\tfrac{1}{2}(\theta + \sqrt{t}(1+i\lambda))^2 - \exp\tfrac{1}{2}(\theta + \sqrt{t}(1-i\lambda))^2$$

from which it follows easily, since $\lambda = \pi/t$, that $h(k\sqrt{t}) = 0$, for $k \in \mathbb{Z}$.

7. Some connections with hyperbolic Brownian motion

As announced in Section 1.2, we sketch here how we might deduce the joint law of

$$\left(A_t^{(v)} \equiv \int_0^t ds \exp 2(B_s + vs),\ B_t\right)$$

from the knowledge of the semigroup of the hyperbolic Brownian motion.

In order to compute the joint law of $(A_t^{(v)}, B_t)$ for fixed t, it is obviously sufficient to compute the conditional law $a_t^v(x, u)\,du$ of $A_t^{(v)}$, given $B_t = x$.

From the relation (6.a′), we deduce

$$a_t^v(x, u) = a_t(x + vt, u), \qquad du \text{ a.s.} \tag{7.a}$$

In the following, we shall compute (via a Laplace transform) $a_t^v(x, u)$ for $v = -\frac{1}{2}$, hence for all v's, thanks to (7.a). We note, for simplicity, $\tilde{a}(x, u)$ for $a_t^{-\frac{1}{2}}(x, u)$. Now, the semigroup associated with the hyperbolic Laplacian operator

$$\Delta = \frac{y^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

on Poincaré's half-plane is known to be [3], [6]:

$$p_t(z;l)=\frac{\sqrt{2}\exp\left(-\frac{t}{2}\right)}{(2\pi t)^{\frac{3}{2}}}\int_d^\infty r\exp\left(-\frac{r^2}{2t}\right)\frac{dr}{(\cosh r-\cosh d)^{\frac{1}{2}}} \tag{7.b}$$

where $d=d(z,l)$ is the hyperbolic distance between z and l.

A stochastic differential equation satisfied by the hyperbolic Brownian motion $((X_t,Y_t);t\geqq 0)$, starting at $(x,y)\in\mathbb{R}\times\mathbb{R}_+$, with infinitesimal generator $\boldsymbol{\Delta}$, is:

$$X_t=x+\int_0^t Y_s\,dU_s;\qquad Y_t=y+\int_0^t Y_s\,dB_s\equiv y\exp\left(B_t-\frac{t}{2}\right), \tag{7.c}$$

where U and B are two independent real-valued Brownian motions started at 0. Thanks to the independence of U and B, $(X_t,t\geqq 0)$ may also be represented as

$$X_t=x+\gamma(A_t^{(-\frac{1}{2})}),\quad\text{where}\quad A_t^{(-\frac{1}{2})}=\int_0^t ds\exp(2B_s-s) \tag{7.d}$$

and $(\gamma(u),u\geqq 0)$ is a one-dimensional Brownian motion starting at 0, which is independent of B. Taking $x=0$ and $y=1$, one obtains, for any Borel function $f:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$,

- on the one hand:

$$\begin{aligned}\boldsymbol{E}[f(X_t,Y_t)]&=\int_{\mathbb{R}\times\mathbb{R}_+}p_t(i;(h,k))f(h,k)\,dh\,dk\\&=\int_{\mathbb{R}^2}p_t\left(i;\left(h,\exp\left(x-\frac{t}{2}\right)\right)\right)\exp\left(x-\frac{t}{2}\right)f\left(h,\exp\left(x-\frac{t}{2}\right)\right)dh\,dx;\end{aligned}$$

- on the other hand, from (7.c) and (7.d):

$$\boldsymbol{E}[f(X_t,Y_t)]=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^\infty dx\exp\left(-\frac{x^2}{2t}\right)\int_0^\infty du\,\tilde{a}(x,u)\boldsymbol{E}\left[f\left(\gamma(u),\exp\left(x-\frac{t}{2}\right)\right)\right].$$

From those two expressions for $\boldsymbol{E}[f(X_t,Y_t)]$, we deduce

$$\begin{aligned}&p_t\left(i;\left(h,\exp\left(x-\frac{t}{2}\right)\right)\right)\exp\left(x-\frac{t}{2}\right)\\&=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\int_{-\infty}^\infty du\,\tilde{a}(x,u)\frac{1}{\sqrt{2\pi u}}\exp\left(-\frac{h^2}{2u}\right).\end{aligned} \tag{7.e}$$

Changing u to $(1/u)$ in the last integral makes it a Laplace transform (in $h^2/2$) from which $\tilde{a}(x,u)$ can—at least in theory—be deduced.

Acknowledgements

The author is grateful to Marc Chesney, Helyette Geman, Jean-Claude Gruet, Saul Jacka and Nicole El Karoui for their continued interest in this work and many

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms

stimulating discussions. J. C. Gruet (personal communication) checked that (7.e) and (6.e) agree, and also developed further computations linking Bessel processes and hyperbolic Brownian motions.

References

[1] BOUAZIZ, L., BRIYS, E. AND CROUHY, M. (1991) The pricing of forward-starting Asian options. Preprint.

[2] BOUGEROL, Ph. (1983) Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré* **19,** 369–391.

[3] GETOOR, R. K. (1961) Infinitely divisible probabilities on the hyperbolic plane. *Pacific J. Math.* **11,** 1287–1308.

[4] HARTMAN, P. AND WATSON, G. S. (1974) 'Normal' distribution functions on spheres and the modified Bessel functions. *Ann. Prob.* **2,** 593–607.

[5] ITÔ, K. AND MCKEAN, H. P. (1965) *Diffusion Processes and their Sample Paths.* Springer-Verlag, Berlin.

[6] KARPALEVICH, F. I., TUTUBALIN, V. N. AND SHUR, M. (1959) Limit theorems for the compositions of distributions in the Lobachevsky plane and space. *Theory Prob. Appl.* **4,** 399–402.

[7] KEMNA, A. G. Z. AND VORST, A. C. F. (1990) A pricing method for options based on average asset values. *J. Banking and Finance* **14,** 113–129.

[8] LEBEDEV, N. N. (1972) *Special Functions and their Applications.* Dover, New York.

[9] PITMAN, J. D. AND YOR, M. (1981) Bessel processes and infinitely divisible laws. in *Stochastic Integrals,* ed. D. Williams. Lecture Notes in Mathematics **851,** Springer-Verlag, Berlin.

[10] REVUZ, D. AND YOR, M. (1991) *Continuous Martingales and Brownian Motion.* Springer-Verlag, Berlin.

[11] STOYANOV, J. (1987) *Counterexamples in Probability.* Wiley, New York.

[12] VORST, A. C. F. (1990) Prices and hedge ratios of average exchange rate options. Preliminary report.

[13] WATSON, G. N. (1966) *A Treatise on the Theory of Bessel Functions.* 2nd paperback edn. Cambridge University Press.

[14] WILLIAMS, D. (1974) Path decomposition and continuity of local time for one-dimensional diffusions I. *Proc. London Math. Soc.* (3) **28,** 738–768.

[15] YOR, M. (1980) Loi de l'indice du lacet brownien et distribution de Hartman-Watson. *Z. Wahrscheinlichkeitsth.* **53,** 71–95.

[16] YOR, M. (1992) Sur certaines fonctionnelles exponentielles du mouvement brownien réel. *J. Appl. Prob.* **29,** 202–208.

This content downloaded from 132.174.255.86 on Fri, 03 Feb 2017 23:40:21 UTC
All use subject to http://about.jstor.org/terms