On the Distribution of Extended CIR Model

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Abstract

We provide a complete representation of the interest rate in the extended Cox-Ingersoll-Ross model. The model we consider is an extension of the traditonal Cox-Ingersoll-Ross model to the case where all the parameters are time-varying, and no specific relationship exists between them. The rate can be represented as the sum of a non-central chi-square random variable, and of a weighted series of central chi-square random variables, where all random variables are independent.

1 Introduction

We consider the extended Cox-Ingersoll-Ross term structure model (ECIR model), namely, the spot interest rate $r(t)$ is assumed to follow a squared Bessel process $\{r(t)\}_{t\geq 0}$ which satisfies the following stochastic differential equation:

$$\begin{cases} \mathrm{d}r(t) = (-b(t)r(t) + \theta(t))\,\mathrm{d}t + \sigma(t)\sqrt{r(t)}\,\mathrm{d}W(t); \\ r(0) = r_0 \geq 0, \end{cases} \tag{1.1}$$

where $b(t) \geq 0$, $\sigma(t) > 0$ and $\theta(t) \geq 0$ are time dependent continuous functions and W is a standard Wiener process. The Cox-Ingersoll-Ross term structure model (CIR model), was first introduced in Cox et al. (1985a), (1985b). In its original specification, the speed of mean reversion $b \geq 0$, the volatility $\sigma > 0$ and the parameter $\theta > 0$ are assumed constant.

Several features of the CIR model are particularly attractive. Firstly, it can be justified by general equilibrium considerations, see Cox et al (1985a). Secondly, the interest rate is always positive and stationary. Cox et al. found that its distribution follows a noncentral chi-square distribution. Finally, there is a closed form formula for the bond price. For practitioners however, the main shortcoming of the constant parameters version of the model is that it cannot reproduce the original term structure of interest rates. This fact was highlighted by several authors (Hull, 1990; Keller-Ressel and Steiner, 2008; Yang, 2006 and all the references therein): yield curves can be only normal, inverse, or humped.

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The extended CIR model, however, has enough parameters to be fitted to the original yield curve.

Maghsoodi (1996), Jamshidian (1995), and Rogers (1995) propose a representation of the extended CIR model as a sum of squares of Ornstein-Uhlenbeck processes when the dimension $d(t) \equiv 4\theta(t)/\sigma^2(t)$ is constant and integer. As a consequence, the interest rate follows the generalized chi-square distribution. Maghsoodi (1996), and Shirakawa (2002) also propose a representation of the interest rate as a time-changed lognormal process. However, as the latter author states, "it is difficult to derive the probability distribution of the squared Bessel processes with time-varying dimensions explicitly". Brigo and Mercurio (2006) state that no solution to that problem has been found.

In our paper, we determine this distribution explicitly. We show that, for each, the rate can be represented as the sum of a non-central chi-square random variable X_0, and of a weighted series of central chi-square random variables $\{X_1, X_2, \ldots\}$. The variables X_0, $X_1, X_2, \ldots$ are all independent. The random variable X_0 has a number of degrees of freedom equal to $d(0)$.When the dimension d is constant, the random variables $\{X_1, X_2, \ldots\}$ are all zero, and we recover the traditional result. Thus the latter random variables represent deviations of the rate from the constant dimension case.

There are alternate approaches to extend the CIR model. Brigo and Mercurio (2001) find that a deterministic shift of the CIR model is analytically tractable. The obvious drawback of making the parameters of the CIR model functions of time is the problem of overparameterization: the parameters will not be robust in a change of regime. Several authors consider instead a CIR model of interest rates with constant parameters and stochastic volatility. For instance, Longstaff and Schwartz (1992), and Duffie and Kan (1996), consider a generalized two-factor CIR model, where one factor is the interest rate, and the other one is its volatility. The volatility in that model is a variation of the volatility in the popular Heston (1993) model. Cotton et al. (2004) calculate an asymptotic expansion of the bond price in such a model (with constant parameters) when the speed of mean reversion is fast. Fouque and Lorig (2011) generalize this model to a model with a volatility of volatility.

Finally, we note that several authors have generalized the CIR model in a different way using multiple factors. We refer the reader to the references contained in Chen, Filipovic and Poor (2004) and Gourieroux and Monfort (2011).

In the same way that it is not too difficult to generalize the one-factor CIR model to multiple factors (see Duffie and Kan, 1996), we believe it is not difficult to generalize our results on the extended CIR model to multiple factors. However, we leave this for future research.

2 Characteristic Function of ECIR Model

Theorem 2.1 *For $t \geq 0$, the characteristic function of $r(t)$ is given by*

$$\mathbb{E}[e^{i\omega r(t)}] = \exp\left(i\omega\Big(\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)} + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(s,t)}\,\mathrm{d}s\Big)\right), \tag{2.1}$$

where

$$\Sigma(s,t) := \frac{1}{4}\int_s^t e^{-\int_v^t b(u)\,\mathrm{d}u}\sigma^2(v)\,\mathrm{d}v.$$

Proof. Let $\{X_t^{(\gamma)}\}_{t\geq 0}$ be a squared Bessel process with initial value $X_0^{(\gamma)} = x$. On one hand, from Proposition 3.4 in Carmona (1996), we derive the characteristic function of $X_t^{(\gamma)}$ as

$$\mathbb{E}[e^{i\omega X_t^{(\gamma)}}] = \exp\left(i\omega\Big(\frac{x}{1-2i\omega t} + \int_0^t \frac{\gamma(u)}{1-2i\omega(t-u)}\,\mathrm{d}u\Big)\right). \tag{2.2}$$

On the other hand, it is shown that $(r(t))_{t\geq 0}$ in (2.1) follows the same distribution as a squared Bessel process with time and state changes (see Lemma 2.4 and Corollary 3.1 in Shirakawa, 2002):

$$\{r(t)\}_{t\geq 0} \sim \left\{\nu(t)X_{\tau(t)}^{(\gamma)}\right\}_{t\geq 0}, \tag{2.3}$$

with

- The state change parameter $\nu(t) = \exp\big(-\int_0^t b(u)\,\mathrm{d}u\big)$.
- The time change parameter $\tau(t) = \frac{1}{4}\int_0^t \frac{\sigma^2(u)}{\nu(u)}\,\mathrm{d}u$.
- $\gamma(t) = \frac{4(\theta\circ\tau^{-1})(t)}{(\sigma^2\circ\tau^{-1})(t)}$, where $f\circ g$ denotes the composed function of f and g.
- $\{X_t^{(\gamma)}\}_{t\geq 0}$ denotes the squared Bessel process with time-varying dimension $\gamma(t)$ and initial value $X_0^{(\gamma)} = r_0$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}\mathbb{E}[e^{i\omega r(t)}] &= \mathbb{E}\big[e^{i\omega\nu(t)X_{\tau(t)}^{(\gamma)}}\big]\\ &= \exp\left(i\omega\nu(t)\Big(\frac{r_0}{1-2i\omega\nu(t)\tau(t)} + \int_0^{\tau(t)} \frac{4(\theta\circ\tau^{-1})(u)/(\sigma^2\circ\tau^{-1})(u)}{1-2i\omega\nu(t)(\tau(t)-u)}\,\mathrm{d}u\Big)\right).\end{aligned} \tag{2.4}$$

By the change of variable $u = \tau(s)$ and the fact that $\tau'(s) = \frac{1}{4}\frac{\sigma^2(s)}{\nu(s)}$,

$$\begin{aligned}&\int_0^{\tau(t)} \frac{4(\theta\circ\tau^{-1})(u)/(\sigma^2\circ\tau^{-1})(u)}{1-2i\omega\nu(t)(\tau(t)-u)}\,\mathrm{d}u = \int_0^t \frac{4\theta(s)/\sigma^2(s)}{1-2i\omega\nu(t)(\tau(t)-\tau(s))}\tau'(s)\,\mathrm{d}s\\ &= \int_0^t \frac{\theta(s)/\nu(s)}{1-2i\omega\nu(t)(\tau(t)-\tau(s))}\,\mathrm{d}s.\end{aligned} \tag{2.5}$$

Then plugging (2.5) into (2.4) leads to (2.1). ∎

Note that another proof of Theorem 2.1 by using the Fokker-Planck equation is provided in Liu, Peng and Schellhorn (2013).

3 Probability Density of ECIR Model

In the following theorem we derive the transition probability density of $r(t)$, starting from $r(0) = r_0$. Define

$$d(t) = \frac{4\theta(t)}{\sigma(t)^2}$$

and assume $d \in C^1(0, \infty)$.

Theorem 3.1 *Let $r(t)$ satisfy Equation (1.1), then there is a sequence of independent random variables $\{X_0, X_{0,N}, \ldots, X_{N-1,N},\ N \geq 1\}$ verifying*

$$X_0 \sim \chi^2_{d(0)}\Big(\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{\Sigma(0,t)}\Big);$$
$$X_{j,N} \sim \chi^2_{\frac{t}{N}d'(\frac{jt}{N})} \text{ for } j = 0, \ldots, N-1;$$

and

$$\Sigma(0,t)X_0 + \sum_{j=0}^{N-1} \Sigma\Big(\frac{jt}{N}, t\Big) X_{j,N} \xrightarrow[N\to\infty]{\text{in distribution}} r(t), \tag{3.1}$$

where we denote by $\chi^2_q(\lambda)$ $(\lambda \geq 0)$ the noncentral chi-square distribution with $q \geq 0$ (q is not necessarily an integer) degrees of freedom and by χ^2_q with $q \geq 0$ a central chi-square distribution with $q \geq 0$ degrees of freedom.

A straightforward result of Theorem 3.1 is: the solution of Equation (1.1) is distributed as the limit of sum of independent scaled chi-square random variables. More precisely, the following statement holds true:

Corollary 3.2 *For $t \geq 0$, the random variable $r(t)$ in (1.1) has probability density (with support $(0, +\infty)$)*

$$f_{r(t)}(z) = \lim_{N\to\infty} h * h_{0,N} * \ldots * h_{N-1,N}(z), \tag{3.2}$$

where $f * g$ denotes the convolution of f, g and the functions $\{h, h_{j,N} : j = 0, 1, \ldots, N-1\}$ are defined by: with support $x > 0$,

$$\begin{aligned} h(x) :=& \frac{1}{2\Sigma(0,t)} e^{-\frac{r_0 \exp(-\int_0^t b(u)\,\mathrm{d}u) + x}{2\Sigma(0,t)}} \Big(\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{x}\Big)^{1/2 - d(0)/4} \\ &\times I_{d(0)/2-1}\Big(\frac{r_0 \exp(-\int_0^t b(u)\,\mathrm{d}u)x}{\Sigma(0,t)^2}\Big)^{1/2} \end{aligned} \tag{3.3}$$

and

$$h_{j,N}(x) := \frac{(x/(2\Sigma(jt/N,t)))^{\frac{t}{2N}d'(\frac{jt}{N})-1}\, e^{-\frac{x}{2\Sigma(jt/N,t)}}}{2\Sigma(jt/N,t)\Gamma\left(\frac{t}{2N}d'(\frac{jt}{N})\right)}, \tag{3.4}$$

where

- I_α, *with* α *being integer or positive, is the modified Bessel function of the first kind defined by (see e.g. Abramowitz and Stegun, 1965)*

$$I_\alpha(x) := \sum_{l=0}^{\infty} \frac{(\frac{x}{2})^{2l+\alpha}}{l!\Gamma(l+\alpha+1)}.$$

Now we prove Theorem 3.1.

Proof. First we observe that, by using integration by parts,

$$\begin{aligned}&\frac{1}{2}\int_0^t d'(s)\log\left(1-2i\omega\Sigma(s,t)\right)\,\mathrm{d}s\\&= -\frac{d(0)}{2}\log\left(1-2i\omega\Sigma(0,t)\right) - i\omega\int_0^t \frac{d(s)e^{-\int_s^t b(u)\,\mathrm{d}u}\sigma^2(s)/4}{1-2i\omega\Sigma(s,t)}\,\mathrm{d}s.\end{aligned} \tag{3.5}$$

Since the mappings $s \mapsto d'(s)$ and $s \mapsto \Sigma(s,t)$ are continuous, we use (2.1), (3.5) and the Riemann sum representation of integral to obtain

$$\begin{aligned}\mathbb{E}[e^{i\omega r(t)}] &= \frac{\exp\left(\frac{i\omega r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)} - \frac{1}{2}\int_0^t d'(s)\log\left(1-2i\omega\Sigma(s,t)\right)\,\mathrm{d}s\right)}{(1-2i\omega\Sigma(0,t))^{d(0)/2}}\\
&= \frac{\exp\left(\frac{i\omega r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)}\right)}{(1-2i\omega\Sigma(0,t))^{d(0)/2}}\cdot e^{-\frac{1}{2}\int_0^t d'(s)\log(1-2i\omega\Sigma(s,t))\,\mathrm{d}s}\\
&= \frac{\exp\left(\frac{i\omega r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)}\right)}{(1-2i\omega\Sigma(0,t))^{d(0)/2}}\left(\lim_{N\to\infty} e^{-\frac{1}{2}\sum_{j=0}^{N-1}\frac{t}{N}d'(\frac{jt}{N})\log(1-2i\omega\Sigma(\frac{jt}{N},t))}\right)\\
&= \lim_{N\to\infty}\frac{\exp\left(\frac{i\omega r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)}\right)}{(1-2i\omega\Sigma(0,t))^{d(0)/2}}\left(\prod_{j=0}^{N-1}\left(1-2i\omega\Sigma(\frac{jt}{N},t)\right)^{-\frac{t}{2N}d'(\frac{jt}{N})}\right).\end{aligned} \tag{3.6}$$

The first factor on the right-hand side of (3.6):

$$\frac{\exp\left(\frac{i\omega r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{1-2i\omega\Sigma(0,t)}\right)}{(1-2i\omega\Sigma(0,t))^{d(0)/2}}$$

is known as the characteristic function of a scaled noncentral chi-square distribution with noncentral parameter $\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{\Sigma(0,t)}$, $d(0)$ degrees of freedom and scaling parameter $\Sigma(0,t)$, i.e., its corresponding random variable is $\Sigma(0,t)X_0$ with

$$X_0 \sim \chi^2_{d(0)}\Big(\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{\Sigma(0,t)}\Big). \tag{3.7}$$

For the second part on the right-hand side of (3.6), we observe that for $j = 0,1,\dots,N-1$, $\omega \mapsto (1-2i\omega\Sigma(\frac{jt}{N},t))^{-\frac{t}{2N}d'(\frac{jt}{N})}$ is the characteristic function of the random variable $\Sigma(\frac{jt}{N},t)X_{j,N}$, with

$$X_{j,N} \sim \chi^2_{\frac{t}{N}d'(\frac{jt}{N})}. \tag{3.8}$$

Therefore Theorem 3.1 follows from (3.6), (3.7), (3.8) and the Lévy-Cramér continuity theorem. ■

4 An approximation formula of the density function of $r(t)$

In this section, we derive an efficient approximation formula of the explicit probability density of $r(t)$. The result is the following theorem.

Theorem 4.1 *Let $N \geq 2$ and*

$$f_N = h * h_{0,N} * \ldots * h_{N-1,N}. \tag{4.1}$$

We have for $z > 0$,

$$f_N(z) = \int_0^z \frac{1}{2\Sigma(0,t)} e^{-\frac{r_0\exp(-\int_0^t b(u)\,\mathrm{d}u)+x}{2\Sigma(0,t)}} \Big(\frac{r_0 e^{-\int_0^t b(u)\,\mathrm{d}u}}{x}\Big)^{1/2-d(0)/4}$$
$$\times I_{d(0)/2-1}\Big(\frac{(r_0x)^{1/2}\exp(-\frac{1}{2}\int_0^t b(u)\,\mathrm{d}u)}{\Sigma(0,t)}\Big)\Big(\sum_{k=0}^{\infty} b_{k,N}\psi_{k,N}(z-x)\Big)\,\mathrm{d}x, \tag{4.2}$$

where

- *for $N \geq 2$ and $k \in \mathbb{N}$,*

$$b_{k,N} = a_{k,N}\prod_{j=1}^{N-1}\big(\frac{\Sigma(0,t)}{\Sigma(\frac{jt}{N},t)}\big)^{\frac{t}{2N}d'(\frac{jt}{N})}$$

  and the sequence $\{a_{k,N}\}_{k\geq 0}$ satisfies

$$a_{k,2} = \Big(\frac{t}{4}d'(\frac{t}{2})\Big)^{(k)}\Big(1-\frac{\Sigma(0,t)}{\Sigma(\frac{t}{2},t)}\Big)^k k!, \text{ for } k \in \mathbb{N}$$

and

$$a_{k,N} = \sum_{l=0}^{k} A_{l,N}^{(N-1)} \Big(\frac{t}{2N} d'(\frac{(N-1)t}{N}) \Big)^{(k-l)} \Big(1 - \frac{\Sigma(0,t)}{\Sigma(\frac{(N-1)t}{N}, t)} \Big)^{k-l} (k-l)!$$

for $N \geq 3$ and $k \in \mathbb{N}$, where $(m)^{(r)} = m(m+1)\ldots(m+r-1)$ denotes the rising factorial.

- $\psi_{k,N}$ *is the density function of the Gamma distribution with shape parameter $\sum_{j=0}^{N-1} \frac{t}{2N} d'(\frac{jt}{N}) + k$ and scale parameter $2\Sigma(0,t)$:*

$$\psi_{k,N}(x) = \frac{x^{\sum_{j=0}^{N-1} \frac{t}{2N} d'(\frac{jt}{N}) + k - 1} e^{-\frac{x}{2\Sigma(0,t)}}}{(2\Sigma(0,t))^{\sum_{j=0}^{N-1} \frac{t}{2N} d'(\frac{jt}{N}) + k} \Gamma\big(\sum_{j=0}^{N-1} \frac{t}{2N} d'(\frac{jt}{N}) + k \big)}. \tag{4.3}$$

Proof. In view of Moschopoulos and Canada (1984), the probability density f_N in (4.1) is given as

$$f_N(z) = \Big(\prod_{j=1}^{N-1} \Big(\frac{\Sigma(0,t)}{\Sigma(\frac{jt}{N}, t)} \Big)^{\frac{t}{2N} d'(\frac{jt}{N})} \Big) \sum_{k=0}^{\infty} a_{k,N} \psi_{k,N}(z), \tag{4.4}$$

where

- the sequence $\{a_{k,N}\}_{k \geq 0, N \geq 2}$ solves

$$\prod_{k=2}^{N} \Big(\sum_{r=0}^{\infty} (\frac{t}{2N} d'(\frac{(k-1)t}{N}))^{(r)} \big(1 - \frac{\Sigma(0,t)}{\Sigma(\frac{(k-1)t}{N}, t)} \big)^r r! x^{-r} \Big) = \sum_{k=0}^{\infty} a_{k,N} x^{-k},$$

  Equivalently (we refer to (2.3), (2.5) in Moschopoulos and Canada, 1984), for $N \geq 2$, $a_{k,N} = A_{k,N}^{(N)}$, with $\{A_{k,N}^{(p)}\}_{k \geq 0, p \geq 2}$ being defined by

$$A_{k,N}^{(2)} = \Big(\frac{t}{2N} d'(\frac{t}{N}) \Big)^{(k)} \Big(1 - \frac{\Sigma(0,t)}{\Sigma(\frac{t}{N}, t)} \Big)^k k!, \text{ for } k \in \mathbb{N}$$

  and

$$A_{k,N}^{(p)} = \sum_{l=0}^{k} A_{l,N}^{(p-1)} \Big(\frac{t}{2N} d'(\frac{(p-1)t}{N}) \Big)^{(k-l)} \Big(1 - \frac{\Sigma(0,t)}{\Sigma(\frac{(p-1)t}{N}, t)} \Big)^{k-l} (k-l)!$$

  for $p \geq 3$ and $k \in \mathbb{N}$.

- The expression of $\psi_{k,N}$ in (4.3) results from (2.4) in Moschopoulos and Canada (1984).

Finally (4.2) follows from (3.3) and (2.6) in Moschopoulos and Canada (1984). ■

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