Proof of HJM No-arbitrage Condition for Extended CIR Model

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1 Proof

The generalized extended CIR model we proposed is

$$dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)\sqrt{R(t)}d\tilde{W}(t).$$

where all the model parameters are time-dependent, and $\tilde{W}(t)$ is a Brownian modtion under a risk-neutral probability measure $\tilde{\mathbb{P}}$.

Let bond price $B(t,T) = f(t, R(t))$. The discounted bond price should be martingale under risk neutral measure, therefore

$$f_t(t,r) + (a(t) - b(t)r)f_r(t,r) + \frac{1}{2}\sigma(t)^2 r f_{rr}(t,r) = rf(t,r). \tag{1.1}$$

This is equation (6.5.4) from Shreve's book.

The zero-coupon bond price under affine model is

$$B(t,T) = e^{-R(t)C(t,T)-A(t,T)}. \tag{1.2}$$

Substitute equation (1.2) into equation (1.1), we have

$$[(-C'(t,T) + b(t)C(t,T) + \frac{1}{2}\sigma(t)^2 C(t,T)^2 - 1)r - A'(t,T) - aC(t,T)]f(t,r) = 0.$$

where $C'(t,T) = \frac{\partial}{\partial t}C(t,T)$ and $A'(t,T) = \frac{\partial}{\partial t}A(t,T)$.

The above equation should hold for every r, and we obtain the SDEs

$$\begin{cases} C'(t,T) = b(t)C(t,T) + \frac{1}{2}\sigma(t)^2 R(t) C(t,T)^2 - 1, \\ A'(t,T) = -aC(t,T). \end{cases} \tag{1.3}$$

Differentiate both sides by T, and we yield

$$\begin{cases} \frac{\partial}{\partial T}C'(t,T) = b(t)\frac{\partial}{\partial t}C(t,T) + \sigma(t)^2 R(t) C(t,T)\frac{\partial}{\partial T}C(t,T), \\ \frac{\partial}{\partial T}A'(t,T) = -a\frac{\partial}{\partial T}C(t,T). \end{cases} \tag{1.4}$$

A general affine interest rate model is

$$dR(t) = \beta(t, R(t))dt + \gamma(r, R(t))d\tilde{W}(t)$$

Following section 10.3.5 of Shreve's book, the HJM no-arbitrage condition (Shreve's Theorem 10.3.1) for affine model is equivalent to the validation of

$$\frac{\partial}{\partial T}C(t,T)\beta(t,R(t))+R(t)\frac{\partial}{\partial T}C'(t,T)+\frac{\partial}{\partial T}A'(t,T)=\left(\frac{\partial}{\partial T}C(t,T)\right)C(t,T)\gamma(t,R(t))^2. \tag{1.5}$$

Substitute equation (1.4) into the left-hand side of equation (1.5). We have

$$\begin{aligned}
\frac{\partial}{\partial T}C(t,T)\beta(t,R(t)) \quad &+ \quad R(t)\frac{\partial}{\partial T}C'(t,T)+\frac{\partial}{\partial T}A'(t,T) \\
&= \quad \frac{\partial}{\partial T}C(t,T)(a(t)-b(t)R(t)) \\
&\quad\quad +R(t)[b(t)\frac{\partial}{\partial t}C(t,T)+\sigma(t)^2C(t,T)\frac{\partial}{\partial T}C(t,T)]-a\frac{\partial}{\partial T}C(t,T) \\
&= \quad R(t)\sigma(t)^2C(t,T)\frac{\partial}{\partial T}C(t,T) \\
&= \quad \left(\frac{\partial}{\partial T}C(t,T)\right)C(t,T)\gamma(t,R(t))^2.
\end{aligned}$$

Hence we completed the proof of HJM no-arbitrage condition for extended CIR model.