1 Definitions

ω_c: Cutoff Frequency
$\omega_{c,h}$: High Cutoff Frequency (Bandpass)
$\omega_{c,l}$: Low Cutoff Frequency (Bandpass)
ω_0: Center Frequency
B_p: $\omega_{c,h} - \omega_{c,l}$
sbd_{dB}: Stopband Deviation in Decibles
pbd_{dB}: Passband Deviation in Decibles
G_0: Passband Gain
$F(s)$: Filter Equation
n: Filter Order
k_f: Scaling Factor for the Cuttoff Frequency
Δ: Difference / Derivative
x^*: Complex Conjugate of x
N: Window Length

1.1 Group Delay

$$\frac{\Delta p}{\Delta f}$$

Where $-180 \leq p \leq 180$ degrees and f in Hertz

2 Analog Filters

$$\text{Narrower Transition} \downarrow \quad \begin{matrix} \text{Bessel} \\ \text{Butterworth} \\ \text{Chebyshev} \\ \text{Elliptic} \end{matrix} \quad \uparrow \text{More Linear Passband Phase}$$

2.1 Bessel

Choose n then create as follows:

$$F(s) = \frac{\theta_n(0)}{\theta_n(s)}$$

$$\theta_n(x) = \sum_{k=0}^{n} \frac{(2n-k)!}{2^{n-k}k!(n-k)!} x^k$$

2.2 Butterworth

Choose n and G_0 then create as follows:

$$F(s) = \frac{G_0}{B_n(\frac{s}{w_c})}$$

For even n:

$$B_n(x) = \prod_{k=1}^{n/2} x^2 - 2cos(\frac{\pi(2k+n-1)}{2n})x + 1$$

For odd n:

$$B_n(x) = (x+1) \prod_{k=1}^{(n-1)/2} x^2 - 2cos(\frac{\pi(2k+n-1)}{2n})x + 1$$

2.2.1 Chebyshev

Choose G_0, n, ω_c, and pbd_{dB} then create as follows:

$$F(s) = \frac{G_0 X_n(s)}{V_n(s)}$$

$$\epsilon = \sqrt{10^{\frac{-pbd_{dB}}{10}} - 1}$$

$$\alpha = \frac{1}{\epsilon} + \sqrt{1 + \epsilon^{-2}}$$

$$A = \frac{\alpha^{\frac{1}{n}} - \alpha^{\frac{-1}{n}}}{2}$$

$$B = \frac{\alpha^{\frac{1}{n}} + \alpha^{\frac{-1}{n}}}{2}$$

For even n:

$$p_k = \omega_c(Acos(\frac{(2k-1)\pi}{2n}) + jBsin(\frac{(2k-1)\pi}{2n}))$$

$$V_n(s) = \prod_{k=1}^{n/2} s^2 + s(p_k + p_k^*) + p_k p_k^*$$

$$X_n(s) = 10^{-pbd_{dB}/20} \prod_{k=1}^{n/2} p_k p_k^*$$

For odd n:

$$p_k = \omega_c(Acos(\frac{k\pi}{n}) + jBsin(\frac{k\pi}{n}))$$

$$V_n(s) = (s + A\omega_c) \prod_{k=1}^{(n-1)/2} s^2 + s(p_k + p_k^*) + p_k p_k^*$$

$$X_n(s) = A\omega_c \prod_{k=1}^{(n-1)/2} p_k p_k^*$$

Side Note:
Chebyshev Polynomials:

$$T_n(x) = \begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \end{cases} = \begin{cases} cos(ncos^{-1}(x)), |x| \leq 1 \\ cosh(ncosh^{-1}(x)), x \geq 1 \\ -1^n cosh(ncosh^{-1}(-x)), x \leq -1 \end{cases}$$

$$|F(s)| = \sqrt{\frac{1}{1 + \epsilon^2 T_n^2(\frac{\omega}{\omega_c})}}$$

2.2.2 Elliptic / Cauer

2.2.3 Inverse Chebyshev / Type II Chebyshev

Choose G_0, n, ω_c, and sbd_{dB} then create as follows:

$$F(s) = \frac{G_0 X_n(s)}{V_n(s)}$$

$$A_s = 10^{\frac{sbd_{dB}}{10}}$$

$$\epsilon = \sqrt{\frac{A_s}{1 - A_s}}$$

$$u = \frac{sinh^{-1}(\frac{1}{\epsilon})}{n}$$

$$z_k = j\omega_c sec(\frac{(2k-1)\pi}{2n})$$

$$p_k = \frac{\omega_c}{sinh(u)sin(\frac{(2k-1)\pi}{2n}) + jcosh(u)cos(\frac{(2k-1)\pi}{2n})}$$

For even n:

$$V_n(s) = \prod_{k=1}^{k=n/2} s^2 + s(p_k p_k^*) + p_k p_k^*$$

$$X_n(s) = \prod_{k=1}^{k=n/2} \frac{p_k p_k^*}{z_k z_k^*}(s^2 + s(z_k + z_k^*) + z_k z_k^*)$$

For odd n:

$$V_n(s) = (s + p_n) \prod_{k=1}^{k=(n-1)/2} s^2 + s(p_k p_k^*) + p_k p_k^*$$

$$X_n(s) = \sqrt{p_n p_n^*} \prod_{k=1}^{k=(n-1)/2} \frac{p_k p_k^*}{z_k z_k^*}(s^2 + s(z_k + z_k^*) + z_k z_k^*)$$

2.3 Filter Transformations

2.3.1 Frequency Scaling

Replace s with $\frac{s}{k_f}$

2.3.2 Lowpass to Highpass

Replace s with $\frac{1}{s}$

2.3.3 Lowpass to Bandpass

Replace s with $\frac{s^2+\omega_0^2}{B_p s}$

2.3.4 Lowpass to Bandstop

Replace s with $\frac{B_p s}{s^2+\omega_0^2}$

2.3.5 Lowpass to Notch

Notch is synonymous with 'Bandstop' and 'Band-reject'
Replace s with $\frac{B_p s}{s^2+\omega_0^2}$

3 Digital Filters

3.1 Bilinear Transformation

Transforms any analog filter into a IIR digital filter. Phase response will differ from the analog version.
Form desired analog filter then replace s as follows: $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$
where T is the sample rate and z^{-1} is the previous data point

3.2 Finite Impulse Response (FIR)

No feedback and linear phase

3.2.1 Window Method

$F(n) = [Window] \times [Impulse\ Type]$

Window	**Peak Sidelobe**	**Main Lobe Width**	**Peak Approximation Error**
Rectangular	0 dB	2 π/N	Large
Hanning	-31.5 dB	2 π/N	Moderate
Hamming	-42.7 dB	2.2 π/N	Small
Blackman	-58.1 dB	3 π/N	Very Small

Window	**Equation**
Rectangular	0
Hanning	$sin^2(\frac{n\pi}{N})$
Hamming	$0.54 - 0.46cos(\frac{2\pi n}{N})$
Blackmann	$0.42 - 0.5cos(\frac{2\pi n}{N}) + 0.08cos(\frac{4\pi n}{N})$

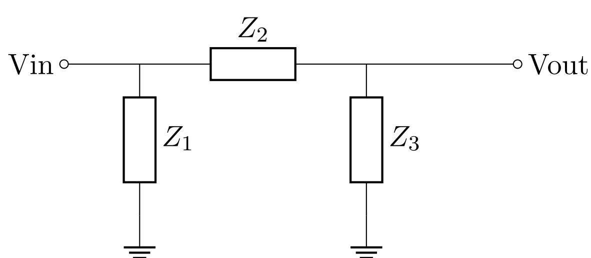
Impulse Type	**Equation**
Low Pass	$\frac{\omega_c}{\pi} sinc(\frac{n\omega_c}{\pi})$
High Pass	$sinc(n) - \frac{\omega_c}{\pi} sinc(\frac{n\omega_c}{\pi})$
Band Pass	$\frac{\omega_{c,h}}{\pi} sinc(\frac{n\omega_{c,h}}{\pi}) - \frac{\omega_{c,l}}{\pi} sinc(\frac{n\omega_{c,l}}{\pi})$

4 Implementations

4.1 Passive

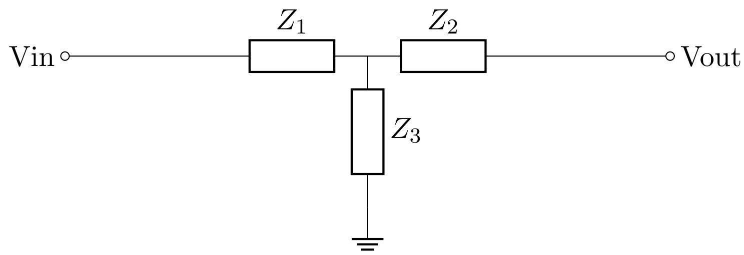
4.1.1 Pi Filter

$\frac{V_{OUT}}{V_{IN}} =$



4.1.2 T FIlter

$\frac{V_{OUT}}{V_{IN}} =$



4.2 Active

4.2.1 Sallen-Key

$\frac{V_{OUT}}{V_{IN}} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_3(Z_1+Z_2) + Z_3 Z_4}$
For Lowpass: Z_1, Z_2 become resistors and Z_3, Z_4 become capacitors
For Highpass: Z_3, Z_4 become resistors and Z_1, Z_2 become capacitors

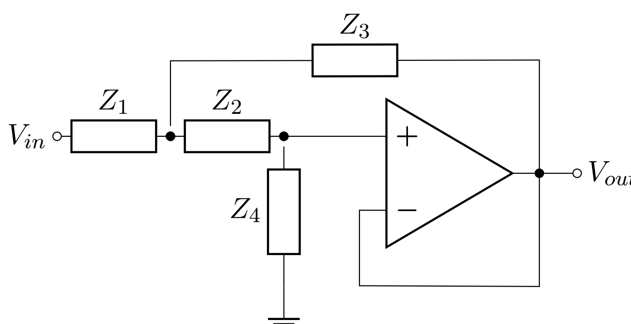


Figure 1: Sallen-Key Generic