Chapter 1

Literature Review and Methodological Foundations

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1.0.1 Skellam distribution: construction and moments

Let $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Pois}(\mu)$ independent. The difference $D = X - Y$ has the Skellam pmf

$$(D = d) = e^{-(\lambda+\mu)} \left(\frac{\lambda}{\mu}\right)^{d/2} I_{|d|}\big(2\sqrt{\lambda\mu}\big), \qquad d \in \mathbb{Z}, \tag{1.1}$$

with $I_\nu(\cdot)$ the modified Bessel function of the first kind. Using pgfs and coefficient extraction yields (**??**). Its mean/variance are

$$[D] = \lambda - \mu, \qquad (D) = \lambda + \mu. \tag{1.2}$$

It is often convenient to reparametrize by $(\mu_D, \sigma_D^2) = (\lambda - \mu, \lambda + \mu)$ with $\lambda = \frac{1}{2}(\sigma_D^2 + \mu_D)$ and $\mu = \frac{1}{2}(\sigma_D^2 - \mu_D)$.

Gaussian limit. For large (λ, μ) with fixed (μ_D, σ_D^2), the Skellam distribution approaches $\mathcal{N}(\mu_D, \sigma_D^2)$, motivating probit links between spreads and win probability.

1.0.2 Stern's spread-to-win map: full derivation

Assume the realized margin M satisfies $M = \mu + \varepsilon$ with $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. If the posted spread is p (favorite $-p$), then

$$(\text{favorite wins}) = (M > 0) = \Phi\Big(\frac{\mu}{\sigma}\Big), \qquad (\text{favorite covers}) = \Phi\Big(\frac{\mu - p}{\sigma}\Big). \tag{1.3}$$

Under efficiency for the mean $\mu \approx p$ we get the classical approximation $(\text{win}) \approx \Phi(p/\sigma)$. Empirically we estimate σ with a probit regression of win indicators on posted spreads; we report $\hat{\sigma}$ by season/era.

1.0.3 Key-number reweighting as constrained projection

Let $q(d)$ be a baseline integer pmf (Skellam or discretized Gaussian) for the margin D and let $\mathcal{K} = \{3, 6, 7, 10\}$. We seek nonnegative weights $\{w_d\}$ s.t. $\tilde{q}(d) = w_d q(d)$ is a pmf matching empirical key masses $\{m_k\}_{k\in\mathcal{K}}$:

$$\begin{aligned}
\min_{\{w_d\geq 0\}} \quad & \sum_{k\in\mathcal{K}} \big(w_k q(k) - m_k\big)^2 && (1.4)\\
\text{s.t.} \quad & \sum_d w_d q(d) = 1, \quad \sum_d d\, w_d q(d) = \mu_D, \quad \sum_d (d-\mu_D)^2 w_d q(d) = \sigma_D^2.
\end{aligned}$$

The last two constraints preserve mean/variance so reweighting changes shape, not location/scale. In matrix form, (**??**) is a small convex QP; if only normalization and key masses are imposed it has a closed form via Lagrange multipliers. We use $\tilde{q}$ for teaser/middle pricing in Chapter **??**.

1.1 Paired-comparison and Dynamic Rating Models

1.1.1 Spread–total dependence via Gaussian copula

Let $(U, V) \sim \mathrm{Cop}_\rho$ be a Gaussian copula with correlation ρ: $(Z_1, Z_2) \sim \mathcal{N}(\mathbf{0}, 1\rho$ $\rho 1)$, $U = \Phi(Z_1)$, $V = \Phi(Z_2)$. Let M be margin and T the total with marginal CDFs F_M, F_T. Define $(M, T) = \big(F_M^{-1}(U),\, F_T^{-1}(V)\big)$. Then for teaser legs $A = \{M > p_1\}$ and $B = \{T > q_1\}$,

$$(A \cap B) = \iint 1\{F_M^{-1}(u) > p_1,\ F_T^{-1}(v) > q_1\}\, c_\rho(u, v)\, u\, v,$$

where c_ρ is the Gaussian copula density. We estimate ρ by rank correlation on historical (M, T) and evaluate $(A\cap B)$ by quasi-MC. This yields teaser/SGP prices under calibrated dependence.

1.1.2 Edgeworth and saddlepoint tail refinement

Let M be integer margin with mean μ_D, variance σ_D^2, standardized $Z = (M - \mu_D)/\sigma_D$, skewness γ_1 and kurtosis γ_2. The Edgeworth approximation to $(M \leq m)$ is

$$\Phi(z) + \phi(z)\left(\frac{\gamma_1}{6}(z^2-1) + \frac{\gamma_2}{24}(z^3-3z) + \frac{\gamma_1^2}{72}(z^5-10z^3+15z)\right),$$

$z = (m + \frac{1}{2} - \mu_D)/\sigma_D$ (continuity-corrected). For lattice accuracy at extreme tails we also use the saddlepoint approximation with cumulant generator $K(t) = \log[e^{tM}]$:

$$(M = m) \approx \frac{1}{\sqrt{2\pi K''(\hat{t})}} \exp\big(K(\hat{t}) - \hat{t}\, m\big), \quad \text{with } K'(\hat{t}) = m.$$

1.1.3 Restricted EM for Skellam under key constraints

Let D_i be observed margins and (λ, μ) the Skellam parameters. Define a pseudo-complete representation with latent (X_i, Y_i) s.t. $D_i = X_i - Y_i$, $X_i \sim \text{Pois}(\lambda)$, $Y_i \sim \text{Pois}(\mu)$. The E-step computes $[X_i \mid D_i]$ and $[Y_i \mid D_i]$ via Bessel identities; the M-step sets

$$\lambda^{\text{new}} = \frac{1}{n} \sum_i [X_i \mid D_i], \qquad \mu^{\text{new}} = \frac{1}{n} \sum_i [Y_i \mid D_i].$$

To enforce key masses $\tilde{q}(k) = m_k$ $(k \in \mathcal{K})$, project (λ, μ) after the M-step onto the feasible set $\{(\lambda, \mu) : \sum_{d \in \mathbb{Z}} w_d(\lambda, \mu) q(d) = 1,\ \tilde{q}(k) = m_k\}$.

1.1.4 Bradley–Terry / Thurstone Foundations

Let each team i have latent skill θ_i. The Bradley–Terry model gives

$$P(i \text{ beats } j) = \frac{\exp(\theta_i)}{\exp(\theta_i) + \exp(\theta_j)} = \text{logit}^{-1}(\theta_i - \theta_j).$$

While elegant, this static model cannot capture temporal dynamics or strength drift.

1.1.5 Elo and Margin-Aware Variants

Rating systems such as Elo provide a pragmatic alternative with online updates after each game. We summarize connections between Elo updates and Bayesian posteriors under logistic likelihoods, and discuss margin-aware adjustments that temper updates in blowouts and amplify them in close, high-information games.

1.1.6 Harville's Linear Mixed Models for NFL

Harville (1980) models score margin Y_{ij} as

$$Y_{ij} = \alpha_i - \alpha_j + \delta\, \text{HFA}_{ij} + \mathbf{x}_{ij}^\top \beta + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2),$$

with α as random effects and shrinkage (BLUP) estimation. (**?**) This approach is interpretable and computationally efficient, but lacks temporal adaptability.

1.1.7 Glickman–Stern State-Space Model

Glickman & Stern (1998) model latent strengths $\theta_{i,t}$ evolving via

$$\theta_{i,t} = \gamma\, \theta_{i,t-1} + \eta_{i,t}, \quad \eta_{i,t} \sim N(0, \tau^2),$$

and margins:

$$Y_{ij,t} = (\theta_{i,t} - \theta_{j,t}) + \delta\, \text{HFA}_{ij,t} + \varepsilon_{ij,t}, \quad \varepsilon_{ij,t} \sim N(0, \sigma^2).$$

They link spread to win probability via the normal approximation $P(\text{win}) = \Phi(p/\sigma)$. (**??**) Strengths: credible intervals, smooth drift, extensions for covariates. Weaknesses: Gaussian residuals may misfit extreme margins, MCMC scaling, linear covariate assumptions.

Stochastic Volatility Extensions. Some works allow σ_t to vary via

$$\log \sigma_t = \phi \log \sigma_{t-1} + \zeta_t,$$

which captures heteroskedasticity across games or seasons.

Inference Details. Two complementary estimators are considered: (i) conjugate Kalman filtering/smoothing for linear-Gaussian special cases; (ii) MCMC or particle methods when the observation model departs from normality. We examine the trade-offs between exact filtering speed and the fidelity of heavy-tailed observation models for margins.

1.2 Score / Margin Distributions

1.2.1 Independent Poisson + Dixon-Coles Adjustment

Model goals/scores as independent Poissons with rates λ, μ. Dixon & Coles (1997) introduce an adjustment $\tau(x, y; \rho)$ to correct low-score dependence:

$$P(X = x, Y = y) \propto \tau(x, y; \rho) \,\text{Poisson}(x; \lambda) \,\text{Poisson}(y; \mu).$$

This improves fit in tightly contested, low-score games. (**?**)

1.2.2 Bivariate Poisson via Shared Component

Karlis & Ntzoufras model

$$X = U + Z, \; Y = V + Z,$$

with U, V, Z independent Poissons. The shared Z introduces positive covariance: $\text{Cov}(X, Y) = \mathbb{E}[Z]$. (**?**) This structure better captures correlated scoring (e.g. fast score pace, momentum).

1.2.3 Dynamic Bivariate Poisson Models

Koopman, Lit & Lucas (2015) evolve scoring intensities over time in a state-space framework, using simulation-based filtering to handle the non-Gaussian likelihood. (**?**) This allows adaptation of attack/defense parameters mid-season.

Mapping Spreads to Win Probability. In practice, decision makers often require an implied win probability. Under a symmetric margin model with variance σ^2, a spread p can be mapped via $\Phi(p/\sigma)$ (**?**). We catalog historical σ estimates by season and demonstrate calibration drift when market regimes change.

1.2.4 Skellam Margins and Key-Number Reweighting

For margins $D = X - Y$, where X, Y are Poisson, D follows a Skellam distribution:

$$P(D = d) = e^{-(\lambda+\mu)} \left(\frac{\lambda}{\mu}\right)^{d/2} I_{|d|}(2\sqrt{\lambda\mu}).$$

In the NFL, margins concentrate on key numbers (3,6,7,10). We reweight the Skellam/Poisson PMF to match empirical key-number frequencies before feeding into teaser/middle simulations (see Chapter **??**).

1.2.5 Zero-Inflated and Hurdle Models

To capture structural zeros (e.g. exceptionally low scoring conditions), we outline zero-inflated/hurdle extensions that mix a point mass with Poisson components. These models improve fit for rare outcomes without distorting the mainstream distribution.

1.2.6 Proper scoring rules and reliability

For Bernoulli Y, a scoring rule $S(p, Y)$ is *proper* if $[S(p^\star, Y)] \le [S(p, Y)]$ with equality iff $p = p^\star$. Log-loss and Brier are strictly proper; minimizing the empirical score with a correctly specified model yields calibrated p. Reliability diagrams visualize $[Y \mid p \in \text{bin}]$ versus bin center; we include binomial Wilson intervals to show uncertainty.

1.3 Calibration, Scoring & Uncertainty

1.3.1 Scoring Rules

We evaluate models by:

- **Brier score:** $\frac{1}{N}\sum(p_i - y_i)^2$
- **Log-loss:** $-\frac{1}{N}\sum[y_i \log p_i + (1 - y_i)\log(1 - p_i)]$
- **Reliability diagrams, ECE:** partition probabilities into bins and check empirical frequency

1.3.2 Uncertainty Quantification

Classical Bayesian/state-space models give posterior predictive distributions by default. For ML models, we will estimate predictive intervals via:

- Bootstrapping over training subsets
- Quantile regression layers
- Ensemble variance

We propagate these intervals into staking decisions: bets with wide uncertainty may be filtered or heavily downweighted.

1.3.3 Evaluation Protocols

Temporal cross-validation, blocked by week and season, avoids leakage from future injuries and market moves. We report Brier score decomposition (reliability, resolution, uncertainty) and reliability diagrams with equal-frequency bins. For margin distributions we report CRPS and PIT histograms to check sharpness and calibration simultaneously.

1.3.4 Robustness Checks

We test sensitivity to era definitions (pre- and post-rule changes), outlier handling (wins above the 99th percentile), and class imbalance between favorites and underdogs. Where necessary, we employ robust losses and quantile calibration to maintain stability.

1.4 Machine Learning Models in NFL Prediction

1.4.1 Feature Sets and Interactions

Key feature families include:

- Efficiency metrics: EPA/play, success rate (offense, defense, by down/distance)
- Play-calling: PROE (pass rate over expected), pace (sec/play), pass vs run splits
- Trench indicators: pressure allowed/created, stuff rate, line yards proxies
- Roster & injuries: QB status, adjusted games lost (AGL), starters out
- Environmental: weather (wind, rain, temp), turf/grass, altitude
- Market microstructure: implied probability, hold, line-move delta, cross-book spreads (CBV)

We use ML (e.g. gradient boosting, neural nets) to capture nonlinear interactions among these features, stacking with classical model outputs as base features.

Feature Interactions and Shifts. We devote special attention to interaction effects (e.g. weather by pass rate, injuries by team form) and to covariate shift between early and late season. Drift monitors track the distribution of CBV, EPA, and pace to trigger recalibration.

1.4.2 Regularization, Calibration & Robustness

We guard against overfitting via:

- Time-based cross-validation (rolling windows)
- Strong regularization (ridge, lasso, elastic net)
- Probability calibration (Platt scaling, isotonic regression) on held-out data
- Ensemble bootstraps and variance reduction

1.5 Reinforcement Learning for Betting

1.5.1 MDP Formulation for Betting

We treat each potential bet (pre-game or intra-game) as a step in an MDP:

$$s_t = (\text{model predictions}, \text{market state}, \text{bankroll}, \text{time}), \quad a_t \in \{\text{no bet}, \text{stake bucket}\}, \quad r_t = \text{PnL}\,(\text{or utility}$$

Actions can include correlated bets across markets (spread + total) or hedges.

1.5.2 RL Algorithms and Offline Training

We experiment with:

- **DQN / Q-learning**: discretized stake buckets, value iteration + experience replay
- **PPO / Actor–Critic**: continuous or stochastic stake policies, clipped updates, entropy regularization
- **Uncertainty-aware gating**: suppress stakes when posterior CI is wide (e.g. if variance too high)

We train offline (historical seasons) and optionally refine online via simulated paper-trading episodes.

1.5.3 Off-Policy Evaluation

Before deploying a learned policy, we estimate its value via inverse-propensity scoring, weighted importance sampling, and doubly robust estimators. We discuss variance control via self-normalization and clipping, and how model-based simulators can bias OPE if mis-specified.

1.6 Betting Market Theory & Microstructure

1.6.1 Economics of Wagering Markets

Sauer (1998) surveys the structure and efficiency of wagering markets, including bookmaker margins, bettor behavior models, and informational asymmetries. (?) Levitt (2004) argues bookmakers sometimes exploit bettor biases (e.g. overbetting favorites) rather than purely balancing books. (?)

1.6.2 Closing-Line Efficiency and Biases

We review evidence that the closing line aggregates information efficiently on average, yet exhibits pockets of bias around key numbers and popular teams. Behavioral patterns (favorite-longshot bias, recency effects) appear in subsets of the market and motivate features that measure retail pressure and line velocity.

1.6.3 Cross-Market Dependence

Spreads, totals, and moneylines are not independent. We discuss correlation structures induced by shared latent team strength and tempo, and implications for correlated parlays and hedging.

1.6.4 Market as Signal and Benchmark

We treat the market (closing lines) as both:

- A performance benchmark: our models must outperform or capture CLV (closing line value) edge
- A feature: cross-book spreads, line velocity, implied vs model delta, push rules

We define **Comparative Book Value (CBV)** as the difference between our fair probability and implied market probability; large CBV signals potential mispricing worth a bet.

1.7 Design Synthesis and Implications

From the literature, our design principles are:

1. Use Bayesian / shrinkage models to generate priors and uncertainty bounds.
2. Use discrete margin / score distributions (bivariate Poisson + reweight) to price spreads, totals, teasers.
3. Use ML meta-models to absorb nonlinear interactions among features.

Table 1.1: Qualitative comparison of modeling families referenced in Chapter **??**.

Model	Uncertainty	Scalability	Interpretability	Deployment notes
Harville LMM	analytic posteriors	high	high	rapid updates; assumes Gaussian
Glickman–Stern	full posterior	moderate	medium	MCMC cost; strong priors clarify
Dixon–Coles	low-score tweaks	high	high	quick to recalibrate key numbers
Bivariate Poisson	parametric draws	moderate	medium	handles correlation; needs careful
ML ensembles	ensemble variance	high	low	monitor drift via SHAP / feature
RL policy	bootstrap + MC	moderate	medium	gating via CVaR; compute intensi

4. Use RL to convert edges into action sequences under risk constraints.

5. Leverage the market as both a signal and benchmark; bet only when CBV passes threshold.

1.8 Annotated Reading List

Add figure: empirical vs model margin PMF reweighted at key integers (3,6,7,10).

We provide brief annotations of representative works that inform the hybrid design:

- **?**: Linear mixed models for NFL margins; interpretable and fast with shrinkage via BLUP. Serves as a reliable baseline and prior.

- **?**: State-space dynamics for team strength with Bayesian inference; enables credible intervals and smooth drift handling.

- **?**: Mapping spread to win probability via normal approximation; practical bridge between margin and moneyline pricing.

- **?**: Independent Poisson with low-score adjustment; cornerstone for discrete score modeling in low-scoring sports.

- **?**: Bivariate Poisson with shared component to capture correlation; essential for teaser/parlay risk modeling.

- **?**: Dynamic Poisson intensities with simulation-based filtering; template for time-varying scoring rates.

- **?**: Random-forest win probability at play-level; illustrates ML gains and calibration considerations for football.

- **?**: Economics of wagering markets; frames bookmaker margins, bettor behavior, and informational asymmetry.

- **?**: Bookmaker objectives and bettor biases; motivates microstructure features and bias-aware evaluation.

- **?**: Collective wisdom of lines; closing prices as an efficient benchmark with pockets of inefficiency.
- **?**: Time-zone effects on performance; supports travel/rest features in predictive models.
- **?**: Bayesian hierarchical football models; demonstrates full-probabilistic inference benefits for uncertainty quantification.

1.9 Canonical Works Integrated

We explicitly compare and implement: Harville (1980), Glickman–Stern (1998), Stern spread mapping (1991), Dixon–Coles (1997), Karlis–Ntzoufras (2003), Koopman dynamic Poisson (2015), Lock & Nettleton (2014), Sauer (1998), Levitt (2004). Implementation, ablation, and critique will occur in Chapter **??**.

1.10 Classical vs Modern: A Comparative Synthesis

Classical models provide structure, interpretability, and tractable uncertainty, while modern ML models absorb nonlinear interactions and idiosyncrasies that generative assumptions miss. The hybrid approach leverages classical models for priors and calibration discipline, layering ML for residual structure and using RL to translate edges into actions under explicit constraints. This division of labor prevents ML from overfitting low-signal regimes and keeps decision-making grounded in uncertainty.

1.10.1 When Classical Wins

In data-scarce or rapidly shifting regimes (e.g., early season, injury turbulence), shrinkage and state-space models dominate due to better calibrated uncertainty and temporal smoothing. Their transparent parameters support operational overrides.

1.10.2 When ML Wins

With stable covariates and rich features (market microstructure, team-form interactions), ML ensembles produce sharper probabilities. Calibration layers (Platt, isotonic) restore reliability while preserving sharpness.

1.10.3 Bridging to Decision Value

Sharpness without calibration harms staking; calibration without sharpness limits EV. We therefore co-optimize for proper scoring rules and enforce economic gates

(CBV thresholds, variance caps) to realize value.

1.11 From Score Distributions to Strategy

Discrete score distributions support actionable constructs: teaser planning around key numbers, middle opportunities when line drift occurs, and hedges conditioned on joint outcomes. Reweighting Skellam/Poisson mass at integers aligns simulated legs with observed push probabilities, preventing systematic teaser mispricing. The bivariate Poisson's shared component parameter governs correlation risk across legs; governance caps adjust as this parameter rises.

1.12 Calibration Theory and Scoring Rules

We review proper scoring rules for probability forecasts (log-loss, Brier) and for full distributions (CRPS), highlighting the trade-off between calibration and sharpness. We discuss reliability diagrams with binning bias corrections and isotonic/probit calibration approaches.

1.13 Mapping Models to Decision Value

We connect statistical metrics to economic outcomes by modeling the relationship between error distributions and realized CLV under market friction. This motivates optimizing for CRPS/log-loss while monitoring EV degradation from slippage and limits.

1.14 Market Efficiency and Bias Tests

We outline simple tests for favorite-longshot bias, key-number mispricing, and cross-book arbitrage signals, emphasizing multiple-testing corrections and robust standard errors.

1.15 Synthesis and Open Questions

The surveyed literature illustrates a continuum from interpretable generative models to flexible discriminative and sequential decision methods. Open challenges include (i) reconciling calibration with sharpness under distribution shift, (ii) integrating market microstructure without double-counting information, and (iii) handling multi-objective trade-offs between growth and risk in an operational setting. We outline how the hybrid approach in later chapters addresses these in a modular way that eases future extensions.

1.16 Related Work Beyond Football

Insights from other sports transfer imperfectly but inform modeling choices, particularly for low-scoring games (soccer, hockey) where Poisson-type models excel and for sequential decision domains (basketball substitutions, baseball bullpen management) where RL ideas have matured. We adapt ideas on tempo, possession value, and injury priors to the NFL context.

1.17 Extended Notes on Calibration

Calibration is both a statistical and an operational concern. A predictor can be perfectly calibrated and yet economically uninteresting if it lacks resolution; conversely, extremely sharp predictions can be economically harmful if they are miscalibrated. We therefore emphasize a portfolio of diagnostics: reliability diagrams with uncertainty bands, calibration slope/intercept for binary outcomes, PIT histograms for distributions, and CRPS to integrate sharpness and calibration into a single score. We also highlight the practical benefits of over-conservative probability outputs in risk-constrained decision problems.

1.18 Liquidity, Limits, and Execution

Modeling performance cannot be divorced from execution. Liquidity varies by book, time to kickoff, and market type. We discuss how to translate an estimated edge into an executable stake given posted limits and depth, and the implications for policy evaluation when some recommended bets cannot be filled at quoted prices. Execution-aware evaluation reduces optimism from paper backtests and promotes policies that scale gracefully.

1.19 Teasers and Parlays

Teasers (point adjustments for changed odds) and parlays (multiple legs) are common strategy components. We interpret them through the lens of joint distributions and correlation risk. A teaser can be attractive when key-number probabilities are underpriced; correlated parlays can be rational when the joint distribution assigns high mass to specific co-movements (e.g. low totals and underdogs). We caution that naive independence assumptions can be severely misleading.