$\boxed{\text{Bessel differential equ}^n}$ – $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-p^2)y=0 \Rightarrow p\geq 0$

We know $x=0$ is a RSP, so we can use Frobenius method

Let $y(x)=\sum_{n=0}^{\infty} a_n x^{n+\sigma}$, $\Rightarrow y'(x)=\sum_{n=0}^{\infty}(n+\sigma)a_n x^{n+\sigma-1}$,

$y''(x)=\sum_{n=0}^{\infty}(n+\sigma-1)(n+\sigma)a_n x^{n+\sigma-2}$ $\Rightarrow$ plug these into given equ.

$$\Rightarrow x^2\left[\sum_{n=0}^{\infty}(n+\sigma-1)(n+\sigma)a_n x^{n+\sigma-2}\right] + x\left[\sum_{n=0}^{\infty}(n+\sigma)a_n x^{n+\sigma-1}\right] + (x^2-p^2)\sum_{n=0}^{\infty}a_n x^{n+\sigma}=0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+\sigma-1)(n+\sigma)a_n x^{n+\sigma} + \sum_{n=0}^{\infty}(n+\sigma)a_n x^{n+\sigma} + \sum_{n=0}^{\infty}a_n x^{n+\sigma+2} - p^2\sum_{n=0}^{\infty}a_n x^{n+\sigma}=0$$

$\Rightarrow$ Making all exp to $x^{n+\sigma}$

$$\Rightarrow \sum_{n=0}^{\infty}(n+\sigma-1)(n+\sigma)a_n x^{n+\sigma} + \sum_{n=0}^{\infty}(n+\sigma)a_n x^{n+\sigma} + \sum_{n=2}^{\infty}a_{n-2}x^{n+\sigma} - p^2\sum_{n=0}^{\infty}a_n x^{n+\sigma}=0$$

$\Rightarrow$ Making all series start at $n=2$

$$\Rightarrow (\sigma-1)(\sigma)a_0x^{\sigma} + \sigma(\sigma+1)a_1x^{\sigma+1} + \sum_{n=2}^{\infty}(n+\sigma-1)(n+\sigma)a_n x^{n+\sigma} + \sigma a_0 x^{\sigma} +$$
$$(\sigma+1)a_1x^{\sigma+1} + \sum_{n=2}^{\infty}(n+\sigma)a_n x^{n+\sigma} + \sum_{n=2}^{\infty}a_{n-2}x^{n+\sigma} - p^2a_0x^{\sigma} - p^2a_1x^{\sigma+1} -$$
$$p^2\sum_{n=2}^{\infty}a_n x^{n+\sigma}=0$$

$\Rightarrow$ Equating the coefficients x^{σ}, $x^{\sigma+1}$, $x^{n+\sigma}$ for $n\geq 2$

$\Rightarrow x^{\sigma}$:- $(\sigma-1)\sigma a_0 + \sigma a_0 - p^2a_0 = 0$

$\Rightarrow \sigma^2a_0 - p^2a_0=0 \Rightarrow a_0\neq 0 \Rightarrow (\sigma^2-p^2)=0 \Rightarrow \sigma=\pm p.$

$\Rightarrow \sigma_1=p,\ \sigma_2=-p$

$\Rightarrow$ for first sol, as usual using the larger root $\sigma_1 = p$ (96)

$\Rightarrow x^{\sigma+1}$: $\sigma(\sigma+1)a_1 + (\sigma+1)a_1 - p^2 a_1 = 0$

$$\Rightarrow \left((\sigma+1)^2 - p^2\right) = 0$$

$\Rightarrow$ if p is an integer, we have following two cases

(1) if $p = 0 \Rightarrow$ equal roots

(2) if $p \neq 0 \Rightarrow \sigma_1 \neq \sigma_2$ but $\sigma_1 - \sigma_2 = k \in \mathbb{Z}$ (difference of roots is an integer.

$\Rightarrow$ if p is not an integer (excluding $p = 1/2$) $\Rightarrow \sigma_1 \neq \sigma_2$ and $\sigma_1 - \sigma_2 \notin \mathbb{Z}$

$\Rightarrow$ First let's do $\boxed{p \text{ is not an integer (excluding } p = 1/2)}$.

$\Rightarrow x^{\sigma+1}$:- $\sigma(\sigma+1)a_1 + (\sigma+1)a_1 - p^2 a_1 = 0$

$$= \left((\sigma+1)^2 - p^2\right) a_1 = 0$$

with the $\sigma_1 = p$ or $\sigma_2 = -p$ we can't have above equation equal to zero.

$\therefore$ we have $a_1 = 0$

$\Rightarrow x^{\sigma+n}$:- $(n+\sigma-1)(n+\sigma)a_n + (n+\sigma)a_n + a_{n-2} - p^2 a_n = 0$

$$\Rightarrow (n+\sigma)^2 a_n + a_{n-2} - p^2 a_n = 0$$

$$\Rightarrow a_n = \frac{-a_{n-2}}{(n+\sigma)^2 - p^2}, \quad n \geq 2$$

$$\Rightarrow a_n = \frac{-a_{n-2}}{(n+\sigma+p)(n+\sigma-p)}, \quad n \geq 2.$$

$$\Rightarrow \quad a_2 = \frac{-a_0}{(2+\sigma+p)(2+\sigma-p)}$$

$$a_3 = \frac{-a_1}{(3+\sigma+p)(3+\sigma-p)} = 0 \text{ because } a_1 = 0$$

$$a_1 = a_3 = a_5 = \dots = 0$$

$$a_4 = \frac{-a_2}{(4+\sigma+p)(4+\sigma-p)} = \frac{a_0}{(2+\sigma+p)(2+\sigma-p)(4+\sigma+p)(4+\sigma-p)}$$

Similarly $a_6 = \dfrac{-a_0}{(2+\sigma+p)(2+\sigma-p)(4+\sigma+p)(4+\sigma-p)(6+\sigma+p)(6+\sigma-p)}$

$$a_{2m} = \frac{(-1)^m a_0}{(2+\sigma+p)(2+\sigma-p)(4+\sigma+p)(4+\sigma-p)\cdots(2m+\sigma+p)(2m+\sigma-p)}, \ m \geq 1$$

$\Rightarrow$ for first sol, as usual using the bigger root $\sigma_1 = p$.

$$a_{2m} = \frac{(-1)^m a_0}{(2+2p)2\,(4+2p)4\,(6+2p)6\cdots(2m+2p)\,2m}, \ m \geq 1$$

$$= \frac{(-1)^m a_0}{2(p+1)\,2(p+2)\,2(p+3)\cdots 2(p+m)\ 2\cdot 4\cdot 6\cdots\cdot 2m}, \ m \geq 1$$

$$\Rightarrow \frac{(-1)^m a_0}{2^m\cdot 2^m\cdot m!\ (p+1)(p+3)\cdots(p+m)}, \ m \geq 1$$

$$\Rightarrow \frac{(-1)^m a_0}{2^{2m}\, m!\ (p+1)(p+2)\cdots(p+m)}, \ m \geq 1$$

$$y_1(x) = a_0 x^p + \sum_{n=1}^{\infty} a_n x^{n+\sigma}$$

$$= a_0 x^p + a_0 \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}\, m!\ (p+1)(p+2)\cdots(p+m)} \cdot x^{2m+p}$$

$\Rightarrow$ In the series, divide and multiply by $\Gamma(p+1)$ and we know if $p \notin \mathbb{Z}^+$, we can write $\Gamma(p+1) = p\Gamma(p)$.

$$y_1(x) = a_0 x^p + a_0 \sum_{m=1}^{\infty} \frac{\Gamma(p+1)\,(-1)^m\, x^{2m+p}}{2^{2m}\cdot m!\ \Gamma(p+1)\,(p+1)(p+2)\cdots(p+m)}$$

$$= a_0 x^p + a_0 \sum_{m=1}^{\infty} \frac{(-1)^m\, \Gamma(p+1)\, x^{2m+p}}{2^{2m}\cdot m!\ (p+1)\Gamma(p+1)\,(p+2)\cdots(p+m)}$$

$\Rightarrow$ we can write $(p+1)\Gamma(p+1) = \Gamma(p+2)$

$$(p+2)\Gamma(p+2) = \Gamma(p+3)$$

$$\vdots$$

$$(p+m)\Gamma(p+m) = \Gamma(p+m+1)$$

$$\Rightarrow y_1(x) = a_0 x^p + a_0 \sum_{m=1}^{\infty} \frac{(-1)^m \Gamma(p+1)\, x^{2m+p}}{2^{2m} \cdot m!\, \Gamma(p+m+1)}$$

let $a_0 = \dfrac{1}{\Gamma(p+1) 2^p}$

$$\Rightarrow y_1(x) = \frac{1}{\Gamma(p+1)2^p} x^p + \frac{1}{\cancel{\Gamma(p+1)}2^p} \sum_{m=1}^{\infty} \frac{(-1)^m \cancel{\Gamma(p+1)}\, x^{2m+p}}{2^{2m} \cdot m!\, \Gamma(p+m+1)}$$

$$= \frac{1}{\Gamma(p+1)2^p} x^p + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \cdot \left(\frac{x}{2}\right)^{2m+p}$$

$$y_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p}.$$

let's define $y_1(x) = \boxed{J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p}.}$

$J_p(x)$ is known as Bessel function of the first kind of order p.

$\Rightarrow$ for second sol $\sigma_2 = -p$, first check $\Gamma(-p+m+1)$ is defined. because we know $\Gamma(p) = \infty$ if $p=0$ or $p \in \mathbb{Z}^-$. In the given equation we have $p \geq 0$, $\therefore p \notin \mathbb{Z}^-$. To check $\Gamma(p) = \infty$ for other case when $p=0$, we can consider following.

$m=0 \Rightarrow \Gamma(-p+1) \Rightarrow (-p+1) = 0$ iff $p=1$

$m=1 \Rightarrow \Gamma(-p+2) \Rightarrow (-p+2) = 0$ iff $p=2$

$\vdots$

$m=m \Rightarrow \Gamma(-p+m+1) \Rightarrow (-p+m+1) = 0$ iff $p = m+1$

for this case we are assuming $p \notin \mathbb{Z}^+$ on pg 96

$\Rightarrow$ ∴ we have $\Gamma(-p+m+1)$ defined for our case in consideration

$\Rightarrow$ To obtain the second sol, $\sigma_2 = -p$, just replacing p with $-p$ in the first sol.

$$y_2(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-p}$$

$$J_{-p}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-p}$$ $\left\{ J_{-p}(x) = \text{Bessel function of } \cancel{\text{second}} \text{ first kind of order-}p \right\}$

∴ If $p \notin \mathbb{Z}^+$, we have $y(x) = c_1 J_p(x) + c_2 J_{-p}(x)$

We must always check if sol are independent. by Wronskian

$$W(x)(J_p, J_{-p}) = \begin{vmatrix} J_p(x) & J_{-p}(x) \\ J_p'(x) & J_{-p}'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(+p+m+1)} \left(\frac{x}{2}\right)^{2m+p} & \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-p} \\ \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} (2m+p)\left(\frac{x}{2}\right)^{2m+p-1} \cdot \frac{1}{2} & \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m (2m-p)}{m!\,\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-p-1} \cdot \frac{1}{2} \end{vmatrix}$$

$$\Rightarrow \left| \sum_{m=0}^{\infty} \frac{1}{2} \cdot \frac{(2m-p)(-1)^{m+1}}{m!\,\Gamma(p+m+1)\Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{2m-1} (-1) \right.$$

$$\left. \sum_{m=0}^{\infty} \frac{1}{2} \frac{(2m+p)(-1)^{m+1}}{m!\,\Gamma(-p+m+1)\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m-1} \right|$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^{2m-1} \frac{(-1)^{m+1}}{m!\,\Gamma(p+m+1)\,\Gamma(-p+m+1)} \cdot \left(2m-p(-)\,2m+p\right)$$

$$\Rightarrow \sum_{m=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^{2m-1} \frac{(-1)^{m+1}}{m!\,\Gamma(p+m+1)\,\Gamma(-p+m+1)} \cdot \left[\cancel{2m}-p-\cancel{2m}-p\right]$$

$$\Rightarrow \sum_{m=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^{2m-1} \frac{(-1)^{m+1}}{m!\,\Gamma(p+m+1)\,\Gamma(-p+m+1)} \cdot (-2p)$$

This expression cannot be equal to zero because

(i) $p \neq 0$ (case in consideration)

(2) $x \neq 0$ (looking for non-trivial sol)

(3) $\Gamma(-p+m+1) \neq 0$

Thus $W(x)\left(J_p(x), J_{-p}(x)\right) \neq 0$, these two sol are independent.

Now, let's $\boxed{\text{consider } p = \frac{1}{2}}$, case we have excluded in this case.

for $y_1(x) = \Gamma\left(m+\frac{3}{2}\right)$ is defined for all $m = 0, 1, 2, \ldots$

for $y_2(x) = \Gamma(-p+m+1) = \Gamma\left(m+\frac{1}{2}\right)$ is defined for all $m = 0, 1, 2, \ldots$

Thus, for $p = \frac{1}{2}$, we have the general sol:-

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x)$$

$\boxed{\text{Considering } p = 0}$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p} \Bigg|_{p=0}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m} \Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,m!} \left(\frac{x}{2}\right)^{2m} \quad \left\{\begin{array}{l}\text{since}\\ \Gamma(n+1) = n! \\ \text{when } n=1,2,3,\ldots\end{array}\right\}$$

$$\boxed{\Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}.}$$

Bessel function of first kind of order 0.

For the second independent sol, we use case 2 of Frobenius method for equal roots from pg # 82Q

$\Rightarrow$ By comparing the coeff of x^{σ}, $x^{\sigma+1}$, $x^{\sigma+n}$, we have :-

x^{σ} : $(\sigma-1)\sigma a_0 + \sigma a_0 = 0$

$\Rightarrow \sigma^2 a_0 = 0 \Rightarrow a_0 \neq 0, \quad \sigma_1 = \sigma_2 = 0$

$x^{\sigma+1}$: $\sigma(\sigma+1) a_1 + (\sigma+1) a_1 = 0$

$\Rightarrow (\sigma+1)^2 a_1 = 0 \Rightarrow$ for $\sigma = 0$ we can't have $(\sigma+1)^2 = 0$

$\therefore a_1 = 0$

$x^{\sigma+n}$: $(n+\sigma-1)(n+\sigma) a_n + (n+\sigma) a_n + a_{n-2} = 0$

$\Rightarrow (n+\sigma)^2 a_n + a_{n-2} = 0$

$$\Rightarrow a_n = \frac{-a_{n-2}}{(n+\sigma)^2}, \quad n \geq 2$$

$$a_2 = \frac{-a_0}{(2+\sigma)^2}$$

$$a_3 = \frac{-a_1}{(3+\sigma)^2} = 0, \text{ because } a_1 = 0$$

$$a_1 = a_3 = a_5 = \cdots = 0$$

$$a_4 = \frac{-a_2}{(4+\sigma)^2} = \frac{a_0}{(2+\sigma)^2(4+\sigma)^2}$$

$$a_6 = \frac{-a_0}{(2+\sigma)^2(4+\sigma)^2(6+\sigma)^2}$$

$$a_{2m} = \frac{(-1)^m a_0}{(2+\sigma)^2(4+\sigma)^2(6+\sigma)^2\cdots(2m+\sigma)^2}, \quad m \geq 1$$

for second sol, we need a_{2m}'. Differentiate wrt σ

$$\Rightarrow a_{2m}' = (-1)^m a_0 \left[\frac{1}{(2+\sigma)^2}\cdot\frac{1}{(4+\sigma)^2}\cdot\frac{1}{(6+\sigma)^2}\cdot\cdots\cdot\frac{1}{(2m+\sigma)^2}\right]'$$

$\Rightarrow$ let $t = \frac{1}{(2+\sigma)^2}\cdot\frac{1}{(4+\sigma)^2}\cdot\frac{1}{(6+\sigma)^2}\cdot\cdots\frac{1}{(2m+\sigma)^2}$

$\Rightarrow$ taking ln both sides

$$\ln t = -2\left[\ln(2+\sigma) + \ln(4+\sigma) + \ln(6+\sigma) + \cdots + \ln(2m+\sigma)\right]$$

differentiat wrt σ

$$\frac{1}{t}\cdot t' = -2\left[\frac{1}{2+\sigma} + \frac{1}{4+\sigma} + \frac{1}{6+\sigma} + \cdots + \frac{1}{2m+\sigma}\right]$$

$$t' = -2t\left[\frac{1}{2+\sigma} + \frac{1}{4+\sigma} + \frac{1}{6+\sigma} + \cdots + \frac{1}{2m+\sigma}\right]$$

$$t' = \frac{-2}{(2+\sigma)^2(4+\sigma^2)(6+\sigma)^2\cdots(2m+\sigma)^2}\left[\frac{1}{2+\sigma} + \frac{1}{4+\sigma} + \frac{1}{6+\sigma} + \cdots + \frac{1}{2m+\sigma}\right]$$

Put it back in the equation, we have

$$a'_{2m} = \frac{(-1)^m a_0 (-2)}{(2+\sigma)^2 (4+\sigma)^2 \cdots (2m+\sigma)^2} \left[\frac{1}{2+\sigma} + \frac{1}{4+\sigma} + \frac{1}{6+\sigma} + \cdots + \frac{1}{2m+\sigma} \right]$$

Now evaluate $a'_{2m}(\sigma)\Big|_{\sigma=0}$

$$\Rightarrow \frac{(-1)^m a_0 (-2)}{2^2 \cdot 4^2 \cdot 6^2 \cdots 2m^2} \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2m} \right]$$

$$\Rightarrow \frac{(-1)^m a_0 (-2)}{(2 \cdot 4 \cdot 6 \cdots 2m)^2} \left[\frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \right) \right]$$

let $g(m) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{m}$

$$\Rightarrow \frac{(-1)^m a_0 (-2)}{(2^m \cdot m!)^2} \frac{1}{2} \cdot g(m) \Rightarrow \frac{(-1)^m a_0 (-\cancel{2})}{2^{2m} \cdot (m!)^2} \cdot \frac{1}{\cancel{2}} g(m).$$

$$y_2(x) = y_1(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} a_0}{2^{2m} \cdot (m!)^2} g(m) x^{2m}$$

$$= y_1(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} a_0}{2^{2m} \cdot (m!)^2} g(m) x^{2m}, \quad \text{let } a_0 = 1$$

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2^{2m} (m!)^2} g(m) x^{2m}$$

$$= J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cdot g(m)}{(m!)^2} \left(\frac{x}{2} \right)^{2m}$$

let $\boxed{Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2) J_0(x)\right]}$

$$= \frac{2}{\pi}\left[J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(m!)^2}\left(\frac{x}{2}\right)^{2m} g(m) + (\gamma - \ln 2) J_0(x)\right]$$

$$= \frac{2}{\pi}\left[J_0(x)\left[\ln x - \ln 2 + \gamma\right] + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(m!)^2}\left(\frac{x}{2}\right)^{2m} g(m)\right]$$

$$= \frac{2}{\pi}\left[J_0(x)\left[\gamma + \ln\left(\frac{x}{2}\right)\right] + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(m!)^2}\left(\frac{x}{2}\right)^{2m} g(m)\right]$$

where γ is Euler constant. $\gamma = \lim_{m\to\infty}\left(g(m) - \ln m\right) = 0.5772\ldots$

Since $Y_0(x)$ is a linear combination of $y_2(x)$ and $J_0(x)$, it is also a sol of equation. $Y_0(x)$ is known as Bessel function of the second kind of order zero.

Thus, $y(x) = c_1 J_0(x) + c_2 Y_0(x)$, if $p = 0$

$\boxed{\text{Considering } p \neq 0 \Rightarrow \sigma_1 \neq \sigma_2 \text{ but } \sigma_1 - \sigma_2 = k \in \mathbb{Z}^+}$

$\Rightarrow \sigma_1 = p,\ \sigma_2 = -p,\quad \sigma_1 - \sigma_2 = p - (-p) = 2p$ is an integer

two cases :- $2p = 1, 2, 3, 4, \ldots,$

$\boxed{\text{Case A}} \rightarrow 2p = 1, 3, 5, \ldots \Rightarrow p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

we know $\Gamma(p+m+1)$ is defined for $p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

$\Gamma(-p+m+1) =$ for $p = \frac{1}{2} = \Gamma(m+\frac{1}{2})$ is defined

$p = \frac{3}{2} = \Gamma(m-\frac{1}{2})$ is defined

$p = \frac{5}{2} = \Gamma(m-\frac{3}{2})$ is defined

$\Gamma(-p+m+1)$ is defined for $p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ----$

Thus the sol is same as we have obtained ~~in~~ when p is not an integer.

$$y(x) = C_1 J_p(x) + C_2 J_{-p}(x)$$

$\boxed{\text{Case B}}$ $2p = 2, 4, 6, ---- \Rightarrow p = 1, 2, 3, ---$

$\Gamma(p+m+1)$ is defined for $p = 1, 2, 3, ---.$

But

$\Gamma(-p+m+1)$ is not defined for all $p = 1, 2, 3, ---.$

because for $p=1 \Rightarrow \Gamma(m) =$ undefined at $m=0$

$p=2 \Rightarrow \Gamma(m-1) =$ undefined at $m = 0, 1,$

$p=3 \Rightarrow \Gamma(m-2) =$ undefined at $m = 0, 1, 2,$

and so on.

$\Rightarrow$ first sol when $\sigma_1 = p$, the larger root is given by

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p}, \quad \text{Since } p = 1, 2, 3, ---$$

we can write $\Gamma(p+m+1) = (p+m)!$

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(p+m)!} \left(\frac{x}{2}\right)^{2m+p}$$

$\Rightarrow$ for second sol, as we did in Frobenius method. Case 3.

we need to check $a_{2p}(-p)$ for unboundedness.

We had $\cancel{a_{2m} = \frac{(-1)^m a_0}{2^{2m}\, m!\,(p+1)(p+2)---(p+m)}, \; m \geq 1}$

Similarly

$$\cancel{a_{2p} = \frac{(-1)^p a_0}{2^{2p} p!\,(p+1)(p+2)\cdots(p+p)}}, \quad \cancel{p \geq 1}$$

We had $a_{2m} = \dfrac{(-1)^m a_0}{(2+\sigma+p)(2+\sigma-p)(4+\sigma+p)(4+\sigma-p)\cdots(2m+\sigma+p)(2m+\sigma-p)}, \quad m \geq 1$

$$a_{2p} = \left. \frac{(-1)^p a_0}{(2+\sigma+p)(2+\sigma-p)(4+\sigma+p)(4+\sigma-p)\cdots(3p+\sigma)(p+\sigma)} \right|_{\sigma=-p}$$

$$a_{2p}(-p) = \frac{(-1)^p a_0}{2(2-2p)\,4(4-2p)\cdots 2p \cdot 0} \Rightarrow \infty$$

. Thus $a_{2p}(-p) \to \infty$ (unbounded) when $m = p$. [Pg 87 first two lines]

So, we need to use Case 3 part 2 of frobenius method as on pg # 85.

$$y_2(x) = \sum_{m=0}^{p-1} a_{2m}(-p)\, x^{2m-p} + \sum_{m=p}^{\infty} \left[(\sigma+p)\, a_{2m}(\sigma) \right]'_{\sigma=-p} x^{2m-p} +$$

$$\ln x \sum_{m=p}^{\infty} \left[(\sigma+p)\, a_{2m}(\sigma) \right]_{\sigma=-p} x^{2m-p}.$$

using first series $\sum_{m=0}^{p-1} a_{2m}(-p)\, x^{2m-p}$.:- We already calculated

$$a_{2m}(\sigma) = \frac{(-1)^m a_0}{(2+\sigma-p)(4+\sigma-p)\cdots(2m+\sigma-p)\,(2+\sigma+p)(4+\sigma+p)\cdots(2m+\sigma+p)}$$

$$a_{2m}(-p) = \frac{(-1)^m a_0}{(2-2p)(4-2p)\cdots(2m-2p)\; 2\cdot 4 \cdots 2m}, \quad m \geq 1$$

$$= \frac{(-1)^m a_0}{2^m (1-p)(2-p)\cdots(m-p)\; 2^m \cdot m!}, \quad m \geq 1$$

$$= \frac{\cancel{(-1)^m}\, a_0}{2^{2m}\, \cancel{(-1)^m} (p-1)(p-2)\cdots(p-m)\, m!}, \quad m \geq 1$$

$\Rightarrow$ multiply and divide by $(p-m-1)!$

$$\Rightarrow \frac{a_0 (p-m-1)!}{2^{2m} (p-1)(p-2)\cdots(p-m)(p-m-1)!} \Rightarrow \frac{a_0 (p-m-1)!}{2^{2m}\, m!\, (p-1)!}$$

$\Rightarrow$ Let $a_0 = \frac{-1}{2^{-p+1}} \cdot (p-1)!$

$$\Rightarrow \frac{-1\, \cancel{(p-1)!}\, (p-m-1)!}{2^{2m}\, m!\, \cancel{(p-1)!}\, 2^{-p+1}} \Rightarrow \frac{-(p-m-1)!}{2^{2m-p+1}\, m!}, \quad m \geq 1$$

$$\Rightarrow \sum_{m=0}^{p-1} -\frac{1}{2} \frac{(p-m-1)!}{m!} \left(\frac{x}{2}\right)^{2m-p} \equiv S_1$$

using 3rd series: $\sum_{m=p}^{\infty} \left[(\sigma+p)\, a_{2m}(\sigma) \right]_{\sigma=-p} x^{2m-p}$

$$(\sigma+p)\, a_{2m}(\sigma) = \frac{\cancel{(\sigma+p)}(-1)^m a_0}{\left[(2+\sigma-p)(4+\sigma-p)(6+\sigma-p)\cdots(\sigma+2p-4-p)(\sigma+2p-2-p)\cancel{(\sigma+2p-p)}\right] \left[(\sigma+2p+2-p)(\sigma+2p+4-p)\cdots(\sigma+2m-2-p)(\sigma+2m-p)\right] \left[(2+\sigma+p)(4+\sigma+p)\cdots(2m+\sigma+p)\right]}$$

{ the series in the denominator $= (2+\sigma-p)\cdots(2m+\sigma-p)$ is broken up into 2 parts. First part ends when $m=p$ and second part begins at $(\sigma+2p+2-p)$ because each successive term is increased by 2 over previous term }

$\Rightarrow$ evaluate it $\sigma=-p$, we get

$$\Rightarrow \frac{(-1)^m a_0}{\left[(2-2p)(4-2p)(6-2p)\cdots(-4)(-2)\right]\left[2\cdot4\cdot6\cdots(2m-2-2p)(2m-2p)\right]\left[2\cdot4\cdot6\cdots2m\right]}$$

$$\Rightarrow \frac{(-1)^m a_0}{(-1)^{p-1}\left[2\cdot4\cdots(2p-4)(2p-2)\right]\left[2^{m-p}(1\cdot2\cdot3\cdots(m-1-p)(m-p))\right]\left[2^m(1\cdot2\cdot3\cdots m)\right]}$$

$$\Rightarrow \frac{(-1)^m a_0}{(-1)^{p-1}2^{p-1}\left[1\cdot2\cdots(p-2)(p-1)\right]\left[2^{m-p}(m-p)!\right]\left[2^m\cdot m!\right]}$$

$$\Rightarrow \frac{(-1)^m a_0}{(-1)^{p-1}2^{2m-1}(p-1)!\,(m-p)!\,m!}, \quad \text{Here also } a_0 = \frac{-1\,(p-1)!}{2^{-p+1}}$$

$$\Rightarrow \frac{(-1)^m(-1)\cancel{(p-1)!}}{(-1)^{p-1}2^{2m-1}\cdot\cancel{(p-1)!}\,(m-p)!\,m!\,2^{-p+1}} \Rightarrow \frac{(-1)^{m+1-p+1}}{2^{2m-p}(m-p)!\,m!}$$

$$\Rightarrow \frac{(-1)^{m-p}}{2^{2m-p}(m-p)!\,m!} \quad \{\text{because } (-1)^2=1\}$$

$$\Rightarrow \sum_{m=p}^{\infty}\frac{(-1)^{m-p}}{2^{2m-p}(m-p)!\,m!}x^{2m-p}$$

⇒ Let's make the series starts at $m=0$ by $m \to m+p$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^{m+p-p}}{2^{2m+2p-p}(m+p-p)!_0\,(m+p)!_0}\, x^{2m+2p-p}$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{m!_0\,(m+p)!_0}\left(\frac{x}{2}\right)^{2m+p} = J_p(x)$$

$$\Rightarrow \ln x \sum_{m=p}^{\infty}\Big[(\sigma+p)\,a_{2m}(\sigma)\Big]_{\sigma=-p} x^{2m-p} = \ln x\, J_p(x) \equiv S_2$$

using 2^{nd} series $\displaystyle\sum_{m=p}^{\infty}\Big[(\sigma+p)\,a_{2m}(\sigma)\Big]'_{\sigma=-p} x^{2m-p}$

$$\Rightarrow (\sigma+p)\,a_{2m}(\sigma) = \frac{(-1)^m a_0}{\big[(2+\sigma-p)(4+\sigma-p)\cdots(\sigma+2p-4-p)(\sigma+2p-2-p)\big]\big[(\sigma+2p+2-p)(\sigma+2p+4-p)\cdots(\sigma+2m-2-p)(\sigma+2m-p)\big]\big[(2+\sigma+p)(4+\sigma+p)\cdots(2m+\sigma+p)\big]}$$

{as we have obtained it in 3^{rd} series on pg # 107} $\equiv A$

⇒ Taking ln both sides

$$\Rightarrow \ln\Big[(\sigma+p)\,a_{2m}(\sigma)\Big] = \ln(A)$$

$$\Rightarrow \ln\Big[(\sigma+p)\,a_{2m}(\sigma)\Big] = \ln\big((-1)^m a_0\big) - \Big\{\big[\ln(2+\sigma-p)+\ln(4+\sigma-p)+\cdots+\ln(\sigma+p-4)+\ln(\sigma+p-2)\big]+\big[\ln(\sigma+p+2)+\ln(\sigma+p+4)+\cdots+\ln(\sigma+2m-2-p)+\ln(\sigma+2m-p)\big]+\big[\ln(2+\sigma+p)+\ln(4+\sigma+p)+\cdots+\ln(2m+\sigma+p)\big]\Big\}$$

⇒ differentiating both sides wrt σ

$$\Rightarrow \frac{1}{(\sigma+p)q_{2m}(\sigma)}\left[(\sigma+p)q_{2m}(\sigma)\right]' = 0 - \left\{\left[\frac{1}{2+\sigma-p}+\frac{1}{4+\sigma-p}+\dots+\frac{1}{\sigma+p-4}+\frac{1}{\sigma+p-2}\right]\right.$$

$$+\left[\frac{1}{\sigma+p+2}+\frac{1}{\sigma+p+4}+\dots+\frac{1}{\sigma+2m-2-p}+\frac{1}{\sigma+2m-p}\right]+\left[\frac{1}{2+\sigma+p}+\right.$$

$$\left.\left.\frac{1}{4+\sigma+p}+\dots+\frac{1}{2m+\sigma+p}\right]\right\} \equiv B$$

$$\Rightarrow \left[(\sigma+p)q_{2m}(\sigma)\right]' = -(\sigma+p)q_{2m}(\sigma)\left\{B\right\}$$ $\Rightarrow$ Put the value of $(\sigma+p)q_{2m}(\sigma)$ from previous page.

$$\Rightarrow \left[(\sigma+p)q_{2m}(\sigma)\right]' = \left\{\frac{(-1)^m a_0}{\left[(2+\sigma-p)\cdots(\sigma+p-2)\right]\left[(\sigma+p+2)\cdots(\sigma+2m-p)\right]}\right.$$

continuing in the denominator

$$\left.\left[(2+\sigma+p)\cdots(2m+\sigma+p)\right]\right\} B.$$

$\Rightarrow$. evaluate this at $\sigma=-p$

$$\Rightarrow \left[(\sigma+p)q_{2m}(\sigma)\right]'_{\sigma=-p} = \left\{\frac{(-1)^m a_0}{\left[(2-2p)(4-2p)\cdots(-4)(-2)\right]\left[2\cdot4\cdot\cdots\cdot(2m-2-2p)(2m-2p)\right]}\right.$$

continue in the denominator

$$\left.\left[2\cdot4\cdot\cdots\cdot(2m-2)(2m)\right]\right\}(-1)\left\{\left[\frac{1}{2-2p}+\frac{1}{4-2p}+\dots+\frac{1}{-4}+\frac{1}{-2}\right]+\left[\frac{1}{2}+\right.\right.$$

$$\left.\left.\frac{1}{4}+\dots+\frac{1}{2m-2-p}+\frac{1}{2m-2p}\right]+\left[\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2m-2}+\frac{1}{2m}\right]\right\}$$

$$\Rightarrow \left[(\sigma+p)q_{2m}(\sigma)\right]'_{\sigma=-p} = \left\{\frac{(-1)^{m+1} a_0}{\left[(-1)^{p-1}(2\cdot4\cdot\cdots\cdot(2p-4)(2p-2)\right]\left[2^{m-p}(1\cdot2\cdot\cdots\cdot(m-p-1)\right.}\right.$$

in the denominator

$$\left.\left.(m-p))\right]\left[2^m(1\cdot2\cdot\cdots\cdot m)\right]\right\}\left\{-\left[\frac{1}{2p-2}+\frac{1}{2p-4}+\dots+\frac{1}{4}+\frac{1}{2}\right]+\right.$$

$$\left.\frac{1}{2}\left[1+\frac{1}{2}+\dots+\frac{1}{m-p-1}+\frac{1}{m-p}\right]+\frac{1}{2}\left[1+\frac{1}{2}+\dots+\frac{1}{m-1}+\frac{1}{m}\right]\right\}$$

$$\Rightarrow \left[(\sigma+p)a_{2m}(\sigma)\right]'_{\sigma=-p} = \left\{ \frac{(-1)^{m+1} a_0}{\left[(-1)^{p-1} 2^{p-1} (1\cdot 2 \cdots (p-2)(p-1))\right]\left[2^{m-p}(m-p)!\right]\left[2^m \cdot m!\right]} \right\}$$

$$\left\{ \frac{-1}{2}\left[-1+\frac{1}{2}+\cdots+\frac{1}{p-2}+\frac{1}{p-1}\right] + \frac{1}{2}\left[g(m-p)\right] + \frac{1}{2}\left[g(m)\right] \right\}$$

where $g(m-p) = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m-p}$, $g(m) = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}$

$$\Rightarrow \left[(\sigma+p)a_{2m}(\sigma)\right]'_{\sigma=-p} = \left\{ \frac{(-1)^{m+1-p+1} a_0}{2^{\cancel{p}-\cancel{1}+m-\cancel{p}+m+\cancel{1}} (p-1)!\,(m-p)!\,m!} \right\} \left\{ -(g(p-1))\right.$$

$$\left. + g(m-p) + g(m) \right\}$$

where $g(p-1) = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}$

$\Rightarrow$ Let $a_0 = \dfrac{(-1)(p-1)!}{2^{-p+1}}$

$$\Rightarrow \left[(\sigma+p)a_{2m}(\sigma)\right]'_{\sigma=-p} = \left\{ \frac{(-1)^{m-p}\cancel{(-1)^2}^{\,1}(-1)\cancel{(p-1)!}}{2^{-p+1}\cdot 2^{2m}\cancel{(p-1)!}\,(m-p)!\,m!} \right\} \left\{ -g(p-1) + \right.$$

$$\left. g(m-p) + g(m) \right\}$$

$$\Rightarrow \left[(\sigma+p)a_{2m}(\sigma)\right]'_{\sigma=-p} = \left\{ \frac{(-1)^{m-p+1}}{2^{2m-p+1}(m-p)!\,m!} \right\} \left\{ -g(p-1) + g(m-p) + g(m) \right\}$$

$$\Rightarrow \sum_{m=p}^{\infty} \left[(\sigma+p)a_{2m}(\sigma)\right]'_{\sigma=-p} x^{2m-p} = \frac{1}{2}\sum_{m=p}^{\infty} \left\{ \frac{(-1)^{m-p+1}}{2^{2m-p}(m-p)!\,m!} \right\} \left\{ -g(p-1) + g(m-p) + g(m) \right\} x^{2m-p}$$

$\Rightarrow$ Make the start at $m=0$ by $m \to m+p$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(\sigma+p)a_{2m+2p}(\sigma)\right]_{\sigma=-p} x^{2m+p} = \frac{1}{2}\sum_{m=0}^{\infty} \left\{ \frac{(-1)^{m+1}}{2^{2m+p}(m+p)!\,(m)!} \right\} \left\{ -g(p-1) + g(m) + g(m+p) \right\} x^{2m+p}$$

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$$\Rightarrow \sum_{m=0}^{\infty}\left[(\sigma+p)\,a_{2m+2p}(\sigma)\right]'_{\sigma=-p} x^{2m+p} = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^{m+2}}{(m+p)!\,m!}\,g(p-1)\left(\frac{x}{2}\right)^{2m+p} - \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+p)!\,m!}\left[g(m)+g(m+p)\right]\left(\frac{x}{2}\right)^{2m+p}$$

$\Rightarrow$ The first series on RHS can be written as $\frac{1}{2}g(p-1)J_p(x)$.

Thus, it is redundant, we can remove it from the sol.

$$\Rightarrow \sum_{m=0}^{\infty}\left[(\sigma+p)\,a_{2m+2p}(\sigma)\right]'_{\sigma=-p} x^{2m+p} = -\frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+p)!\,m!}\left[g(m)+g(m+p)\right]\left(\frac{x}{2}\right)^{2m+p} = S_3$$

Putting these three series together, the second independent sol is :- $(S_1+S_2+S_3)$

$$y_2(x) = -\frac{1}{2}\sum_{m=0}^{p-1}\frac{(p-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-p} + \ln x\, J_p(x) - \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+p)!\,m!}\left[g(m)+g(m+p)\right]\left(\frac{x}{2}\right)^{2m+p}$$

Let's define $Y_p(x) = \frac{2}{\pi}\left[(\gamma-\ln 2)J_p(x) + y_2(x)\right]$

$$\Rightarrow \frac{2}{\pi}\left\{(\gamma-\ln 2)J_p(x) + \ln x\,J_p(x) - \frac{1}{2}\sum_{m=0}^{p-1}\frac{(p-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-p} - \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+p)!\,m!}\left[g(m)+g(m+p)\right]\left(\frac{x}{2}\right)^{2m+p}\right\}$$

$$\Rightarrow \frac{2}{\pi}\left\{J_p(x)\left[\gamma+\ln\left(\frac{x}{2}\right)\right] - \frac{1}{2}\sum_{m=0}^{p-1}\frac{(p-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-p} - \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+p)!\,m!}\left[g(m)+g(m+p)\right]\left(\frac{x}{2}\right)^{2m+p}\right\}$$

$\therefore$ $Y_n(x)$ is Bessel function of the second kind of order p.

$\Rightarrow y(x) = C_1 J_p(x) + C_2 Y_p(x)$. if $p = 1, 2, 3, \ldots$

$\boxed{\underline{\underline{Ex}} \quad x^2y'' + xy' + (x^2-1)y = 0}$ (Bessel equation of order 1) (113)

$\Rightarrow y'' + \frac{y'}{x} + \left(1 - \frac{1}{x^2}\right) y = 0$

clearly $x=0$ is RSP. Using Frobenius method

Let $y(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$, $y'(x) = \sum_{n=0}^{\infty} (n+\sigma) a_n x^{n+\sigma-1}$,

$y''(x) = \sum_{n=0}^{\infty} (n+\sigma-1)(n+\sigma) a_n x^{n+\sigma-2}$ $\Rightarrow$ plug all these into given eq

$\Rightarrow x^2\left[\sum_{n=0}^{\infty} (n+\sigma-1)(n+\sigma) a_n x^{n+\sigma-2}\right] + x\left[\sum_{n=0}^{\infty} (n+\sigma) a_n x^{n+\sigma-1}\right] + (x^2-1)\sum_{n=0}^{\infty} a_n x^{n+\sigma} = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+\sigma-1)(n+\sigma) a_n x^{n+\sigma} + \sum_{n=0}^{\infty} (n+\sigma) a_n x^{n+\sigma} + \sum_{n=0}^{\infty} a_n x^{n+\sigma+2} - \sum_{n=0}^{\infty} a_n x^{n+\sigma} = 0$

$\Rightarrow$ Making all exponents to $x^{n+\sigma}$

$\Rightarrow \sum_{n=0}^{\infty} (n+\sigma-1)(n+\sigma) a_n x^{n+\sigma} + \sum_{n=0}^{\infty} (n+\sigma) a_n x^{n+\sigma} + \sum_{n=2}^{\infty} a_{n-2} x^{n+\sigma} - \sum_{n=0}^{\infty} a_n x^{n+\sigma} = 0$

$\Rightarrow$ Making all the series start at $n=2$

$\Rightarrow (\sigma-1)(\sigma) a_0 x^{\sigma} + (\sigma)(\sigma+1) a_1 x^{\sigma+1} + \sum_{n=2}^{\infty} (n+\sigma-1)(n+\sigma) a_n x^{n+\sigma} +$

$(\sigma) a_0 x^{\sigma} + (\sigma+1) a_1 x^{\sigma+1} + \sum_{n=2}^{\infty} (n+\sigma) a_n x^{n+\sigma} + \sum_{n=2}^{\infty} a_{n-2} x^{n+\sigma} - a_0 x^{\sigma} - a_1 x^{\sigma+1} -$

$\sum_{n=2}^{\infty} a_n x^{n+\sigma} = 0$

$\Rightarrow$ Equating the coefficients of $x^{\sigma}, x^{\sigma+1}, x^{n+\sigma}$ to RHS

$\Rightarrow x^{\sigma}$:- $(\sigma-1)\sigma a_0 + \sigma a_0 - a_0 = 0$

$\Rightarrow \sigma^2 a_0 - a_0 = 0 \Rightarrow a_0 \neq 0 \Rightarrow \sigma_1 = 1, \sigma_2 = -1$

$x^{\sigma+1}$:- $(\sigma)(\sigma+1) a_1 + (\sigma+1) a_1 - a_1 = 0$

$\Rightarrow (\sigma+1)^2 a_1 - a_1 = 0 \Rightarrow a_1\left[(\sigma+1)^2 - 1\right] = 0$

with $\sigma = 1$ or -1, we can't get $[(\sigma+1)^2-1]=0$, $\therefore a_1 = 0$

$x^{\sigma+n}$: $-(n+\sigma-1)(n+\sigma)a_n + (n+\sigma)a_n + a_{n-2} - a_n = 0$

$\Rightarrow (n+\sigma)^2 a_n + a_{n-2} - a_n = 0$

$$\Rightarrow a_n = \frac{-a_{n-2}}{(n+\sigma)^2 - 1}, \quad n \geq 2$$

$$a_n = \frac{-a_{n-2}}{(n+\sigma-1)(n+\sigma+1)}, \quad n \geq 2$$

for $n=2$, $a_2 = \dfrac{-a_0}{(\sigma+1)(\sigma+3)}$

$n=3$, $a_3 = \dfrac{-a_1}{(\sigma+2)(\sigma+4)} = 0$, because $a_1 = 0$

$\Rightarrow a_1 = a_3 = a_5 = \cdots = 0$

$n=4$, $a_4 = \dfrac{-a_2}{(\sigma+3)(\sigma+5)} = \dfrac{a_0}{(\sigma+1)(\sigma+3)(\sigma+3)(\sigma+5)}$

$n=6$, $a_6 = \dfrac{-a_4}{(\sigma+5)(\sigma+7)} = \dfrac{-a_0}{(\sigma+1)(\sigma+3)(\sigma+5)(\sigma+3)(\sigma+5)(\sigma+7)}$

$n=2m$, $a_{2m} = \dfrac{(-1)^m a_0}{(\sigma+1)(\sigma+3)\cdots(\sigma+2m-1)(\sigma+3)(\sigma+5)\cdots(\sigma+2m+1)}$, $m \geq 1$

$\Rightarrow$ for first we bigger root $\sigma_1 = 1$

$n=2m$, $a_{2m} = \dfrac{(-1)^m a_0}{2\cdot4\cdot6\cdots(2m)(4)(6)(8)\cdots(2m+2)}$, $m \geq 1$

$$= \frac{(-1)^m a_0}{2^m \cdot m!\; 2^m (m+1)!}, \quad m \geq 1$$

$\Rightarrow$ first sol ie $y_1(x) = a_0 x + \sum_{m=1}^{\infty} \frac{(-1)^m a_0}{2^{2m} m!\,(m+1)!} x^{2m+1}$

$= \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{2^{2m} m!\,(m+1)!} x^{2m+1}$ $\Rightarrow$ let $a_0 = \frac{1}{2}$

$J_1(x) \Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \left(\frac{x}{2}\right)^{2m+1}$ = Bessel function of first kind of order 1.

$\Rightarrow$ checking $\sigma_1 - \sigma_2 = 1-(-1) = 2$, we have $\sigma_1 \neq \sigma_2$, but $\sigma_1 - \sigma_2 = k \in \mathbb{Z}^+$

$\Rightarrow \therefore$ we need to check $a_2(\sigma=-1)$ for unboundedness

2) $a_2(\sigma)$ we have :- $a_2(\sigma) = \left.\frac{-a_0}{(\sigma+1)(\sigma+3)}\right|_{\sigma=-1} = \frac{-a_0}{0\cdot(\sigma+3)} \to \infty$

but numerator $\neq 0$.

$\Rightarrow$ So, we need to use Case 3 part 2 of Frobenius method on pg # 85 [First series ends at a_1 because $a_2 \to \infty$] [but $a_1 = 0$, so $m = 0 \to 0$]

$\Rightarrow y_2(x) = \sum_{m=0}^{0} a_{2m}(\sigma=-1) x^{2m-1} + \sum_{m=1}^{\infty} \left[(\sigma+1) a_{2m}(\sigma)\right]'_{\sigma=-1} x^{2m-1} +$

$\ln x \sum_{m=1}^{\infty} \left[(\sigma+1) a_{2m}(\sigma)\right]_{\sigma=-1} x^{2m-1}$

using first series :- $\sum_{m=0}^{0} a_{2m}(\sigma=-1) x^{2m-1}$

$\Rightarrow a_0 x^{2m-1} = \frac{1}{2} x^{2m-1}$ by letting $a_0 = \frac{1}{2}$

using 3rd series :- $\ln x \sum_{m=1}^{\infty} \left[(\sigma+1) a_{2m}(\sigma)\right]_{\sigma=-1} x^{2m-1}$

$\Rightarrow (\sigma+1) a_{2m}(\sigma) = \frac{\cancel{(\sigma+1)}(-1)^m a_0}{\cancel{(\sigma+1)}(\sigma+3)(\sigma+5)\cdots(\sigma+2m-1)(\sigma+3)(\sigma+5)\cdots(\sigma+2m+1)}$

evaluate it $\sigma = -1$, we get

$$(\sigma+1)a_{2m}(\sigma)\Big|_{\sigma=-1} = \frac{(-1)^m a_0}{2\cdot4\cdot6\cdots(2m-2)\,(2\cdot4\cdot6\cdots 2m)}, m\geq 1$$

$$= \frac{(-1)^m a_0}{2^{m-1}(1\cdot2\cdot3\cdots(m-1))\ 2^m(1\cdot2\cdot3\cdots m)}, m\geq 1$$

$$= \frac{(-1)^m a_0}{2^{2m-1}(m-1)!\,m!}$$, same again, let $a_0 = \frac{1}{2}$

$$S_0 = \sum_{m=1}^{\infty}\frac{(-1)^m}{2^{2m}(m-1)!\,m!}x^{2m-1}$$ ⇒ let make the series start at $m=0$ by $m\to m+1$

$$\Rightarrow \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2^{2m+2}\,m!\,(m+1)!}x^{2m+1} \Rightarrow \frac{-1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\cdot\left(\frac{x}{2}\right)^{2m+1}$$

$$\Rightarrow \frac{-1}{2}J_1(x) \Rightarrow \frac{-1}{2}\ln x\, J_1(x)$$

Now, using the second series $\sum_{m=1}^{\infty}\left[(\sigma+1)a_{2m}(\sigma)\right]'_{\sigma=-1}x^{2m-1}$

$$\Rightarrow (\sigma+1)a_{2m}(\sigma) = \frac{(-1)^m a_0}{(\sigma+3)(\sigma+5)\cdots(\sigma+2m-1)(\sigma+3)(\sigma+5)\cdots(\sigma+2m+1)}$$

⇒ Taking ln both sides

$$\Rightarrow \ln\left[(\sigma+1)a_{2m}(\sigma)\right] = \ln\left[(-1)^m a_0\right] - \left[\ln(\sigma+3)+\ln(\sigma+5)+\cdots \ln(\sigma+2m-1)+\ln(\sigma+3)+\ln(\sigma+5)+\cdots+\ln(\sigma+2m+1)\right]$$

⇒ differentiate wrt σ

$$\Rightarrow \frac{1}{(\sigma+1)a_{2m}(\sigma)}\cdot\left[(\sigma+1)a_{2m}(\sigma)\right]' = -\left[\frac{1}{\sigma+3}+\frac{1}{\sigma+5}+\cdots+\frac{1}{\sigma+2m-1}+\frac{1}{\sigma+3}+\frac{1}{\sigma+5}+\cdots+\frac{1}{\sigma+2m+1}\right]$$

$$\Rightarrow \left[(\sigma+1)a_{2m}(\sigma)\right]' = (\sigma+1)a_{2m}(\sigma)(-1)\left[\frac{1}{\sigma+3}+\frac{1}{\sigma+5}+\dots+\frac{1}{\sigma+2m-1}+\frac{1}{\sigma+3}+\frac{1}{\sigma+5}+\dots+\frac{1}{\sigma+2m+1}\right]$$

$\Rightarrow$ Put the value of $(\sigma+1)a_{2m}(\sigma)$

$$\Rightarrow \left[(\sigma+1)a_{2m}(\sigma)\right]' = \left\{\frac{(-1)^m a_0}{((\sigma+3)(\sigma+5)\dots(\sigma+2m-1)(\sigma+3)(\sigma+5)\dots(\sigma+2m+1)}\right\}(-1)$$

$$\left\{\frac{1}{\sigma+3}+\frac{1}{\sigma+5}+\dots+\frac{1}{\sigma+2m-1}+\frac{1}{\sigma+3}+\frac{1}{\sigma+5}+\dots+\frac{1}{\sigma+2m+1}\right\}$$

$\Rightarrow$ evaluated at $\sigma=-1$

$$\Rightarrow \left[(\sigma+1)a_{2m}(\sigma)\right]'_{\sigma=-1} = \left\{\frac{(-1)^m a_0}{(2\cdot4\cdot6\dots(2m-2))(2\cdot4\cdot6\dots2m)}\right\}(-1)$$

$$\left\{\left[\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2m-2}\right]+\left[\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2m}\right]\right\}$$

$$\Rightarrow \left[(\sigma+1)a_{2m}(\sigma)\right]'_{\sigma=-1} = \left\{\frac{(-1)^m a_0}{(2^{m-1}\cdot(m-1)!)(2^m\cdot m!)}\right\}(-1)\left\{\frac{1}{2}\left(1+\frac{1}{2}+\dots+\frac{1}{m-1}\right)+\frac{1}{2}\left(1+\frac{1}{2}+\dots+\frac{1}{m}\right)\right\}$$

$$\Rightarrow \left[(\sigma+1)a_{2m}(\sigma)\right]'_{\sigma=-1} = \left\{\frac{(-1)^{m+1} a_0}{2^{2m-1}(m-1)!\,m!}\right\}\left\{\frac{1}{2}g(m-1)+\frac{1}{2}g(m)\right\}$$

where $g(m-1) = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m-1}$, $g(m) = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m}$.

$$\Rightarrow \left[(\sigma+1)a_{2m}(\sigma)\right]'_{\sigma=-1} = \frac{(-1)^{m+1} a_0}{2^{2m}(m-1)!\,m!}\cdot\left[g(m-1)+g(m)\right]$$

$$\text{So} \Rightarrow \sum_{m=1}^{\infty}\frac{(-1)^{m+1} a_0}{2^{2m}(m-1)!\,m!}\left[g(m-1)+g(m)\right]x^{2m-1}$$

Let's make the series start at $m=0$ by $m \to m+1$ (18)

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+2} a_0}{2^{2m+2} m!_0 (m+1)!_0} \left[g(m) + g(m+1)\right] x^{2m+1}, \quad \text{let } a_0 = \frac{1}{2}$$

$$\Rightarrow \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!_0 (m+1)!_0} \left[g(m) + g(m+1)\right] \left(\frac{x}{2}\right)^{2m+1}$$

Put together all these three series

$$y_2(x) = \frac{1}{2} x^{2m-1} - \frac{1}{2} \ln x \, J_1(x) + \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!_0 (m+1)!_0} \left[g(m) + g(m+1)\right] \left(\frac{x}{2}\right)^{2m+1}$$

define $Y_1(x) = \frac{2}{\pi} \left[(\gamma - \ln 2) J_1(x) + y_2(x) \right]$

$$\Rightarrow \frac{2}{\pi} \left[(\gamma - \ln 2) J_1(x) + -\frac{1}{2} \ln x \, J_1(x) + \frac{1}{2} x^{2m-1} + \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!_0 (m+1)!_0} \left[g(m) + g(m+1)\right] \left(\frac{x}{2}\right)^{2m+1} \right]$$

$$\Rightarrow \frac{2}{\pi} \left\{ J_1(x) \left[\gamma - \ln 2 - \frac{\ln x}{2} \right] + \frac{1}{2} x^{2m-1} + \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!_0 (m+1)!_0} \left[g(m) + g(m+1)\right] \left(\frac{x}{2}\right)^{2m+1} \right\}$$

$\Rightarrow$ $Y_1(x)$ is Bessel function of second kind of order 1.