Project No. 5
"Pricing of a call option"

Pricing of Derivatives

Multiple-Time step Trinomial Tree

February 2, 2023

Submitted by Liteshwar Rao
MA Mathematics, Yeshiva University
Mathematics of Finance, Fall 2022, MAT-5640-1

Contents

1 Abstract

This project is based on the paper written by *Paul Clifford and Oleg Zaboronski* titled *Pricing Options using Trinomial Trees* dated *17.11.2008*. It is an extension of Binomial model that we used earlier to price the call option. We will see that in addition of up and down movements in stock price in time $T = t+1$ over $T = t$, we consider the possibility of stock price remains same. So, in this trinomial model we have three branches, one for each scenario i.e. up, down, and same. In this project we explore the idea of trinomial model to price the call option , and we see that if we increase the number of time steps to maturity to a significant number, then it converges to Black-Scholes model.

2 Introduction

Previously, we have seen two-jump process for the asset prices over each discrete time step in the binomial model. Phelim Boyle (1986) expanded this framework to trinomial model by allowing for an extra-jump in the stochastic process. In its simplified form, the third jump was seen as keeping the asset prices same, while the other two are up and down movements in asset prices. [1]

2.1 Assumptions

Similar to the Binomial model, this model has same assumptions, some of which are listed below:

(a) No arbitrage opportunity.

(b) Volatility denoted by σ of the underlying asset is constant.

(c) Asset prices follow geometric Brownian motion.

(d) r is the rate of risk free investment exists.

2.2 List of variables

The following is the list of variables we use in this project:

(a) S_0 and S_t is price of underlying asset at time $T = 0$ and $T = t$ respectively.

(b) C_0 and C_t is price of a call option at time $T = 0$ and $T = t$ respectively.

(c) $T = T$ is time to maturity of a call option.

(d) n is number of time steps to maturity.

(e) σ is the historic volatility of asset prices.

(f) K is the strike price.

(g) Δt is the change in small change in time period.

(h) u and d are rates by which asset prices changes in a subsequent period.

(i) p_u, p_d, and p_m are corresponding probabilities for asset prices moving up, down, and remains same respectively.

(j) r is the risk free rate.

[1] https://sites.google.com/view/vinegarhill-financelabs/trinomial-model

2.3 List of Formula

The following is the list of variables we use in this project:

$$C_t = max(S_t - K, 0) \tag{1}$$

$$u = e^{\sigma\sqrt{2\Delta t}} \tag{2}$$

$$d = e^{-\sigma\sqrt{2\Delta t}} \tag{3}$$

$$p_u = \left(\frac{e^{\frac{r\Delta t}{2}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}\right)^2 \tag{4}$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{\frac{r\Delta t}{2}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}\right)^2 \tag{5}$$

$$p_m = 1 - p_u - p_d \tag{6}$$

3 Methodology

The methodology when pricing options using a trinomial tree is exactly the same as when using a binomial tree. Once the share price tree is built, the option payoffs at maturity time T are calculated. After that it remains to apply the following backward induction algorithm, where n represents the time position and j the space position:

$$C_{n,j} = e^{-r\Delta t}\left[p_u C_{n+1,j+1} + p_m C_{n+1,j} + p_d C_{n+1,j-1}\right] \tag{7}$$

The backward induction algorithm can be derived from the risk-neutrality principle and is the same for put and call options. When applied in the context of a trinomial tree (using the exact same methodology as the binomial tree), we can calculate the option value at interior nodes of the tree by considering it as a weighting of the option value at the future nodes, discounted by one time step. Thus we can calculate the option price at time n, C_n, as the option price of an up move $p_u C_{n+1}$ plus the option price of the middle move by $p_m C_{n+1}$ plus the option price of a down move by $p_d C_{n+1}$, discounted by one time step, $e^{-r\Delta t}$. So at any node on the tree our backward induction formula, (7), is applied to give us the option prices at any node in the tree. The name of the algorithm should now be clear since we only need to value the option at maturity, i.e the leaf nodes, and then work our way backwards through the tree calculating option values at all the nodes until we reach the root S_0, C_0.

4 Simulation

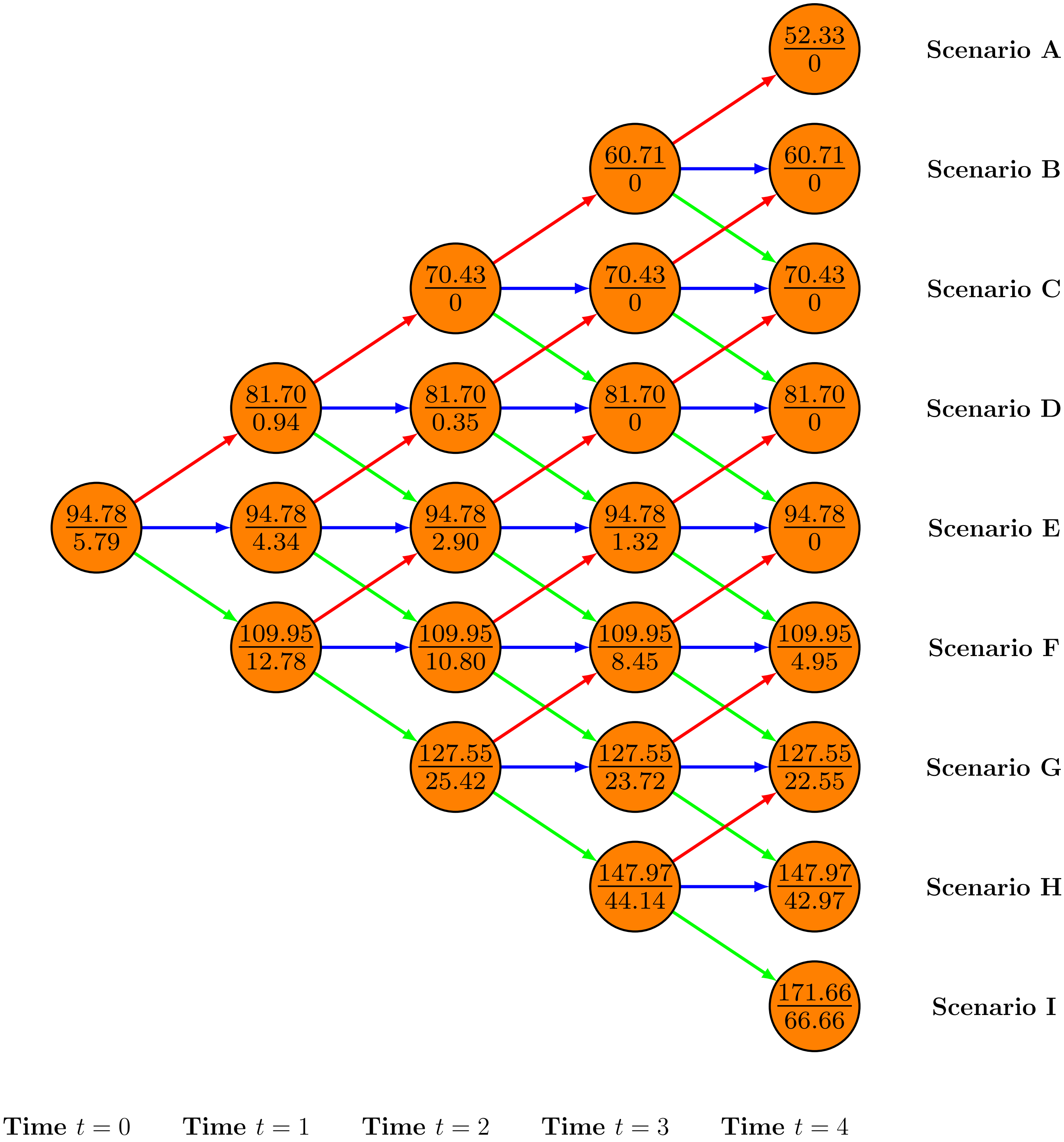
We simulate option pricing with the following parameters.

- Underlying security is of The Walt Disney company.
- Time steps to maturity $N = 4$ (assumed).
- $T = 1$ in year is time to maturity.
- Stock price at time $T = 0$ is $S_0 = 94.78$ taken from *yahoo finance* as on 12-13-2022

- Strike price is $K = 105$ taken from *yahoo finance* as on 12-13-2022.
- Historical volatility denoted by $\sigma = 0.21$ calculated on the daily closing adjusted asset prices for the period 12-12-2021 to 12-13-2022.
- $r = 0.045$ taken from *Bloomberg website*.
- $\Delta t = \frac{T}{N} = 0.25$.
- By plugging in the values of $r, \sigma, \Delta t$ in equations 2, 3, 4, 5, and 6 we get u, d, p_u, p_d, and p_m as follows. $u = 1.16, d = 0.86, p_u = 0.26, p_d = 0.23, p_m = 0.49$.

The Trinomial price tree with corresponding option price at each node under the given parameters are shown below in Fig 1. Our objective is to find the value of call option C_0 at time $\boldsymbol{t = 0}$ by using equation 7. Using the strategy mentioned in methodology section we calculate option price at time $\boldsymbol{t = 0}$ is $\boldsymbol{C_0 = 5.79}$. The red, blue, green edges denote down, same, and up movements respectively in asset prices from a previous node.

Fig 1: Stock price, value of call option under four time steps Trinomial model



5 Convergence

Now, we use the following **Black-Scholes** (BSM) formula to find the price of call option.

$$C_0 = S_0.N(d_1) - K.e^{-r.T}.N(d_2) \tag{8}$$

$$d_j = \frac{\ln(\frac{S_0}{K}) + (r + (-1)^{j-1}\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{9}$$

$$N(d_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_j} e^{-t^2/2} dt \tag{10}$$

$$\text{where } j = 1, 2$$

By plugging in the values of S_0, K, r, T, σ in equations 8, 9, and 10 we get $C_0 = 5.66$.

As per section no. 5 of the referenced paper we attempted to see the convergence between Trinomial and BSM as we increase the step size from 4 to $5, 6, \ldots, 50$. We found that Trinomial model converged to BSM. A few values of option price are given below as we increase the step size.

Table 1: Values of call option

Step Size	C_0
4	5.768441317316525
5	5.768168434686909
6	5.7657936688239335
$\vdots$	$\vdots$
48	5.681407995096132
49	5.681303049833277
50	5.68121643163421

The convergence also shows in the below figure 2. We can see that as increase out time steps from 4 to $5, 6, \ldots, 50$, the option price is converging to the value we find with BSM i.e. 5.66.

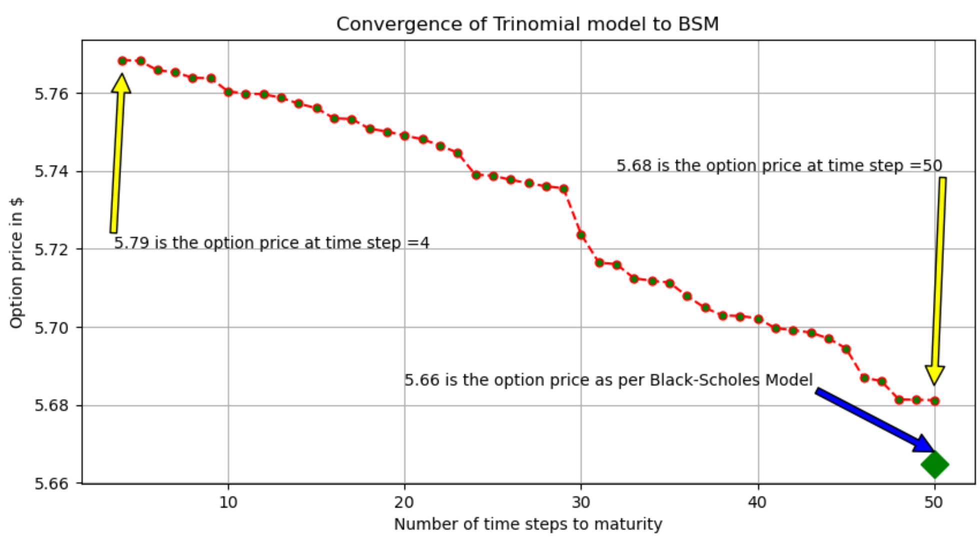


Figure 2 Convergence of Trinomial to BSM

References

[1] Referenced Paper
http://www2.warwick.ac.uk/fac/sci/maths/people/staff/oleg_zaboronski/fm/trinomial_tree_2008.pdf