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A Literature Review of the Theory and Applications of Convex Lifts

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Abstract

This literature review provides a brief overview of the theory and applications of lifts of convex sets. A lift of a convex set is a higher dimensional convex set that linearly projects onto the given set. Many convex sets have dramatically simpler structures when lifted to a higher dimension. Such convex lifts have significant implications for the field of optimization. In this paper we motivate the need for such convex lifts and provide examples to build an intuitive understanding of the structures of these sets.

Given a convex set, the question of whether a polyhedral or spectrahedral lift exists is a fundamental and quite challenging problem. We would ideally either like to find a (low-complexity) polyhedral or spectrahedral lift, or alternatively find an "obstruction" proving that no such lift exists. In this paper we explain the connection between the existence of lifts of a convex set and certain structured factorizations of its associated slack operator, providing some theoretical bounds on the dimensions of the lifts possible. This review will strictly limit itself to the lifts of convex set and mainly focus on the lifts of polytopes and spectrahedra.

Keywords: Convex Lifts, Semidefinite Programming, Conic Programming, Non-Negative Factorization

1. Introduction

The representation of convex sets is a crucial aspect of algorithmic primitives for cases such as conve programs, membership testing and volume calculation. The study of convex sets is of key importance in optimization though it finds uses in several varied domains. On well established idea in the literature that has been of great academic interest is methods to represent convex sets as a linear projection of "simpler", higher dimensional convex sets.

The idea is quite powerful yet elegant as we can show through several examples how sets with highly complicated descriptions in their ambient space of then admit lifts to convex sets that provide more concise descriptions. These lifted sets, linearly projected onto to the set in question. We describe this method pictorially in Figure 1. Lifting to such concise sets can lead to considerable algorithmic efficiencies for applications such as conic programs over the initial set. These higher dimensional descriptions effectively exploit the non-uniqueness of possible preimages of a projection, to find a more convenient description. In this literature review we will give the reader a brief overview of the theory and applications of convex lifts, by providing a intuitive motivation and discussing some key results in the field. The following excellent survey on convex lifts served as our primary source fot this report (Fawzi et al., 2022).

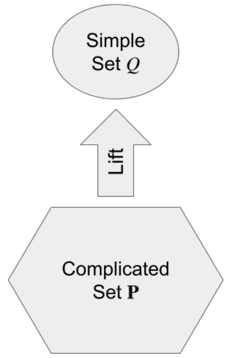


Figure 1: Illustration of a Lift

1.1. Motivation

Why perform a lift? Let $Q \in \mathbb{R}^l$ and $P \in \mathbb{R}^n$, where $l > n$. Then Q is a lift of P if P can be written as a linear projection of Q as $P = \pi(Q)$. Such a Q is called a *lift* of P. In addition, if both l and the number of facets of Q are small in size (polynomial in n) then the given lift is called a small/compact lift. So a convex problem $\min_{x \in P}\langle c, x\rangle$ is lifted as

$$\min_{x \in P}\langle c, x\rangle = \min_{y \in Q}\langle c, \pi(y)\rangle.$$

The right hand side is a optimization problem in Q which having a more concise construction maybe much easier to solve. To further motivate this point, consider linear programming, which is an optimization problem that optimizes a linear function over a convex polytope. Each facet of this polytope has an associated half-space constraint and the number of facets can be thought of as its complexity as it affects the efficiency of algorithms. Take for example the test of membership for a polytope. Its complexity would depend on the number of facets as a point would have satisfy each of the constraints associate with the facets. So a polytope lift Q with a lower number of facets would greatly improve the efficiency of algorithm.

1.2. Motivating Examples

Now we show two well known examples in the literature of convex lifts that illustrate the simplification in structure achieved by lifting.

Example 1.1 (l_1 **norm ball).** Consider the l_1 norm ball in $\mathbb{R}^n$ which is also the n dimensional cross-polytope, defined as the convex hull of the $2n$ vectors $\pm e_i$, where $e_i \forall i \in \{1, 2 \ldots n\}$ are the standard basis vectors. The 3-dimensional cross-polytope C_3 is illustrated in Figure 2 Note this polytope has 2^n facets and $2n$ vertices and is described as,

$$C_n = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} \pm x_i \leq 1\}.$$

Let $\pi(\cdot) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ be the projection $(x, y) \rightarrow x$ and $Q = \{(x, y) \in \mathbb{R}^{2n} | \sum_{i=1}^{n} y_i = 1, -y_i \leq x_i \leq y_i\}$. Note Q is a polytope with only $2n$ facets and a dimension of $2n$, increasing our dimension only by a factor of 2. $\square$

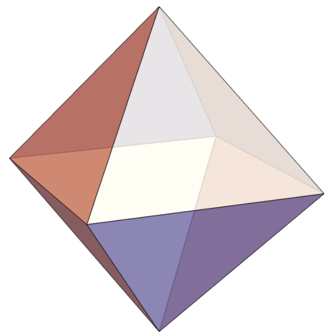
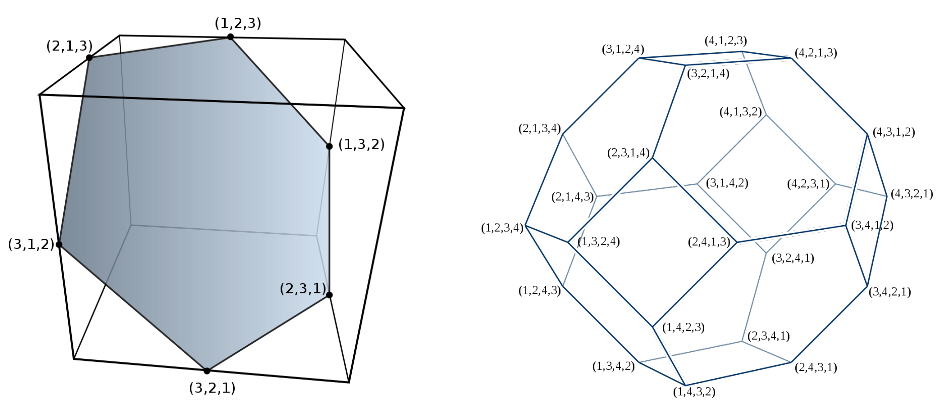


Figure 2: The cross-polytope $C_3 \subset \mathbb{R}^3$

The above example of the cross-polytop admits a rather compact lift. This however is not surprising as the initial polytope had an exponential number of facets 2^n but the number of vertices was quite small as $2n$, allowing for a very compact lift.

Example 1.2 (**Permutahedron** Π_n**).** The permutahedron $\Pi_n \subset \mathbb{R}^n$ is the convex hull of the $n!$ vectors that are permutations of $(1, 2 \ldots n)$. Note that Π_n always has a dimension of $n-1$,2^n-2 facets and $n!$ vertices. Clearly making the permutahedron a more complicated structure than the cross-polytope. You can find illustrations of Π_3 and Π_4 in Figure 3.



(a) The Permutahedron $\Pi_3 \subset \mathbb{R}^2$ (b) The Permutahedron $\Pi_4 \subset \mathbb{R}^3$

Figure 3: Illustration of Permutahedron Π_n

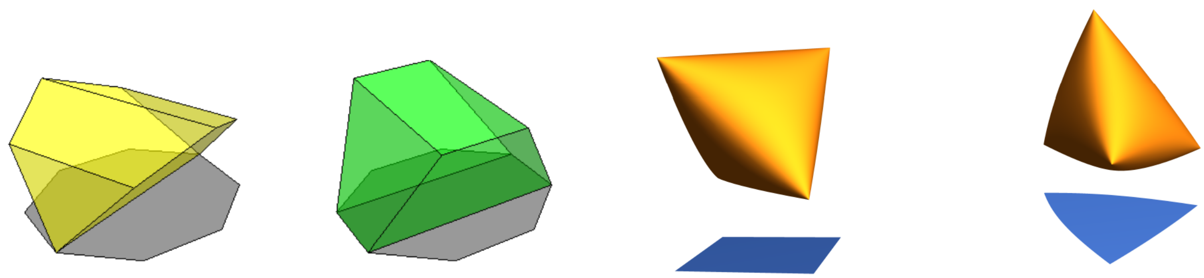
The Birkhoff polytope $B_n \subset \mathbb{R}^{n\times n}$, which is the convex hull of the set of permutation matrices in $\mathbb{R}^{n\times n}$, forms a lift for Π_n. The Birkhoff-von Neumann theorem states that the Birkhoff polytope coincides with the set of doubly stochastic matrices in $\mathbb{R}^{n\times n}$ (Birkhof, 1946; Neumann, 1963).

$$B_n = \{X \in \mathbb{R}^{n\times n} | \sum_{i=1}^{n} X_{ij} = 1, \sum_{j=1}^{n} X_{ij} = 1, X_{ij} \geq 0\}$$

The linear projection $\pi(X) : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ is defined as $x = [1, 2 \dots n]X$. Note that $l = n^2$ and number of facets (from the non-negativity condition) is n^2. $\square$

This gives the lift to the Birkhoff polytope a linear extension complexity of $o(n^2)$. Interestingly Goemans proved that the linear extension complexity of $\Pi_n$is $o(n \log(n))$ (Goemans, 2015). The proof involves an explicit construction of a lift using an Ajtai-Koml´os-Szemer´edi (AKS) sorting network (M. Ajtai and Szemerédi, 1983).

Remark 1.1 (**Lifts are Non-Isomorphic**). *Lifts in general are far from unique (isomorphic) and can be transformed into any class of convex sets. Take the following examples.*



(a) Two non isomorphic lifts for an octagon (b) Two different projections of the elliptope in $\mathbb{R}^3$

In Figure 4(a)subfigure the underlying regular octagon admits to very different polytope lifts. These two lifts are non-isomorphic, since the combinatorial structure of the polytope is different but both have the minimal number possible facets which in this case is 6. So they are equally efficient lifts.

In Figure 4(b)subfigure we illustrate how to different linear images of the elliptope in $\mathbb{R}^3$ leads to very different classes of sets. The projection on the left is a 2-dimensional rectangle and hence a polytope while the image on the right is a convex shape that can only be classified as SDP representable. $\square$

2. The Conic Point of View and General Definitions

This paper will focus on the formal setting of *conic lifts*, which will permit a generalized treatment of polytopes and spectrahedra. A conic lift will permit a lift into any class of sets that can be described as an affine intersection of a proper cone K. When K is $\mathbb{R}^n_+$ the lift is to a polytope, and when K is S^n_+ the lift is to a spectrahedra. We now make some formal definitions which we will use to prove some key results for conic lifts.

Definition 2.1 (The K lift). Let $C \subset \mathbb{R}^n$ be a convex set and $K \subset \mathbb{R}^l$ be a proper cone. We say C has a K lift if $C = \pi(K \cap L)$ where L is a affine space in $\mathbb{R}^l$ and $\pi(\cdot) : \mathbb{R}^l \to \mathbb{R}^n$ is a linear map. The convex set $K \cap L$ is called the K-lift of C.

If K is $\mathbb{R}^l_+$ (respectively, $\mathbb{S}^l_+$) then the we that C has a linear (respectively, spectrahedral) lift.

Definition 2.2 (The K-extension complexity)**.** The smallest m for which a convex set $C = \pi(\mathbb{R}_+^m \cup L)$ is called the *linear extension complexity of* C, and the smallest m for which $C = \pi(\mathbb{S}_+^m \cup L)$ is called the *semidefinite extension complexity* of C.

The K-extension complexity though has a simple definition is quite challenging to evaluate. We will see later in this paper that the evaluation is NP-hard.

Remark 2.1 (Inclusion Relationship)**.** *The inclusion relationship between the different types of convex sets admits the same structure as conic programs. We describe this relationship through an illustration in Figure 4.*

Convex sets having a polyhedral lift = polyhedra	$\subsetneq$	Convex sets having a second-order cone lift	$\subsetneq$	Convex sets having a spectrahedral lift	$\subsetneq$	All convex semialgebraic sets

Figure 4: Inclusion Relationship of Convex Lifts

3. Slack Matrices and Linear Extension Complexity

3.1. Slack Matrix

Definition 3.1 (Slack Matrix). Let a polytope P have the set of vertices $\mathcal{V}$ and half-spaces (associated with each facet) $\mathcal{H}$. The slack matrix is the substitution of each vertex in to each half-space inequality. Typically with vertices column indexed and half-spaces row indexed.

Example 3.1 (Slack Matrix)**.** Take the convex polytope $P \subset \mathbb{R}^3$ defined by the convex hull of its seven vertices $(\pm 1, \pm 1, -1/2)$, $(\pm 1, -1, 1/2)$ and $(0, -1, 3/2)$. This polytope can also be alternatively defined by its seven affine inequalities which are,

$$1 - x \geq 0, \quad 1 + x \geq 0, \quad 1 + 2z \geq 0, \quad 1 + y \geq 0, \quad 1 - 2y - 2z \geq 0,$$
$$1 - x - \frac{1}{2}y - z \geq 0, \quad 1 + x - \frac{1}{2} - z \geq 0.$$

The slack matrix combines the $\mathcal{V}$ and $\mathcal{H}$ description by calculating the slacks of each vertex on each constraint. The elements that are 0 imply those constraints are tight and that the respective vertex lies on the respective half plane. The associated 7×7 slack matrix we calculate is,

$$S_P = \begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 & 4 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 4 & 0 & 2 & 4 & 0 & 2 & 0 \\ 3 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 2 & 2 & 0 \end{bmatrix}$$

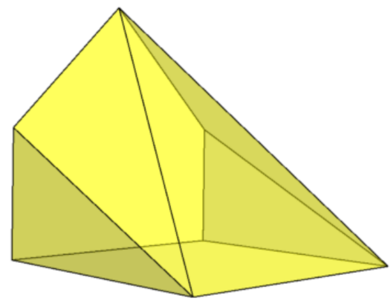


Figure 5: A visualization of the polytope P

3.2. Non-Negative Factorization and Yannakakis' Theorem

Definition 3.2 (Non-Negative Factorization and Rank)**.** A non-negative factorization of a non-negative matrix $M \in \mathbb{R}^{n \times k}$ is $M = A^T B$ where $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times k}$. The inner dimension m is called the size of the factorization and the smallest admissible m is the non-negative rank of M.

We now show how the linear extension complexity of a polytope P is in-extrinsically linked to the factorization of its slack matrix S_P. This one the most fundamental result in the field of convex lifts.

Theorem 3.1 (Yannakakis' Theorem). *The linear extension complexity of a polytope P with slack matrix S_P is equal to the non-negative rank of its slack matrix rank$_+(S_P)$(Yannakakis, 1991).*

Proof Let a polytope P be cut by the inequalities $h_i^T x \leq \beta_i$ for $i = 1, ...f$ .And we know that $S_P = A^T B$ is a non-negative factorization of the slack matrix with inner dimension m. We let a_i for $i = 1, ..., m$ be the columns of the matrix A we will show that:

$$P = \{x \in \mathbb{R} : \exists y \in \mathbb{R}_+^m s.t. a_i^T y = \beta_i - h_i^T x, i = 1, ..., m\} \tag{1}$$

First we see why (1) implies the existence of a lift of dimension m.This equality rewrites P as the projection of a polytope with m facets. P has a size m polyhedral lift. To see why the equality holds, we note that if p is in the right hand side we must have $h_i p = \beta_i a_i y$ for some nonnegative y. Then, it follows that since $a_i y$ is nonnegative, we have that $h_i p \leq \beta_i$ for all i, and so $p \in \mathbb{P}$. Alternatively ,if v_j is a vertex of $\mathbb{P}$,we let $y = b_j$ be the corresponding column of B which is non-negative. Then we have that $a_i y = a_i b_j = [S_P]_{ij} = \beta_i h_i v_j$. Consequently v_j , and indeed all vertices of P, are in the right hand side set. Since the right hand side set is convex, all of P must be contained in it, and we have equality between the two sets.

Now consider that P can be written as the projection of a polytope Q with m facets. Take the slack matrix S_Q and keep only columns that correspond to vertices that project to vertices of P, and we call this reduced $m \times |\mathcal{V}|$ dimensional matrix, $\hat{S}_Q$. Now any facet inequality on P can be pulled back by the projection to a valid inequality on Q, so by a version of Farkas Lemma, it can be written as a non-negative combination of facet inequalities of Q. Record the coefficients as column vectors, and form a matrix $A \in \mathbb{R}_+^{m \times f}$ with those columns. Then $S_P = A\hat{S}_Q$ is a non-negative factorization of S_P with inner dimension m. ■

Theorem 3.2 (**Vavasis' Theorem**). *The computation of the non-negative rank of a matrix $M \in \mathbb{R}^{n\times k}$ is NP-hard. (Vavasis, 2010).*

Vavasis' theorem shows us why finding the linear extension complexity of a polytope P is so challenging and computationally intensive, since the non-negative rank of its slack matrix is NP-hard.

Example 3.2 (**Slack Matrix Continued**). Continuing the example 3.1, take the convex polytope P defined by its vertices $(\pm 1, \pm 1, -1/2)$, $(\pm 1, -1, 1/2)$ and $(0, -1, 3/2)$. Which can also be alternatively defined by its seven affine inequalities which are,

$$1 - x \geq 0, \quad 1 + x \geq 0, \quad 1 + 2z \geq 0, \quad 1 + y \geq 0, \quad 1 - 2y - 2z \geq 0,$$
$$1 - x - \frac{1}{2}y - z \geq 0, \quad 1 + x - \frac{1}{2} - z \geq 0.$$

The associated 7×7 slack matrix has a non-negative factorization as

$$S_P = \begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 & 4 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 4 & 0 & 2 & 4 & 0 & 2 & 0 \\ 3 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The linear extension complexity of P in this case is 6. There by showing us that a linear lift into $\mathbb{R}^6_+$ exists. □

4. Slack Operators

4.1. Slack Operator Definitions and Examples

Now to study the existence of lifts for convex sets in general we need to generalize our definition of the slack matrix to a slack operator. This slack operator relates the geometries of the extreme points of the set in question to the extreme points of its polar. Note that to simplify the results we assume C is compact. Recall that an extreme point of a convex set C is a point that cannot be represented as a convex combination of any other points. We first start by defining the polar set of a convex set.

Definition 4.1 (**Polar Set**). Given a convex set C, its polar set C° is defined as,

$$C^\circ = \{y \in \mathbb{R}^N \ : \ \langle x, y \rangle \leq 1, \forall x \in C\}$$

.

We also use the notation ext(C) to denote the set of extreme points of C. With the extreme points and polar sets defined we now describe the slack operator of a convex set.

Definition 4.2 (Slack Operator). Given a compact convex set C, its slack operator is the map $s_C : \text{ext}(C) \times \text{ext}(C^\circ) \to \mathbb{R}_+$ defined as

$$s_C(x, y) = 1 - \langle x, y \rangle, \ x \in \text{ext}(C) \ y \in \text{ext}(C^\circ).$$

When C is a polytope, both C ad C° have a finite number of extreme points (their vertices) and the slack operator collapses to the slack matrix.

We now show a example of how to sonctruct the slack operator.

Example 4.1 (Slack Operator). Let $C = \{(x, y) \in \mathbb{R}^2 : (1 + x)^2(x - 1) + y^2 \leq 0, x \geq -1\}$. This convex set is a bounded by a cubic curve and its boundary is paramterized by

$$(1 - v^2, 2v - v^3)$$

with $v \in [-\sqrt{2}, \sqrt{2}]$ and every boundary point is extreme. The polar of C is defined as the convex hull of the cardiod defined by

$$4x^4 + 32y^4 + 13x^2y^2 + 18xy^2 - 4x^3 - 27y^2 = 0.$$

As illustrated in Figure 6, not all boundary points of C° are extreme points.

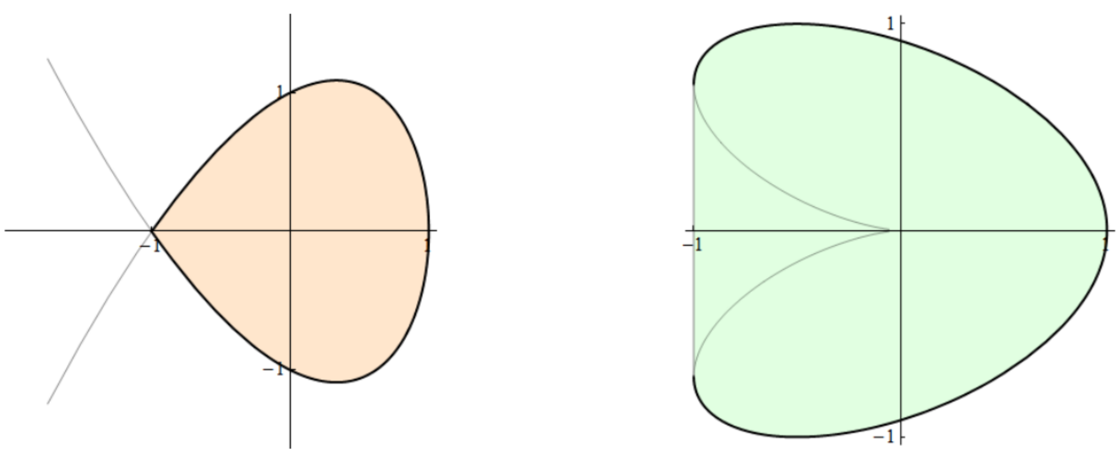


Figure 6: Illustrations of the convex set C (on the left) and C° (on the right) with their extreme points higlighted.

The extreme points of C° can be parameterized with $u \in [-\sqrt{2}, \sqrt{2}]$ as,

$$\left(\frac{2 - 3u^2}{2 - u^2 + u^4}, \frac{2u}{2 - u^2 + u^4} \right).$$

Therefore we paramterize the slack operator $s_C(u, v) : [-\sqrt{2}, \sqrt{2}]^2 \to \mathbb{R}_+$ where ,

$$s_C(u, v) = 1 - \frac{(2 - 3u^2)(1 - v^2) + 2u(2v - v^3)}{2 - u^2 + u^4} = \frac{(2 - v^2)(v - u)^2 + (u^2 - v^2)^2}{u^4 + (2 - u^2)},$$

it is obvious from the last expression that s_C is non-negative. $\square$

4.2. Cone Factorizations of Slack Operators

The goal of the generalization to the slack operator was to study the geometric interpretations of its factorizations. So now as we proceed from polyhedral lifts to a general K- cone lift, we need to also redefine what a non-negative factorization is. To that end recall that a dual of a cone K is defined as

$$K^* = \{y \in \mathbb{R}^l \,:\, \langle x, y\rangle \geq 0 \forall x \in K\}.$$

We now can define the non-negative cone factorizations of the slack operator with respect to the geometries of the cone K and its dual K^*.

Definition 4.3 (K-Factorization of Slack Operators)**.** Let K be a proper cone and K^* its dual and C be a compact convex set with its slack operator s_C. A factorization of s_C *through* K or K*-factorization of* s_C is the pair of maps $A : \text{ext}(C) \to K$ and $B : \text{ext}(C^\circ) \to K^*$ such that $s_C(y, x) = \langle A(x), B(y)\rangle$ for all $x \in \text{ext}(C)$ and $y \in \text{ext}(C^\circ)$.

This generalization now allows us to factorize any compact convex set C through a cone K (if such a factorization exists). We now present the theorem that can be viewed as a generaliztion of Yannakakis' theorem for the existence of linear lifts certified through non-negative factorizations.

Theorem 4.1 (Generalization of Yannakakis' Theorem)**.** *If a convex body C has a proper K-left then its slack operator, s_C, has a K-factorization. Reciprocally, if s_C has a K -factorization then C admits a K-lift (Gouveia et al., 2013)*

Proof This proof is just a generalization of the proof of Theorem 3.1. Let A and B form a K-factorization of s_C. Then it is trivial to see that the following holds true,

$$C = \{x \in \mathbb{R}^n \,:\exists X \in K \text{ s.t } \langle X, B(y)\rangle = s_C(y, x), \forall y \in \text{ext}(C^\circ)\},$$

since the non-negativity of the slack operator forces $s_c(y, x) \geq 0$ for all $y \in \text{ext}(C^\circ)$, therefore the right hand side is a subset of C. However plugging in $X = A(X)$ give us $\text{ext}(C)$ is a subset of the right hand side, there by validating the above equation. Although there are (potentially) infinitely many linear inequalities, they necessarily cut out a finite dimensional space and we can eliminate x and explicitly get C to be a projection of a slice of K.

Let C have a K-left, i.e., $C = \pi(K \cap L)$, for some linear map π and affine space L. Now for $x \in \text{ext}(C)$, define $A(x)$ to be a transform that picks any element in the pre-image $\pi^{-1}(x) \cap (K \cap L)$ (since π is a many to one map). Now for any $y \in \text{ext}(C^\circ)$ we can pull back the inequality $1 - \langle x, y\rangle$ (through π^{-1}) to a linear inequality on an element in $K \cap L$ and y. The hypothesis that $K \cap L$ has a non-empty relative interior (Slater's condition) can be shown to guarantee that any valid inequality on $K \cap L$ is equal to an inequality $\langle Y, \cdot\rangle$ for some $Y \in K^*$ (strong duality). We can define a map that picks such a Y as $B(y)$. We can now represent the slack operator $s_C(x, y) = 1 - \langle x, y\rangle$ as $\langle A(x), B(y)\rangle$. Hence completing the proof. ■

By setting K to be $\mathbb{S}^n_+$ the above theorem lets use ascertain the spectrahedral extension complexity of the set or similarly by taking K to be $\mathcal{L}^n$ we can ascertain the second order cone extension complexity of the set. Note how the linear extension complexity for a spectrahedra will always be ∞ due to the inclusion relationship, discussed previously. So note that extension complexity of a set need not be finite. We now illustrate this powerful theorem by revisiting an example and factorizing its slack operator.

Example 4.2 (Slack Operator Continued)**.** We continue Example 4.1 to show that factorizing its slack operator give us a certificate that it admits a K-lift. Recall that the slack operator of $C = \{(x, y) \in \mathbb{R}^2 : (1 + x)^2(x - 1) + y^2 \leq 0, x \geq -1\}$ was given by the function $s_C(u, v)$ over the parameterized domain $[-\sqrt{2}, \sqrt{2}]^2$ and

$$s_C(u,v) = 1 - \frac{(2-3u^2)(1-v^2) + 2u(2v - v^3)}{2 - u^2 + u^4} = \frac{(2-v^2)(v-u)^2 + (u^2 - v^2)^2}{u^4 + (2 - u^2)}.$$

In this case we have parameterized both points in $\mathrm{ext}(C)$ and $\mathrm{ext}(C^\circ)$ by the set $[-\sqrt{2}, \sqrt{2}]^2$. We can therefore think of (v, u) as a labeling of the points in $ext(C) \times ext(C^\circ)$. We now claim we can factorize $s_C(u, v)$ through the cone $\mathbb{S}^3_+$. We define,

$$A(v) = \begin{bmatrix} 1 & 0 & 1-v^2 \\ 0 & 2-v^2 & v(2-v^2) \\ 1-v^2 & v(2-v^2) & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1-v^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1-v^2 \end{bmatrix}^T + (2-v^2) \begin{bmatrix} 0 \\ 1 \\ v \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ v \end{bmatrix}^T$$

and

$$B(u) = \frac{1}{2-u^2+u^4} \begin{bmatrix} (u^2-1)^2 & -u(u^2-1) & u^2-1 \\ -u(u^2-1) & u^2 & -u \\ u^2-1 & -u & 1 \end{bmatrix} = \frac{1}{2-u^2+u^4} \begin{bmatrix} u^2-1 \\ -u \\ 1 \end{bmatrix} \begin{bmatrix} u^2-1 \\ -u \\ 1 \end{bmatrix}^T.$$

Since $A(v)$ and $B(u)$ are always positive semi-definite and S^3_+ is a self dual cone, the above transformation give us a proper $\mathbb{S}^3_+$ factorization of s_C. This factorization is a certificate for the existence of a spectrahedral lift of C to $\mathbb{S}^3_+$. □

5. Conclusion

In this review paper we provided a brief tour of the theory and applications of convex lifts. We started of by motivating the need for such convex lifts, how it affects the algorithmic efficiency of convex programs and provided well-known examples to give an intuitive understanding of convex lifts. We then defined the general concept of a cone lift and provided an intuitive analogue to conic programs. We then showed how the existence of polytope lifts is related to the non-negative factorization of the sets slack matrix and proved the key result (in short), while providing examples for each. Finally, we extended the idea of slack matrices to slack operators and non-negative matrix factorizations to K- factorizations of operators to prove key results on the existence of K-lifts. We then provided an example to illustrated how slack operators could be evaluated and K-factorizations could be achieved.

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