The Local Randomization Approach

Northwestern Causal Inference Workshop

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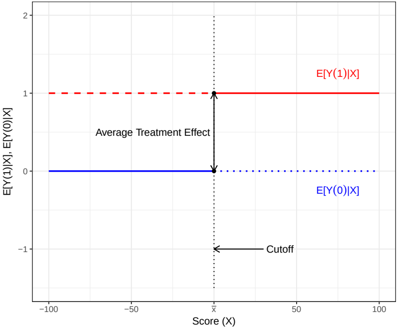
Overview: RD as a randomized experiment

- Informally, RDDs are interpreted as local experiments
- Idea: close enough to the cutoff, some units were "lucky"
  - Units slightly above and below the cutoff are comparable
- Our framework so far does not formalize this interpretation
  - Based on continuity of regression functions only
- When can we interpret an RD as a local experiment?

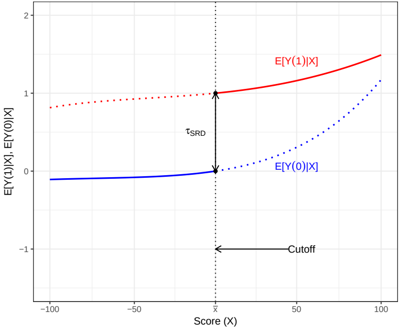
Overview: RD as a randomized experiment

- Any experiment can be thought of as an RD where:
  - The score is a uniform random variable
  - The cutoff is chosen to ensure a given probability of treatment
- Example:
  - Each unit is assigned a score $X_i \sim U[0,1]$
  - $D_i = \mathbb{1}(X_i \geq c) \Longrightarrow \mathbb{P}[D_i = 1] = 1 - c$
- Key issue: score unrelated to potential outcomes **by construction**

Experiments versus RD designs

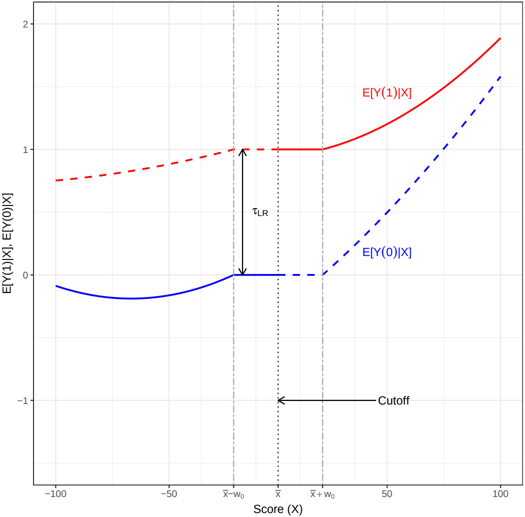


(a) Randomized Experiment



(b) RD Design

Local randomization RD



Local randomization approach to RD

- There is a window $W_0 = [c - w, c + w]$ in which:
  - Probability distribution of X_i is unrelated to individual characteristics

$$\mathbb{P}[X_i \leq x | X_i \in W_0] = F_0(x), \quad \forall\, i$$

  - Potential outcomes not affected by value of the score:

$$Y_i(d, x) = Y_i(d)$$

- Note: stronger assumption than continuity
  - Potential outcomes are a constant function of the score

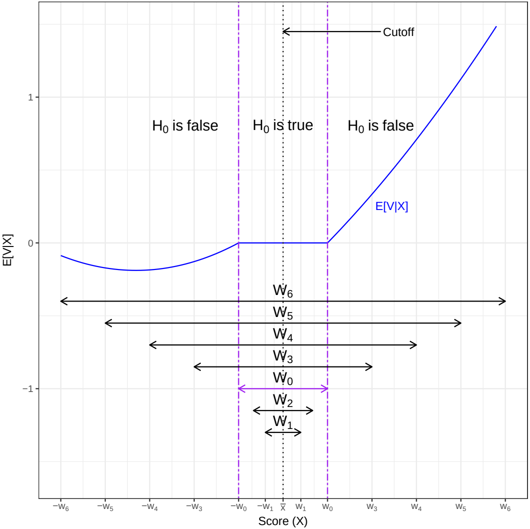
Window selection

- Under random assignment, covariates should be balanced:

$$\mathbb{P}[Z_i \leq z | D_i = 1] = \mathbb{P}[Z_i \leq z | D_i = 0]$$

- Can use this idea as a window selection criterion:
  - Find window in which all covariates are balanced
- Iterative procedure:
  1. Choose a test statistic (diff. means, Kolmogorov-Smirnov,...)
  2. Choose an initial "small" window $W_0^{(1)} = [c - w_{(1)}, c + w_{(1)}]$
  3. Test null that covariates are balanced above and below c
  4. Repeat until null hypothesis is rejected

Window selection procedure



Estimation and inference

- Once W_0 is found, proceed as in a randomized experiment

$$\hat{\tau} = \bar{Y}_1 - \bar{Y}_0$$

- Covariate-balance criterion may yield windows with few obs
- Inference based on large-sample approximations may not be reliable
- Alternative approach: Fisherian randomization inference
    - Local randomization approach

Fisher's approach to inference

- Alternative approach to inference in randomized experiments
- Finite-sample exact
  - Valid for any sample size
  - No distributional assumptions or approximations
- Conditional on a particular sample
  - Potential outcomes are fixed
  - All randomness due to treatment assignment
- Tests a specific type of hypotheses: *sharp null*
  - E.g. no treatment effect on any unit

The Lady Tasting Tea

The Design of Experiments (Sir Ronald A. Fisher, 1935)

- *"A lady declares that by tasting a cup of tea made with milk she can discriminate whether the milk or the tea infusion was first added to the cup."*
- *"We will consider the problem of designing an experiment by means of which this assertion can be tested."*
- *"Our experiment consists in mixing eight cups of tea, four in one way and four in the other, and presenting them to the subject in a random order."*
- *"Her task is to divide the 8 cups into two sets of 4, agreeing, if possible, with the treatments received."*

The Lady Tasting Tea

- There are $\binom{8}{4} = 70$ ways to arrange the cups
- H_0: the lady is guessing at random
- Statistic: number of correct guesses
- Suppose she guesses all cups correctly
  - Probability of all guesses correct under $\mathsf{H}_0 : \frac{1}{70} \approx 0.014$
  - 0.014 is a finite-sample exact p-value
  - No distributional assumptions or approximations

Randomization inference

- Consider a randomly assigned treatment D_i, $\mathbb{P}[D_i = 1] = 1/2$
- Suppose potential outcomes are non-random
    - Inference conditional on the observed sample
- All randomness comes through the assignment mechanism:

$$Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i)$$

- Sharp null hypothesis of no treatment effect:

$$\mathsf{H}_0 : \quad Y_i(1) = Y_i(0), \quad \forall\, i = 1, \ldots, n$$

Randomization inference

$$\mathsf{H}_0: \quad Y_i(1) = Y_i(0), \quad \forall\, i = 1, \ldots, n$$

- Under H_0 we can impute all the missing potential outcomes
- For any test statistic, T, under H_0

$$T(\mathbf{Y}, \mathbf{D}) = T(\mathbf{Y}_0, \mathbf{D})$$

- The distribution of T is given by the **known** distribution of $\mathbf{D}$

Randomization inference

$$\mathsf{H}_0: \quad Y_i(1) = Y_i(0), \quad \forall\, i = 1, \ldots, n$$

- Inference procedure:
  - Calculate the observed value of the statistic, $T_{\textsf{obs}}$
  - Calculate T for all possible permutations of $\mathbf{D} = (D_1, \ldots, D_n)$
  - Randomization inference p-value:

$$p = \mathbb{P}[T(\mathbf{Y}_0, \mathbf{D}) \geq T_{\textsf{obs}}]$$

- If the number of permutations is too large, use a random sample

Empirical illustration: incumbency advantage

- Cattaneo, Frandsen and Titiunik (2015, JCI)
  - Incumbency advantage in U.S. Senate
- Data:
  - $Y_i =$ election outcome at $t+1$ ($=1$ if party wins)
  - $D_i =$ election outcome at t ($=1$ if party wins)
  - $X_i =$ margin of victory at t ($c=0$)
  - Additional covariates

Software implementation

- Cattaneo, Titiunik and Vazquez-Bare (2016, Stata Journal)
- `rdlocrand` package:
  - `rdwinselect`: window selection
  - `rdrandinf`: randomization inference
  - `rdsensitivity`: sensitivity analysis
  - `rdrbounds`: Rosenbaum bounds

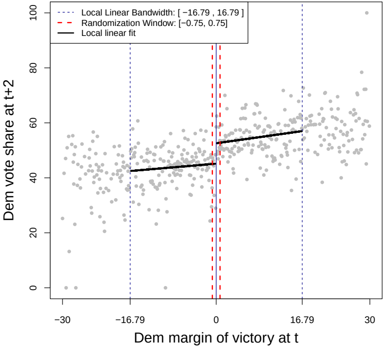
Choosing the window with `rdwinselect`

```
. rdwinselect demmv $covariates, wmin(.5) wstep(.125) reps(10000)

Window selection for RD under local randomization

Cutoff c = 0.00   | Left of c   Right of c          Number of obs   =        1390
------------------+------------------------         Order of poly   =           0
    Number of obs |       640          750          Kernel type     =     uniform
   1th percentile |         6            8          Reps            =       10000
   5th percentile |        32           38          Testing method  =   rdrandinf
  10th percentile |        64           75          Balance test    =       ttest
  20th percentile |       128          150

                  |  Bal. test        Var. name        Bin. test
Window length /2  |   p-value      (min p-value)        p-value    Obs<c    Obs>=c
------------------+---------------------------------------------------------------
            0.500 |     0.268      demvoteshlag2         0.230        9         16
            0.625 |     0.435          dopen             0.377       13         19
            0.750 |     0.268          dopen             0.200       15         24
            0.875 |     0.150          dopen             0.211       16         25
            1.000 |     0.069          dopen             0.135       17         28
            1.125 |     0.037          dopen             0.119       19         31
            1.250 |     0.062          dopen             0.105       21         34
            1.375 |     0.141        dmidterm            0.539       30         36
            1.500 |     0.092        dmidterm            0.640       34         39
            1.625 |     0.113        dmidterm            0.734       37         41
Variable used in binomial test (running variable): demmv
Covariates used in balance test: presdemvoteshlag1 population demvoteshlag1 demvoteshlag2
> demwinprv1 demwinprv2 dopen dmidterm

Largest recommended window is [-.75; .75] with 39 observations (15 below, 24 above).
```

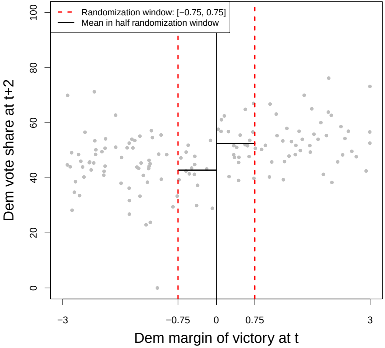
Randomization inference with rdrandinf

```
. rdrandinf demvoteshfor2 demmv, wl(-.75) wr(.75)

Selected window = [-.75 ; .75]

Running permutation test...
Permutation test complete.

Inference for sharp design

Cutoff c = 0.00   | Left of c  Right of c          Number of obs =          1390
------------------+------------------------        Order of poly =             0
    Number of obs |       595         702          Kernel type   =       uniform
Eff. Number of obs|        15          22          Reps          =          1000
  Mean of outcome |    42.808      52.497          Window        =   set by user
  S.D. of outcome |     7.042       7.742          H0:       tau =         0.000
           Window |    -0.750       0.750          Randomization = fixed margins

Outcome: demvoteshfor2. Running variable: demmv.
------------------------------------------------------------------------------------
                  |                 Finite sample          Large sample
                  |                 -------------   ------------------------------
        Statistic |        T            P>|T|        P>|T|    Power vs d =     3.52
------------------+-----------------------------------------------------------------
   Diff. in means |    9.689            0.001        0.000                    0.300
------------------------------------------------------------------------------------
```

Continuity-based vs local randomization analysis

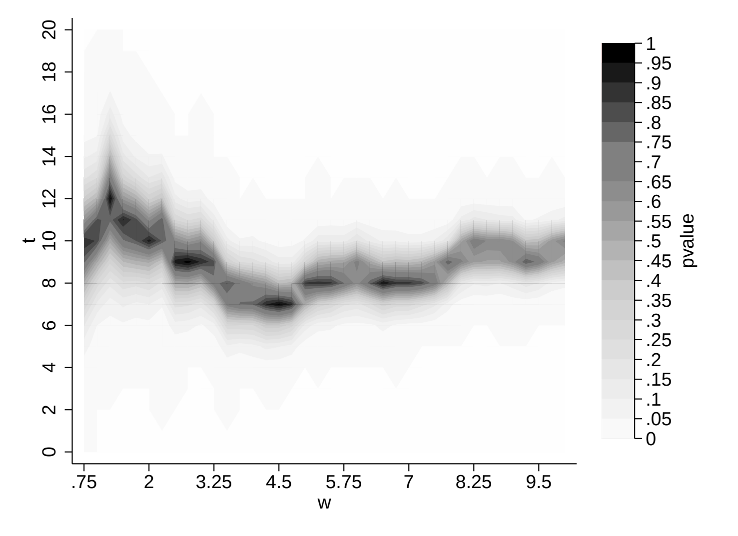


(a) Continuity-based analysis



(b) Local randomization analysis

Sensitivity analysis with `rdsensitivity`



Rosenbaum bounds with rdrbounds

```
. rdrbounds demvoteshfor2 demmv, gammalist(.8 1 1.2) wlist(.5 .75 1) reps(1000)
Calculating randomization p-values...
                                  w =       0.500        0.750        1.000
---------------------------------------------------------------------------
               Bernoulli p-value |         0.012        0.001        0.000
---------------------------------------------------------------------------

Running sensitivity analysis...
gamma  exp(gamma)                 w =       0.500        0.750        1.000
---------------------------------------------------------------------------
 0.80       2.23      lower bound |         0.006        0.001        0.000
                      upper bound |         0.068        0.015        0.002
---------------------------------------------------------------------------
 1.00       2.72      lower bound |         0.004        0.001        0.000
                      upper bound |         0.106        0.034        0.006
---------------------------------------------------------------------------
 1.20       3.32      lower bound |         0.003        0.001        0.000
                      upper bound |         0.168        0.060        0.017
---------------------------------------------------------------------------
```