(6) Show that:

$$\log(n!) = \Theta(n \log n).$$

We gotta show,

$$c_1 g(n) \leq \log(n!) \leq c_2 g(n)$$

For upper Bound $O(g(n))$.

$$\log(n!) = \log(1 \cdot 2 \cdot 3 \ldots (n-1)\, n)$$

$$= \underbrace{\log(1)}_{\leq \log(n)} + \underbrace{\log(2)}_{\leq \log(n)} + \underbrace{\log(3)}_{\leq \log(n)} + \ldots + \underbrace{\log(n)}_{\leq \log(n)}$$

$$\log(n!) \leq \log(n) + \log(n) + \log(n) + \ldots + \log(n)$$

$$\log(n!) \leq n \log(n)$$

Thus, $\log(n!) = O(n \log(n))$ has been proved,

Now for the lower bound,

$\log(n!) = \Omega(n \log(n))$

Consider taking $(n!)^2$ for this problem

$[n!]^2 = n! \times n!$

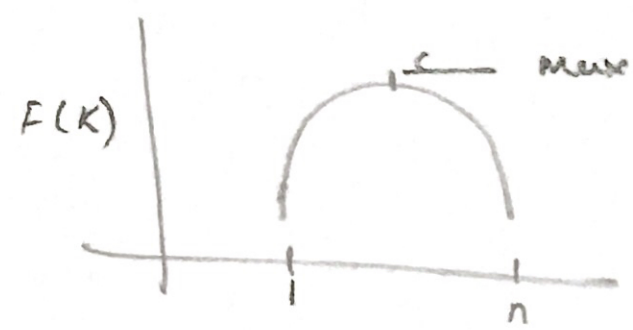
$$= [n \times 1] \times [(n-1) \times 2] \times [(n-2) \times 3] \times \dots \times (1 \times n)$$

$$= \prod_{K=1}^{n} (n-K+1) \times K$$

$$= \prod_{K=1}^{n} (-K^2 + nK + K)$$

Now, Let. $\boxed{F(K) = -K^2 + nK + K.}$ where $(1 \le K \le n)$

$F(K)_{min} = -1^2 + n \times 1 + n$ —— (i)



Now we know that,

$$(n!)^2 = \prod_{K=1}^{n} F(K) \geq \prod_{K=1}^{n} F(K)_{min}$$ —— (ii)

Sub (ii) by (i)

$$(n!)^2 \geq \prod_{K=1}^{n} n$$

$$(n!)^2 \geq n^n$$

log on both sides

$$2\log(n!) \geq n\log(n).$$

$$\boxed{\log(n!) \geq \frac{1}{2} n \log(n)}$$

This proves $\log(n!)$ $\Theta(n \log n)$

Thus, $\log(n!) = \Theta(n \log n)$