Multivariable & Vector Calculus

[Interactive] Class Notes

Boise State University, Math 275

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Introduction

Multivariable & Vector Calculus: the Mathematics of Space

Multivariable and Vector Calculus generalizes the theory and computational techniques from algebra, geometry, and single-variable calculus to higher dimensional settings.

Applications of the material introduced in this class (and further developed in later classes) include analyzing motion in 2- and 3-dimensional space, studying forces and fluid dynamics, and modeling processes that involve multiple parameters. Newtonian mechanics, special and general relativity, fluid dynamics, and electromagnetic theory all utilize concepts and methods from multivariable and vector calculus. If you're a math major; this class is preliminary to some of the most important areas of study in the field, including differential geometry and partial differential equations.

This course will present different challenges than you have encountered in previous math classes.

Lower-level classes primarily exist to introduce you to computational techniques and basic concepts. The majority of your work as a student is carrying out computational techniques introduced by your teacher, at various levels of difficulty. From this experience, you may have been led to believe that mathematics is a computational process that leads to a numeric solution.

As you progress to higher-level classes, however, you will see that computation is only one small part of what makes math so important and powerful.

In this class, most computational methods will be directly related to techniques learned in previous classes. So, instead of learning completely new computational methods, the goals of this class are:

- **Applying previously learned concepts and computational techniques** in more complex situations.
- **Exploring connections** between mathematics and the physical/spatial world.
- **Developing conceptual understanding** that will allow you to interpret and apply the mathematics of this class in your primary area of study.

A few of the big questions you will be exploring in this class are:

- **How do I generalize concepts** from single-variable calculus to multivariable calculus?
- **How do I apply computational techniques** from algebra, geometry, trigonometry and calculus in new contexts?
- **How do I use mathematics** to model (formulate, analyze, and solve) problems requiring either multi-variable input or output — in particular, mathematics applied to 2- and 3-dimensional space?
- **How do I connect** intuitive understanding of spatial phenomena to a formal mathematical framework?

Over the course of this semester, how you think about mathematics will change.

You will begin to understand that math is more than a collection of algorithms and computational techniques

You may run into challenges that have not experienced in previous math classes. If this feels frustrating or uncomfortable, don't be discouraged! It's a normal part of the learning process.

There is a commonly held belief that being "good" at math means you never run into any difficulties — but this is not the case. Success in math comes from a commitment to keep working on a problem until you understand it. This means putting in time and effort, but it also means that success is a result of work, not "natural ability" (whatever that means).

Keep working, talk to your classmates and to me, and always ask questions. This is a great subject, and worth taking the time to really understand what's going on.

— Shari

How to Use These Notes

The purpose of these Notes is twofold: they are the method by which you learn the concepts and computational methods of this course, and once completed, they provide a reference for future classes.

This is not a "teach yourself" class. These Notes are designed to provide clear guidance as you work through them. If at any time you are uncertain about what you are supposed to be learning, or what the instructions or goals are, please reach out to me for clarification. You can ask me questions about the material, and you can work with your classmates.

In order to use these Notes successfully, you need to **actively** engage with them. This is why I prefer to teach using this format (called "guided inquiry") rather than lecture. Lecture-based learning (including videos) lends itself to reliance on information received from an "expert". By contrast, these Notes guide you through a process of reading, inquiry, and discovery. My experience using them is that students see important points more clearly, ask more relevant and deeper questions, and remember information better.

Because it requires active participation, this process will be challenging at times — but by the end of this semester, you will develop more confidence and independence, and build critical thinking skills. And whenever you feel like you do need help, you can reach out to me or your classmates.

As you work on these Notes:

If something doesn't make sense, keep questioning and exploring until you understand. And don't hesitate to ask for help.

Differences between BSU Math 170/175 and BSU Math 275

If you have taken Math 170 and/or Math 175 at Boise State University, it is important that you understand a fundamental difference in the design of these classes:

Math 170/175:

- The focus of the course is learning new computational methods
- You learn most of the material from the online homework; the worksheets and notes in those classes are supplemental.

Math 275:

- The focus of the course is learning how to generalize and apply computational methods from previous classes to higher-dimensional settings. In other words, you are not learning many new computational methods; you are leaning new *concepts* where you apply computational methods you already know.
- You learn most of the material from these [Interactive] Course Notes; the online homework supports and reenforces what you learn here.

It is a natural instinct for students to want to work through the Course Notes as quickly as possible in order to get the homework on WebAssign, since previous classes centered the homework as the "important" part of class. **Do not rush through the Notes in this class.** The Notes are the primary way you learn course material, so give yourself time to work through them. Re-read as needed, and make sure you understand each question before moving on to assignments on WebAssign. Future-you will thank you.

The Structure of the Notes

Each topic has its own section in the notes, which includes:

An Introduction — broken down into what you need to review (**From the Toolbox**), objectives (**Goals**), an outline/summary (**Big Picture**) and a list of new **Definitions & Notation**.

Readings — (indicated by the heading **Read Me**), which introduce concepts and computational techniques, highlight important points, and provide additional information and commentary.

Models — which provide specific examples for comparison and analysis.

Questions — which refer back to the models and/or readings. In each section, questions progress from straightforward to more challenging:

- **Questions at the beginning of each section will be easy to answer** by referring directly to either the reading or the models.
- **Later questions will expand on concepts** introduced in earlier questions, or require deeper analysis of the models.
- **Challenging questions** are indicated by a "$\oplus$".

Course Prerequisites

Algebra & Arithmetic These are the basis for our fundamental computational tools. You need to be comfortable with order of operations, adding fractions, and working with exponents.

Trigonometry We will be working extensively with the sine and cosine functions. You need to know the values of sine, cosine, and tangent at the "special angles" on the unit circle, and how to express sine, cosine, and tangent using the sides of a right triangle. Other topics from trig include: the Pythagorean theorem and Pythagorean identity; polar coordinates; writing tangent, cotangent, secant, and cosecant in terms of sine and cosine; inverse sine, cosine, and tangent functions. (Trig identities like the double angle formula are sometimes useful, but you don't need to have them memorized.)

Single Variable Calculus Math 275 requires a solid foundation in the calculus of functions of a single variable (at Boise State, these are covered in Math 170/175). The computational techniques of differentiation and integration are used extensively in Math 275, and many concepts in Math 275 expand on those introduced in earlier Calculus classes.

You should know the following well:

Differentiation Rules: constant rule; power rule; derivatives of sine & cosine, the exponential function with the natural base e, and the natural logarithm; product, quotient, and chain rules

Interpretations of the Derivative: slopes of tangent lines; instantaneous rates of change

Applications of the Derivative: velocity & acceleration; optimization — identifying local maxima & minima, and inflection points

Integration Rules: Know the antiderivatives for constant functions, polynomials, sine, cosine, tangent, and the exponential function with the natural base e; be apply use u-substitution. (Integration by parts and trig substitution will show up on homework, but you may use a calculator or other computational app for these.)

Interpretations & Applications of Definite Integrals: area, mass.

Chapter 1: Vectors & Vector Functions

Single variable calculus deals with "one-dimensional" problems. In multivariable and vector calculus, we learn how to generalize calculus to problems in higher-dimensional space; primarily, the plane $\mathbb{R}^2$, which is 2-dimensional, and 3-space $\mathbb{R}^3$, which is 3-dimensional.

These problems often deal with situations involving both numerical values and directions. For example, think of a force, which has both strength and direction in which it acts; or velocity, which accounts for both speed and direction of motion. Vectors are the mathematical tools used to represent and analyze these situations.

Applications of vectors include analysis of motion, forces, electromagnetism, and fluid dynamics. We will be using vectors throughout the remainder of this class. Vectors are also used in differential equations, linear algebra, statics, dynamics, physics, computer graphics, and many other classes.

This chapter introduces vectors and vector operations — the mathematical manipulation of vectors. Many of the computational techniques are basic, building on what you already know about working with real numbers. The really new, important ideas will be how vectors relate to geometric concepts like length, direction, position, and angles — which is what makes vectors so useful in applied areas like engineering, physics, and chemistry.

The concepts and computational methods introduced this chapter are the foundation for all following work with vectors for the rest of the course.

Topic 1.1 - Introduction to Vectors

From the Toolbox (what you need from previous classes)

- ☐ Basic arithmetic & algebra.
- ☐ Plotting points in the plane ($\mathbb{R}^2$) using Cartesian coordinates.

Goals In this set of class notes, you will:

- ☐ Compute the algebraic representation of a vector given its initial and terminal points.
- ☐ Determine when two vectors are equivalent.
- ☐ Compare vectors represented algebraically and geometrically.
- ☐ Sketch vectors in 2- and 3-dimensions.
- ☐ Compare position vectors and displacement vectors.

Big Picture

In this class, vectors will be represented in two different ways:

- **Geometric/Physical:** Vectors can be drawn as arrows from an initial (starting) point, to a terminal (ending) point. These arrows are defined by their magnitude (length) and direction.
- **Algebraic/Computational:** Vectors can be expressed numerically, by subtracting the coordinates of the terminal point from the coordinates of the initial point.

The idea that the same vector can be represented both ways is what makes vectors such useful tools in the analysis of physical phenomena. You can begin with a real-world physical system, assign numbers to it, perform computations, and then draw conclusions about the original physical system.

Section 1 You will sketch vectors (in 2-dimensions) based on their initial and terminal points, find their algebraic representation, and compare vectors to determine whether they are equivalent.

Section 2 You will compare two applications of vectors: position (where an object is located) and displacement (the total change in position of a moving object).

Section 3 You will learn a method for sketching vectors in 3-dimensions.

Definitions & Notation

Scalar A number (in this class, a real number). Usually denoted by numbers, or by lower-case roman letters or Greek letters. Examples of scalars:

$$5, \quad -1/2, \quad 0, \quad \sqrt{2}, \quad \pi, \quad a, \quad \lambda$$

Point A location in space. Usually denoted by upper-case letter (non-bolded), with the coordinates enclosed using parentheses. Examples of points:

$$P = (1, 6), \quad Q = (a, b, c)$$

Vector A directed line segment (arrow) from one point to a second point. Usually denoted by bold letters or letters with arrows over them. Algebraic representation is given with the components enclosed by angle brackets. Examples of vectors:

$$\vec{\boldsymbol{v}} = \langle 1, 6 \rangle, \quad \boldsymbol{u} = \langle u_1, u_2 \rangle, \quad \vec{w} = \langle a, b, c \rangle, \quad \overrightarrow{PQ} = \langle -0.25, 0.5 \rangle, \quad \vec{\mathbf{0}} = \langle 0, 0 \rangle$$

Initial Point (of a Vector) The point where a vector originates, also called the **tail** of the vector. A vector with initial point P is said to be **based** at P.

Terminal Point (of a Vector) The point where a vector terminates, also called the **head** of the vector.

Geometric Representation (of a Vector) The vector is represented as an arrow.

Algebraic Representation (of a Vector) The vector is represented as a sequence of scalar values. This is also called the **Component Form**, and the scalars entries called the **components** of the vector.

Equivalent Vectors Two vectors are **equivalent** if they have the same algebraic and geometric representations.

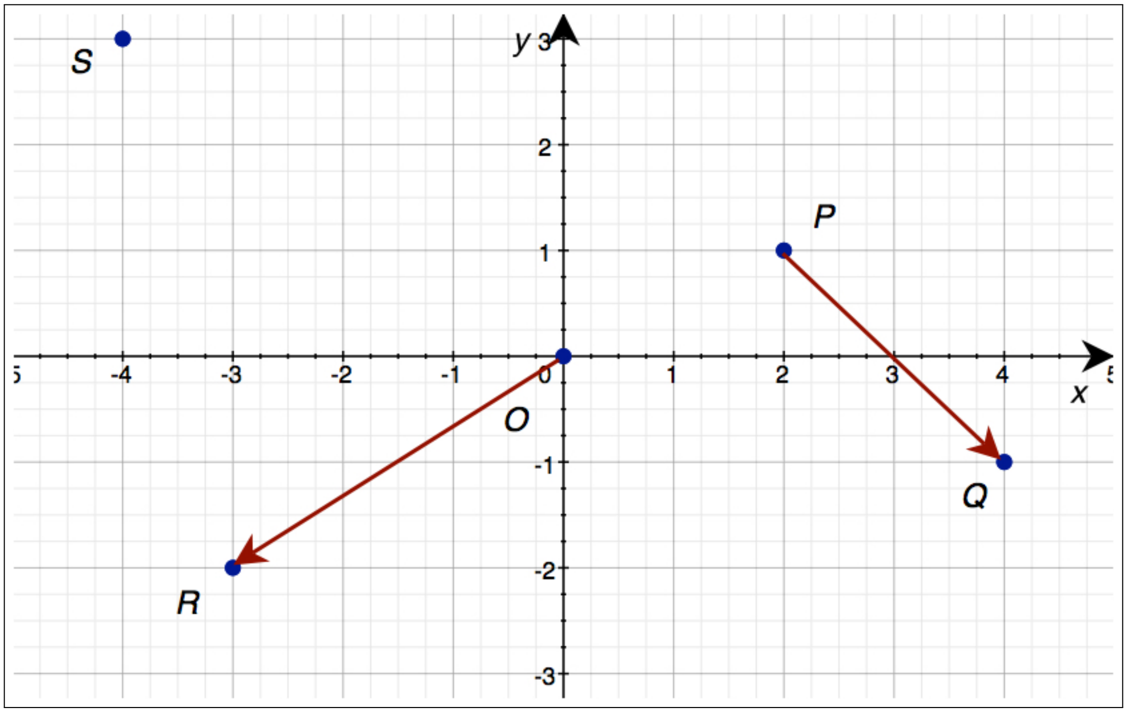
Position Vector — denoted $\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{s}}$ — A vector based at the origin, and indicating the location of an object.

Displacement Vector — denoted $\Delta\vec{\boldsymbol{r}}$ or $\Delta\vec{\boldsymbol{s}}$ — A vector indicating the total change in position of a moving object.

Zero Vector — denoted $\vec{\mathbf{0}}$ — The vector that has zeros in all of it's components. For example, the 2-dimensional zero vector is $\vec{\mathbf{0}} = \langle 0, 0 \rangle$, and the 3-dimensional zero vector is $\vec{\mathbf{0}} = \langle 0, 0, 0 \rangle$.

Section 1: Algebraic & Geometric Representations of Vectors, and Equivalent Vectors

Model 1A



Critical Thinking Questions

Read Me 1 — Computing the Algebraic Form of a Vector

Notation: The vector $\overrightarrow{P_1P_2}$ originates at the **initial point** P_1 and ends at the **terminal point** P_2.

To compute the **algebraic representation** of a vector, subtract the coordinates of the initial point from the coordinates of the terminal point.

For example, if $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

(Q1) Look at Model 1A at the top of the page, and locate the points P and Q.

The coordinates of the points P and Q are: $P = \left(\_\_\_\_\_ , \_\_\_\_\_ \right)$ and $Q = \left(\_\_\_\_\_ , \_\_\_\_\_ \right)$, so the algebraic representation of $\overrightarrow{PQ}$ is:

$$\overrightarrow{PQ} = \langle 4 - 2, (-1) - 1 \rangle$$

$$= \langle \_\_\_\_\_ , \_\_\_\_\_ \rangle$$

(Q2) Look again at Model 1A, and locate the points O and R. Use the coordinates of these two points to compute the algebraic representation of the vector $\overrightarrow{OR}$.

(Q3) Suppose the vector $\vec{\boldsymbol{v}}$ has initial point $(-4, 0)$ and terminal point $(3, -2)$. Sketch this vector on Model 1A, then find its algebraic form.

$$\vec{\boldsymbol{v}} = \left\langle \_\_\_\_\_\_\_ \, , \, \_\_\_\_\_\_\_ \right\rangle$$

(Q4) On Model 1A, locate the points P, Q, R, and S. Which (if any) of the following vectors has algebraic representation $\langle -6, 2 \rangle$?

$\overrightarrow{RP}$ $\overrightarrow{QR}$ $\overrightarrow{SP}$ $\overrightarrow{PS}$ None of these.

(Q5) On Model1A, compare the directions of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$, then find their algebraic representations:

$$\overrightarrow{PQ} = \left\langle \_\_\_\_\_\_\_ \, , \, \_\_\_\_\_\_\_ \right\rangle \qquad \overrightarrow{QP} = \left\langle \_\_\_\_\_\_\_ \, , \, \_\_\_\_\_\_\_ \right\rangle$$

Based on these, you can see that multiplying all components of the original vector by _______ results in a vector with the same magnitude, but opposite direction.

(Q6) Use your conclusion from (Q5) to answer the following:

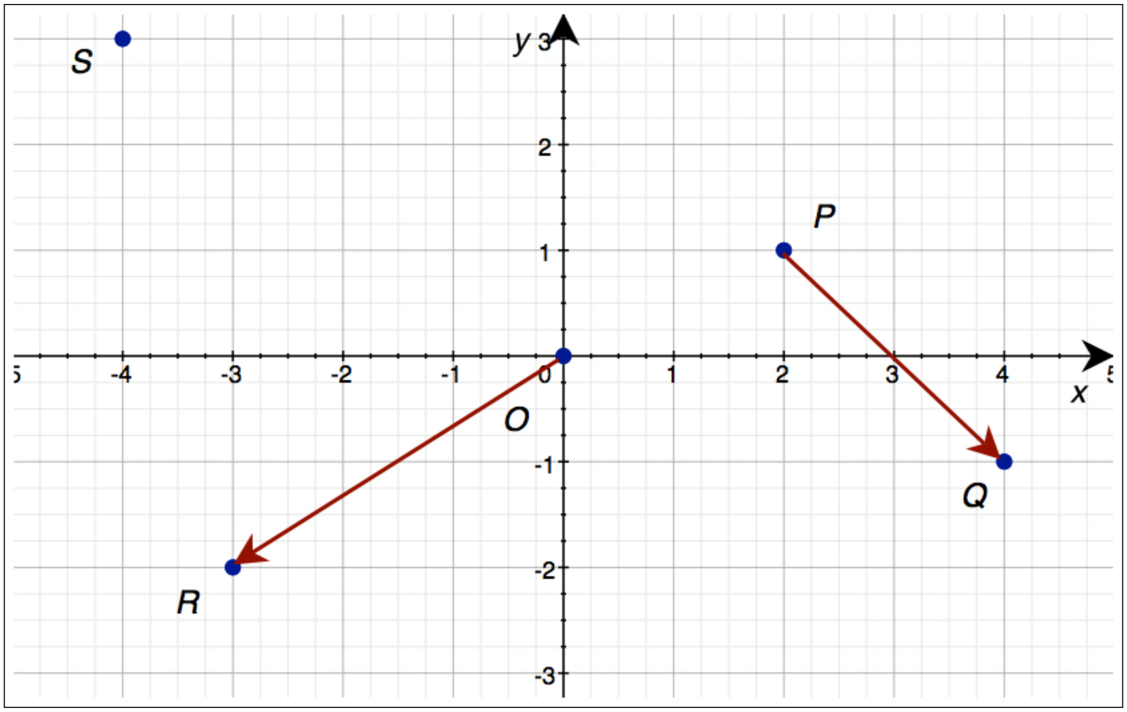
What is the algebraic representation of a vector $\vec{\boldsymbol{w}}$ that has the same magnitude as the vector $\vec{\boldsymbol{v}} = \langle a, -b \rangle$, but points in the opposite direction?

$$\vec{\boldsymbol{w}} = \left\langle \_\_\_\_\_\_\_ \, , \, \_\_\_\_\_\_\_ \right\rangle$$

Read Me 2 — Multiplication by -1 Reverses Direction

This is an important connection between algebra and geometry that we will see repeatedly throughout this course: multiplying the algebraic representation of an object by -1 reverses the direction of the object. (This is related to a property called **orientation**.)

Model 1B



Read Me 3 — Equivalent Vectors

Two vectors that have the same algebraic representation are **equivalent**.

(Q7) Suppose $\overrightarrow{AB}$ is the vector that has initial point $A = (-4, 2)$ and terminal point $B = (-2, 0)$. On Model 1B, sketch $\overrightarrow{AB}$, and locate the points P, Q, R, S, and O. Then circle all vectors in the list below that are equivalent to $\overrightarrow{AB}$:

$$\overrightarrow{SO} \qquad \overrightarrow{OQ} \qquad \overrightarrow{PQ} \qquad \overrightarrow{OR} \qquad \overrightarrow{RS}$$

(Q8) On Model 1B, locate the point O. Find the terminal point C so that the vectors $\overrightarrow{OC}$ is equivalent to the vector $\overrightarrow{AB}$ from (Q7), and sketch $\overrightarrow{OC}$ on Model 1B.

(Q9) In (Q7)-(Q9), you should've found that the vectors $\overrightarrow{AB}$, the vector $\overrightarrow{PQ}$, and the vector $\overrightarrow{OC}$ all have the same algebraic representation, so they are all equivalent.

Compare your sketches of these three vectors on Model 1B. Based on these sketches, which of the following statements do you think is true?

a) Equivalent vectors have the same magnitudes (lengths), but may have different directions.

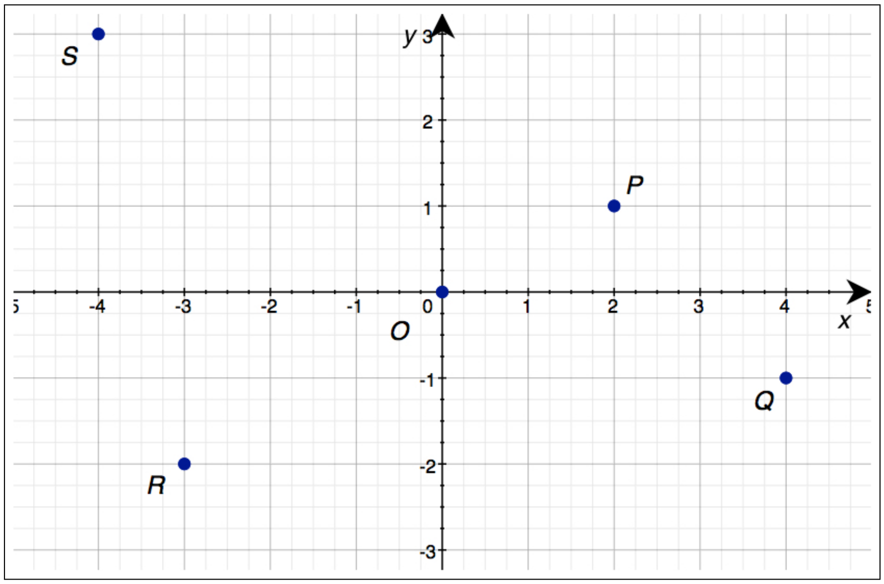
b) Equivalent vectors have the same direction, but may have different magnitudes (lengths).

c) Equivalent vectors have the same direction and the same magnitude.

d) Being equivalent doesn't tell you anything about the direction or magnitude of two vectors.

Section 2: Application of Vectors - Position & Displacement

MODEL 2



Critical Thinking Questions

Read Me 4 — Position vs. Displacement Vectors

A **position vector** — denoted $\vec{r}$ or $\vec{s}$ — starts at the origin, and ends at a point P. It gives the position of an object located at position P relative to the origin O (it "points to" the point P).

A **displacement vector** — denoted $\Delta\vec{r}$ or $\Delta\vec{s}$ — begins at a point P and ends at another point Q. It indicates the total change in position (displacement) of an object that moves from location P to location Q.

Note: The capital Greek letter delta — Δ — is often used to denote the change in a quantity. Since $\vec{r}$ is position, then displacement $\Delta\vec{r}$ means "change in position".

(Q10) On Model 2 at the top of this page, locate the points P, Q, R, and S, and sketch their position vectors. Then complete this table with the algebraic representations for each of the position vectors.

Point	Position Vector
$P = (2, 1)$	$\vec{r}_P = \langle 2, 1 \rangle$
$Q = (4, -1)$	$\vec{r}_Q = \langle \quad , \quad \rangle$
$\_\_\_\_ = (\quad , \quad)$	$\vec{r}_{\_\_\_} = \langle -3, -2 \rangle$
$S = (\quad , \quad)$	$\vec{r}_S = \langle \quad , \quad \rangle$

(Q11) Complete this table of displacement vectors using the points from Model 2, then sketch and label these displacement vectors on Model 2.

Initial Point	Terminal Point	Displacement Vector
$P = (2, 1)$	$Q = (4, -1)$	$\Delta\vec{\boldsymbol{r}}_{PQ} = \overrightarrow{PQ} = \langle 2, -2 \rangle$
$P = (2, 1)$	$S = (-4, 3)$	$\Delta\vec{\boldsymbol{r}}_{PS} = \overrightarrow{PS} = \langle \quad , \quad \rangle$
$O = (0, 0)$	$\_\_\_\_ = (\quad , \quad)$	$\Delta\vec{\boldsymbol{r}}_{O\_\_} = \overrightarrow{O\_\_} = \langle -3, -2 \rangle$
$P = (2, 1)$	$P = (2, 1)$	$\Delta\vec{\boldsymbol{r}}_{PP} = \overrightarrow{PP} = \langle \quad , \quad \rangle$

(Q12) From (Q10): The position vector of the point R is $\vec{\boldsymbol{r}}_R = \langle -3, -2 \rangle$.

From (Q11): The displacement vector from the origin O to the point R is $\Delta\vec{\boldsymbol{r}}_{OR} = \langle -3, -2 \rangle$.

What can you say about the two vectors $\vec{\boldsymbol{r}}_R$ and $\Delta\vec{\boldsymbol{r}}_{OR}$?

(a) They have $\boxed{\text{the same} \;\|\; \text{different}}$ algebraic representation(s).

(b) They have $\boxed{\text{the same} \;\|\; \text{different}}$ initial and terminal points.

(c) They have $\boxed{\text{the same} \;\|\; \text{different}}$ direction(s).

(d) They have $\boxed{\text{the same} \;\|\; \text{different}}$ magnitude(s).

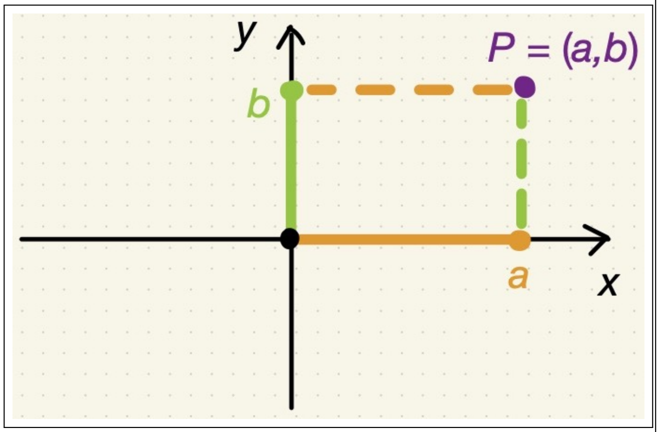
($\oplus$ Q13) From (Q12): The position vector $\vec{\boldsymbol{r}} = \langle -3, -2 \rangle$ and the displacement vector $\Delta\vec{\boldsymbol{r}}_{OR} = \langle -3, -2 \rangle$ from (Q12) have the same algebraic representations, the same geometric representations (the same directions and magnitudes), and the same initial and terminal points. **Are they the same vector, or are they different? Explain.**

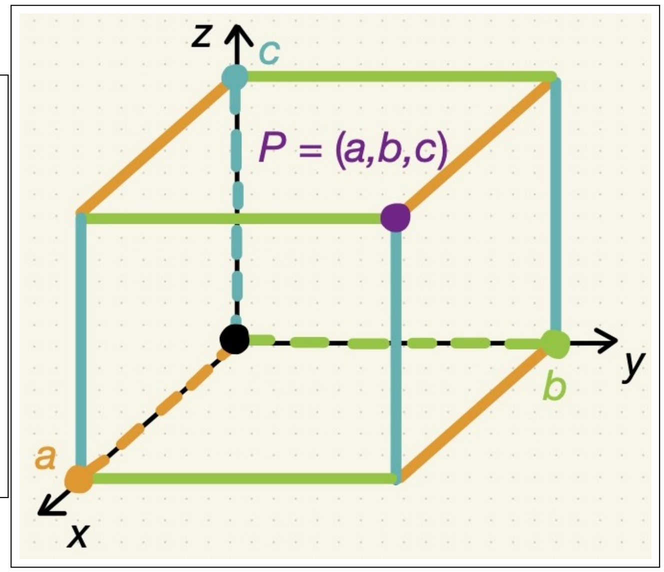
Section 3: Cartesian Coordinates &Sketching Vectors in 3-Dimensions

Read Me 5 — Sketching Vectors in $\mathbb{R}^3$ (Standard Position)

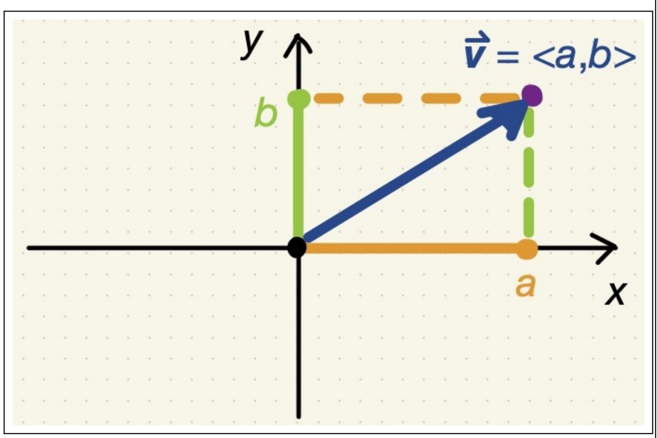
A vector is in **standard position** if its initial point is the origin.

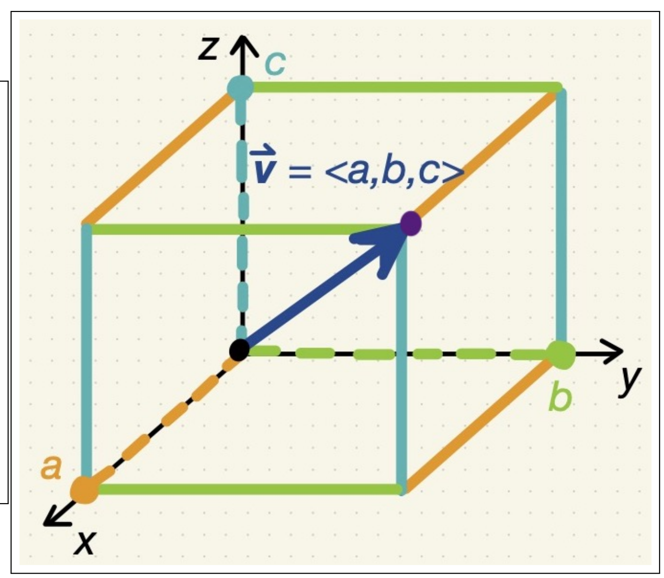
You can think of a point P in Cartesian coordinates as the far corner of a box with one vertex at the origin:





If P is the terminal point of a vector $\vec{v}$ in standard position, then the vector $\vec{v}$ forms the diagonal of the box:





(Q14) Sketch the vector $\vec{\boldsymbol{v}} = \langle 2, 3, 1 \rangle$ (in standard position) by following these steps:

Step 1 Sketch the x,y,z-coordinate axes the same way they are drawn on the previous page:

- The positive x-axis pointing "towards" you.
- The positive y-axis pointing to the right.
- The positive z-axis pointing up.

Did you sketch a teeny-tiny little set of axes? If you did, make them bigger.

Step 2 Sketch a box with sides of length 2 running parallel to the x-axis, length 3 running parallel to the y axis, and length 1 running parallel to z axis.

Step 3 Sketch the vector $\vec{\boldsymbol{v}}$ so that it forms the diagonal of the box with initial point at the origin, $O = (0, 0, 0)$. The terminal point of $\vec{\boldsymbol{v}}$ is the point $P = (2, 3, 1)$.

Step 4 Label! The! Axes! And the vector $\vec{\boldsymbol{v}}$! Did you put an arrow $\vec{}$ over the $\boldsymbol{v}$ for the vector? If not, do it now.

Read Me 6 — Sketching Coordinate Axes

Two notes on sketching coordinate axes for 3-dimensions

1) I usually only include the positive axes in my sketches. The negative axes exist, they just are not explicitly drawn. You are allowed to include the negative axes, but 3-d sketches can quickly become cluttered.

2) I will usually sketch the axes with the xy-plane in the horizontal position and the positive z axis pointing up (see Diagram A). This is because I often use the z axis to represent height above (or below) the xy-plane.

   In some other classes, you may see the xy-plane drawn "in the page", with the z axis pointing towards you (see Diagram B).

   What you **do not want** is to switch the relative positions of two axes (see Diagrams C & D). This is again related to the property of orientation.

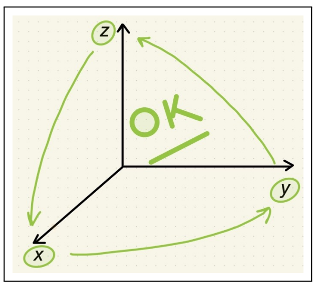


Diagram A

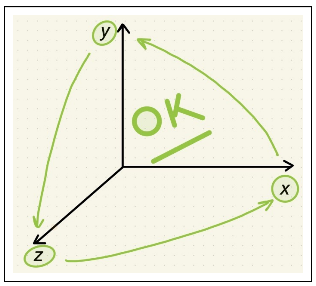


Diagram B

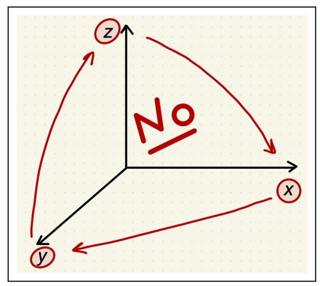


Diagram C

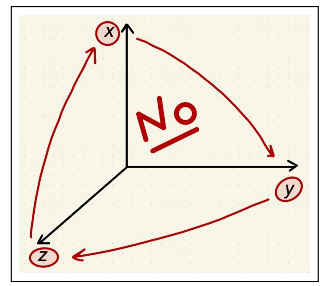


Diagram D

Read Me 7 — Cartesian Coordinates

Cartesian Coordinates (also called **rectangular coordinates**) is the coordinate system you are used to using. Each coordinate represents signed distance relative to the origin along each coordinate axis.

Two notes about Cartesian coordinates:

1) They are named after their originator, René Descartes — known for the quote: "I think, therefore I am".

2) We pretty much take them for granted, but they revolutionized mathematics. Before the introduction of the coordinate system, geometry was carried out axiomatically (remember the compass-and-straight-edge constructions you worked on in high school geometry?). But the existence of coordinates assigns numbers to points, which allows us to apply computational methods to geometric problems (and vice versa). Mathematics as we know it today only exist because of this.

Topic 1.2 - Introduction to Vector Operations

From the Toolbox (what you need from previous classes)

- ☐ Basic arithmetic & algebra.
- ☐ Trigonometry: Pythagorean theorem, polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$.
- ☐ Vectors: Finding the algebraic representation (aka the component form) of a vector given two points.

Goals In this set of class notes, you will:

- ☐ Compute vector sums (vector addition), scalar/vector products (scalar/vector multiplication), and the magnitude of vectors.
- ☐ Sketch the vectors resulting from vector addition and scalar/vector multiplication.
- ☐ Apply vector operations to manipulate the magnitude and direction of the resulting vectors.
- ☐ Compare the geometric aspects of vectors resulting from vector addition and scalar/vector multiplication to those of the original vectors, in order to determine the geometric effects of these operations.
- ☐ Derive additional ways of representing for vectors (besides angle-bracket notation) using vector operations and their geometric properties.

Big Picture

In the same way numbers can be manipulated using arithmetic operations (eg: addition/subtraction, multiplication, absolute value), vectors can be manipulated using vector operations. Today, you will focus on computing the magnitude of vectors, vector addition, scalar/vector multiplication, and their geometric properties.

Section 1 You will derive and apply formulae (that's the plural of formula) to compute the magnitude (length) of 2- and 3-dimensional vectors. These formulae are based on the Pythagorean theorem.

Section 2 You will compute vector sums both algebraically (adding component-wise) and geometrically (using the parallelogram law).

Section 3 You will compute scalar/vector products algebraically (multiplying component-wise) and determine how the scalar multiplier is related to the magnitude and direction of the resultant vector.

Section 4 You will combine computations of vector magnitude and scalar/vector multiplication to produce unit vectors (vectors with magnitude equal to one unit). You will apply this to finding vectors of specified magnitude that have the same direction as a given vector, and representing vectors in terms of magnitude and direction.

Section 5 You will learn a new way to denote the algebraic representation of 2- and 3-dimensional vectors using special sets of unit vectors — the $\{\,\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\,\}$ basis vectors.

Definitions & Notation

2-space — denoted $\mathbb{R}^2$ — 2-dimensional space, also called the plane. The notation comes from the fact that there are two coordinate axes, each a copy of the real number line $\mathbb{R}$:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

3-space — denoted $\mathbb{R}^3$ — 3-dimensional space, also called the plane. Since there are three coordinate axes (copies of $\mathbb{R}$):

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

Magnitude — denoted $|\vec{\boldsymbol{v}}|$ or $\|\vec{\boldsymbol{v}}\|$ — The distance between a vector's initial and terminal points — the length of the vector. It is the vector equivalent of the absolute value of a number.

Unit Vector A vector with magnitude equal to 1 (unit): $|\vec{\boldsymbol{v}}| = 1$. Unit vectors are often denoted using a "hat", for example: $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{e}}$, $\hat{\boldsymbol{\imath}}$.

Vector Addition Two vectors are added component-wise. If $\vec{\boldsymbol{v}} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{\boldsymbol{w}} = \langle w_1, w_2, \ldots, w_n \rangle$, then:

$$\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} = \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$$

For example, if $\vec{\boldsymbol{v}} = \langle 1, 2 \rangle$ and $\vec{\boldsymbol{w}} = \langle 3, 4 \rangle$, then $\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}} = \langle 1+3, 2+4 \rangle = \langle 4, 7 \rangle$.

Scalar Multiplication A vector and a scalar are multiplied component-wise. If $\vec{\boldsymbol{v}} = \langle v_1, v_2, \ldots, v_n \rangle$ and $c \in \mathbb{R}$ is a scalar, then:

$$c\vec{\boldsymbol{v}} = c\langle v_1, v_2, \ldots, v_n \rangle = \langle cv_1, cv_2, \ldots, cv_n \rangle$$

For example, if $\vec{\boldsymbol{v}} = \langle 2, 3 \rangle$ and $c = 5$, then $c\vec{\boldsymbol{v}} = 5\langle 2, 3 \rangle = \langle 10, 15 \rangle$.

Magnitude/Direction Representation of a Vector Every (non-zero) vector can be represented in terms of its direction — expressed as a unit vector — and its magnitude:

$$\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|\hat{\boldsymbol{v}}, \text{ where } \hat{\boldsymbol{v}} = \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$$

$\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$-Basis Representation of a Vector The $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$-basis is a collection of three unit vectors that point in the directions of the positive coordinate axes:

$$\hat{\boldsymbol{\imath}} = \langle 1, 0, 0 \rangle, \quad \hat{\boldsymbol{\jmath}} = \langle 0, 1, 0 \rangle, \quad \hat{\boldsymbol{\imath}} = \langle 0, 0, 1 \rangle$$

Every 2- and 3-dimensional vector can be represented in terms of the $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$-basis, for example:

$$\vec{\boldsymbol{v}} = \langle 2, -3 \rangle = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{w}} = \langle a, b, c \rangle = a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}, \quad \vec{\boldsymbol{0}} = \langle 0, 0, 0 \rangle = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$$

Section 1: Vector Magnitude

MODEL 1

DIAGRAM 1A:

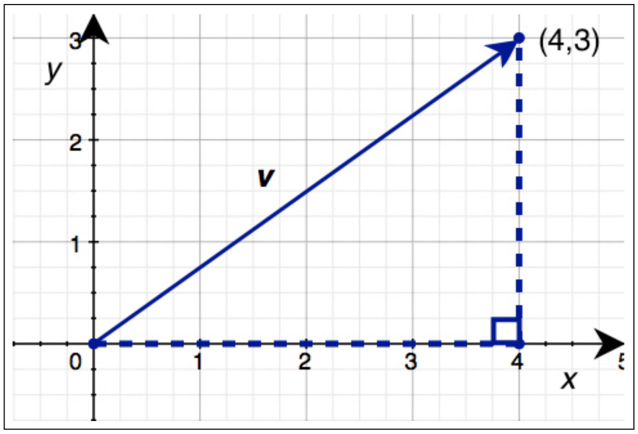


Diagram 1A(i)

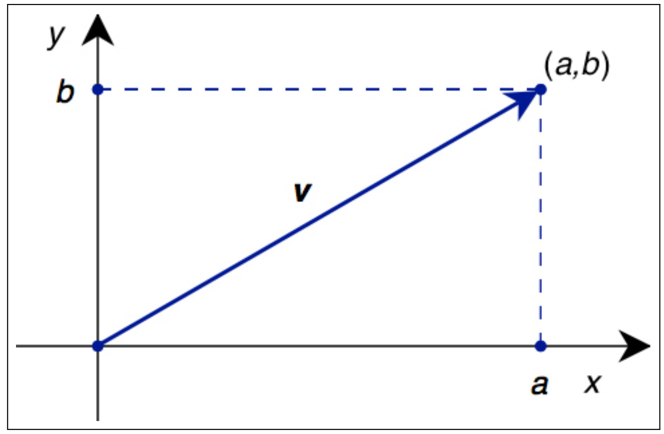


Diagram 1A(ii)

DIAGRAM 1B:

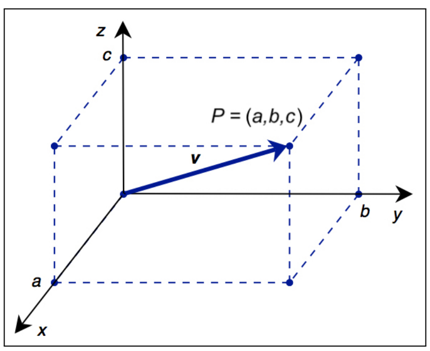


Diagram 1B(i)

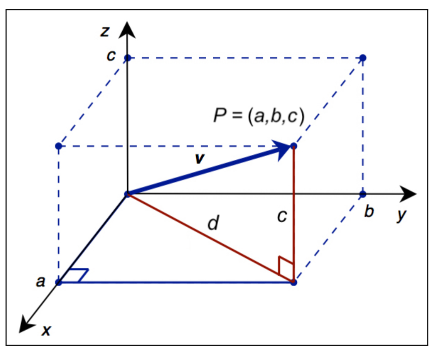


Diagram 1B(ii)

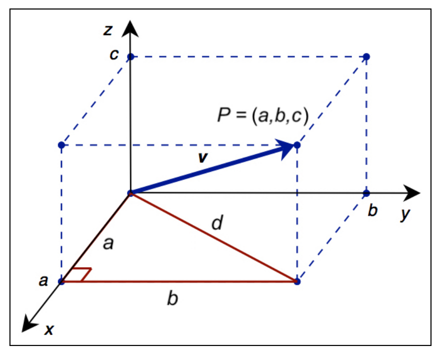


Diagram 1B(iii)

Critical Thinking Questions

Read Me 1 — Vector Magnitude (Length) & the Pythagorean Theorem

Vector magnitude (length) is computed by applying the Pythagorean theorem. For vectors with dimension greater than 2, you need to apply it more than once.

(Q1) To find the formula for computing the magnitude of a vector in $\mathbb{R}^2$, use the Pythagorean theorem:

(a) In Model 1, Diagram 1A(i): Write the algebraic representation for the vector $\vec{\boldsymbol{v}}$.

$$\vec{\boldsymbol{v}} = \langle \underline{\qquad\quad}, \underline{\qquad\quad} \rangle$$

(b) In Model 1, Diagram 1A(i): Because the vector $\vec{v}$ forms the hypotenuse of a right triangle, you can use the Pythagorean theorem to compute the magnitude (length) of this vector:

$$|\vec{v}| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(c) In Model 1, Diagram 1A(ii): Write the algebraic representation for the vector $\vec{v}$. *(a and b are arbitrary numbers).*

$$\vec{v} = \langle \_\_\_\_\_\_, \_\_\_\_\_\_ \rangle$$

(d) In Model 1, Diagram 1A(ii): Again, this vector forms the hypotenuse of a right triangle, you can use the Pythagorean theorem to compute its magnitude:

$$|\vec{v}| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

This is the general formula for computing the magnitude of a vector in $\mathbb{R}^2$.

(Q2) To find the formula for computing the magnitude of a vector in $\mathbb{R}^3$, use the Pythagorean theorem *twice*:

(a) In Model 1, Diagram 1B(i): The vector $\vec{v}$ forms the diagonal of a box. The initial point of $\vec{v}$ is the origin $O = (0, 0, 0)$, and the terminal point is the point $P = (a, b, c)$, so the algebraic representation for the vector $\vec{v}$ is:

$$\vec{v} = \langle \_\_\_\_\_\_, \_\_\_\_\_\_, \_\_\_\_\_\_ \rangle$$

(b) In Model 1, Diagram 1B(ii): The vector $\vec{v}$ forms the hypotenuse of a right triangle with sides c and d, so by the Pythagorean theorem, the *square* of the magnitude of $\vec{v}$ in terms of these two sides is:

$$|\vec{v}|^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(c) In Model 1, Diagram 1B(iii): But wait - the side d of the triangle from (Q2b) is the hypotenuse of a right triangle with sides a and b! So, using the Pythagorean theorem a second time:

$$d^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(d) Substitute your expression for d^2 from (Q2c) into your expression for $|\vec{v}|$ from (Q2b), and take the square roots of both sides to get the magnitude of $\vec{v}$:

$$|\vec{v}|^2 = c^2 + d^2$$

$$|\vec{v}|^2 = c^2 + \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_$$

so:

$$|\vec{v}| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

This is the general formula for computing the magnitude of a vector in $\mathbb{R}^3$.

($\oplus$ e) Suppose $\vec{v} = \langle v_1, v_2, v_3, v_4 \rangle$. What do you think is the formula for computing the magnitude of $\vec{v}$?

$$|\vec{v}| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(Q3) Compute the magnitudes of the vectors:

(a) $\vec{v} = \langle 3, 4 \rangle$

(b) $\vec{w} = \langle 2, -1, 5 \rangle$

Section 2: Vector Addition & the Parallelogram Law

MODEL 2

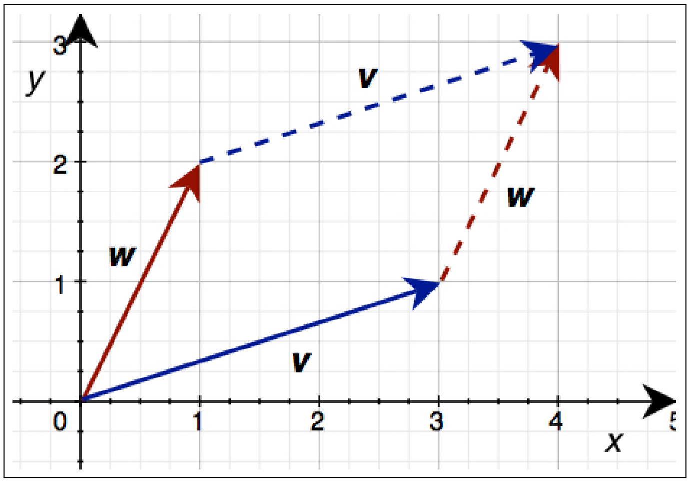
$\vec{v} = \langle 3, 1 \rangle$

$\vec{w} = \langle 1, 2 \rangle$

$\vec{v} + \vec{w} = \langle$ _______ , _______ $\rangle \leftarrow$ *from* (Q4)

$\vec{w} + \vec{v} = \langle$ _______ , _______ $\rangle \leftarrow$ *from* (Q4)

$\vec{D} = \langle$ _______ , _______ $\rangle \leftarrow$ *from* (Q6)



Critical Thinking Questions

Read Me 2 — Vector Addition - Algebraic & Geometric Expressions

Recall from Topic 1.1 that vectors have both algebraic and geometric representations. Similarly vector operations like vector addition and scalar/vector multiplication have both algebraic definitions, which are used computationally, and geometric expressions, which reflect how vector operations relate to the magnitude and direction of the original vectors.

In fact, if you have a diagram of the original vector(s), you can sketch vector sums or scalar products without performing computations.

(Q4) In Model 2: Use the definition of vector addition from the Definitions & Notation section at the beginning of the worksheet to compute the vector sums $\vec{v} + \vec{w}$ and $\vec{w} + \vec{v}$:

$\vec{v} + \vec{w} =$ ____________________ $\vec{w} + \vec{v} =$ ____________________

(Q5) Based on your answers from (Q4):

True ‖ False	Vector addition is commutative (that is, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$).

(Q6) In Model 2: Sketching a copy of $\vec{v}$ based at the terminal point of $\vec{w}$, and a copy of $\vec{w}$ based at the terminal point of $\vec{v}$, forms a parallelogram.

(a) On the diagram, sketch the vector $\vec{D}$ that forms the diagonal of this parallelogram, with initial point at the origin.

(b) Write down the algebraic representation of this vector:

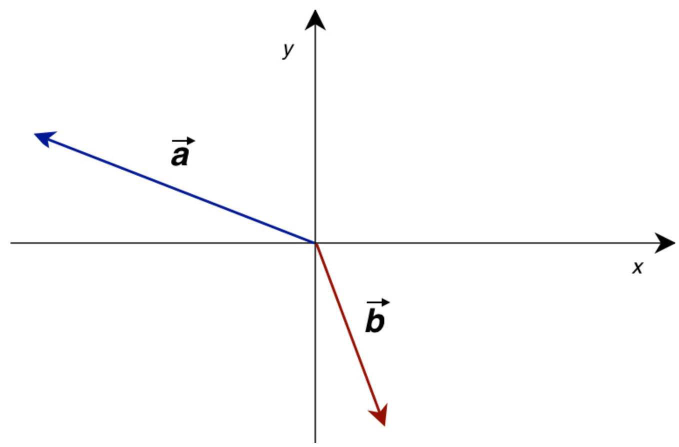
$$\vec{D} = \langle \underline{\qquad\quad} , \underline{\qquad\quad} \rangle$$

(Q7) Compare the vector sum of $\vec{v} + \vec{w}$ from (Q4) to the vector $\vec{D}$ from (Q5). Explain why the geometric expression of vector addition can be described as the **parallelogram law**.

(Q8) You can use the parallelogram law to sketch the sum or difference of two vectors without performing computations.

(a) On the diagram at the right, sketch $\vec{a} + \vec{b}$.

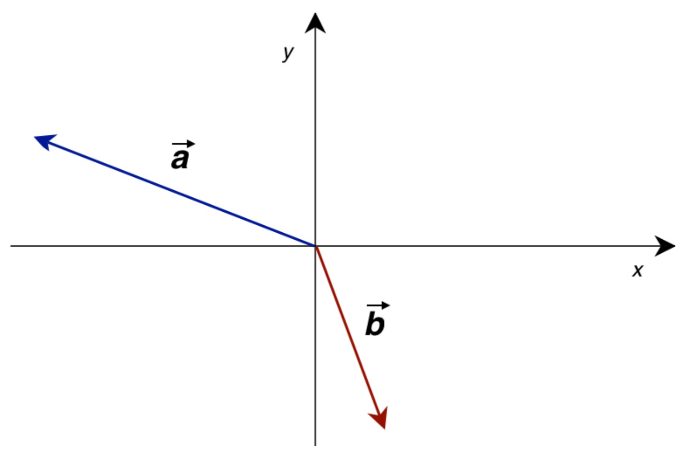
You don't have algebraic representations for $\vec{a}$ and $\vec{b}$, but you can use the parallelogram law.



($\oplus$ b) On the diagram at the right, sketch $\vec{a} - \vec{b}$.

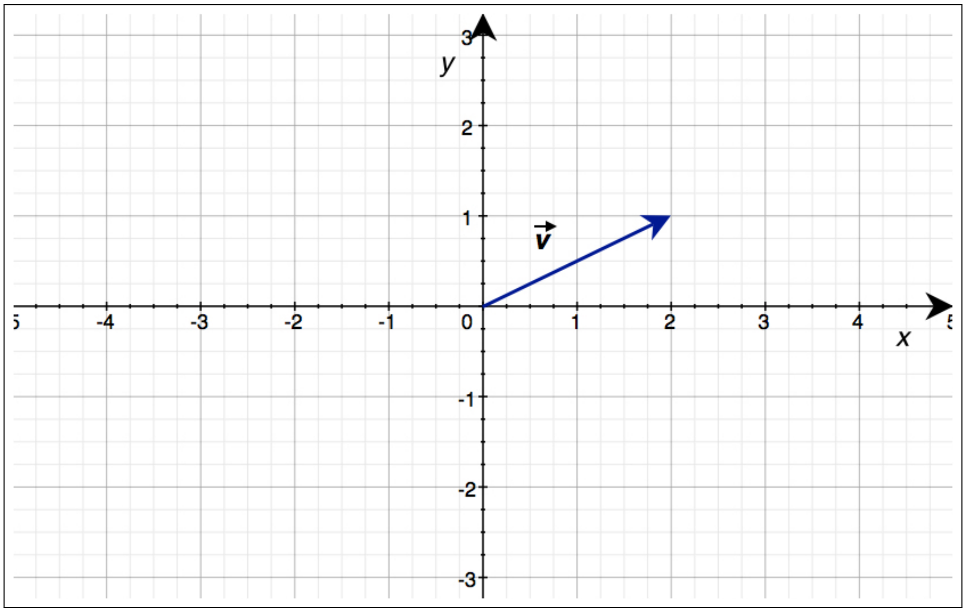
To do this, you need to put together a couple of steps:

- First: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$
- Now, recall from Topic 1.1 that $-\vec{b}$ has the same magnitude as $\vec{b}$, but the opposite direction.
- So: Sketch $-\vec{b}$, then use the parallelogram law for $\vec{a}$ and $-\vec{b}$.



Section 3: Scalar Multiplication, Lines Through the Origin, and Parallel Vectors

MODEL 3



Critical Thinking Questions

Read Me 3 — Scalar Multiplication

The geometric expression of scalar/vector multiplication is to change the magnitude of the original vector, and, if the scalar is negative, to reverse the direction.

This is where scalars get their name: when you multiply a vector by a scalar, the scalar "scales" (changes the size/magnitude) of the vector.

This space intentionally left blank - tricky formatting issue.

(Q9) In Model 3, the pictured vector is $\vec{\boldsymbol{v}} = \langle 2, 1 \rangle$.

(a) Use the definition of scalar multiplication from the Definitions & Notation section at the beginning of the worksheet to compute the scalar/vector product $c\vec{\boldsymbol{v}}$ for each of the following values of c:

$c = 2:$ $\quad 2\vec{\boldsymbol{v}} = 2\langle 2, 1 \rangle = \langle 4, 2 \rangle$

$c = 1:$ $\quad 1\vec{\boldsymbol{v}} = 1\langle 2, 1 \rangle = \langle 2, 1 \rangle$

$c = 1/2:$ $\quad 1/2\vec{\boldsymbol{v}} = \langle$ ______ , ______ $\rangle$

$c = 0:$ $\quad 0\vec{\boldsymbol{v}} = \langle$ ______ , ______ $\rangle$

$c = -1:$ $\quad -1\vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}} = \langle$ ______ , ______ $\rangle$

$c = -2:$ $\quad -2\vec{\boldsymbol{v}} = \langle$ ______ , ______ $\rangle$

(b) Sketch and label each of the scalar/vector products from part (a) on Model 3.

(Q10) Use your sketches of the vectors from (Q9) to answer the following:

(a) **If the absolute value of the scalar c is greater than 1,** the magnitude of the vector $c\vec{\boldsymbol{v}}$ is $\boxed{\text{less than} \;\|\; \text{equal to} \;\|\; \text{greater than}}$ the magnitude of the vector $\vec{\boldsymbol{v}}$.

(b) **If the absolute value of the scalar c is less than 1,** the magnitude of the vector $c\vec{\boldsymbol{v}}$ is $\boxed{\text{less than} \;\|\; \text{equal to} \;\|\; \text{greater than}}$ the magnitude of the vector $\vec{\boldsymbol{v}}$.

(c) **There are two values of scalar c such that** $\vec{v}$ and $c\vec{v}$ have the same magnitude. These two values are:

$c =$ ______ and $c =$ ______

(d) **If the scalar c is negative,** describe the relationship between the directions of the vectors $\vec{\boldsymbol{v}}$ and $c\vec{\boldsymbol{v}}$.

(e) If you look at the terminal points of all vectors $c\vec{\boldsymbol{v}}$, what "shape" do they form?

Read Me 4 — Scalar Multiplication & Parallel Vectors

An important application of scalar multiplication is defining when two non-zero vectors are parallel.

Geometrically, two non-zero vectors are parallel if they span parallel lines (that is, the lines drawn through their initial and terminal points are parallel).

Algebraically, **two non-zero vectors are parallel if their algebraic representations are scalar multiples of each other.**

(Q11) (a) Which of the following vectors (if any) are parallel to $\vec{w} = \langle 4, -1 \rangle$?

$$\vec{v}_1 = \langle 8, -2 \rangle \qquad \vec{v}_2 = \langle -12, 3 \rangle \qquad \vec{v}_3 = \langle -1, 4 \rangle \qquad \vec{v}_4 = \left\langle -2, \frac{1}{2} \right\rangle \qquad \vec{0} = \langle 0, 0 \rangle$$

($\oplus$ b) Only three of the vectors in part (a) are parallel to $\vec{w}$. Go back and re-read Read Me 4.

If you thought four of the vectors were parallel to $\vec{w}$: After re-reading Read Me 4, can you identify your mistake?

If you thought three of the vectors were parallel to $\vec{w}$ (and you are pretty sure you chose the correct three): After re-reading Read Me 4, can you identify the common mistake made by people who choose four vectors ?

Read Me 5 — Computational Tools

Two useful computational facts (these are easy to prove by writing out the algebraic representations and showing both sides of the equations are equal).

Scalar/Vector Multiplication and Vector Addition Scalar multiplication distributes over vector addition:

$$c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$$

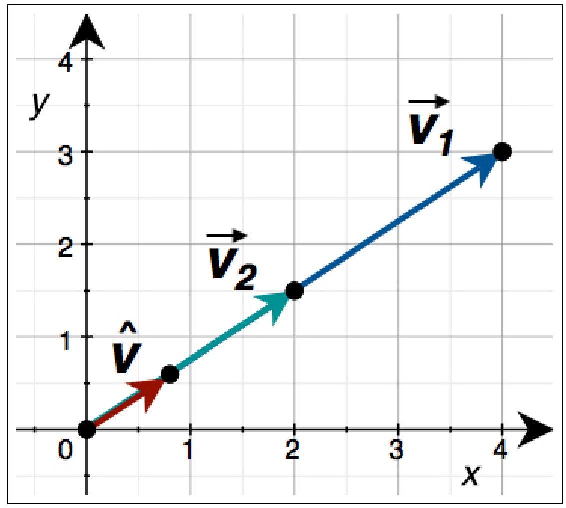
Scalar/Vector Multiplication and Magnitude: The magnitude of a scalar/vector product equals the product of the absolute value of the scalar and the magnitude of the vector:

$$|c\vec{v}| = |c||\vec{v}|$$

Section 4: Unit Vectors and Magnitude/Direction Representation of Vectors

MODEL 4

Vector:	Magnitude:
$\vec{v}_1 = \langle 4, 3 \rangle$	$\lvert\vec{v}_1\rvert = 5$
$\vec{v}_2 = \left\langle 2, \dfrac{3}{2} \right\rangle$	$\lvert\vec{v}_2\rvert = 5/2$
$\hat{v} = \left\langle \dfrac{4}{5}, \dfrac{3}{5} \right\rangle$	$\lvert\hat{v}\rvert = 1$



Critical Thinking Questions

Read Me 6 — Unit Vectors

The word **unit** in geometry refers to an object with length or distance equal to 1 — for example, the unit circle is the circle with radius $r = 1$, which is the set of all point at distance 1 from the origin.

So a **unit vector** is a vector whose magnitude equals 1.

Notation: Unit vectors are often denoted wearing a "hat", for example: $\hat{v}$, $\hat{w}$, $\hat{e}$, $\hat{\imath}$.

If $\vec{v}$ is a non-zero vector ($\vec{v} \neq \vec{0}$), then **the unit vector that has the same direction as $\vec{v}$ is:**

$$\hat{v} = \frac{1}{|\vec{v}|}\vec{v}$$

To save space (and because mathematicians are naturally lazy), this is often denoted as $\hat{v} = \vec{v}/|\vec{v}|$ or $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$.

(Q12) In Model 4: Which of the three vectors is a unit vector? $\boxed{\vec{v}_1 \ \S \ \vec{v}_2 \ \S \ \hat{v}}$.

(Q13) Find the unit vector $\hat{v}_1$ that has the same direction as the vector $\vec{v}_1$ in Model 4.

(a) Multiply the vector $\vec{v}_1 = \langle 4, 3 \rangle$ by the scalar $\frac{1}{|\vec{v}_1|} = 1/5$:

$$\hat{v}_1 = \frac{1}{|\vec{v}_1|}\vec{v}_1 = \frac{1}{5}\langle 4, 3 \rangle = \Big\langle \qquad , \qquad \Big\rangle$$

Your answer should be the same as the unit vector $\hat{v}$ from Model 4.

(b) Now find the unit vector $\hat{\boldsymbol{v}}_2 = \frac{1}{\vec{\boldsymbol{v}}_2}\vec{\boldsymbol{v}}_2$. You should get the same vector as $\hat{\boldsymbol{v}}$ in Model 4 and $\hat{\boldsymbol{v}}_1$ from part (a).

$$\hat{\boldsymbol{v}}_2 = \Big\langle \qquad\qquad , \qquad\qquad \Big\rangle$$

(c) In Model 4, you were told that the magnitude of $\hat{\boldsymbol{v}} = \langle 4/5, 3/5 \rangle$ equals 1.
Now, compute the magnitude by hand to confirm that what you were told, is in fact true.

($\oplus$ Q14) Find the two unit vectors parallel to $\vec{\boldsymbol{w}} = \langle 2, -5 \rangle$. *(Hint: One points in the same direction as* $\vec{\boldsymbol{w}}$ *, the other points in the opposite direction.)*

($\oplus$ Q15) Students often make the mistake of thinking that the vector $\langle 1, 1 \rangle$ is a unit vector.

(a) Why do you think this is a common mistake?

(b) How do you know that $\langle 1, 1 \rangle$ is **not** a unit vector?

($\oplus$ Q16) If you look at the terminal points of all unit vectors based at the origin in the plane ($\mathbb{R}^2$), what "shape" do they form? What about all unit vectors based at the origin in 3-space ($\mathbb{R}^3$)?

Read Me 7 — Magnitude/Direction Representation of Vectors

Every non-zero vector $\vec{v}$ can be represented as the product of its magnitude $|\vec{v}|$ and the unit vector $\hat{v}$ that has the same direction as $\vec{v}$:

$$\vec{v} = |\vec{v}|\hat{v}$$

This is relatively easy to see, since:

$$|\vec{v}|\hat{v} = |\vec{v}|\frac{\vec{v}}{|\vec{v}|} = \cancel{|\vec{v}|}\frac{\vec{v}}{\cancel{|\vec{v}|}} = \vec{v}$$

(Q17) Find the magnitude/unit vector representations for the following vectors:

(a) $\vec{v} = \langle 2, -5 \rangle$

$|\vec{v}| = \sqrt{4+25} = \sqrt{29}$

$\hat{v} = \langle 2/\sqrt{29}, -5/\sqrt{29} \rangle$

$$\vec{v} = |\vec{v}|\ \hat{v} = \underline{\qquad\qquad} \Big\langle \qquad\qquad , \qquad\qquad \Big\rangle$$

(b) $\vec{w} = \langle 3, 0, 4 \rangle$

$|\vec{w}| =$

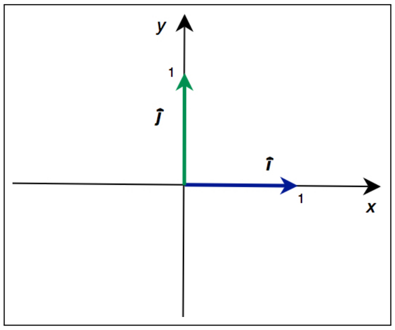
$\hat{w} =$

$$\vec{w} = |\vec{w}|\ \hat{w} = \underline{\qquad\qquad} \Big\langle \qquad\qquad , \qquad\qquad , \qquad\qquad \Big\rangle$$

($\oplus$ Q18) Find the vector $\vec{u}$ that has magnitude $|\vec{u}| = 7$ and points in the direction opposite to the vector $\vec{v} = \langle 1, 1, -2 \rangle$.

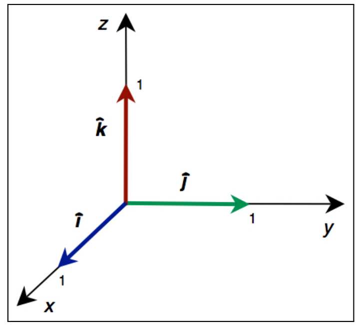
Section 5: The $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$-Basis

Model 5



Basis Vectors in $\mathbb{R}^2$

$$\hat{\imath} = \langle 1, 0 \rangle$$
$$\hat{\jmath} = \langle 0, 1 \rangle$$



Basis Vectors in $\mathbb{R}^3$

$$\hat{\imath} = \langle 1, 0, 0 \rangle$$
$$\hat{\jmath} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

Critical Thinking Questions

Read Me 8 — $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$-Basis

Every vector in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be represented using the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$-basis.

This representation might seem redundant, since we already have angle-bracket notation. In fact, the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$-basis is actually extremely interesting mathematically, and has a multitude of applications. For example, see the Wikipedia article https://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation.

(Q19) $\hat{\imath}$ is the **unit vector** that points in the direction of the $\boxed{\text{positive § negative}}$ $\boxed{\text{x § y § z}}$-axis.
$\hat{\jmath}$ is the **unit vector** that points in the direction of the $\boxed{\text{positive § negative}}$ $\boxed{\text{x § y § z}}$-axis.
$\hat{k}$ is the **unit vector** that points in the direction of the $\boxed{\text{positive § negative}}$ $\boxed{\text{x § y § z}}$-axis.

(Q20) Consider the vector $\vec{v} = \langle a, b, c \rangle$:

$$\begin{aligned} \vec{v} &= \langle a, b, c \rangle \\ &= \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, \underline{\hspace{2em}} \rangle \\ &= \underline{\hspace{2em}}\langle 1, 0, 0 \rangle + \underline{\hspace{2em}}\langle 0, 1, 0 \rangle + \underline{\hspace{2em}}\langle 0, 0, 1 \rangle \\ &= a\hat{\imath} + b\hat{\jmath} + c\hat{k} \end{aligned}$$

(Q21) Write the vector $\vec{v} = \langle 3, 0, -7 \rangle$ using $\hat{\imath}, \hat{\jmath}, \hat{k}$-notation.

(Q22) Write the vector $\vec{v} = 3\hat{\jmath} - \hat{k}$ using angle-bracket notation.

Topic 1.3 - Introduction to Dot & Cross Products

From the Toolbox (what you need from previous classes)

- ☐ Basic arithmetic & algebra (assume this for all topics for the rest of the semester).
- ☐ Trigonometry: Cosine function — especially, the values of cosine at the "special angles" you learn on the unit circle.
- ☐ Vectors: Algebraic representation using angle-bracket & $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$-basis vector notation; vector magnitude.

Goals In this set of class notes, you will:

- ☐ Compute the dot product of two vectors using both the algebraic and geometric definitions.
- ☐ Compute the cross product of two vectors in $\mathbb{R}^3$ using both the algebraic and geometric definitions.
- ☐ Identify the specific value of the dot product associated with perpendicular vectors.

Big Picture

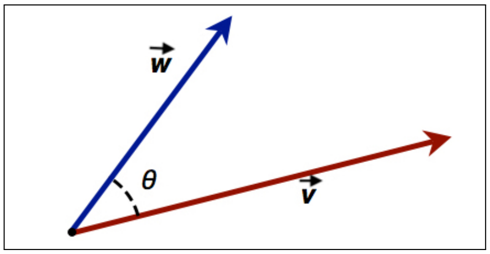
The dot and cross products are vector operations with many important applications. We will use them throughout the semester. Today, you will focus on computation.

Section 1 You will compute the dot product of two vectors using both the algebraic and the geometric definitions.

Section 2 You will determine how the dot product can be used as a "test" to determine when two vectors are perpendicular. This is related to a more general properties of vectors called **orthogonality**.

Section 3 You will compute the cross product of two vectors in $\mathbb{R}^3$ using the algebraic definition.

Definitions & Notation



$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\,\hat{\imath} + v_2\,\hat{\jmath} + v_3\,\hat{k}$$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle = w_1\,\hat{\imath} + w_2\,\hat{\jmath} + w_3\,\hat{k}$$

$$0 \le \theta \le \pi$$

Dot Product — denoted $\vec{v} \cdot \vec{w}$ — A vector operation that is defined for any two vectors that have the same number of components.

As we have seen previously with vectors and vector operations, the dot product has both an **algebraic definition**, which involves the component form of the vectors, and a **geometric definition**, which involves the magnitudes of the vectors and the angle between them.

Algebraic Definition: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

Geometric Definition: $\vec{v} \cdot \vec{w} = |\vec{v}\,||\vec{w}\,| \cos\theta$

The example given above for the algebraic definition uses two 3-dimensional vectors, but the dot product works for pairs of vectors in any dimension — as long as they both have the same number of components. A more general form would be:

$$\langle v_1, v_2, \ldots, v_n \rangle \cdot \langle w_1, w_2, \ldots, w_n \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

Note: The dot product of two vectors is a scalar.

Orthogonal Vectors Two vectors are **orthogonal** if their dot product equals zero: $\vec{v} \cdot \vec{w} = 0$.

Cross Product — denoted $\vec{v} \times \vec{w}$ — A vector operation that is defined for two vectors in $\mathbb{R}^3$ (both vectors have three components).

The cross product also has both an **algebraic** and a **geometric** definition, but for this topic we will only be using the algebraic definition. The geometric definition will be given in Topic 1.5.

Algebraic Definition: $\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$$= (v_2 w_3 - v_3 w_2)\,\hat{\imath} - (v_1 w_3 - v_3 w_1)\,\hat{\jmath} + (v_1 w_2 - v_2 w_1)\,\hat{k}$$

Note: The cross product of two vectors is a vector.

Section 1: Computing the Dot Product - Algebraic vs. Geometric Definitions

SECTION 1 WARM UP

Read Me 1 — Computing the Dot Product

There are two definitions for the dot product:

Algebraic Definition: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$

Geometric Definition: $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta$

These two definitions are equivalent — computationally, they will both give you the same answer. Which one you use depends on the information you are given.

Example: If $\vec{v} = \langle 4, -3 \rangle$ and $\vec{w} = \langle 5, 2 \rangle$, then:

$$\vec{v} \cdot \vec{w} = \langle 4, -3 \rangle \cdot \langle 5, 2 \rangle = (4)(5) + (-3)(2) = 20 - 6 = 14$$

Example: If $|\vec{v}| = \sqrt{7}$, $|\vec{w}| = \sqrt{6}$, and the angle between $\vec{v}$ and $\vec{w}$ is $\theta = 180^\circ$, then:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(180^\circ) = \sqrt{7}\sqrt{6}(-1) = -\sqrt{42}$$

(WU-1) Compute the dot products of the following vectors *(the answers are given at the end of this topic's notes)*:

(a) $\vec{v} = \langle 2, 6 \rangle$, $\vec{w} = \langle 10, 3 \rangle$:

$\vec{v} \cdot \vec{w} =$

(b) $\vec{v} = \langle 2, 1, -1 \rangle$, $\vec{w} = \langle 5, 2, 3 \rangle$:

$\vec{v} \cdot \vec{w} =$

(c) $\vec{v} = \langle 1, 1, 1, 1 \rangle$, $\vec{w} = \langle -1, 0, -1, 0 \rangle$:

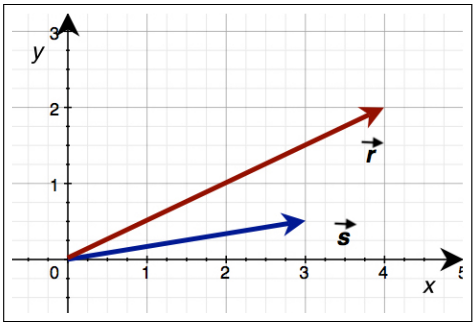
$\vec{v} \cdot \vec{w} =$

(WU-2) Compute the dot product of $\vec{v}$ and $\vec{w}$ if $|\vec{v}| = \sqrt{2}$, $|\vec{w}| = \sqrt{45}$, and the angle between them is $\theta = 60^\circ$ *(the answer is given after the "Summary" page)*:

$\vec{v} \cdot \vec{w} =$

Model 1

Diagram 1A:

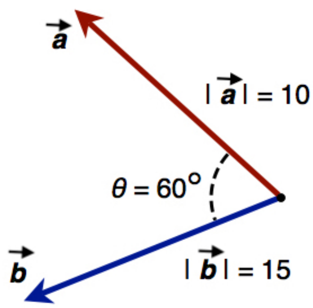


Vectors: $\vec{r}$, $\vec{s}$

$$\vec{r} = \langle 4, 2 \rangle$$

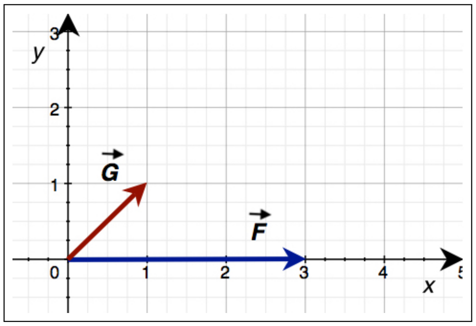
$$\vec{s} = \left\langle 3, \frac{1}{2} \right\rangle$$

Diagram 1B:



Vectors $\vec{a}$, $\vec{b}$

Diagram 1C:



Vectors $\vec{F}$, $\vec{G}$

Critical Thinking Questions

(Q1) In Model 1: Two out of the three Diagrams 1A, 1B, 1C contain the information you need to find the component forms of their vectors. Which Diagrams are they?

Diagram 1A ‖ Diagram 1B ‖ Diagram 1C

(Q2) In Model 1: One of the three Diagrams 1A, 1B, 1C gives explicit information about the magnitudes of the vectors and the angle between them. Which one is it?

Diagram 1A ‖ Diagram 1B ‖ Diagram 1C

(Q3) In Diagram 1A: There are two definitions for the dot product — algebraic and geometric. Choose one of them to compute the dot product $\vec{r} \cdot \vec{s}$. Which definition did you use, and why did you choose that definition?

(Q4) In Diagram 1B: Compute the dot product $\vec{a} \cdot \vec{b}$. Which definition did you use, and why did you choose that definition?

(Q5) In Diagram 1C: Compute the dot product $\vec{F} \cdot \vec{G}$. Which definition did you use, and why did you choose that definition?

(Q6) In Diagram 1C: Now, compute the dot product $\vec{F} \cdot \vec{G}$ for the vectors in using the *other* definition for the dot product. You should get the same answer as you did in (Q5).

($\oplus$ Q7) Suppose $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} < 0$. Is the angle θ between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ acute or obtuse? Or is there not enough information given to answer this question?

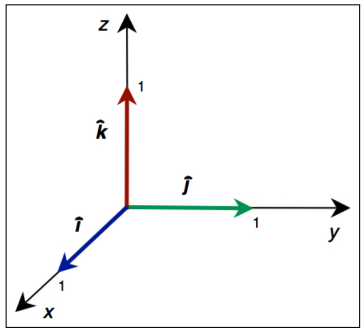
($\oplus$ Q8) Suppose you know that two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are perpendicular. What is the value of the dot product $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$? Which definition for the dot product (algebraic or geometric) did you use to determine your answer?

($\oplus$ Q9) What is the value of the dot product $\vec{\mathbf{0}} \cdot \vec{\boldsymbol{w}}$? (*Recall that* $\vec{\mathbf{0}}$ *is the* **zero vector** *whose components all equal zero.*) Which definition for the dot product (algebraic or geometric) did you use to determine your answer?

Section 2: The Dot Product & Orthogonality

MODEL 2

DIAGRAM 2A:

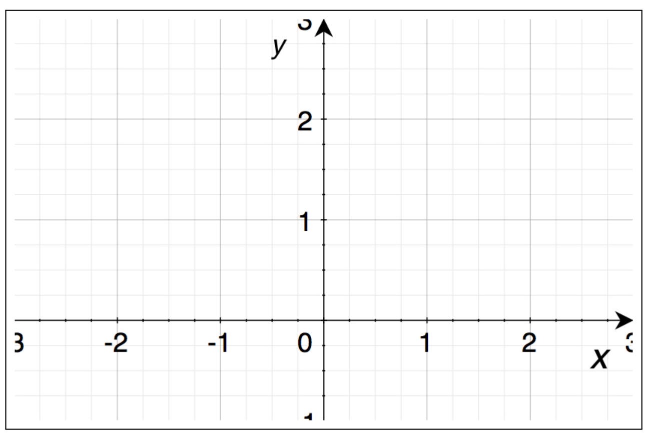


$\hat{\imath} = \langle 1, 0, 0 \rangle$
$\hat{\jmath} = \langle 0, 1, 0 \rangle$
$\hat{k} = \langle 0, 0, 1 \rangle$

Pair of Vectors	Angle Between Vectors	Value of Dot Product
$\hat{\imath}, \hat{\jmath}$	$\theta_{\hat{\imath},\hat{\jmath}} = 90°$	$\hat{\imath} \cdot \hat{\jmath} = \langle 1,0,0 \rangle \cdot \langle 0,1,0 \rangle =$
$\hat{\imath}, \hat{k}$	$\theta_{\hat{\imath},\hat{k}} =$	$\hat{\imath} \cdot \hat{k} = \langle 1,0,0 \rangle \cdot \langle 0,0,1 \rangle = 0$
$\hat{\jmath}, \hat{k}$	$\theta_{\hat{\jmath},\hat{k}} =$	$\hat{\jmath} \cdot \hat{k} = \langle 0,1,0 \rangle \cdot \langle 0,0,1 \rangle =$

Table 2A *Fill in this table after reading (Q10)&(Q11).*

DIAGRAM 2B:



$\vec{u} = \langle 0, 2 \rangle$
$\vec{v} = \langle 1, 1 \rangle$
$\vec{w} = \langle -2, 2 \rangle$

Pair of Vectors	Angle Between Vectors	Value of Dot Product
$\vec{u}, \vec{v}$	$\theta_{\vec{u},\vec{v}} = 45°$	$\vec{u} \cdot \vec{v} =$
$\vec{u}, \vec{w}$	$\theta_{\vec{u},\vec{w}} =$	$\vec{u} \cdot \vec{w} = \langle 0,2 \rangle \cdot \langle -2,2 \rangle = 4$
$\vec{v}, \vec{w}$	$\theta_{\vec{v},\vec{w}} =$	$\vec{v} \cdot \vec{w} =$

Table 2B *Fill in this table after reading (Q12).*

Critical Thinking Questions

(Q10) In Diagram 2A: The angles between the three pairs of vectors $\{\hat{\imath}, \hat{\jmath}\}$, $\{\hat{\imath}, \hat{k}\}$, and $\{\hat{\jmath}, \hat{k}\}$ are all the same. Write these angles in the middle column of Table 2A.

(Q11) In Diagram 2A: Compute the values of the dot products $\hat{\imath} \cdot \hat{\jmath}$, $\hat{\imath} \cdot \hat{k}$, and $\hat{\jmath} \cdot \hat{k}$ — you will see that they are all the same. Write these values in the right column of Table 2A.

(Q12) In Diagram 2B: Sketch and label the vectors $\vec{u} = \langle 0, 2 \rangle$, $\vec{v} = \langle 1, 1 \rangle$, $\vec{w} = \langle -2, 2 \rangle$ on the coordinate axes, and complete Table 2B. *The angles between vectors and values of the dot products will not all be the same.*

(Q13) In Diagram 2B: Using the information from the center column of Table 2B, which pairs of vectors are perpendicular? $\boxed{\vec{u} \perp \vec{v} \quad \| \quad \vec{u} \perp \vec{w} \quad \| \quad \vec{v} \perp \vec{w}}$

(Q14) In Diagram 2B: Using the information from the right column of Table 2B, which vectors have a dot product equal to zero? $\boxed{\vec{u} \cdot \vec{v} = 0 \ \| \ \vec{u} \cdot \vec{w} = 0 \ \| \ \vec{v} \cdot \vec{w} = 0}$

(Q15) Based on the results of questions (Q10-14), complete the following statement.

Statement: If two vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} =$ ________.

(Q16) Use the geometric definition of the dot product, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, to explain why the statement in (Q15) is true for all pairs of perpendicular vectors.

(Q17) **Orthogonal Vectors:** Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.

Compute the dot products to determine which pairs of vectors in the following set are orthogonal.

$$\vec{v} = \langle 1, -1, 2 \rangle, \quad \vec{w} = \langle 1, 1, -1 \rangle, \quad \vec{u} = \langle 3, 1, -1 \rangle, \quad \vec{0} = \langle 0, 0, 0 \rangle$$

Dot Product:	$\vec{v} \cdot \vec{w} =$	$\vec{v} \cdot \vec{u} =$	$\vec{v} \cdot \vec{0} =$	$\vec{w} \cdot \vec{u} =$	$\vec{w} \cdot \vec{0} =$	$\vec{u} \cdot \vec{0} =$
Orthogonal?	Y ‖ N	Y ‖ N	Y ‖ N	Y ‖ N	Y ‖ N	Y ‖ N

(Q18) The following statement is true, but incomplete. Complete the statement, based on your answer to (Q17).

Statement: If two vectors are **orthogonal**, then either they are **perpendicular** (the angle between them is 90°), **or** one or both of the vectors is the ____________________ **vector**.

Read Me 2 — Summary of Orthogonality

Orthogonality is one of the big ideas in this class. It expands on what it means to be perpendicular.

Two non-zero vectors are **perpendicular** if they span perpendicular lines, or, equivalently, if the angle between them is 90^{deg}.

Two vectors are **orthogonal** if their dot product equals zero. This happens when:

- the vectors are perpendicular, or:
- one of the vectors is the zero vector.

For this class, it is helpful to focus on the perpendicular aspect of orthogonality.

In future classes (for example, in Linear Algebra), you will start using the fact that the zero vector is orthogonal to all other vectors — even though it isn't perpendicular to anything (since it doesn't have a direction).

Read Me 3 — A Note on Angles - Degrees vs. Radians

You should be able to express angles and evaluate trig functions using both degrees and radians.

- In general, **degrees** are more useful when you want to **visualize** the angle.
- When **differentiating** or **integrating**, use **radians**.

Section 3: Computing the Cross Product — Algebraic Definition

Read Me 4 — Computing Cross Products - Determinant Method

There are several methods for computing the cross product. This method is called the determinant method, because it uses the same technique as computing determinants for 3×3 matrices. This will look familiar if you have taken Differential Equations with Matrix Theory, or Linear Algebra).

(If you have learned another way of computing the cross product: you may use any method you prefer, as long as you carry it out correctly.)

Step 1 - The determinant method begins with computing a 2×2 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

When you compute a cross product, you will compute three of these 2×2 determinants.

Step 2 - Setting up the Cross Product: Beginning with two vectors in $\mathbb{R}^3$:

$$\vec{\boldsymbol{v}} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}} \qquad \vec{\boldsymbol{w}} = w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$$

Set up a 3×3 array:

$$\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \begin{array}{l} \leftarrow \textit{top row - basis vectors} \\ \leftarrow \textit{middle row - components of the first vector} \\ \leftarrow \textit{bottom row - components of the second vector} \end{array}$$

Step 3 - Reduce the 3×3 array to three 2×2 arrays & evaluate:

$$\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{\imath}} - \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{\jmath}} + \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{k}} \qquad \leftarrow \textit{cross out rows \& columns}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \hat{\boldsymbol{\imath}} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \hat{\boldsymbol{\jmath}} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \hat{\boldsymbol{k}} \qquad \leftarrow \textit{use Step 1 to evaluate}$$

$$= \Big(v_2 w_3 - v_3 w_2\Big)\,\hat{\boldsymbol{\imath}} - \Big(v_1 w_3 - v_3 w_1\Big)\,\hat{\boldsymbol{\jmath}} + \Big(v_1 w_2 - v_2 w_1\Big)\,\hat{\boldsymbol{k}}$$

EXAMPLE

If $\vec{v} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{w} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$, then:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \hat{\imath} - \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \hat{k}$$

$$= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \hat{k}$$

$$= \Big((2)(6) - (3)(5)\Big)\hat{\imath} - \Big((1)(6) - (3)(4)\Big)\hat{\jmath} + \Big((1)(5) - (2)(4)\Big)\hat{k}$$

$$= (12 - 15)\hat{\imath} - (6 - 12)\hat{\jmath} + (5 - 8)\hat{k}$$

$$= -3\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$$

SECTION 3 WARM UP

(WU-3) Compute $\vec{v} \times \vec{w}$ for $\vec{v} = 3\hat{\imath} + \hat{\jmath} - 4\hat{k}$ and $\vec{w} = 5\hat{\imath} + 2\hat{k} = 5\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$ *(the answer is given at the end of this topic's notes)*.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \quad & \quad & \quad \\ & & \\ & & \end{vmatrix}$$

$$= \begin{vmatrix} \quad & \quad & \quad \\ & & \\ & & \end{vmatrix} \hat{\imath} - \begin{vmatrix} \quad & \quad & \quad \\ & & \\ & & \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \quad & \quad & \quad \\ & & \\ & & \end{vmatrix} \hat{k}$$

$$= \begin{vmatrix} \quad & \quad \\ & \end{vmatrix} \hat{\imath} - \begin{vmatrix} \quad & \quad \\ & \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \quad & \quad \\ & \end{vmatrix} \hat{k}$$

$$= \Big(\qquad - \qquad \Big)\hat{\imath} - \Big(\qquad - \qquad \Big)\hat{\jmath} + \Big(\qquad - \qquad \Big)\hat{k}$$

$$= \underline{\qquad\qquad}\,\hat{\imath}\ \underline{\qquad\qquad}\,\hat{\jmath}\ \underline{\qquad\qquad}\,\hat{k}$$

Critical Thinking Questions

(Q19) In the Section 3 Warm-Up, you computed $\vec{v} \times \vec{w}$ for $\vec{v} = 3\hat{\imath} + \hat{\jmath} - 4\hat{k}$ and $\vec{w} = 5\hat{\imath} + 2\hat{k} = 5\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$. Now, compute the cross product of the same vectors, but in the opposite order.

$\vec{w} \times \vec{v} =$

(Q20) Based on your answers in the Section 3 Warm-Up and in (Q19), you should have found that $\vec{v} \times \vec{w} = 2\hat{\imath} - 26\hat{\jmath} - 5\hat{k}$ and $\vec{w} \times \vec{v} = -2\hat{\imath} + 26\hat{\jmath} + 5\hat{k}$.

In other words, when you reverse the order of the vectors in a cross product, you multiply

the components of the original cross product by __________.

Recall from Topic 1.2: this means $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ have the ____________________ magnitude,

but _________________________ directions.

Read Me 5 — Reversing the Order of the Cross Product

Algebraically: $\vec{w} \times \vec{v} = -\left(\vec{v} \times \vec{w}\right)$ (reversing the order is the same as multiplying by -1).

Geometrically: $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ have the same magnitude, but opposite directions.

(Q21) Compare this equation to the last line in Read Me 4. Circle and correct the errors:

$$\vec{v} \times \vec{w} = \left(v_2 w_3 - v_3 w_2\right)\hat{\imath} + \left(v_1 w_3 - v_3 w_1\right)\hat{\jmath} + \left(v_1 w_2 - v_2 w_1\right)\hat{k}$$

(Q22) Compare this equation to the last line in Read Me 4. Circle and correct the errors:

$$\vec{v} \times \vec{w} = \left(v_2 w_3 + v_3 w_2\right)\hat{\imath} - \left(v_1 w_3 + v_3 w_1\right)\hat{\jmath} + \left(v_1 w_2 + v_2 w_1\right)\hat{k}$$

($\oplus$ Q23) Compute the cross product $\vec{v} \times \vec{0}$, where $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ is any vector, and $\vec{0} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$.

$\vec{v} \times \vec{0} =$

($\oplus$ Q24) **The cross product of parallel vectors is the zero vector.**

Recall from Topic 1.2: Two (non-zero) vectors are parallel if and only if one is a scalar multiple of the other.

For example, the vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{A} = ca_1\hat{\imath} + ca_2\hat{\jmath} + ca_3\hat{k}$ are parallel, since $\vec{A} = c\vec{a}$.

Compute $\vec{A} \times \vec{a}$ to show that the cross product of parallel vectors is the zero vector.

$$\vec{A} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ ca_1 & ca_2 & ca_3 \\ a_1 & a_2 & a_3 \end{vmatrix} =$$

Read Me 6 — Computational Tools

Some useful computational facts:

Dot & Cross Products and the Zero Vector

$$\vec{v} \cdot \vec{0} = \vec{0} \cdot \vec{v} = 0$$
$$\vec{v} \times \vec{0} = \vec{0} \times \vec{v} = \vec{0}$$

Dot & Cross Products and Scalar Multiplication

$$(c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$$
$$(c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) = c(\vec{v} \times \vec{w})$$

Dot & Cross Products and Perpendicular & Parallel Vectors

$$\text{If } \vec{v} \perp \vec{w}\text{, then } \vec{v} \cdot \vec{w} = 0$$
$$\text{If } \vec{v} \parallel \vec{w}\text{, then } \vec{v} \times \vec{w} = \vec{0}$$

Answers to Warm-Up Questions

(WU-1) (a) $\langle 2,6\rangle \cdot \langle 10,3\rangle = (2)(10)+(6)(3) = 38$

(b) $\langle 2,1,-1\rangle \cdot \langle 5,2,3\rangle = (2)(5)+(1)(2)+(-1)(3) = 9$

(c) $\langle 1,1,1,1\rangle \cdot \langle -1,0,-1,0\rangle = (1)(-1)+(1)(0)+(1)(-1)+(1)(0) = -2$

(WU-2) $\vec{v} \cdot \vec{w} = \sqrt{2}\sqrt{45}\cos(60^\circ) = \sqrt{90}\left(\frac{1}{2}\right) = \frac{\sqrt{(2)(5)(9)}}{2} = \frac{3\sqrt{10}}{2}$

(WU-3) $\vec{v} = 3\hat{\imath} + \hat{\jmath} - 4\hat{k}$ and $\vec{w} = 5\hat{\imath} + 2\hat{k} = 5\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -4 \\ 5 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -4 \\ 5 & 0 & 2 \end{vmatrix}\hat{\imath} - \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -4 \\ 5 & 0 & 2 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -4 \\ 5 & 0 & 2 \end{vmatrix}\hat{k}$$

$$= \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 3 & 1 \\ 5 & 0 \end{vmatrix}\hat{k}$$

$$= \Big((1)(2) - (-4)(0)\Big)\hat{\imath} - \Big((3)(2) - (-4)(5)\Big)\hat{\jmath} + \Big((3)(0) - (1)(5)\Big)\hat{k}$$

$$= (2-0)\hat{\imath} - (6+20)\hat{\jmath} + (0-5)\hat{k}$$

$$= 2\hat{\imath} - 26\hat{\jmath} - 5\hat{k}$$

More Coming Soon . . .

Watch This Space

Appendix A: Equations

Vectors & Vector Operations

For vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$:

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = |\vec{v}||\vec{w}| \cos\theta$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$$

$$\operatorname{proj}_{\vec{w}} \vec{v} = (\vec{v} \cdot \hat{w})\hat{w} = \left(\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\right)\vec{w}$$

$$\operatorname{comp}_{\vec{w}} \vec{v} = \vec{v} \cdot \hat{w} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$$

Vector Functions

For the vector function $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$:

$$\vec{r}\,'(t) = x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath} + z'(t)\,\hat{k}$$

$$\hat{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\vec{v}(t) = \vec{r}\,'(t)$$

$$v(t) = |\vec{v}(t)|$$

$$\vec{a}(t) = \vec{r}\,''(t)$$

Derivatives of Multivariable Functions

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}$$

$$D_{\hat{u}} f(x,y) = \vec{\nabla} f \cdot \hat{u}$$

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

critical points: $\vec{\nabla} f = 0$ or $\vec{\nabla} f$ does not exist

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

Polar, Cylindrical & Spherical Coordinates — Including Area & Volume Elements

$x = r\cos\theta$	$x = r\cos\theta$	$x = \rho\sin\varphi\cos\theta$
$y = r\sin\theta$	$y = r\sin\theta$	$y = \rho\sin\varphi\sin\theta$
$dA = r\,dr\,d\theta = r\,d\theta\,dr$	$z = z$	$z = \rho\cos\varphi$
	$dV = r\,dr\,dz\,d\theta = \dots$	$dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \dots$

Line Integrals

For the vector function $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$:

$$d\vec{r} = \vec{r}\,'(t)\,dt$$

$$ds = \left|d\vec{r}\right| = \left|\vec{r}\,'(t)\right|\,dt$$

Gradient, Curl & Divergence

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$$

$$\begin{aligned} \mathrm{grad}(f) &= \vec{\nabla} f \\ \mathrm{curl}\,(\vec{F}\,) &= \vec{\nabla} \times \vec{F} \\ \mathrm{div}\,(\vec{F}\,) &= \vec{\nabla} \cdot \vec{F} \end{aligned}$$

Integral Theorems

FTC : $$f(b) - f(a) = \int_a^b f'(t)dt$$

FTLI : $$f(B) - f(A) = \int_C \vec{\nabla} f \cdot d\vec{r}$$

Green's Theorem : $$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

A Few Applications of Integrals

Applications of Double Integrals

Area: $\iint_D dA = \iint_D 1\, dA$	Volume: $\iint_D f(x,y)\, dA$	Mass: $\iint_D \sigma(x,y)dA,\ \sigma(x,y) \geq 0$

Applications of Triple Integrals

Volume: $\iiint_W dV = \iiint_W 1\, dV$	Mass: $\iiint_W \rho(x,y,z)dV,\ \rho(x,y,z) \geq 0$

Applications of Scalar Line Integrals

Length: $\int_C ds = \int_C 1\, ds$	Mass: $\int_C \delta(x,y,z)ds,\ \delta(x,y,z) \geq 0$

Application of Vector Line Integrals

Work: $\int_C \vec{F} \cdot d\vec{r}$

Derivatives & Integrals

$$\frac{d}{dx}c = 0 \qquad \frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}a^x = a^x \ln a \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$

scalar multiple rule:

$$\frac{d}{dx}\Big[cf(x)\Big] = cf'(x)$$

sum rule:

$$\frac{d}{dx}\Big[f(x) + g(x)\Big] = f'(x) + g'(x)$$

product rule:

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$$

quotient rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

chain rule:

$$\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'(g(x))g'(x)$$

n and c are constants

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \ln x\,dx = x\ln x - x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$
$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$
$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

scalar multiple rule:

$$\int cf(x)\,dx = c\int f(x)\,dx$$

sum rule:

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

u-substitution:

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$
$$u = g(x) \qquad du = g'(x)\,dx$$

integration by parts:

$$\int f(x)g(x)\,dx = uv - \int v\,du$$
$$u = f(x) \qquad dv = g(x)\,dx$$
$$du = f'(x)\,dx \qquad v = \int g(x)\,dx$$

c, n, and C are constants

Appendix B: Parametric Curves

Some common vector parametrizations for frequently used curves.

Curves in $\mathbb{R}^2$

Line passing through the point $P = (x_0, y_0)$, parallel to $\vec{\boldsymbol{v}} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}}$:

$$\begin{aligned}\vec{\boldsymbol{r}}(t) &= t\vec{\boldsymbol{v}} + \vec{\boldsymbol{r}}_0\\ &= t\Big(v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}}\Big) + \Big(x_0\,\hat{\boldsymbol{\imath}} + y_0\,\hat{\boldsymbol{\jmath}}\Big)\\ &= (tv_1 + x_0)\,\hat{\boldsymbol{\imath}} + (tv_2 + y_0)\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

Graphs of Functions

$$\begin{array}{lll}\boldsymbol{y = f(x)} & (y \text{ is a function of } x): & \vec{\boldsymbol{r}}(t) = t\,\hat{\imath} + f(t)\,\hat{\jmath}\\ \boldsymbol{x = g(y)} & (x \text{ is a function of } y): & \vec{\boldsymbol{r}}(t) = g(t)\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}}\end{array}$$

Circle $\boldsymbol{x^2 + y^2 = R^2}$ centered at the origin; R, ω constant:

$$\vec{\boldsymbol{r}}(t) = R\cos(\omega t)\,\hat{\boldsymbol{\imath}} + R\sin(\omega t)\,\hat{\boldsymbol{\jmath}}$$

Circle $\boldsymbol{(x - x_0)^2 + (y - y_0)^2 = R^2}$, centered at the point $P = (x_0, y_0)$; R, x_0, y_0, ω constant:

$$\vec{\boldsymbol{r}}(t) = \Big(R\cos(\omega t) + x_0\Big)\,\hat{\boldsymbol{\imath}} + \Big(R\sin(\omega t) + y_0\Big)\,\hat{\boldsymbol{\jmath}}$$

Ellipse $\boldsymbol{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$ in standard position; a, b, ω constant:

$$\vec{\boldsymbol{r}}(t) = a\cos(\omega t)\,\hat{\boldsymbol{\imath}} + b\sin(\omega t)\,\hat{\boldsymbol{\jmath}}$$

Curves in $\mathbb{R}^3$

Line passing through the point $P = (x_0, y_0, z_0)$, parallel to $\vec{\boldsymbol{v}} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$:

$$\begin{aligned}\vec{\boldsymbol{r}}(t) &= t\vec{\boldsymbol{v}} + \vec{\boldsymbol{r}}_0\\ &= t\Big(v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}\Big) + \Big(x_0\,\hat{\boldsymbol{\imath}} + y_0\,\hat{\boldsymbol{\jmath}} + z_0\,\hat{\boldsymbol{k}}\Big)\\ &= (tv_1 + x_0)\,\hat{\boldsymbol{\imath}} + (tv_2 + y_0)\,\hat{\boldsymbol{\jmath}} + (tv_3 + z_0)\,\hat{\boldsymbol{k}}\end{aligned}$$

Helix centered about z-axis; R, k, ω constant:

$$\vec{\boldsymbol{r}}(t) = R\cos(\omega t)\,\hat{\boldsymbol{\imath}} + R\sin(\omega t)\,\hat{\boldsymbol{\jmath}} + kt\,\hat{\boldsymbol{k}}$$