Topics in Set Theory Sheet 4

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March 12, 2019

34.

Fix $A \in M[G]$, a formula $\varphi(x, y, x_1, ..., x_n)$ which specifies the function that we want to use for replacement, and fix parameters $a_1, ..., a_n \in M[G]$. We want a name for

$$B := \{y : M[G] \vDash \exists x \in A\varphi(x, y, a_1, ..., a_n)\}$$

Take a name σ for A and some names $\tau_1, ..., \tau_n$ for $a_1, ..., a_n$. Define a formula

$$\psi(x, y, x_1, ..., x_n, A) := x \in A \wedge \varphi(x, y, x_1, ..., x_n)$$

and now consider the name

$$\rho := \{(\pi, p) : p \Vdash^* \exists x\psi(x, \pi, \tau_1, ..., \tau_n, \sigma)\}$$

We claim that $val(\rho, G) = B$.
$\subseteq$: suppose $y \in val(\rho, G)$. So there is $(\pi, p) \in \rho$ with $p \in G$ and $val(\pi, G) = y$. So $p \Vdash^* \pi \in \exists x\psi(x, \pi, \tau_1, ..., \tau_n, \sigma)$. By definition, this means that the set

$$D := \{r : \exists \mu \in M^{\mathbb{P}}(r \Vdash^* \psi(\mu, \pi, \tau_1, ..., \tau_n, \sigma))\}$$

is dense below p. In particular $G \cap D \neq \phi$, so take some $q \in G \cap D$. $q \in D$ means there exists a name μ s.t.

$$q \Vdash^* \psi(\mu, \pi, \tau_1, ..., \tau_n, \sigma)$$

But $q \in G$ as well, so by FT,

$$M[G] \vDash \psi(\mu, \pi, \tau_1, ..., \tau_n, \sigma)$$

which translates to: there exists x $(:= val(\mu, G))$ that

$$\begin{aligned} &M[G] \vDash \psi(x, y, a_1, ..., a_n, A) \\ \iff &M[G] \vDash x \in A \wedge \varphi(x, y, x_1, ..., x_n) \end{aligned}$$

we can rewrite this as

$$M[G] \vDash \exists x(x \in A \wedge \varphi(x, y, x_1, ..., x_n))$$

i.e. $y \in B$.
$\supseteq$: suppose $y \in B$. So $M[G] \vDash \exists x\psi(x, y, a_1, ...a_n, A)$. Now take a name π for y; so by FT, there exists $p \in G$ s.t. $p \Vdash^* \exists x\psi(x, \pi, \tau_1, ..., \tau_n, \sigma)$. But this is exactly the requirement for $(\pi, p) \in \rho$. So $y \in val(\rho, G)$.

35.

Fix $A \in M[G]$, and $\phi \notin A$. We want to prove that there exists $f \in M[G]$ s.t. f is a function from $A \to \cup A$, and $x \in A \implies f(x) \in x$.

Take a name σ of A. Now $dom(\sigma) \in M$, and $M \vDash AC$, so there exists a choice function g for $dom(\sigma)$. In other words, for each $(\pi, p) \in \sigma$, we have $g(\pi) \in \pi$.