

Representation Theory

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0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over \mathbb{C} . Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise:

- F is a field, usually \mathbb{C} , \mathbb{R} or \mathbb{Q} . When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or \bar{F}_p (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
- V is a vector space over F , always finite dimensional;
 $GL(V) = \{\theta : V \rightarrow V, \theta \text{ linear, invertible}\}$, i.e. $\det \theta \neq 0$.

Recall from Linear Algebra:

If $\dim_F V = n < \infty$, choose basis e_1, \dots, e_n over F , so we can identify it with F^n . Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \rightarrow A_\theta$ ($A_{\theta_1 \theta_2} = A_{\theta_1} A_{\theta_2}$ and bijection).
 Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = X A_1 X^{-1}$.

Recall that $\text{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A .

(1.3) $\text{tr}(X A X^{-1}) = \text{tr}(A)$, hence we can define $\text{tr}(\theta) = \text{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V in f.d. over \mathbb{C} , with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $\text{End} V$ is the set of all linear maps $V \rightarrow V$, e.g. $\text{End}(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over \mathbb{C} , $\alpha \in \text{End}(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1} (X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a \mathbb{C} -vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:

(1.6) The symmetric group, $S_n = \text{Sym}(X)$ on the set $X = \{1, \dots, n\}$ is the set of all permutations of X . $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m : $C_m = \langle x : x^m = 1 \rangle$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in \mathbb{C} (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of \mathbb{C}). We also have the group of rotations, centre O of regular m -gon in \mathbb{R}^2 (inside $GL_2(\mathbb{R})$).

(1.8) Dihedral groups D_{2m} of order $2m = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$. Think of this as the set of rotations and reflections preserving a regular m -gon.

(1.9) Quaternion group, $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \rightarrow X$ by $(g, x) \rightarrow g \cdot x$ for $g \in G, x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X , we obtain a homomorphism $\theta : G \rightarrow \text{Sym}(X)$, called the *permutation representation* of G .

Proof. For $g \in G$, the function $\theta_g : X \rightarrow X$ by $x \rightarrow gx$ is a permutation on X , with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G, \theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2}$ since $(g_1 g_2)x = g_1(g_2 x)$ for $x \in X$. \square

2 Basic Definitions

2.1 Representations

Let G be finite, F be a field, usually \mathbb{C} .

Definition. (2.1)

Let V be a f.d. vector space over F . A (linear, in some books) *representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \rightarrow GL(V)$$

Write ρ_g for the image $\rho_V(g)$; so for each $g \in G$, $\rho_g \in GL(V)$, and $\rho_{g_1 g_2} = \rho_{g_1} \rho_{g_2}$, and $(\rho_g)^{-1} = \rho_{g^{-1}}$.

The *dimension* (or *degree*) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \triangleleft G$ (kernel is a normal subgroup), and $G/\ker \rho \cong \rho(G) \leq GL(V)$ (1st isomorphism theorem). We say ρ is *faithful* if $\ker \rho = 1$.

An alternative (and equivalent) approach is to observe that a representation of G on V is "the same as" a *linear action* of G :

Definition. (2.3)

G *acts linearly* on V if there exists a *linear action*

$$\begin{aligned} G \times V &\rightarrow V \\ (g, v) &\rightarrow gv \end{aligned}$$

By linear action we mean: (action) $(g_1 g_2)v = g_1(g_2 v)$, $1v = v \ \forall g_1, g_2 \in G, v \in V$, and (linear) $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv \ \forall g \in G, v_1, v_2 \in V, \lambda \in F$.

Now if G acts linearly on V , the map

$$\begin{aligned} G &\rightarrow GL(V) \\ g &\rightarrow \rho_g \end{aligned}$$

with $\rho_g : v \rightarrow gv$ is a representation of G . Conversely, given a representation $\rho : G \rightarrow GL(V)$, we have a linear action of G on V via $g \cdot v := \rho(g)v \ \forall v \in V, g \in G$.

(2.4) In (2.3) we also say that V is a G -space or that V is a G -module. In fact if we define the *group algebra* FG , or $F[G]$, to be $\{\sum \alpha_j g : \alpha_j \in F\}$ with natural addition and multiplication, then V is actually a FG -module (in the sense from GRM).

(2.5) R is a *matrix representation* of G of degree n if R is a homomorphism $G \rightarrow GL_n(F)$. Given representation $\rho : G \rightarrow GL(V)$ with $\dim_F V = n$, fix basis B ; we get matrix representation

$$\begin{aligned} G &\rightarrow GL_n(F) \\ g &\rightarrow [\rho(g)]_B \end{aligned}$$

Conversely, given matrix representation $R : G \rightarrow GL_n(F)$, we get representation

$$\begin{aligned}\rho : G &\rightarrow GL(F^n) \\ g &\rightarrow \rho_g\end{aligned}$$

via $\rho_g(v) = R_g v$ where R_g is the matrix of g .

Example. (2.6)

Given any group G , take $V = F$ the 1-dimensional space, and

$$\begin{aligned}\rho : G &\rightarrow GL(F) \\ g &\rightarrow (id : F \rightarrow F)\end{aligned}$$

is known as the trivial representation of G . So $\deg \rho = 1$ ($\dim_F F = 1$).

Example. (2.7)

Let $G = C_4 = \langle x : x^4 = 1 \rangle$. Let $n = 2$, and $F = \mathbb{C}$. Note that any $R : x \rightarrow X$ will determine $x^j \rightarrow X^j$ as it is a homomorphism, and also we need $X^4 = I$. So we can take X to be diagonal matrix – any such with diagonal entries a root to $x^4 = 1$, i.e. $\{\pm 1, \pm i\}$, or if X is not diagonal then it will be similar to a diagonal matrix by (1.4) ($X^4 = I$).

2.2 Equivalent representations

Definition. (2.8)

Fix G, F . Let V, V' be F -spaces, and $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$ which are representations of G . The linear map $\phi : V \rightarrow V'$ is a G -homomorphism if

$$\phi \rho(g) = \rho'(g) \phi \forall g \in G(*)$$

We can understand this more by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \phi \downarrow & \searrow & \downarrow \phi \\ V' & \xrightarrow{\rho'_{g'}} & V' \end{array}$$

We say ϕ *intertwines* ρ, ρ' . Write $\text{Hom}_G(V, V')$ for the F -space of all these. ϕ is a G -isomorphism if it is also bijective; if such ϕ exists, ρ, ρ' are isomorphic/equivalent representations. If ϕ is a G -isomorphism, we can write (*) as $\rho' = \phi\rho\phi^{-1}$.

Lemma. (2.9)

The relation "being isomorphic" is an equivalent relation on the set of all representations of G (over F).

Remark. (2.10)

If ρ, ρ' are isomorphic representations, they have the same dimension.

The converse may be false: C_4 has four non-isomorphic 1-dimensional representations: if $\omega = e^{2\pi i/4}$ then they are $\rho_j(x^i) = \omega^{ij}$ ($0 \leq i \leq 3$).

Remark. (2.11)

Given G, V over F of dimension n and $\rho : G \rightarrow GL(V)$. Fix basis B for V : we get a linear isomorphism

$$\begin{aligned} \phi : V &\rightarrow F^n \\ v &\rightarrow [v]_B \end{aligned}$$

and we get a representation $\rho' : G \rightarrow GL(F^n)$ isomorphic to ρ :

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \\ \downarrow \phi & & \downarrow \phi \\ F^n & \xrightarrow{\rho'} & F^n \end{array}$$

(2.12) In terms of matrix representations, we have

$$\begin{aligned} R : G &\rightarrow GL_n(F), \\ R' : G &\rightarrow GL_n(F) \end{aligned}$$

are (G) -isomorphic or equivalent if there exists a nonsingular matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1} \forall g \in G$.

In terms of linear G -actions, the actions of G on V, V' are G -isomorphic if there exists isomorphisms $\phi : V \rightarrow V'$ such that $g : \phi(v) = \phi(gv) \forall v \in V, g \in G$.

2.3 Subrepresentations

Definition. (2.13)

Let $\rho : G \rightarrow GL(V)$ be a representation of G . We say $W \leq V$ is a G -subspace if it's a subspace and it is $\rho(G)$ -invariant, i.e. $\rho_g(W) \leq W \forall g \in G$. Obviously $\{0\}$ and V are G -subspaces, however.

ρ is *irreducible/simple* representation if there are no proper G -subspaces.

Example. (2.14)

Any 1-dimensional representation of G is irreducible, but not conversely, e.g. D_8 has 2-dimensional \mathbb{C} -irreducible representation.

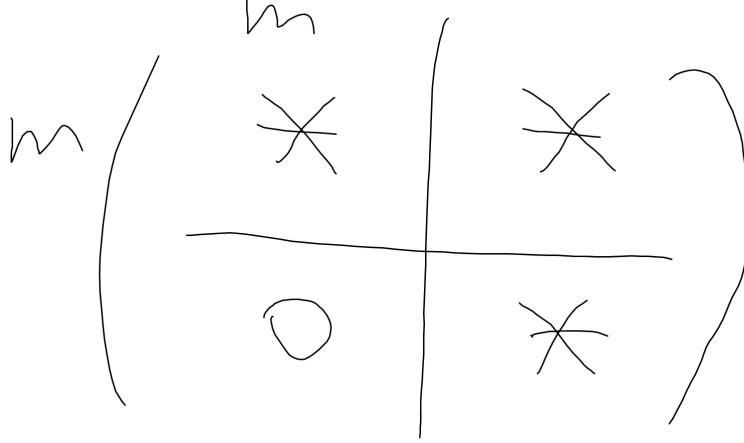
(2.15) In definition (2.13), if W is a G -subspace, then the corresponding map

$$\begin{aligned} G &\rightarrow GL(W) \\ g &\rightarrow \rho(g)|_W \end{aligned}$$

is a representation of G , a *subrepresentation* of ρ .

Lemma. (2.16)

In definition (2.13), given $\rho : G \rightarrow GL(V)$, if W is a G -subspace of V and if $B = \{v_1, \dots, v_n\}$ is a basis containing basis $B_1 = \{v_1, \dots, v_m\}$ of W ($0 < m < n$) then the matrix of $\rho(g)$ w.r.t. B has block upper triangular form as the graph below, for



each $g \in G$.

Example. (2.17)

(i) The irreducible representations of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1-dimensional and four of these are $x \rightarrow i, x \rightarrow -1, x \rightarrow -i, x \rightarrow 1$. In general, $C_m = \langle x : x^m = 1 \rangle$ has precisely m irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)* or see (4.4) below).

(ii) $G = D_6$: any irreducible C -representation has dimension ≤ 2 .

Let $\rho : G \rightarrow GL(V)$ be irreducible G -representation. Let r, s be rotation and reflection in D_6 respectively. Let V be eigenvector of $\rho(r)$. So $\rho(r)v = \lambda v$

for some $\lambda \neq 0$. Let $W = \text{span}\{v, \rho(s)v\} \leq V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, both of which are in W ; so W is G -invariant, i.e. a G -subspace. Since V is irreducible, $W = V$.

Definition. (2.18)

We say that $\rho : G \rightarrow GL(V)$ is *decomposable* if there are proper G -invariant subspaces U, W with $V = U \oplus W$. Say ρ is direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists, we say that ρ is *indecomposable*.

Lemma. (2.19)

Suppose $\rho : G \rightarrow GL(V)$ is decomposable with G -invariant decomposition $V = U \oplus W$. If B is a basis $\{\underbrace{u_1, \dots, u_k}_{B_1}, \underbrace{w_1, \dots, w_l}_{B_2}\}$ of V consisting of basis of U and basis of W , then w.r.t. B , $\rho(g)_B$ is a block diagonal matrix $\forall g \in G$ as

$$\rho(g)_B = \begin{pmatrix} [\rho_U(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

Definition. (2.20)

If $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$, the *direct sum* of ρ, ρ' is

$$\rho \oplus \rho' : G \rightarrow GL(V \oplus V')$$

where $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$, a *block diagonal action*. For matrix representations $R : G \rightarrow GL_n(F)$, $R' : G \rightarrow GL_{n'}(F)$, define $R \oplus R' : G \rightarrow GL_{n+n'}(F)$:

$$g \rightarrow \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Definition. (3.1)

A representation $\rho : G \rightarrow GL(V)$ is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

Remark. (3.2)

- (1) The converse is false;
- (2) See sheet 1 Q3: \mathbb{C} -representation of \mathbb{Z} is not completely reducible and also representation of C_p over \mathbb{F}_p is not c.r..

From now on, take G finite and $\text{char } F = 0$.

Theorem. (3.3)

Every f.d. representation V of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus \dots \oplus V_r$$

is a direct sum of representations, each V_i irreducible.

It is enough to prove:

Theorem. (3.4 Maschke's theorem, 1899)

Let G be finite, $\rho : G \rightarrow GL(V)$ a f.d. representation, $\text{char } F = 0$. If W is a G -subspace of V , then there exists a G -subspace U of V s.t. $V = W \oplus U$, a direct sum of G -subspaces.

Proof. (1)

Let W' be any *vector subspace* complement of W in V , i.e. $V = W \oplus W'$ as vector spaces, and $W \cap W' = 0$. Let $q : V \rightarrow W$ be the projection of V onto W along W' ($\ker q = W'$), i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\bar{q} : v \rightarrow \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of q over G . Note that in order for $\frac{1}{|G|}$ to exist, we need $\text{char } F = 0$.

It still works if $\text{char } F \nmid |G|$.

Claim (1): $\bar{q} : V \rightarrow W$: For $v \in V$, $g(q(g^{-1}v)) \in W$ and $gW \leq W$;

Claim (2): $\bar{q}(w) = w$ for $w \in W$:

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w = w$$

So these two claims imply that \bar{q} projects V onto W .

Claim (3) If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ ($v \in V$):

$$\begin{aligned}
 h\bar{q}(v) &= h \frac{1}{|G|} \sum_g g \cdot q(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g hgq(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g (hg)q((hg)^{-1}hv) \\
 &= \frac{1}{|G|} \sum_g gq(g^{-1}(hv)) \\
 &= \bar{q}(hv) \\
 &= \bar{q}(hv)
 \end{aligned}$$

We'll then show that the kernel of this map is G -invariant, so this gives a G -summand on Thursday.

Let's now show $\ker \bar{q}$ is G -invariant. If $v \in \ker \bar{q}$, then $h\bar{q}(v) = 0 = \bar{q}(hv)$, so $hv \in \ker \bar{q}$. Thus $V = \text{im } \bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$ is a G -subspace decomposition.

We can deduce (3.3) from (3.4) by induction on $\dim V$. If $\dim V = 0$ or V is irreducible, then result is clear. Otherwise, V has non-trivial G -invariant subspace, W . Then by (3.4), there exists G -invariant complement U s.t. $V = U \oplus W$ as representations of G . But $\dim U, \dim W < \dim V$. So by induction they can be broken up into direct sum of irreducible subrepresentations. \square

The second proof uses inner products, hence we need to take $F = \mathbb{C}$ and can be generalised to compact groups in section 15.

Recall, for V a \mathbb{C} -space, \langle, \rangle is a *Hermitian inner product* if

- (a) $\langle w, v \rangle = \overline{\langle v, w \rangle} \ \forall v, w$ (Hermitian);
- (b) linear in RHS (sesquilinear);
- (c) $\langle v, v \rangle > 0$ iff $v \neq 0$ (positive definite).

Additionally, \langle, \rangle is *G -invariant* if

- (d) $\langle gv, gw \rangle = \langle v, w \rangle \ \forall v, w \in V, g \in G$.

Note if W is G -invariant subspace of V , with G -invariant inner product, then W^\perp is also G -invariant, and $V \oplus W^\perp$. For all $v \in W^\perp, g \in G$, we have to show that $gv \in W^\perp$. But $v \in W^\perp \iff \langle v, w \rangle = 0 \ \forall w \in W$. Thus by (d), $\langle gv, gw \rangle = 0 \ \forall g \in G \ \forall w \in W$. Hence $\langle gv, w' \rangle = 0 \ \forall w' \in W$. Since we can choose $w = g^{-1}w' \in W$ by G -invariance of W . Thus $gv \in W^\perp$ since g was arbitrary.

Hence if there is a G -invariant inner product on any G -space, we get another proof of Maschke's theorem:

(3.4*) (Weyl's unitary trick)

Let ρ be a complex representation of the finite group G on the \mathbb{C} -space V . Then there is a G -invariant Hermitian inner product on V .

Remark. Recall the *unitary group* $U(V)$ on V : $\{f \in GL(V) : (fu, fv) = (u, v) \ \forall u, v \in V\} = \{A \in GL_n(\mathbb{C}) : A\bar{A}^T = I\} (= U(n))$ by choosing orthonormal

basis.

Sheet 1 Q.12: any finite subgroup of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Proof. (2)

There exist an inner product on V : take basis e_1, \dots, e_n and define $(e_i, e_j) = \delta_{ij}$, extended sesquilinearly. Now

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} (gv, gw)$$

we claim that \langle, \rangle is sesquilinear, positive definite and G -invariant: if $h \in G$, then

$$\begin{aligned} \langle hv, hw \rangle &= \frac{1}{|G|} \sum_{g \in G} ((gh)v, (gh)w) \\ &= \frac{1}{|G|} \sum_{g' \in G} (g'v, g'w) \\ &= \langle v, w \rangle \end{aligned}$$

for all $v, w \in V$. □

Definition. (3.5, the regular representation)

Recall *group algebra* of G is F -space $FG = \text{span}\{e_g : g \in G\}$. There is a linear G -action

$$h \in G, h \sum_{g \in G} a_g e_g = \sum_{g \in G} a_g e_{hg} (= \sum_{g' \in G} a_{h^{-1}g'} e_{g'})$$

ρ_{reg} is the corresponding representation, the *regular representation* of G . This is faithful of $\dim |G|$. FG is the *regular module*.

Proposition. Let ρ be an irreducible representation of G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg} .

Proof. Take $\rho : G \rightarrow GL(V)$ irreducible and let $0 \neq v \in V$. Let $\theta : FG \rightarrow V$ by $\sum a_g e_g \rightarrow \sum a_g gv$. Check this is a G -homomorphism. Now V is irreducible so $\text{im } \theta = V$ (since $\text{im } \theta$ is a G -subspace).

Also $\ker \theta$ is G -subspace of FG . Let W be G -complement of $\ker \theta$ in FG (Maschke), so that $W < FG$ is G -subspace and $FG = \ker \theta \oplus W$. Thus $W \cong FG / \ker \theta \cong (G\text{-isomorphism}) \text{im } \theta \cong V$. □

More generally,

Definition. (3.7)

Let F be a field. Let G act on set X . Let $FX = \text{span}\{e_x : x \in X\}$ with G -action

$$g(\sum a_x e_x) = \sum a_x e_{gx}$$

The representation $G \rightarrow GL(V)$ where $V = FX$ is the corresponding *permutation representation*. See section 7.

4 Schur's lemma

It's really unfair that such an important result is only remembered by a lemma, so we shall call it a theorem.

Theorem. (4.1, Schur)

- (a) Assume V, W are irreducible G -spaces over field F . Then any G -homomorphism $\theta : V \rightarrow W$ is either 0 or an isomorphism.
- (b) Assume F is algebraically closed, and let V be an irreducible G -space. Then any G -endomorphism $V \rightarrow V$ is a scalar multiple of the identity map ι_V .

Proof. (a) Let $\theta : V \rightarrow W$ be a G -homomorphism. Then $\ker \theta$ is G subspace of V and, since V is irreducible, we get $\ker \theta = 0$ or $\ker \theta = V$.

And $\text{im} \theta$ is G -subspace of W , so as W is irreducible, $\text{im} \theta$ is either 0 or W . Hence, either $\theta = 0$ or θ is injective and surjective, hence isomorphism.

(b) Since F is algebraically closed, θ has an eigenvalue, λ . Then $\theta - \lambda \iota$ is singular G -endomorphism of V , but it cannot be an isomorphism, so it is 0 (by (a)). So $\theta = \lambda \iota_V$. \square

Recall from (2.8), the F -space $\text{Hom}_G(V, W)$ of all G -homomorphisms $V \rightarrow W$. Write $\text{End}_G(V)$ for the G -endomorphisms of V .

Corollary. (4.2)

If V, W are irreducible complex G -spaces, then

$$\dim_{\mathbb{C}} \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V, W are not G -isomorphic then the only G -homomorphism $V \rightarrow W$ is 0 by (4.1). Assume $v \cong_G W$ and $\theta_1, \theta_2 \in \text{Hom}_G(V, W)$, both non-zero. Then θ_2 is invertible by (4.1), and $\theta_2^{-1}\theta_1 \in \text{End}_G(V)$, and non-zero, so $\theta_2^{-1}\theta_1 = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$. Hence $\theta_1 = \lambda \theta_2$. \square

Corollary. (4.3)

If finite group G has a faithful complex irreducible representation, then $Z(G)$, the centre of the group, is cyclic.

Note that the converse is false (Sheet 1, Q10).

Proof. Let $\rho : G \rightarrow GL(V)$ be faithful irreducible complex representation. Let $z \in Z(G)$, so $zg = gz \forall g \in G$, hence the map $\phi_z : v \rightarrow z(v)$ ($v \in V$) is G -endomorphism of V , hence is multiplication by scalar μ_z , say.

By Schur's lemma, $z(v) = \mu_z v \forall v$. Then the map

$$\begin{aligned} Z(G) &\rightarrow \mathbb{C}^* \text{ (multiplicative group)} \\ z &\rightarrow \mu_z \end{aligned}$$

is a representation of Z and is faithful, since ρ is. Thus $Z(G)$ is isomorphic to some finite subgroup of \mathbb{C}^* , so is cyclic. \square

Let's now consider representation of finite abelian groups.

Corollary. (4.4)

The irreducible \mathbb{C} -representations of a finite abelian group are all 1-dimensional.

Proof. Either: use (1.4)* to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$, and if V is irreducible, then $V = \langle v \rangle$.

Or: Let V be an irreducible \mathbb{C} -representation. For $g \in G$, the map

$$\begin{array}{ccc} \theta_g : V & \rightarrow & v \\ v & \rightarrow & gv \end{array}$$

is a G -endomorphism of V , and as V irreducible, $\theta_g = \lambda_g \text{id}_V$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any $g \in G$ (so $\langle v \rangle$ is a G -subspace of V). Thus as $0 \neq V$ is irreducible, $V = \langle v \rangle$, which is 1-dimensional. \square

Remark. Schur's lemma fails over non-algebraically closed field, in particular, over \mathbb{R} . For example, let's consider the cyclic group C_3 . It has 2 irreducible \mathbb{R} -representations, one of dimension 1 (maps everything to 1) and one of dimension 2 (imo consider \mathbb{C} as a dimension 2 space over \mathbb{R} , then map the generator to the 3rd root of unity?) (so 'contradicting' with Schur's lemma via the corollary above).

Recall that every finite abelian group G is isomorphic to a product of cyclic groups (see GRM). For example, $C_6 = C_2 \times C_3$. In fact, it can be written as a product of C_{p^α} for various primes p and $\alpha \geq 1$, and the factors are uniquely determined up to reordering.

Proposition. (4.5)

The finite abelian group $G = C_{n_1} \times \dots \times C_{n_r}$ has precisely $|G|$ irreducible \mathbb{C} -representations, as described below:

Proof. Write $G = \langle x_1 \rangle \times \dots \times \langle x_r \rangle$ where $|x_j| = n_j$. Suppose ρ is irreducible, so by (4.4), it's 1-dimensional: $\rho : G \rightarrow \mathbb{C}^*$.

Let $\rho(1, \dots, x_j, \dots, 1)$ (all 1 apart from the j^{th} entry) be λ_j . Then $\lambda_j^{n_j} = 1$, so λ_j is a n_j -th root of unity. Now, the values $(\lambda_1, \dots, \lambda_r)$ determine ρ :

$$\rho(x_1^{j_1}, \dots, x_r^{j_r}) = \lambda_1^{j_1} \dots \lambda_r^{j_r}$$

thus $\rho \leftrightarrow (\lambda_1, \dots, \lambda_r)$ with $\lambda_j^{n_j} = 1 \forall j$; we have $n_1 \dots n_r$ such r -tuples, each giving 1-dimensional representation. \square

Example. (4.6)

Consider $G = C_4 = \langle x \rangle$. We could have $\rho_1(x) = 1, \rho_2(x) = i, \rho_3(x) = -1, \rho_4(x) = -i$.

Warning: There is no "natural" 1-1 correspondence between the elements of G and the representations of G (G -finite abelian). If you choose an isomorphism $G \cong C_{a_1} \times \dots \times C_{a_r}$, then we can identify the two sets (elements of groups and representations of G), but it depends on the choice of isomorphism.

Isotypical decomposition:

Recall any diagonalisable endomorphism $\alpha : V \rightarrow V$ gives eigenspace decomposition of $V \cong \oplus_\lambda V(\lambda)$, where $V(\lambda) = \{v : \alpha v = \lambda v\}$. This is *caconical* (one of the three useless words: *arbitrary*(anything), *canonical*(only one choice), *uniform*(you can choose, but it doesn't really matter)), in the sense that it depends on α alone (and nothing else).

There is no canonical eigenbasis of V : must choose basis in each $V(\lambda)$.

We know that in *char* 0 every representation V decomposes as $\oplus n_i V_i$, V_i irreducible, $n_i \geq 0$. How unique is this?

We have this wishlist (4.7):

- (a) Uniqueness: for each V there is only one way to decompose V as above. However, this doesn't work obviously.
- (b) Isotypes: for each V , there exists a unique collection of subrepresentations U_1, \dots, U_k s.t. $V = \oplus U_i$ and, if $V_i \subseteq U_i$ and $V_j' \subseteq U_j$ are irreducible subrepresentations, then $V_i \cong V_j'$ iff $i = j$.
- (c) Uniqueness of factors: If $\oplus_{i=1}^k V_i \cong \oplus_{i=1}^{k'} V_i'$ with V_i, V_i' irreducible, then $k = k'$, and $\exists \pi \in S_k$ such that $V_{\pi(i)}' \cong V_i$ (Krull-Schmidt theorem). For (b),(c) see Teleman section 5.

Lemma. (4.8)

Let V, V_1, V_2 be G -spaces over F .

- (i) $\text{Hom}_G(V, V_1 \oplus V_2) \cong \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2)$;
- (ii) $\text{Hom}_G(V_1 \oplus V_2, V) \cong \text{Hom}_G(V_1, V) \oplus \text{Hom}_G(V_2, V)$;

Proof. (i) Let $\pi_i : V_1 \oplus V_2 \rightarrow V_i$ be G -linear projections onto V_i , with kernel V_{3-i} ($i = 1, 2$).

Consider

$$\begin{aligned} \text{Hom}_G(V, V_1 \oplus V_2) &\rightarrow \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2) \\ \phi &\rightarrow (\pi_1 \phi, \pi_2 \phi) \end{aligned}$$

This map has inverse $(\psi_1, \psi_2) \rightarrow \psi_1 + \psi_2$. Check details.

- (ii) The map $\phi \rightarrow (\phi|_{V_1}, \phi|_{V_2})$ has inverse $(\psi_1, \psi_2) \rightarrow \psi_1 \pi_1 + \psi_2 \pi_2$. □

Lemma. Let F be algebraically closed, $V = \oplus_1^n V_i$ a decomposition of G -space into irreducible summands. Then, for each irreducible representation S of G ,

$$\#\{j : V_j \cong S\} = \dim \text{Hom}_G(S, V)$$

where $\#$ means 'number of times'. This is called the *multiplicity* of S in V .

Proof. Induction on n . $n = 0, 1$ are trivial.

If $n > 1$, $V = \oplus_1^{n-1} V_i \oplus V_n$. By (4.8) we have

$$\dim \text{Hom}_G(S, \oplus_1^{n-1} V_i \oplus V_n) = \dim \text{Hom}(S, \oplus_1^{n-1} V_i) + \underbrace{\dim \text{Hom}_G(S, V_n)}_{\text{Schur's lemma}}$$

□

Definition. (4.10)

A decomposition of V as $\oplus W_j$ where each $W_j \cong n_j$ copies of irreducible representations S_j (each non-isomorphic for each j) is the *canonical decomposition* or the decomposition into *isotypical components* W_j . For F algebraically closed, $n_j = \dim \operatorname{Hom}_G(S_j, V)$.

5 Character theory

We want to attach invariants to representation ρ of a finite group G on V . Matrix coefficients of $\rho(g)$ are basis dependent, so not true invariants.

Let's take $F = \mathbb{C}$, G finite, $\rho = \rho_V : G \rightarrow GL(V)$ be a representation of G .

Definition. (5.1)

The *character* $\chi_\rho = \chi_V = \chi$ is defined as $\chi(g) = \text{tr } \rho(g) = \text{tr } R(g)$ where $R(g)$ is any matrix representation of $\rho(g)$ w.r.t. any basis.

The degree of χ_V is $\dim_{\mathbb{C}} V$.

Thus χ is a function $G \rightarrow \mathbb{C}$. χ is *linear* (not a universal name) if $\dim V = 1$, in which case χ is a homomorphism $G \rightarrow \mathbb{C}^*$ ($= GL_1(\mathbb{C})$).

χ is irreducible if ρ is; χ is faithful if ρ is; and, χ is trivial, or principal, if ρ is the trivial representation (2.6). We write $\chi = 1_G$ in that case.

χ is a complete invariant in the sense that it determines ρ up to isomorphism – see (5.7).

Theorem. (5.2, first properties)

- (i) $\chi_V(1) = \dim_{\mathbb{C}} V$; (clear: $\text{tr } I_n = n$)
- (ii) χ_V is a *class function*, via it is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$$

Thus χ_V is constant on conjugacy classes.

- (iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, the complex conjugate;

- (iv) For two representations V, W , $\chi_{V \oplus W} = \chi_V + \chi_W$.

Proof. (ii) $\chi(hgh^{-1}) = \text{tr}(R_h R_g R_h^{-1}) = \text{tr}(R_g) = \chi(g)$.

(iii) Recall $g \in G$ has finite order, so we can assume $\rho(g)$ is represented by a diagonal matrix $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Then $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by the matrix $\text{Diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$, and hence $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\chi(g)}$ (since λ_i 's are roots of unity – since $g^k = 1$ for some k ! (I mean an exclamation mark here to express surprise) and by homomorphism we know that).

(iv) Suppose $V = V_1 \oplus V_2$, $\rho_i : G \rightarrow GL(V_i)$, $\rho : G \rightarrow GL(V)$. Take basis $B = B_1 \cup B_2$ of V w.r.t. B , $\rho(g)$ has matrix of block form $\text{Diag}([\rho_1(g)]_{B_1}, [\rho_2(g)]_{B_2})$ and as $\chi(g)$ is the trace of the above matrix, it is equal to $\text{tr } \rho_1(g) + \text{tr } \rho_2(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$. \square

Remark. We see later that χ_1, χ_2 character of G implies that $\chi_1 \chi_2$ is also a character of G : uses tensor products, see (9.6).

Lemma. (5.3)

Let $\rho : G \rightarrow GL(V)$ be a complex representation *affording* the character χ (i.e. χ is a character of ρ). Then $|\chi(g)| \leq \chi(1)$, with equality iff $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ iff $g \in \ker \rho$.

Proof. Fix g . W.r.t. basis of V of eigenvalues $\rho(g)$, the matrix of $\rho(g)$ is $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Hence $|\chi(g)| = |\sum \lambda_j| \leq \sum |\lambda_j| = \sum 1 = \dim V = \chi(1)$. Equality holds iff all λ_j are equal (to λ , say).

If $\chi(g) = \chi(1)$, then $\rho(g) = \lambda I$ has $\chi(g) = \lambda \chi(1)$. \square

Lemma. (5.4)

- (a) If χ is a complex irreducible character of G , so is $\bar{\chi}$;
- (b) Under the same assumption, so is $\varepsilon\chi$ for any linear character ε of G .

Proof. If $R : G \rightarrow GL_n(\mathbb{C})$ is a complex irreducible representation then so is $\bar{R} : G \rightarrow GL_n(\mathbb{C})$ by $g \rightarrow \bar{R}(g)$. Similarly for $R' : g \rightarrow \varepsilon(g)R(g)$ for $g \in G$. Check the details. \square

Definition. (5.5)

$\mathcal{C}(G) = \{f : G \rightarrow \mathbb{C} : f(hgh^{-1}) = f(g) \forall h, g \in G\}$, the \mathbb{C} -space of class functions (we call it a space since $f_1 + f_2 : g \rightarrow f_1(g) + f_2(g)$, $\lambda f : g \rightarrow \lambda f(g)$ are still in $\mathcal{C}(G)$), so this is a vector space.

Let $k = k(G)$ be the number of ccls of G . List the ccls $\mathcal{C}_1, \dots, \mathcal{C}_k$. Conventionally we choose $g_1 = 1, g_2, \dots, g_k$, representatives of the ccls (hence $\mathcal{C}_1 = \{1\}$). Note that $\dim_{\mathbb{C}} \mathcal{C}(G) = k$ (the characteristic functions δ_j of each ccl which maps any element in the ccl to 1 and others to 0 form a basis).

We define Hermitian inner product on $\mathcal{C}(G)$:

$$\begin{aligned} \langle f, f' \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} f'(g) \\ &= \frac{1}{|G|} \sum_{j=1}^k |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) \\ &= \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \overline{f(g_j)} f'(g_j) \end{aligned}$$

using $|\mathcal{C}_x| = |G : C_G(x)|$, where \mathcal{C}_x is the ccl of x , $C_G(x)$ is the centraliser of x . For characters

$$\langle \chi, \chi' \rangle = \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

is a real symmetric form (in fact, $\langle \chi, \chi' \rangle \in \mathbb{Z}$ – see later).

Theorem. (5.6)

The \mathbb{C} -irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$. Moreover,

- (a) If $\rho : G \rightarrow GL(V), \rho' : G \rightarrow GL(V')$ are irreducible representations of G affording characters χ, χ' respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho, \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

we call this 'row orthogonality'.

- (b) Each class function of G can be expressed as a linear combination of G . This will be proved later in section 6.

Corollary. (5.7)

Complex representations of *finite* groups are characterised by their characters. We emphasise on finiteness here: for example, $G = \mathbb{Z}$, consider $1 \rightarrow I_2, 1 \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are non-isomorphic but have same character.

Proof. Let $\rho : G \rightarrow GL(V)$ be representation affording χ (G finite over \mathbb{C}). (3.3) says

$$\rho = m_1 \rho_1 \oplus \dots \oplus m_k \rho_k$$

where ρ_1, \dots, ρ_k are irreducible, and $m_j \geq 0$. Then $m_j = \langle \chi, \chi_j \rangle$ where χ_j is afforded by ρ_j : we have $\chi = m_1 \chi_1 + \dots + m_k \chi_k$, but the ρ_i 's are orthonormal. \square

Corollary. (5.8, irreducibility criterion)

If ρ is \mathbb{C} -representation of G affording χ , then ρ irreducible $\iff \langle \chi, \chi \rangle = 1$.

Proof. Forward is just the statement of orthonormality. Conversely, assume $\langle \chi, \chi \rangle = 1$. Now take a (complete) decomposition of ρ and take characters of it we get $\chi = \sum m_j \chi_j$ with χ_j irreducible and $m_j \geq 0$. Then $\sum m_j^2 = 1$. Hence $\chi = \chi_j$ for some j (since the m_j 's are obviously integers), so is irreducible. \square

Corollary. (5.9)

If the irreducible \mathbb{C} -representations of G are ρ_1, \dots, ρ_k have dimensions n_1, \dots, n_k , then

$$|G| = \sum_{i=1}^k n_i^2$$

Proof. Recall from (3.5), $\rho_{reg} : G \rightarrow GL(\mathbb{C}G)$, the regular representation G of dimension $|G|$ (where $\mathbb{C}G$ is just a G -space with basis $\{e_g : g \in G\}$ and any $h \in G$ permutes the e_g : $e_g \rightarrow e_{hg}$).

Let π_{reg} be its character, the *regular character* of G .

Claim 1: $\pi_{reg}(1) = |G|$, $\pi_{reg}(h) = 0$ if $h \neq 1$.

This is clear: take $h \in G, h \neq 1$, then we always have 0 down the diagonal since h permutes things around, so the trace is 0; if $h = 1$ then we have an identity matrix so trace is $\dim \rho = |G|$.

Claim 2: $\pi_{reg} = \sum n_j \chi_j$ with $n_j = \chi_j(1)$.

This is because

$$\begin{aligned} n_j &= \langle \pi_{reg}, \chi_j \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\pi_{reg}(g)} \chi_j(g) \\ &= \frac{1}{|G|} \cdot |G| \chi_j(1) = \chi_j(1) \end{aligned}$$

(all the other $\pi_{reg}(g)$ are zero by claim 1).

Our corollary is then obvious by just calculating $|G| = \pi_{reg}(1)$. \square

Corollary. (5.10)

Number of irreducible characters of G (up to equivalence) = k (=number of ccls).

Corollary. (5.11)

Elements $g_1, g_2 \in G$ are conjugate iff $\chi(g_1) = \chi(g_2)$ for all irreducible characters of G .

Proof. Forward: characters are class functions;

Backward: Let δ be the characteristic function of the class of g_1 . In particular, δ is a class function, so can be written as a linear combination of the irreducible characters of G . Hence $\delta(g_2) = \delta(g_1) = 1$, so $g_2 \in \mathcal{C}_G(g_1)$. \square

In the end let's introduce a good friend which will be around for the next few lectures:

Recall from (5.5), the inner product on $\mathcal{C}(G)$ and the real symmetric form \langle, \rangle on characters:

Definition. The *character table* of G is the $k \times k$ matrix (where k is the number of ccls) $X = [\chi_i(g_j)]$, the i^{th} character on the j^{th} class, where we let $\chi_1 = 1_G, \chi_2, \dots, \chi_k$ are the irreducible characters of G , and $\mathcal{C}_1 = \{1\}, \dots, \mathcal{C}_k$ are the ccls with $g_j \in \mathcal{C}_j$ (as we defined in 5.5).

So the $(i, j)^{th}$ entry of X is just $\chi_i(g_j)$.

Example. (5.13)

(a) $C_3 = \langle x : x^3 = 1 \rangle$. The character table is

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = e^{2\pi i/3}$.

(b) $G = D_6 \cong S_3 = \langle r, s : r^3 = s^2 = 1, sr^{-1} = r^{-1} \rangle$.

ccls of G : $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r, r^{-1}\}$, $\mathcal{C}_3 = \{s, sr, sr^2\}$. We have 3 irreducible representations over \mathbb{C} : 1_G (trivial); \mathcal{S} (sign): $x \rightarrow 1$ for x even, $x \rightarrow -1$ for x odd; and W (2-dimensional): sr^i acts by matrix with eigenvalues ± 1 ; r^k acts by the matrix

$$\begin{pmatrix} \cos 2k\pi/3 & -\sin 2k\pi/3 \\ \sin 2k\pi/3 & \cos 2k\pi/3 \end{pmatrix}$$

so $\chi_w(sr^i) = 0 \ \forall j$, $\chi_w(r^k) = 2 \cos 2k\pi/3 = -1 \ \forall k$. So the charactable is:

	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3
1_G	1	1	1
χ_s	1	-1	1
χ_w	2	0	-1

6 Proofs and orthogonality

We want to prove(5.6): irreducible characters form orthonormal basis for the space of \mathbb{C} -class functions.

Proof. (of 5.6 (a))

Fix bases of V and V' . Write $R(g)$, $R'(g)$ for matrices of $\rho(g)$, $\rho'(g)$ w.r.t. these bases, respectively. Then

$$\begin{aligned}\langle \chi', \chi \rangle &= \frac{1}{|G|} \chi'(g^{-1}) \chi(g) \\ &= \frac{1}{|G|} \sum_{g \in G, i, j \text{ s.t. } 1 \leq i \leq n', 1 \leq j \leq n} R'(g^{-1})_{ii} R(g)_{jj}\end{aligned}$$

the trick is to define something that annihilates almost the whole thing. Let $\phi : V \rightarrow V'$ be linear and define

$$\begin{aligned}\tilde{\phi} : V &\rightarrow V' \\ v &\rightarrow \frac{1}{|G|} \sum_{g \in G} \rho'(g^{-1}) \phi \rho(g) v\end{aligned}$$

We claim that this is a G -homomorphism: if $h \in G$, let's calculate

$$\begin{aligned}\rho'(h^{-1}) \tilde{\phi} \rho(h)(v) &= \frac{1}{|G|} \sum_{g \in G} \rho'(gh)^{-1} \phi \rho(gh)(v) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho'(g'^{-1}) \phi \rho(g')(v) \\ &= \tilde{\phi}(v)\end{aligned}$$

(when g runs through G , gh runs through G as well). So (2.8) is satisfied, i.e. ϕ is a G -homomorphism.

Case 1: ρ, ρ' are not isomorphic. Schur's lemma says $\tilde{\phi} = 0$ for any given linear $\phi : V \rightarrow V'$. Take $\phi = \varepsilon_{\alpha\beta}$, having matrix $E_{\alpha\beta}$ (w.r.t our basis). This is 0 everywhere except 1 in the (α, β) -position. Then $\varepsilon_{\alpha\beta} = 0$. So $\frac{1}{|G|} \sum_{g \in G} (R'(g^{-1}) E_{\alpha\beta} R(g))_{ij} = 0$. So $\frac{1}{|G|} \sum R'(G^{-1})_{i\alpha} R(g)_{\beta j} = 0 \forall i, j$, with $\alpha = i, \beta = j$. Now $\frac{1}{|G|} \sum_{g \in G} R'(g^{-1})_{ii} R(g)_{jj} = 0$ sum over i, j . Then $\langle \chi', \chi \rangle = 0$. Case 2: ρ, ρ' isomorphic. So $\chi = \chi'$; take $V = V'$, $\rho = \rho'$. If $\phi : V \rightarrow V$ is linear endomorphism, we claim $\text{tr } \phi = \text{tr } \tilde{\phi}$:

$$\text{tr } \tilde{\phi} = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\rho(g)^{-1} \phi \rho(g)) = \frac{1}{|G|} \sum_{g \in G} \text{tr } \phi = \text{tr } \phi$$

By Schur's lemma, $\tilde{\phi} = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$ (depending on ϕ). Then $\lambda = \frac{1}{n} \text{tr } \phi$. Let $\phi = \varepsilon_{\alpha\beta}$. So $\text{tr } \phi = \delta_{\alpha\beta}$. Hence $\varepsilon_{\alpha\beta} = \frac{1}{n} \delta_{\alpha\beta} \iota_v = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \varepsilon_{\alpha\beta} \rho(g)$. In terms of matrices, take (i, j) -entry: $\frac{1}{|G|} \sum_j R(g^{-1})_{i\alpha} R(g)_{\beta j} = \frac{1}{n} \delta_{\alpha\beta} \delta_{ij} \forall i, j$. Put $\alpha = i, \beta = j$ to get $\frac{1}{|G|} \sum_g R(g^{-1})_{ii} R(g)_{jj} = \frac{1}{n} \delta_{ij}$. Finally sum over i, j to get $\langle \chi, \chi \rangle = 1$. \square

Before proving (b), let's prove column orthogonality:

Theorem. (6.1, column orthogonality relations)

$$\sum_{i=1}^k \overline{\chi_i(g_j)} \chi_i(g_l) = \delta_{jl} |C_G(g_j)|$$

having an easy corollary

Corollary. (6.2)

$$|G| = \sum_{i=1}^k \chi_i^2(1).$$

Proof. (of (6.1))

$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum \overline{\chi_i(g_l)} \chi_j(g_l) / |C_G(g_l)|$. Consider the character table $X = (\chi_i(g_j))$. Then $\bar{X} D^{-1} X^T = I_{k \times k}$ where $D = \text{Diag}(|C_G(g_1)|, \dots, |C_G(g_k)|)$.

Since X is square, it follows that $D^{-1} \bar{X}^T$ is the inverse of X , so $\bar{X}^T X = D$. \square

Proof. (of (5.6(b)))

The χ_i generate \mathcal{C}_G . Let all the irreducible characters χ_1, \dots, χ_l of G : claim these generate \mathcal{C}_G , the \mathbb{C} -space of class functions on G . It's enough to show that the orthogonal complement to $\text{span}\{\chi_1, \dots, \chi_l\}$ in \mathcal{C}_G is $\{0\}$. To see this, assume $f \in \mathcal{C}_G$ with $\langle f, \chi_j \rangle = 0 \forall j$. Let $\rho : G \rightarrow GL(V)$ be irreducible representation affording $\chi \in \{\chi_1, \dots, \chi_l\}$. Then $\langle f, \chi \rangle = 0$.

Consider

$$\frac{1}{|G|} \sum_G \overline{f(g)} \rho(g) : V \rightarrow V$$

This is a G -homomorphism, so as ρ is irreducible, it must be λ_i for some $\lambda \in \mathbb{C}$.

Now

$$\begin{aligned} n\lambda &= \text{tr} \frac{1}{|G|} \sum_g \overline{f(g)} \rho(g) \\ &= \frac{1}{|G|} \sum_g \overline{f(g)} \chi(g) = 0 = \langle f, \chi \rangle \end{aligned}$$

So $\lambda = 0$. Hence $\sum \overline{f(g)} \rho(g) = 0$, the zero endomorphism on V for all representations ρ (complete reducibility).

Take $\rho = \rho_{\text{reg}}$ where $\rho_{\text{reg}}(g) : e_1 \rightarrow e_g$ ($g \in G$). So

$$\sum_g \overline{f(g)} \rho_{\text{reg}}(g) : e_1 \rightarrow \sum_g \overline{f(g)} e_g$$

So it follows $\sum_g \overline{f(g)} e_g = 0$. So $\overline{f(g)} = 0 \forall g \in G$, so $f \equiv 0$. \square

Various corollaries now follow:

- The number of irreducible representations of G = number of ccls; (5.10)
- Column orthogonality (6.1);
- $|G| = \sum n_i^2$ (6.2);
- $g_1 \sim g_2 \iff \chi(g_1) = \chi(g_2)$ for all irreducible χ (5.11);
- If $g \in G$, $g \sim g^{-1} \iff \chi(g) \in \mathbb{R}$ for all irreducible χ .

7 Permutation representations

Preview was given in (3.7). Recall: • G finite group acting on finite set $X = \{x_1, \dots, x_n\}$;

• $\mathbb{C}X = \mathbb{C}$ -space, with basis $\{e_{x_1}, \dots, e_{x_n}\}$ of dimension $|X|$, so is $\{\sum_j a_j e_{x_j} : a_j \in \mathbb{C}\}$;

• corresponding permutation representation $\rho_X : G \rightarrow GL(\mathbb{C}X)$ by $g \rightarrow \rho(g)$, where $\rho(g)$ sends $e_{x_j} \rightarrow e_{gx_j}$, extending linearly.

• ρ_X is the *permutation representation* corresponding to the action of G on X .

• matrices representing $\rho_X(g)$ w.r.t. basis $\{e_x\}_{x \in X}$ are permutation matrices: 0 except for one 1 in each row and column, and $(\rho(g))_{ij} = 1$ iff $gx_j = x_i$. Consider its character:

(7.1) Permutation character, π_X , is

$$\pi_X(g) = |\text{Fix}_X(g)| = |\{x \in X : gx = x\}|.$$

(7.2) ρ_X always contains 1_G : $\text{span}\{e_{x_1} + \dots + e_{x_n}\}$ is a trivial G -subspace of $\mathbb{C}X$ with G -invariant complement $\text{span}\{\sum a_x e_x : \sum a_x = 0\}$.

Lemma. (7.3, Burnside's lemma, after Cauchy, Frobenius) $\langle \pi_X, 1 \rangle =$ number of orbits of G on X .

Proof. If $X = X_1 \cup \dots \cup X_l$ disjoint union of orbits, then $\pi_X = \pi_{X_1} + \dots + \pi_{X_l}$, with π_{X_j} permutation character of G on X_j , so to prove the claim it's enough to show that if G is transitive on X then $\langle \pi_X, 1 \rangle = 1$. Assume G is transitive on X . Now

$$\begin{aligned} \langle \pi_X, 1 \rangle &= \frac{1}{|G|} \sum_g \pi_X(g) = \frac{1}{|G|} |\{(g, x) \in G \times X : gx = x\}| \\ &= \frac{1}{|G|} \sum_{x \in X} |G_x| = \frac{1}{|G|} |X| |G_x| = \frac{1}{|G|} |G| = 1 \end{aligned}$$

(Note the use of orbit-stabilizer theorem). □

Lemma. (7.4)

Let G act on the sets X_1, X_2 . Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1 \times X_2} = \pi_{X_1} \pi_{X_2}$ and so $\langle \pi_{X_1}, \pi_{X_2} \rangle =$ number of orbits of G on $X_1 \times X_2$.

Proof. If $g \in G$ then $\pi_{X_1 \times X_2}(g) = \pi_{X_1}(g) \pi_{X_2}(g)$. And we have

$$\langle \pi_{X_1}, \pi_{X_2} \rangle = \langle \pi_{X_1} \pi_{X_2}, 1 \rangle = \langle \pi_{X_1 \times X_2}, 1 \rangle = (7.3) \text{ number of orbits of } G \text{ on } X_1 \times X_2.$$

□

Definition. (7.5)

Let G act on X , $|X| > 2$. Then G is *2-transitive* on X if G has precisely two orbits on $X \times X : \{(x, x) : x \in X\}$ and $\{(x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

Lemma. (7.6)

Let G act on X , $|X| > 2$. Then $\pi_X = 1 + \chi$ with χ irreducible $\iff G$ is 2-transitive on X .

Proof. $\pi_X = m_1 1 + m_2 \chi_2 + \dots + m_l \chi_l$ with $1, \chi_2, \dots, \chi_l$ distinct irreducible characters and $m_i \in \mathbb{N}$. Then

$$\langle \pi_X, \pi_X \rangle = \sum_{i=1}^l m_i^2$$

hence G is 2-transitive on $X \iff l = 2, m_1 = m_2 = 1$. \square

Example. (7.7)

Consider S_n acting on $X = \{1, \dots, n\}$ which is 2-transitive. Hence $\pi_X = 1 + \chi$ with χ irreducible of degree $n - 1$. Similarly for A_n ($n > 3$).

Example. (7.8)

Consider $G = S_4$.

$\subset \subset 1$	1	3	8	6	6
rep	1	$(1, 2)(3, 4)$	$(1, 2, 3)$	$(1, 2, 3, 4)$	$(1, 2)$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	3	-1	0	-1	1
χ_4	3	-1	0	1	-1
χ_5	2

Get by column orthogonality \leftarrow

Last lecture we were talking about using column orthogonality to find χ_5 . Indeed we have

$$\chi_{reg} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5$$

So we can use this to find χ_5 . Also, $S_4/V_4 \cong S_3$ by 'lifting' – see next chapter.

7.1 Alternating groups

Suppose $g \in A_n$. In 1A we've known that $|C_{S_n}(g)| = |S_n : C_{S_n}(g)|$ and $|C_{A_n}(g)| = |A_n : C_{A_n}(g)|$.

These are not necessarily equal. For example, $\sigma = (123) \in A_3$, $\mathcal{A}_3(\sigma) = \{\sigma\}$, but $\mathcal{S}_\exists(\sigma) = \{\sigma, \sigma^{-1}\}$.

Lemma. (7.9)

Let $g \in A_n$. Then if g commutes with some odd permutation in S_n then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$; otherwise $\mathcal{C}_{S_n}(g)$ splits into two ccls in A_n of equal size.

For example, consider $G = A_4$, so $|G| = 12$.

	1	$(12)(34)$	(123)	$(123)^{-1}$
$1\bar{G}$ $= \chi_1$	1	1	1	1
$\bar{1}\bar{G}$ $= \chi_2$	3	-1	0	0
χ_3	1	1	ω	ω^2
χ_4	1	1	ω^2	ω

Note that if we ignore the second row and first column, the table becomes identical to that of $C_3 \cong G/V_4$. This is not a coincidence, and is actually called *lifting*.

8 Normal subgroups and lifting characters

Lemma. (8.1)

Let $N \triangleleft G$. Let $\tilde{\rho} : G/N \rightarrow GL(V)$ be a representation of G/N . Then

$$\begin{array}{ccc} \rho : G & \xrightarrow{\text{canonical}} & G/N & \xrightarrow{\tilde{\rho}} & GL(V) \\ g & \rightarrow & & & \tilde{\rho}(gN) \end{array}$$

is a representation of G , where $\rho(g) := \tilde{\rho}(gN)$. Moreover, ρ is irreducible iff $\tilde{\rho}$ is irreducible.

The corresponding characters satisfy $\chi(g) = \tilde{\chi}(gN)$. We say that $\tilde{\chi}$ *lifts* to χ . The lifting $\tilde{\chi} \rightarrow \chi$ is a bijection between irreducible representations of G/N and irreducible representations of G with N in \ker .

Well this looks like Q4/Q12 in the first example sheet.

Proof. Note $\chi(g) = \text{tr}(\rho(g)) = \text{tr}(\tilde{\rho}(gN)) = \tilde{\chi}(gN) \forall g$, and $\chi(1) = \tilde{\chi}(N)$. SO have some degree (?).

Bijection: if $\tilde{\chi}$ is a charcter of G/N -representaiton and χ is its lift to G , then $\chi(N) = \chi(1)$. Also, if $k \in N$ then

$$\chi(k) = \tilde{\chi}(kN) = \tilde{\chi}(N) = \chi(1)$$

So $N \leq \ker \chi$.

Now let χ be character of G with $N \leq \ker \chi$. Suppose $\rho : G \rightarrow GL(V)$ affords χ . Define

$$\begin{array}{ccc} \tilde{\rho} : G/N & \rightarrow & GL(V) \\ gN & \rightarrow & \rho(g) \end{array}$$

Check this is well-defined (uses $N \leq \ker \chi$) and $\tilde{\rho}$ is homomorphism, hence gives representaiton of G/N . If $\tilde{\chi}$ is the character of $\tilde{\rho}$ then $\tilde{\chi}(gN) = \chi(g) \forall g \in G$. So $\tilde{\chi}$ lifts to χ .

Check irreducibility. □

Lemma. (8.2)

The derived subgroup, $G' = \langle [a, b], a, b \in G \rangle$ of G is the unique minimal normal subgroup of G s.t. G/G' is abelian, i.e. G/N is abelian $\implies G' \leq N$ and $G^{ab} = G/G'$ is abelian, where G^{ab} is the *abelianisation* of G .

G has precisely $l = |G/G'|$ representaitons of $\dim 1$, all with kernel containing G' and obtained by lifting from G/G' . In particular, $l \mid |G|$.

Proof. $G' \triangleleft G$ is an easy exercise.

Let $N \triangleleft G$. Let $h, g \in G$, so

$$\begin{aligned} g^{-1}h^{-1}gh \in N &\iff (gh)N = (hg)N \\ [g, h] &\iff (gN)(hN) = (hN)(gN) \end{aligned}$$

So $G' \leq N \iff G/N$ is abelian. Since $G' \triangleleft G$ we deduce G/G' is abelian.

By (4.5), G/G' has exactly l irreducible characters $\tilde{\chi}_1, \dots, \tilde{\chi}_l$ all of degree 1. The lifts of these to G also have degree 1 and by (8.1) these are precisely the irreducible characters χ_i of G s.t. $G' \leq \ker \chi_i$. But any linear character of G is a homomorphism $\chi : G \rightarrow \mathbb{C}^*$, hence $G' \leq \ker \chi$ ($\chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g^{-1})\chi(h)^{-1} = 1$), so the χ_1, \dots, χ_l are all the linear characters of G . \square

Examples:

(a) If $G = S_n$, show $s'_n = A_n$. Thus since $G/G' \cong C_2$, S_n must have exactly two linear characters.

(b) Consider $G = A_4$. We've seen previously that this can be lifted from C_3 using (8.1), (8.2).

Lemma. (8.4)

G is not simple iff $\chi(g) = \chi(1)$ for some irreducible character $\chi \neq 1_G$ and some $1 \neq g \in G$.

Any normal subgroup of G is the intersection of the kernels of some of the irreducible characters of G :

$$N = \bigcap_i \ker \chi_i$$

Proof. If $\chi(g) = \chi(1)$ for some non-trivial irreducible character χ (afforded by ρ , say). Then $g \in \ker \rho$ (5.3), so if $g \neq 1$, then $1 \neq \ker \rho \triangleleft G$.

If $1 \neq N \triangleleft G$, take irreducible $\tilde{\chi}$ of G/N , $\tilde{\chi}$ non-trivial. Lift to get an irreducible χ , afforded by ρ of G , then $N \leq \ker \rho \triangleleft G$. So $\chi(g) = \chi(1)$ for $g \in N$.

We claim that, if $1 \neq N \triangleleft G$, then N is the intersection of the kernels of the lifts of all the irreducibles of G/N .

\leq is clear from (8.1). If $g \in G \setminus N$, then $gN \neq N$. so $\tilde{\chi}(gN) \neq \tilde{\chi}(N)$ for some irreducible $\tilde{\chi}$ of G/N . Lifting $\tilde{\chi}$ to χ , we have $\chi(g) \neq \chi(1)$. \square

Recall $\ker \chi = \{g \in G : \chi(g) = \chi(1)\}$. (5.3) : $g \in \ker \chi \iff g \in \ker \rho$.

9 Dual spaces and tensor products of representations

Recall (5.5):

- $\mathcal{C}(G)$ is \mathbb{C} -space of class functions on G ;
- endowed with irreducible product, $\dim \mathcal{C}(G) = k$, orthonormal basis of irreducible characters of G (5.6)1
- there exists an involution (ring homomorphism of order 2): $f \rightarrow f^*$ where $f^*(g) = f(g^{-1})$.

Lemma. (9.1)

Let $\rho : G \rightarrow GL(V)$, representation over F , and let $V^* = Hom_F(V, F)$, dual space of V . Then V^* is a G -space under

$$(\rho^*(g)\phi)(v) = \phi(\rho(g^{-1})v)$$

called the *dual representation* to ρ . Its character is $\chi_{\rho^*}(g) = \chi_{\rho}(g^{-1})$.

Proof.

$$\begin{aligned} \rho^*(g_1)(\rho^*(g_2)\phi)(v) &= (\rho^*(g_2)\phi)(\rho(g_1^{-1})v) \\ &= \phi(\rho(g_2^{-1})\rho(g_1^{-1})v) \\ &= \phi(\rho(g_1g_2)^{-1}v) \\ &= (\rho^*(g_1g_2)\phi)(v) \end{aligned}$$

So this is a representation. For its character, fix $g \in G$ and let e_1, \dots, e_n be basis of V of eigenvectors of $\rho(g)$, say $\rho(g)e_j = \lambda_j e_j$. Let $\varepsilon_1, \dots, \varepsilon_n$ be dual basis. We claim that $\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j$:

$$(\rho^*(g)\varepsilon_j)(e_i) = \varepsilon_j(\rho(g^{-1})e_i) = \varepsilon_j\lambda_i^{-1}e_i = \lambda_j^{-1}\varepsilon_j e_i \forall i$$

So $\chi_{\rho^*}(g) = \sum \lambda_j^{-1} = \chi_{\rho}(g^{-1})$. □

Definition. (9.2)

$\rho : G \rightarrow GL(V)$ is *self-dual* if $V \cong V^*$ (as G -spaces). Over \mathbb{C} , this holds iff $\chi_{\rho}(g) = \chi_{\rho}(g^{-1})$ ($= \chi_{\rho}(g)$) $\forall g$, iff $\chi_{\rho}(g) \in \mathbb{R}$ for all g .

Exercise: all irreducible representations of S_n are self-dual (the ccls are determined by cycle type, so g, g^{-1} are always S_n -conjugate. Not always true for A_n).

9.1 tensor products

Let V, W be F -spaces, $\dim V = m$, $\dim W = n$. Fix bases v_1, \dots, v_m and w_1, \dots, w_n of V, W respectively. The *tensor product space* $V \otimes_F W$ is an nm -dimensional F -space with basis $\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Thus

(a) $V \otimes W = \{\sum_{i,j} \lambda_{ij} v_i \otimes w_j : \lambda_{ij} \in F\}$ with 'obvious' addition and scalar multiplication;

(b) If $v = \sum_i \alpha_i v_i \in V$, $w = \sum_j \beta_j w_j \in W$, define $v \otimes w := \sum_{i,j} \alpha_i \beta_j (v_i \otimes w_j)$.

Remark. Not all elements of $V \otimes W$ are of this form: some are combinations, e.g. $v_1 \otimes w_1 + v_2 \otimes w_2$, which can't be further simplified. (like entangled)

Lemma. (9.3)

- (i) For $v \in V$, $w \in W$, $\lambda \in F$, $(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$;
 (ii) If $x_1, x_2, x \in V$, $y_1, y_2, y \in W$, then

$$\begin{aligned}(x_1 + x_2) \otimes y &= (x_1 \otimes y) + (x_2 \otimes y), \\ x \otimes (y_1 + y_2) &= (x \otimes y_1) + (x \otimes y_2)\end{aligned}$$

Proof. (i) $v = \sum \alpha_i v_i$, $w = \sum \beta_j w_j$. Then just multiply out everything we get the desired equality. (ii) is similar. \square

Lemma. (9.4)

If $\{e_1, \dots, e_m\}$ is a basis of V , $\{f_1, \dots, f_n\}$ is a basis of W , then $\{e_i \otimes f_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $V \otimes W$.

Proof. Writing $v_k = \sum_i \alpha_{ik} e_i$, $w_l = \sum_j \beta_{jl} f_j$, we have

$$v_k \otimes w_l = \sum_{i,j} \alpha_{ik} \beta_{jl} e_i \otimes f_j$$

Hence $\{e_i \otimes f_j\}$ spans $V \otimes W$ and, since we have nm of them, they form a basis. \square

Remark. One can define $V \otimes W$ in a basis-independent way in the first place, see Teleman chapter 6.

Proposition. (9.5)

Let $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$ be representations of G . Define $\rho \otimes \rho' : G \rightarrow GL(V \otimes V')$ by

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij} v_i \otimes w_j \rightarrow \sum \lambda_{ij} \rho(g) v_i \otimes \rho'(g) w_j$$

Then $\rho \otimes \rho'$ is a representation of G with character

$$\chi_{\rho \otimes \rho'}(g) = \chi_\rho(g) \chi_{\rho'}(g) \forall g \in G$$

Hence product of two characters of G is still a character of G .

Proof. On Tuesday. \square

(After lecture 11: this is the first notes to get beyond 1000 lines!)

Remark. (9.6)

Sheet 1, Q2 says ρ irreducible, ρ' of degree 1, then $\rho \otimes \rho'$ irreducible; if ρ' is not of deg 1 this is usually false.

Proof. (of 9.5)

It's clear that $(\rho \otimes \rho')(g) \in GL(V \otimes V') \forall g \in G$ and so $\rho \otimes \rho'$ is a homomorphism $G \rightarrow GL(V \otimes V')$. Let $g \in G$. Let v_1, \dots, v_m be basis of V of eigenvectors of $\rho(g)$; let w_1, \dots, w_n be a basis of V' . Say:

$$\rho(g)v_j = \lambda_j v_j, \rho'(g)w_j = \mu_j w_j$$

Then

$$\begin{aligned} (\rho \otimes \rho')(g)(v_i \otimes w_j) &= \rho(g)v_i \otimes \rho'(g)w_j \\ &= \lambda_i v_i \otimes \mu_j w_j \\ &= (\lambda_i \mu_j)(v_i \otimes w_j) \end{aligned}$$

$$\text{So } \chi_{\rho \otimes \rho'}(g) = \sum_{i,j} \lambda_i \mu_j = (\sum \lambda_i)(\sum \mu_j) = \chi_\rho(g) \chi_{\rho'}(g) \quad \square$$

Now work over \mathbb{C} . Take $V = V'$ and define $V^{\otimes 2} = V \otimes V$.

Let

$$\tau : \sum \lambda_{ij} v_i \otimes v_j \rightarrow \sum \lambda_{ij} \lambda_j \otimes v_i$$

which is a linear G -endomorphism of $V^{\otimes 2}$, s.t. $\tau^2 = 1$ (so eigenvalues ± 1).

Definition. (9.7)

$$\begin{aligned} S^2 V &= \{v \in V^{\otimes 2} : \tau(x) = x\}, \\ \wedge^2 V &= \{x \in V^{\otimes 2} : \tau(x) = -x\} \end{aligned}$$

known as the *symmetric square* of V and *exterior square* of V respectively.

Lemma. (9.8)

$S^2 V$ and $\wedge^2 V$ are G -subspaces of $V^{\otimes 2}$ and $V^{\otimes 2} \cong S^2 V \oplus \wedge^2 V$. $S^2 V$ has basis $\{v_i v_j := v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$, and $\wedge^2 V$ has basis $\{v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$. Hence we have $\dim S^2 V = \frac{1}{2}n(n+1)$ and $\dim \wedge^2 V = \frac{1}{2}n(n-1)$.

Proof. Exercise in linear algebra.

To show $V^{\otimes 2}$ is reducible, write $x \in V^{\otimes 2}$ as $x = \frac{1}{2}(x + \tau(x)) + \frac{1}{2}(x - \tau(x))$, which is in $S^2 V$ and $\wedge^2 V$ respectively. \square

In fact, $V^{\otimes 2}, V^{\otimes 3} = V \otimes V \otimes V, \dots$, etc. are never irreducible if $\dim V > 1$.

Lemma. (9.9)

If $\rho : G \rightarrow GL(V)$ is a representation affording character χ , then $\chi^2 = \chi_S + \chi_\wedge$ where $\chi_S (= S^2 \chi)$ is the character of G in the subrepresentation $S^2 V$, and $\chi_\wedge (= \wedge^2 \chi)$ is the character of G in the subrepresentation $\wedge^2 V$. Moreover, for $g \in G$,

$$\chi_S(g) = \frac{1}{2}(\chi(g^2) + \chi(g)^2), \chi_\wedge(g) = \frac{1}{2}(\chi(g^2) - \chi(g)^2).$$

Proof. Let's compute the characters χ_S, χ_\wedge . Fix $g \in G$. Let v_1, \dots, v_n be a basis of eigenvectors of $\rho(g)$, say $\rho(g)v_i = \lambda_i v_i$ (we drop the ρ to write $gv_i = \lambda_i v_i$ for simplicity below). Then

$$\begin{aligned} gv_i v_j &= \lambda_i \lambda_j v_i v_j \\ gv_i \wedge v_j &= \lambda_i \lambda_j v_i \wedge v_j \end{aligned}$$

Hence $\chi_s(g) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j$ and $\chi_\wedge(g) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j$. Now,

$$\begin{aligned} (\chi(g))^2 &= \left(\sum \lambda_i \right)^2 \\ &= \sum \lambda_i^2 + 2 \sum_{i < j} \lambda_i \lambda_j \\ &= \chi(g^2) + 2 \sum_{i < j} \lambda_i \lambda_j \\ &= \chi(g^2) + 2\chi_\wedge(g) \end{aligned}$$

So $\chi_\wedge(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$. But $\chi^2 = \chi_s + \chi_\wedge$ so we get the expression for $\chi_s(g)$. \square

Example. (9.10)

Consider our usual example $G = S_4$ (see 7.8).

	1	(12)(34)	(123)	(12)	(1234)
$\downarrow \mathbb{R}$	1	1	1	1	1
sign	1	1	1	-1	-1
$\chi_1 = \overline{\chi_1}$	3	-1	0	1	-1
$\overline{\chi_3}$	3	-1	0	-1	1
χ_5	2	2	-1	0	0

χ_3^2	9	1	6	1	1
$\chi_3 \chi_5$	3	3	0	3	-1
$S^2 \chi_3$	6	2	0	2	0
$\wedge^2 \chi_3$	3	-1	0	-1	1

Notice that $\wedge^2 \chi_3 = \bar{\chi}_3$ (irreducible since $\langle \chi_\wedge, \chi_\wedge \rangle = 1$),
 $S^2 \chi_3 = 1 + \chi_3 + \chi_5$: The inner product is 3 and it contains $1, \chi_3$, so the one left is χ_5 .

Characters of $G \times H$ (seen in (4.5) for abelian groups):

Proposition. (9.11)

If G, H are finite groups with irreducible characters χ_1, \dots, χ_k and ψ_1, \dots, ψ_r respectively, then the irreducible characters of the direct product $G \times H$ are precisely $\{\chi_i \psi_j : 1 \leq i \leq k, 1 \leq j \leq r\}$, where $\chi_i \psi_j(g, h) = \chi_i * g(\psi_j(h))$.

Proof. If $\rho : G \rightarrow GL(V)$, $\rho' : H \rightarrow GL(W)$ affording χ and ψ respectively, then

$$\begin{aligned} \rho \otimes \rho' : G \times H &\rightarrow GL(V \otimes W) \\ (g, h) &\rightarrow \rho(g) \otimes \rho'(h) \quad v_i \otimes w_j \rightarrow \rho(g)v_i \otimes \rho'(h)w_j \end{aligned}$$

is a representation of $G \times H$ on $V \otimes W$ by (9.5), and $\chi_{\rho \otimes \rho'} = \chi \psi$, again by (9.5). We claim that $\chi_i \psi_j$ are distinct and irreducible:

$$\begin{aligned} \langle \chi_i \psi_j, \chi_r \psi_s \rangle_{G \times H} &= \frac{1}{|G \times H|} \sum_{(g, h)} \overline{\chi_i \psi_j(g, h)} \chi_r \psi_s(g, h) \\ &= \left(\frac{1}{|G|} \overline{\chi_i(g)} \chi_r(g) \right) \left(\frac{1}{|H|} \sum_h \overline{\psi_j(h)} \psi_s(h) \right) \\ &= \delta_{ir} \delta_{js} \end{aligned}$$

...tbc.

Let's complete on $\chi_i \psi_j$ being distinct and irreducible:

Complete set: $\sum_{i,j} (\chi_i \psi_j)(1)^2 = \sum_i \chi_i(1)^2 \sum_j \psi_j(1)^2 = |G||H| = |G \times H| \quad \square$

9.2 Symmetric and exterior powers

Let V be a vector space, $\dim_F V = d$, with basis $\{v_1, \dots, v_d\}$. Let $V^{\otimes n} = V \otimes \dots \otimes V$, with basis $\{v_{i_1} \otimes \dots \otimes v_{i_n} : (i_1, \dots, i_n) \in \{1, \dots, d\}^n\}$, so $\dim V^{\otimes n} = d^n$.

S_n -action: for any $\sigma \in S_n$, we can define linear map

$$\begin{aligned} \sigma : V^{\otimes n} &\rightarrow V^{\otimes n} \\ v_1 \otimes \dots \otimes v_n &\rightarrow v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)} \end{aligned}$$

for $v_1, \dots, v_n \in V$, permuting positions of vectors in a tensor.

For example, (12)($v_1 \otimes v_2 \otimes v_3$) = $v_2 \otimes v_1 \otimes v_3$, (13)($v_2 \otimes v_1 \otimes v_3$) = $v_3 \otimes v_1 \otimes v_2$.

Check that this defines a representation of S_n on $V^{\otimes n}$ (extended linearly).

G -action: given representation $\rho : G \rightarrow GL(V)$, then the action of G on $V^{\otimes n}$ is

$$\rho^{\otimes n}(g) : v_1 \otimes \dots \otimes v_n \mapsto \rho(g)v_1 \otimes \dots \otimes \rho(g)v_n$$

extended linearly, and this commutes with the S_n -action. We can decompose $V^{\otimes n}$ as S_n -module, and each isotypical component (4.?) is G -invariant subspace of $V^{\otimes n}$. In particular:

Definition. (9.12)

For G -space V , define

- (i) the n th symmetric power of V , $S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \forall \sigma \in S_n\}$;
 - (ii) the n th exterior power of V , $\wedge^n V = \{x \in V^{\otimes n} : \sigma(x) = \text{sign}(\sigma)x \forall \sigma \in S_n\}$.
- Both are G -subspaces of $V^{\otimes n}$, but for $n > 2$, $S^n V \oplus \wedge^n V \subsetneq V^{\otimes n}$, so in general there are lots of others for the S_n -action.

(9.13) See Sheet 3 Q7 for bases of $S^n V$, $\wedge^n V$ and their characters.

9.3 Tensor algebra

Take $\text{char } F = 0$.

Definition. (9.14)

Let $T^n V = V^{\otimes n}$. The tensor algebra of V is $TV := \bigoplus_{n \geq 0} T^n V$, $T^0 V = F$.

This is F -space and is a (non-commutative) graded ring with product $x \in T^n V$, $y \in T^m V$, $x \cdot y = x \otimes y \in T^{n+m} V$.

There are two graded quotient rings

$$\begin{aligned} SV &= TV / (\text{ideal generated by all } U \otimes V - V \otimes U) \\ \wedge V &= TV / \text{ideal generated by all } V \otimes V \end{aligned}$$

called the symmetric algebra and exterior algebra respectively.

Definition. (9.15)

The 2-submodule of $\mathcal{C}(G)$ spanned by irreducible characters of G is the character

ring of G , $R(G)$. Elements of $R(G)$ are called generalised/virtual characters if $\psi = \sum n_\chi \chi$, $n_\chi \in \mathbb{Z}$ correspondingly.

- $R(G)$ is a commutative ring and any generalised character is a difference of two characters, $\psi = \alpha - \beta$:

$$\alpha = \sum_{n_\chi \geq 0} n_\chi \chi, \beta = - \sum_{n_\chi < 0} n_\chi \chi.$$

The $\{\chi_i\}$ form a \mathbb{Z} -basis for $R(G)$ as a free \mathbb{Z} -module.

- Suppose ψ is virtual character and $\langle \psi, \psi \rangle = 1$ and $\psi(1) > 0$. Then ψ is actually the character of an irreducible representation of G .

List irreducible characters of G : χ_1, \dots, χ_k , $\psi = \sum n_i \chi_i$; orthonormality says $\langle \psi, \psi \rangle = \sum n_i^2$, so $\sum n_i^2 = 1$, meaning $n_i = \pm 1$ for exactly one i and $n_j = 0$ for $j \neq i$. Since $\psi(1) > 0$, we must have $n_i = +1$.

- Henceforth we don't distinguish between a character and its negative and we often study generalised characters of norm 1 rather than irreducible characters.

10 Restriction and induction

Throughout we set $H \leq G$, $F = \mathbb{C}$.

Definition. (10.1, restriction)

Let $\rho : G \rightarrow GL(V)$ be representation affording χ . We can think of V as a H -space by restricting attention to $h \in H$. We then get

$$Res_H^G \rho : H \rightarrow GL(V)$$

This is sometimes written as ρ_H or $\rho \downarrow_H$, the restriction of ρ to H . It affords the character $Res_H^G \chi = \chi_H = \chi \downarrow_H$.

Lemma. (10.2)

If ψ is any non-zero character of $H \leq G$, then there exists irreducible character χ of G s.t. $\langle Res_H^G \chi, \psi \rangle_H \neq 0$. We say ψ is a constituent of $Res_H^G \chi$.

Proof.

$$0 \neq \frac{|G|}{|H|} \psi(1) = \langle \pi_{reg} \downarrow_H, \psi \rangle = \sum_1^k \deg \chi_i \langle \chi_i \downarrow_H, \psi \rangle$$

where ψ_i are irreducible characters of G . □

Lemma. (10.3)

Let χ be irreducible character of G , and let $Res_H^G \chi = \sum c_i \chi_i$ with χ_i irreducible characters of H , $c_i \in \mathbb{Z}_{\geq 0}$. Then

$$\sum c_i^2 \leq |G : H|$$

with equality iff $\chi(g) = 0 \forall g \in G \setminus H$.

Proof.

$$\sum c_i^2 = \langle Res_H^G \chi, Res_H^G \chi \rangle_H = \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2$$

But

$$\begin{aligned} 1 &= \langle \chi, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\ &= \frac{1}{|G|} \left(\sum_{h \in H} |\chi(h)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2 \right) \\ &= \frac{|H|}{|G|} \sum c_i^2 + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2}_{\geq 0} \end{aligned}$$

So $\sum c_i^2 \leq |G : H|$, with equality holds iff $\chi(g) = 0 \forall g \in G \setminus H$. □

Example. Let $G = S_5$, $H = A_5$. This has 7 representations of degree 1, 1, 4, 4, 5, 5, 6 respectively, where if we restrict to H , the two representations of degree 1, 4, 5 combines into one of the same degree respectively; however, the

degree 6 representation splits into two irreducible representations of degree 3. In the first case we have $\chi(g) \neq 0$ somewhere outside H ; for the degree 6 representation, $\chi(g) = 0 \ \forall g \in S_5 \setminus A_5$. All restrictions are irreducible if $|G : H| = 2$ which is the case here. Fact: $\chi \downarrow_H$ all constituents have same degree if $H \triangleleft G$ (Janes-Liebeck, chapter 20).

Let's talk about induced characters.

Definition. (10.4)

If $\psi \in \mathcal{C}(H)$, define $Ind_H^G \psi(g) = \frac{1}{|G|} \sum_{\chi \in G} \psi(x^{-1}gx)$, where

$$\psi(g) = \begin{cases} \psi(g) & g \in H \\ 0 & g \notin H \end{cases}$$

We also write $Ind_H^G \psi(g)$ as $\psi \uparrow^G = \psi^G$.

Lemma. (10.5)

If $\psi \in \mathcal{C}(H)$ then $Ind_H^G \psi \in \mathcal{C}(G)$ and $Ind_H^G \psi(1) = |G : H| \psi(1)$.

Proof. This is clear, noting that $Ind_H^G \psi(1) = \frac{1}{|H|} \sum \psi(1) = |G : H| \psi(1)$. \square

Let $n = |G : H|$. Let $1 = t_1, t_2, \dots, t_n$ be a *left transversal* of H in G (complete set of coset representatives), so that $t_1H = H, t_2H, \dots, t_nH$ are precisely the n left cosets of H in G .

Lemma. (10.6)

Given left transversal as above,

$$Ind_H^G \psi(g) = \sum_{i=1}^n \psi(t_i^{-1}gt_i)$$

Proof. For $h \in H$, $\psi((t_i h)^{-1}g(t_i h)) = \psi(t_i^{-1}gt_i)$ as ψ is a class function on H . \square

Theorem. (10.7, Frobenius reciprocity)

$H \leq G$. ψ is a class function for H , ϕ is a class function for G . Then

$$\langle \underbrace{Res_H^G \phi}_{in \ \mathcal{C}(H)}, \psi \rangle_H = \langle \phi, \underbrace{Ind_H^G \psi}_{in \ \mathcal{C}(G)} \rangle_G$$

Proof. We want to show $\langle \phi_H, \psi \rangle_H = \langle \phi, \psi^G \rangle_G$:

$$\langle \phi, \psi^G \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi^G(g) = \frac{1}{|G||H|} \sum_{g, x \in G} \overline{\phi(g)} \psi(x^{-1}gx)$$

Put $y = x^{-1}gx$. The above then equals

$$\frac{1}{|G||H|} \sum_{x, y \in G} \overline{\phi(y)} \psi(y) = \frac{1}{|H|} \sum_{y \in G} \overline{\phi(y)} \psi(y)$$

which is independent of x , and then equals

$$\frac{1}{|H|} \sum_{y \in H} \overline{\phi(y)} \psi(y) = \langle \phi_H, \psi \rangle_H$$

□

Corollary. (10.8)

If ψ is a character of H , then $\text{Ind}_H^G \psi$ is a character of G .

Proof. Let χ be an irreducible character of G . Then

$$\langle \text{Ind}_H^G \psi, \chi \rangle = \langle \psi, \text{Res}_H^G \chi \rangle \in \mathbb{Z}_{\geq 0}$$

since ψ and $\text{Res}_H^G \chi$ are characters. Hence $\text{Ind}_H^G \psi$ is a linear combination of irreducible characters with non-negative coefficients, hence a character. □

Lemma. (10.9)

Let ψ be a character of $H \leq G$, and let $g \in G$. Let

$$\mathcal{C}_G(g) \cup H = \bigcup_{i=1}^m \mathcal{C}_H(x_i)$$

(disjoint union), where the x_i are representatives of the H -ccls of elements of H conjugate to g .

If $m = 0$, then $\text{Ind}_H^G \psi(g) = 0$. Otherwise

$$\text{Ind}_H^G \psi(g) = |\mathcal{C}_G(g)| \cdot \sum_{i=1}^m \frac{\psi(x_i)}{|\mathcal{C}_H(x_i)|}$$

Proof. Assume $m > 0$. Let $X_i = \{x \in G : x^{-1}gx \in H \text{ and is conjugate in } H \text{ to } x_i\} \forall 1 \leq i \leq m$. The X_i are pairwise disjoint, and their union is $\{x \in G : x^{-1}gx \in H\}$. By definition,

$$\begin{aligned} \text{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{\alpha \in G} \psi(\alpha^{-1}g\alpha) \\ &= \frac{1}{|H|} \sum_{i=1}^m \sum_{x \in X_i} \psi(x^{-1}gx) \\ &= \frac{1}{|H|} \sum_{i=1}^m \sum_{x \in X_i} \psi(x_i) \\ &= \sum_{i=1}^m \frac{|X_i|}{|H|} \psi(x_i) \end{aligned}$$

and evaluate $\frac{|X_i|}{|H|}$ to get what we want... although a bit tedious. □