Model Theory

October 28, 2018

Contents

1 Langauges and structures

Definition. (1.1) A language L consists of:
•(i) a set $\mathcal{F}$ of function symbols, and for each $f \in \mathcal{F}$, a positive integer n_f, the arity of f;
•(ii) a set $\mathcal{R}$ of relation symbols, and for each $R \in \mathcal{R}$, a positive integer n_R, the arity of R;
•(iii) a set $\mathcal{C}$ of constant symbols.
Note that each of the above three sets can be empty.

Example. $L = \{\{\cdot, -1\}, \{1\}\}$ where $\cdot$ is a binary function, -1 is a unary function, and 1 is a constant. We call this L_{gp} (language of groups);
$L_{lo} = \{<\}$, where $<$ is a binary relation (linear order).

Definition. (1.2)
Given a language L, say, an L-structure consists of:
(i) a set M, the *domain*;
(ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \to M$;
(iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$;
(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the *interpretation* of f, R, c respectively.

Notation. (1.3)
We often fail to distinguish between the symbols in the language L and their interpretations in a L-structure, if the context allows.

We may write $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C}\rangle$.

Example. (1.4)
(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, -1\}, 1\rangle$ is an L_{gp}-structure.
$\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0\rangle$ is also an L_{gp}-structure (here $+$ is a binary and $-$ is the unary negation function).
$\mathcal{Q} = \langle \mathbb{Q}, <\rangle$ is an L_{lo} structure ($<$ is the interpretation of relation).

Definition. (1.5)
Let L be a language, let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.
An *embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an injection $\alpha : M \to N$ that preserves the structure:
(i) For all $f \in \mathcal{F}$, and $a_1, ..., a_{n_f} \in M$,

$$\alpha(f^M(a_1, ..., a_{n_f})) = f^N(\alpha(a_1), ..., \alpha(a_{n_f}))$$

(ii) For all $R \in \mathcal{R}$, and $a_1, ..., a_{n_R} \in M$,

$$(a_1, ..., a_{n_R}) \in R^M \iff (\alpha(a_1), ..., \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if.
(iii) For all $c \in \mathcal{C}$, we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding $\mathcal{M} \to \mathcal{N}$ is also called an *isomorphism* of $\mathcal{M}$ onto $\mathcal{N}$.

(1.6) Exercise. Let G_1, G_2 be groups, regarded as L_{gp}-structures.
Check that $G_1 \cong G_2$ in the usual algebra sense, if and only if there is an isomprhism $\alpha : G_1 \to G_2$ in the sense of above definition 1.5.

2 Terms, formulae, and their interpretations

In addition to the symbols of L, we also have:
(i) infinitely many variables, $\{x_i\}_{i\in I}$;
(ii) logical connectives, $\wedge, \neg$ (also express $\vee, \to, \leftrightarrow$);
(iii) quantifier $\exists$ (also express $\forall$);
(iv) punctuations (,).

Definition. (2.1)
L-terms are defined recursively as follows:
• any variable x_i is a term;
• any constant symbol is a term;
• for any $f \in \mathcal{F}$,

$$f(t_1, ..., t_{n_f})$$

for any terms $t_1, ..., t_{n_f}$ is a term;
• nothing else is a term.

Notation: we write $t(x_1, ..., x_n)$ to mean that the variables appearing in t are among $x_1, .., x_n$.

Example. In $\mathcal{R} =< \mathbb{R}, \cdot, -1, 1 >$,
• $(\cdot(x_1, x_2), x_3)$ is a term $(x_1 \cdot x_2) \cdot x_3)$;
• $(\cdot(1, x_1))^{-1}$ is a term $(1 \cdot x)^{-1}$.

Definition. (2.2)
If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, ..., x_k)$ we assign a function

$$t^M : M^k \to M$$

defined as follows:
(i) If $t = x_i, t^M[a_1, ..., a_k] = a_i$;
(ii) If $t = c$ is a constant, $t^M[a_1, ..., a_k] = c^m$;
(iii) If $t = f(t_1(x_1, ..., x_k), ..., t_{n_f}(x_1, ..., x_k))$,

$$t^M(a_1, ..., a_k) = f^M(t_1^M(a_1, ..., a_k), ..., t_{n_f}^M(a_1, ..., a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on $http:$
$users.mct.open.ac.uk/sb27627/MT.html$ (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

Remark. (The lecture forgot about this last time) Any language L includes an equality symbol $=$.

Last time we assigned a function t^m. In L_{gp}, the term $x_2 \cdot x_3$ can be described as, say $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), \ldots$.
Then the term $x_2 \cdot x_3$ can be assigned to functions $t_1^M : M^3 \to M : (a_1, a_2, a_3) \to (a_2 \cdot a_3)$, or $t_2^M : M^4 \to M : (a_1, a_2, a_3, a_4) \to (a_2 \cdot a_3)$. These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term t to be the number of symbols of L occuring in t.

Fact (2.3): Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any L-term $t(x_1, \ldots, x_k)$ and $a_1, \ldots, a_k \in M$, we have

$$\alpha(t^M(a_1, \ldots, a_k)) = t^N(\alpha(a_1), \ldots, \alpha(a_k))$$

Proof. Prove by induction on complexity of t.
Let $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{x} = (x_1, \ldots, x_l)$. Then:
(i) if $t = x_i$a a variable, then $t^M(\bar{a}) = a_i$, and $t^N(\alpha(a_1), \ldots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds;
(ii) if $t = c$ is a constant, then $t^M(\bar{a}) = c^M$, and $t^N(\alpha(\bar{a})) = c^N$ by definition of a term. The key here is that, since α is an embedding we have $\alpha(c^M) = c^N$;
(iii) if $t = f(t_1(\bar{x}, \ldots, t_{n_f}(\bar{x})))$, then

$$\alpha(f^M(t_1^M(\bar{a}), \ldots, t_{n_f}(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), \ldots, \alpha(t_{n_f}^M(\bar{a})))$$

as α is an embedding. But $t_1(\bar{x}), \ldots, t_{n_f}(\bar{x})$ have lower complexity than t, so the inductive hypothesis applies. $\square$

Exercise (2.4): conclude the proof of the above fact.
(Actually is it not done?)

Definition. (2.5)
The set of *atmoic formulas* of L is defined as follows:
(i) if t_1, t_2 are L-terms, then $t_1 = t_2$ is an atomic formula;
(ii) if R is a relation symbol, and $t_1, \ldots, t_{n_R}$ are L-terms, then $R(t_1, \ldots, t_{n_R})$ is an atomic formula;
(iii) nothing else is an atomic formula.

Definition. (2.6)
The set of L-formulas is defined as follows:
(i) any atomic formula is an L-formula;
(ii) if ϕ is an L-formula, then so is $\neg\phi$;
(iii) if ϕ and ψ are L-formulas, then so is $\phi \wedge \psi$;
(iv) if ϕ is an L-formula, for any $i \geq 1$, $\exists x_i \phi$ is a formula;
(v) nothing else is a formula (note that $\forall$ can be constructed by $\neg$ and $\exists$).

Example. In L_{gp}, $x_1 \cdot x_1 = x_2$, or $x_1 \cdot x_2 = 1$ are both atomic formulas;
$\exists x_1 (x_1 \cdot x_2) = 1$ is an L-formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier $\exists$. We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example, $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp}-sentence.

Notation: $\phi(x_1, ..., x_k)$ means that the free variables in ϕ are among $x_1, ..., x_k$.

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

Definition. (2.7)
Let $\phi(x_1, ..., x_k)$ be an L-formula, let $\mathcal{M}$ be an L-structure, and let $\bar{a} = a_1, ..., a_k$ be elements of $\mathcal{M}$.
We define $\mathcal{M} \vDash \phi(\bar{a})$ (syntactic implication, read as *M models* $\phi(\bar{a})$) as follows:
(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$;
(ii) if ϕ is $R(t_1, ..., t_{n_R})$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff

$$\left(t_1^M(\bar{a}), ..., t_{n_R}^M(\bar{a})\right) \in R^M$$

(iii) if ϕ is a conjunction, say $\psi \wedge \chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \chi(\bar{a})$;
(iv) if ϕ is $\exists x_j \chi(x_1, ..., x_k, x_j)$ (where we'll assume that x_j is not one of the free variables $x_1, ..., x_k$), then $\mathcal{M} \vDash \phi(\bar{a})$ iff there exists $b \in \mathcal{M}$ s.t. $\mathcal{M} \vDash \chi(a_1, ..., a_k, b)$;
(v) (lecture forgets this, this should probably be more in front rather than in the end) if ϕ is $\neg\psi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \psi(\bar{a})$.

Example. Consider $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1 \rangle$, the multiplicative group of non-negative reals, and suppose we have $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$, then $\mathcal{R} \vDash \phi(1)$, but $\mathcal{R} \nvDash \phi(-1)$.

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):
- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$;
- $\phi \to \psi$ for $\neg\phi \vee \psi$;
- $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$;
- $\forall x_i \phi$ for $\neg\exists x_i (\neg\phi)$.

Proposition. (2.9)
Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding.
Let $\phi(\bar{x})$ be an atomic(!) formula, and $\bar{a} \in M^{|\bar{x}|}$, here $|\bar{x}|$ means the length of the tuple $\bar{x}$ (from now on, when we write a tuple like $\bar{a}$, we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a}))$$

Question: if ϕ is an L-formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.
Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.
In definition 2.7 we defined what it means for $\mathcal{M} \vDash \phi(\bar{a})$, in particular we defined: if $\phi \equiv \neg\chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \chi(\bar{a})$. Here by $\mathcal{M} \vDash \phi(\bar{a})$ we mean $\mathcal{M} \vDash \neg\chi(\bar{a})$, and $\chi(\bar{a})$ is *shorter* than $\phi(\bar{a})$, so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

Proof. There are two cases:
• $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. Use Fact (2.3). (exercise on example sheet)
• $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), ..., t_{n_R}(\bar{x}))$. Then $\mathcal{M} \vDash R(t_1(\bar{a}), ..., t_{n_R}(\bar{a}))$ if and only if ... (lecturer says work this out by yourself. Basically the induction step). □

Proposition. (2.10)
Exercise: show that prop (2.9) holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).
(I guess that also suggests when does it not hold for general formulas – see below).

Example. (2.11, Do embeddings preserve all formulas? No.)
Let $\mathcal{Z} = (\mathbb{Z}, <)$ an $L_{lo}-$structure, $\mathcal{Q} = (\mathbb{Q}, <)$ also an $L_{lo}-$structure. Then

$$\begin{aligned} \alpha : &\mathbb{Z} \to \mathbb{Q} \\ &n \to n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3(x_1 < x_3 \wedge x_3 < x_2)$$

Now $\mathcal{Q} \vDash \phi(1, 2)$ but $\mathcal{Z} \nvDash \phi(1, 2)$.

Fact (2.12) (From now on we'll stop saying that $\mathcal{M}, \mathcal{N}$ are L-structures etc to save time) Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L-formula, and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).

3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let L be a language, $\mathcal{M}, \mathcal{N}$ be L-structures.

Definition. (3.1)
An L-*theory* T is a set of L-sentences.
$\mathcal{M}$ is a *model* of T if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$.
The class of all the models of T is written $Mod(T)$.
The *theory of* $\mathcal{M}$ is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L - \text{sentence and } \mathcal{M} \vDash \sigma\}$$

Example. (3.2)
Let T_{gp} be the set of L_{gp}-sentences:
(i) $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$;
(ii) $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$;
(iii) $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$.
Clearly, for a group G, $G \vDash T_{gp}$ (as they are just the group axioms). However, for a specific group G, clearly the theory of it, $Th(G)$ is lartger than T_{gp}.

Definition. (3.3)
$\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if $Th(\mathcal{M}) = Th(\mathcal{N})$.
We write $\mathcal{M} \equiv \mathcal{N}$.
Clearly, if $\mathcal{M} \simeq \mathcal{N}$ ($\simeq$ means isomorphism), then $\mathcal{M} \equiv \mathcal{N}$.
But if $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!
We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as $L_{lo}-$structures(!).

Definition. (3.4)
(i) An embedding $\beta : \mathcal{M} \to \mathcal{N}$ is *elementary* if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\beta(\bar{a}))$$

(ii) If $M \subseteq N$, and $id : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.
(iii) If $M \subseteq N$ and $id : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then $\mathcal{M}$ is said to be an *elementary substructure* of $\mathcal{N}$, written as $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example. (3.5)
Let $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an $L_{lo}-$structure where $<$ is the usual order;
Let $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$, also an $L_{lo}-$structure with the same $<$.
Then $\mathcal{M} \simeq \mathcal{N}$ as $L_{lo}-$structures. So $\mathcal{M} \equiv \mathcal{N}$ (since they are isomoprhic).
Also, $\mathcal{M} \subseteq \mathcal{N}$ (read as *is a substructure of*), since the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$. *However*, $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if we pick the formula $\phi(x) \equiv \exists y (x < y)$, then $\mathcal{N} \vDash \phi(1)$, but $\mathcal{M} \nvDash \phi(1)$.

Definition. (3.6)
Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where c_a are constant symbols). An interpretation of $\mathcal{M}$ as an L-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way, i.e. $c_a^M = a$.
In this context, the elements of A are called *parameters*.
If $\mathcal{M}$ and $\mathcal{N}$ are two structures, and $A \subseteq M \cap N$, then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$ sentences.

—Lecture 4—

Reminder: we have a lecture next Monday (22nd Oct)!

Proposition. It turns out that, $\mathcal{M} \preccurlyeq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where M is the domain of $\mathcal{M}$).

Lemma. (3.8, Tarski-Vaught test)
Let $\mathcal{N}$ be an L-structure, let $A \subseteq N$. The follwing are equivalent:
(i) A is the domain of a structure $\mathcal{M}$ s.t. $\mathcal{M} \preccurlyeq \mathcal{N}$;
(ii) if $\phi(x) \in L(A)$ (with an abuse of notations $\phi(x, c_{a_1}, ..., c_{a_n}) = \phi(x, a_1, ..., a_n)$), if $\mathcal{N} \vDash \exists x \phi(x)$, then $\mathcal{N} \vDash \phi(b)$ for some $b \in A$.

Proof. (i) $\implies$ (ii): Suppose $\mathcal{N} \vDash \exists x \phi(x)$. Then by elementarity, $\mathcal{M} \vDash \exists x \phi(x)$, and so $\mathcal{M} \vDash \phi(b)$ for $b \in \mathcal{M}$. So (again by elementarity), $\mathcal{N} \vDash \phi(b)$.
(ii) $\implies$ (i): This is the harder direction. First we prove that A is the domain of a substructure $\mathcal{M} \subseteq \mathcal{N}$.
By Sheet 1 Q4, it suffices to check:
(a) For each constant c, $c^N \in A$;
(b) For each function symbol f, $f^N(\bar{a}) \in A$ (for all $\bar{a} \in A^{n_R}$);
For (a), use property (ii) with $\exists x(x = c)$.
For (b), use property (ii) with the formula $\exists x((\bar{a}) = x)$.
So we now have $\mathcal{M} \subseteq \mathcal{N}$, and domain of $\mathcal{M}$ is A. But we actually want to prove that $\mathcal{M} \preccurlyeq \mathcal{N}$. Now let $\chi(\bar{x})$ be an L-formula.
We want to show that for $\bar{a} \in A^{|\bar{x}|}$ $\mathcal{M} \vDash \chi(\bar{a}) \iff \mathcal{N} \vDash \chi(\bar{a})$ (*).
By induction on the complexity of $\chi(\bar{x})$:
- if $\chi(\bar{x})$ is atomic, (*) follows from $\mathcal{M} \subseteq \mathcal{N}$ (since $\mathcal{M}$ is a substructure!);
- if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$, it's a straightforward induction;
- (interesting case) if $\chi(\bar{x}) = \exists y \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an L-formula, suppose that $\mathcal{M} \vDash \chi(\bar{a})$, then $\mathcal{M} \vDash \exists y \psi(\bar{a}, y)$, hence $\mathcal{M} \vDash \psi(\bar{a}, b)$ for some $b \in A = \text{dom}(\mathcal{M})$ (this is the definition of truth).

But then $\mathcal{N} \vDash \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \vDash \chi(\bar{a})$.
Now let $\mathcal{N} \vDash \chi(\bar{a})$, i.e. $\mathcal{N} \vDash \exists y \psi(\bar{a}, y)$ (we find a *witness* for it). By property (ii), $\mathcal{N} \vDash \psi(\bar{a}, b)$ for some $b \in A = \text{dom}(\mathcal{M})$.
Again by inductive hypothesis, we have $\mathcal{M} \vDash \psi(\bar{a}, b)$, and so in particular, $\mathcal{M} \vDash \chi(\bar{a})$ as it has got a witness there. ☐

Remark. (3.9)
Even more assumptions: let's assume that the set of variables is countably infinite. Then:
- the cardinality of the set of L-formulas is $|L| + \omega$ (where by $|L|$ we mean

the number of symbols. For example, $|L_{gp}| = 3$, $|L_{lo}| = 1$), where we abuse another notation that we use ω as cardinals (rather than ordinals) (note that the formulas are just strings of finite length);
• if A is a set of parameters in some structure, the cardinality of the set $L(A)$ is $|A| + |L| + \omega$, where by $+$ here we merely mean $\max\{|L|, |A|, \omega\}$ (instead of addition), and same for the $+$ above.

Definition. (3.10)
Let λ be an ordinal. Then *a chain of length* λ of sets is a sequence $\langle M_i : i < \lambda\rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$.
A chain of L-structures is a seequence: $\langle \mathcal{M}_i : i < \lambda\rangle$ s.t. $\mathcal{M}_i \subseteq \mathcal{M}_j$ (note that it's substructure here) for $i \leq j < \lambda$.
The *union* of this chain is the L-structure $\mathcal{M}$ defined as follows:
• the domain is $\bigcup_{i<\lambda} M_i$ (when you think of this, you can always start with the case $\lambda = \omega$);
• for constants c, $c^M = c^{M_i}$ for any $i < \lambda$ (this is well defined, because of the substructure condition above);
• if f is a function symbol, $\bar{a} \in M^{|n_f|}$ (why the mod sign here), $f^M\bar{a} = f^{M_i}\bar{a}$ where i is s.t. $\bar{a} \in M_i^{|n_f|}$;
• if R is a relation synbol, then $R^M = \bigcup_{i<\lambda} R^{M_i}$.

Theorem. (3.11, Downward Löwenheim-Skolem theorem)
(Recall that in part II Logic and Set Theory we had the countable version of this)
Let $\mathcal{N}$ be an L-structure, and $|\mathcal{N}| \geq |L| + \omega$. Let $A \subseteq N$. Then for every cardinal λ s.t. $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preccurlyeq \mathcal{N}$ s.t.
(i) $A \subseteq M$;
(ii) $|\mathcal{M}| = \lambda$.

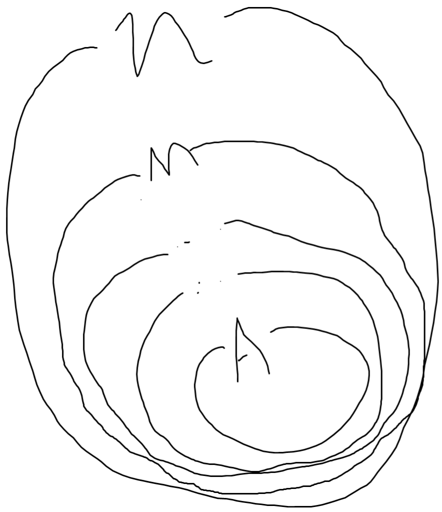
(It helps to think about the case $|A| = \omega$ and $|N|$ is uncountable.)
A quick example how this could be useful (we'll go very sloppy here): think of $(\mathbb{C}, +, \cdot, -, \cdot^{-1}, 0, 1)$ as a field. Consider $\mathbb{Q} \subseteq \mathbb{C}$ (both as subset and substructure). Note that algebraic closeness is a property of $\mathbb{C}$. By downward Löwenheim-

Skolem, there is a substructure in $\mathcal{C}$ that contains $\mathbb{Q}$ that is also algebraically closed (apparently, the set of algebraic numbers).

Proof. We build a chain $\langle A_i : i < \lambda \rangle$, with $A_i \subseteq N$, s.t. $|A_i| = \lambda$.
(our goal: define an elementary substructure with domain $M = \bigcup_{i<\omega} A_i$).
Base case: Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$.
Successors: At stage $i+1$, assume A_i has been built, with $|A_i| = \lambda$.
Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$-formulas such that $\mathcal{N} \models \exists x \phi_k(x)$. Let a_k be such that $\mathcal{N} \models \phi_k(a_k)$, and let $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$ (basically, with those witnesses added). Then $|A_{i+1}| = \lambda$ (note that we haven't increased the size).
Now let $M = \bigcup_{i<\omega} A_i$ (note the subscript range). We use lemma (3.8) to show that M is the domain of $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|M| = \lambda$. We're running out of time, so we'll continue next Monday.

—Lecture 5—
Solutions to worksheet 1: either take along to lecture on Friday, or email them to *silvia.barbina@open.ac.uk*.

Let's continue with the proof:



Start with $A_0 \subset N$, $A \subseteq A_0$, $|A_0| = l$. The idea is to define $\langle A_i : i < \omega \rangle$ so that $M = \cup_{i<\omega} A_i$ satisfies (ii) via the TV test (3.8).
List all formulas $\phi(x, \bar{a})$ ($\bar{a}$ is a tuple in A_0), and $\mathcal{N} \models \phi(b, \bar{a})$ for some b.
Add each such b to A_0 (one for each such ϕ).
Let $A_1 = A_0 \cup \{$ all thes b's$\}$.
Repeat for formulas $\phi(x, \bar{a})$ where $\bar{a}$ is in A_1,...
Eventually, $\langle A_i : i < \omega \rangle$ is such that $M = \cup_{i<\omega} A_i$ is as required (i.e. M is the domain of some elementary substrucutre of $\mathcal{N}$ that we need).
We claim that M satisfies condition (ii) in Lemma (3.8): Let $\mathcal{N} \models \exists x \psi(x, \bar{a})$, where $\bar{a}$ is a tuple in M. Then $\bar{a}$ is a *finite* tuple, so there is an i s.t. $\bar{a}$ is in A_i.

Then A_{i+1}, by construction, contains b s.t. $\mathcal{N} \vDash \phi(b, \bar{a})$. But $A_{i+1} \subseteq M, b \in M$. Then apply (3.8) we're done. $\square$

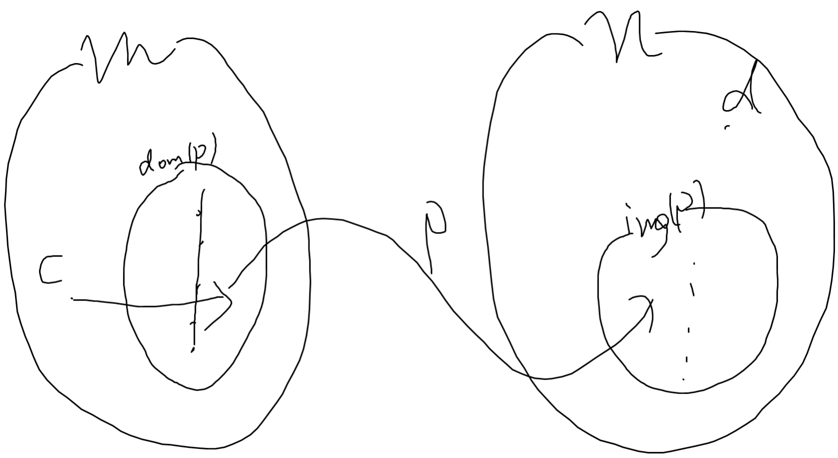
4 Two relational structures

Definition. (4.1, dense linear orders)
A *linear order* is an $L_{lo} = \{<\}$-structure such that:
(i) $\forall x \neg(x < x)$;
(ii) $\forall xyz((x < y \wedge y < z) \rightarrow x < z)$;
(iii) $\forall xy((x < y) \vee (y < x) \vee x = y)$ (total).
A linear order is *dense* if, in addition, it also satisfies:
(iv) $\exists xy(x < y)$;
(v) $\forall xy, (x < y \rightarrow \exists z(x < z \wedge z < y))$ (density).
A linear order has no endpoints if, in addition,
(vi) $\forall x(\exists y(x < y) \wedge \exists z(z < x))$.
We use T_{dlo} to denote the theory that includes all axioms (i) to (vi), and T_{lo} is the theory that includes axioms (i) to (iii) only.

Remark. (iv) and (v) imply that if $\mathcal{M} \vDash T_{dlo}$, then $|\mathcal{M}| \geq \omega$.

Definition. (4.2)
If $\mathcal{M}, \mathcal{N} \vDash T_{lo}$, then an *injective* map $p : A \subseteq M \rightarrow N$ is a *partial embedding* if $\mathcal{M} \vDash a < b \implies \mathcal{N} \vDash p(a) < p(b)$.
In particular, if $|\operatorname{dom}(p)| < \omega$, then p is a *finite* partial embedding.

Lemma. (4.3, extension lemma)
Take a linear order $\mathcal{M} \vDash T_{lo}$, and a dense linear endpoints $\mathcal{N} \vDash T_{dlo}$, and let $p : M \rightarrow N$ be a finite partial embedding. Then if $c \in \mathcal{M}$, there is a finite partial embedding $\hat{p}$ s.t. $p \subseteq \hat{p}$ and $c \in \operatorname{dom}(\hat{p})$.
(*we can always add one extra element in our embedding.*)



Proof.

Case 1: c is greater than all elements in $\operatorname{dom}(p)$. In that case, pick an element $d \in \mathcal{N}$ s.t. $d > b$ for all $b \in img(p)$;
Case 2: $a_i < c < a_{i+1}$ where $a_i, a_{i+1} \in \operatorname{dom}(p)$. Then we choose $\mathcal{N} \vDash p(a_i) < d < p(a_{i+1})$, where d is chosen appropriately by density (here's the case why we

need $\mathcal{N}$ to be dense;
Case 3: c is less than all elements in $\text{dom}(p)$. This is similar to case 1.

Note that the ability to extend by one point allows us to embed any finite linear order into a dense linear order without endpoints. $\square$

Theorem. (4.4)
Let $\mathcal{M}, \mathcal{N} \vDash T_{dlo}$ s.t. $|\mathcal{M}| = |\mathcal{N}| = \omega$. Let $p : A \subseteq M \to N$ be a finite partial embedding.
Then there is an isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ s.t. $p \subseteq \pi$.

Proof. Enumerate M, N, say $M = \langle a_i : i < \omega \rangle$, $N = \langle b_i : i < \omega \rangle$ (sequences of elements).
We define, inductively, a chain of finite partial embedding $\langle p_i : i < \omega \rangle$ (idea: $\pi = \cup_{i<\omega} p_i$).
Let's start with $p_0 = p$. At stage $i + 1$, suppose we are given p_i. We want to include a_i in $\text{dom}\, p_{i+1}$, and b_i in the $img(p_{i+1})$.
(Lecturer calls this a *back and forth* method) Forth step: By lemma 4.3, we can extend p_i to $p_{i+\frac{1}{2}}$ such that $a_i \in \text{dom}(p_{i+\frac{1}{2}})$;
Back step: By lemma 4.3 again applied to $(p_{i+\frac{1}{2}})^{-1}$ to include $b_i \in \text{dom}(p_{i+1}^{-1})$ (i.e. in the range of p_{i+1}).
We claim that p_{i+1} extends p_i as required.
Let $\pi = \cup_{i<\omega} p_i$. Then (check) π is an isomorphism (i.e. order-preserving bijection). $\square$

Definition. (4.5)
An L-theory is *consistent* if there is L-structure $\mathcal{M}$ s.t. $\mathcal{M} \vDash T$.
If T is a theory in L and ϕ is an L-sentence, then $T \vdash \phi$ (read as T *entails* ϕ, note that this has nothing to do with syntactic implication) if for all $\mathcal{M}$ such that $\mathcal{M} \vDash T$, we have $\mathcal{M} \vDash \phi$.
Finally, an L-theory T is *complete* if for all L-sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$ (see part II Logic and Set Theory).
For example, T_{dlo} is complete.

—Lecture 6—

Definition. (4.6)
A theory T in a countable language with a (infinitely) countable model is *ω-categorical* if any two countable models of T are isomorphic.

Corollary. (4.7 of theorem (4.4))
T_{dlo} is ω-categorical.

Proof. If $\mathcal{M}, \mathcal{N} \vDash T_{dlo}$, $|\mathcal{M}| = |\mathcal{N}| = \omega$, then ϕ (the empty map) is a finite partial embedding. But by theorem (4.4) we get $\mathcal{M} \simeq \mathcal{N}$.
(We can also use any $\{\langle a, b \rangle\}$ where $a \in \mathcal{M}$ and $b \in \mathcal{M}$ as initial finite partial embedding). $\square$

Theorem. (4.8)
(erratum 26th Oct 2018: lecturer wants to add a condition T *has no finite models.*

Then the problem with (4.11) is fixed.)
If T is an ω-categorical theory in a countable language, then T is complete.

Proof. Let $\mathcal{M} \vDash T$ and ϕ be an L-sentence.
If $\mathcal{M} \vDash \phi$, suppose $\mathcal{N} \vDash T$. Then by theorem (3.11) (Downward Lowenheim-Skolem), there are $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$ s.t. $|\mathcal{M}'| = |\mathcal{N}'| = \omega$.
But $\mathcal{M}' \simeq \mathcal{N}'$ (by ω-categoricity), so in particular $\mathcal{M}' \equiv \mathcal{N}'$, and so $\mathcal{N}' \vDash \phi$. By elementarity, $\mathcal{N} \vDash \phi$.
The case $\mathcal{M} \vDash \neg\phi$ is similar.
(Think about if T could have a finite model.) □

Corollary. (4.9)
T_{dlo} is complete.

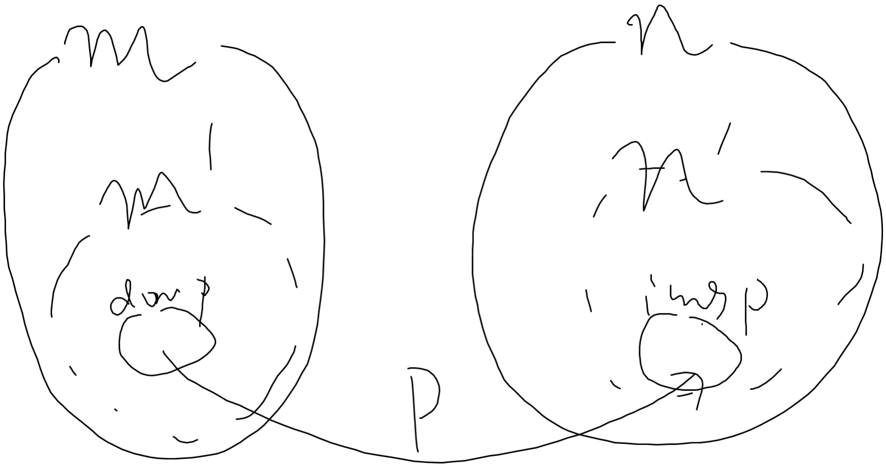
Definition. (4.10)
If $\mathcal{M}, \mathcal{N}$ are L-structures, a map f such that $\text{dom}(f) \subseteq M$ (the domain of $\mathcal{M}$), and $img(f) \subseteq N$ is a (partial) *elementary map* if for all L-formulas $\phi(\bar{x})$ and $\bar{a} \in (\text{dom}(f))^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(f(\bar{a}))$$

Remark. (4.11)
A map f is elementary iff every finite restriction of f is elementary.
(Why? For forward, if $f_0 \subseteq f$ is a finite restriction that is not elementary, then for some formula $\phi(\bar{x})$, $\bar{a} \in \text{dom}(f_0)$, the above equivalence doesn't hold; but then that equivalence doesn't hold for f either; contradiction; for backward, if f is not elementary, then the above equivalence fails on a finite tuple, so the above equivalence fails on some finite restriction.)

Proposition. (4.12)
Let $\mathcal{M}, \mathcal{N} \vDash T_{dlo}$, and let $p : A \subseteq M \to N$ be a partial embedding. Then p is elementary.

Proof. By remark (4.11), it suffices to consider p finite.
By Downward L-S theoem (3.11), we choose $\mathcal{M}', \mathcal{N}'$ such that
(i) $|\mathcal{M}'| = |\mathcal{N}'| = \omega$;
(ii) $\mathcal{M}' \preccurlyeq \mathcal{M}$, $\mathcal{N}' \preccurlyeq \mathcal{N}$;
(iii) $\text{dom}(p) \subseteq M', img(p) \subseteq N'$.



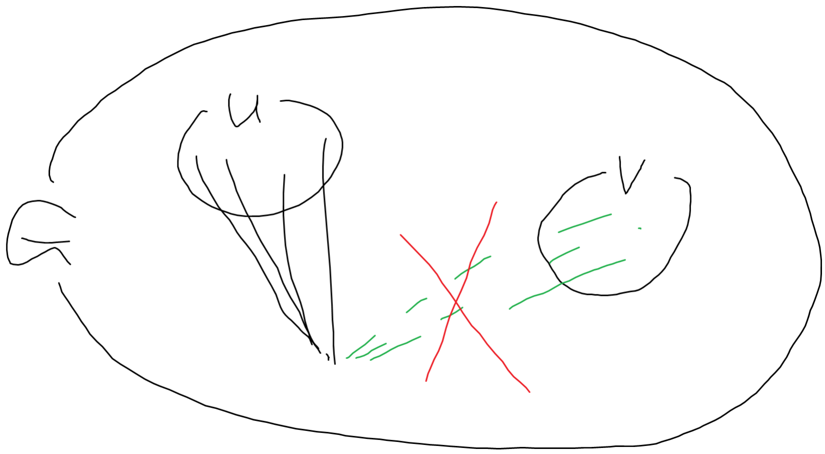
Now p is a finite partial embedding between countable models, so p extends to an isomorphism $\pi : \mathcal{M}' \to \mathcal{N}'$.
In particular, π is an elemntary map between $\mathcal{M}$ and $\mathcal{N}$. □

Corollary. (4.13)
$(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

Proof. Use proposition (4.12) with $id : \mathbb{Q} \to \mathbb{R}$. □

Definition. (4.14)
(See Part II Logic and Set Theory)
Let $L_{gph} = \{R\}$, where R is a binary relation symbol.
An L_{gph}-structure is a graph if
(i) $\forall x \, \neg R(x, x)$;
(ii) $\forall xy (R(x, y) \leftrightarrow R(y, x))$.

An L_{gph}-structure is a random graph if it is a graph such that the following axiom-schema (r_n) hold:



$$\forall x_0 ... x_n, y_0 ... y_n, (\bigwedge_{i,j=0}^{n} x_i \neq y_j \rightarrow \exists z(\bigwedge_{i=0}^{n} (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i)))$$

(iii) $\exists xy(x \neq y)$.

Remark. (similar to what is mentioned in the link above)
A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

Fact. (4.15)
There is a random graph.

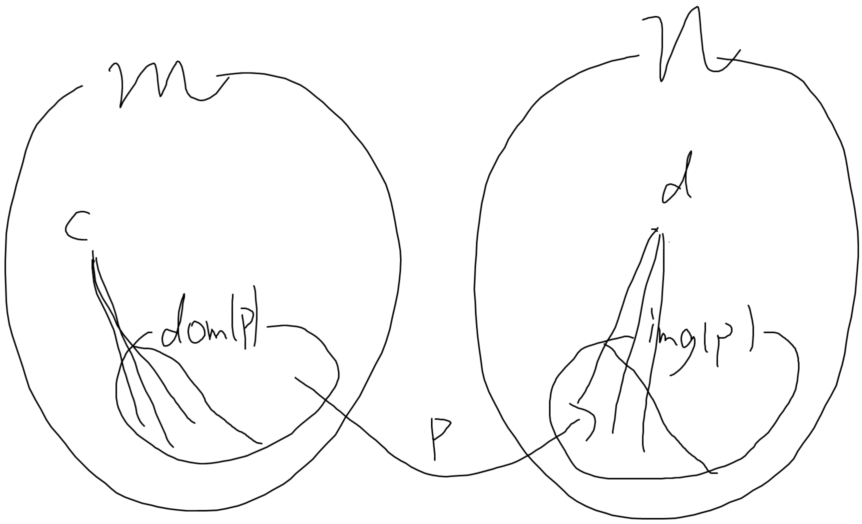
Proof. Let the domain be ω, let $i, j \in \omega$ such that $i < j$. Write j as a sum of distinct powers of 2. Then $\{i, j\}$ is an edge iff 2^i appears in the sum.
As an exercise, prove that ω with this definition of R is indeed a random graph. □

Definition. (4.16, or more precisely just notations)
$T_{gph} = \{\text{axioms (i), (ii)}\}$, $T_{rg} = T_{gph} \cup \{\text{(iii)}, (r_n) : n \in \omega\}$.
If $\mathcal{M}, \mathcal{N} \vDash T_{gph}$, a partial embedding is an injection $p : A \subseteq M \rightarrow N$ s.t. $\mathcal{M} \vDash R(p(a), p(b)) \iff \mathcal{N} \vDash R(a, b)$ for all a, b in the domain.

Lemma. (4.17)
Let $\mathcal{M} \vDash T_{gph}, \mathcal{N} \vDash T_{rg}$, let $p : A \subseteq M \rightarrow N$ be a finite partial embedding, and let $c \in M$.
Then there is a map $\hat{p} : \hat{A} \subseteq M \rightarrow N$ such that $\hat{p}$ is a partial embedding, $c \in \operatorname{dom} \hat{p}$, $p \subseteq \hat{p}$.
(This is like another extension lemma.)
We'll prove this next time.

—Lecture 7—

Last time we defined what a random graph is (in this course). We also defined what is a partial embedding in this theory (just preserves all edges).
Let's continue with the proof of the lemma now. Let $c \in M$, $c \notin \mathrm{dom}(p)$.



Find $d \in N$ such that $\mathcal{N} \models R(d, p(a)) \iff \mathcal{M} \models R(c, a)$.

Theorem. (4.18)
Let $\mathcal{M}, \mathcal{N} \models T_{rg}$ and $|\mathcal{M}| = |\mathcal{N}| = \omega$, and $P : A \subseteq M \to N$ is a finite partial embedding.
Then $\mathcal{M} \simeq \mathcal{N}$, by an isomorphism that extends p.

Proof. Same as proof of Theorem (4.4) (there is only one model of T_{dlo} up to isomorphism), but with lemma (4.17) instead of lemma (4.3). □

Corollary. (4.19)
T_{rg} is ω-categorical (see definition (4.6) – this is just a restatement of the theorem above) and complete.
In particular, every finite partial embedding between models of T_{rg} is an elementary map.

Remark. The unique (up to isomorphism) model of T_{rg} is *the* countable random graph, or the *Rado* graph.
It is universal w.r.t. finite and countable graphs (i.e. it embeds all).
Another nice property (which you are not required to see this immediately – it is far from trivial) *ultrahomogeneous*, i.e. every isomorphism between finite substructures extends to an automorphism of the whole graph.
Google *David Marker's* book, or *Tent-Ziegler*. Warning: both of them contain a lot of typos.

5 Compactness

Definition. (5.1)
Suppose we have a L-theory T.
(i) T is *finitely satisfiable* if every finite subset of sentences in T has a model.
(ii) T is *maximal* if for all L-sentences σ, either $\sigma \in T$ or $\neg\sigma \in T$.
(iii) T has the *witness property* (WP): if for all $\phi(x)$ (L-formula with 1 free variable), there is a constant $c \in \mathcal{C}$ s.t.

$$(\exists x \phi(x) \to \phi(c)) \in T$$

Lemma. (5.2)
If T is maximal and finitely satisfiable (we'll sometimes use *f.s.* from now onwards), and ϕ is an L-sentence, and $\triangle \overset{fin}{\subseteq} T$ and $\triangle \vdash \phi$, then $\phi \in T$.
(Prove it by yourself)

Lemma. (5.3)
Let T be a maximal, f.s. theory with WP. Then T has a model.
Moreover, if λ is a cardinal and $|\mathcal{C}| \leq L$ ($\mathcal{C}$ is the set of constants in L), then T has a model of size at most λ.

Proof. Let $\mathcal{C}$ be the constants of L. Let $c, d \in \mathcal{C}$, define $c \sim d$ iff $c = d \in T$.
We claim that $\sim$ is an equivalence relation: reflexivity and symmetry are trivial; for transitivity, let $c \sim d$ and $d \sim e$. Then $c = d \in T$ and $d = e \in T$. Then by the lemma $c = e \in T$ as it is implied by the two sentences. So $c \sim e$.
Notation: we'll use $c/\sim = c^*$ to denote the equivalence class of c.
Now define a structure $\mathcal{M}$ whose domain is $\mathcal{C}/\sim = M$. Clearly, $|M| \leq \lambda$ if $|\mathcal{C}| \leq \lambda$.
We must define interpretations in $\mathcal{M}$ for symbols for L:
• If $c \in \mathcal{C}$, then $c^m = c^*(= c/\sim)$;
• If $R \in \mathcal{R}$ is a relation symbol, we define $R^{\mathcal{M}} = \{(c_1^*, ..., c_{n_R}^*) : R(c_1, ..., c_{n_R}) \in T\}$.
We have to check that $R^{\mathcal{M}}$ is well-defined: suppose $\bar{c}, \bar{d} \in \mathcal{C}^{n_R}$ and suppose $c_i \sim d_i$ for each i, i.e. $c_i = d_i \in T$ for every $i = 1, ..., n_R$. However, now

$$R(\bar{c}) \in T \iff R(\bar{d}) \in T$$

by maximality of T and the previous lemma. So that $R^{\mathcal{M}}$ is well defined.
• If $f \in \mathcal{F}$ is a function symbol, then $f\bar{c} = d \in T$ for some $d \in \mathcal{C}$: this is because $\exists x (f(\bar{c} = x) \in T$ by maximality and f.s..
Then define $f^{\mathcal{M}}(\bar{c}^*) = \bar{d}^*$ (obvious notation).
We also have to check this is well-defined. Lecturer decides to left this to us.
Now we claim that the terms behave nicely as what the theory says, i.e. if $t(x_1, ..., x_n)$ is an L-term and $c_1, ..., c_n, d \in \mathcal{C}$, then $t(c_1, ..., c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, ..., c_n^*) = d^*$.
• $\implies$: by induction on the complexity of T (lecturer decided to leave this as another exercise).
• $\Leftarrow$: Assume $t^{\mathcal{M}}(c_1^*, ..., c_n^*) = d^*$. Then $t(c_1, ..., c_n) = e \in T$ for some constant e (why? As our theory is maximal, it has to say what the result is when we apply

t to these terms). We then use $\implies$ to get that $t^{\mathcal{M}}(c_1^*, ..., c_n^*) = e^*$.
But then $d^* = e^*$, i.e. $d = e \in T$. So by lemma (5.2), the sentence implied by these two sentences, $t(c_1, ..., c_n) = d \in T$.
The last massive claim: for all L-formulas $\phi(\bar{x})$ and $\bar{c} \in \mathcal{C}^{|\bar{x}|}$, we have

$$\mathcal{M} \vDash \phi(\bar{c}) \iff \phi(\bar{c}) \in T$$

The proof is by induction on complexity of $\phi(\bar{x})$ (The lecturer decided to leave yet another proof to us – lots of work to be done here. Lecturer is speeding up!). $\square$