Combinatorics

October 9, 2018

Contents

0 Introduction

In this course we'll discuss three main aspects:
- Set systems;
- Isoperimetric Inequalities;
- Projections (combinatorics in continuous settings).

References:
Combinatorics, Bocabas, Cambridge University Press, 1986 (chapter 1,2);
Combinatorics and finite sets, Anderson, Oxford University Press, 1987 (chapter 1).

1 Set Systems

Let X be a set. A *set system* on X (or family of subsets of X) is a family $\mathcal{A} \subset \mathbb{P}(X)$.
For example, we define $X^{(r)} = \{A \subset X : |A| = r\}$.

Unless otherwise stated, $X = [n] = \{1, 2, ..., n\}$. For example, $|X^{(r)}| = \binom{n}{r}$ (assume finiteness). So $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$.

We often make $\mathbb{P}(x)$ into a graph, called Q_n, by joining A to B if $|A \triangle B| = 1$ (symmetric difference).

(examples of Q_3, Q_n)

If we identify a set $A \subset X$ with a 0-1 sequence of length n via $A \leftrightarrow 1_A$ (characteristic function), then Q_3 acn be thought of as a cube. In general, Q_n is an n-dimensional cube (hypercube/discretecube/n-cube/...).

1.1 Chains and antichains

A family $\mathcal{A} \subset \mathbb{P}(X)$ is a *chain* if $\forall A, B \in \mathcal{A}, A \subset B$ or $B \subset A$. It is an antichain if $\forall A \neq B \in \mathcal{A}$, $A \not\subset B$.

Obviously the maximum size of a chain in X is $n + 1$.

For antichains, we can take $X^{\lfloor \frac{n}{2} \rfloor}$, which has size $\binom{n}{\lfloor n/2 \rfloor}$. The result is that wee can't beat this, but the proof is not trivial.

—Lecture 2—

No lecture this thursday (11 Oct 2018)!

Idea: inspired by *each chain meets each level $X^{(r)}$ in at most one place* – try to decompose Q_n into chains.

Theorem. (Sperner's Lemma)
Let $A \subset \mathbb{P}(X)$ be an antichain. Then $|A| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof. It's sufficient to partitoin $\mathbb{P}(X)$ into that many chains (since an anti-chain and a chain can have at most one common vertex).
For this, it's sufficient to show:
• $\forall r < n/2$, there exists a matching (set of disjoint edges) from $X^{(r)}$ to $X^{(r+1)}$;
• $\forall r > n/2$, there exists a matching from $X^{(r)}$ to $X^{(r-1)}$.
(Then put these matchings together to form chains, each passing through $X^{(\lfloor n/2 \rfloor)}$), so the result.
By taking complements it's sufficient to prove (i).
Consider subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$ which is bipartite. For any $B \subset X^{(r)}$, we have:
• number of $B - \mathbb{P}(B)$ edges $= |B|(n - r)$; (each point in $X^{(r)}$ has degree $(n - r)$)

• number of $B - \mathbb{P}(B)$ edges $\leq |\mathbb{P}(B)|(r+1)$. (each point in $X^{(r+1)}$) has degree $r+1$)
Thus $|\mathbb{P}(B)| \geq |B|\frac{n-r}{r+1} \geq |B|$, as $r < n/2$.
Hence by Hall's theorem there exists a matching. $\square$

Remark. • 1. $\binom{n}{\lfloor n/2 \rfloor}$ is achievable by just taking $X^{(\lfloor n/2 \rfloor)}$.
• 2. This proof says nothing about extremal cases: which antichains have size $\binom{n}{\lfloor n/2 \rfloor}$?

Aim: For $\mathcal{A}$ an antichain, $\sum_{r=0}^n \frac{|\mathcal{A} \cap X^{(r)}}{\binom{n}{r}} \leq 1$. Note that this trivally implies Sperner's lemma.

Let $\mathcal{A} \subset X^{(r)}$ for some $1 \leq r \leq n$. The *shadow* or *lower shadow* of $\mathcal{A}$ is

$$\partial A = \partial^- A = \{A - \{i\} : A \in \mathcal{A}, i \in A\}$$

So $\partial A \subset X^{(r-1)}$.

For example, if $\mathcal{A} = \{123, 124, 134, 135\} \subset X^{(3)}$, then $\partial A = \{12, 13, 23, 14, 24, 34, 15, 35\} \subset X^{(2)}$.

Lemma. (Local LYM)
Let $\mathcal{A} \subset X^{(r)}$, $1 \leq r \leq n$. Then

$$\frac{|\partial \mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}$$

(*the fraction of the layer occupied increases when we take the shadow.*)

Proof. • Number of $\mathcal{A} - \partial\mathcal{A}$ edges (in Q_n) $= r|\mathcal{A}|$ (counting from above);
• Number of $\mathcal{A} - \partial\mathcal{A}$ edges $\leq (n-r+1)|\partial\mathcal{A}|$ (counting from below).
So

$$\frac{|\partial \mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1}$$

However RHS is the ratio of size between the two layers. $\square$

Let's consider when is equality achieved in local LYM. we need $A - \{i\} \cup \{j\} \in \mathcal{A}$ $\forall a \in \mathcal{A}, i \in A, j \notin A$.
Hence $\mathcal{A} = X^{(r)}$ or ϕ.

Theorem. (Lubell-Yamamoto-Meshalkin inequality)
Let $\mathcal{A} \subset \mathbb{P}(X)$ be an antichain. Then $\sum_{r=0}^n \frac{|\mathcal{A} \cap X^{(r)}}{\binom{n}{r}} \leq 1$.

Proof. (1, *Bubble down with local LYM*)
Let's start with $X^{(r)}$. Write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$.
We have $\frac{|\mathcal{A}_n|}{\binom{n}{n}} \leq 1$ (trivially).
Also, $\partial\mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are disjoint (as $\mathcal{A}$ is an antichain). So

$$\frac{|\partial\mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial\mathcal{A}_n \cup \mathcal{A}_{n-1}}{\binom{n}{n-1}} \leq 1$$

So

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1$$

by local LYM. Note that we have successfully expanded LHS to two terms. Also, $\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})$ is disjoint from $\mathcal{A}_{n-2}$ again since $\mathcal{A}$ is an antichain. So

$$\frac{|\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1$$

So

$$\frac{|\partial\mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1$$

So

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1$$

Keep going and we obtain the desired result. □

When is equality achieved in LYM? We must have equality in each use of local LYM, so the first r with $\mathcal{A}_r \neq \phi$ must have $\mathcal{A}_r = X^{(r)}$, i.e. $\mathcal{A} = X^{(r)}$.

Hence equality in Sperner's lemma is only achieved when $\mathcal{A} = X^{\lfloor n/2 \rfloor}$ for n even, or also $X^{\lceil n/2 \rceil}$ when n is odd.