

# Model Theory

October 26, 2018

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## 1 Langauges and structures

**Definition.** (1.1) A language  $L$  consists of:

- (i) a set  $\mathcal{F}$  of function symbols, and for each  $f \in \mathcal{F}$ , a positive integer  $n_f$ , the arity of  $f$ ;
- (ii) a set  $\mathcal{R}$  of relation symbols, and for each  $R \in \mathcal{R}$ , a positive integer  $n_R$ , the arity of  $R$ ;
- (iii) a set  $\mathcal{C}$  of constant symbols.

Note that each of the above three sets can be empty.

**Example.**  $L = \{\{ \cdot, -1 \}, \{ 1 \} \}$  where  $\cdot$  is a binary function,  $-1$  is a unary function, and  $1$  is a constant. We call this  $L_{gp}$  (language of groups);  $L_{lo} = \{ < \}$ , where  $<$  is a binary relation (linear order).

**Definition.** (1.2)

Given a language  $L$ , say, an  $L$ -structure consists of:

- (i) a set  $M$ , the *domain*;
- (ii) for each  $f \in \mathcal{F}$ , a function  $f^M : M^{n_f} \rightarrow M$ ;
- (iii) for each  $R \in \mathcal{R}$ , a relation  $R^M \subseteq M^{n_R}$ ;
- (iv) for each  $c \in \mathcal{C}$ , an element  $c^M \in M$ .

$f^M, R^M, c^M$  are called the *interpretation* of  $f, R, c$  respectively.

**Notation.** (1.3)

We often fail to distinguish between the symbols in the language  $L$  and their interpretations in a  $L$ -structure, if the context allows.

We may write  $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$ .

**Example.** (1.4)

(a)  $\mathcal{R} = \langle \mathbb{R}^+, \{ \cdot, -1 \}, 1 \rangle$  is an  $L_{gp}$ -structure.

$\mathcal{Z} = \langle \mathbb{Z}, \{ +, - \}, 0 \rangle$  is also an  $L_{gp}$ -structure (here  $+$  is a binary and  $-$  is the unary negation function).

$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$  is an  $L_{lo}$  structure ( $<$  is the interpretation of relation).

**Definition.** (1.5)

Let  $L$  be a language, let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures.

An *embedding* of  $\mathcal{M}$  into  $\mathcal{N}$  is an injection  $\alpha : M \rightarrow N$  that preserves the structure:

- (i) For all  $f \in \mathcal{F}$ , and  $a_1, \dots, a_{n_f} \in M$ ,

$$\alpha(f^M(a_1, \dots, a_{n_f})) = f^N(\alpha(a_1), \dots, \alpha(a_{n_f}))$$

- (ii) For all  $R \in \mathcal{R}$ , and  $a_1, \dots, a_{n_R} \in M$ ,

$$(a_1, \dots, a_{n_R}) \in R^M \iff (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if.

- (iii) For all  $c \in \mathcal{C}$ , we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding  $\mathcal{M} \rightarrow \mathcal{N}$  is also called an *isomorphism* of  $\mathcal{M}$  onto  $\mathcal{N}$ .

(1.6) Exercise. Let  $G_1, G_2$  be groups, regarded as  $L_{gp}$ -structures. Check that  $G_1 \cong G_2$  in the usual algebra sense, if and only if there is an isomorphism  $\alpha : G_1 \rightarrow G_2$  in the sense of above definition 1.5.

## 2 Terms, formulae, and their interpretations

In addition to the symbols of  $L$ , we also have:

- (i) infinitely many variables,  $\{x_i\}_{i \in I}$ ;
- (ii) logical connectives,  $\wedge, \neg$  (also express  $\vee, \rightarrow, \leftrightarrow$ );
- (iii) quantifier  $\exists$  (also express  $\forall$ );
- (iv) punctuations  $(, )$ .

**Definition.** (2.1)

$L$ -terms are defined recursively as follows:

- any variable  $x_i$  is a term;
- any constant symbol is a term;
- for any  $f \in \mathcal{F}$ ,

$$f(t_1, \dots, t_{n_f})$$

for any terms  $t_1, \dots, t_{n_f}$  is a term;

- nothing else is a term.

Notation: we write  $t(x_1, \dots, x_n)$  to mean that the variables appearing in  $t$  are among  $x_1, \dots, x_n$ .

**Example.** In  $\mathcal{R} = \langle \mathbb{R}, \cdot, -1, 1 \rangle$ ,

- $(\cdot(x_1, x_2), x_3)$  is a term  $(x_1 \cdot x_2) \cdot x_3$ ;
- $(\cdot(1, x_1))^{-1}$  is a term  $(1 \cdot x)^{-1}$ .

**Definition.** (2.2)

If  $\mathcal{M}$  is an  $L$ -structure, to each  $L$ -term  $t(x_1, \dots, x_k)$  we assign a function

$$t^M : M^k \rightarrow M$$

defined as follows:

- (i) If  $t = x_i$ ,  $t^M[a_1, \dots, a_k] = a_i$ ;
- (ii) If  $t = c$  is a constant,  $t^M[a_1, \dots, a_k] = c^M$ ;
- (iii) If  $t = f(t_1(x_1, \dots, x_k), \dots, t_{n_f}(x_1, \dots, x_k))$ ,

$$t^M(a_1, \dots, a_k) = f^M(t_1^M(a_1, \dots, a_k), \dots, t_{n_f}^M(a_1, \dots, a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on [http](http://users.mct.open.ac.uk/sb27627/MT.html) :

[users.mct.open.ac.uk/sb27627/MT.html](http://users.mct.open.ac.uk/sb27627/MT.html) (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

**Remark.** (The lecture forgot about this last time) Any language  $L$  includes an equality symbol  $=$ .

Last time we assigned a function  $t^m$ . In  $L_{gp}$ , the term  $x_2 \cdot x_3$  can be described as, say  $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), \dots$

Then the term  $x_2 \cdot x_3$  can be assigned to functions  $t_1^M : M^3 \rightarrow M : (a_1, a_2, a_3) \rightarrow (a_2 \cdot a_3)$ , or  $t_2^M : M^4 \rightarrow M : (a_1, a_2, a_3, a_4) \rightarrow (a_2 \cdot a_3)$ . These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term  $t$  to be the number of symbols of  $L$  occurring in  $t$ .

Fact (2.3): Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures, and let  $\alpha : \mathcal{M} \rightarrow \mathcal{N}$  be an embedding. For any  $L$ -term  $t(x_1, \dots, x_k)$  and  $a_1, \dots, a_k \in M$ , we have

$$\alpha(t^M(a_1, \dots, a_k)) = t^N(\alpha(a_1), \dots, \alpha(a_k))$$

*Proof.* Prove by induction on complexity of  $t$ .

Let  $\bar{a} = (a_1, \dots, a_k)$  and  $\bar{x} = (x_1, \dots, x_l)$ . Then:

- (i) if  $t = x_i$  a variable, then  $t^M(\bar{a}) = a_i$ , and  $t^N(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$ , so the conclusion holds;
- (ii) if  $t = c$  is a constant, then  $t^M(\bar{a}) = c^M$ , and  $t^N(\alpha(\bar{a})) = c^N$  by definition of a term. The key here is that, since  $\alpha$  is an embedding we have  $\alpha(c^M) = c^N$ ;
- (iii) if  $t = f(t_1(\bar{x}, \dots, t_{n_f}(\bar{x})))$ , then

$$\alpha(f^M(t_1^M(\bar{a}), \dots, t_{n_f}^M(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), \dots, \alpha(t_{n_f}^M(\bar{a})))$$

as  $\alpha$  is an embedding. But  $t_1(\bar{x}), \dots, t_{n_f}(\bar{x})$  have lower complexity than  $t$ , so the inductive hypothesis applies.  $\square$

Exercise (2.4): conclude the proof of the above fact.  
(Actually is it not done?)

**Definition.** (2.5)

The set of *atomic formulas* of  $L$  is defined as follows:

- (i) if  $t_1, t_2$  are  $L$ -terms, then  $t_1 = t_2$  is an atomic formula;
- (ii) if  $R$  is a relation symbol, and  $t_1, \dots, t_{n_R}$  are  $L$ -terms, then  $R(t_1, \dots, t_{n_R})$  is an atomic formula;
- (iii) nothing else is an atomic formula.

**Definition.** (2.6)

The set of  $L$ -formulas is defined as follows:

- (i) any atomic formula is an  $L$ -formula;
- (ii) if  $\phi$  is an  $L$ -formula, then so is  $\neg\phi$ ;
- (iii) if  $\phi$  and  $\psi$  are  $L$ -formulas, then so is  $\phi \wedge \psi$ ;
- (iv) if  $\phi$  is an  $L$ -formula, for any  $i \geq 1$ ,  $\exists x_i \phi$  is a formula;
- (v) nothing else is a formula (note that  $\forall$  can be constructed by  $\neg$  and  $\exists$ ).

**Example.** In  $L_{gp}$ ,  $x_1 \cdot x_1 = x_2$ , or  $x_1 \cdot x_2 = 1$  are both atomic formulas;  $\exists x_1(x_1 \cdot x_2) = 1$  is an  $L$ -formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier  $\exists$ . We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example,  $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$  is an  $L_{gp}$ -sentence.

Notation:  $\phi(x_1, \dots, x_k)$  means that the free variables in  $\phi$  are among  $x_1, \dots, x_k$ .

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

**Definition.** (2.7)

Let  $\phi(x_1, \dots, x_k)$  be an  $L$ -formula, let  $\mathcal{M}$  be an  $L$ -structure, and let  $\bar{a} = a_1, \dots, a_k$  be elements of  $\mathcal{M}$ .

We define  $\mathcal{M} \models \phi(\bar{a})$  (syntactic implication, read as  $\mathcal{M}$  models  $\phi(\bar{a})$ ) as follows:

- (i) if  $\phi$  is  $t_1 = t_2$ , then  $\mathcal{M} \models \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$ ;
- (ii) if  $\phi$  is  $R(t_1, \dots, t_{n_R})$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff

$$(t_1^M(\bar{a}), \dots, t_{n_R}^M(\bar{a})) \in R^M$$

- (iii) if  $\phi$  is a conjunction, say  $\psi \wedge \chi$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{M} \models \psi(\bar{a})$  and  $\mathcal{M} \models \chi(\bar{a})$ ;
- (iv) if  $\phi$  is  $\exists x_j \chi(x_1, \dots, x_k, x_j)$  (where we'll assume that  $x_j$  is not one of the free variables  $x_1, \dots, x_k$ ), then  $\mathcal{M} \models \phi(\bar{a})$  iff there exists  $b \in \mathcal{M}$  s.t.  $\mathcal{M} \models \chi(a_1, \dots, a_k, b)$ ;
- (v) (lecture forgets this, this should probably be more in front rather than in the end) if  $\phi$  is  $\neg\psi$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{M} \not\models \psi(\bar{a})$ .

**Example.** Consider  $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1 \rangle$ , the multiplicative group of non-negative reals, and suppose we have  $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$ , then  $\mathcal{R} \models \phi(1)$ , but  $\mathcal{R} \not\models \phi(-1)$ .

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):

- $\phi \vee \psi$  for  $\neg(\neg\phi \wedge \neg\psi)$ ;
- $\phi \rightarrow \psi$  for  $\neg\phi \vee \psi$ ;
- $\phi \leftrightarrow \psi$  for  $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ ;
- $\forall x_i \phi$  for  $\neg \exists x_i (\neg\phi)$ .

**Proposition.** (2.9)

Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures, and let  $\alpha : \mathcal{M} \rightarrow \mathcal{N}$  be an embedding.

Let  $\phi(\bar{x})$  be an atomic(!) formula, and  $\bar{a} \in M^{|\bar{x}|}$ , here  $|\bar{x}|$  means the length of the tuple  $\bar{x}$  (from now on, when we write a tuple like  $\bar{a}$ , we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

Question: if  $\phi$  is an  $L$ -formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.

Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.

In definition 2.7 we defined what it means for  $\mathcal{M} \models \phi(\bar{a})$ , in particular we defined: if  $\phi \equiv \neg\chi$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{M} \not\models \chi(\bar{a})$ . Here by  $\mathcal{M} \models \phi(\bar{a})$  we mean  $\mathcal{M} \models \neg\chi(\bar{a})$ , and  $\chi(\bar{a})$  is *shorter* than  $\phi(\bar{a})$ , so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

*Proof.* There are two cases:

- $\phi(\bar{x})$  is of the form  $t_1(\bar{x}) = t_2(\bar{x})$  where  $t_1, t_2$  are terms. Use Fact (2.3). (exercise on example sheet)
- $\phi(\bar{x})$  is of the form  $R(t_1(\bar{x}), \dots, t_{n_R}(\bar{x}))$ . Then  $\mathcal{M} \models R(t_1(\bar{a}), \dots, t_{n_R}(\bar{a}))$  if and only if ... (lecturer says work this out by yourself. Basically the induction step).  $\square$

**Proposition.** (2.10)

Exercise: show that prop (2.9) holds if  $\phi(\bar{x})$  is a formula without quantifiers (a quantifier-free formula).

(I guess that also suggests when does it not hold for general formulas – see below).

**Example.** (2.11, Do embeddings preserve all formulas? No.)

Let  $\mathcal{Z} = (\mathbb{Z}, <)$  an  $L_{lo}$ -structure,  $\mathcal{Q} = (\mathbb{Q}, <)$  also an  $L_{lo}$ -structure. Then

$$\begin{aligned} \alpha : \mathbb{Z} &\rightarrow \mathbb{Q} \\ n &\rightarrow n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3 (x_1 < x_3 \wedge x_3 < x_2)$$

Now  $\mathcal{Q} \models \phi(1, 2)$  but  $\mathcal{Z} \not\models \phi(1, 2)$ .

Fact (2.12) (From now on we'll stop saying that  $\mathcal{M}, \mathcal{N}$  are  $L$ -structures etc to save time) Let  $\alpha : \mathcal{M} \rightarrow \mathcal{N}$  be an isomorphism. Then if  $\phi(\bar{x})$  is an  $L$ -formula, and  $\bar{a} \in M^{|\bar{x}|}$ , then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).



### 3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let  $L$  be a language,  $\mathcal{M}, \mathcal{N}$  be  $L$ -structures.

**Definition.** (3.1)

An  $L$ -theory  $T$  is a set of  $L$ -sentences.

$\mathcal{M}$  is a *model* of  $T$  if  $\mathcal{M} \models \sigma$  for all  $\sigma \in T$ . We write  $\mathcal{M} \models T$ .

The class of all the models of  $T$  is written  $Mod(T)$ .

The *theory of*  $\mathcal{M}$  is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L\text{-sentence and } \mathcal{M} \models \sigma\}$$

**Example.** (3.2)

Let  $T_{gp}$  be the set of  $L_{gp}$ -sentences:

(i)  $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$ ;

(ii)  $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$ ;

(iii)  $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$ .

Clearly, for a group  $G$ ,  $G \models T_{gp}$  (as they are just the group axioms). However, for a specific group  $G$ , clearly the theory of it,  $Th(G)$  is larger than  $T_{gp}$ .

**Definition.** (3.3)

$\mathcal{M}$  and  $\mathcal{N}$  are *elementarily equivalent* if  $Th(\mathcal{M}) = Th(\mathcal{N})$ .

We write  $\mathcal{M} \equiv \mathcal{N}$ .

Clearly, if  $\mathcal{M} \simeq \mathcal{N}$  ( $\simeq$  means isomorphism), then  $\mathcal{M} \equiv \mathcal{N}$ .

But if  $\mathcal{M}$  and  $\mathcal{N}$  are not isomorphic, establishing whether  $\mathcal{M} \equiv \mathcal{N}$  can be highly non-trivial!

We'll see  $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$  as  $L_{lo}$ -structures(!).

**Definition.** (3.4)

(i) An embedding  $\beta : \mathcal{M} \rightarrow \mathcal{N}$  is *elementary* if for all formulas  $\phi(\bar{x})$  and  $\bar{a} \in M^{|\bar{x}|}$ ,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\beta(\bar{a}))$$

(ii) If  $M \subseteq N$ , and  $id : \mathcal{M} \rightarrow \mathcal{N}$  is an embedding, then  $\mathcal{M}$  is a *substructure* of  $\mathcal{N}$ .

(iii) If  $M \subseteq N$  and  $id : \mathcal{M} \rightarrow \mathcal{N}$  is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then  $\mathcal{M}$  is said to be an *elementary substructure* of  $\mathcal{N}$ , written as  $\mathcal{M} \preceq \mathcal{N}$ .

**Example.** (3.5)

Let  $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$ , an  $L_{lo}$ -structure where  $<$  is the usual order;

Let  $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$ , also an  $L_{lo}$ -structure with the same  $<$ .

Then  $\mathcal{M} \simeq \mathcal{N}$  as  $L_{lo}$ -structures. So  $\mathcal{M} \equiv \mathcal{N}$  (since they are isomorphic).

Also,  $\mathcal{M} \subseteq \mathcal{N}$  (read as *is a substructure of*), since the ordering  $<$  coincides on  $\mathcal{M}$  and  $\mathcal{N}$ . However,  $\mathcal{M} \not\preceq \mathcal{N}$ , since if we pick the formula  $\phi(x) \equiv \exists y (x < y)$ , then  $\mathcal{N} \models \phi(1)$ , but  $\mathcal{M} \not\models \phi(1)$ .

**Definition.** (3.6)

Let  $\mathcal{M}$  be an  $L$ -structure,  $A \subseteq M$ , then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where  $c_a$  are constant symbols). An interpretation of  $\mathcal{M}$  as an  $L$ -structure extends to an interpretation of  $\mathcal{M}$  as an  $L(A)$ -structure in the obvious way, i.e.  $c_a^M = a$ .

In this context, the elements of  $A$  are called *parameters*.

If  $\mathcal{M}$  and  $\mathcal{N}$  are two structures, and  $A \subseteq M \cap N$ , then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean  $\mathcal{M}, \mathcal{N}$  satisfy exactly the same  $L(A)$  structures.

—Lecture 4—

Reminder: we have a lecture next Monday (22nd Oct)!

**Proposition.** It turns out that,  $\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$  (where  $M$  is the domain of  $\mathcal{M}$ ).

**Lemma.** (3.8, Tarski-Vaught test)

Let  $\mathcal{N}$  be an  $L$ -structure, let  $A \subseteq N$ . The following are equivalent:

- (i)  $A$  is the domain of a structure  $\mathcal{M}$  s.t.  $\mathcal{M} \preceq \mathcal{N}$ ;
- (ii) if  $\phi(x) \in L(A)$  (with an abuse of notations  $\phi(x, c_{a_1}, \dots, c_{a_n}) = \phi(x, a_1, \dots, a_n)$ ), if  $\mathcal{N} \models \exists x \phi(x)$ , then  $\mathcal{N} \models \phi(b)$  for some  $b \in A$ .

*Proof.* (i)  $\implies$  (ii): Suppose  $\mathcal{N} \models \exists x \phi(x)$ . Then by elementarity,  $\mathcal{M} \models \exists x \phi(x)$ , and so  $\mathcal{M} \models \phi(b)$  for  $b \in M$ . So (again by elementarity),  $\mathcal{N} \models \phi(b)$ .

(ii)  $\implies$  (i): This is the harder direction. First we prove that  $A$  is the domain of a substructure  $\mathcal{M} \subseteq \mathcal{N}$ .

By Sheet 1 Q4, it suffices to check:

- (a) For each constant  $c$ ,  $c^N \in A$ ;
- (b) For each function symbol  $f$ ,  $f^N(\bar{a}) \in A$  (for all  $\bar{a} \in A^{n_R}$ );

For (a), use property (ii) with  $\exists x(x = c)$ .

For (b), use property (ii) with the formula  $\exists x((\bar{a} = x))$ .

So we now have  $\mathcal{M} \subseteq \mathcal{N}$ , and domain of  $\mathcal{M}$  is  $A$ . But we actually want to prove that  $\mathcal{M} \preceq \mathcal{N}$ . Now let  $\chi(\bar{x})$  be an  $L$ -formula.

We want to show that for  $\bar{a} \in A^{|\bar{x}|}$   $\mathcal{M} \models \chi(\bar{a}) \iff \mathcal{N} \models \chi(\bar{a})$  (\*).

By induction on the complexity of  $\chi(\bar{x})$ :

- if  $\chi(\bar{x})$  is atomic, (\*) follows from  $\mathcal{M} \subseteq \mathcal{N}$  (since  $\mathcal{M}$  is a substructure!);
- if  $\chi(\bar{x})$  is  $\neg\psi(\bar{x})$  or  $\chi(\bar{x})$  is  $\psi(\bar{x}) \wedge \xi(\bar{x})$ , it's a straightforward induction;
- (interesting case) if  $\chi(\bar{x}) = \exists y \psi(\bar{x}, y)$  where  $\psi(\bar{x}, y)$  is an  $L$ -formula, suppose that  $\mathcal{M} \models \chi(\bar{a})$ , then  $\mathcal{M} \models \exists y \psi(\bar{a}, y)$ , hence  $\mathcal{M} \models \psi(\bar{a}, b)$  for some  $b \in A = \text{dom}(\mathcal{M})$  (this is the definition of truth).

But then  $\mathcal{N} \models \psi(\bar{a}, b)$  by inductive hypothesis, so  $\mathcal{N} \models \chi(\bar{a})$ .

Now let  $\mathcal{N} \models \chi(\bar{a})$ , i.e.  $\mathcal{N} \models \exists y \psi(\bar{a}, y)$  (we find a *witness* for it). By property (ii),  $\mathcal{N} \models \psi(\bar{a}, b)$  for some  $b \in A = \text{dom}(\mathcal{M})$ .

Again by inductive hypothesis, we have  $\mathcal{M} \models \psi(\bar{a}, b)$ , and so in particular,  $\mathcal{M} \models \chi(\bar{a})$  as it has got a witness there.  $\square$

**Remark.** (3.9)

Even more assumptions: let's assume that the set of variables is countably infinite. Then:

- the cardinality of the set of  $L$ -formulas is  $|L| + \omega$  (where by  $|L|$  we mean

the number of symbols. For example,  $|L_{gp}| = 3$ ,  $|L_{lo}| = 1$ ), where we abuse another notation that we use  $\omega$  as cardinals (rather than ordinals) (note that the formulas are just strings of finite length);

- if  $A$  is a set of parameters in some structure, the cardinality of the set  $L(A)$  is  $|A| + |L| + \omega$ , where by  $+$  here we merely mean  $\max\{|L|, |A|, \omega\}$  (instead of addition), and same for the  $+$  above.

**Definition.** (3.10)

Let  $\lambda$  be an ordinal. Then a *chain of length  $\lambda$*  of sets is a sequence  $\langle M_i : i < \lambda \rangle$ , where  $M_i \subseteq M_j$  for all  $i \leq j < \lambda$ .

A chain of  $L$ -structures is a sequence:  $\langle \mathcal{M}_i : i < \lambda \rangle$  s.t.  $\mathcal{M}_i \subseteq \mathcal{M}_j$  (note that it's substructure here) for  $i \leq j < \lambda$ .

The *union* of this chain is the  $L$ -structure  $\mathcal{M}$  defined as follows:

- the domain is  $\bigcup_{i < \lambda} M_i$  (when you think of this, you can always start with the case  $\lambda = \omega$ );
- for constants  $c$ ,  $c^{\mathcal{M}} = c^{\mathcal{M}_i}$  for any  $i < \lambda$  (this is well defined, because of the substructure condition above);
- if  $f$  is a function symbol,  $\bar{a} \in M^{|\bar{a}|}$  (why the mod sign here),  $f^{\mathcal{M}} \bar{a} = f^{\mathcal{M}_i} \bar{a}$  where  $i$  is s.t.  $\bar{a} \in M_i^{|\bar{a}|}$ ;
- if  $R$  is a relation symbol, then  $R^{\mathcal{M}} = \bigcup_{i < \lambda} R^{\mathcal{M}_i}$ .

**Theorem.** (3.11, Downward Löwenheim-Skolem theorem)

(Recall that in part II Logic and Set Theory we had the countable version of this)

Let  $\mathcal{N}$  be an  $L$ -structure, and  $|\mathcal{N}| \geq |L| + \omega$ . Let  $A \subseteq N$ . Then for every cardinal  $\lambda$  s.t.  $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$ , there is  $\mathcal{M} \preceq \mathcal{N}$  s.t.

- $A \subseteq M$ ;
- $|\mathcal{M}| = \lambda$ .



(It helps to think about the case  $|A| = \omega$  and  $|N|$  is uncountable.)

A quick example how this could be useful (we'll go very sloppy here): think of  $(\mathbb{C}, +, \cdot, -, \cdot^{-1}, 0, 1)$  as a field. Consider  $\mathbb{Q} \subseteq \mathbb{C}$  (both as subset and substructure). Note that algebraic closeness is a property of  $\mathbb{C}$ . By downward Löwenheim-

Skolem, there is a substructure in  $\mathcal{C}$  that contains  $\mathbb{Q}$  that is also algebraically closed (apparently, the set of algebraic numbers).

*Proof.* We build a chain  $\langle A_i : i < \lambda \rangle$ , with  $A_i \subseteq N$ , s.t.  $|A_i| = \lambda$ .

(our goal: define an elementary substructure with domain  $M = \bigcup_{i < \omega} A_i$ ).

Base case: Let  $A_0 \subseteq N$  be such that  $A \subseteq A_0$  and  $|A_0| = \lambda$ .

Successors: At stage  $i + 1$ , assume  $A_i$  has been built, with  $|A_i| = \lambda$ .

Let  $\langle \phi_k(x) : k < \lambda \rangle$  be an enumeration of those  $L(A_i)$ -formulas such that  $\mathcal{N} \models \exists x \phi_k(x)$ . Let  $a_k$  be such that  $\mathcal{N} \models \phi_k(a_k)$ , and let  $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$  (basically, with those witnesses added). Then  $|A_{i+1}| = \lambda$  (note that we haven't increased the size).

Now let  $M = \bigcup_{i < \omega} A_i$  (note the subscript range). We use lemma (3.8) to show that  $M$  is the domain of  $\mathcal{M} \preceq \mathcal{N}$ , and  $|M| = \lambda$ . We're running out of time, so we'll continue next Monday.

—Lecture 5—

Solutions to worksheet 1: either take along to lecture on Friday, or email them to [silvia.barbina@open.ac.uk](mailto:silvia.barbina@open.ac.uk).

Let's continue with the proof:



Start with  $A_0 \subset N$ ,  $A \subseteq A_0$ ,  $|A_0| = \lambda$ . The idea is to define  $\langle A_i : i < \omega \rangle$  so that  $M = \bigcup_{i < \omega} A_i$  satisfies (ii) via the TV test (3.8).

List all formulas  $\phi(x, \bar{a})$  ( $\bar{a}$  is a tuple in  $A_0$ ), and  $\mathcal{N} \models \phi(b, \bar{a})$  for some  $b$ .

Add each such  $b$  to  $A_0$  (one for each such  $\phi$ ).

Let  $A_1 = A_0 \cup \{ \text{all these } b\text{'s} \}$ .

Repeat for formulas  $\phi(x, \bar{a})$  where  $\bar{a}$  is in  $A_1, \dots$

Eventually,  $\langle A_i : i < \omega \rangle$  is such that  $M = \bigcup_{i < \omega} A_i$  is as required (i.e.  $M$  is the domain of some elementary substructure of  $\mathcal{N}$  that we need).

We claim that  $M$  satisfies condition (ii) in Lemma (3.8): Let  $\mathcal{N} \models \exists x \psi(x, \bar{a})$ , where  $\bar{a}$  is a tuple in  $M$ . Then  $\bar{a}$  is a *finite* tuple, so there is an  $i$  s.t.  $\bar{a}$  is in  $A_i$ .

Then  $A_{i+1}$ , by construction, contains  $b$  s.t.  $\mathcal{N} \models \phi(b, \bar{a})$ . But  $A_{i+1} \subseteq M, b \in M$ . Then apply (3.8) we're done.  $\square$

## 4 Two relational structures

**Definition.** (4.1, dense linear orders)

A *linear order* is an  $L_{lo} = \{<\}$ -structure such that:

- (i)  $\forall x \neg(x < x)$ ;
- (ii)  $\forall xyz((x < y \wedge y < z) \rightarrow x < z)$ ;
- (iii)  $\forall xy((x < y) \vee (y < x) \vee x = y)$  (total).

A linear order is *dense* if, in addition, it also satisfies:

- (iv)  $\exists xy(x < y)$ ;
- (v)  $\forall xy, (x < y \rightarrow \exists z(x < z \wedge z < y))$  (density).

A linear order has no endpoints if, in addition,

- (vi)  $\forall x(\exists y(x < y) \wedge \exists z(z < x))$ .

We use  $T_{dlo}$  to denote the theory that includes all axioms (i) to (vi), and  $T_{lo}$  is the theory that includes axioms (i) to (iii) only.

**Remark.** (iv) and (v) imply that if  $\mathcal{M} \models T_{dlo}$ , then  $|\mathcal{M}| \geq \omega$ .

**Definition.** (4.2)

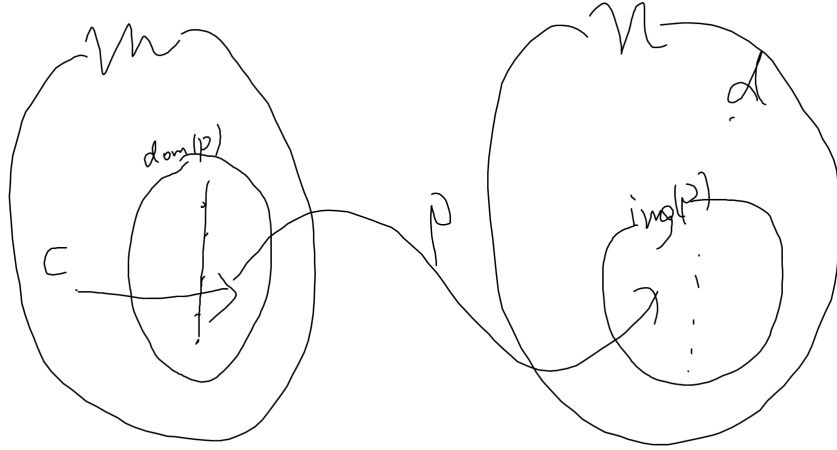
If  $\mathcal{M}, \mathcal{N} \models T_{lo}$ , then an *injective* map  $p : A \subseteq M \rightarrow N$  is a *partial embedding* if  $\mathcal{M} \models a < b \implies \mathcal{N} \models p(a) < p(b)$ .

In particular, if  $|\text{dom}(p)| < \omega$ , then  $p$  is a *finite* partial embedding.

**Lemma.** (4.3, extension lemma)

Take a linear order  $\mathcal{M} \models T_{lo}$ , and a dense linear endpoints  $\mathcal{N} \models T_{dlo}$ , and let  $p : M \rightarrow N$  be a finite partial embedding. Then if  $c \in \mathcal{M}$ , there is a finite partial embedding  $\hat{p}$  s.t.  $p \subseteq \hat{p}$  and  $c \in \text{dom}(\hat{p})$ .

(we can always add one extra element in our embedding.)



*Proof.*

Case 1:  $c$  is greater than all elements in  $\text{dom}(p)$ . In that case, pick an element  $d \in \mathcal{N}$  s.t.  $d > b$  for all  $b \in \text{img}(p)$ ;

Case 2:  $a_i < c < a_{i+1}$  where  $a_i, a_{i+1} \in \text{dom}(p)$ . Then we choose  $\mathcal{N} \models p(a_i) < d < p(a_{i+1})$ , where  $d$  is chosen appropriately by density (here's the case why we

need  $\mathcal{N}$  to be dense;

Case 3:  $c$  is less than all elements in  $\text{dom}(p)$ . This is similar to case 1.

Note that the ability to extend by one point allows us to embed any finite linear order into a dense linear order without endpoints.  $\square$

**Theorem.** (4.4)

Let  $\mathcal{M}, \mathcal{N} \models T_{dlo}$  s.t.  $|\mathcal{M}| = |\mathcal{N}| = \omega$ . Let  $p : A \subseteq M \rightarrow N$  be a finite partial embedding.

Then there is an isomorphism  $\pi : \mathcal{M} \rightarrow \mathcal{N}$  s.t.  $p \subseteq \pi$ .

*Proof.* Enumerate  $M, N$ , say  $M = \langle a_i : i < \omega \rangle$ ,  $N = \langle b_i : i < \omega \rangle$  (sequences of elements).

We define, inductively, a chain of finite partial embedding  $\langle p_i : i < \omega \rangle$  (idea:  $\pi = \bigcup_{i < \omega} p_i$ ).

Let's start with  $p_0 = p$ . At stage  $i + 1$ , suppose we are given  $p_i$ . We want to include  $a_i$  in  $\text{dom } p_{i+1}$ , and  $b_i$  in the  $\text{img}(p_{i+1})$ .

(Lecturer calls this a *back and forth* method) Forth step: By lemma 4.3, we can extend  $p_i$  to  $p_{i+\frac{1}{2}}$  such that  $a_i \in \text{dom}(p_{i+\frac{1}{2}})$ ;

Back step: By lemma 4.3 again applied to  $(p_{i+\frac{1}{2}})^{-1}$  to include  $b_i \in \text{dom}(p_{i+1}^{-1})$  (i.e. in the range of  $p_{i+1}$ ).

We claim that  $p_{i+1}$  extends  $p_i$  as required.

Let  $\pi = \bigcup_{i < \omega} p_i$ . Then (check)  $\pi$  is an isomorphism (i.e. order-preserving bijection).  $\square$

**Definition.** (4.5)

An  $L$ -theory is *consistent* if there is  $L$ -structure  $\mathcal{M}$  s.t.  $\mathcal{M} \models T$ .

If  $T$  is a theory in  $L$  and  $\phi$  is an  $L$ -sentence, then  $T \vdash \phi$  (read as  $T$  entails  $\phi$ , note that this has nothing to do with syntactic implication) if for all  $\mathcal{M}$  such that  $\mathcal{M} \models T$ , we have  $\mathcal{M} \models \phi$ .

Finally, an  $L$ -theory  $T$  is *complete* if for all  $L$ -sentences  $\phi$ , either  $T \vdash \phi$  or  $T \vdash \neg\phi$  (see part II Logic and Set Theory).

For example,  $T_{dlo}$  is complete.

—Lecture 6—

**Definition.** (4.6)

A theory  $T$  in a countable language with a (infinitely) countable model is  $\omega$ -categorical if any two countable models of  $T$  are isomorphic.

**Corollary.** (4.7 of theorem (4.4))

$T_{dlo}$  is  $\omega$ -categorical.

*Proof.* If  $\mathcal{M}, \mathcal{N} \models T_{dlo}$ ,  $|\mathcal{M}| = |\mathcal{N}| = \omega$ , then  $\phi$  (the empty map) is a finite partial embedding. But by theorem (4.4) we get  $\mathcal{M} \simeq \mathcal{N}$ .

(We can also use any  $\{\langle a, b \rangle\}$  where  $a \in \mathcal{M}$  and  $b \in \mathcal{M}$  as initial finite partial embedding).  $\square$

**Theorem.** (4.8)

If  $T$  is an  $\omega$ -categorical theory in a countable language, then  $T$  is complete.

*Proof.* Let  $\mathcal{M} \models T$  and  $\phi$  be an  $L$ -sentence.

If  $\mathcal{M} \models \phi$ , suppose  $\mathcal{N} \models T$ . Then by theorem (3.11) (Downward Lowenheim-Skolem), there are  $\mathcal{M}' \preceq \mathcal{M}$ ,  $\mathcal{N}' \preceq \mathcal{N}$  s.t.  $|\mathcal{M}'| = |\mathcal{N}'| = \omega$ .

But  $\mathcal{M}' \simeq \mathcal{N}'$  (by  $\omega$ -categoricity), so in particular  $\mathcal{M}' \equiv \mathcal{N}'$ , and so  $\mathcal{N}' \models \phi$ . By elementarity,  $\mathcal{N} \models \phi$ .

The case  $\mathcal{M} \models \neg\phi$  is similar.

(Think about if  $T$  could have a finite model.) □

**Corollary.** (4.9)

$T_{dlo}$  is complete.

**Definition.** (4.10)

If  $\mathcal{M}, \mathcal{N}$  are  $L$ -structures, a map  $f$  such that  $\text{dom}(f) \subseteq M$  (the domain of  $\mathcal{M}$ ), and  $\text{img}(f) \subseteq N$  is a (partial) *elementary map* if for all  $L$ -formulas  $\phi(\bar{x})$  and  $\bar{a} \in (\text{dom}(f))^{|\bar{x}|}$ , then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(f(\bar{a}))$$

**Remark.** (4.11)

A map  $f$  is elementary iff every finite restriction of  $f$  is elementary.

(Why? For forward, if  $f_0 \subseteq f$  is a finite restriction that is not elementary, then for some formula  $\phi(\bar{x})$ ,  $\bar{a} \in \text{dom}(f_0)$ , the above equivalence doesn't hold; but then that equivalence doesn't hold for  $f$  either; contradiction; for backward, if  $f$  is not elementary, then the above equivalence fails on a finite tuple, so the above equivalence fails on some finite restriction.)

**Proposition.** (4.12)

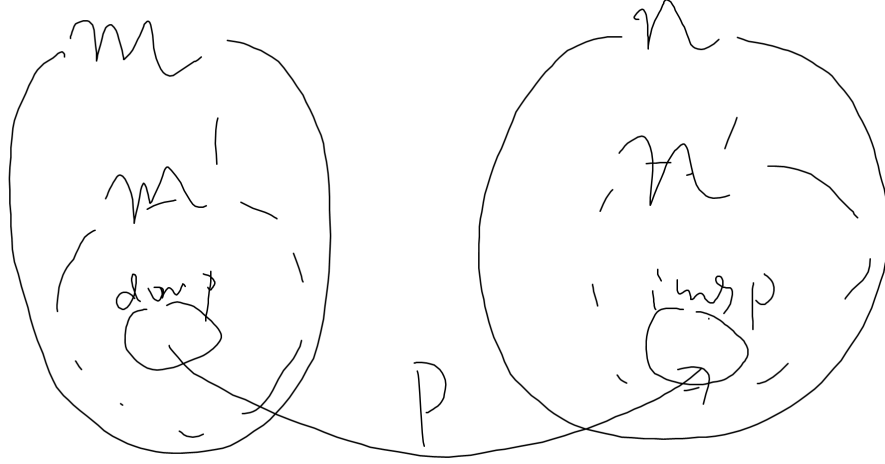
Let  $\mathcal{M}, \mathcal{N} \models T_{dlo}$ , and let  $p : A \subseteq M \rightarrow N$  be a partial embedding. Then  $p$  is elementary.

*Proof.* By remark (4.11), it suffices to consider  $p$  finite.

By Downward L-S theoem (3.11), we choose  $\mathcal{M}', \mathcal{N}'$  such that

- (i)  $|\mathcal{M}'| = |\mathcal{N}'| = \omega$ ;
- (ii)  $\mathcal{M}' \preceq \mathcal{M}$ ,  $\mathcal{N}' \preceq \mathcal{N}$ ;
- (iii)  $\text{dom}(p) \subseteq M'$ ,  $\text{img}(p) \subseteq N'$ .





Now  $p$  is a finite partial embedding between countable models, so  $p$  extends to an isomorphism  $\pi : \mathcal{M}' \rightarrow \mathcal{N}'$ .

In particular,  $\pi$  is an elementary map between  $\mathcal{M}$  and  $\mathcal{N}$ . □

**Corollary.** (4.13)

$(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$ .

*Proof.* Use proposition (4.12) with  $id : \mathbb{Q} \rightarrow \mathbb{R}$ . □

**Definition.** (4.14)

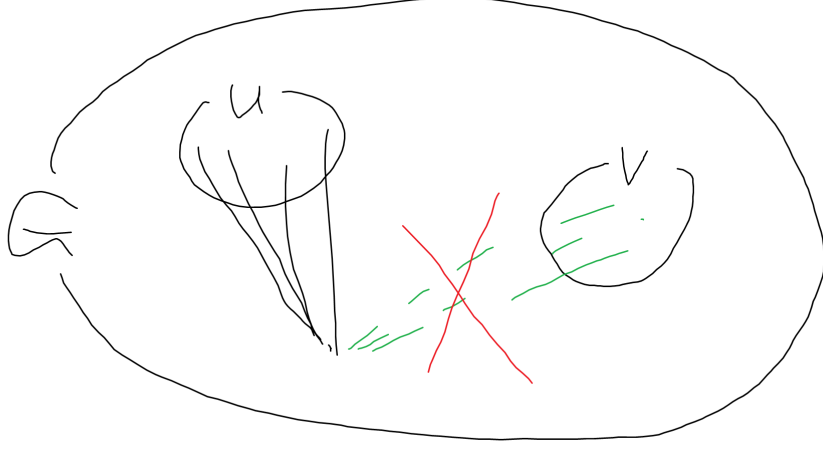
(See Part II Logic and Set Theory)

Let  $L_{gph} = \{R\}$ , where  $R$  is a binary relation symbol.

An  $L_{gph}$ -structure is a graph if

- (i)  $\forall x \neg R(x, x)$ ;
- (ii)  $\forall xy (R(x, y) \leftrightarrow R(y, x))$ .

An  $L_{gph}$ -structure is a **random graph** if it is a graph such that the following axiom-schema  $(r_n)$  hold:



$$\forall x_0 \dots x_n, y_0 \dots y_n, \left( \bigwedge_{i,j=0}^n x_i \neq y_j \rightarrow \exists z \left( \bigwedge_{i=0}^n (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

(iii)  $\exists xy (x \neq y)$ .

**Remark.** (similar to what is mentioned in the link above)

A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

**Fact.** (4.15)

There is a random graph.

*Proof.* Let the domain be  $\omega$ , let  $i, j \in \omega$  such that  $i < j$ . Write  $j$  as a sum of distinct powers of 2. Then  $\{i, j\}$  is an edge iff  $2^i$  appears in the sum.

As an exercise, prove that  $\omega$  with this definition of  $R$  is indeed a random graph.  $\square$

**Definition.** (4.16, or more precisely just notations)

$T_{gph} = \{\text{axioms (i), (ii)}\}$ ,  $T_{rg} = T_{gph} \cup \{(iii), (r_n) : n \in \omega\}$ .

If  $\mathcal{M}, \mathcal{N} \models T_{gph}$ , a partial embedding is an injection  $p : A \subseteq M \rightarrow N$  s.t.  $\mathcal{M} \models R(p(a), p(b)) \iff \mathcal{N} \models R(a, b)$  for all  $a, b$  in the domain.

**Lemma.** (4.17)

Let  $\mathcal{M} \models T_{gph}, \mathcal{N} \models T_{rg}$ , let  $p : A \subseteq M \rightarrow N$  be a finite partial embedding, and let  $c \in M$ .

Then there is a map  $\hat{p} : \hat{A} \subseteq M \rightarrow N$  such that  $\hat{p}$  is a partial embedding,  $c \in \text{dom } \hat{p}$ ,  $p \subseteq \hat{p}$ .

(This is like another extension lemma.)

We'll prove this next time.