

Model Theory

November 9, 2018

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1 Langauges and structures

Definition. (1.1) A language L consists of:

- (i) a set \mathcal{F} of function symbols, and for each $f \in \mathcal{F}$, a positive integer n_f , the arity of f ;
- (ii) a set \mathcal{R} of relation symbols, and for each $R \in \mathcal{R}$, a positive integer n_R , the arity of R ;
- (iii) a set \mathcal{C} of constant symbols.

Note that each of the above three sets can be empty.

Example. $L = \{\{\cdot, -1\}, \{1\}\}$ where \cdot is a binary function, -1 is a unary function, and 1 is a constant. We call this L_{gp} (language of groups); $L_{lo} = \{<\}$, where $<$ is a binary relation (linear order).

Definition. (1.2)

Given a language L , say, an L -structure consists of:

- (i) a set M , the *domain*;
- (ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \rightarrow M$;
- (iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$;
- (iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the *interpretation* of f, R, c respectively.

Notation. (1.3)

We often fail to distinguish between the symbols in the language L and their interpretations in a L -structure, if the context allows.

We may write $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$.

Example. (1.4)

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, -1\}, 1 \rangle$ is an L_{gp} -structure.

$\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is also an L_{gp} -structure (here $+$ is a binary and $-$ is the unary negation function).

$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo} structure ($<$ is the interpretation of relation).

Definition. (1.5)

Let L be a language, let \mathcal{M} and \mathcal{N} be L -structures.

An *embedding* of \mathcal{M} into \mathcal{N} is an injection $\alpha : M \rightarrow N$ that preserves the structure:

- (i) For all $f \in \mathcal{F}$, and $a_1, \dots, a_{n_f} \in M$,

$$\alpha(f^M(a_1, \dots, a_{n_f})) = f^N(\alpha(a_1), \dots, \alpha(a_{n_f}))$$

- (ii) For all $R \in \mathcal{R}$, and $a_1, \dots, a_{n_R} \in M$,

$$(a_1, \dots, a_{n_R}) \in R^M \iff (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if.

- (iii) For all $c \in \mathcal{C}$, we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding $\mathcal{M} \rightarrow \mathcal{N}$ is also called an *isomorphism* of \mathcal{M} onto \mathcal{N} .

(1.6) Exercise. Let G_1, G_2 be groups, regarded as L_{gp} -structures. Check that $G_1 \cong G_2$ in the usual algebra sense, if and only if there is an isomorphism $\alpha : G_1 \rightarrow G_2$ in the sense of above definition 1.5.

2 Terms, formulae, and their interpretations

In addition to the symbols of L , we also have:

- (i) infinitely many variables, $\{x_i\}_{i \in I}$;
- (ii) logical connectives, \wedge, \neg (also express $\vee, \rightarrow, \leftrightarrow$);
- (iii) quantifier \exists (also express \forall);
- (iv) punctuations $(,)$.

Definition. (2.1)

L -terms are defined recursively as follows:

- any variable x_i is a term;
- any constant symbol is a term;
- for any $f \in \mathcal{F}$,

$$f(t_1, \dots, t_{n_f})$$

for any terms t_1, \dots, t_{n_f} is a term;

- nothing else is a term.

Notation: we write $t(x_1, \dots, x_n)$ to mean that the variables appearing in t are among x_1, \dots, x_n .

Example. In $\mathcal{R} = \langle \mathbb{R}, \cdot, -1, 1 \rangle$,

- $(\cdot(x_1, x_2), x_3)$ is a term $(x_1 \cdot x_2) \cdot x_3$;
- $(\cdot(1, x_1))^{-1}$ is a term $(1 \cdot x)^{-1}$.

Definition. (2.2)

If \mathcal{M} is an L -structure, to each L -term $t(x_1, \dots, x_k)$ we assign a function

$$t^M : M^k \rightarrow M$$

defined as follows:

- (i) If $t = x_i$, $t^M[a_1, \dots, a_k] = a_i$;
- (ii) If $t = c$ is a constant, $t^M[a_1, \dots, a_k] = c^M$;
- (iii) If $t = f(t_1(x_1, \dots, x_k), \dots, t_{n_f}(x_1, \dots, x_k))$,

$$t^M(a_1, \dots, a_k) = f^M(t_1^M(a_1, \dots, a_k), \dots, t_{n_f}^M(a_1, \dots, a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on [http](http://users.mct.open.ac.uk/sb27627/MT.html) :

users.mct.open.ac.uk/sb27627/MT.html (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

Remark. (The lecture forgot about this last time) Any language L includes an equality symbol $=$.

Last time we assigned a function t^m . In L_{gp} , the term $x_2 \cdot x_3$ can be described as, say $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), \dots$

Then the term $x_2 \cdot x_3$ can be assigned to functions $t_1^M : M^3 \rightarrow M : (a_1, a_2, a_3) \rightarrow (a_2 \cdot a_3)$, or $t_2^M : M^4 \rightarrow M : (a_1, a_2, a_3, a_4) \rightarrow (a_2 \cdot a_3)$. These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term t to be the number of symbols of L occurring in t .

Fact (2.3): Let \mathcal{M} and \mathcal{N} be L -structures, and let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an embedding. For any L -term $t(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in M$, we have

$$\alpha(t^M(a_1, \dots, a_k)) = t^N(\alpha(a_1), \dots, \alpha(a_k))$$

Proof. Prove by induction on complexity of t .

Let $\bar{a} = (a_1, \dots, a_k)$ and $\bar{x} = (x_1, \dots, x_l)$. Then:

- (i) if $t = x_i$ a variable, then $t^M(\bar{a}) = a_i$, and $t^N(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds;
- (ii) if $t = c$ is a constant, then $t^M(\bar{a}) = c^M$, and $t^N(\alpha(\bar{a})) = c^N$ by definition of a term. The key here is that, since α is an embedding we have $\alpha(c^M) = c^N$;
- (iii) if $t = f(t_1(\bar{x}), \dots, t_{n_f}(\bar{x}))$, then

$$\alpha(f^M(t_1^M(\bar{a}), \dots, t_{n_f}^M(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), \dots, \alpha(t_{n_f}^M(\bar{a})))$$

as α is an embedding. But $t_1(\bar{x}), \dots, t_{n_f}(\bar{x})$ have lower complexity than t , so the inductive hypothesis applies. \square

Exercise (2.4): conclude the proof of the above fact.
(Actually is it not done?)

Definition. (2.5)

The set of *atomic formulas* of L is defined as follows:

- (i) if t_1, t_2 are L -terms, then $t_1 = t_2$ is an atomic formula;
- (ii) if R is a relation symbol, and t_1, \dots, t_{n_R} are L -terms, then $R(t_1, \dots, t_{n_R})$ is an atomic formula;
- (iii) nothing else is an atomic formula.

Definition. (2.6)

The set of L -formulas is defined as follows:

- (i) any atomic formula is an L -formula;
- (ii) if ϕ is an L -formula, then so is $\neg\phi$;
- (iii) if ϕ and ψ are L -formulas, then so is $\phi \wedge \psi$;
- (iv) if ϕ is an L -formula, for any $i \geq 1$, $\exists x_i \phi$ is a formula;
- (v) nothing else is a formula (note that \forall can be constructed by \neg and \exists).

Example. In L_{gp} , $x_1 \cdot x_1 = x_2$, or $x_1 \cdot x_2 = 1$ are both atomic formulas; $\exists x_1(x_1 \cdot x_2) = 1$ is an L -formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier \exists . We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example, $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp} -sentence.

Notation: $\phi(x_1, \dots, x_k)$ means that the free variables in ϕ are among x_1, \dots, x_k .

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

Definition. (2.7)

Let $\phi(x_1, \dots, x_k)$ be an L -formula, let \mathcal{M} be an L -structure, and let $\bar{a} = a_1, \dots, a_k$ be elements of \mathcal{M} .

We define $\mathcal{M} \models \phi(\bar{a})$ (syntactic implication, read as \mathcal{M} models $\phi(\bar{a})$) as follows:

- (i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$;
- (ii) if ϕ is $R(t_1, \dots, t_{n_R})$, then $\mathcal{M} \models \phi(\bar{a})$ iff

$$(t_1^M(\bar{a}), \dots, t_{n_R}^M(\bar{a})) \in R^M$$

- (iii) if ϕ is a conjunction, say $\psi \wedge \chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \chi(\bar{a})$;
- (iv) if ϕ is $\exists x_j \chi(x_1, \dots, x_k, x_j)$ (where we'll assume that x_j is not one of the free variables x_1, \dots, x_k), then $\mathcal{M} \models \phi(\bar{a})$ iff there exists $b \in \mathcal{M}$ s.t. $\mathcal{M} \models \chi(a_1, \dots, a_k, b)$;
- (v) (lecture forgets this, this should probably be more in front rather than in the end) if ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \psi(\bar{a})$.

Example. Consider $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1 \rangle$, the multiplicative group of non-negative reals, and suppose we have $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$, then $\mathcal{R} \models \phi(1)$, but $\mathcal{R} \not\models \phi(-1)$.

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):

- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$;
- $\phi \rightarrow \psi$ for $\neg\phi \vee \psi$;
- $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$;
- $\forall x_i \phi$ for $\neg \exists x_i (\neg\phi)$.

Proposition. (2.9)

Let \mathcal{M} and \mathcal{N} be L -structures, and let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an embedding.

Let $\phi(\bar{x})$ be an atomic(!) formula, and $\bar{a} \in M^{|\bar{x}|}$, here $|\bar{x}|$ means the length of the tuple \bar{x} (from now on, when we write a tuple like \bar{a} , we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

Question: if ϕ is an L -formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.

Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.

In definition 2.7 we defined what it means for $\mathcal{M} \models \phi(\bar{a})$, in particular we defined: if $\phi \equiv \neg\chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \chi(\bar{a})$. Here by $\mathcal{M} \models \phi(\bar{a})$ we mean $\mathcal{M} \models \neg\chi(\bar{a})$, and $\chi(\bar{a})$ is *shorter* than $\phi(\bar{a})$, so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

Proof. There are two cases:

- $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. Use Fact (2.3). (exercise on example sheet)
- $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), \dots, t_{n_R}(\bar{x}))$. Then $\mathcal{M} \models R(t_1(\bar{a}), \dots, t_{n_R}(\bar{a}))$ if and only if ... (lecturer says work this out by yourself. Basically the induction step). \square

Proposition. (2.10)

Exercise: show that prop (2.9) holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

(I guess that also suggests when does it not hold for general formulas – see below).

Example. (2.11, Do embeddings preserve all formulas? No.)

Let $\mathcal{Z} = (\mathbb{Z}, <)$ an L_{lo} -structure, $\mathcal{Q} = (\mathbb{Q}, <)$ also an L_{lo} -structure. Then

$$\begin{aligned} \alpha : \mathbb{Z} &\rightarrow \mathbb{Q} \\ n &\rightarrow n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3 (x_1 < x_3 \wedge x_3 < x_2)$$

Now $\mathcal{Q} \models \phi(1, 2)$ but $\mathcal{Z} \not\models \phi(1, 2)$.

Fact (2.12) (From now on we'll stop saying that \mathcal{M}, \mathcal{N} are L -structures etc to save time) Let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L -formula, and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).

3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let L be a language, \mathcal{M}, \mathcal{N} be L -structures.

Definition. (3.1)

An L -theory T is a set of L -sentences.

\mathcal{M} is a *model* of T if $\mathcal{M} \models \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \models T$.

The class of all the models of T is written $Mod(T)$.

The *theory of* \mathcal{M} is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L\text{-sentence and } \mathcal{M} \models \sigma\}$$

Example. (3.2)

Let T_{gp} be the set of L_{gp} -sentences:

(i) $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$;

(ii) $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$;

(iii) $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$.

Clearly, for a group G , $G \models T_{gp}$ (as they are just the group axioms). However, for a specific group G , clearly the theory of it, $Th(G)$ is larger than T_{gp} .

Definition. (3.3)

\mathcal{M} and \mathcal{N} are *elementarily equivalent* if $Th(\mathcal{M}) = Th(\mathcal{N})$.

We write $\mathcal{M} \equiv \mathcal{N}$.

Clearly, if $\mathcal{M} \simeq \mathcal{N}$ (\simeq means isomorphism), then $\mathcal{M} \equiv \mathcal{N}$.

But if \mathcal{M} and \mathcal{N} are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!

We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as L_{lo} -structures(!).

Definition. (3.4)

(i) An embedding $\beta : \mathcal{M} \rightarrow \mathcal{N}$ is *elementary* if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\beta(\bar{a}))$$

(ii) If $M \subseteq N$, and $id : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding, then \mathcal{M} is a *substructure* of \mathcal{N} .

(iii) If $M \subseteq N$ and $id : \mathcal{M} \rightarrow \mathcal{N}$ is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then \mathcal{M} is said to be an *elementary substructure* of \mathcal{N} , written as $\mathcal{M} \preceq \mathcal{N}$.

Example. (3.5)

Let $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an L_{lo} -structure where $<$ is the usual order;

Let $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$, also an L_{lo} -structure with the same $<$.

Then $\mathcal{M} \simeq \mathcal{N}$ as L_{lo} -structures. So $\mathcal{M} \equiv \mathcal{N}$ (since they are isomorphic).

Also, $\mathcal{M} \subseteq \mathcal{N}$ (read as *is a substructure of*), since the ordering $<$ coincides on \mathcal{M} and \mathcal{N} . However, $\mathcal{M} \not\preceq \mathcal{N}$, since if we pick the formula $\phi(x) \equiv \exists y (x < y)$, then $\mathcal{N} \models \phi(1)$, but $\mathcal{M} \not\models \phi(1)$.

Definition. (3.6)

Let \mathcal{M} be an L -structure, $A \subseteq M$, then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where c_a are constant symbols). An interpretation of \mathcal{M} as an L -structure extends to an interpretation of \mathcal{M} as an $L(A)$ -structure in the obvious way, i.e. $c_a^M = a$.

In this context, the elements of A are called *parameters*.

If \mathcal{M} and \mathcal{N} are two structures, and $A \subseteq M \cap N$, then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean \mathcal{M}, \mathcal{N} satisfy exactly the same $L(A)$ sentences.

—Lecture 4—

Reminder: we have a lecture next Monday (22nd Oct)!

Proposition. It turns out that, $\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where M is the domain of \mathcal{M}).

Lemma. (3.8, Tarski-Vaught test)

Let \mathcal{N} be an L -structure, let $A \subseteq N$. The following are equivalent:

- (i) A is the domain of a substructure \mathcal{M} s.t. $\mathcal{M} \preceq \mathcal{N}$;
- (ii) if $\phi(x) \in L(A)$ (with an abuse of notations $\phi(x, c_{a_1}, \dots, c_{a_n}) = \phi(x, a_1, \dots, a_n)$), if $\mathcal{N} \models \exists x \phi(x)$, then $\mathcal{N} \models \phi(b)$ for some $b \in A$.

Proof. (i) \implies (ii): Suppose $\mathcal{N} \models \exists x \phi(x)$. Then by elementarity, $\mathcal{M} \models \exists x \phi(x)$, and so $\mathcal{M} \models \phi(b)$ for $b \in M$. So (again by elementarity), $\mathcal{N} \models \phi(b)$.

(ii) \implies (i): This is the harder direction. First we prove that A is the domain of a substructure $\mathcal{M} \subseteq \mathcal{N}$.

By Sheet 1 Q4, it suffices to check:

- (a) For each constant c , $c^N \in A$;
- (b) For each function symbol f , $f^N(\bar{a}) \in A$ (for all $\bar{a} \in A^{n_R}$);

For (a), use property (ii) with $\exists x(x = c)$.

For (b), use property (ii) with the formula $\exists x((\bar{a}) = x)$.

So we now have $\mathcal{M} \subseteq \mathcal{N}$, and domain of \mathcal{M} is A . But we actually want to prove that $\mathcal{M} \preceq \mathcal{N}$. Now let $\chi(\bar{x})$ be an L -formula.

We want to show that for $\bar{a} \in A^{|\bar{x}|}$ $\mathcal{M} \models \chi(\bar{a}) \iff \mathcal{N} \models \chi(\bar{a})$ (*).

By induction on the complexity of $\chi(\bar{x})$:

- if $\chi(\bar{x})$ is atomic, (*) follows from $\mathcal{M} \subseteq \mathcal{N}$ (since \mathcal{M} is a substructure!);
- if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$, it's a straightforward induction;
- (interesting case) if $\chi(\bar{x}) = \exists y \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an L -formula, suppose that $\mathcal{M} \models \chi(\bar{a})$, then $\mathcal{M} \models \exists y \psi(\bar{a}, y)$, hence $\mathcal{M} \models \psi(\bar{a}, b)$ for some $b \in A = \text{dom}(\mathcal{M})$ (this is the definition of truth).

But then $\mathcal{N} \models \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \models \chi(\bar{a})$.

Now let $\mathcal{N} \models \chi(\bar{a})$, i.e. $\mathcal{N} \models \exists y \psi(\bar{a}, y)$ (we find a *witness* for it). By property (ii), $\mathcal{N} \models \psi(\bar{a}, b)$ for some $b \in A = \text{dom}(\mathcal{M})$.

Again by inductive hypothesis, we have $\mathcal{M} \models \psi(\bar{a}, b)$, and so in particular, $\mathcal{M} \models \chi(\bar{a})$ as it has got a witness there. \square

Remark. (3.9)

Even more assumptions: let's assume that the set of variables is countably infinite. Then:

- the cardinality of the set of L -formulas is $|L| + \omega$ (where by $|L|$ we mean

the number of symbols. For example, $|L_{gp}| = 3$, $|L_{lo}| = 1$), where we abuse another notation that we use ω as cardinals (rather than ordinals) (note that the formulas are just strings of finite length);

- if A is a set of parameters in some structure, the cardinality of the set $L(A)$ is $|A| + |L| + \omega$, where by $+$ here we merely mean $\max\{|L|, |A|, \omega\}$ (instead of addition), and same for the $+$ above.

Definition. (3.10)

Let λ be an ordinal. Then a *chain of length λ* of sets is a sequence $\langle M_i : i < \lambda \rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$.

A chain of L -structures is a sequence: $\langle \mathcal{M}_i : i < \lambda \rangle$ s.t. $\mathcal{M}_i \subseteq \mathcal{M}_j$ (note that it's substructure here) for $i \leq j < \lambda$.

The *union* of this chain is the L -structure \mathcal{M} defined as follows:

- the domain is $\bigcup_{i < \lambda} M_i$ (when you think of this, you can always start with the case $\lambda = \omega$);
- for constants c , $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any $i < \lambda$ (this is well defined, because of the substructure condition above);
- if f is a function symbol, $\bar{a} \in M^{|\bar{a}|}$ (why the mod sign here), $f^{\mathcal{M}}\bar{a} = f^{\mathcal{M}_i}\bar{a}$ where i is s.t. $\bar{a} \in M_i^{|\bar{a}|}$;
- if R is a relation symbol, then $R^{\mathcal{M}} = \bigcup_{i < \lambda} R^{\mathcal{M}_i}$.

Theorem. (3.11, Downward Löwenheim-Skolem theorem)

(Recall that in part II Logic and Set Theory we had the countable version of this)

Let \mathcal{N} be an L -structure, and $|\mathcal{N}| \geq |L| + \omega$. Let $A \subseteq N$. Then for every cardinal λ s.t. $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preceq \mathcal{N}$ s.t.

- $A \subseteq M$;
- $|\mathcal{M}| = \lambda$.



(It helps to think about the case $|A| = \omega$ and $|N|$ is uncountable.)

A quick example how this could be useful (we'll go very sloppy here): think of $(\mathbb{C}, +, \cdot, -, \cdot^{-1}, 0, 1)$ as a field. Consider $\mathbb{Q} \subseteq \mathbb{C}$ (both as subset and substructure). Note that algebraic closeness is a property of \mathbb{C} . By downward Löwenheim-

Skolem, there is a substructure in \mathcal{C} that contains \mathbb{Q} that is also algebraically closed (apparently, the set of algebraic numbers).

Proof. We build a chain $\langle A_i : i < \lambda \rangle$, with $A_i \subseteq N$, s.t. $|A_i| = \lambda$.
(our goal: define an elementary substructure with domain $M = \bigcup_{i < \omega} A_i$).

Base case: Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$.

Successors: At stage $i + 1$, assume A_i has been built, with $|A_i| = \lambda$.

Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$ -formulas such that $\mathcal{N} \models \exists x \phi_k(x)$. Let a_k be such that $\mathcal{N} \models \phi_k(a_k)$, and let $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$ (basically, with those witnesses added). Then $|A_{i+1}| = \lambda$ (note that we haven't increased the size).

Now let $M = \bigcup_{i < \omega} A_i$ (note the subscript range). We use lemma (3.8) to show that M is the domain of $\mathcal{M} \preceq \mathcal{N}$, and $|M| = \lambda$. We're running out of time, so we'll continue next Monday.

—Lecture 5—

Solutions to worksheet 1: either take along to lecture on Friday, or email them to silvia.barbina@open.ac.uk.

Let's continue with the proof:



Start with $A_0 \subset N$, $A \subseteq A_0$, $|A_0| = \lambda$. The idea is to define $\langle A_i : i < \omega \rangle$ so that $M = \bigcup_{i < \omega} A_i$ satisfies (ii) via the TV test (3.8).

List all formulas $\phi(x, \bar{a})$ (\bar{a} is a tuple in A_0), and $\mathcal{N} \models \phi(b, \bar{a})$ for some b .

Add each such b to A_0 (one for each such ϕ).

Let $A_1 = A_0 \cup \{ \text{all the } b\text{'s} \}$.

Repeat for formulas $\phi(x, \bar{a})$ where \bar{a} is in A_1, \dots

Eventually, $\langle A_i : i < \omega \rangle$ is such that $M = \bigcup_{i < \omega} A_i$ is as required (i.e. M is the domain of some elementary substructure of \mathcal{N} that we need).

We claim that M satisfies condition (ii) in Lemma (3.8): Let $\mathcal{N} \models \exists x \psi(x, \bar{a})$, where \bar{a} is a tuple in M . Then \bar{a} is a *finite* tuple, so there is an i s.t. \bar{a} is in A_i .

Then A_{i+1} , by construction, contains b s.t. $\mathcal{N} \models \phi(b, \bar{a})$. But $A_{i+1} \subseteq M, b \in M$. Then apply (3.8) we're done. \square

4 Two relational structures

Definition. (4.1, dense linear orders)

A *linear order* is an $L_{lo} = \{<\}$ -structure such that:

- (i) $\forall x \neg(x < x)$;
- (ii) $\forall xyz((x < y \wedge y < z) \rightarrow x < z)$;
- (iii) $\forall xy((x < y) \vee (y < x) \vee x = y)$ (total).

A linear order is *dense* if, in addition, it also satisfies:

- (iv) $\exists xy(x < y)$;
- (v) $\forall xy, (x < y \rightarrow \exists z(x < z \wedge z < y))$ (density).

A linear order has no endpoints if, in addition,

- (vi) $\forall x(\exists y(x < y) \wedge \exists z(z < x))$.

We use T_{dlo} to denote the theory that includes all axioms (i) to (vi), and T_{lo} is the theory that includes axioms (i) to (iii) only.

Remark. (iv) and (v) imply that if $\mathcal{M} \models T_{dlo}$, then $|\mathcal{M}| \geq \omega$.

Definition. (4.2)

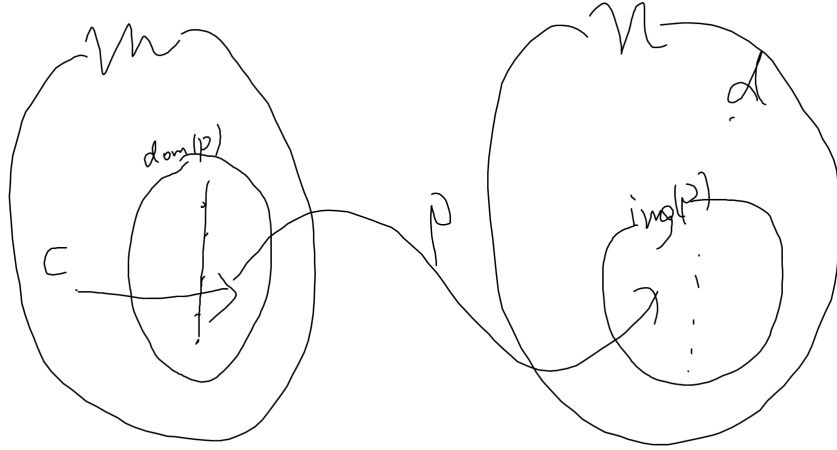
If $\mathcal{M}, \mathcal{N} \models T_{lo}$, then an *injective* map $p : A \subseteq M \rightarrow N$ is a *partial embedding* if $\mathcal{M} \models a < b \implies \mathcal{N} \models p(a) < p(b)$.

In particular, if $|\text{dom}(p)| < \omega$, then p is a *finite* partial embedding.

Lemma. (4.3, extension lemma)

Take a linear order $\mathcal{M} \models T_{lo}$, and a dense linear endpoints $\mathcal{N} \models T_{dlo}$, and let $p : M \rightarrow N$ be a finite partial embedding. Then if $c \in \mathcal{M}$, there is a finite partial embedding \hat{p} s.t. $p \subseteq \hat{p}$ and $c \in \text{dom}(\hat{p})$.

(we can always add one extra element in our embedding.)



Proof.

Case 1: c is greater than all elements in $\text{dom}(p)$. In that case, pick an element $d \in \mathcal{N}$ s.t. $d > b$ for all $b \in \text{img}(p)$;

Case 2: $a_i < c < a_{i+1}$ where $a_i, a_{i+1} \in \text{dom}(p)$. Then we choose $\mathcal{N} \models p(a_i) < d < p(a_{i+1})$, where d is chosen appropriately by density (here's the case why we

need \mathcal{N} to be dense;

Case 3: c is less than all elements in $\text{dom}(p)$. This is similar to case 1.

Note that the ability to extend by one point allows us to embed any finite linear order into a dense linear order without endpoints. \square

Theorem. (4.4)

Let $\mathcal{M}, \mathcal{N} \models T_{dlo}$ s.t. $|\mathcal{M}| = |\mathcal{N}| = \omega$. Let $p : A \subseteq M \rightarrow N$ be a finite partial embedding.

Then there is an isomorphism $\pi : \mathcal{M} \rightarrow \mathcal{N}$ s.t. $p \subseteq \pi$.

Proof. Enumerate M, N , say $M = \langle a_i : i < \omega \rangle$, $N = \langle b_i : i < \omega \rangle$ (sequences of elements).

We define, inductively, a chain of finite partial embedding $\langle p_i : i < \omega \rangle$ (idea: $\pi = \bigcup_{i < \omega} p_i$).

Let's start with $p_0 = p$. At stage $i + 1$, suppose we are given p_i . We want to include a_i in $\text{dom } p_{i+1}$, and b_i in the $\text{img}(p_{i+1})$.

(Lecturer calls this a *back and forth* method) Forth step: By lemma 4.3, we can extend p_i to $p_{i+\frac{1}{2}}$ such that $a_i \in \text{dom}(p_{i+\frac{1}{2}})$;

Back step: By lemma 4.3 again applied to $(p_{i+\frac{1}{2}})^{-1}$ to include $b_i \in \text{dom}(p_{i+1}^{-1})$ (i.e. in the range of p_{i+1}).

We claim that p_{i+1} extends p_i as required.

Let $\pi = \bigcup_{i < \omega} p_i$. Then (check) π is an isomorphism (i.e. order-preserving bijection). \square

Definition. (4.5)

An L -theory is *consistent* if there is L -structure \mathcal{M} s.t. $\mathcal{M} \models T$.

If T is a theory in L and ϕ is an L -sentence, then $T \vdash \phi$ (read as T entails ϕ , note that this has nothing to do with syntactic implication) if for all \mathcal{M} such that $\mathcal{M} \models T$, we have $\mathcal{M} \models \phi$.

Finally, an L -theory T is *complete* if for all L -sentences ϕ , either $T \vdash \phi$ or $T \vdash \neg\phi$ (see part II Logic and Set Theory).

For example, T_{dlo} is complete.

—Lecture 6—

Definition. (4.6)

A theory T in a countable language with a (infinitely) countable model is *ω -categorical* if any two countable models of T are isomorphic.

Corollary. (4.7 of theorem (4.4))

T_{dlo} is ω -categorical.

Proof. If $\mathcal{M}, \mathcal{N} \models T_{dlo}$, $|\mathcal{M}| = |\mathcal{N}| = \omega$, then ϕ (the empty map) is a finite partial embedding. But by theorem (4.4) we get $\mathcal{M} \simeq \mathcal{N}$.

(We can also use any $\{a, b\}$ where $a \in \mathcal{M}$ and $b \in \mathcal{M}$ as initial finite partial embedding). \square

Theorem. (4.8)

(erratum 26th Oct 2018: lecturer wants to add a condition T has no finite models.

Then the problem with (4.11) is fixed.)

If T is an ω -categorical theory in a countable language, then T is complete.

Proof. Let $\mathcal{M} \models T$ and ϕ be an L -sentence.

If $\mathcal{M} \models \phi$, suppose $\mathcal{N} \models T$. Then by theorem (3.11) (Downward Lowenheim-Skolem), there are $\mathcal{M}' \preceq \mathcal{M}$, $\mathcal{N}' \preceq \mathcal{N}$ s.t. $|\mathcal{M}'| = |\mathcal{N}'| = \omega$.

But $\mathcal{M}' \simeq \mathcal{N}'$ (by ω -categoricity), so in particular $\mathcal{M}' \equiv \mathcal{N}'$, and so $\mathcal{N}' \models \phi$. By elementarity, $\mathcal{N} \models \phi$.

The case $\mathcal{M} \models \neg\phi$ is similar.

(Think about if T could have a finite model.) □

Corollary. (4.9)

T_{dlo} is complete.

Definition. (4.10)

If \mathcal{M}, \mathcal{N} are L -structures, a map f such that $\text{dom}(f) \subseteq M$ (the domain of \mathcal{M}), and $\text{img}(f) \subseteq N$ is a (partial) *elementary map* if for all L -formulas $\phi(\bar{x})$ and $\bar{a} \in (\text{dom}(f))^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(f(\bar{a}))$$

Remark. (4.11)

A map f is elementary iff every finite restriction of f is elementary.

(Why? For forward, if $f_0 \subseteq f$ is a finite restriction that is not elementary, then for some formula $\phi(\bar{x})$, $\bar{a} \in \text{dom}(f_0)$, the above equivalence doesn't hold; but then that equivalence doesn't hold for f either; contradiction; for backward, if f is not elementary, then the above equivalence fails on a finite tuple, so the above equivalence fails on some finite restriction.)

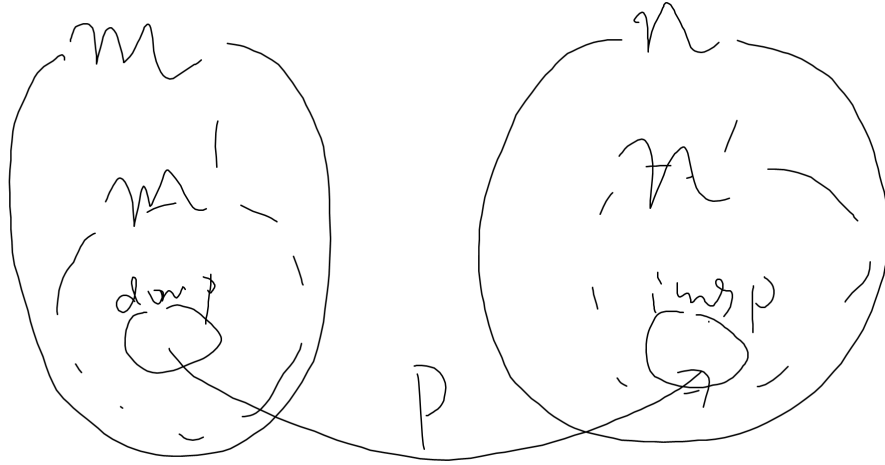
Proposition. (4.12)

Let $\mathcal{M}, \mathcal{N} \models T_{dlo}$, and let $p : A \subseteq M \rightarrow N$ be a partial embedding. Then p is elementary.

Proof. By remark (4.11), it suffices to consider p finite.

By Downward L-S theoem (3.11), we choose $\mathcal{M}', \mathcal{N}'$ such that

- (i) $|\mathcal{M}'| = |\mathcal{N}'| = \omega$;
- (ii) $\mathcal{M}' \preceq \mathcal{M}$, $\mathcal{N}' \preceq \mathcal{N}$;
- (iii) $\text{dom}(p) \subseteq M'$, $\text{img}(p) \subseteq N'$.



Now p is a finite partial embedding between countable models, so p extends to an isomorphism $\pi : \mathcal{M}' \rightarrow \mathcal{N}'$.

In particular, π is an elementary map between \mathcal{M} and \mathcal{N} . □

Corollary. (4.13)

$(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$.

Proof. Use proposition (4.12) with $id : \mathbb{Q} \rightarrow \mathbb{R}$. □

Definition. (4.14)

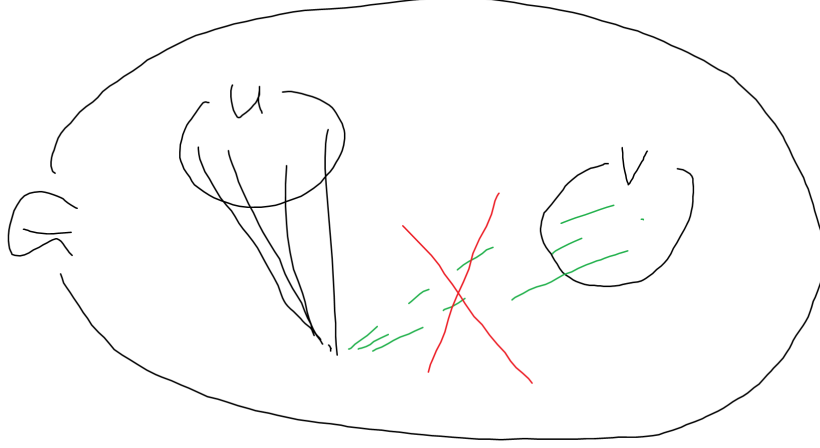
(See Part II Logic and Set Theory)

Let $L_{gph} = \{R\}$, where R is a binary relation symbol.

An L_{gph} -structure is a graph if

- (i) $\forall x \neg R(x, x)$;
- (ii) $\forall xy (R(x, y) \leftrightarrow R(y, x))$.

An L_{gph} -structure is a **random graph** if it is a graph such that the following axiom-schema (r_n) hold:



$$\forall x_0 \dots x_n, y_0 \dots y_n, \left(\bigwedge_{i,j=0}^n x_i \neq y_j \rightarrow \exists z \left(\bigwedge_{i=0}^n (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

(iii) $\exists xy (x \neq y)$.

Remark. (similar to what is mentioned in the link above)

A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

Fact. (4.15)

There is a random graph.

Proof. Let the domain be ω , let $i, j \in \omega$ such that $i < j$. Write j as a sum of distinct powers of 2. Then $\{i, j\}$ is an edge iff 2^i appears in the sum.

As an exercise, prove that ω with this definition of R is indeed a random graph. \square

Definition. (4.16, or more precisely just notations)

$T_{gph} = \{\text{axioms (i), (ii)}\}$, $T_{rg} = T_{gph} \cup \{(iii), (r_n) : n \in \omega\}$.

If $\mathcal{M}, \mathcal{N} \models T_{gph}$, a partial embedding is an injection $p : A \subseteq M \rightarrow N$ s.t. $\mathcal{M} \models R(p(a), p(b)) \iff \mathcal{N} \models R(a, b)$ for all a, b in the domain.

Lemma. (4.17)

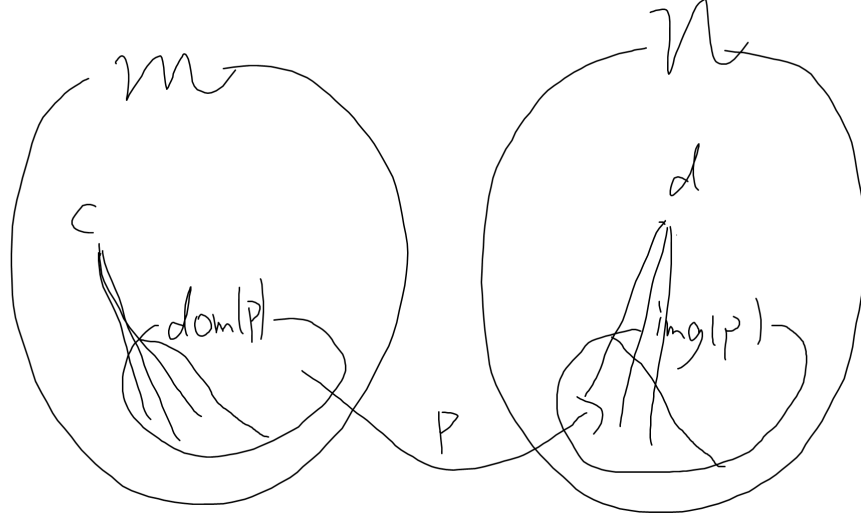
Let $\mathcal{M} \models T_{gph}, \mathcal{N} \models T_{rg}$, let $p : A \subseteq M \rightarrow N$ be a finite partial embedding, and let $c \in M$.

Then there is a map $\hat{p} : \hat{A} \subseteq M \rightarrow N$ such that \hat{p} is a partial embedding, $c \in \text{dom } \hat{p}$, $p \subseteq \hat{p}$.

(This is like another extension lemma.)

We'll prove this next time.

Last time we defined what a random graph is (in this course). We also defined what is a partial embedding in this theory (just preserves all edges). Let's continue with the proof of the lemma now. Let $c \in M$, $c \notin \text{dom}(p)$.



Find $d \in N$ such that $\mathcal{N} \models R(d, p(a)) \iff \mathcal{M} \models R(c, a)$.

Theorem. (4.18)

Let $\mathcal{M}, \mathcal{N} \models T_{rg}$ and $|\mathcal{M}| = |\mathcal{N}| = \omega$, and $P : A \subseteq M \rightarrow N$ is a finite partial embedding.

Then $\mathcal{M} \simeq \mathcal{N}$, by an isomorphism that extends p .

Proof. Same as proof of Theorem (4.4) (there is only one model of T_{dlo} up to isomorphism), but with lemma (4.17) instead of lemma (4.3). \square

Corollary. (4.19)

T_{rg} is ω -categorical (see definition (4.6) – this is just a restatement of the theorem above) and complete.

In particular, every finite partial embedding between models of T_{rg} is an elementary map.

Remark. The unique (up to isomorphism) model of T_{rg} is *the* countable random graph, or the *Rado* graph.

It is universal w.r.t. finite and countable graphs (i.e. it embeds all).

Another nice property (which you are not required to see this immediately – it is far from trivial) *ultrahomogeneous*, i.e. every isomorphism between finite substructures extends to an automorphism of the whole graph.

Google *David Marker's* book, or *Tent-Ziegler*. Warning: both of them contain a lot of typos.

5 Compactness

Definition. (5.1)

Suppose we have a L -theory T .

- (i) T is *finitely satisfiable* if every finite subset of sentences in T has a model.
- (ii) T is *maximal* if for all L -sentences σ , either $\sigma \in T$ or $\neg\sigma \in T$.
- (iii) T has the *witness property* (WP): if for all $\phi(x)$ (L -formula with 1 free variable), there is a constant $c \in \mathcal{C}$ s.t.

$$(\exists x \phi(x) \rightarrow \phi(c)) \in T$$

Lemma. (5.2)

If T is maximal and finitely satisfiable (we'll sometimes use *f.s.* from now onwards), and ϕ is an L -sentence, and $\Delta \stackrel{fin}{\subseteq} T$ and $\Delta \vdash \phi$, then $\phi \in T$.
(Prove it by yourself)

Lemma. (5.3)

Let T be a maximal, f.s. theory with WP. Then T has a model.
Moreover, if λ is a cardinal and $|\mathcal{C}| \leq \lambda$ (\mathcal{C} is the set of constants in L), then T has a model of size at most λ .

Proof. Let \mathcal{C} be the constants of L . Let $c, d \in \mathcal{C}$, define $c \sim d$ iff $c = d \in T$.

We claim that \sim is an equivalence relation: reflexivity and symmetry are trivial; for transitivity, let $c \sim d$ and $d \sim e$. Then $c = d \in T$ and $d = e \in T$. Then by the lemma $c = e \in T$ as it is implied by the two sentences. So $c \sim e$.

Notation: we'll use $c / \sim = c^*$ to denote the equivalence class of c .

Now define a structure \mathcal{M} whose domain is $\mathcal{C} / \sim = M$. Clearly, $|M| \leq \lambda$ if $|\mathcal{C}| \leq \lambda$.

We must define interpretations in \mathcal{M} for symbols for L :

- If $c \in \mathcal{C}$, then $c^m = c^* (= c / \sim)$;
- If $R \in \mathcal{R}$ is a relation symbol, we define $R^{\mathcal{M}} = \{(c_1^*, \dots, c_{n_R}^*) : R(c_1, \dots, c_{n_R}) \in T\}$.

We have to check that $R^{\mathcal{M}}$ is well-defined: suppose $\bar{c}, \bar{d} \in \mathcal{C}^{n_R}$ and suppose $c_i \sim d_i$ for each i , i.e. $c_i = d_i \in T$ for every $i = 1, \dots, n_R$. However, now

$$R(\bar{c}) \in T \iff R(\bar{d}) \in T$$

by maximality of T and the previous lemma. So that $R^{\mathcal{M}}$ is well defined.

- If $f \in \mathcal{F}$ is a function symbol, then $f\bar{c} = d \in T$ for some $d \in \mathcal{C}$: this is because $\exists x(f(\bar{c}) = x) \in T$ by maximality and f.s..

Then define $f^{\mathcal{M}}(\bar{c}^*) = \bar{d}^*$ (obvious notation).

We also have to check this is well-defined. Lecturer decides to left this to us.

Now we claim that the terms behave nicely as what the theory says, i.e. if $t(x_1, \dots, x_n)$ is an L -term and $c_1, \dots, c_n, d \in \mathcal{C}$, then $t(c_1, \dots, c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$.

- \implies : by induction on the complexity of T (lecturer decided to leave this as another exercise).
- \Leftarrow : Assume $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$. Then $t(c_1, \dots, c_n) = e \in T$ for some constant e (why? As our theory is maximal, it has to say what the result is when we apply

t to these terms). We then use \implies to get that $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$.

But then $d^* = e^*$, i.e. $d = e \in T$. So by lemma (5.2), the sentence implied by these two sentences, $t(c_1, \dots, c_n) = d \in T$.

The last massive claim: for all L -formulas $\phi(\bar{x})$ and $\bar{c} \in \mathcal{C}^{|\bar{x}|}$, we have

$$\mathcal{M} \models \phi(\bar{c}) \iff \phi(\bar{c}) \in T$$

The proof is by induction on complexity of $\phi(\bar{x})$ (The lecturer decided to leave yet another proof to us – lots of work to be done here. Lecturer is speeding up!). \square

—Lecture 8—

Lemma. (5.4)

Let T be a f.s. L -theory. Then there are $L^* \supseteq L$ and a f.s. $T^* \supseteq T$ such that:

- (i) $|L^*| = |L| + \omega$;
- (ii) any L^* -theory extending T^* has WP.

Proof. Define $\langle L_i : i < \omega \rangle$ a chain of languages containing L and s.t. $|L_i| = |L| + \omega$, and $\langle T_i : i < \omega \rangle$ of f.s. theories s.t. $\forall i$ T_i is an L_i -theory, and $T_i \supseteq T$.

• Base step: $L_0 = L$ and $T_0 = T$.

• At stage $i + 1$, L_i and T_i are given. List all L_i -formulas $\phi(x)$ (one free variable x) and let $L_{i+1} = L_i \cup \{c_\phi : \phi(x) \text{ is an } L_i\text{-formula}\}$.

For all $\phi(x)$ (L_i -formula), let Φ_ϕ be the L_{i+1} -sentence $\exists x \phi(x) \rightarrow \phi(c_\phi)$.

Then $T_{i+1} := T_i \cup \{\Phi_\phi : \phi(x) \text{ is } L_i\text{-formula}\}$.

Claim: T_{i+1} is f.s.. Why is that? Let's take a finite subset $\Delta \stackrel{fin}{\subseteq} T_{i+1}$. Then $\Delta = \Delta_0 \cup \{\Phi_{\phi_2}, \dots, \Phi_{\phi_n}\}$ where $\Delta_0 \subseteq T_i$.

Let $\mathcal{M} \models \Delta_0$ (\mathcal{M} is an L_i -structure; it exists because T_i is f.s.).

We define an L_{i+1} -structure \mathcal{M}' with domain M (of \mathcal{M}). Define the interpretation of new constants as follows: if $\mathcal{M} \models \exists x \phi(x)$, then $c_\phi^{\mathcal{M}'} = a$ for any $a \in \mathcal{M}$ s.t. $\mathcal{M} \models \phi(a)$.

Otherwise, $c_\phi^{\mathcal{M}'}$ is arbitrary.

Then $\mathcal{M}' \models \Delta$.

Now let $L^* = \bigcup_{i < \omega} L_i$, $T^* = \bigcup_{i < \omega} T_i$.

By our construction, any extension of T^* has WP (check), and T^* is f.s. ($\Delta \stackrel{fin}{\subseteq} T^*$, then $\Delta \subseteq T_i$ for some i). \square

Lemma. (5.5)

If T is f.s., there exists a maximal f.s. $T' \supseteq T$ (of underlying language L).

Proof. Let $I := \{S : S \text{ is a f.s. } L\text{-theory s.t. } T \subseteq S\}$.

I is partially ordered by inclusion.

If $\langle C_i : i < \lambda \rangle$ is a chain in I , then $\bigcup_{i < \lambda} C_i$ is an upper bound for the chain: it is f.s. ($\Delta \subseteq \bigcup C_i$ is s.t. $\Delta \subseteq C_i$).

Then by Zorn's lemma, I has a maximal element (w.r.t. \subseteq).

We claim that the maximal element T' of I is the required extension of T (check that $\forall L$ -sentences σ , $\sigma \in T'$ or $\neg \sigma \in T'$). \square

Theorem. (5.6, Compactness)

If T is a f.s. L -theory, and $\lambda \geq |L| + \omega$, then there is $\mathcal{M} \models T$ s.t. $|\mathcal{M}| \leq \lambda$.
(See part II Logic and Set Theory for the finite case).

Proof. (sketch)

Extend T to T^* , an L^* -theory that is f.s., and s.t. any $S \supseteq T^*$ has WP (by lemma (5.4)).

By lemma (5.5), we can extend this T^* to a maximal f.s. theory T' ; and by the lemma T' would have WP since it is an extension of T^* .

Now T' is maximal and f.s., so we can use lemma (5.3) to show that there is a model $\mathcal{M} \models T'$. Then in particular, $\mathcal{M} \models T$ (check $\mathcal{M} \leq \lambda$). \square

Definition. (5.7)

Let L be a language. Then an L -type $p(\bar{x})$ is a set of L -formulas whose free variables are in \bar{x} (and $\bar{x} = \langle x_i : i < \lambda \rangle$).

- An L -type is *satisfiable* if there is an L -structure \mathcal{M} and an assignment $\bar{a} \in \mathcal{M}^{|\bar{x}|}$ s.t. $\mathcal{M} \models \phi(\bar{a})$ for all $\phi(\bar{x}) \in p(\bar{x})$ (we also say $p(\bar{x})$ is *consistent*, and \bar{a} *realizes* $p(\bar{x})$ in \mathcal{M}).

We write $\mathcal{M} \models p(\bar{a})$, or $\mathcal{M}, \bar{a} \models p(\bar{x})$.

We also say that $p(\bar{x})$ is *satisfied in* \mathcal{M} .

- A type $p(\bar{x})$ is *finitely satisfiable* if every finite subset of $p(\bar{x})$ is satisfiable (we may say $p(\bar{x})$ is *finitely consistent*).

Remark. An L -type may be finitely satisfiable in a model \mathcal{M} (by this we mean every finite subset is satisfiable in \mathcal{M}), but not satisfiable in \mathcal{M} .

Example. Let $\mathcal{M} = (\mathbb{N}, <)$. Let $\phi_n(x)$ say *there are at least n elements less than x* , and $p(x) := \{\phi_n(x) : n < \omega\}$.

Is $p(x)$ finitely satisfiable in \mathcal{M} ? Yes; but obviously $p(x)$ is not satisfiable in \mathcal{M} .

Theorem. (5.8)

Every finitely satisfiable L -type $p(\bar{x})$ is satisfiable (not necessarily in the original model, of course).

Proof. Let $\bar{x} = \langle x_i : i < \lambda \rangle$, let $\langle c_i : i < \lambda \rangle$ be new constants (not in L).

Expand L to $L' = L \cup \{c_i : i < \lambda\}$. Then $p(\bar{c})$ is an L' -theory, and theorem (5.6) applied to $p(\bar{c})$ gives the desired conclusion (think). \square

—Lecture 9—

Example class 2: 19th November, Monday.

Lemma. (5.9)

Let \mathcal{M} be a structure, let $\bar{a} = \langle a_i : i < \lambda \rangle$ enumerate \mathcal{M} . Let

$$q(\bar{x}) = \{\phi(\bar{x}) : \mathcal{M} \models \phi(\bar{a})\}$$

where $|\bar{x}| = \lambda$.

Then $q(\bar{x})$ is satisfiable in \mathcal{N} iff there is $\beta : \mathcal{M} \rightarrow \mathcal{N}$ that is an elementary embedding.

Proof. If $q(\bar{x})$ is satisfiable in \mathcal{N} , there is $\bar{b} \in N^{|\bar{x}|}$ s.t. $\mathcal{N} \models \phi(\bar{b})$ for all $\phi(\bar{x}) \in q(\bar{x})$.

Then $\beta : a_i \rightarrow b_i$ ($i < \lambda$) is an elementary embedding (β preserves, for example, atomic formulas of the form $f(a_{i_1}, \dots, a_{i_n}) = a_{i_{n+1}}$, so it is an embedding.)

More generally, for any $\phi(\bar{x})$ (L -formula),

$\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{N} \models \phi(\bar{b})$, but $\beta(\bar{a}) = \bar{b}$.

Conversely, if $\beta : \mathcal{M} \rightarrow \mathcal{N}$ is elementary, then $\beta(\bar{a})$ satisfies $q(\bar{x})$ in \mathcal{N} . \square

Remark. The above is usually stated as (diagram lemma): Let $Th(\mathcal{M}_M)$ be a theory in $L(M)$, $\mathcal{N} \models Th(\mathcal{M}_M)$, then \mathcal{M} embeds elementarily in \mathcal{N} .

Remark. (5.10)

We can consider types in $L(A)$, where $A \subseteq \mathcal{M}$.

In particular, we can consider $A = M$ the whole domain of \mathcal{M} .

Types of this kind are said to *have parameters in A* (or to be *over A*).

If $p(\bar{x})$ is a type over M , then there is a \bar{a} , an enumeration of M , and a type $p'(\bar{x}, \bar{z})$ where the \bar{z} are *new constants*, $|\bar{z}| = |\bar{a}|$, and $p(\bar{x}) = p'(\bar{x}, \bar{a})$.

Theorem. (5.11)

If \mathcal{M} is a structure, and $p(x)$ is a type in $L(M)$ that is finitely satisfiable in \mathcal{M} , then $p(\bar{x})$ is satisfiable in some \mathcal{N} such that $\mathcal{M} \preceq \mathcal{N}$.

Example. Consider $\mathcal{M} = (\mathbb{Q}, <)$, and let $\langle a_i : i < \omega \rangle$ a sequence in \mathbb{Q} that converges to $\sqrt{2}$ from below, and let $\langle b_i : i < \omega \rangle \subseteq \mathbb{Q}$ tend to $\sqrt{2}$ from above.

Let $\phi_n(x) \equiv a_n < x < b_n$. Then let $p(x) = \{\phi_n(x) \text{ s.t. } n < \omega\}$.

Then $p(x)$ is an $L(\mathbb{Q})$ -type which is finitely satisfiable in \mathbb{Q} . But $p(x)$ is *not* satisfiable in \mathcal{M} .

It is, however, satisfiable in $(\mathbb{R}, <)$ which contains $(\mathbb{Q}, <)$ as an elementary substructure.

(we can actually also use the example about natural numbers in last lecture – just need to add an ∞ .)

Proof. (of 5.11)

Let $\langle a_i : i < \lambda \rangle$ enumerate \mathcal{M} , and let $q(\bar{x}) := \{\phi(\bar{z}) : \mathcal{M} \models \phi(\bar{a})\}$, where $|\bar{z}| = \lambda$ and the z_i are new variables (so not among the \bar{x}).

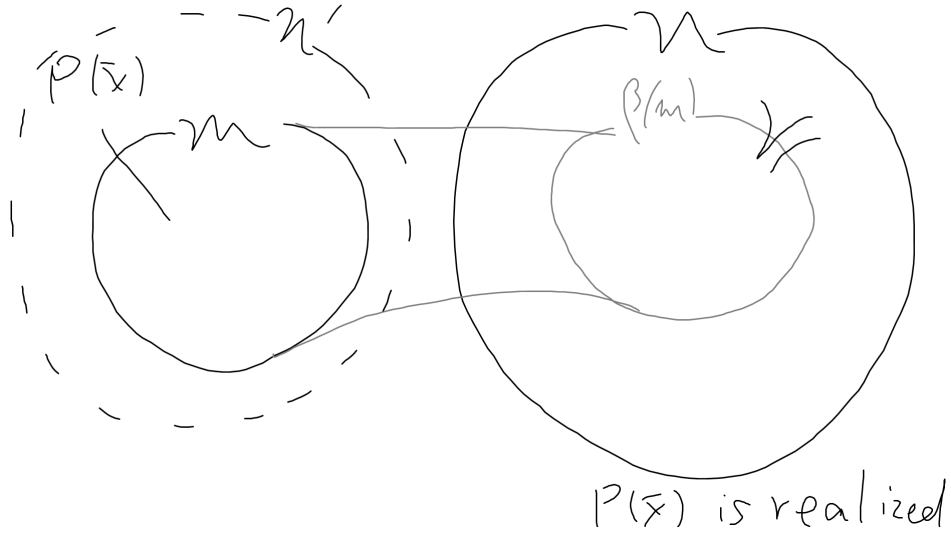
Write $p(\bar{x})$ as $p'(\bar{x}, \bar{a})$ for some $p'(\bar{x}, \bar{z})$ (an L -type).

We claim that $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is finitely satisfiable in \mathcal{M} : this is because $p'(\bar{x}, \bar{a})$ is finitely satisfiable by hypothesis and $q(\bar{z})$ is realized by \bar{a} .

Then, by theorem (5.8) (compactness for types), $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$ is satisfiable.

That is, there is \mathcal{N} and $\bar{b} \in \mathcal{N}^{|\bar{z}|}$ and $\bar{c} \in \mathcal{N}^{|\bar{a}|}$ such that $\mathcal{N} \models p'(\bar{c}, \bar{b}) \cup q(\bar{b})$.

In particular, $\mathcal{N} \models q(\bar{b})$. Then by lemma (5.9), $\beta : \mathcal{M} \rightarrow \mathcal{N}$ by $a_i \rightarrow b_i$ is an elementary embedding.



(Get $\mathcal{N}' \simeq \mathcal{N}$ s.t. $\mathcal{M} \preccurlyeq \mathcal{N}'$, $\mathcal{M} \simeq \beta(\mathcal{M})$.) □

Theorem. (5.12, Upward Löwenheim-Skolem theorem)

(See part II Logic and Set theory for the (countable?) case)

Let \mathcal{M} be s.t. $|\mathcal{M}| \geq \omega$. Then for any $\lambda \geq |\mathcal{M}| + |L|$, there is \mathcal{N} s.t. $\mathcal{M} \preccurlyeq \mathcal{N}$ with $|\mathcal{N}| = \lambda$.

Proof. Let $\bar{x} = \langle x_i : i < \lambda \rangle$ a tuple of distinct variables.

Let $p(\bar{x}) = \{x_i \neq x_j : i < j < \lambda\}$. Then $p(\bar{x})$ is finitely consistent in \mathcal{M} . So by theorem (5.11), $p(\bar{x})$ is realized in some \mathcal{N} that is an elementary extension of \mathcal{M} , and $|\mathcal{N}| \geq \lambda$.

Then by downward L-S theorem (3.11) we could have $|\mathcal{N}| = \lambda$. □

6 Saturation

Definition. (6.1)

Let λ be an infinite cardinal, let $|\mathcal{M}| \geq \omega$. Then \mathcal{M} is λ -saturated if \mathcal{M} realizes every type $p(x)$ (with one free variable) such that:

- (i) $p(x)$ has parameters in $A \subseteq \mathcal{M}$ and $|A| < \lambda$;
- (ii) $p(x)$ is finitely consistent in \mathcal{M} .

We say \mathcal{M} is *saturated* if it is $|\mathcal{M}|$ -saturated.

Question: can \mathcal{M} be λ -saturated if $\lambda > |\mathcal{M}|$? If so, \mathcal{M} would satisfy f.s. types in $L(M)$.

(Hint: we could use $p(x) = \{x \neq a_i : i < |\mathcal{M}|\}$ where $\langle a_i : i < |\mathcal{M}|\rangle$ enumerates \mathcal{M} . This could not be satisfiable.)

—Lecture 10—

Definition. (6.2)

Let \mathcal{M} be an L -structure, $A \subseteq M$, \bar{b} a tuple of in M , possibly infinite. The type of \bar{b} over A is the following $L(A)$ -type:

$$tp_{\mathcal{M}}(\bar{b}/A) = \{\phi(\bar{x}) \in L(A) : \mathcal{M} \models \phi(\bar{b})\}$$

(i.e. all the $L(A)$ -formulas that, when plugged in \bar{b} , can be implied by \mathcal{M}).

\mathcal{M} is often omitted if it's clear from the context.

Remark. (6.3)

- (i) $tp_{\mathcal{M}}(\bar{b}/A)$ is complete, i.e. for every $L(A)$ formula $\phi(\bar{x})$, either $\phi(\bar{x}) \in tp(\bar{b}/A)$ or $\neg\phi(x) \in tp(\bar{b}/A)$ (clear).
- (ii) If $\mathcal{M} \preceq \mathcal{N}$, then for $A \subseteq M$, and \bar{b} a tuple,

$$tp_{\mathcal{M}}(\bar{b}/A) = tp_{\mathcal{N}}(\bar{b}/A)$$

(elementary embeddings preserve truth of formulas.)

Fact. (6.4, elementary maps and types)

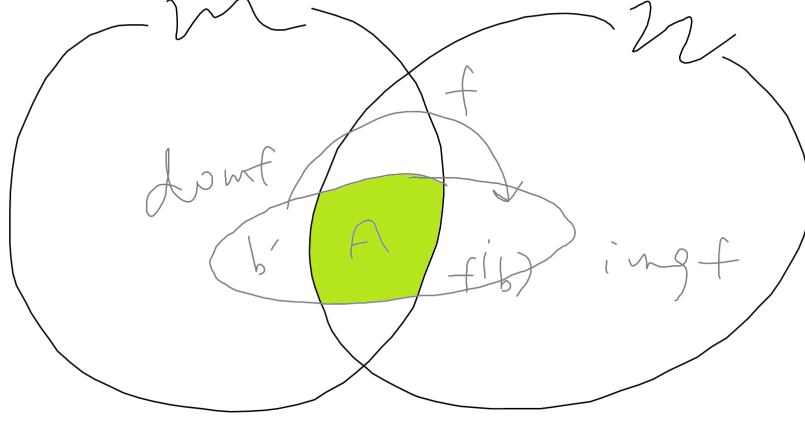
Recall definition (4.10):

- (i) If $f : A \subseteq \mathcal{M} \rightarrow \mathcal{N}$ is a (partial) elementary map, then in particular, f preserves L -sentences, so $\mathcal{M} \equiv \mathcal{N}$.
- (ii) If $\mathcal{M} \equiv \mathcal{N}$, then ϕ , the empty map, is an elementary map (only required to preserve sentences).
- (iii) If $f : A \subseteq \mathcal{M} \rightarrow \mathcal{N}$ is elementary, and \bar{a} is an enumeration of $A = \text{dom}(f)$, then (obviously, ϕ denotes the empty set here)

$$tp(\bar{a}/\phi) = tp(f(\bar{a})/\phi)$$

More generally: if $f : \mathcal{M} \rightarrow \mathcal{N}$ (we'll stop writing $f : A \subseteq \mathcal{M} \rightarrow \mathcal{N}$ for elementary maps now) is elementary and there is $A \subseteq M \cap N$ s.t. $A \subseteq \text{dom}(f)$, $f|_A = \text{id}$ (f fixes A pointwise), then for every tuple \bar{b} in $\text{dom}(f)$,

$$tp_{\mathcal{M}}(\bar{b}/A) = tp_{\mathcal{N}}(f(\bar{b}), A)$$



(iv) Let \bar{a} enumerate $A \subseteq M$, $A = \text{dom}(f)$ where $f : \mathcal{M} \rightarrow \mathcal{N}$ is elementary. Let $p(\bar{x}, \bar{a})$ be a type in $L(A)$ that is finitely satisfiable in \mathcal{M} . Then $p(\bar{x}, f(\bar{a}))$ is finitely satisfiable in \mathcal{N} : with massive abuse of notations, let $\{\phi_1(\bar{x}, \bar{a}), \dots, \phi_n(\bar{x}, \bar{a})\} \subseteq p(\bar{x}, \bar{a})$. By f.s. of $p(\bar{x}, \bar{a})$, $\mathcal{M} \models \exists \bar{x} \bigwedge_{i=1}^n \phi_i(\bar{x}, \bar{a})$. Then $\mathcal{N} \models \exists \bar{x} \bigwedge_{i=1}^n \phi_i(\bar{x}, f(\bar{a}))$ by elementarity of f . (Does $p(\bar{x}, \bar{a})$ satisfiable in \mathcal{M} imply $p(\bar{x}, f(\bar{a}))$ satisfiable in \mathcal{N} ? The answer is no – this doesn't hold in the infinite case.)

(Note that we haven't really discussed about the existence of saturated models. The general result is that given any model with size λ we could extend it elementarily to a model that is λ^+ saturated, but we'll have no control over the size of the extended model unless we assume some more subtle things (Lecturer was unsure of that)).

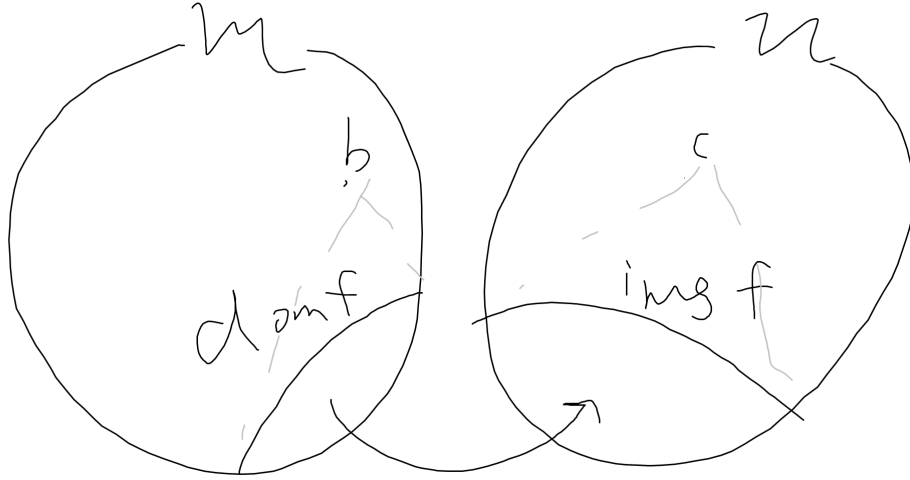
Theorem. (6.5)

Let \mathcal{N} be s.t. $|\mathcal{N}| \geq \lambda \geq |L| + \omega$. The following are equivalent:

- (i) \mathcal{N} is λ -saturated;
- (ii) if $\mathcal{M} \equiv \mathcal{N}$, $b \in M$ and $f : \mathcal{M} \rightarrow \mathcal{N}$ partial elementary s.t. $|\text{dom } f| < \lambda$, then there is partial elementary $\bar{f} \supseteq f$ such that $b \in \text{dom}(\bar{f})$;
- (iii) If $p(\bar{z})$ is an $L(A)$ -type where $|\bar{z}| \leq \lambda$, and $|A| < \lambda$ and $p(\bar{z})$ is finitely satisfiable in \mathcal{N} , then $p(\bar{z})$ is satisfiable in \mathcal{N} .

Proof. (i) \implies (ii): Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be as in (ii), and let $b \in M$. The idea is the following: look at the type of b over M , and then prove that the corresponding type in \mathcal{N} is realized in \mathcal{N} .

Let \bar{a} be an enumeration of $\text{dom}(f)$, so $|\bar{a}| < \lambda$. Let $p(x/\bar{a}) = tp_{\mathcal{M}}(b/\bar{a})$.

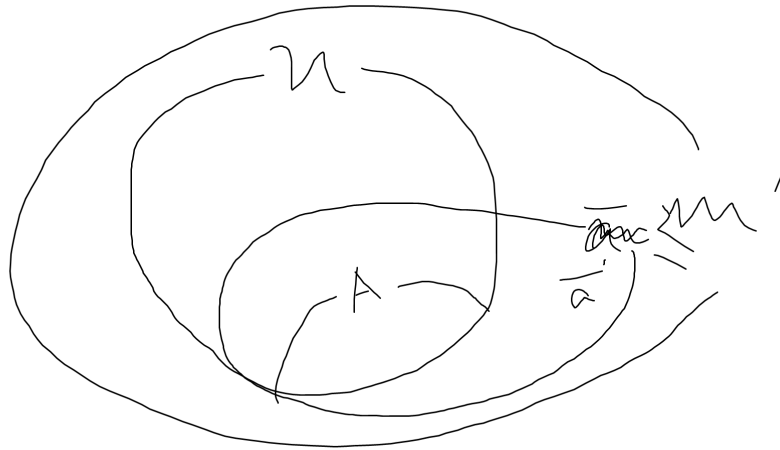


Then $p(x/\bar{a})$ is finitely satisfiable in \mathcal{M} , hence $tp(x/f(\bar{a}))$ is f.s. in \mathcal{N} (6.4 (iv)). Since $|f(\bar{a})| < \lambda$ and \mathcal{N} is λ -saturated, $tp(x/f(\bar{a}))$ is realized in \mathcal{N} by some c . Then $f \cup \{ \langle b, c \rangle \}$ is the required extension.

($\mathcal{M} \models \phi(b, \bar{a}) \iff \mathcal{N} \models \phi(c, f(\bar{a}))$), so f preserves the truth).

(ii) \implies (iii): Let $p(\bar{z})$ be as in (iii), $p(\bar{z})$ is f.s. in \mathcal{N} . Then by (5.11), $p(\bar{z})$ is realized in some elementary extension \mathcal{M}' of \mathcal{N} , by some tuple \bar{a} (where $|\bar{a}| = |\bar{z}|$).

But now we can shrink $|\mathcal{M}'|$ by downward L-S theorem: there is $\mathcal{M} \preceq \mathcal{M}'$ such that $A \cup \bar{a} \subseteq M$, where A is the parameter set of the type.



□

7 Example class 1

7.1 Question 3

- (i) if $\phi(\bar{x})$ is universal (i.e. $\forall \bar{y} \psi(\bar{x}, \bar{y})$ where $\psi(\bar{x}, \bar{y})$ is quantifier-free), then $\mathcal{N} \models \phi(\beta(\bar{a})) \implies \mathcal{M} \models \phi(\bar{a})$.
- (ii) Consider $\mathcal{N} = (\{\frac{n}{2} : n \in \mathbb{Z}\}, +)$. Assume for a X that $\phi(x)$ exists. Then $\mathcal{M} \models \phi(1)$ as 1 is odd. Now consider $\beta : M \rightarrow N$ defined by $n \rightarrow \frac{n}{2}$; it's an isomorphism between \mathcal{M} and \mathcal{N} ; Also, $id : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding. In particular, if $\mathcal{M} \models \phi(1)$, then $\mathcal{N} \models \phi(1)$; but $\mathcal{M} \equiv \mathcal{N}$ by the isomorphism, so $\mathcal{M} \models \forall x(\phi(x) \leftrightarrow \forall y(y + y \neq x))$, however that is not true in \mathcal{N} .

7.2 Question 4

- (i) \implies (ii): $id : A \rightarrow N$ is an embedding.
- (ii) \implies (i): Define structure on A : $c^{\mathcal{M}} = c^{\mathcal{N}}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap A^{n^R}$, and similar for functions.

7.3 Question 5

Prove that $\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$, where $L(M) = L \cup \{c_a : a \in M\}$.
It should be clear that everything follows from the fact that, $\mathcal{M} \models \phi(\bar{a})$ as an L -structure iff $\mathcal{M} \models \phi(c_{a_1}, \dots, c_{a_n})$ as an $L(M)$ -structure.

7.4 Question 6

We have $[0, 1] \equiv [0, 2]$ (seen in lectures, as they are isomorphic as L_{lo} -structures. However, $[0, 1] \not\equiv [0, 2]$ as 1 has a property in LHS which it doesn't have in RHS. Instead, $(0, 1) \preceq (0, 2)$. It's best to use part (i): If \mathcal{M}, \mathcal{N} are s.t. $A \subseteq M \cap N$,

then $\mathcal{M} \equiv_A \mathcal{N} \iff \mathcal{M} \equiv_B \mathcal{N}$ for every $B \stackrel{fin}{\subseteq} A$.

Select finite $B \subseteq (0, 1) \cap (0, 2)$, and show $(0, 1) \equiv_B (0, 2)$.

Induction on complexity of formulas are not strictly necessary; instead if we can find an isomorphism that fixes the elements of B then we're done.

Say $B = \{b_0, \dots, b_n\}$, where $b_m \geq b_i \forall b_i \in B$. We can just send every $x \leq b_n$ in $(0, 1)$ to the same x in $(0, 2)$, and rescale the interval $(x, 1)$ to match $(x, 2)$, which gives an isomorphism.

7.5 Question 7

Let $\phi(\bar{x})$ be a formula and $\bar{a} \in M_i^{|\bar{x}|}$.

It's required to prove that $\mathcal{M}_i \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$. By induction on $\phi(\bar{x})$:

if $\phi(\bar{x})$ is atomic, claim follows from $\mathcal{M}_i \subseteq \mathcal{N}$ (substructure); \wedge and \neg are also easy induction;

For existential formula $\phi(\bar{x}) \equiv \exists y \psi(\bar{x}, y)$, if $\mathcal{M} \models \exists y \psi(\bar{a}, y)$, then $\mathcal{M} \models \psi(\bar{a}, b)$ for b in some \mathcal{M}_i , so $\mathcal{N} \models \psi(\bar{a}, b)$;

Conversely, if $\mathcal{N} \models \exists y \psi(\bar{a}, y)$, then $\mathcal{N} \models \psi(\bar{a}, b)$ for $b \in \mathcal{N}$. But then \bar{a}, b are in some \mathcal{M}_j for some finite j ; as a result, by inductive hypothesis $\mathcal{M}_j \models \exists y \psi(\bar{a}, y)$. But then $\mathcal{M}_i \models \exists y \psi(\bar{a}, y)$ by elementarity.

Alternatively, use T-V test.

7.6 Question 8

(Lecturer is assuming CH here.....)

First show that $(\mathbb{Q} + \mathbb{R}) \not\cong (\mathbb{R}, <)$. Then for any $U \models T_{dlo}$ with $|U| = \lambda > 2^{\aleph_0}$. Then prove $(\mathbb{Q} + \mathbb{R} + U) \not\cong \mathbb{R} + U$.