Model Theory

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0 Reviews

0.1 Langauges and structures

Definition. (1.1) A language L consists of:
•(i) a set $\mathcal{F}$ of function symbols, and for each $f \in \mathcal{F}$, a positive integer n_f, the arity of f;
•(ii) a set $\mathcal{R}$ of relation symbols, and for each $R \in \mathcal{R}$, a positive integer n_R, the arity of R;
•(iii) a set $\mathcal{C}$ of constant symbols.
Note that each of the above three sets can be empty.

Example. $L = \{\{\cdot, -1\}, \{1\}\}$ where $\cdot$ is a binary function, -1 is a unary function, and 1 is a constant. We call this L_{gp} (language of groups);
$L_{lo} = \{<\}$, where $<$ is a binary relation (linear order).

Definition. (1.2) Given a language L, say, an L-structure consists of:
(i) a set M, the *domain*;
(ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \to M$;
(iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$;
(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the *interpretation* of f, R, c respectively.

Notation. (1.3)
We often fail to distinguish between the symbols in the language L and their interpretations in a L-structure, if the context allows.

We may write $\mathcal{M} =< M, \mathcal{F}, \mathcal{R}, \mathcal{C} >$.

Example. (1.4)
(a) $\mathcal{R} =< \mathbb{R}^+, \{\cdot, -1\}, 1 >$ is an L_{gp}-structure.
$\mathcal{Z} =< \mathbb{Z}, \{+, -\}, 0 >$ is also an L_{gp}-structure (here $+$ is a binary and $-$ is the unary negation function).
$\mathcal{Q} =< \mathbb{Q}, <>$ is an L_{lo} structure ($<$ is the interpretation of relation).

Definition. (1.5)
Let L be a language, let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.
An *embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an injection $\alpha : M \to N$ that preserves the structure:
(i) For all $f \in \mathcal{F}$, and $a_1, ..., a_{n_f} \in M$,

$$\alpha(f^M(a_1, ..., a_{n_f})) = f^N(\alpha(a_1), ..., \alpha(a_{n_f}))$$

(ii) For all $R \in \mathcal{R}$, and $a_1, ..., a_{n_R} \in M$,

$$(a_1, ..., a_{n_R}) \in R^M \iff (\alpha(a_1), ..., \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if. (iii) For all $c \in \mathcal{C}$, we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding $\mathcal{M} \to \mathcal{N}$ is also called an *isomorphism* of $\mathcal{M}$ onto $\mathcal{N}$.

(1.6) Exercise. Let G_1, G_2 be groups, regarded as L_{gp}-structures.
Check that $G_1 \cong G_2$ in the usual algebra sense, if and only if there is an isomprhism $\alpha : G_1\ toG_2$ in the sense of above definition 1.5.

0.2 Terms, formulae, and their interpretations

In addition to the symbols of L, we also have:
(i) infinitely many variables, $\{x_i\}_{i\in I}$;
(ii) logical connectives, $\wedge, \neg$ (also express $\vee, \rightarrow, \leftrightarrow$);
(iii) quantifier $\exists$ (also express $\forall$);
(iv) punctuations $(,)$.

Definition. (2.1)
L-terms are defined recursively as follows:
• any variable x_i is a term;
• any constant symbol is a term;
• for any $f \in \mathcal{F}$,

$$f(t_1, ..., t_{n_f})$$

for any terms $t_1, ..., t_{n_f}$ is a term;
• nothing else is a term.

Notation: we write $t(x_1, ..., x_n)$ to mean that the variables appearing in t are among $x_1, .., x_n$.

Example. In $\mathcal{R} =< \mathbb{R}, \cdot, -1, 1 >$,
• $(\cdot(x_1, x_2), x_3)$ is a term $(x_1 \cdot x_2) \cdot x_3)$;
• $(\cdot(1, x_1))^{-1}$ is a term $(1 \cdot x)^{-1}$.

Definition. (2.2)
If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, ..., x_k)$ we assign a function

$$t^M : M^k \rightarrow M$$

defined as follows:
(i) If $t = x_i, t^M[a_1, ..., a_k] = a_i$;
(ii) If $t = c$ is a constant, $t^M[a_1, ..., a_k] = c^m$;
(iii) If $t = f(t_1(x_1, ..., x_k), ..., t_{n_f}(x_1, ..., x_k))$,

$$t^M(a_1, ..., a_k) = f^M(t_1^M(a_1, ..., a_k), ..., t_{n_f}^M(a_1, ..., a_k))$$