

Representation Theory

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0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over \mathbb{C} . Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise:

- F is a field, usually \mathbb{C} , \mathbb{R} or \mathbb{Q} . When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or \bar{F}_p (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
- V is a vector space over F , always finite dimensional;
 $GL(V) = \{\theta : V \rightarrow V, \theta \text{ linear, invertible}\}$, i.e. $\det \theta \neq 0$.

Recall from Linear Algebra:

If $\dim_F V = n < \infty$, choose basis e_1, \dots, e_n over F , so we can identify it with F^n . Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \rightarrow A_\theta$ ($A_{\theta_1 \theta_2} = A_{\theta_1} A_{\theta_2}$ and bijection). Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = X A_1 X^{-1}$.

Recall that $\text{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A .

(1.3) $\text{tr}(X A X^{-1}) = \text{tr}(A)$, hence we can define $\text{tr}(\theta) = \text{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V is f.d. over \mathbb{C} , with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $\text{End} V$ is the set of all linear maps $V \rightarrow V$, e.g. $\text{End}(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over \mathbb{C} , $\alpha \in \text{End}(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1} (X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a \mathbb{C} -vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:

(1.6) The symmetric group, $S_n = \text{Sym}(X)$ on the set $X = \{1, \dots, n\}$ is the set of all permutations of X . $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m : $C_m = \langle x : x^m = 1 \rangle$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in \mathbb{C} (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of \mathbb{C}). We also have the group of rotations, centre O of regular m -gon in \mathbb{R}^2 (inside $GL_2(\mathbb{R})$).

(1.8) Dihedral groups D_{2m} of order $2m = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$. Think of this as the set of rotations and reflections preserving a regular m -gon.

(1.9) Quaternion group, $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \rightarrow X$ by $(g, x) \rightarrow g \cdot x$ for $g \in G, x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X , we obtain a homomorphism $\theta : G \rightarrow \text{Sym}(X)$, called the *permutation representation* of G .

Proof. For $g \in G$, the function $\theta_g : X \rightarrow X$ by $x \rightarrow gx$ is a permutation on X , with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G, \theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2}$ since $(g_1 g_2)x = g_1(g_2 x)$ for $x \in X$. \square

2 Basic Definitions

2.1 Representations

Let G be finite, F be a field, usually \mathbb{C} .

Definition. (2.1)

Let V be a f.d. vector space over F . A (linear, in some books) *representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \rightarrow GL(V)$$

Write ρ_g for the image $\rho_V(g)$; so for each $g \in G$, $\rho_g \in GL(V)$, and $\rho_{g_1 g_2} = \rho_{g_1} \rho_{g_2}$, and $(\rho_g)^{-1} = \rho_{g^{-1}}$.

The *dimension* (or *degree*) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \triangleleft G$ (kernel is a normal subgroup), and $G/\ker \rho \cong \rho(G) \leq GL(V)$ (1st isomorphism theorem). We say ρ is *faithful* if $\ker \rho = 1$.

An alternative (and equivalent) approach is to observe that a representation of G on V is "the same as" a *linear action* of G :

Definition. (2.3)

G *acts linearly* on V if there exists a *linear action*

$$\begin{aligned} G \times V &\rightarrow V \\ (g, v) &\rightarrow gv \end{aligned}$$

By linear action we mean: (action) $(g_1 g_2)v = g_1(g_2 v)$, $1v = v \ \forall g_1, g_2 \in G, v \in V$, and (linear) $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv \ \forall g \in G, v_1, v_2 \in V, \lambda \in F$.

Now if G acts linearly on V , the map

$$\begin{aligned} G &\rightarrow GL(V) \\ g &\rightarrow \rho_g \end{aligned}$$

with $\rho_g : v \rightarrow gv$ is a representation of G . Conversely, given a representation $\rho : G \rightarrow GL(V)$, we have a linear action of G on V via $g \cdot v := \rho(g)v \ \forall v \in V, g \in G$.

(2.4) In (2.3) we also say that V is a G -space or that V is a G -module. In fact if we define the *group algebra* FG , or $F[G]$, to be $\{\sum \alpha_j g : \alpha_j \in F\}$ with natural addition and multiplication, then V is actually a FG -module (in the sense from GRM).

(2.5) R is a *matrix representation* of G of degree n if R is a homomorphism $G \rightarrow GL_n(F)$. Given representation $\rho : G \rightarrow GL(V)$ with $\dim_F V = n$, fix basis B ; we get matrix representation

$$\begin{aligned} G &\rightarrow GL_n(F) \\ g &\rightarrow [\rho(g)]_B \end{aligned}$$

Conversely, given matrix representation $R : G \rightarrow GL_n(F)$, we get representation

$$\begin{aligned}\rho : G &\rightarrow GL(F^n) \\ g &\rightarrow \rho_g\end{aligned}$$

via $\rho_g(v) = R_g v$ where R_g is the matrix of g .

Example. (2.6)

Given any group G , take $V = F$ the 1-dimensional space, and

$$\begin{aligned}\rho : G &\rightarrow GL(F) \\ g &\rightarrow (id : F \rightarrow F)\end{aligned}$$

is known as the trivial representation of G . So $\deg \rho = 1$ ($\dim_F F = 1$).

Example. (2.7)

Let $G = C_4 = \langle x : x^4 = 1 \rangle$. Let $n = 2$, and $F = \mathbb{C}$. Note that any $R : x \rightarrow X$ will determine $x^j \rightarrow X^j$ as it is a homomorphism, and also we need $X^4 = I$. So we can take X to be diagonal matrix – any such with diagonal entries a root to $x^4 = 1$, i.e. $\{\pm 1, \pm i\}$, or if X is not diagonal then it will be similar to a diagonal matrix by (1.4) ($X^4 = I$).

2.2 Equivalent representations

Definition. (2.8)

Fix G, F . Let V, V' be F -spaces, and $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$ which are representations of G . The linear map $\phi : V \rightarrow V'$ is a G -homomorphism if

$$\phi \rho(g) = \rho'(g) \phi \forall g \in G(*)$$

We can understand this more by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \phi \downarrow & \searrow & \downarrow \phi \\ V' & \xrightarrow{\rho'_{g'}} & V' \end{array}$$

We say ϕ *intertwines* ρ, ρ' . Write $\text{Hom}_G(V, V')$ for the F -space of all these. ϕ is a G -isomorphism if it is also bijective; if such ϕ exists, ρ, ρ' are isomorphic/equivalent representations. If ϕ is a G -isomorphism, we can write (*) as $\rho' = \phi\rho\phi^{-1}$.

Lemma. (2.9)

The relation "being isomorphic" is an equivalent relation on the set of all representations of G (over F).

Remark. (2.10)

If ρ, ρ' are isomorphic representations, they have the same dimension.

The converse may be false: C_4 has four non-isomorphic 1-dimensional representations: if $\omega = e^{2\pi i/4}$ then they are $\rho_j(x^i) = \omega^{ij}$ ($0 \leq i \leq 3$).

Remark. (2.11)

Given G, V over F of dimension n and $\rho : G \rightarrow GL(V)$. Fix basis B for V : we get a linear isomorphism

$$\begin{aligned} \phi : V &\rightarrow F^n \\ v &\rightarrow [v]_B \end{aligned}$$

and we get a representation $\rho' : G \rightarrow GL(F^n)$ isomorphic to ρ :

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \\ \downarrow \phi & & \downarrow \phi \\ F^n & \xrightarrow{\rho'} & F^n \end{array}$$

(2.12) In terms of matrix representations, we have

$$\begin{aligned} R : G &\rightarrow GL_n(F), \\ R' : G &\rightarrow GL_n(F) \end{aligned}$$

are (G) -isomorphic or equivalent if there exists a nonsingular matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1} \forall g \in G$.

In terms of linear G -actions, the actions of G on V, V' are G -isomorphic if there exists isomorphisms $\phi : V \rightarrow V'$ such that $g : \phi(v) = \phi(gv) \forall v \in V, g \in G$.

2.3 Subrepresentations

Definition. (2.13)

Let $\rho : G \rightarrow GL(V)$ be a representation of G . We say $W \leq V$ is a G -subspace if it's a subspace and it is $\rho(G)$ -invariant, i.e. $\rho_g(W) \leq W \forall g \in G$. Obviously $\{0\}$ and V are G -subspaces, however.

ρ is *irreducible/simple* representation if there are no proper G -subspaces.

Example. (2.14)

Any 1-dimensional representation of G is irreducible, but not conversely, e.g. D_8 has 2-dimensional \mathbb{C} -irreducible representation.

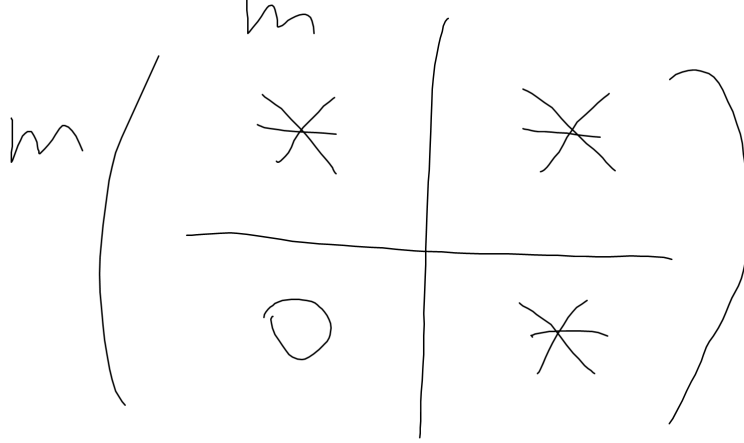
(2.15) In definition (2.13), if W is a G -subspace, then the corresponding map

$$\begin{aligned} G &\rightarrow GL(W) \\ g &\rightarrow \rho(g)|_W \end{aligned}$$

is a representation of G , a *subrepresentation* of ρ .

Lemma. (2.16)

In definition (2.13), given $\rho : G \rightarrow GL(V)$, if W is a G -subspace of V and if $B = \{v_1, \dots, v_n\}$ is a basis containing basis $B_1 = \{v_1, \dots, v_m\}$ of W ($0 < m < n$) then the matrix of $\rho(g)$ w.r.t. B has block upper triangular form as the graph below, for



each $g \in G$.

Example. (2.17)

(i) The irreducible representations of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1-dimensional and four of these are $x \rightarrow i, x \rightarrow -1, x \rightarrow -i, x \rightarrow 1$. In general, $C_m = \langle x : x^m = 1 \rangle$ has precisely m irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)* or see (4.4) below).

(ii) $G = D_6$: any irreducible C -representation has dimension ≤ 2 .

Let $\rho : G \rightarrow GL(V)$ be irreducible G -representation. Let r, s be rotation and reflection in D_6 respectively. Let V be eigenvector of $\rho(r)$. So $\rho(r)v = \lambda v$

for some $\lambda \neq 0$. Let $W = \text{span}\{v, \rho(s)v\} \leq V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, both of which are in W ; so W is G -invariant, i.e. a G -subspace. Since V is irreducible, $W = V$.

Definition. (2.18)

We say that $\rho : G \rightarrow GL(V)$ is *decomposable* if there are proper G -invariant subspaces U, W with $V = U \oplus W$. Say ρ is direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists, we say that ρ is *indecomposable*.

Lemma. (2.19)

Suppose $\rho : G \rightarrow GL(V)$ is decomposable with G -invariant decomposition $V = U \oplus W$. If B is a basis $\{\underbrace{u_1, \dots, u_k}_{B_1}, \underbrace{w_1, \dots, w_l}_{B_2}\}$ of V consisting of basis of U and basis of W , then w.r.t. B , $\rho(g)_B$ is a block diagonal matrix $\forall g \in G$ as

$$\rho(g)_B = \begin{pmatrix} [\rho_U(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

Definition. (2.20)

If $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$, the *direct sum* of ρ, ρ' is

$$\rho \oplus \rho' : G \rightarrow GL(V \oplus V')$$

where $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$, a *block diagonal action*. For matrix representations $R : G \rightarrow GL_n(F)$, $R' : G \rightarrow GL_{n'}(F)$, define $R \oplus R' : G \rightarrow GL_{n+n'}(F)$:

$$g \rightarrow \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Definition. (3.1)

A representation $\rho : G \rightarrow GL(V)$ is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

Remark. (3.2)

- (1) The converse is false;
- (2) See sheet 1 Q3: \mathbb{C} -representation of \mathbb{Z} is not completely reducible and also representation of C_p over \mathbb{F}_p is not c.r..

From now on, take G finite and $\text{char } F = 0$.

Theorem. (3.3)

Every f.d. representation V of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus \dots \oplus V_r$$

is a direct sum of representations, each V_i irreducible.

It is enough to prove:

Theorem. (3.4 Maschke's theorem, 1899)

Let G be finite, $\rho : G \rightarrow GL(V)$ a f.d. representation, $\text{char } F = 0$. If W is a G -subspace of V , then there exists a G -subspace U of V s.t. $V = W \oplus U$, a direct sum of G -subspaces.

Proof. (1)

Let W' be any *vector subspace* complement of W in V , i.e. $V = W \oplus W'$ as vector spaces, and $W \cap W' = 0$. Let $q : V \rightarrow W$ be the projection of V onto W along W' ($\ker q = W'$), i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\bar{q} : v \rightarrow \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of q over G . Note that in order for $\frac{1}{|G|}$ to exist, we need $\text{char } F = 0$.

It still works if $\text{char } F \nmid |G|$.

Claim (1): $\bar{q} : V \rightarrow W$: For $v \in V$, $g(q(g^{-1}v)) \in W$ and $gW \leq W$;

Claim (2): $\bar{q}(w) = w$ for $w \in W$:

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w = w$$

So these two claims imply that \bar{q} projects V onto W .

Claim (3) If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ ($v \in V$):

$$\begin{aligned}
 h\bar{q}(v) &= h \frac{1}{|G|} \sum_g g \cdot q(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g hgq(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g (hg)q((hg)^{-1}hv) \\
 &= \frac{1}{|G|} \sum_g gq(g^{-1}(hv)) \\
 &= \bar{q}(hv) \\
 &= \bar{q}(hv)
 \end{aligned}$$

We'll then show that the kernel of this map is G -invariant, so this gives a G -summand on Thursday.

Let's now show $\ker \bar{q}$ is G -invariant. If $v \in \ker \bar{q}$, then $h\bar{q}(v) = 0 = \bar{q}(hv)$, so $hv \in \ker \bar{q}$. Thus $V = \text{im } \bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$ is a G -subspace decomposition.

We can deduce (3.3) from (3.4) by induction on $\dim V$. If $\dim V = 0$ or V is irreducible, then result is clear. Otherwise, V has non-trivial G -invariant subspace, W . Then by (3.4), there exists G -invariant complement U s.t. $V = U \oplus W$ as representations of G . But $\dim U, \dim W < \dim V$. So by induction they can be broken up into direct sum of irreducible subrepresentations. \square

The second proof uses inner products, hence we need to take $F = \mathbb{C}$ and can be generalised to compact groups in section 15.

Recall, for V a \mathbb{C} -space, \langle, \rangle is a *Hermitian inner product* if

- (a) $\langle w, v \rangle = \overline{\langle v, w \rangle} \ \forall v, w$ (Hermitian);
- (b) linear in RHS (sesquilinear);
- (c) $\langle v, v \rangle > 0$ iff $v \neq 0$ (positive definite).

Additionally, \langle, \rangle is *G -invariant* if

- (d) $\langle gv, gw \rangle = \langle v, w \rangle \ \forall v, w \in V, g \in G$.

Note if W is G -invariant subspace of V , with G -invariant inner product, then W^\perp is also G -invariant, and $V \oplus W^\perp$. For all $v \in W^\perp, g \in G$, we have to show that $gv \in W^\perp$. But $v \in W^\perp \iff \langle v, w \rangle = 0 \ \forall w \in W$. Thus by (d), $\langle gv, gw \rangle = 0 \ \forall g \in G, w \in W$. Hence $\langle gv, w' \rangle = 0 \ \forall w' \in W$. Since we can choose $w = g^{-1}w' \in W$ by G -invariance of W . Thus $gv \in W^\perp$ since g was arbitrary.

Hence if there is a G -invariant inner product on any G -space, we get another proof of Maschke's theorem:

(3.4*) (Weyl's unitary trick)

Let ρ be a complex representation of the finite group G on the \mathbb{C} -space V . Then there is a G -invariant Hermitian inner product on V .

Remark. Recall the *unitary group* $U(V)$ on V : $\{f \in GL(V) : (fu, fv) = (u, v) \ \forall u, v \in V\} = \{A \in GL_n(\mathbb{C}) : A\bar{A}^T = I\} (= U(n))$ by choosing orthonormal

basis.

Sheet 1 Q.12: any finite subgroup of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Proof. (2)

There exist an inner product on V : take basis e_1, \dots, e_n and define $(e_i, e_j) = \delta_{ij}$, extended sesquilinearly. Now

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} (gv, gw)$$

we claim that \langle, \rangle is sesquilinear, positive definite and G -invariant: if $h \in G$, then

$$\begin{aligned} \langle hv, hw \rangle &= \frac{1}{|G|} \sum_{g \in G} ((gh)v, (gh)w) \\ &= \frac{1}{|G|} \sum_{g' \in G} (g'v, g'w) \\ &= \langle v, w \rangle \end{aligned}$$

for all $v, w \in V$. □

Definition. (3.5, the regular representation)

Recall *group algebra* of G is F -space $FG = \text{span}\{e_g : g \in G\}$. There is a linear G -action

$$h \in G, h \sum_{g \in G} a_g e_g = \sum_{g \in G} a_g e_{hg} (= \sum_{g' \in G} a_{h^{-1}g'} e_{g'})$$

ρ_{reg} is the corresponding representation, the *regular representation* of G . This is faithful of $\dim |G|$. FG is the *regular module*.

Proposition. Let ρ be an irreducible representation of G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg} .

Proof. Take $\rho : G \rightarrow GL(V)$ irreducible and let $0 \neq v \in V$. Let $\theta : FG \rightarrow V$ by $\sum a_g e_g \rightarrow \sum a_g gv$. Check this is a G -homomorphism. Now V is irreducible so $\text{im } \theta = V$ (since $\text{im } \theta$ is a G -subspace).

Also $\ker \theta$ is G -subspace of FG . Let W be G -complement of $\ker \theta$ in FG (Maschke), so that $W < FG$ is G -subspace and $FG = \ker \theta \oplus W$. Thus $W \cong FG / \ker \theta \cong (G\text{-isomorphism}) \text{im } \theta \cong V$. □

More generally,

Definition. (3.7)

Let F be a field. Let G act on set X . Let $FX = \text{span}\{e_x : x \in X\}$ with G -action

$$g(\sum a_x e_x) = \sum a_x e_{gx}$$

The representation $G \rightarrow GL(V)$ where $V = FX$ is the corresponding *permutation representation*. See section 7.

4 Schur's lemma

It's really unfair that such an important result is only remembered by a lemma, so we shall call it a theorem.

Theorem. (4.1, Schur)

- (a) Assume V, W are irreducible G -spaces over field F . Then any G -homomorphism $\theta : V \rightarrow W$ is either 0 or an isomorphism.
- (b) Assume F is algebraically closed, and let V be an irreducible G -space. Then any G -endomorphism $V \rightarrow V$ is a scalar multiple of the identity map ι_V .

Proof. (a) Let $\theta : V \rightarrow W$ be a G -homomorphism. Then $\ker \theta$ is G subspace of V and, since V is irreducible, we get $\ker \theta = 0$ or $\ker \theta = V$.

And $\text{im} \theta$ is G -subspace of W , so as W is irreducible, $\text{im} \theta$ is either 0 or W . Hence, either $\theta = 0$ or θ is injective and surjective, hence isomorphism.

(b) Since F is algebraically closed, θ has an eigenvalue, λ . Then $\theta - \lambda \iota$ is singular G -endomorphism of V , but it cannot be an isomorphism, so it is 0 (by (a)). So $\theta = \lambda \iota_V$. \square

Recall from (2.8), the F -space $\text{Hom}_G(V, W)$ of all G -homomorphisms $V \rightarrow W$. Write $\text{End}_G(V)$ for the G -endomorphisms of V .

Corollary. (4.2)

If V, W are irreducible complex G -spaces, then

$$\dim_{\mathbb{C}} \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V, W are not G -isomorphic then the only G -homomorphism $V \rightarrow W$ is 0 by (4.1). Assume $v \cong_G W$ and $\theta_1, \theta_2 \in \text{Hom}_G(V, W)$, both non-zero. Then θ_2 is invertible by (4.1), and $\theta_2^{-1}\theta_1 \in \text{End}_G(V)$, and non-zero, so $\theta_2^{-1}\theta_1 = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$. Hence $\theta_1 = \lambda \theta_2$. \square

Corollary. (4.3)

If finite group G has a faithful complex irreducible representation, then $Z(G)$, the centre of the group, is cyclic.

Note that the converse is false (Sheet 1, Q10).

Proof. Let $\rho : G \rightarrow GL(V)$ be faithful irreducible complex representation. Let $z \in Z(G)$, so $zg = gz \forall g \in G$, hence the map $\phi_z : v \rightarrow z(v)$ ($v \in V$) is G -endomorphism of V , hence is multiplication by scalar μ_z , say.

By Schur's lemma, $z(v) = \mu_z v \forall v$. Then the map

$$\begin{aligned} Z(G) &\rightarrow \mathbb{C}^* \text{ (multiplicative group)} \\ z &\rightarrow \mu_z \end{aligned}$$

is a representation of Z and is faithful, since ρ is. Thus $Z(G)$ is isomorphic to some finite subgroup of \mathbb{C}^* , so is cyclic. \square

Let's now consider representation of finite abelian groups.

Corollary. (4.4)

The irreducible \mathbb{C} -representations of a finite abelian group are all 1-dimensional.

Proof. Either: use (1.4)* to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$, and if V is irreducible, then $V = \langle v \rangle$.

Or: Let V be an irreducible \mathbb{C} -representation. For $g \in G$, the map

$$\begin{array}{ccc} \theta_g : V & \rightarrow & v \\ v & \rightarrow & gv \end{array}$$

is a G -endomorphism of V , and as V irreducible, $\theta_g = \lambda_g \text{id}_V$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any $g \in G$ (so $\langle v \rangle$ is a G -subspace of V). Thus as $0 \neq V$ is irreducible, $V = \langle v \rangle$, which is 1-dimensional. \square

Remark. Schur's lemma fails over non-algebraically closed field, in particular, over \mathbb{R} . For example, let's consider the cyclic group C_3 . It has 2 irreducible \mathbb{R} -representations, one of dimension 1 (maps everything to 1) and one of dimension 2 (imo consider \mathbb{C} as a dimension 2 space over \mathbb{R} , then map the generator to the 3rd root of unity?) (so 'contradicting' with Schur's lemma via the corollary above).

Recall that every finite abelian group G is isomorphic to a product of cyclic groups (see GRM). For example, $C_6 = C_2 \times C_3$. In fact, it can be written as a product of C_{p^α} for various primes p and $\alpha \geq 1$, and the factors are uniquely determined up to reordering.

Proposition. (4.5)

The finite abelian group $G = C_{n_1} \times \dots \times C_{n_r}$ has precisely $|G|$ irreducible \mathbb{C} -representations, as described below:

Proof. Write $G = \langle x_1 \rangle \times \dots \times \langle x_r \rangle$ where $|x_j| = n_j$. Suppose ρ is irreducible, so by (4.4), it's 1-dimensional: $\rho : G \rightarrow \mathbb{C}^*$.

Let $\rho(1, \dots, x_j, \dots, 1)$ (all 1 apart from the j^{th} entry) be λ_j . Then $\lambda_j^{n_j} = 1$, so λ_j is a n_j -th root of unity. Now, the values $(\lambda_1, \dots, \lambda_r)$ determine ρ :

$$\rho(x_1^{j_1}, \dots, x_r^{j_r}) = \lambda_1^{j_1} \dots \lambda_r^{j_r}$$

thus $\rho \leftrightarrow (\lambda_1, \dots, \lambda_r)$ with $\lambda_j^{n_j} = 1 \forall j$; we have $n_1 \dots n_r$ such r -tuples, each giving 1-dimensional representation. \square

Example. (4.6)

Consider $G = C_4 = \langle x \rangle$. We could have $\rho_1(x) = 1, \rho_2(x) = i, \rho_3(x) = -1, \rho_4(x) = -i$.

Warning: There is no "natural" 1-1 correspondence between the elements of G and the representations of G (G -finite abelian). If you choose an isomorphism $G \cong C_{a_1} \times \dots \times C_{a_r}$, then we can identify the two sets (elements of groups and representations of G), but it depends on the choice of isomorphism.

Isotypical decomposition:

Recall any diagonalisable endomorphism $\alpha : V \rightarrow V$ gives eigenspace decomposition of $V \cong \oplus_\lambda V(\lambda)$, where $V(\lambda) = \{v : \alpha v = \lambda v\}$. This is *canonical* (one of the three useless words: *arbitrary*(anything), *canonical*(only one choice), *uniform*(you can choose, but it doesn't really matter)), in the sense that it depends on α alone (and nothing else).

There is no canonical eigenbasis of V : must choose basis in each $V(\lambda)$.

We know that in *char* 0 every representation V decomposes as $\oplus n_i V_i$, V_i irreducible, $n_i \geq 0$. How unique is this?

We have this wishlist (4.7):

- (a) Uniqueness: for each V there is only one way to decompose V as above. However, this doesn't work obviously.
- (b) Isotypes: for each V , there exists a unique collection of subrepresentations U_1, \dots, U_k s.t. $V = \oplus U_i$ and, if $V_i \subseteq U_i$ and $V_j' \subseteq U_j$ are irreducible subrepresentations, then $V_i \cong V_j'$ iff $i = j$.
- (c) Uniqueness of factors: If $\oplus_{i=1}^k V_i \cong \oplus_{i=1}^{k'} V_i'$ with V_i, V_i' irreducible, then $k = k'$, and $\exists \pi \in S_k$ such that $V_{\pi(i)}' \cong V_i$ (Krull-Schmidt theorem). For (b),(c) see Teleman section 5.

Lemma. (4.8)

Let V, V_1, V_2 be G -spaces over F .

- (i) $\text{Hom}_G(V, V_1 \oplus V_2) \cong \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2)$;
- (ii) $\text{Hom}_G(V_1 \oplus V_2, V) \cong \text{Hom}_G(V_1, V) \oplus \text{Hom}_G(V_2, V)$;

Proof. (i) Let $\pi_i : V_1 \oplus V_2 \rightarrow V_i$ be G -linear projections onto V_i , with kernel V_{3-i} ($i = 1, 2$).

Consider

$$\begin{aligned} \text{Hom}_G(V, V_1 \oplus V_2) &\rightarrow \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2) \\ \phi &\rightarrow (\pi_1 \phi, \pi_2 \phi) \end{aligned}$$

This map has inverse $(\psi_1, \psi_2) \rightarrow \psi_1 + \psi_2$. Check details.

- (ii) The map $\phi \rightarrow (\phi|_{V_1}, \phi|_{V_2})$ has inverse $(\psi_1, \psi_2) \rightarrow \psi_1 \pi_1 + \psi_2 \pi_2$. □

Lemma. Let F be algebraically closed, $V = \oplus_1^n V_i$ a decomposition of G -space into irreducible summands. Then, for each irreducible representation S of G ,

$$\#\{j : V_j \cong S\} = \dim \text{Hom}_G(S, V)$$

where $\#$ means 'number of times'. This is called the *multiplicity* of S in V .

Proof. Induction on n . $n = 0, 1$ are trivial.

If $n > 1$, $V = \oplus_1^{n-1} V_i \oplus V_n$. By (4.8) we have

$$\dim \text{Hom}_G(S, \oplus_1^{n-1} V_i \oplus V_n) = \dim \text{Hom}(S, \oplus_1^{n-1} V_i) + \underbrace{\dim \text{Hom}_G(S, V_n)}_{\text{Schur's lemma}}$$

□

Definition. (4.10)

A decomposition of V as $\oplus W_j$ where each $W_j \cong n_j$ copies of irreducible representations S_j (each non-isomorphic for each j) is the *canonical decomposition* or the decomposition into *isotypical components* W_j . For F algebraically closed, $n_j = \dim \operatorname{Hom}_G(S_j, V)$.

5 Character theory

We want to attach invariants to representation ρ of a finite group G on V . Matrix coefficients of $\rho(g)$ are basis dependent, so not true invariants.

Let's take $F = \mathbb{C}$, G finite, $\rho = \rho_V : G \rightarrow GL(V)$ be a representation of G .

Definition. (5.1)

The *character* $\chi_\rho = \chi_V = \chi$ is defined as $\chi(g) = \text{tr } \rho(g) = \text{tr } R(g)$ where $R(g)$ is any matrix representation of $\rho(g)$ w.r.t. any basis.

The degree of χ_V is $\dim_{\mathbb{C}} V$.

Thus χ is a function $G \rightarrow \mathbb{C}$. χ is *linear* (not a universal name) if $\dim V = 1$, in which case χ is a homomorphism $G \rightarrow \mathbb{C}^*$ ($= GL_1(\mathbb{C})$).

χ is irreducible if ρ is; χ is faithful if ρ is; and, χ is trivial, or principal, if ρ is the trivial representation (2.6). We write $\chi = 1_G$ in that case.

χ is a complete invariant in the sense that it determines ρ up to isomorphism – see (5.7).

Theorem. (5.2, first properties)

- (i) $\chi_V(1) = \dim_{\mathbb{C}} V$; (clear: $\text{tr } I_n = n$)
- (ii) χ_V is a *class function*, via it is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$$

Thus χ_V is constant on conjugacy classes.

- (iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, the complex conjugate;

- (iv) For two representations V, W , $\chi_{V \oplus W} = \chi_V + \chi_W$.

Proof. (ii) $\chi(hgh^{-1}) = \text{tr}(R_h R_g R_h^{-1}) = \text{tr}(R_g) = \chi(g)$.

(iii) Recall $g \in G$ has finite order, so we can assume $\rho(g)$ is represented by a diagonal matrix $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Then $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by the matrix $\text{Diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$, and hence $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\chi(g)}$ (since λ_i 's are roots of unity – since $g^k = 1$ for some k ! (I mean an exclamation mark here to express surprise) and by homomorphism we know that).

(iv) Suppose $V = V_1 \oplus V_2$, $\rho_i : G \rightarrow GL(V_i)$, $\rho : G \rightarrow GL(V)$. Take basis $B = B_1 \cup B_2$ of V w.r.t. B , $\rho(g)$ has matrix of block form $\text{Diag}([\rho_1(g)]_{B_1}, [\rho_2(g)]_{B_2})$ and as $\chi(g)$ is the trace of the above matrix, it is equal to $\text{tr } \rho_1(g) + \text{tr } \rho_2(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$. \square

Remark. We see later that χ_1, χ_2 character of G implies that $\chi_1 \chi_2$ is also a character of G : uses tensor products, see (9.6).

Lemma. (5.3)

Let $\rho : G \rightarrow GL(V)$ be a complex representation *affording* the character χ (i.e. χ is a character of ρ). Then $|\chi(g)| \leq \chi(1)$, with equality iff $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ iff $g \in \ker \rho$.

Proof. Fix g . W.r.t. basis of V of eigenvalues $\rho(g)$, the matrix of $\rho(g)$ is $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Hence $|\chi(g)| = |\sum \lambda_j| \leq \sum |\lambda_j| = \sum 1 = \dim V = \chi(1)$. Equality holds iff all λ_j are equal (to λ , say).

If $\chi(g) = \chi(1)$, then $\rho(g) = \lambda I$ has $\chi(g) = \lambda \chi(1)$. \square

Lemma. (5.4)

- (a) If χ is a complex irreducible character of G , so is $\bar{\chi}$;
- (b) Under the same assumption, so is $\varepsilon\chi$ for any linear character ε of G .

Proof. If $R : G \rightarrow GL_n(\mathbb{C})$ is a complex irreducible representation then so is $\bar{R} : G \rightarrow GL_n(\mathbb{C})$ by $g \rightarrow \bar{R}(g)$. Similarly for $R' : g \rightarrow \varepsilon(g)R(g)$ for $g \in G$. Check the details. \square

Definition. (5.5)

$\mathcal{C}(G) = \{f : G \rightarrow \mathbb{C} : f(hgh^{-1}) = f(g) \forall h, g \in G\}$, the \mathbb{C} -space of class functions (we call it a space since $f_1 + f_2 : g \rightarrow f_1(g) + f_2(g)$, $\lambda f : g \rightarrow \lambda f(g)$ are still in $\mathcal{C}(G)$), so this is a vector space.

Let $k = k(G)$ be the number of ccls of G . List the ccls $\mathcal{C}_1, \dots, \mathcal{C}_k$. Conventionally we choose $g_1 = 1, g_2, \dots, g_k$, representatives of the ccls (hence $\mathcal{C}_1 = \{1\}$). Note that $\dim_{\mathbb{C}} \mathcal{C}(G) = k$ (the characteristic functions δ_j of each ccl which maps any element in the ccl to 1 and others to 0 form a basis).

We define Hermitian inner product on $\mathcal{C}(G)$:

$$\begin{aligned} \langle f, f' \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} f'(g) \\ &= \frac{1}{|G|} \sum_{j=1}^k |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) \\ &= \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \overline{f(g_j)} f'(g_j) \end{aligned}$$

using $|\mathcal{C}_x| = |G : C_G(x)|$, where \mathcal{C}_x is the ccl of x , $C_G(x)$ is the centraliser of x . For characters

$$\langle \chi, \chi' \rangle = \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

is a real symmetric form (in fact, $\langle \chi, \chi' \rangle \in \mathbb{Z}$ – see later).

Theorem. (5.6)

The \mathbb{C} -irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$. Moreover,

- (a) If $\rho : G \rightarrow GL(V), \rho' : G \rightarrow GL(V')$ are irreducible representations of G affording characters χ, χ' respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho, \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

we call this 'row orthogonality'.

- (b) Each class function of G can be expressed as a linear combination of G . This will be proved later in section 6.

Corollary. (5.7)

Complex representations of *finite* groups are characterised by their characters. We emphasise on finiteness here: for example, $G = \mathbb{Z}$, consider $1 \rightarrow I_2, 1 \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are non-isomorphic but have same character.

Proof. Let $\rho : G \rightarrow GL(V)$ be representation affording χ (G finite over \mathbb{C}). (3.3) says

$$\rho = m_1 \rho_1 \oplus \dots \oplus m_k \rho_k$$

where ρ_1, \dots, ρ_k are irreducible, and $m_j \geq 0$. Then $m_j = \langle \chi, \chi_j \rangle$ where χ_j is afforded by ρ_j : we have $\chi = m_1 \chi_1 + \dots + m_k \chi_k$, but the ρ_i 's are orthonormal. \square

Corollary. (5.8, irreducibility criterion)

If ρ is \mathbb{C} -representation of G affording χ , then ρ irreducible $\iff \langle \chi, \chi \rangle = 1$.

Proof. Forward is just the statement of orthonormality. Conversely, assume $\langle \chi, \chi \rangle = 1$. Now take a (complete) decomposition of ρ and take characters of it we get $\chi = \sum m_j \chi_j$ with χ_j irreducible and $m_j \geq 0$. Then $\sum m_j^2 = 1$. Hence $\chi = \chi_j$ for some j (since the m_j 's are obviously integers), so is irreducible. \square

Corollary. (5.9)

If the irreducible \mathbb{C} -representations of G are ρ_1, \dots, ρ_k have dimensions n_1, \dots, n_k , then

$$|G| = \sum_{i=1}^k n_i^2$$

Proof. Recall from (3.5), $\rho_{reg} : G \rightarrow GL(\mathbb{C}G)$, the regular representation G of dimension $|G|$ (where $\mathbb{C}G$ is just a G -space with basis $\{e_g : g \in G\}$ and any $h \in G$ permutes the e_g : $e_g \rightarrow e_{hg}$).

Let π_{reg} be its character, the *regular character* of G .

Claim 1: $\pi_{reg}(1) = |G|$, $\pi_{reg}(h) = 0$ if $h \neq 1$.

This is clear: take $h \in G, h \neq 1$, then we always have 0 down the diagonal since h permutes things around, so the trace is 0; if $h = 1$ then we have an identity matrix so trace is $\dim \rho = |G|$.

Claim 2: $\pi_{reg} = \sum n_j \chi_j$ with $n_j = \chi_j(1)$.

This is because

$$\begin{aligned} n_j &= \langle \pi_{reg}, \chi_j \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\pi_{reg}(g)} \chi_j(g) \\ &= \frac{1}{|G|} \cdot |G| \chi_j(1) = \chi_j(1) \end{aligned}$$

(all the other $\pi_{reg}(g)$ are zero by claim 1).

Our corollary is then obvious by just calculating $|G| = \pi_{reg}(1)$. \square

Corollary. (5.10)

Number of irreducible characters of G (up to equivalence) = k (=number of ccls).

Corollary. (5.11)

Elements $g_1, g_2 \in G$ are conjugate iff $\chi(g_1) = \chi(g_2)$ for all irreducible characters of G .

Proof. Forward: characters are class functions;

Backward: Let δ be the characteristic function of the class of g_1 . In particular, δ is a class function, so can be written as a linear combination of the irreducible characters of G . Hence $\delta(g_2) = \delta(g_1) = 1$, so $g_2 \in \mathcal{C}_G(g_1)$. \square

In the end let's introduce a good friend which will be around for the next few lectures:

Recall from (5.5), the inner product on $\mathcal{C}(G)$ and the real symmetric form \langle, \rangle on characters:

Definition. The *character table* of G is the $k \times k$ matrix (where k is the number of ccls) $X = [\chi_i(g_j)]$, the i^{th} character on the j^{th} class, where we let $\chi_1 = 1_G, \chi_2, \dots, \chi_k$ are the irreducible characters of G , and $\mathcal{C}_1 = \{1\}, \dots, \mathcal{C}_k$ are the ccls with $g_j \in \mathcal{C}_j$ (as we defined in 5.5). So the $(i, j)^{th}$ entry of X is just $\chi_i(g_j)$.

Example. (5.13)

(a) $C_3 = \langle x : x^3 = 1 \rangle$. The character table is

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = e^{2\pi i/3}$.

(b) $G = D_6 \cong S_3 = \langle r, s : r^3 = s^2 = 1, sr^{-1} = r^{-1} \rangle$.

ccls of G : $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r, r^{-1}\}$, $\mathcal{C}_3 = \{s, sr, sr^2\}$. We have 3 irreducible representations over \mathbb{C} : 1_G (trivial); \mathcal{S} (sign): $x \rightarrow 1$ for x even, $x \rightarrow -1$ for x odd; and W (2-dimensional): sr^i acts by matrix with eigenvalues ± 1 ; r^k acts by the matrix

$$\begin{pmatrix} \cos 2k\pi/3 & -\sin 2k\pi/3 \\ \sin 2k\pi/3 & \cos 2k\pi/3 \end{pmatrix}$$

so $\chi_w(sr^i) = 0 \ \forall j$, $\chi_w(r^k) = 2 \cos 2k\pi/3 = -1 \ \forall k$. So the charactable is:

	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3
1_G	1	1	1
χ_s	1	-1	1
χ_w	2	0	-1

6 Proofs and orthogonality

We want to prove(5.6): irreducible characters form orthonormal basis for the space of \mathbb{C} -class functions.

Proof. (of 5.6 (a))

Fix bases of V and V' . Write $R(g)$, $R'(g)$ for matrices of $\rho(g)$, $\rho'(g)$ w.r.t. these bases, respectively. Then

$$\begin{aligned}\langle \chi', \chi \rangle &= \frac{1}{|G|} \chi'(g^{-1}) \chi(g) \\ &= \frac{1}{|G|} \sum_{g \in G, i, j \text{ s.t. } 1 \leq i \leq n', 1 \leq j \leq n} R'(g^{-1})_{ii} R(g)_{jj}\end{aligned}$$

the trick is to define something that annihilates almost the whole thing. Let $\phi : V \rightarrow V'$ be linear and define

$$\begin{aligned}\tilde{\phi} : V &\rightarrow V' \\ v &\rightarrow \frac{1}{|G|} \sum_{g \in G} \rho'(g^{-1}) \phi \rho(g) v\end{aligned}$$

We claim that this is a G -homomorphism: if $h \in G$, let's calculate

$$\begin{aligned}\rho'(h^{-1}) \tilde{\phi} \rho(h)(v) &= \frac{1}{|G|} \sum_{g \in G} \rho'(gh)^{-1} \phi \rho(gh)(v) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho'(g'^{-1}) \phi \rho(g')(v) \\ &= \tilde{\phi}(v)\end{aligned}$$

(when g runs through G , gh runs through G as well). So (2.8) is satisfied, i.e. ϕ is a G -homomorphism.

Case 1: ρ, ρ' are not isomorphic. Schur's lemma says $\tilde{\phi} = 0$ for any given linear $\phi : V \rightarrow V'$. Take $\phi = \varepsilon_{\alpha\beta}$, having matrix $E_{\alpha\beta}$ (w.r.t our basis). This is 0 everywhere except 1 in the (α, β) -position. Then $\varepsilon_{\tilde{\alpha}\tilde{\beta}} = 0$. So $\frac{1}{|G|} \sum_{g \in G} (R'(g^{-1}) E_{\alpha\beta} R(g))_{ij} = 0$. So $\frac{1}{|G|} \sum R'(G^{-1})_{i\alpha} R(g)_{\beta j} = 0 \forall i, j$, with $\alpha = i, \beta = j$. Now $\frac{1}{|G|} \sum_{g \in G} R'(g^{-1})_{ii} R(g)_{jj} = 0$ sum over i, j . Then $\langle \chi', \chi \rangle = 0$. Case 2: ρ, ρ' isomorphic. So $\chi = \chi'$; take $V = V'$, $\rho = \rho'$. If $\phi : V \rightarrow V$ is linear endomorphism, we claim $\text{tr } \phi = \text{tr } \tilde{\phi}$:

$$\text{tr } \tilde{\phi} = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\rho(g)^{-1} \phi \rho(g)) = \frac{1}{|G|} \sum_{g \in G} \text{tr } \phi = \text{tr } \phi$$

By Schur's lemma, $\tilde{\phi} = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$ (depending on ϕ). Then $\lambda = \frac{1}{n} \text{tr } \phi$. Let $\phi = \varepsilon_{\alpha\beta}$. So $\text{tr } \phi = \delta_{\alpha\beta}$. Hence $\varepsilon_{\tilde{\alpha}\tilde{\beta}} = \frac{1}{n} \delta_{\alpha\beta} \iota_v = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \varepsilon_{\alpha\beta} \rho(g)$. In terms of matrices, take (i, j) -entry: $\frac{1}{|G|} \sum_j R(g^{-1})_{i\alpha} R(g)_{\beta j} = \frac{1}{n} \delta_{\alpha\beta} \delta_{ij} \forall i, j$. Put $\alpha = i, \beta = j$ to get $\frac{1}{|G|} \sum_g R(g^{-1})_{ii} R(g)_{jj} = \frac{1}{n} \delta_{ij}$. Finally sum over i, j to get $\langle \chi, \chi \rangle = 1$. \square

Before proving (b), let's prove column orthogonality:

Theorem. (6.1, column orthogonality relations)

$$\sum_{i=1}^k \overline{\chi_i(g_j)} \chi_i(g_l) = \delta_{jl} |C_G(g_j)|$$

having an easy corollary

Corollary. (6.2)

$$|G| = \sum_{i=1}^k \chi_i^2(1).$$

Proof. (of (6.1))

$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum \overline{\chi_i(g_l)} \chi_j(g_l) / |C_G(g_l)|$. Consider the character table $X = (\chi_i(g_j))$. Then $\bar{X} D^{-1} X^T = I_{k \times k}$ where $D = \text{Diag}(|C_G(g_1)|, \dots, |C_G(g_k)|)$.

Since X is square, it follows that $D^{-1} \bar{X}^T$ is the inverse of X , so $\bar{X}^T X = D$. \square

Proof. (of (5.6(b)))

The χ_i generate \mathcal{C}_G . Let all the irreducible characters χ_1, \dots, χ_l of G : claim these generate \mathcal{C}_G , the \mathbb{C} -space of class functions on G . It's enough to show that the orthogonal complement to $\text{span}\{\chi_1, \dots, \chi_l\}$ in \mathcal{C}_G is $\{0\}$. To see this, assume $f \in \mathcal{C}_G$ with $\langle f, \chi_j \rangle = 0 \forall j$. Let $\rho : G \rightarrow GL(V)$ be irreducible representation affording $\chi \in \{\chi_1, \dots, \chi_l\}$. Then $\langle f, \chi \rangle = 0$.

Consider

$$\frac{1}{|G|} \sum_G \overline{f(g)} \rho(g) : V \rightarrow V$$

This is a G -homomorphism, so as ρ is irreducible, it must be λ_i for some $\lambda \in \mathbb{C}$.

Now

$$\begin{aligned} n\lambda &= \text{tr} \frac{1}{|G|} \sum_g \overline{f(g)} \rho(g) \\ &= \frac{1}{|G|} \sum_g \overline{f(g)} \chi(g) = 0 = \langle f, \chi \rangle \end{aligned}$$

So $\lambda = 0$. Hence $\sum \overline{f(g)} \rho(g) = 0$, the zero endomorphism on V for all representations ρ (complete reducibility).

Take $\rho = \rho_{\text{reg}}$ where $\rho_{\text{reg}}(g) : e_1 \rightarrow e_g$ ($g \in G$). So

$$\sum_g \overline{f(g)} \rho_{\text{reg}}(g) : e_1 \rightarrow \sum_g \overline{f(g)} e_g$$

So it follows $\sum_g \overline{f(g)} e_g = 0$. So $\overline{f(g)} = 0 \forall g \in G$, so $f \equiv 0$. \square

Various corollaries now follow:

- The number of irreducible representations of G = number of ccls; (5.10)
- Column orthogonality (6.1);
- $|G| = \sum n_i^2$ (6.2);
- $g_1 \sim g_2 \iff \chi(g_1) = \chi(g_2)$ for all irreducible χ (5.11);
- If $g \in G$, $g \sim g^{-1} \iff \chi(g) \in \mathbb{R}$ for all irreducible χ .

7 Permutation representations

Preview was given in (3.7). Recall: • G finite group acting on finite set $X = \{x_1, \dots, x_n\}$;

• $\mathbb{C}X = \mathbb{C}$ -space, with basis $\{e_{x_1}, \dots, e_{x_n}\}$ of dimension $|X|$, so is $\{\sum_j a_j e_{x_j} : a_j \in \mathbb{C}\}$;

• corresponding permutation representation $\rho_X : G \rightarrow GL(\mathbb{C}X)$ by $g \rightarrow \rho(g)$, where $\rho(g)$ sends $e_{x_j} \rightarrow e_{gx_j}$, extending linearly.

• ρ_X is the *permutation representation* corresponding to the action of G on X .

• matrices representing $\rho_X(g)$ w.r.t. basis $\{e_x\}_{x \in X}$ are permutation matrices: 0 except for one 1 in each row and column, and $(\rho(g))_{ij} = 1$ iff $gx_j = x_i$. Consider its character:

(7.1) Permutation character, π_X , is

$$\pi_X(g) = |\text{Fix}_X(g)| = |\{x \in X : gx = x\}|.$$

(7.2) ρ_X always contains 1_G : $\text{span}\{e_{x_1} + \dots + e_{x_n}\}$ is a trivial G -subspace of $\mathbb{C}X$ with G -invariant complement $\text{span}\{\sum a_x e_x : \sum a_x = 0\}$.

Lemma. (7.3, Burnside's lemma, after Cauchy, Frobenius) $\langle \pi_X, 1 \rangle =$ number of orbits of G on X .

Proof. If $X = X_1 \cup \dots \cup X_l$ disjoint union of orbits, then $\pi_X = \pi_{X_1} + \dots + \pi_{X_l}$, with π_{X_j} permutation character of G on X_j , so to prove the claim it's enough to show that if G is transitive on X then $\langle \pi_X, 1 \rangle = 1$. Assume G is transitive on X . Now

$$\begin{aligned} \langle \pi_X, 1 \rangle &= \frac{1}{|G|} \sum_g \pi_X(g) = \frac{1}{|G|} |\{(g, x) \in G \times X : gx = x\}| \\ &= \frac{1}{|G|} \sum_{x \in X} |G_x| = \frac{1}{|G|} |X| |G_x| = \frac{1}{|G|} |G| = 1 \end{aligned}$$

(Note the use of orbit-stabilizer theorem). □

Lemma. (7.4)

Let G act on the sets X_1, X_2 . Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1 \times X_2} = \pi_{X_1} \pi_{X_2}$ and so $\langle \pi_{X_1}, \pi_{X_2} \rangle =$ number of orbits of G on $X_1 \times X_2$.

Proof. If $g \in G$ then $\pi_{X_1 \times X_2}(g) = \pi_{X_1}(g) \pi_{X_2}(g)$. And we have

$$\langle \pi_{X_1}, \pi_{X_2} \rangle = \langle \pi_{X_1} \pi_{X_2}, 1 \rangle = \langle \pi_{X_1 \times X_2}, 1 \rangle = (7.3) \text{ number of orbits of } G \text{ on } X_1 \times X_2.$$

□

Definition. (7.5)

Let G act on X , $|X| > 2$. Then G is *2-transitive* on X if G has precisely two orbits on $X \times X : \{(x, x) : x \in X\}$ and $\{(x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

Lemma. (7.6)

Let G act on X , $|X| > 2$. Then $\pi_X = 1 + \chi$ with χ irreducible $\iff G$ is 2-transitive on X .

Proof. $\pi_X = m_1 1 + m_2 \chi_2 + \dots + m_l \chi_l$ with $1, \chi_2, \dots, \chi_l$ distinct irreducible characters and $m_i \in \mathbb{N}$. Then

$$\langle \pi_X, \pi_X \rangle = \sum_{i=1}^l m_i^2$$

hence G is 2-transitive on $X \iff l = 2, m_1 = m_2 = 1$. \square

Example. (7.7)

Consider S_n acting on $X = \{1, \dots, n\}$ which is 2-transitive. Hence $\pi_X = 1 + \chi$ with χ irreducible of degree $n - 1$. Similarly for A_n ($n > 3$).

Example. (7.8)

Consider $G = S_4$.

$\subset \subset 1$	1	3	8	6	6
rep	1	$(1,2)(3,4)$	(123)	(1234)	$(1,2)$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	3	-1	0	-1	1
χ_4	3	-1	0	1	-1
χ_5	2

Get by column orthogonality \leftarrow

Last lecture we were talking about using column orthogonality to find χ_5 . Indeed we have

$$\chi_{reg} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5$$

So we can use this to find χ_5 . Also, $S_4/V_4 \cong S_3$ by 'lifting' – see next chapter.

7.1 Alternating groups

Suppose $g \in A_n$. In 1A we've known that $|C_{S_n}(g)| = |S_n : C_{S_n}(g)|$ and $|C_{A_n}(g)| = |A_n : C_{A_n}(g)|$.

These are not necessarily equal. For example, $\sigma = (123) \in A_3$, $\mathcal{A}_3(\sigma) = \{\sigma\}$, but $\mathcal{S}_\exists(\sigma) = \{\sigma, \sigma^{-1}\}$.

Lemma. (7.9)

Let $g \in A_n$. Then if g commutes with some odd permutation in S_n then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$; otherwise $\mathcal{C}_{S_n}(g)$ splits into two ccls in A_n of equal size.

For example, consider $G = A_4$, so $|G| = 12$.

	1	$(12)(34)$	(123)	$(123)^{-1}$
$1\bar{G}$ $= \chi_1$	1	1	1	1
$\bar{1}\bar{G}$ $= \chi_2$	3	-1	0	0
χ_3	1	1	ω	ω^2
χ_4	1	1	ω^2	ω

Note that if we ignore the second row and first column, the table becomes identical to that of $C_3 \cong G/V_4$. This is not a coincidence, and is actually called *lifting*.

8 Normal subgroups and lifting characters

Lemma. (8.1)

Let $N \triangleleft G$. Let $\tilde{\rho} : G/N \rightarrow GL(V)$ be a representation of G/N . Then

$$\begin{array}{ccc} \rho : G & \xrightarrow{\text{canonical}} & G/N & \xrightarrow{\tilde{\rho}} & GL(V) \\ g & \rightarrow & & & \tilde{\rho}(gN) \end{array}$$

is a representation of G , where $\rho(g) := \tilde{\rho}(gN)$. Moreover, ρ is irreducible iff $\tilde{\rho}$ is irreducible.

The corresponding characters satisfy $\chi(g) = \tilde{\chi}(gN)$. We say that $\tilde{\chi}$ *lifts* to χ . The lifting $\tilde{\chi} \rightarrow \chi$ is a bijection between irreducible representations of G/N and irreducible representations of G with N in \ker .

Well this looks like Q4/Q12 in the first example sheet.

Proof. Note $\chi(g) = \text{tr}(\rho(g)) = \text{tr}(\tilde{\rho}(gN)) = \tilde{\chi}(gN) \forall g$, and $\chi(1) = \tilde{\chi}(N)$. SO have some degree (?).

Bijection: if $\tilde{\chi}$ is a charcter of G/N -representation and χ is its lift to G , then $\chi(N) = \chi(1)$. Also, if $k \in N$ then

$$\chi(k) = \tilde{\chi}(kN) = \tilde{\chi}(N) = \chi(1)$$

So $N \leq \ker \chi$.

Now let χ be character of G with $N \leq \ker \chi$. Suppose $\rho : G \rightarrow GL(V)$ affords χ . Define

$$\begin{array}{ccc} \tilde{\rho} : G/N & \rightarrow & GL(V) \\ gN & \rightarrow & \rho(g) \end{array}$$

Check this is well-defined (uses $N \leq \ker \chi$) and $\tilde{\rho}$ is homomorphism, hence gives representation of G/N . If $\tilde{\chi}$ is the character of $\tilde{\rho}$ then $\tilde{\chi}(gN) = \chi(g) \forall g \in G$. So $\tilde{\chi}$ lifts to χ .

Check irreducibility. □

Lemma. (8.2)

The derived subgroup, $G' = \langle [a, b], a, b \in G \rangle$ of G is the unique minimal normal subgroup of G s.t. G/G' is abelian, i.e. G/N is abelian $\implies G' \leq N$ and $G^{ab} = G/G'$ is abelian, where G^{ab} is the *abelianisation* of G .

G has precisely $l = |G/G'|$ representations of $\dim 1$, all with kernel containing G' and obtained by lifting from G/G' . In particular, $l \mid |G|$.

Proof. $G' \triangleleft G$ is an easy exercise.

Let $N \triangleleft G$. Let $h, g \in G$, so

$$\begin{aligned} g^{-1}h^{-1}gh \in N &\iff (gh)N = (hg)N \\ [g, h] &\iff (gN)(hN) = (hN)(gN) \end{aligned}$$

So $G' \leq N \iff G/N$ is abelian. Since $G' \triangleleft G$ we deduce G/G' is abelian.

By (4.5), G/G' has exactly l irreducible characters $\tilde{\chi}_1, \dots, \tilde{\chi}_l$ all of degree 1. The lifts of these to G also have degree 1 and by (8.1) these are precisely the irreducible characters χ_i of G s.t. $G' \leq \ker \chi_i$. But any linear character of G is a homomorphism $\chi : G \rightarrow \mathbb{C}^*$, hence $G' \leq \ker \chi$ ($\chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g^{-1})\chi(h)^{-1} = 1$), so the χ_1, \dots, χ_l are all the linear characters of G . \square

Examples:

(a) If $G = S_n$, show $s'_n = A_n$. Thus since $G/G' \cong C_2$, S_n must have exactly two linear characters.

(b) Consider $G = A_4$. We've seen previously that this can be lifted from C_3 using (8.1), (8.2).

Lemma. (8.4)

G is not simple iff $\chi(g) = \chi(1)$ for some irreducible character $\chi \neq 1_G$ and some $1 \neq g \in G$.

Any normal subgroup of G is the intersection of the kernels of some of the irreducible characters of G :

$$N = \bigcap_i \ker \chi_i$$

Proof. If $\chi(g) = \chi(1)$ for some non-trivial irreducible character χ (afforded by ρ , say). Then $g \in \ker \rho$ (5.3), so if $g \neq 1$, then $1 \neq \ker \rho \triangleleft G$.

If $1 \neq N \triangleleft G$, take irreducible $\tilde{\chi}$ of G/N , $\tilde{\chi}$ non-trivial. Lift to get an irreducible χ , afforded by ρ of G , then $N \leq \ker \rho \triangleleft G$. So $\chi(g) = \chi(1)$ for $g \in N$.

We claim that, if $1 \neq N \triangleleft G$, then N is the intersection of the kernels of the lifts of all the irreducibles of G/N .

\leq is clear from (8.1). If $g \in G \setminus N$, then $gN \neq N$. so $\tilde{\chi}(gN) \neq \tilde{\chi}(N)$ for some irreducible $\tilde{\chi}$ of G/N . Lifting $\tilde{\chi}$ to χ , we have $\chi(g) \neq \chi(1)$. \square

Recall $\ker \chi = \{g \in G : \chi(g) = \chi(1)\}$. (5.3) : $g \in \ker \chi \iff g \in \ker \rho$.

9 Dual spaces and tensor products of representations

Recall (5.5):

- $\mathcal{C}(G)$ is \mathbb{C} -space of class functions on G ;
- endowed with irreducible product, $\dim \mathcal{C}(G) = k$, orthonormal basis of irreducible characters of G (5.6)1
- there exists an involution (ring homomorphism of order 2): $f \rightarrow f^*$ where $f^*(g) = f(g^{-1})$.

Lemma. (9.1)

Let $\rho : G \rightarrow GL(V)$, representation over F , and let $V^* = Hom_F(V, F)$, dual space of V . Then V^* is a G -space under

$$(\rho^*(g)\phi)(v) = \phi(\rho(g^{-1})v)$$

called the *dual representation* to ρ . Its character is $\chi_{\rho^*}(g) = \chi_{\rho}(g^{-1})$.

Proof.

$$\begin{aligned} \rho^*(g_1)(\rho^*(g_2)\phi)(v) &= (\rho^*(g_2)\phi)(\rho(g_1^{-1})v) \\ &= \phi(\rho(g_2^{-1})\rho(g_1^{-1})v) \\ &= \phi(\rho(g_1g_2)^{-1}v) \\ &= (\rho^*(g_1g_2)\phi)(v) \end{aligned}$$

So this is a representation. For its character, fix $g \in G$ and let e_1, \dots, e_n be basis of V of eigenvectors of $\rho(g)$, say $\rho(g)e_j = \lambda_j e_j$. Let $\varepsilon_1, \dots, \varepsilon_n$ be dual basis. We claim that $\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j$:

$$(\rho^*(g)\varepsilon_j)(e_i) = \varepsilon_j(\rho(g^{-1})e_i) = \varepsilon_j\lambda_i^{-1}e_i = \lambda_j^{-1}\varepsilon_j e_i \forall i$$

So $\chi_{\rho^*}(g) = \sum \lambda_j^{-1} = \chi_{\rho}(g^{-1})$. □

Definition. (9.2)

$\rho : G \rightarrow GL(V)$ is *self-dual* if $V \cong V^*$ (as G -spaces). Over \mathbb{C} , this holds iff $\chi_{\rho}(g) = \chi_{\rho}(g^{-1})$ ($= \chi_{\rho}(g)$) $\forall g$, iff $\chi_{\rho}(g) \in \mathbb{R}$ for all g .

Exercise: all irreducible representations of S_n are self-dual (the ccls are determined by cycle type, so g, g^{-1} are always S_n -conjugate. Not always true for A_n).

9.1 tensor products

Let V, W be F -spaces, $\dim V = m$, $\dim W = n$. Fix bases v_1, \dots, v_m and w_1, \dots, w_n of V, W respectively. The *tensor product space* $V \otimes_F W$ is an nm -dimensional F -space with basis $\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Thus

(a) $V \otimes W = \{\sum_{i,j} \lambda_{ij} v_i \otimes w_j : \lambda_{ij} \in F\}$ with 'obvious' addition and scalar multiplication;

(b) If $v = \sum_i \alpha_i v_i \in V$, $w = \sum_j \beta_j w_j \in W$, define $v \otimes w := \sum_{i,j} \alpha_i \beta_j (v_i \otimes w_j)$.

Remark. Not all elements of $V \otimes W$ are of this form: some are combinations, e.g. $v_1 \otimes w_1 + v_2 \otimes w_2$, which can't be further simplified. (like entangled)

Lemma. (9.3)

- (i) For $v \in V$, $w \in W$, $\lambda \in F$, $(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$;
 (ii) If $x_1, x_2, x \in V$, $y_1, y_2, y \in W$, then

$$\begin{aligned}(x_1 + x_2) \otimes y &= (x_1 \otimes y) + (x_2 \otimes y), \\ x \otimes (y_1 + y_2) &= (x \otimes y_1) + (x \otimes y_2)\end{aligned}$$

Proof. (i) $v = \sum \alpha_i v_i$, $w = \sum \beta_j w_j$. Then just multiply out everything we get the desired equality. (ii) is similar. \square

Lemma. (9.4)

If $\{e_1, \dots, e_m\}$ is a basis of V , $\{f_1, \dots, f_n\}$ is a basis of W , then $\{e_i \otimes f_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $V \otimes W$.

Proof. Writing $v_k = \sum_i \alpha_{ik} e_i$, $w_l = \sum_j \beta_{jl} f_j$, we have

$$v_k \otimes w_l = \sum_{i,j} \alpha_{ik} \beta_{jl} e_i \otimes f_j$$

Hence $\{e_i \otimes f_j\}$ spans $V \otimes W$ and, since we have nm of them, they form a basis. \square

Remark. One can define $V \otimes W$ in a basis-independent way in the first place, see Teleman chapter 6.

Proposition. (9.5)

Let $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$ be representations of G . Define $\rho \otimes \rho' : G \rightarrow GL(V \otimes V')$ by

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij} v_i \otimes w_j \rightarrow \sum \lambda_{ij} \rho(g) v_i \otimes \rho'(g) w_j$$

Then $\rho \otimes \rho'$ is a representation of G with character

$$\chi_{\rho \otimes \rho'}(g) = \chi_\rho(g) \chi_{\rho'}(g) \forall g \in G$$

Hence product of two characters of G is still a character of G .

Proof. On Tuesday. \square

(After lecture 11: this is the first notes to get beyond 1000 lines!)

Remark. (9.6)

Sheet 1, Q2 says ρ irreducible, ρ' of degree 1, then $\rho \otimes \rho'$ irreducible; if ρ' is not of deg 1 this is usually false.

Proof. (of 9.5)

It's clear that $(\rho \otimes \rho')(g) \in GL(V \otimes V') \forall g \in G$ and so $\rho \otimes \rho'$ is a homomorphism $G \rightarrow GL(V \otimes V')$. Let $g \in G$. Let v_1, \dots, v_m be basis of V of eigenvectors of $\rho(g)$; let w_1, \dots, w_n be a basis of V' . Say:

$$\rho(g)v_j = \lambda_j v_j, \rho'(g)w_j = \mu_j w_j$$

Then

$$\begin{aligned} (\rho \otimes \rho')(g)(v_i \otimes w_j) &= \rho(g)v_i \otimes \rho'(g)w_j \\ &= \lambda_i v_i \otimes \mu_j w_j \\ &= (\lambda_i \mu_j)(v_i \otimes w_j) \end{aligned}$$

$$\text{So } \chi_{\rho \otimes \rho'}(g) = \sum_{i,j} \lambda_i \mu_j = (\sum \lambda_i)(\sum \mu_j) = \chi_\rho(g) \chi_{\rho'}(g) \quad \square$$

Now work over \mathbb{C} . Take $V = V'$ and define $V^{\otimes 2} = V \otimes V$.

Let

$$\tau : \sum \lambda_{ij} v_i \otimes v_j \rightarrow \sum \lambda_{ij} \lambda_j \otimes v_i$$

which is a linear G -endomorphism of $V^{\otimes 2}$, s.t. $\tau^2 = 1$ (so eigenvalues ± 1).

Definition. (9.7)

$$\begin{aligned} S^2 V &= \{v \in V^{\otimes 2} : \tau(x) = x\}, \\ \wedge^2 V &= \{x \in V^{\otimes 2} : \tau(x) = -x\} \end{aligned}$$

known as the *symmetric square* of V and *exterior square* of V respectively.

Lemma. (9.8)

$S^2 V$ and $\wedge^2 V$ are G -subspaces of $V^{\otimes 2}$ and $V^{\otimes 2} \cong S^2 V \oplus \wedge^2 V$. $S^2 V$ has basis $\{v_i v_j := v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$, and $\wedge^2 V$ has basis $\{v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$. Hence we have $\dim S^2 V = \frac{1}{2}n(n+1)$ and $\dim \wedge^2 V = \frac{1}{2}n(n-1)$.

Proof. Exercise in linear algebra.

To show $V^{\otimes 2}$ is reducible, write $x \in V^{\otimes 2}$ as $x = \frac{1}{2}(x + \tau(x)) + \frac{1}{2}(x - \tau(x))$, which is in $S^2 V$ and $\wedge^2 V$ respectively. \square

In fact, $V^{\otimes 2}, V^{\otimes 3} = V \otimes V \otimes V, \dots$, etc. are never irreducible if $\dim V > 1$.

Lemma. (9.9)

If $\rho : G \rightarrow GL(V)$ is a representation affording character χ , then $\chi^2 = \chi_S + \chi_\wedge$ where $\chi_S (= S^2 \chi)$ is the character of G in the subrepresentation $S^2 V$, and $\chi_\wedge (= \wedge^2 \chi)$ is the character of G in the subrepresentation $\wedge^2 V$. Moreover, for $g \in G$,

$$\chi_S(g) = \frac{1}{2}(\chi(g^2) + \chi(g)^2), \chi_\wedge(g) = \frac{1}{2}(\chi(g^2) - \chi(g)^2).$$

Proof. Let's compute the characters χ_S, χ_\wedge . Fix $g \in G$. Let v_1, \dots, v_n be a basis of eigenvectors of $\rho(g)$, say $\rho(g)v_i = \lambda_i v_i$ (we drop the ρ to write $gv_i = \lambda_i v_i$ for simplicity below). Then

$$\begin{aligned} gv_i v_j &= \lambda_i \lambda_j v_i v_j \\ gv_i \wedge v_j &= \lambda_i \lambda_j v_i \wedge v_j \end{aligned}$$

Hence $\chi_s(g) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j$ and $\chi_\wedge(g) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j$. Now,

$$\begin{aligned} (\chi(g))^2 &= \left(\sum \lambda_i \right)^2 \\ &= \sum \lambda_i^2 + 2 \sum_{i < j} \lambda_i \lambda_j \\ &= \chi(g^2) + 2 \sum_{i < j} \lambda_i \lambda_j \\ &= \chi(g^2) + 2\chi_\wedge(g) \end{aligned}$$

So $\chi_\wedge(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$. But $\chi^2 = \chi_s + \chi_\wedge$ so we get the expression for $\chi_s(g)$. \square

Example. (9.10)

Consider our usual example $G = S_4$ (see 7.8).

	1	(12)(34)	(123)	(12)	(1234)
$\downarrow \chi$	1	1	1	1	1
sign	1	1	1	-1	-1
$\chi_1 = \overline{\chi_1}$	3	-1	0	1	-1
$\overline{\chi_3}$	3	-1	0	-1	1
χ_5	2	2	-1	0	0

χ_3^2	9	1	6	1	1
$\chi_3 \chi_5$	3	3	0	3	-1
$S^2 \chi_3$	6	2	0	2	0
$\wedge^2 \chi_3$	3	-1	0	-1	1

Notice that $\wedge^2 \chi_3 = \bar{\chi}_3$ (irreducible since $\langle \chi_\wedge, \chi_\wedge \rangle = 1$),
 $S^2 \chi_3 = 1 + \chi_3 + \chi_5$: The inner product is 3 and it contains $1, \chi_3$, so the one left is χ_5 .

Characters of $G \times H$ (seen in (4.5) for abelian groups):

Proposition. (9.11)

If G, H are finite groups with irreducible characters χ_1, \dots, χ_k and ψ_1, \dots, ψ_r respectively, then the irreducible characters of the direct product $G \times H$ are precisely $\{\chi_i \psi_j : 1 \leq i \leq k, 1 \leq j \leq r\}$, where $\chi_i \psi_j(g, h) = \chi_i(g) \psi_j(h)$.

Proof. If $\rho : G \rightarrow GL(V)$, $\rho' : H \rightarrow GL(W)$ affording χ and ψ respectively, then

$$\begin{aligned} \rho \otimes \rho' : G \times H &\rightarrow GL(V \otimes W) \\ (g, h) &\rightarrow \rho(g) \otimes \rho'(h) \quad v_i \otimes w_j \rightarrow \rho(g)v_i \otimes \rho'(h)w_j \end{aligned}$$

is a representation of $G \times H$ on $V \otimes W$ by (9.5), and $\chi_{\rho \otimes \rho'} = \chi \psi$, again by (9.5). We claim that $\chi_i \psi_j$ are distinct and irreducible:

$$\begin{aligned} \langle \chi_i \psi_j, \chi_r \psi_s \rangle_{G \times H} &= \frac{1}{|G \times H|} \sum_{(g, h)} \overline{\chi_i \psi_j(g, h)} \chi_r \psi_s(g, h) \\ &= \left(\frac{1}{|G|} \overline{\chi_i(g)} \chi_r(g) \right) \left(\frac{1}{|H|} \sum_h \overline{\psi_j(h)} \psi_s(h) \right) \\ &= \delta_{ir} \delta_{js} \end{aligned}$$

...tbc.

Let's complete on $\chi_i \psi_j$ being distinct and irreducible:

Complete set: $\sum_{i,j} (\chi_i \psi_j)(1)^2 = \sum_i \chi_i(1)^2 \sum_j \psi_j(1)^2 = |G||H| = |G \times H| \quad \square$

9.2 Symmetric and exterior powers

Let V be a vector space, $\dim_F V = d$, with basis $\{v_1, \dots, v_d\}$. Let $V^{\otimes n} = V \otimes \dots \otimes V$, with basis $\{v_{i_1} \otimes \dots \otimes v_{i_n} : (i_1, \dots, i_n) \in \{1, \dots, d\}^n\}$, so $\dim V^{\otimes n} = d^n$.

S_n -action: for any $\sigma \in S_n$, we can define linear map

$$\sigma : V^{\otimes n} \rightarrow V^{\otimes n}$$

$$v_1 \otimes \dots \otimes v_n \rightarrow v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)}$$

for $v_1, \dots, v_n \in V$, permuting positions of vectors in a tensor.

For example, (12)($v_1 \otimes v_2 \otimes v_3$) = $v_2 \otimes v_1 \otimes v_3$, (13)($v_2 \otimes v_1 \otimes v_3$) = $v_3 \otimes v_1 \otimes v_2$.

Check that this defines a representation of S_n on $V^{\otimes n}$ (extended linearly).

G -action: given representation $\rho : G \rightarrow GL(V)$, then the action of G on $V^{\otimes n}$ is

$$\rho^{\otimes n}(g) : v_1 \otimes \dots \otimes v_n = \rho(g)v_1 \otimes \dots \otimes \rho(g)v_n$$

extended linearly, and this commutes with the S_n -action. We can decompose $V^{\otimes n}$ as S_n -module, and each isotypical component (4.?) is G -invariant subspace of $V^{\otimes n}$. In particular:

Definition. (9.12)

For G -space V , define

- (i) the n th symmetric power of V , $S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \forall \sigma \in S_n\}$;
 - (ii) the n th exterior power of V , $\wedge^n V = \{x \in V^{\otimes n} : \sigma(x) = \text{sign}(\sigma)x \forall \sigma \in S_n\}$.
- Both are G -subspaces of $V^{\otimes n}$, but for $n > 2$, $S^n V \oplus \wedge^n V \subsetneq V^{\otimes n}$, so in general there are lots of others for the S_n -action.

(9.13) See Sheet 3 Q7 for bases of $S^n V$, $\wedge^n V$ and their characters.

9.3 Tensor algebra

Take $\text{char } F = 0$.

Definition. (9.14)

Let $T^n V = V^{\otimes n}$. The tensor algebra of V is $TV := \bigoplus_{n \geq 0} T^n V$, $T^0 V = F$.

This is F -space and is a (non-commutative) graded ring with product $x \in T^n V$, $y \in T^m V$, $x \cdot y = x \otimes y \in T^{n+m} V$.

There are two graded quotient rings

$$SV = TV / (\text{ideal generated by all } U \otimes V - V \otimes U)$$

$$\wedge V = TV / \text{ideal generated by all } V \otimes V$$

called the symmetric algebra and exterior algebra respectively.

Definition. (9.15)

The 2-submodule of $\mathcal{C}(G)$ spanned by irreducible characters of G is the character

ring of G , $R(G)$. Elements of $R(G)$ are called generalised/virtual characters if $\psi = \sum n_\chi \chi$, $n_\chi \in \mathbb{Z}$ correspondingly.

- $R(G)$ is a commutative ring and any generalised character is a difference of two characters, $\psi = \alpha - \beta$:

$$\alpha = \sum_{n_\chi \geq 0} n_\chi \chi, \beta = - \sum_{n_\chi < 0} n_\chi \chi.$$

The $\{\chi_i\}$ form a \mathbb{Z} -basis for $R(G)$ as a free \mathbb{Z} -module.

- Suppose ψ is virtual character and $\langle \psi, \psi \rangle = 1$ and $\psi(1) > 0$. Then ψ is actually the character of an irreducible representation of G .

List irreducible characters of G : χ_1, \dots, χ_k , $\psi = \sum n_i \chi_i$; orthonormality says $\langle \psi, \psi \rangle = \sum n_i^2$, so $\sum n_i^2 = 1$, meaning $n_i = \pm 1$ for exactly one i and $n_j = 0$ for $j \neq i$. Since $\psi(1) > 0$, we must have $n_i = +1$.

- Henceforth we don't distinguish between a character and its negative and we often study generalised characters of norm 1 rather than irreducible characters.

10 Restriction and induction

Throughout we set $H \leq G$, $F = \mathbb{C}$.

Definition. (10.1, restriction)

Let $\rho : G \rightarrow GL(V)$ be representation affording χ . We can think of V as a H -space by restricting attention to $h \in H$. We then get

$$Res_H^G \rho : H \rightarrow GL(V)$$

This is sometimes written as ρ_H or $\rho \downarrow_H$, the restriction of ρ to H . It affords the character $Res_H^G \chi = \chi_H = \chi \downarrow_H$.

Lemma. (10.2)

If ψ is any non-zero character of $H \leq G$, then there exists irreducible character χ of G s.t. $\langle Res_H^G \chi, \psi \rangle_H \neq 0$. We say ψ is a constituent of $Res_H^G \chi$.

Proof.

$$0 \neq \frac{|G|}{|H|} \psi(1) = \langle \pi_{reg} \downarrow_H, \psi \rangle = \sum_1^k \deg \chi_i \langle \chi_i \downarrow_H, \psi \rangle$$

where ψ_i are irreducible characters of G . □

Lemma. (10.3)

Let χ be irreducible character of G , and let $Res_H^G \chi = \sum c_i \chi_i$ with χ_i irreducible characters of H , $c_i \in \mathbb{Z}_{\geq 0}$. Then

$$\sum c_i^2 \leq |G : H|$$

with equality iff $\chi(g) = 0 \ \forall g \in G \setminus H$.

Proof.

$$\sum c_i^2 = \langle Res_H^G \chi, Res_H^G \chi \rangle_H = \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2$$

But

$$\begin{aligned} 1 = \langle \chi, \chi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\ &= \frac{1}{|G|} \left(\sum_{h \in H} |\chi(h)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2 \right) \\ &= \frac{|H|}{|G|} \sum c_i^2 + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2}_{\geq 0} \end{aligned}$$

So $\sum c_i^2 \leq |G : H|$, with equality holds iff $\chi(g) = 0 \ \forall g \in G \setminus H$. □

Example. Let $G = S_5$, $H = A_5$. This has 7 representations of degree 1, 1, 4, 4, 5, 5, 6 respectively, where if we restrict to H , the two representations of degree 1, 4, 5 combines into one of the same degree respectively; however, the

degree 6 representation splits into two irreducible representations of degree 3. In the first case we have $\chi(g) \neq 0$ somewhere outside H ; for the degree 6 representation, $\chi(g) = 0 \ \forall g \in S_5 \setminus A_5$. All restrictions are irreducible if $|G : H| = 2$ which is the case here. Fact: $\chi \downarrow_H$ all constituents have same degree if $H \triangleleft G$ (Janes-Liebeck, chapter 20).

Let's talk about induced characters.

Definition. (10.4)

If $\psi \in \mathcal{C}(H)$, define $Ind_H^G \psi(g) = \frac{1}{|G|} \sum_{\chi \in G} \psi(x^{-1}gx)$, where

$$\psi(g) = \begin{cases} \psi(g) & g \in H \\ 0 & g \notin H \end{cases}$$

We also write $Ind_H^G \psi(g)$ as $\psi \uparrow^G = \psi^G$.

Lemma. (10.5)

If $\psi \in \mathcal{C}(H)$ then $Ind_H^G \psi \in \mathcal{C}(G)$ and $Ind_H^G \psi(1) = |G : H| \psi(1)$.

Proof. This is clear, noting that $Ind_H^G \psi(1) = \frac{1}{|H|} \sum \psi(1) = |G : H| \psi(1)$. \square

Let $n = |G : H|$. Let $1 = t_1, t_2, \dots, t_n$ be a *left transversal* of H in G (complete set of coset representatives), so that $t_1H = H, t_2H, \dots, t_nH$ are precisely the n left cosets of H in G .

Lemma. (10.6)

Given left transversal as above,

$$Ind_H^G \psi(g) = \sum_{i=1}^n \psi(t_i^{-1}gt_i)$$

Proof. For $h \in H$, $\psi((t_i h)^{-1}g(t_i h)) = \psi(t_i^{-1}gt_i)$ as ψ is a class function on H . \square

Theorem. (10.7, Frobenius reciprocity)

$H \leq G$. ψ is a class function for H , ϕ is a class function for G . Then

$$\langle \underbrace{Res_H^G \phi}_{in \ \mathcal{C}(H)}, \psi \rangle_H = \langle \phi, \underbrace{Ind_H^G \psi}_{in \ \mathcal{C}(G)} \rangle_G$$

Proof. We want to show $\langle \phi_H, \psi \rangle_H = \langle \phi, \psi^G \rangle_G$:

$$\langle \phi, \psi^G \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi^G(g) = \frac{1}{|G||H|} \sum_{g, x \in G} \overline{\phi(g)} \psi(x^{-1}gx)$$

Put $y = x^{-1}gx$. The above then equals

$$\frac{1}{|G||H|} \sum_{x, y \in G} \overline{\phi(y)} \psi(y) = \frac{1}{|H|} \sum_{y \in G} \overline{\phi(y)} \psi(y)$$

which is independent of x , and then equals

$$\frac{1}{|H|} \sum_{y \in H} \overline{\phi(y)} \psi(y) = \langle \phi_H, \psi \rangle_H$$

□

Corollary. (10.8)

If ψ is a character of H , then $\text{Ind}_H^G \psi$ is a character of G .

Proof. Let χ be an irreducible character of G . Then

$$\langle \text{Ind}_H^G \psi, \chi \rangle = \langle \psi, \text{Res}_H^G \chi \rangle \in \mathbb{Z}_{\geq 0}$$

since ψ and $\text{Res}_H^G \chi$ are characters. Hence $\text{Ind}_H^G \psi$ is a linear combination of irreducible characters with non-negative coefficients, hence a character. □

Lemma. (10.9)

Let ψ be a character of $H \leq G$, and let $g \in G$. Let

$$C_G(g) \cup H = \bigcup_{i=1}^m C_H(x_i)$$

(disjoint union), where the x_i are representatives of the H -ccls of elements of H conjugate to g .

If $m = 0$, then $\text{Ind}_H^G \psi(g) = 0$. Otherwise

$$\text{Ind}_H^G \psi(g) = |C_G(g)| \cdot \sum_{i=1}^m \frac{\psi(x_i)}{|C_H(x_i)|}$$

Proof. Assume $m > 0$. Let $X_i = \{x \in G : x^{-1}gx \in H \text{ and is conjugate in } H \text{ to } x_i\} \forall 1 \leq i \leq m$. The X_i are pairwise disjoint, and their union is $\{x \in G : x^{-1}gx \in H\}$. By definition,

$$\begin{aligned} \text{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{\alpha \in G} \psi(\alpha^{-1}g\alpha) \\ &= \frac{1}{|H|} \sum_{i=1}^m \sum_{x \in X_i} \psi(x^{-1}gx) \\ &= \frac{1}{|H|} \sum_{i=1}^m \sum_{x \in X_i} \psi(x_i) \\ &= \sum_{i=1}^m \frac{|X_i|}{|H|} \psi(x_i) \end{aligned}$$

and evaluate $\frac{|X_i|}{|H|}$ to get what we want... although a bit tedious: Fix $1 \leq i \leq m$ and choose some $g_i \in G$ s.t. $g_i^{-1}gg_i = x_i$ so $\forall c \in C_G(g)$ and $h \in H$,

$$\begin{aligned} (cg_ih)^{-1}g(cg_ih) &= h^{-1}g_i^{-1}c^{-1}gcg_ih \\ &= h^{-1}g_i^{-1}c^{-1}cgg_ih \\ &= h^{-1}g_i^{-1}gg_ih \\ &= h^{-1}x_ih \in H \end{aligned}$$

i.e. $cg_ih \in X_i$, hence $C_G(g)g_iH \subseteq X_i$;

Conversely, if $x \in X_i$ then $x^{-1}gx = h^{-1}x_ih = h^{-1}(g_i^{-1}gg_i)h$ for some $h \in H$; thus $xh^{-1}g_i^{-1} \in C_G(g)$. So $x \in C_G(g)g_iH \subseteq C_G(g)g_iH$. Conclude $X_i = C_G(g)g_iH$, thus

$$|X_i| = |C_G(g)g_iH| = \frac{|C_G(g)||H|}{|H \cap g_i^{-1}C_G(g)g_i|}$$

(see notes at end). Finally $g_i^{-1}C_G(g)g_i = C_G(g_i^{-1}gg_i) = C_G(x_i)$. Thus

$$\begin{aligned} |X_i| &= |H : H \cup C_G(x_i)||C_G(g)| \\ &= |H : C_H(x_i)||C_G(g)| \end{aligned}$$

Thus,

$$\begin{aligned} \frac{|X_i|}{|H|} &= \frac{|H : C_H(x_i)||C_G(g)|}{|H|} \\ &= \frac{|C_G(g)|}{|C_H(x_i)|} \end{aligned}$$

for each $1 \leq i \leq m$. □

Note: if $H, K \leq G$, a double coset of H and K in G is a set $HgK = \{h g k : h \in H, k \in K\}$ for some $g \in G$.

Facts:

- two double cosets are either disjoint or equal;
- $|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}|} = \frac{|H||K|}{|g^{-1}Hg \cap K|}$ (prove this: it's a bit like $|HK|$).

Example. Consider $H = C_4 = \langle (1234) \rangle \leq G = S_4$, of index 6. Char of induced representation $\text{Ind}_H^G(\alpha)$ where α is faithful 1-dim representation of C_4 . If $\alpha((1234)) = i$, then char of α is $(1 \ i \ -1 \ i)$ for $(1), (1234), (13)(24), (1432)$. The induced representation of S_4 , we know $\text{Ind}_{C_4}^{S_4}\chi_\alpha$ evaluates to 6 at (1) (by (10.5)) and to 0 at (12) and (123) .

For $(12)(34)$ only one of the three elements of S_4 it's conjugate to, lies in H , namely $(13)(24)$. So $\text{Ind}_H^G\chi_\alpha((12)(34)) = 8(-1/4) = -2$.

For (1234) , it is conjugate to 6 elements of S_4 of which two are in C_4 , namely (1234) and (1432) . So $\text{Ind}_H^G\chi_\alpha(1234) = 4(\frac{i}{4} - \frac{i}{4}) = 0$.

10.1 Induced representations

Let $H \leq G$, of index n . Let $1 = t_1, t_2, \dots, t_n$ transversal, i.e. H, t_2H, \dots, t_nH are left cosets of H . Let W be a H -space.

Lemma. (10.10)

$$\text{Ind}_{\{1\}}^G 1 = \rho_{\text{reg}}.$$

Definition. (10.11) Let $V := W \oplus t_2 \otimes W \oplus \dots \oplus t_n \otimes W = \bigoplus_{t_i} t_i \otimes W$, where $t_i \otimes W = \{t_i \otimes w : w \in W\}$. So $\dim V = n \dim W$. We write $V = \text{Ind}_H^G W$.

G-action: Let $g \in G$. $\forall i \exists$ unique j with $t_j^{-1}gt_i \in H$ (namely t_jH is the coset containing gt_i). You got to understand where did this g come from, otherwise you can't make progress. Define

$$g(t_i \otimes W) = t_j \otimes ((t_j^{-1}gt_i)w)$$

We drop \otimes from now. Check this is a G -action. Then

$$\begin{aligned} g_1(g_2t_iw) &= g_1(t_j(t_j^{-1}g_2t_i)w) \\ &= t_l((t_l^{-1}g_1t_j)(t_j^{-1}g_2t_i)w) \\ &= t_l(t_l^{-1}(g_1g_2)t_i)w = (g_1)(g_2)(t_iw) \end{aligned}$$

where j and l are the unique ones such that $g_2t_iH = t_jH$ and $g_1t_jH = t_lH$.

It has the 'right' character: $g : t_iw \rightarrow t_j \underbrace{(t_j^{-1}gt_i)}_{\in H}w$, so the contribution to the character is 0 unless $j = i$, i.e. if $t_i^{-1}gt_i \in H$, in which case it contributes $\psi(t_i^{-1}gt_i)$. So

$$Ind_H^G \psi(g) = \sum_1^m \psi(t_i^{-1}gt_i) \quad (10.6)$$

Remark. (10.12)

There is Frobenius Reciprocity,

$$Hom_H(W, Res_H^G V) \cong Hom_G(Ind_H^G W, V)$$

naturally as vector spaces (W is a H -space, V is a G -space).

Lemma. (10.13)

(i) $Ind_H^G(W_1 \oplus W_2) \cong Ind_H^G W_1 \oplus Ind_H^G W_2$;

(ii) $\dim Ind_H^G W = |G : H| \dim W$.

(iii) If $H \leq K \leq G$, then $Ind_K^G Ind_H^K W \cong Ind_H^G W$.

(lecture had (10.10) here because he missed it previously, and labelled (iii) as

(iv) while (10.10) as (iii)).

Proof. (10.10):

$$\begin{aligned} Ind_H^G \psi(g) &= \sum_{i=1}^n \psi(t_i^{-1}gt_i) \\ &= \sum_{i=1}^n \mathbf{1}_H(e_i^{-1}gt_i) \\ &= |\{i : t_i^{-1}gt_i \in H\}| \\ &= |\{i : g \in t_i H t_i^{-1}\}| = |fix_X(g)| = \pi_X \end{aligned}$$

□

Remark. $\langle \psi_X, 1_G \rangle_G = \langle Ind_H^G 1_H, 1_G \rangle_G = \langle 1_H, 1_H \rangle = 1$ as predicted in chapter 7.

11 Frobenius groups

Theorem. (11.1, Frobenius theorem, 1891)

Let G be a transitive permutation group on a finite X , say $|X| = n$. Assume that each non-identity element of G fixes at most one element of X . Then

$$K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \forall \alpha \in X\}$$

is a normal subgroup of G of order n .

Note that G is necessarily finite, being isomorphic to a subgroup of S_X .

Proof. (method of exceptional characters, due to M. Isaacs - chapter 7 books)
We have to show $K \triangleleft G$. Let $H = G_\alpha$ the stabiliser of $\alpha \in X$ for some $\alpha \in X$, i.e. $gG_\alpha g^{-1} = G_{g\alpha}$. Conjugates of H are stabilisers of single elements of X . No two conjugates can share a non-identity element (by hypothesis), so H has n distinct conjugate, and G itself has $n(|H| - 1)$ elements that fix exactly one element of X . But $|G| = |X||H| = n|H|$ (X and G/H are isomorphic (because transitive action) as G -sets). Hence $|K| = |G| - n(|H| - 1) = n$. Let $1 \neq h \in H$. Suppose $h = ghg^{-1}$ for some $g \in G, h' \in H$. Then h lies in both $H = G_\alpha$ and $gHg^{-1} = G_{g\alpha}$; by hypothesis $g\alpha = \alpha$, hence $g \in H$. Therefore, the ccls in G of h is precisely the ccls in H . Similarly oif $g \in C_G(h)$, then $h = ghg^{-1} \in G_{g\alpha}$ and hence $g \in H$. We conclude $C_G(h) = C_H(h)$ ($1 \neq h \in H$). Every element of G either belongs to K or lies in one of the n stabilisers, each of which is conjugate to H . So every element of $G \setminus K$ is conjugate with a non-identity element of H . So $\{1, h_2, \dots, h_t, y_1, \dots, y_u\}$ (the representations of H -ccls and representations of ccls of G which comprise $K \setminus \{1\}$ respectively) is a set of ccls reps for G .

Take $\theta_1 = 1_G$. $\{1_H = \psi_1, \dots, \psi_t\}$ be irreducible characters of H . Fix $1 \leq i \leq t$. Then, if $g \in G$, we know

$$\text{Ind}_H^G \psi_i(g) = \begin{cases} |G:H|\psi_i(1) = n\psi_i(1) & g = 1 \\ \psi_i(h_j) & g = h_j (2 \leq j \leq t) \\ 0 & g = y_k (1 \leq k \leq u) \end{cases}$$

where in the second case we appeal to $C_G(h_j) = C_H(h_j)$ and (10.9). Now fix some $2 \leq i \leq t$ and put $\theta_i = \psi_i^G - \psi_i(1)\psi_1^G + \psi_i(1)\theta_1 \in R(G)$ by (9.15). Values for $2 \leq j \leq t, 1 \leq k \leq u$ lequ:

	1	h_j	y_k
ψ_i^G	$n\psi_i(1)$	$\psi_i(h_j)$	0
ψ_i^G	$n\psi_i(1)$	$\psi_i(1)$	0
ψ_i^G	$\psi_i(1)$	$\psi_i(1)$	$\psi_i(1)$
θ_i	$\psi_i(1)$	$\psi_i(h_j)$	$\psi_i(1)$

Now calculate

$$\begin{aligned}
 \langle \theta_i, \theta_i \rangle &= \frac{1}{|G|} \sum_{g \in G} |\theta_i(g)|^2 \\
 &= \frac{1}{|G|} \left(\sum_{g \in K} |\theta_i(g)|^2 + \sum_{\alpha \in X} \sum_{1 \neq g \in G_\alpha} |\theta_i(g)|^2 \right) \\
 &= \frac{1}{|G|} (n\psi_i^2(1) + n \sum_{1 \neq h \in H} |\psi_i(h)|^2) \\
 &= \frac{1}{|H|} \sum_{h \in H} |\psi_i(h)|^2 \\
 &= \langle \psi_i, \psi_i \rangle \\
 &= 1
 \end{aligned}$$

As ψ_i is irreducible. So (by (9.15)), either θ_i or $-\theta_i$ is a character. Since $\theta_i(1) > 0$, it's $+\theta_i$, an actual character. Let $\theta = \sum_{i=1}^t \theta_i(1)\theta_i$. Column orthogonality gives $\theta(h) = \sum_{i=1}^t \psi_i(1)\psi_i(h) = 0$ ($1 \neq h \in H$), and for any $y \in K$, $\theta(y) = \sum_{i=1}^t \psi_i^2(1) = |H|$. Hence

$$\theta(g) = \begin{cases} |H| & g \in K \\ 0 & g \notin K \end{cases}$$

So $K = \{g \in G : \theta(g) = \theta(1)\} \triangleleft G$. □

Definition. (11.2)

A Frobenius group is a group G having subgroup H s.t. $H \cap gHg^{-1} = 1 \forall g \notin H$. H is the Frobenius complement of G .

Proposition. (11.3)

Any finite Frobenius group satisfies the hypothesis of (11.1). The normal subgroup K is a Frobenius Kernel of G .

Proof. Let G be Frobenius, with complement H . Then action of G on G/H is transitive and faithful. Furthermore, if $1 \neq g \in G$ fixes both xH and yH , then $g \in xHx^{-1} \cap yHy^{-1} \implies H \cap (y^{-1}x)H(y^{-1}x)^{-1} \neq 1 \implies xH = yH$. \square

Example: If p, q distinct primes, $p \equiv 1 \pmod{q}$, the unique non-abelian group of order pq is a Frobenius group (see James-Liebeck chapter 25 or Teleman chapter 11).

Remarks:

- Thompson (thesis, 1959) proved any finite group having fixed point free automorphism of prime power order is nilpotent. This implied that in finite Frobenius group, K is nilpotent (iff K is a direct product of its sylow subgroups).
- There is no profo of (11.1) known in which character theory is not required.

12 The missing lecture: Mackey Theory

Let's work over \mathbb{C} . Mackey Theory describes restriction to a subgroup $K \leq G$ of an irreducible representation $\text{Ind}_H^G W$. Here K, H are unrelated, but usually we take $K = H$, in which case we can characterise when $\text{Ind}_H^G W$ is irreducible. (?)

Special case: $W = 1_H$ (trivial H -space of dimension 1). Then $\text{Ind}_H^G W$ is the permutation representation of G on G/H (by 10.10, action on left cosets of H in G).

Recall: if G is transitive on a set X and $H = G_\alpha$ for some $\alpha \in X$, then the action of G on X is isomorphic to the action of G on G/H , namely

$$\begin{array}{ccc} g \cdot \alpha & \leftrightarrow & gH \\ \in X & & \in G/H \end{array} \quad (12.1)$$

is a well-defined bijection and commutes with G -actions ($x(g\alpha) = (xg)\alpha \leftrightarrow x(gH) = (xg)H$).

Consider the action of G on G/H and let $K \leq G$. G/H splits into K -orbits: these correspond to *double cosets* $KgH = \{KgH : k \in K, h \in H\}$, namely the K -orbit containing gH contains precisely all kgH with $k \in K$ (bunches of some gH cosets together).

Notation. (12.2)

$K \backslash G/H$ is the set of (K, H) -double cosets; they partition G . Note that $|K \backslash G/H| = \langle \pi G/K, \pi G/H \rangle$ as in (7.4). Let S be the set of representations.

Clearly $G_{gH} = gHg^{-1}$, so $K_{gH} = gHg^{-1} \cap K = Hg$.

So by (12.1), the action of K on the orbit containing gH is isomorphic to the action of K on K/Hg . From this, using $\text{Ind}_H^G 1_H = \mathbb{C}(G/H)$ and, if $X = \cup X_i$ a decomposition into orbits, then $\mathbb{C}X = \oplus_i \mathbb{C}X_i$, we get

Proposition. (12.3)

G is a finite group, $H, K \leq G$. Then

$$\text{Res}_K^G \text{Ind}_H^G 1 \cong \oplus_{g \in S} \text{Ind}_{Hg}^K 1$$

I think this is some application:

Let $S = \{g_1 = 1, g_2, \dots, g_r\}$ be s.t. $G = \cup_i Kg_iH$. Write $H_g = gHg^{-1} \cap K$ ($\leq K$). (ρ, W) is representation of H . For $g \in G$, define (ρ_g, W_g) to be the representation of Hg with the same underlying vector space W , but now the Hg -action is $\rho_g(x) = \rho(h)$, where $x \in gHg^{-1}$. Since $H_g \leq K$, we obtain an induced representation $\text{Ind}_{H_g}^K W_g$ from this.

Theorem. (12.4) (Mackey's restriction formula)

G finite, $H, K \leq G$ and W H -space. Then

$$\text{Res}_K^G \text{Ind}_H^G W = \oplus_{g \in S} \text{Ind}_{H_g}^K W_g$$

as K -modules.

We'll prove this later.

Corollary. (12.5, character version of (12.4))

If ψ is a character of a representation of H , then

$$es_K^G Ind_H^G \psi = \sum_{g \in S} Ind_{H_g}^K \psi_g$$

where ψ_g is the character of H_g given as $\psi_g(x) = \psi(g^{-1}xg)$.

Corollary. (12.6, Mackey's irreducibility criterion)

Let $H \leq G$, W be a H -vector space. Then $V = Ind_H^G W$ is irreducible iff

- (i) W is irreducible;
- (ii) for each $g \in S \setminus H$, the two Hg -spaces Wg and $Res_{H_g}^H W$ have no irreducible constituents in common (they're 'disjoint' representations).

Proof. Let W afford character ψ . Recall W irreducible $\iff \langle \psi, \psi \rangle = 1$. Take $K = H$ in (12.4), so $Hg = gHg^{-1} \cap H$. Then

$$\langle Ind_H^G \psi, Ind_H^G \psi \rangle_G = \langle \psi, Res_H^G Ind_H^G \psi \rangle_H$$

by (10.7), then by (12.5) is equal to

$$\begin{aligned} \sum_{g \in S} \langle \psi Ind_{H_g}^H \psi_g \rangle_H &= \sum_{g \in S} \langle Res_{H_g}^H \psi, \psi_g \rangle_{H_g} \\ &= \langle \psi, \psi \rangle_H + \sum_{g \in S, g \notin H} d_g \end{aligned}$$

where $d_g = \langle Res_{H_g}^H \psi, \psi_g \rangle$ ($g \neq 1$). □

For $g = 1$ we have $H_g = H$, hence we get a sum of non-negative integers which is ≥ 1 . So $Ind_H^G \psi$ is irreducible iff $\langle \psi, \psi \rangle = 1$ and all the other terms in the sum are 0. In other words, W is irreducible representation of H and $\forall g \notin H$, W and Wg are disjoint representations of $H \cap gHg^{-1}$.

Remark. Set S of representations was arbitrary, so could demand $g \in G \setminus H$ in (ii) but in fact suffices to check for $g \in S \setminus H$.

Corollary. (12.7)

If $H \triangleleft G$, assume ψ is an irreducible character of H . Then $Ind_H^G \psi$ is irreducible $\iff \psi$ is distinct from all its conjugates ψ_g for all $g \in G \setminus H$ ($\psi_g(h) = \psi(ghg^{-1})$).

Proof. Again take $K = H$, noting double cosets \equiv left cosets. Also, $Hg = H \forall g$ (as $H \triangleleft G$). Moreover, Wg is irreducible since W is irreducible. So by (12.6), $Ind_H^G W$ is irreducible precisely when $W \not\cong Wg \forall g \in G \setminus H$. This is equivalent to $\psi \neq \psi_g$. □

Remark. Again could check conditions on a set of representatives.

Proof. (of 12.4)

Write $V = Ind_H^G W$. Fix $g \in G$. Now V is a direct sum of $x \oplus W$ with x running

through representations of left cosets of H in G (10.11). $V = \oplus_{x \in J} x \otimes W$. Consider a particular coset $KgH = K \backslash G/H$. The terms

$$V(g) = \oplus_{x \text{ rep of } H \text{ in } G, x \in KgH} x \otimes W$$

forms a subspace invariant under the action of K (it's a direct sum of an orbit of subspaces permuted by K). Now viewing V as a K -space (forget G -structure), $\text{Res}_K^G V = \oplus_{g \in S} V(g)$, so we need to show $V(g) = \text{Ind}_{H_g}^K W_g$ as K -spaces for each $g \in S$.

Now, $\text{Stab}_K(g \otimes W) = \{k \in K : kg \otimes W = g \otimes W\} = \{k \in K : g^{-1}kg \in \text{Stab}_G(1 \otimes W) = H\} = K \cap gHg^{-1} (= Hg)$. This implies if $x = kgh, x' = k'gh'$, then $x \otimes W = x' \otimes W$ iff k, k' lie in same coset in K/Hg , hence $V(g)$ is direct sum $\oplus_{k \in K/Hg} k \otimes (g \otimes W)$. Therefore, as a representation of K , this subspace is

$$V(g) \cong \text{Ind}_{H_g}^K (g \otimes W)$$

□

But $g \otimes W \cong Wg$ as a representation of Hg ($w \rightarrow g \otimes W$ is an isomorphism). Putting everything together we are done.

13 Integrality in the group algebra

Definition. (13.1)

$a \in \mathbb{C}$ is an algebraic integer if: a is a root of a monic polynomial in $\mathbb{Z}[x]$. Equivalently, the subring of \mathbb{C} generated by $\mathbb{Z}[a] = \{f(a) : f(x) \in \mathbb{Z}[x]\}$ is a finitely generated \mathbb{Z} -module.

Fact 1: The algebraic integers form a subring of \mathbb{C} (see number fields);

Fact 2: If $a \in \mathbb{C}$ is both an algebraic integer and a rational number, then it's an integer (see number fields);

Fact 3: Any subring of \mathbb{C} which is a finitely-generated \mathbb{Z} -module consists of algebraic integers.

Proposition. (13.2)

If χ is character of G and $g \in G$, then $\chi(g)$ is an algebraic integer.

Proof. $\chi(g)$ is a sum of n th roots of unity ($n = |g|$). Each root of unity is an algebraic integer, and any sum of algebraic integers is an algebraic integer by fact 1. \square

Corollary. There are no entries in the character tables of any finite group which are rational but not integers, by Fact 2.

13.1 The centre of $\mathbb{C}G$

Recall from (2.4), the group algebra $\mathbb{C}G = \{\sum \alpha_g g : \alpha_g \in \mathbb{C}\}$ of finite group, the \mathbb{C} -space with basis G . Also a ring, hence a finite-dimensional \mathbb{C} -algebra.

List $\{1\} = \mathcal{C}_1, \dots, \mathcal{C}_k$ the G -ccls. Define the class sums:

$$C_j = \sum_{g \in \mathcal{C}_j} g \in \mathbb{C}G$$

Claim, each $C_j \in Z(\mathbb{C}G)$, the centre of $\mathbb{C}G$ (Note: this is not the same as $\mathbb{C}(Z(G))!$).

Proposition. (13.3)

C_1, \dots, C_k is a basis of $Z(\mathbb{C}G)$. There exist non-negative integers a_{ijl} ($1 \leq i, j, l \leq k$) with

$$C_i C_j = \sum_l a_{ijl} C_l$$

These are called the class algebra constants for $Z(\mathbb{C}G)$.