

Topics in Set Theory Sheet 4

Xiaojia Rao

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Fix $A \in M[G]$, a formula $\varphi(x, y, x_1, \dots, x_n)$ which specifies the function that we want to use for replacement, and fix parameters $a_1, \dots, a_n \in M[G]$. We want a name for

$$B := \{y : M[G] \models \exists x \in A \varphi(x, y, a_1, \dots, a_n)\}$$

Take a name σ for A and some names τ_1, \dots, τ_n for a_1, \dots, a_n . Define a formula

$$\psi(x, y, x_1, \dots, x_n, A) := x \in A \wedge \varphi(x, y, x_1, \dots, x_n)$$

and now consider the name

$$\rho := \{(\pi, p) : p \Vdash^* \exists x \psi(x, \pi, \tau_1, \dots, \tau_n, \sigma)\}$$

We claim that $\text{val}(\rho, G) = B$.

\subseteq : suppose $y \in \text{val}(\rho, G)$. So there is $(\pi, p) \in \rho$ with $p \in G$ and $\text{val}(\pi, G) = y$. So $p \Vdash^* \pi \in \exists x \psi(x, \pi, \tau_1, \dots, \tau_n, \sigma)$. By definition, this means that the set

$$D := \{r : \exists \mu \in M^{\mathbb{P}}(r \Vdash^* \psi(\mu, \pi, \tau_1, \dots, \tau_n, \sigma))\}$$

is dense below p . In particular $G \cap D \neq \emptyset$, so take some $q \in G \cap D$. $q \in D$ means there exists a name μ s.t.

$$q \Vdash^* \psi(\mu, \pi, \tau_1, \dots, \tau_n, \sigma)$$

But $q \in G$ as well, so by FT,

$$M[G] \models \psi(\mu, \pi, \tau_1, \dots, \tau_n, \sigma)$$

which translates to: there exists x ($:= \text{val}(\mu, G)$) that

$$\begin{aligned} M[G] &\models \psi(x, y, a_1, \dots, a_n, A) \\ \iff M[G] &\models x \in A \wedge \varphi(x, y, x_1, \dots, x_n) \end{aligned}$$

we can rewrite this as

$$M[G] \models \exists x (x \in A \wedge \varphi(x, y, x_1, \dots, x_n))$$

i.e. $y \in B$.

\supseteq : suppose $y \in B$. So $M[G] \models \exists x \psi(x, y, a_1, \dots, a_n, A)$. Now take a name π for y ; so by FT, there exists $p \in G$ s.t. $p \Vdash^* \exists x \psi(x, \pi, \tau_1, \dots, \tau_n, \sigma)$. But this is exactly the requirement for $(\pi, p) \in \rho$. So $y \in \text{val}(\rho, G)$.

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