

Representation Theory

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0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over \mathbb{C} . Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise:

- F is a field, usually \mathbb{C} , \mathbb{R} or \mathbb{Q} . When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or \bar{F}_p (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
- V is a vector space over F , always finite dimensional;
 $GL(V) = \{\theta : V \rightarrow V, \theta \text{ linear, invertible}\}$, i.e. $\det \theta \neq 0$.

Recall from Linear Algebra:

If $\dim_F V = n < \infty$, choose basis e_1, \dots, e_n over F , so we can identify it with F^n . Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \rightarrow A_\theta$ ($A_{\theta_1 \theta_2} = A_{\theta_1} A_{\theta_2}$ and bijection). Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = X A_1 X^{-1}$.

Recall that $\text{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A .

(1.3) $\text{tr}(XAX^{-1}) = \text{tr}(A)$, hence we can define $\text{tr}(\theta) = \text{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V in f.d. over \mathbb{C} , with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $\text{End} V$ is the set of all linear maps $V \rightarrow V$, e.g. $\text{End}(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over \mathbb{C} , $\alpha \in \text{End}(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1} (X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a \mathbb{C} -vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:

(1.6) The symmetric group, $S_n = \text{Sym}(X)$ on the set $X = \{1, \dots, n\}$ is the set of all permutations of X . $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m : $C_m = \langle x : x^m = 1 \rangle$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in \mathbb{C} (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of \mathbb{C}). We also have the group of rotations, centre O of regular m -gon in \mathbb{R}^2 (inside $GL_2(\mathbb{R})$).

(1.8) Dihedral groups D_{2m} of order $2m = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$. Think of this as the set of rotations and reflections preserving a regular m -gon.

(1.9) Quaternion group, $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \rightarrow X$ by $(g, x) \rightarrow g \cdot x$ for $g \in G, x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X , we obtain a homomorphism $\theta : G \rightarrow \text{Sym}(X)$, called the *permutation representation* of G .

Proof. For $g \in G$, the function $\theta_g : X \rightarrow X$ by $x \rightarrow gx$ is a permutation on X , with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G, \theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2}$ since $(g_1 g_2)x = g_1(g_2 x)$ for $x \in X$. \square

2 Basic Definitions

2.1 Representations

Let G be finite, F be a field, usually \mathbb{C} .

Definition. (2.1)

Let V be a f.d. vector space over F . A (linear, in some books) *representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \rightarrow GL(V)$$

Write ρ_g for the image $\rho_V(g)$; so for each $g \in G$, $\rho_g \in GL(V)$, and $\rho_{g_1 g_2} = \rho_{g_1} \rho_{g_2}$, and $(\rho_g)^{-1} = \rho_{g^{-1}}$.

The *dimension* (or *degree*) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \triangleleft G$ (kernel is a normal subgroup), and $G/\ker \rho \cong \rho(G) \leq GL(V)$ (1st isomorphism theorem). We say ρ is *faithful* if $\ker \rho = 1$.

An alternative (and equivalent) approach is to observe that a representation of G on V is "the same as" a *linear action* of G :

Definition. (2.3)

G *acts linearly* on V if there exists a *linear action*

$$\begin{aligned} G \times V &\rightarrow V \\ (g, v) &\rightarrow gv \end{aligned}$$

By linear action we mean: (action) $(g_1 g_2)v = g_1(g_2 v)$, $1v = v \ \forall g_1, g_2 \in G, v \in V$, and (linear) $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv \ \forall g \in G, v_1, v_2 \in V, \lambda \in F$.

Now if G acts linearly on V , the map

$$\begin{aligned} G &\rightarrow GL(V) \\ g &\rightarrow \rho_g \end{aligned}$$

with $\rho_g : v \rightarrow gv$ is a representation of G . Conversely, given a representation $\rho : G \rightarrow GL(V)$, we have a linear action of G on V via $g \cdot v := \rho(g)v \ \forall v \in V, g \in G$.

(2.4) In (2.3) we also say that V is a G -space or that V is a G -module. In fact if we define the *group algebra* FG , or $F[G]$, to be $\{\sum \alpha_j g : \alpha_j \in F\}$ with natural addition and multiplication, then V is actually a FG -module (in the sense from GRM).

(2.5) R is a *matrix representation* of G of degree n if R is a homomorphism $G \rightarrow GL_n(F)$. Given representation $\rho : G \rightarrow GL(V)$ with $\dim_F V = n$, fix basis B ; we get matrix representation

$$\begin{aligned} G &\rightarrow GL_n(F) \\ g &\rightarrow [\rho(g)]_B \end{aligned}$$

Conversely, given matrix representation $R : G \rightarrow GL_n(F)$, we get representation

$$\begin{aligned}\rho : G &\rightarrow GL(F^n) \\ g &\rightarrow \rho_g\end{aligned}$$

via $\rho_g(v) = R_g v$ where R_g is the matrix of g .

Example. (2.6)

Given any group G , take $V = F$ the 1-dimensional space, and

$$\begin{aligned}\rho : G &\rightarrow GL(F) \\ g &\rightarrow (id : F \rightarrow F)\end{aligned}$$

is known as the trivial representation of G . So $\deg \rho = 1$ ($\dim_F F = 1$).

Example. (2.7)

Let $G = C_4 = \langle x : x^4 = 1 \rangle$. Let $n = 2$, and $F = \mathbb{C}$. Note that any $R : x \rightarrow X$ will determine $x^j \rightarrow X^j$ as it is a homomorphism, and also we need $X^4 = I$. So we can take X to be diagonal matrix – any such with diagonal entries a root to $x^4 = 1$, i.e. $\{\pm 1, \pm i\}$, or if X is not diagonal then it will be similar to a diagonal matrix by (1.4) ($X^4 = I$).

2.2 Equivalent representations

Definition. (2.8)

Fix G, F . Let V, V' be F -spaces, and $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$ which are representations of G . The linear map $\phi : V \rightarrow V'$ is a G -homomorphism if

$$\phi \rho(g) = \rho'(g) \phi \forall g \in G(*)$$

We can understand this more by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \phi \downarrow & \searrow & \downarrow \phi \\ V' & \xrightarrow{\rho'_{g'}} & V' \end{array}$$

We say ϕ *intertwines* ρ, ρ' . Write $\text{Hom}_G(V, V')$ for the F -space of all these. ϕ is a G -isomorphism if it is also bijective; if such ϕ exists, ρ, ρ' are isomorphic/equivalent representations. If ϕ is a G -isomorphism, we can write (*) as $\rho' = \phi\rho\phi^{-1}$.

Lemma. (2.9)

The relation "being isomorphic" is an equivalent relation on the set of all representations of G (over F).

Remark. (2.10)

If ρ, ρ' are isomorphic representations, they have the same dimension.

The converse may be false: C_4 has four non-isomorphic 1-dimensional representations: if $\omega = e^{2\pi i/4}$ then they are $\rho_j(x^i) = \omega^{ij}$ ($0 \leq i \leq 3$).

Remark. (2.11)

Given G, V over F of dimension n and $\rho : G \rightarrow GL(V)$. Fix basis B for V : we get a linear isomorphism

$$\begin{aligned} \phi : V &\rightarrow F^n \\ v &\rightarrow [v]_B \end{aligned}$$

and we get a representation $\rho' : G \rightarrow GL(F^n)$ isomorphic to ρ :

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \\ \downarrow \phi & & \downarrow \phi \\ F^n & \xrightarrow{\rho'} & F^n \end{array}$$

(2.12) In terms of matrix representations, we have

$$\begin{aligned} R : G &\rightarrow GL_n(F), \\ R' : G &\rightarrow GL_n(F) \end{aligned}$$

are (G) -isomorphic or equivalent if there exists a nonsingular matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1} \forall g \in G$.

In terms of linear G -actions, the actions of G on V, V' are G -isomorphic if there exists isomorphisms $\phi : V \rightarrow V'$ such that $g : \phi(v) = \phi(gv) \forall v \in V, g \in G$.

2.3 Subrepresentations

Definition. (2.13)

Let $\rho : G \rightarrow GL(V)$ be a representation of G . We say $W \leq V$ is a G -subspace if it's a subspace and it is $\rho(G)$ -invariant, i.e. $\rho_g(W) \leq W \forall g \in G$. Obviously $\{0\}$ and V are G -subspaces, however.

ρ is *irreducible/simple* representation if there are no proper G -subspaces.

Example. (2.14)

Any 1-dimensional representation of G is irreducible, but not conversely, e.g. D_8 has 2-dimensional \mathbb{C} -irreducible representation.

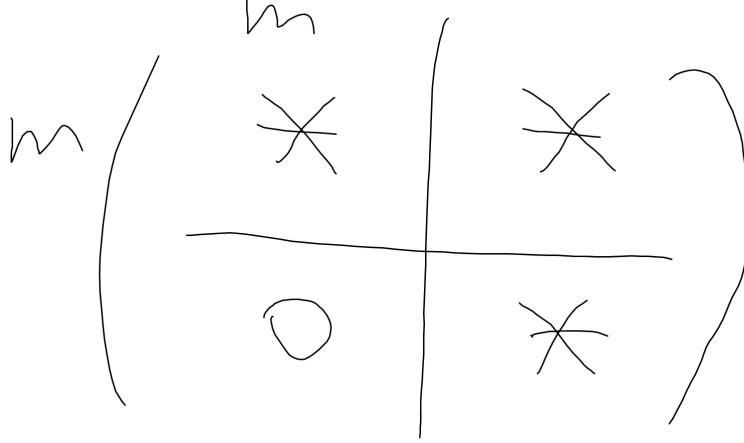
(2.15) In definition (2.13), if W is a G -subspace, then the corresponding map

$$\begin{aligned} G &\rightarrow GL(W) \\ g &\rightarrow \rho(g)|_W \end{aligned}$$

is a representation of G , a *subrepresentation* of ρ .

Lemma. (2.16)

In definition (2.13), given $\rho : G \rightarrow GL(V)$, if W is a G -subspace of V and if $B = \{v_1, \dots, v_n\}$ is a basis containing basis $B_1 = \{v_1, \dots, v_m\}$ of W ($0 < m < n$) then the matrix of $\rho(g)$ w.r.t. B has block upper triangular form as the graph below, for



each $g \in G$.

Example. (2.17)

(i) The irreducible representations of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1-dimensional and four of these are $x \rightarrow i, x \rightarrow -1, x \rightarrow -i, x \rightarrow 1$. In general, $C_m = \langle x : x^m = 1 \rangle$ has precisely m irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)* or see (4.4) below).

(ii) $G = D_6$: any irreducible C -representation has dimension ≤ 2 .

Let $\rho : G \rightarrow GL(V)$ be irreducible G -representation. Let r, s be rotation and reflection in D_6 respectively. Let V be eigenvector of $\rho(r)$. So $\rho(r)v = \lambda v$

for some $\lambda \neq 0$. Let $W = \text{span}\{v, \rho(s)v\} \leq V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, both of which are in W ; so W is G -invariant, i.e. a G -subspace. Since V is irreducible, $W = V$.

Definition. (2.18)

We say that $\rho : G \rightarrow GL(V)$ is *decomposable* if there are proper G -invariant subspaces U, W with $V = U \oplus W$. Say ρ is direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists, we say that ρ is *indecomposable*.

Lemma. (2.19)

Suppose $\rho : G \rightarrow GL(V)$ is decomposable with G -invariant decomposition $V = U \oplus W$. If B is a basis $\{\underbrace{u_1, \dots, u_k}_{B_1}, \underbrace{w_1, \dots, w_l}_{B_2}\}$ of V consisting of basis of U and basis of W , then w.r.t. B , $\rho(g)_B$ is a block diagonal matrix $\forall g \in G$ as

$$\rho(g)_B = \begin{pmatrix} [\rho_U(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

Definition. (2.20)

If $\rho : G \rightarrow GL(V)$, $\rho' : G \rightarrow GL(V')$, the *direct sum* of ρ, ρ' is

$$\rho \oplus \rho' : G \rightarrow GL(V \oplus V')$$

where $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$, a *block diagonal action*. For matrix representations $R : G \rightarrow GL_n(F)$, $R' : G \rightarrow GL_{n'}(F)$, define $R \oplus R' : G \rightarrow GL_{n+n'}(F)$:

$$g \rightarrow \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Definition. (3.1)

A representation $\rho : G \rightarrow GL(V)$ is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

Remark. (3.2)

- (1) The converse is false;
- (2) See sheet 1 Q3: \mathbb{C} -representation of \mathbb{Z} is not completely reducible and also representation of C_p over \mathbb{F}_p is not c.r..

From now on, take G finite and $\text{char } F = 0$.

Theorem. (3.3)

Every f.d. representation V of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus \dots \oplus V_r$$

is a direct sum of representations, each V_i irreducible.

It is enough to prove:

Theorem. (3.4 Maschke's theorem, 1899)

Let G be finite, $\rho : G \rightarrow GL(V)$ a f.d. representation, $\text{char } F = 0$. If W is a G -subspace of V , then there exists a G -subspace U of V s.t. $V = W \oplus U$, a direct sum of G -subspaces.

Proof. (1)

Let W' be any *vector subspace* complement of W in V , i.e. $V = W \oplus W'$ as vector spaces, and $W \cap W' = 0$. Let $q : V \rightarrow W$ be the projection of V onto W along W' ($\ker q = W'$), i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\bar{q} : v \rightarrow \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of q over G . Note that in order for $\frac{1}{|G|}$ to exist, we need $\text{char } F = 0$.

It still works if $\text{char } F \nmid |G|$.

Claim (1): $\bar{q} : V \rightarrow W$: For $v \in V$, $g(q(g^{-1}v)) \in W$ and $gW \leq W$;

Claim (2): $\bar{q}(w) = w$ for $w \in W$:

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w = w$$

So these two claims imply that \bar{q} projects V onto W .

Claim (3) If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ ($v \in V$):

$$\begin{aligned}
 h\bar{q}(v) &= h \frac{1}{|G|} \sum_g g \cdot q(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g hgq(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g (hg)q((hg)^{-1}hv) \\
 &= \frac{1}{|G|} \sum_g gq(g^{-1}(hv)) \\
 &= \bar{q}(hv) \\
 &= \bar{q}(hv)
 \end{aligned}$$

We'll then show that the kernel of this map is G -invariant, so this gives a G -summand on Thursday.

Let's now show $\ker \bar{q}$ is G -invariant. If $v \in \ker \bar{q}$, then $h\bar{q}(v) = 0 = \bar{q}(hv)$, so $hv \in \ker \bar{q}$. Thus $V = \text{im } \bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$ is a G -subspace decomposition.

We can deduce (3.3) from (3.4) by induction on $\dim V$. If $\dim V = 0$ or V is irreducible, then result is clear. Otherwise, V has non-trivial G -invariant subspace, W . Then by (3.4), there exists G -invariant complement U s.t. $V = U \oplus W$ as representations of G . But $\dim U, \dim W < \dim V$. So by induction they can be broken up into direct sum of irreducible subrepresentations. \square

The second proof uses inner products, hence we need to take $F = \mathbb{C}$ and can be generalised to compact groups in section 15.

Recall, for V a \mathbb{C} -space, \langle, \rangle is a *Hermitian inner product* if

- (a) $\langle w, v \rangle = \overline{\langle v, w \rangle} \ \forall v, w$ (Hermitian);
- (b) linear in RHS (sesquilinear);
- (c) $\langle v, v \rangle > 0$ iff $v \neq 0$ (positive definite).

Additionally, \langle, \rangle is *G -invariant* if

- (d) $\langle gv, gw \rangle = \langle v, w \rangle \ \forall v, w \in V, g \in G$.

Note if W is G -invariant subspace of V , with G -invariant inner product, then W^\perp is also G -invariant, and $V \oplus W^\perp$. For all $v \in W^\perp, g \in G$, we have to show that $gv \in W^\perp$. But $v \in W^\perp \iff \langle v, w \rangle = 0 \ \forall w \in W$. Thus by (d), $\langle gv, gw \rangle = 0 \ \forall g \in G \ \forall w \in W$. Hence $\langle gv, w' \rangle = 0 \ \forall w' \in W$. Since we can choose $w = g^{-1}w' \in W$ by G -invariance of W . Thus $gv \in W^\perp$ since g was arbitrary.

Hence if there is a G -invariant inner product on any G -space, we get another proof of Maschke's theorem:

(3.4*) (Weyl's unitary trick)

Let ρ be a complex representation of the finite group G on the \mathbb{C} -space V . Then there is a G -invariant Hermitian inner product on V .

Remark. Recall the *unitary group* $U(V)$ on V : $\{f \in GL(V) : (fu, fv) = (u, v) \ \forall u, v \in V\} = \{A \in GL_n(\mathbb{C}) : A\bar{A}^T = I\} (= U(n))$ by choosing orthonormal

basis.

Sheet 1 Q.12: any finite subgroup of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Proof. (2)

There exist an inner product on V : take basis e_1, \dots, e_n and define $(e_i, e_j) = \delta_{ij}$, extended sesquilinearly. Now

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} (gv, gw)$$

we claim that \langle, \rangle is sesquilinear, positive definite and G -invariant: if $h \in G$, then

$$\begin{aligned} \langle hv, hw \rangle &= \frac{1}{|G|} \sum_{g \in G} ((gh)v, (gh)w) \\ &= \frac{1}{|G|} \sum_{g' \in G} (g'v, g'w) \\ &= \langle v, w \rangle \end{aligned}$$

for all $v, w \in V$. □

Definition. (3.5, the regular representation)

Recall *group algebra* of G is F -space $FG = \text{span}\{e_g : g \in G\}$. There is a linear G -action

$$h \in G, h \sum_{g \in G} a_g e_g = \sum_{g \in G} a_g e_{hg} (= \sum_{g' \in G} a_{h^{-1}g'} e_{g'})$$

ρ_{reg} is the corresponding representation, the *regular representation* of G . This is faithful of $\dim |G|$. FG is the *regular module*.

Proposition. Let ρ be an irreducible representation of G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg} .

Proof. Take $\rho : G \rightarrow GL(V)$ irreducible and let $0 \neq v \in V$. Let $\theta : FG \rightarrow V$ by $\sum a_g e_g \rightarrow \sum a_g gv$. Check this is a G -homomorphism. Now V is irreducible so $\text{im } \theta = V$ (since $\text{im } \theta$ is a G -subspace).

Also $\ker \theta$ is G -subspace of FG . Let W be G -complement of $\ker \theta$ in FG (Maschke), so that $W < FG$ is G -subspace and $FG = \ker \theta \oplus W$. Thus $W \cong FG / \ker \theta \cong (G\text{-isomorphism}) \text{im } \theta \cong V$. □

More generally,

Definition. (3.7)

Let F be a field. Let G act on set X . Let $FX = \text{span}\{e_x : x \in X\}$ with G -action

$$g(\sum a_x e_x) = \sum a_x e_{gx}$$

The representation $G \rightarrow GL(V)$ where $V = FX$ is the corresponding *permutation representation*. See section 7.

4 Schur's lemma

It's really unfair that such an important result is only remembered by a lemma, so we shall call it a theorem.

Theorem. (4.1, Schur)

- (a) Assume V, W are irreducible G -spaces over field F . Then any G -homomorphism $\theta : V \rightarrow W$ is either 0 or an isomorphism.
- (b) Assume F is algebraically closed, and let V be an irreducible G -space. Then any G -endomorphism $V \rightarrow V$ is a scalar multiple of the identity map ι_V .

Proof. (a) Let $\theta : V \rightarrow W$ be a G -homomorphism. Then $\ker \theta$ is G subspace of V and, since V is irreducible, we get $\ker \theta = 0$ or $\ker \theta = V$.

And $\text{im} \theta$ is G -subspace of W , so as W is irreducible, $\text{im} \theta$ is either 0 or W . Hence, either $\theta = 0$ or θ is injective and surjective, hence isomorphism.

(b) Since F is algebraically closed, θ has an eigenvalue, λ . Then $\theta - \lambda \iota$ is singular G -endomorphism of V , but it cannot be an isomorphism, so it is 0 (by (a)). So $\theta = \lambda \iota_V$. \square

Recall from (2.8), the F -space $\text{Hom}_G(V, W)$ of all G -homomorphisms $V \rightarrow W$. Write $\text{End}_G(V)$ for the G -endomorphisms of V .

Corollary. (4.2)

If V, W are irreducible complex G -spaces, then

$$\dim_{\mathbb{C}} \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V, W are not G -isomorphic then the only G -homomorphism $V \rightarrow W$ is 0 by (4.1). Assume $v \cong_G W$ and $\theta_1, \theta_2 \in \text{Hom}_G(V, W)$, both non-zero. Then θ_2 is invertible by (4.1), and $\theta_2^{-1}\theta_1 \in \text{End}_G(V)$, and non-zero, so $\theta_2^{-1}\theta_1 = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$. Hence $\theta_1 = \lambda \theta_2$. \square

Corollary. (4.3)

If finite group G has a faithful complex irreducible representation, then $Z(G)$, the centre of the group, is cyclic.

Note that the converse is false (Sheet 1, Q10).

Proof. Let $\rho : G \rightarrow GL(V)$ be faithful irreducible complex representation. Let $z \in Z(G)$, so $zg = gz \forall g \in G$, hence the map $\phi_z : v \rightarrow z(v)$ ($v \in V$) is G -endomorphism of V , hence is multiplication by scalar μ_z , say.

By Schur's lemma, $z(v) = \mu_z v \forall v$. Then the map

$$\begin{aligned} Z(G) &\rightarrow \mathbb{C}^* \text{ (multiplicative group)} \\ z &\rightarrow \mu_z \end{aligned}$$

is a representation of Z and is faithful, since ρ is. Thus $Z(G)$ is isomorphic to some finite subgroup of \mathbb{C}^* , so is cyclic. \square

Let's now consider representation of finite abelian groups.

Corollary. (4.4)

The irreducible \mathbb{C} -representations of a finite abelian group are all 1-dimensional.

Proof. Either: use (1.4)* to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$, and if V is irreducible, then $V = \langle v \rangle$.

Or: Let V be an irreducible \mathbb{C} -representation. For $g \in G$, the map

$$\begin{array}{ccc} \theta_g : V & \rightarrow & v \\ v & \rightarrow & gv \end{array}$$

is a G -endomorphism of V , and as V irreducible, $\theta_g = \lambda_g \text{id}_V$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any $g \in G$ (so $\langle v \rangle$ is a G -subspace of V). Thus as $0 \neq V$ is irreducible, $V = \langle v \rangle$, which is 1-dimensional. \square

Remark. Schur's lemma fails over non-algebraically closed field, in particular, over \mathbb{R} . For example, let's consider the cyclic group C_3 . It has 2 irreducible \mathbb{R} -representations, one of dimension 1 (maps everything to 1) and one of dimension 2 (imo consider \mathbb{C} as a dimension 2 space over \mathbb{R} , then map the generator to the 3rd root of unity?) (so 'contradicting' with Schur's lemma via the corollary above).

Recall that every finite abelian group G is isomorphic to a product of cyclic groups (see GRM). For example, $C_6 = C_2 \times C_3$. In fact, it can be written as a product of C_{p^α} for various primes p and $\alpha \geq 1$, and the factors are uniquely determined up to reordering.

Proposition. (4.5)

The finite abelian group $G = C_{n_1} \times \dots \times C_{n_r}$ has precisely $|G|$ irreducible \mathbb{C} -representations, as described below:

Proof. Write $G = \langle x_1 \rangle \times \dots \times \langle x_r \rangle$ where $|x_j| = n_j$. Suppose ρ is irreducible, so by (4.4), it's 1-dimensional: $\rho : G \rightarrow \mathbb{C}^*$.

Let $\rho(1, \dots, x_j, \dots, 1)$ (all 1 apart from the j^{th} entry) be λ_j . Then $\lambda_j^{n_j} = 1$, so λ_j is a n_j -th root of unity. Now, the values $(\lambda_1, \dots, \lambda_r)$ determine ρ :

$$\rho(x_1^{j_1}, \dots, x_r^{j_r}) = \lambda_1^{j_1} \dots \lambda_r^{j_r}$$

thus $\rho \leftrightarrow (\lambda_1, \dots, \lambda_r)$ with $\lambda_j^{n_j} = 1 \forall j$; we have $n_1 \dots n_r$ such r -tuples, each giving 1-dimensional representation. \square

Example. (4.6)

Consider $G = C_4 = \langle x \rangle$. We could have $\rho_1(x) = 1, \rho_2(x) = i, \rho_3(x) = -1, \rho_4(x) = -i$.

Warning: There is no "natural" 1-1 correspondence between the elements of G and the representations of G (G -finite abelian). If you choose an isomorphism $G \cong C_{a_1} \times \dots \times C_{a_r}$, then we can identify the two sets (elements of groups and representations of G), but it depends on the choice of isomorphism.

Isotypical decomposition:

Recall any diagonalisable endomorphism $\alpha : V \rightarrow V$ gives eigenspace decomposition of $V \cong \oplus_\lambda V(\lambda)$, where $V(\lambda) = \{v : \alpha v = \lambda v\}$. This is *canonical* (one of the three useless words: *arbitrary*(anything), *canonical*(only one choice), *uniform*(you can choose, but it doesn't really matter)), in the sense that it depends on α alone (and nothing else).

There is no canonical eigenbasis of V : must choose basis in each $V(\lambda)$.

We know that in *char* 0 every representation V decomposes as $\oplus n_i V_i$, V_i irreducible, $n_i \geq 0$. How unique is this?

We have this wishlist (4.7):

- (a) Uniqueness: for each V there is only one way to decompose V as above. However, this doesn't work obviously.
- (b) Isotypes: for each V , there exists a unique collection of subrepresentations U_1, \dots, U_k s.t. $V = \oplus U_i$ and, if $V_i \subseteq U_i$ and $V_j' \subseteq U_j$ are irreducible subrepresentations, then $V_i \cong V_j'$ iff $i = j$.
- (c) Uniqueness of factors: If $\oplus_{i=1}^k V_i \cong \oplus_{i=1}^{k'} V_i'$ with V_i, V_i' irreducible, then $k = k'$, and $\exists \pi \in S_k$ such that $V_{\pi(i)}' \cong V_i$ (Krull-Schmidt theorem). For (b),(c) see Teleman section 5.

Lemma. (4.8)

Let V, V_1, V_2 be G -spaces over F .

- (i) $\text{Hom}_G(V, V_1 \oplus V_2) \cong \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2)$;
- (ii) $\text{Hom}_G(V_1 \oplus V_2, V) \cong \text{Hom}_G(V_1, V) \oplus \text{Hom}_G(V_2, V)$;

Proof. (i) Let $\pi_i : V_1 \oplus V_2 \rightarrow V_i$ be G -linear projections onto V_i , with kernel V_{3-i} ($i = 1, 2$).

Consider

$$\begin{aligned} \text{Hom}_G(V, V_1 \oplus V_2) &\rightarrow \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2) \\ \phi &\rightarrow (\pi_1 \phi, \pi_2 \phi) \end{aligned}$$

This map has inverse $(\psi_1, \psi_2) \rightarrow \psi_1 + \psi_2$. Check details.

- (ii) The map $\phi \rightarrow (\phi|_{V_1}, \phi|_{V_2})$ has inverse $(\psi_1, \psi_2) \rightarrow \psi_1 \pi_1 + \psi_2 \pi_2$. □

Lemma. Let F be algebraically closed, $V = \oplus_1^n V_i$ a decomposition of G -space into irreducible summands. Then, for each irreducible representation S of G ,

$$\#\{j : V_j \cong S\} = \dim \text{Hom}_G(S, V)$$

where $\#$ means 'number of times'. This is called the *multiplicity* of S in V .

Proof. Induction on n . $n = 0, 1$ are trivial.

If $n > 1$, $V = \oplus_1^{n-1} V_i \oplus V_n$. By (4.8) we have

$$\dim \text{Hom}_G(S, \oplus_1^{n-1} V_i \oplus V_n) = \dim \text{Hom}(S, \oplus_1^{n-1} V_i) + \underbrace{\dim \text{Hom}_G(S, V_n)}_{\text{Schur's lemma}}$$

□

Definition. (4.10)

A decomposition of V as $\oplus W_j$ where each $W_j \cong n_j$ copies of irreducible representations S_j (each non-isomorphic for each j) is the *canonical decomposition* or the decomposition into *isotypical components* W_j . For F algebraically closed, $n_j = \dim \operatorname{Hom}_G(S_j, V)$.

5 Character theory

We want to attach invariants to representation ρ of a finite group G on V . Matrix coefficients of $\rho(g)$ are basis dependent, so not true invariants.

Let's take $F = \mathbb{C}$, G finite, $\rho = \rho_V : G \rightarrow GL(V)$ be a representation of G .

Definition. (5.1)

The *character* $\chi_\rho = \chi_V = \chi$ is defined as $\chi(g) = \text{tr } \rho(g) = \text{tr } R(g)$ where $R(g)$ is any matrix representation of $\rho(g)$ w.r.t. any basis.

The degree of χ_V is $\dim_{\mathbb{C}} V$.

Thus χ is a function $G \rightarrow \mathbb{C}$. χ is *linear* (not a universal name) if $\dim V = 1$, in which case χ is a homomorphism $G \rightarrow \mathbb{C}^*$ ($= GL_1(\mathbb{C})$).

χ is irreducible if ρ is; χ is faithful if ρ is; and, χ is trivial, or principal, if ρ is the trivial representation (2.6). We write $\chi = 1_G$ in that case.

χ is a complete invariant in the sense that it determines ρ up to isomorphism – see (5.7).

Theorem. (5.2, first properties)

- (i) $\chi_V(1) = \dim_{\mathbb{C}} V$; (clear: $\text{tr } I_n = n$)
- (ii) χ_V is a *class function*, via it is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$$

Thus χ_V is constant on conjugacy classes.

- (iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, the complex conjugate;

- (iv) For two representations V, W , $\chi_{V \oplus W} = \chi_V + \chi_W$.

Proof. (ii) $\chi(hgh^{-1}) = \text{tr}(R_h R_g R_h^{-1}) = \text{tr}(R_g) = \chi(g)$.

(iii) Recall $g \in G$ has finite order, so we can assume $\rho(g)$ is represented by a diagonal matrix $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Then $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by the matrix $\text{Diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$, and hence $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\chi(g)}$ (since λ_i 's are roots of unity – since $g^k = 1$ for some k ! (I mean an exclamation mark here to express surprise) and by homomorphism we know that).

(iv) Suppose $V = V_1 \oplus V_2$, $\rho_i : G \rightarrow GL(V_i)$, $\rho : G \rightarrow GL(V)$. Take basis $B = B_1 \cup B_2$ of V w.r.t. B , $\rho(g)$ has matrix of block form $\text{Diag}([\rho_1(g)]_{B_1}, [\rho_2(g)]_{B_2})$ and as $\chi(g)$ is the trace of the above matrix, it is equal to $\text{tr } \rho_1(g) + \text{tr } \rho_2(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$. \square

Remark. We see later that χ_1, χ_2 character of G implies that $\chi_1 \chi_2$ is also a character of G : uses tensor products, see (9.6).

Lemma. (5.3)

Let $\rho : G \rightarrow GL(V)$ be a complex representation *affording* the character χ (i.e. χ is a character of ρ). Then $|\chi(g)| \leq \chi(1)$, with equality iff $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ iff $g \in \ker \rho$.

Proof. Fix g . W.r.t. basis of V of eigenvalues $\rho(g)$, the matrix of $\rho(g)$ is $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Hence $|\chi(g)| = |\sum \lambda_j| \leq \sum |\lambda_j| = \sum 1 = \dim V = \chi(1)$. Equality holds iff all λ_j are equal (to λ , say).

If $\chi(g) = \chi(1)$, then $\rho(g) = \lambda I$ has $\chi(g) = \lambda \chi(1)$. \square

Lemma. (5.4)

- (a) If χ is a complex irreducible character of G , so is $\bar{\chi}$;
- (b) Under the same assumption, so is $\varepsilon\chi$ for any linear character ε of G .

Proof. If $R : G \rightarrow GL_n(\mathbb{C})$ is a complex irreducible representation then so is $\bar{R} : G \rightarrow GL_n(\mathbb{C})$ by $g \rightarrow \bar{R}(g)$. Similarly for $R' : g \rightarrow \varepsilon(g)R(g)$ for $g \in G$. Check the details. \square

Definition. (5.5)

$\mathcal{C}(G) = \{f : G \rightarrow \mathbb{C} : f(hgh^{-1}) = f(g) \forall h, g \in G\}$, the \mathbb{C} -space of class functions (we call it a space since $f_1 + f_2 : g \rightarrow f_1(g) + f_2(g)$, $\lambda f : g \rightarrow \lambda f(g)$ are still in $\mathcal{C}(G)$), so this is a vector space.

Let $k = k(G)$ be the number of ccls of G . List the ccls $\mathcal{C}_1, \dots, \mathcal{C}_k$. Conventionally we choose $g_1 = 1, g_2, \dots, g_k$, representatives of the ccls (hence $\mathcal{C}_1 = \{1\}$). Note that $\dim_{\mathbb{C}} \mathcal{C}(G) = k$ (the characteristic functions δ_j of each ccl which maps any element in the ccl to 1 and others to 0 form a basis).

We define Hermitian inner product on $\mathcal{C}(G)$:

$$\begin{aligned} \langle f, f' \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} f'(g) \\ &= \frac{1}{|G|} \sum_{j=1}^k |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) \\ &= \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \overline{f(g_j)} f'(g_j) \end{aligned}$$

using $|\mathcal{C}_x| = |G : C_G(x)|$, where \mathcal{C}_x is the ccl of x , $C_G(x)$ is the centraliser of x . For characters

$$\langle \chi, \chi' \rangle = \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

is a real symmetric form (in fact, $\langle \chi, \chi' \rangle \in \mathbb{Z}$ – see later).

Theorem. (5.6)

The \mathbb{C} -irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$. Moreover,

- (a) If $\rho : G \rightarrow GL(V), \rho' : G \rightarrow GL(V')$ are irreducible representations of G affording characters χ, χ' respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho, \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

we call this 'row orthogonality'.

- (b) Each class function of G can be expressed as a linear combination of G . This will be proved later in section 6.

Corollary. (5.7)

Complex representations of *finite* groups are characterised by their characters. We emphasise on finiteness here: for example, $G = \mathbb{Z}$, consider $1 \rightarrow I_2, 1 \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are non-isomorphic but have same character.

Proof. Let $\rho : G \rightarrow GL(V)$ be representation affording χ (G finite over \mathbb{C}). (3.3) says

$$\rho = m_1 \rho_1 \oplus \dots \oplus m_k \rho_k$$

where ρ_1, \dots, ρ_k are irreducible, and $m_j \geq 0$. Then $m_j = \langle \chi, \chi_j \rangle$ where χ_j is afforded by ρ_j : we have $\chi = m_1 \chi_1 + \dots + m_k \chi_k$, but the ρ_i 's are orthonormal. \square

Corollary. (5.8, irreducibility criterion)

If ρ is \mathbb{C} -representation of G affording χ , then ρ irreducible $\iff \langle \chi, \chi \rangle = 1$.

Proof. Forward is just the statement of orthonormality. Conversely, assume $\langle \chi, \chi \rangle = 1$. Now take a (complete) decomposition of ρ and take characters of it we get $\chi = \sum m_j \chi_j$ with χ_j irreducible and $m_j \geq 0$. Then $\sum m_j^2 = 1$. Hence $\chi = \chi_j$ for some j (since the m_j 's are obviously integers), so is irreducible. \square

Corollary. (5.9)

If the irreducible \mathbb{C} -representations of G are ρ_1, \dots, ρ_k have dimensions n_1, \dots, n_k , then

$$|G| = \sum_{i=1}^k n_i^2$$

Proof. Recall from (3.5), $\rho_{reg} : G \rightarrow GL(\mathbb{C}G)$, the regular representation G of dimension $|G|$ (where $\mathbb{C}G$ is just a G -space with basis $\{e_g : g \in G\}$ and any $h \in G$ permutes the e_g : $e_g \rightarrow e_{hg}$).

Let π_{reg} be its character, the *regular character* of G .

Claim 1: $\pi_{reg}(1) = |G|$, $\pi_{reg}(h) = 0$ if $h \neq 1$.

This is clear: take $h \in G, h \neq 1$, then we always have 0 down the diagonal since h permutes things around, so the trace is 0; if $h = 1$ then we have an identity matrix so trace is $\dim \rho = |G|$.

Claim 2: $\pi_{reg} = \sum n_j \chi_j$ with $n_j = \chi_j(1)$.

This is because

$$\begin{aligned} n_j &= \langle \pi_{reg}, \chi_j \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\pi_{reg}(g)} \chi_j(g) \\ &= \frac{1}{|G|} \cdot |G| \chi_j(1) = \chi_j(1) \end{aligned}$$

(all the other $\pi_{reg}(g)$ are zero by claim 1).

Our corollary is then obvious by just calculating $|G| = \pi_{reg}(1)$. \square

Corollary. (5.10)

Number of irreducible characters of G (up to equivalence) = k (=number of ccls).

Corollary. (5.11)

Elements $g_1, g_2 \in G$ are conjugate iff $\chi(g_1) = \chi(g_2)$ for all irreducible characters of G .

Proof. Forward: characters are class functions;

Backward: Let δ be the characteristic function of the class of g_1 . In particular, δ is a class function, so can be written as a linear combination of the irreducible characters of G . Hence $\delta(g_2) = \delta(g_1) = 1$, so $g_2 \in \mathcal{C}_G(g_1)$. \square

In the end let's introduce a good friend which will be around for the next few lectures:

Recall from (5.5), the inner product on $\mathcal{C}(G)$ and the real symmetric form \langle, \rangle on characters:

Definition. The *character table* of G is the $k \times k$ matrix (where k is the number of ccls) $X = [\chi_i(g_j)]$, the i^{th} character on the j^{th} class, where we let $\chi_1 = 1_G, \chi_2, \dots, \chi_k$ are the irreducible characters of G , and $\mathcal{C}_1 = \{1\}, \dots, \mathcal{C}_k$ are the ccls with $g_j \in \mathcal{C}_j$ (as we defined in 5.5).

So the $(i, j)^{th}$ entry of X is just $\chi_i(g_j)$.