Quantum Computation

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0 Introduction

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Exercise classes: Sat 3 Nov 11am MR4, Sat 24 Nov 11am MR4, early next term (tba).
Thursday 8 November lecture is moved to Saturday 10 November 11am (still MR4).

—Lecture 2—

1 1

Recall that we have an oracle U_f for $f : \mathbb{Z}_M \to \mathbb{Z}_\mathbb{N}$ periodic, with period r, $A = M/r$. We want to find r in $O(poly(m))$ time where $m = \log M$.

1.1 The quantum algorithm

Work on state space $\mathcal{H}_M \otimes \mathcal{N}$ with basis $\{|i\rangle|k\rangle\}_{i\in\mathbb{Z}_M, k\in\mathbb{Z}_N}$.
• Step 1. Make staet $\frac{1}{\sqrt{M}}\sum_{i=0}^{M-1}|i\rangle|0\rangle$.
• Step 2. Apply U_f to get $\frac{1}{\sqrt{M}}\sum_{i=0}^{M-1}|i\rangle|f(i)\rangle$.
• Step 3. Measure the 2nd register to get a result y. By Born rule, the first register collapses to all those i's (and only those) with $f(i)$ equal to the seen y, i.e. $i = x_0, x_0 + r, ..., x_0 + (A-1)r$, where $0 \le x_0 < r$ in 1st period has $f(m) = y$. Discard 2nd register to get $|per\rangle = \frac{1}{\sqrt{A}}\sum_{j=0}^{A-1}|x_0 + jr\rangle$.
Note: each of the r possible function values y occurs with same probability $1/r$, so $0 \le x_0 < r$ has been chosen uniformly at random.
If we now measure $|per\rangle$, we'd get a value $x_0 + jr$ for uniformly random j, i.e. random element (x_0^{th}) of a random period (j^{th}), i.e. random element of $\mathbb{Z}_m$, so we could get no information about r.
• Step 4. Apply quantum Fourier transform mod M (QFT) to $|per\rangle$. Recall the definition of QFT: $QFT : |x\rangle \to \sum_{y=0}^{M-1}\omega^{xy}|y\rangle$ for all $x \in \mathbb{Z}_M$ where $\omega = e^{2\pi i/M}$ is the Mth root of unity. The existing result is that QFT mod M can be implemented in $O(M^2)$ time.
Then we get

$$\begin{aligned} QFT|per\rangle &= \frac{1}{\sqrt{MA}}\sum_{j=0}^{A-1}\left(\sum_{y=0}^{M-1}\omega^{(x_0+jr)y}|y\rangle\right) \\ &= \frac{1}{\sqrt{MA}}\sum_{y=0}^{M-1}\omega^{x_0 y}\left[\sum_{j=0}^{A-1}\omega^{jry}\right]|y\rangle \ (*) \end{aligned}$$

where we group all the terms with the same $|y\rangle$ together. One good thing is that the sum inside the square bracket is a geometric series, with ratio $\alpha = \omega^{ry} = e^{2\pi i r y/M} = (e^{2\pi i/A})^y$.
Hence term inside bracket $= A$ if $\alpha = 1$, i.e. $y = kA = k\frac{M}{r}$, $k = 0, 1, ..., (r-1)$, and equals 0 otherwise when $\alpha \ne 1$. Now

$$QFT|per\rangle = \sqrt{\frac{A}{M}} \sum_{k=0}^{r-1} \omega^{x_0 k \frac{M}{r}} |k\frac{M}{r}\rangle$$

The random shift x_0 now appears only in phase, so measurement probabilities are now independent of x_0!

Measuring $QFT|per\rangle$ gives a value c, where $c = k_0 \frac{M}{r}$ with $0 \leq k_0 \leq r-1$ chosen uniformly at random. Thus $\frac{k_0}{r} = \frac{c}{M}$, note that c, M are known, r is unknown (what we want), and k_0 is unknown but uniformly random.

So note that if we are lucky and get a k_0 that is coprime to r then we could just simplify $\frac{c}{M}$ to get r. Obviously we cannot be always lucky every time, but by theorem in number theory, the number of integers $< r$ coprime to r grows as $O(r/\log\log r)$ for large r, so we know probability of k_0 coprime to r is $O(\frac{1}{\log\log r})$.

Then by some probability calculation we know that $O(1/p)$ trials are enough to achieve $1 - \varepsilon$ probability of success.

So afer Step 4, cancel c/M to the lowest terms a/b, giving r as denominator b (if k_0 is coprime to r). Check b value by computing $f(0)$ and $f(b)$, since $b = r$ iff $f(0) = f(b)$.

Repeating $K = O(\log\log r)$ times gives r with any desired probability.

Further insights into utility of QFT here:
Write $R = \{0, r, 2r, ..., (A-1)r\} \subseteq \mathbb{Z}_M$. $|R\rangle = \frac{1}{\sqrt{A}} \sum_{k=0}^{A-1} |kr\rangle$, and $|per\rangle = |x_0 + R\rangle = \frac{1}{\sqrt{A}} \sum_{k=0}^{A-1} |x_0 + br\rangle$ where x_0 is the random shift that caused problem previously.
For each $x_0 \in \mathbb{Z}_M$, consider mapping $k \to k + x_0$ (shift by x_0) on $\mathbb{Z}_M$, which is a 1-1 invertible map.

So linear map $U(x_0)$ on $\mathcal{H}_M$ defined by $U(x_0) : |k\rangle \to |k + x_0\rangle$ is unitary, and $|x_0 + R\rangle = U(x_0)|R\rangle$.

Since $(\mathbb{Z}_M, +)$ is abelian, $U(x_0)U(x_1) = U(x_0 + x_1) = U(x_1)U(x_0)$ i.e. all $U(x_0)$'s commute as operators on $\mathcal{H}_M$.
So we have orthonormal basis of common eigenvectors $|\chi_k\rangle\}_{k \in \mathbb{Z}_M}$, called *shift invariant states.*

$U(x_0)|\chi_k\rangle = \omega(x_0, k)|\chi_k\rangle$ for all $x_0, k \in \mathbb{Z}_M$ with $|\omega(x_0, k)| = 1$. Now consider $|R\rangle$ written in $|\chi\rangle$ basis,
$|R\rangle = \sum_{k=0}^{M-1} a_k |\chi_k\rangle$ where a_k's depending on r (not x_0).
Then $|per\rangle = U(x_0)|R\rangle = \sum_{k=0}^{M-1} a_k \omega(x_0, k)|\chi_k\rangle$, and measurement in the χ-basis has $prob(k) = |a_k \omega(x_0, k)|^2 = |a_k|^2$ which is independent of x_0, i.e. giving information about r!

—Lecture 3—

Recall last time we had $\mathcal{H}_M$: shift operations $U(x_0)|y\rangle = |y + x_0\rangle$ for $x_0, y \in$

$\mathbb{Z}_M$, which all permute, so have a common eigenbasis (shift invariant states) $\{|\chi_k\rangle\}_{k\in\mathbb{Z}_M}$, $U(x_0)|x_k\rangle = \omega(x_0,k)|\chi_k\rangle$.
Measurement of $|x_0+R\rangle = \frac{1}{\sqrt{A}}\sum_{l=0}^{A-1}|x_0+l_r\rangle = U(x_0)|R\rangle$ in $|\chi\rangle$ basis has output distribution independent of x_0, therefore gives information about r.

Introduce QFT as the unitary mapping that rotates χ-basis to standard basis, i.e. define $QFT|\chi_k\rangle = |k\rangle$. So QFT followed by measurement implements χ-basis measurement.

Explicit form of $|\chi_k\rangle$ eigenspaces (!): consider

$$|\chi_k\rangle = \frac{1}{\sqrt{M}}\sum_{l=0}^{M-1} e^{-2\pi ikl/M}|l\rangle$$

Then

$$\begin{aligned} U(x_0)|\chi_k\rangle &= \frac{1}{\sqrt{M}}\sum_{l=0}^{M-1} e^{-2\pi ikl/M}|l+x_0\rangle \\ &= \frac{1}{\sqrt{M}}\sum_{\tilde{l}=0}^{M-1} e^{-2\pi ik(\tilde{l}-x_0)/M}|\tilde{l}\rangle \text{ where } \tilde{l}=l+x_0 \\ &= e^{2\pi ikx_0/M}\cdot|\chi_k\rangle \end{aligned}$$

i.e. these are the shift invariant staets, eigenvalues $\omega(x_0,k) = e^{2\pi ikx_0/M}$.

Matrix of QFT: So

$$[QFT^{-1}]_{lk} = \frac{1}{\sqrt{M}}e^{-2\pi ilk/M}$$

(componets of $|\chi_k\rangle = QFT^{-1}|k\rangle$ as k^{th} column). So

$$[QFT]_{kl} = \frac{1}{\sqrt{M}}e^{2\pi ilk/M}$$

as expected.

2 The hidden subgroup problem (HSP)

Let G be a finite group of size $|G|$. Given (oracle for) function $f : G \to X$ (X is some set), and promise that there is a subgroup $K < G$ such that f is constant on (left) cosets of K in G, and f is distinct on distinct cosets.
The problem: determine the *hidden subgroup* K (e.g. output a set of generators, or sample uniformly from K).
We want to solve in time $O(poly(\log |G|))$ (an efficient algorithm) with any constant probability $1 - \varepsilon$.

Examples of problems that can be cast(?) as HSPs:
(i) periodicity: $f : \mathbb{Z}_M \to X$, periodic with period r. Let $G = (\mathbb{Z}_m, +)$, the hidden subgroup is $K = \{0, r, 2r, ...\} < G$, cosets $x_0 + K = \{x_0, x_0 + r, x_0 + 2r, ...\}$. The period r is generator of K.
(ii) discrete logarithm: for prime p, $\mathbb{Z}_p^* = \{1, 2, ..., p-1\}$ with multiplication mod p. $g \in \mathbb{Z}_p^*$ is a generator (or primitive root mod p). If powers generate all of $\mathbb{Z}_p^*$, $\mathbb{Z}_p^* = \{g^0 = 1, g^1, ..., g^{p-2}\}$, then also $g^{p-1} \equiv 1 \pmod p$ (easy number theory).
Fact: the generator always exists if p is prime. So any $x \in \mathbb{Z}_p^*$ can be written $x = g^y$ for some $y \in \mathbb{Z}_{p-1}$, write $y = \log_g x$ called the discrete log of x to base g.

Discrete log problem: given a generator g and $x \in \mathbb{Z}_p^*$, compute $y = \log_g x$ (classically hard).
To express as HSP, consider $f : \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1} \to \mathbb{Z}_p^*$: $f(a, b) = g^a x^{-b}\ mod\ p = g^{a-yb}\ mod\ p$.
Then check: $f(a_1, b_1) = f(a_2, b_2)$ iff $(a_2, b_2) = (a_1, b_1) + \lambda(y, 1)$ where $\lambda \in \mathbb{Z}_{p-1}$.

So if $G = \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$, $K = \{\lambda(y, 1) : \lambda \in \mathbb{Z}_{p-1}\} < G$. Then f is constant and distinct on the cosets of K in G, and generator $(y, 1)$ gives $y = \log_g x$.

(iii) graph problems (G non-abelian now): consider undirected graph $A = \{V, E\}$, $|V| = n$, with at most one edge between any two vertices. Label vertices by $[n] = \{1, 2, ..., n\}$.
Introduce the permutation group $\mathcal{P}_n$ of $[n]$. Define $Aut(A)$ to be the group of automorphisms of A, which is a subgroup of $\mathcal{P}_n$, containing exactly the permutations $\pi \in \mathcal{P}_n$ such that for all $i, j \in [n]$, $(i, j) \in E \iff (\pi(i), \pi(j)) \in E$, i.e. the labelled graph $\pi(A)$ obtained by permuting labels of A by π is the same *labelled* graph as A.

Associated HSP: Take $G = \mathcal{P}_n$. Let X be set of all labelled graphs on n vertices. Given A, consider $f_A : \mathcal{P}_n \to X$ by $f_A(\pi) = \pi(A)$, A with labels permuted by π. The associated hiiden subroup is $Aut(A) = K$.

Application: if we can sample uniformly from this K, then we can solve graph isomorphism problem (GI): two labelled graphs A, B are isomorphic if there is 1-1 map $\pi : [n] \to [n]$ such that for all $i, j \in [n]$, i, j is an edge in A iff $\pi(i), \pi(j)$ is an edge in B, i.e. A and B are the same graph but just labelled differently.

—Lecture 4—

Let's come back to the graph isomorphism problem.

Problem: given A, B, decide if $A \cong B$ or not. This can be expressed as anon-abelian HSP (on example sheet), no known classical polynomial time algorithm. However it is in NP, but it is not believed to be NP-complete.
Recent result (2017): a quasi-poly time classical algorithm (L.Babai).

Quantum algorithm for finite *abelian* HSP:
Write group $(G, +)$ additively.

Construction of shift invariant states and FT for G:
Let's introduce some representation theory for abelian group G. Consider mapping $\chi : G \to \mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \cdot)$ satisfying $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$, i.e. χ is a group homomorphism. Such χ's are called *irreducible* representations of G.
We have the following properties (without proof), which we'll call Theorem A later when we refer to it:
(i) any value $\chi(g)$ is a $|G|^{th}$ root of unity (so χ: $G \to S^1$ = unit circle in $\mathbb{C}$);
(ii) (Schur's lemma, orthogonality): If χ_i and χ_j are representations, then $\sum_{g\in G} \chi_i(g)\bar{\chi}_j(g) = \delta_{ij}|G|$;
(iii) there are always exactly $|G|$ different representations χ (well, this is a special case of general representation theory).

By (iii), we can label χ's as χ_g for $g \in G$. For example, $\chi(g) = 1$ for all $g \in G$ is always an irreducible representation (the trivial representation), labelled χ_0;
Then by orthogonality (ii) for any $\chi \neq \chi_0$ gives $\sum_{g\in G} \chi(g) = 0$.

Shift invariant states: in space $\mathcal{H}_{|G|}$ with basis $\{|g\rangle\}_{g\in G}$, introduce *shift operators* $U(k)$ for $k \in G$ defined by $U(k) : |g\rangle \to |g + k\rangle$. Clearly these all commute, so there is simultaneous eigenbasis:
For each χ_k, $k \in G$, consider state $|\chi_k\rangle = \frac{1}{\sqrt{|G|}} \sum_{g\in G} \bar{\chi}_k(g)|g\rangle$. Then theorem A(ii) implies these form orthonormal basis, and $U(g)|\chi_k\rangle = \chi_k(g)|\chi_k\rangle$.

Proof.

$$
\begin{aligned}
U(g)|\chi_k\rangle &= \frac{1}{\sqrt{|G|}} \sum_{h\in G} \chi_k\bar{}(h)|h + g\rangle \\
&\overset{h'=h+g}{=} \frac{1}{\sqrt{|G}} \sum_{h'\in G} \chi_k(h\bar{'} - g)|h'\rangle
\end{aligned}
$$

This implies that

$$
\chi_k * -g) = (\chi_k(g))^{-1} = \chi_k\bar{}(g),
$$
$$
\chi_k(h\bar{'} - g) = \chi_k\bar{}(h')\chi_k\bar{}(-g) = \chi_k(h')\chi_k(g)
$$

So

$$
U(g)|\chi_k\rangle = \frac{1}{\sqrt{|G|}} = \sum_{h'\in G} \chi_k(g)\bar{\chi}_k(h')|h'\rangle = \chi_k(g)|\chi_k\rangle
$$

□

So $|\chi_k\rangle$'s are common eigenspaces, called *shift-invariant states.*
Introduce (define) Fourier transform QFT for group G as the unitary that

$QFT|\chi_g\rangle = |g\rangle$ for all $g \in G$.
In $|g\rangle-$basis matrices, k^{th} column of (QFT^{-1}) =components of $|\chi_k\rangle$, i.e. $\frac{1}{\sqrt{|G|}}\bar{\chi_k}(g) = [QFT^{-1}]_{gk}$.
So $[QFT]^\dagger_{kg} = \frac{1}{\sqrt{|G|}}\chi_k(g)$, and so $QFT|g\rangle = \frac{1}{\sqrt{|G|}}\sum_{k\in G}\chi_k(g)|k\rangle$.

Example. $G = \mathbb{Z}_M$. Check $\chi_a(b) = e^{2\pi i ab/M}$, $a, b \in \mathbb{Z}_M$ is a representation.
Similarly, for $G = \mathbb{Z}_{M_1} \times ... \times \mathbb{Z}_{M_r}$, $(a_1, ..., a_r) = g_1, (b_1, ..., b_r) = g_2$ where $g_1, g_2 \in G$,

$$\chi_{g_1}(g_2) \overset{def}{=} e^{2\pi i\left(\frac{a_1b_1}{M_1}+...+\frac{a_rb_r}{M_r}\right)}$$

is a representation of G. And we get

$$QFT_G = QFT_{M_1} \otimes ... \otimes QFT_{M_r}$$

on $\mathcal{H}_{|G|} = \mathcal{H}_{M_1} \otimes ... \otimes \mathcal{H}_{M_r}$.
This is exhaustive, since by classification theorem, every finite abelian group G is isomorphic to a direct product of the form $G \cong \mathbb{Z}_{M_1} \times ... \times \mathbb{Z}_{M_r}$. Furthermore, we can insist that M_i are prime powers $p_i^{s_i}$, where p_i are not necessarily distinct.

Quantum algorithm for finite abelian HSP:
Let $f : G \to X$, hidden subgroup $K < G$. We have cosets $K = 0 + K, g_2 + K, ..., g_m+K$, where $m = |G|/|K|$. State space as usual, with basis $\{|g\rangle, |x\rangle\}_{g\in G, x\in X}$.
- make the state $\frac{1}{\sqrt{|G|}}\sum_{g\in G}|g\rangle|0\rangle$;
- Apply oracle U_f, get $\frac{1}{\sqrt{|G|}}\sum_{g\in G}|g\rangle|f(g)\rangle$;

measure second register to see a value $f(g_0)$.
Then first register gives coset state (remember the function is constant on each coset). $|g_0 + K\rangle = \frac{1}{\sqrt{|K|}}\sum_{k\in K}|g_0 + K\rangle = U(g_0)|K\rangle$.
Apply QFT and measure to obtain result $g \in G$.

—Lecture 5—

Last time we disccused how to solve the abelian HSP problem. Now how does the output g related to K?
- the output distribution of g is independent of g_0, so same as that obtained from $QFT|K\rangle$ (i.e. $g_0 = 0$) since:

write $|K\rangle$ in shift invariant basis $|\chi_g\rangle$'s, $|K\rangle = \sum_g a_g|\chi_g\rangle$, then $|g_0 + K\rangle = U(g_0)|K\rangle = \sum a_g \underbrace{\chi_g(g_0)|\chi_g\rangle}_{=U(g_0)|\chi_g\rangle}$; but $QFT|\chi_g\rangle = |g\rangle$, so $Prob(g) = |a_g\chi_g(g_0)|^2 = |a_g|^2$ as $\chi_g(g_0)| = 1$.

Thus look at $QFT|K\rangle$. Recall $QFT|k\rangle = \frac{1}{\sqrt{|G|}}\sum_{l\in G}\chi_l(k)|l\rangle$, so $QFT|K\rangle = \frac{1}{\sqrt{|G|}}\frac{1}{\sqrt{|K|}}\sum_{l\in G}\left[\sum_{k\in K}\chi_l(k)\right]|l\rangle$.
The terms in [...] involves irreducible representation χ_l of G restricted to subgroup $K < G$, which is an irreducible representation of K. Hence

$$\sum_{k\in K}\chi_l(k) = \begin{cases} |K| & \chi_l \text{ restricts to trivial irreducible representation on } K \\ 0 & \text{otherwise} \end{cases}$$

and

$$QFT|K\rangle = \sqrt{\frac{|K|}{|G|}} \sum_{l \in G \text{ with } \chi_l \text{ reducing to trivial irreducible representation of } K} |l\rangle$$

So measurement gives a uniformly random choice of l such that $\chi_l(k) = 1$ for all $k \in K$.
e.g. If K has generators $k_1, k_2, ..., k_M$, $M = O(\log|K|) = O(\log|G|)$, then output has $\chi_l(k_i) = 1$ for all i.

It can be shown that if $O(\log|G|)$ such l's are chosen uniformly at random, then with probability $> 2/3$ they suffice to determine a generating set for K via equations $\chi_l(k) = 1$.
(see example sheet 1 for particular examples).

Example. If $G = \mathbb{Z}_{M_1} \times ... \times \mathbb{Z}_{M_q}$.
We had for $l = (l_1, ..., l_q)$, $g \in (b_1, ..., b_q) \in G$,

$$\chi_l(g) = e^{2\pi i(\frac{l_1 k_1}{M_1} + ... + \frac{l_q b_q}{M_q})}$$

So for $k = (k_1, ..., k_q)$, $\chi_l(k) = 1$ becomes

$$\frac{l_1 k_1}{M_1} + ... + \frac{l_q k_q}{M_q} \equiv 0 \pmod 1$$

(i.e. is an integer), a homogeneous linear equation on K, and $O(\log|K|)$ is independent such that equations determine K as null space.

Some remarks on HSP for non-abelian groups G (write multiplicatively):
As before, can easily generate coset states

$$|g_0 K\rangle = \frac{1}{\sqrt{|K|}} \sum_{k \in K} |g_0 K\rangle$$

where g_0's are randomly chosen. But problems arise with QFT construction, because now there's no basis of shift-invariant states exists! (this is since $U(g_0)$'s don't commute anymore, so no common full eigenbasis).

Construction of non-abelian Fourier Transform (some more representation theory):
• d-dimensional representation of G is a group homomorphism $\chi : G \to U(d)$ where $U(d)$ is the space of $d \times d$ unitary matrices acting on $\mathbb{C}^d$, by $\chi(g_1 g_2)\chi(g_1)\chi(g_2)$. (see part II representation theory for the general form)
• χ is irreducible representation if no subspace of $\mathbb{C}^d$ is left invariant under $\chi(g)$ for all $g \in G$ (i.e. cannot simultaneously block diagonalise all $\chi(g)$'s by a basis change).
• a complete set of irreducible representation: set $\chi_1, ..., \chi_m$ such that any irreducible representation is unitarily equivalent to one of them (equivalence $\chi \to \chi' = V\chi V^T$).

Theorem. (non-abelian version of theorem A – properties of representations)
If $d_1, ..., d_m$ are dimensions of a complete set of irreducible representations

$\chi_1, ..., \chi_m$, then:
(i) $d_1^2 + ... + d_m^2 = |G|$;
(ii) Write $\chi_i(g)_{jk}$ for the $(j,k)^{th}$ entry of matrix $\chi_i(g)$, where $j, k = 1, ..., d_i$. Then (Schur orthogonality):

$$\sum_g \chi_i(g)_{jk} \bar{\chi}_{i'}(g)_{j'k'} = |G| \delta_{ii'} \delta j j' \delta k k'$$

Hence states

$$|\chi_{i,jk}\rangle \equiv \frac{1}{\sqrt{|G|}} \sum_{g \in G} \bar{\chi}_i(g)_{jk} |g\rangle$$

is an orthonomal basis.

• QFT on G *defined* to be the unitary that rotates $\{|\chi_{ijk}\rangle\}$ basis into standard basis $\{|g\rangle\}$. However, $|\chi_{ijk}\rangle$ are *not* shift invariant for all $U(g_0)$'s, and consequently measurement of coset state $|g_0 K\rangle$ in $|\chi\rangle$-basis gives an output distribution *not* independent of g_0.

However, *partial* shift invariance survives: Consider the incomplete measurement M_{rep} on $|g_0 K\rangle$ that distinguishes only the irreducible representations (i.e. i values) and not all (i, j, k)'s.
i.e. with measurement outcome i associated to d_i^2-dimensional orthogonal subspaces spanned by $\{|\chi_{(i),jk}\rangle\}_{j,k=1,...,d_i}$.
Then $\chi_i(g_1, g_2) = \chi_i(g_1)\chi_i(g_2)$ implies output distribution of i values is independent of g_0, giving direct, albeit imcomplete, information about K.
E.g. conjugate subgroups K and $= g_0 K g_0^{-1}$ for some $g_0 \in G$ give *same* output distribution.

—Lecture 6—
Non-abelian HSP/FT remarks:
For efficient HSP algorithm, we also need QFT to be efficiently implementable, i.e. $poly(\log |G|)$-time.
This is true for any abelian G and some non-ablien G's (such as $\mathcal{P}_n$), but even in latter case there's no known efficient HSP algorithm.

Some known result:
for normal subgroups, i.e. $gK = Kg$ for all $g \in G$:

Theorem. (Hallgrer, Russell, Tashma, SIAM J.Comp 32 p916-934 (2003))
Suppose G has efficient QFT. Then if hidden subgroup K is normal, then there is an efficient HSP quantum algorithm.
(Construct coset state $|g_0 K\rangle$, perform M_{rep} on it.)
Repeat $O(\log |G|)$ times. Then K normal implies outputs suffice to determine K.

Theorem. (Ettinger, Hoyer, Knill)
For general non-abelian HSP, $M = O(poly(\log |G|))$ random coset states $|g_1 K\rangle$,...,$g_M K\rangle$ suffice to determine K from M coset states, but it's not efficient.
See example sheet for a proof – construct a measurement procedure on $|g_1 K\rangle \otimes ... \otimes g_M K\rangle$ to determine K, but it takes exponential time in $\log |G|$.

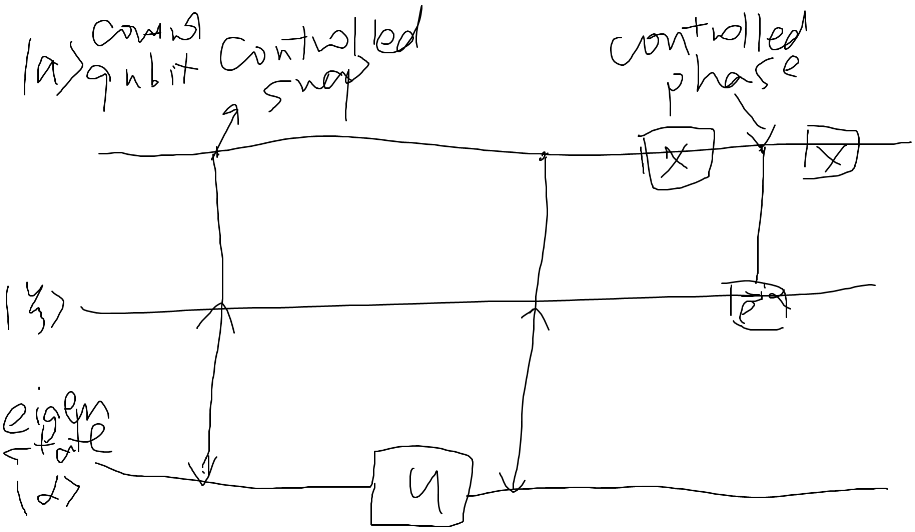
The phase estimation algorithm:
- a unifying principle for quantum algorithms, uses QFT_{2^n} again.
- many applications, e.g. an alternative efficient factoring algorithm (A.Kitaev).

Given unitary operator $\mathcal{U}$ and eigenstate $|v_\phi\rangle \cdot \mathcal{U}|v_\phi\rangle = e^{2\pi i\phi}|v_\phi\rangle$, we want to estimate phase ϕ, where $0 \leq \phi < 1$ (to some precision, say to n binary digits).

We'll need *controlled-U^k* for integers k, writte $C - U^k$, which satisfies $c - U^k|0\rangle|\xi\rangle = |0\rangle|\xi\rangle$, $C - U^k|1\rangle|\xi\rangle = |1\rangle U^k|\xi\rangle$, where $|\xi\rangle$ in general has dimension d.
Note $U^k|v_\phi\rangle = e^{2\pi ik\phi}|v_\phi\rangle$, $C - (U^k) = (C - U)^k$.

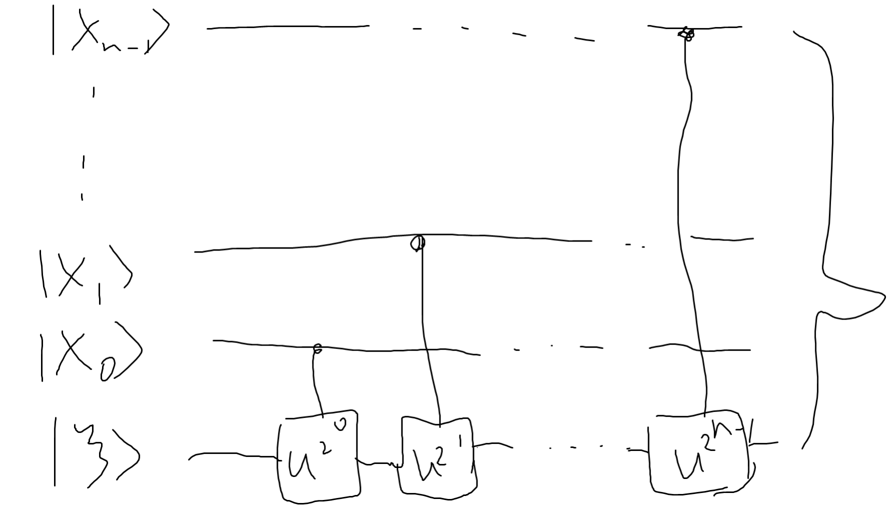
Remark. Given U as a formula or (arant?) description, we can readily implement $C - U$, e.g. just control each gate of U's circuit.
However, if U is given as a *black box*, we need further info:
- it suffices to have an eigenstate $|\alpha\rangle$ with known eigenvalue $U|\alpha\rangle = e^{i\alpha}|\alpha\rangle$:

We can consider



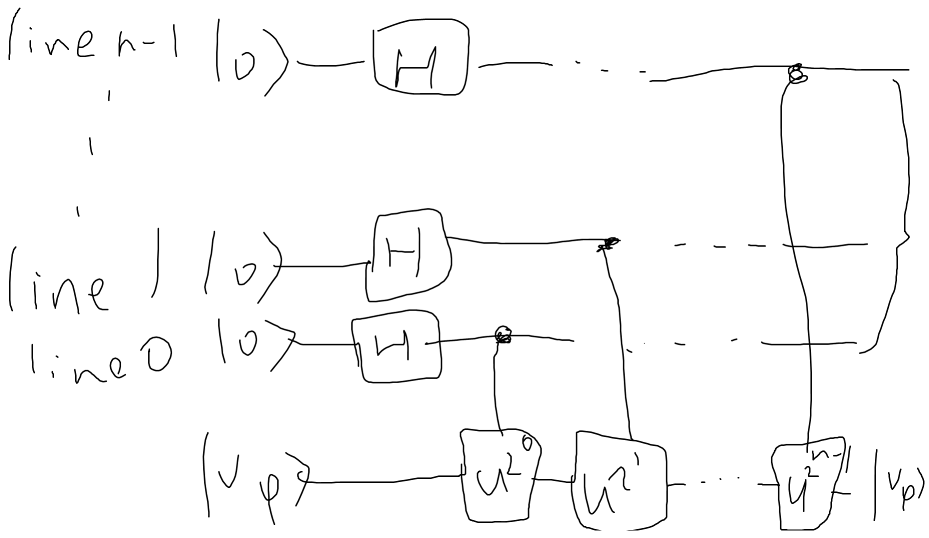
Where we get $CU|a\rangle|\xi\rangle$ at the first two row and the third row $|\alpha\rangle$ is always unchanged.
To see how it works, just check circuit action. (...)

We'll actually want *generalised controlled-U* with $|x\rangle|\xi\rangle \rightarrow |x\rangle U^x|\xi\rangle$, where $|x\rangle$ has n qubits, i.e. $x \in \mathbb{Z}_{2^n}$.
We can make this thing from $C - (U^k)$ as follows:



We get $|x\rangle U^x|\xi\rangle$, where $x = x_{n-1}...x_1x_0$ binary, $U^x = U^{2^{x_{n-1}}}...U^{2^{x_1}}U^{2^{x_0}}$.
Note: if input $|\xi\rangle = |v_\phi\rangle$, then get $e^{2\pi i\phi x}|v_\phi\rangle$.

Now suppose over all $x = 0, 1, ..., 2^{n-1}$ and use $|\xi\rangle = |v_\phi\rangle$,



Where the output is $\frac{1}{\sqrt{2^n}}\sum_x e^{2\pi i\phi x}|x\rangle$, we call this state $|A\rangle$.

Finally apply $QFT_{2^n}^{-1}$ to $|A\rangle$ and measure to see $y_0, ..., y_{n-1}$ on lines $0, 1, ..., n-1$.
Then output $0.y_0...y_{n-1} = \frac{y_0}{2} + ... + \frac{y_{n-1}}{2^{n-1}}$, as the estimate of ϕ.
That's the phase estimation algorithm (for given U and $V_\phi\rangle$).

Suppose ϕ actually had only n binary digits, i.e. ϕ exactly equals $0.z_0z_1...z_{n-1}$ for some $z_k = 0, 1$ for all k.

Then $\phi = \frac{z_0 \dots z_{n-1}}{2^n} = \frac{z}{2^n}$ where z is n-bit integer in $\mathbb{Z}_{2^n}$, and

$$|A\rangle = \frac{1}{\sqrt{2^n}} \sum_x e^{2\pi i x z/2^n} |x\rangle$$

is QFT_{2^n} of $|z\rangle$.
So $QFT^{-1}|A\rangle = |z\rangle$ and get ϕ exactly, with certainty.
In this case the algorithm up to (not including) final measurements is a unitary operation, mapping $|0\rangle...|0\rangle|v_\phi\rangle \to |z_0\rangle...|z_{n-1}\rangle|v_\phi\rangle$.

—Lecture 7— Phase Estimation (continued):

U is a $d \times d$ unitary operation/matrix with eigenstate $U|v_\phi\rangle = 2^{2\pi i\phi}|v_\phi\rangle$, and we want to estimate ϕ.
U as a quantum physical operation is equivalent to $\tilde{U} = e^{i\alpha}U$ for any α and $\tilde{U}$ has $\phi \to \phi + \alpha/2\pi$.
So if U given as quantum physical operation alone, we cannot determine ϕ.
But controlled versions different: $C - U$ and $C - \tilde{U}$ are different as physical operations (set $\{e^{i\alpha}C - U\}_\alpha \neq \{e^{i\alpha}C - \tilde{U}\}_\alpha$), and $C - U/\tilde{U}$ *does* fix ϕ associatied to choice of phase α.
So quantum phase estimation algorithm use $C - U$ ($C - U^{2^k}$) physical operations (not just U's).

We had $\underbrace{|0\rangle...|0\rangle}_{n}|v_\phi\rangle \overset{\text{unitary}}{\underset{C-U's}{\to}} |A\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1} e^{2\pi i\phi x}|x\rangle$ (n qubits).
Apply QFT^{-1} we get $QFT^{-1}|A\rangle$, measure to see $y_0, ..., y_{n-1}$; output $\phi = \frac{(y_0 y_1 \cdots y_{n-1})}{2^n}$, $0 \leq y < 2^{n-1}$, where the numerator is a n-bit integer.
If $\phi = \frac{z}{2^n}$ for integer $0 \leq z < 2^n$, i.e. ϕ has exactly n binary digits, then $|A\rangle = QFT|z\rangle$, so we get z with certainty in the measurement.

Now suppose ϕ has *more* than n bits, say $\phi = 0.z_0 z_1 z_2 ... z_{n-1}|z_n z_{n+1}....$ Then we have:

Theorem. (PE) If measurement in above algorithm give $y_0, ...y_{n-1}$ (so output is $\theta = 0.y_0...y_{n-1}$), then
(a) $\mathbb{P}(\theta$ is closet n binary digit approximate to $\phi) \geq 4\pi^2$;
(b) $\mathbb{P}(|\theta - \phi| \geq \varepsilon)$ is at most $P(\frac{1}{2^n\varepsilon})$ (we'll show it's at most $\frac{1}{2^{n+1}\varepsilon}$).

Remark. In (a), we have probability $\frac{4}{\pi^2}$ that all n lines of n-line QPE process are *good.*
But, if we want ϕ accurate to m bits with probability $1 - \eta$, then we use theorem (PE) (b) with $\varepsilon = 1/2^m$. Then we'll use $n > m$ lines with

$$\frac{1}{2^{n+1}}\varepsilon = \eta, \varepsilon = \frac{1}{2^m}$$

i.e. $n = m + \log(1/\eta) + 1$. In words, number of lines needed is only number of bits wanted with good probability $1 - \eta$ plus a modest polynomial increase for exponetial reduction in η.

Proof. We have

$$QFT^{-1}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i y x/2^n} |y\rangle$$

So

$$QFT^{-1}|A\rangle = \frac{1}{2^n} \sum_y \left[\sum_x e^{2\pi i(\phi - y/2^n)x} \right] |y\rangle$$

So for measurement,

$$\mathbb{P}(\text{see } n- \text{ bit integer } y = y_0 y_1 ... y_{n-1}) = \frac{1}{2^{2n}} \left| \sum_{x=0}^{2^n-1} e^{2\pi i \underbrace{(\phi - \frac{y}{2^n})}_{:=\delta(y)} x} \right|^2$$

Note that this is a geometric series $e^{2\pi i \delta(y)}$, so

$$\mathbb{P}(\text{see } y) = \frac{1}{2^{2n}} \left| \frac{1 - e^{2^n 2\pi i \delta(y)}}{1 - e^{2\pi i \delta(y)}} \right|^2$$

Let's call this equation (P) (maybe for *phase*).
We want to bound/estimate this expression.
For (a): Let $y = a = a_0 a_1 ... a_{n-1}$ give *closest* n-bit approximation to ϕ, i.e. $|\phi - \frac{a}{2^n}| \leq \frac{1}{2^{n+1}}$, i.e. $\delta(a) \leq \frac{1}{2^{n+1}}$.
Now we bounds:
(i) $|1 - e^{i\alpha}| = |2 \sin \frac{\alpha}{2}| \geq \frac{2}{\pi}|\alpha|$ if $|\alpha| < \pi$;
(ii) $|1 - e^{2\pi i \beta}| \leq 2\pi\beta$.

In equation (P), use (i) with $\alpha = 2^n \cdot 2\pi\delta(a) \leq 2^n 2\pi \frac{1}{2^{n+1}} \leq \pi$ to lower bound top line, and (ii) with $\beta = \delta(a)$ to upper bound bottom line, get

$$\mathbb{P}(\text{see } a) \geq \frac{1}{2^{2n}} \left(\frac{2^{n+1}\delta(a)}{2\pi\delta(a)} \right)^2 = \frac{4}{\pi^2}$$

For (b), we want to upper bound equaiton (P): for top line, $|1 - e^{i\alpha}| \leq 2$ for any α; for bottom, use (i) get $|1 - e^{2\pi i \delta(y)}| \geq 4\delta(y)$. So

$$\mathbb{P}(y) \leq \frac{1}{2^{2n}} \left(\frac{2}{4\delta(y)} \right)^2 = \frac{1}{2^{2n+2}} \delta(y)^2$$

Now sum this for all $|\delta(y)| > \varepsilon$, $\delta(y)$ values spaced by $1/2^n$'s. Let δ_+ be first $\delta(y)$ (jumps?) with $\delta(y) \geq \varepsilon$, δ_- be that with $\delta(y) \leq -\varepsilon$. So $|\delta_+|, |\delta_-| \geq \varepsilon$.
Then if $|\delta(y)| \geq \varepsilon$, we have $\delta(y) = \delta_+ + \frac{k}{2^n}$, $k = 0, 1, ...$, or $= \delta_- - \frac{k}{2^n}$, $k = 0, 1, ...$
So $|\delta(y)| \geq \varepsilon + \frac{k}{2^n}$ with $k = 0, 1, 2, ...$ in each case.

So

$$\begin{aligned}\mathbb{P}(|\delta(y)| > \varepsilon) &\le 2\sum_{k=0}^{\infty}\frac{1}{2^{2n+2}}\frac{1}{(\varepsilon+\frac{k}{2^n})^2}\\ &\le \frac{1}{2}\int_0^{\infty}\frac{1}{(2^n\varepsilon+k)^2}dk\\ &= \int_{2^n\varepsilon}^{\infty}\frac{dk}{k^2}\\ &= \frac{1}{2^{n+1}\varepsilon}\end{aligned}$$

□

Further remarks on QPE algorithm:
(1) If $C-U^{2^k}$ is implemented as $(C-U)^{2^k}$, the QPE algorithm needs exponential time in n as we have $1+2+...+2^{n-1} = 2^n-1$ $(C-U)$ gates.
However, for some special U's, $C-U^{2^k}$ can be implemented in $poly(k)$ time, so we get a poly time QPE algorithm.
It can be used to provide alternative facoring (order finding) algorithm (due to A. Kitaev) using PE.

—Lecture 8—

First exercise class: Saturday 3 Nov 11am MR4.

(2) If instead of $|v_\phi\rangle$, use general input state $|\xi\rangle$:

$$|\xi\rangle = \sum_j c_j|v_{\phi_j}\rangle$$

$$U|v_{\phi_j}\rangle = e^{2\pi i\phi_j}|v_{\phi_j}\rangle$$

Then we get in QPE (before final measurement) a unitary process U_{PE} with (lecturer had *that*) effect

$$|0...0\rangle|\xi\rangle \xrightarrow{U_{PE}} \sum_j c_j|\phi_j\rangle|v_{\phi_j}\rangle$$

and final measurement will give a choice of ϕ_j's (or approximation) chosen with probabilities $|c_j|^2$.

Example. Implement $QFT_\mathcal{Q}$ for $\mathcal{Q}$ not a power of 2, with a quantum curcuit of 1− and 2− qubit gates of circuit size $O(poly(\log\mathcal{Q}))$ (Kitaev's method).

Remark. For $\mathcal{Q} = 2^m$, we have explicit known circuit of $O(m^2)$. H and C-phase gate to implement QFT_{2^m} exactly (cf part II QIC Notes).

For $QFT_\mathcal{Q}$: Introduce

$$|\eta_a\rangle = QFT_\mathcal{Q}|a\rangle = \frac{1}{\sqrt{\mathcal{Q}}}\sum_{b=0}^{\mathcal{Q}-1}\omega^{ab}|b\rangle, a\in\mathbb{Z}_\mathcal{Q}, \omega = e^{2\pi i/\mathcal{Q}}$$

It suffices to make circuit hat does $|a\rangle \to |\eta_a\rangle$ (*).
Let $2^{m-1} < \mathcal{Q} < 2^m$, and set $M = 2^m$, view $\mathcal{H}_\mathcal{Q}$ as subspace of m qubits (spanned by $|a\rangle : 0 \leq a < \mathcal{Q} - 1 < 2^m$).
To achieve (*), consider instead on $\mathcal{H}_\mathcal{Q} \otimes \mathcal{H}_\mathcal{Q}$

$$|a\rangle|0\rangle \xrightarrow{(1)} |a\rangle|\eta_a\rangle \xrightarrow{(2)} |0\rangle|\eta_a\rangle$$

(1): get $\eta_a\rangle$ from $|a\rangle$ while *remembering* $|a\rangle$;
(2): *erase/forget* $|a\rangle$.
For (1), first do $|0\rangle \to |\xi\rangle = \frac{1}{\sqrt{\mathcal{Q}}}\sum_{b=0}^{\mathcal{Q}-1}|b\rangle$ as follows:
on m qubits $\mathcal{H}^{\otimes m}$ gives $\frac{1}{\sqrt{M}}\sum_{x=0}^{2^m-1}|x\rangle$. Then consider the step function $f(x) = 0$ if $x < \mathcal{Q}$ and 1 if $x \geq \mathcal{Q}$. It's classically efficiently computable, so can efficiently implement U_f on $(m+1)$ qubits.
So applying U_ρ to $(H^{\otimes m}|0\rangle)|0\rangle$ and measure output $(m + 1^{st})$ qubit to get $|\xi\rangle$ on first n qubits if measurement result is 0.
Note that $prob(0) > 1/2$ as $\mathcal{Q} > 2^{m-1} = 2^m/2$, so we can use multiple trials to give $|\xi\rangle$.
We can do offline: failures/re-tries do not affect state to which we want to apply $QFT_\mathcal{Q}$. So now we have $|\tilde{\xi} = |a\rangle\left(\frac{1}{\sqrt{\mathcal{Q}}}\sum_{b=0}^{\mathcal{Q}-1}|b\rangle\right)$.
Next consider $V|a\rangle|b\rangle = \omega^{ab}|a\rangle|b\rangle$.
Then $V|\tilde{\xi}\rangle = |a\rangle|\eta_a\rangle$ as we want for (1).

To implement V, consider

$$U : |b\rangle \to \omega^b|b\rangle$$

If $|b\rangle$ in m qubits given by $|b_{m-1}\rangle ... |b_0\rangle$, i.e. $b = b_{m-1}...b_0$ in binary, then $\omega^b = \omega^{b_{m-1}2^{m-1}}...\omega^{b_0 2^0}$. So U is product of 1-qubit phase gates

$$P(\omega^{2^{m-1}}) \otimes ... \otimes \mathbb{P}(\omega^{2^0})$$

where $P(\xi) = Diag(1, \xi)$, $|\xi| = 1$ is a phase gate.

Similarly, for $C - U^{2^k}$ (starting with $U \to U^{2^k}$ i.e. $\omega^b \to \omega^{2^k b}$), and $V =$ *generalised* $C - U$:

$$|a\rangle|b\rangle \xrightarrow{V} |a\rangle U^a|b\rangle$$

which is constructed as before, from $C - U^{2^k}$'s.
So now we have $|a\rangle|0\rangle \xrightarrow{(1)} |a\rangle|\eta_a\rangle$.

For (2), i.e. $|a\rangle|\eta_a\rangle \xrightarrow{(2)} |0\rangle|\eta_a\rangle$, *if* we had U with eigenstates $|\eta_a\rangle$, eigenvalues $\omega^a = e^{2\pi i a/\mathcal{Q}}$, then U_{PE} would give

$$|0\rangle|\eta_a\rangle \xrightarrow{U_{PE}} |a\rangle|\eta_a\rangle$$

(we are a bit loose on how information is presented – writing eigenvalue output as a, and note we are assuming that PE works exactly)
Hence U_{PE}^{-1} (*inverse gates taken in reverse order*) would give desired (2)!

Consider $U : |x\rangle \to |x - 1 \ mod \ \mathcal{Q}\rangle$, and check that $U|\eta_a\rangle = \omega^a|\eta_a\rangle$ as wanted.
Now note $x \to x - k \ mod \ \mathcal{Q}$ for $k \in \mathbb{Z}_\mathcal{Q}$ is classically computable in $poly(\log \mathcal{Q})$-time, thus we also have $U^k : |x\rangle \to |x - k \ mod \ \mathcal{Q}\rangle$, and PE algotirhm with

$m = O(\log(Q))$ lines.
Then implementing (1) then (2) gives $poly(\log \mathcal{Q})$ sized circuit for $QFT_{\mathcal{Q}}$.

But PE is not exact. However, using more qubit lines ($O(\log 1/\varepsilon)$ lines), we can achieve (by theorem PE(b))

$$|0\rangle|\eta_a\rangle \xrightarrow{U_{PE}} (\sqrt{1-\varepsilon}|a\rangle + \sqrt{\varepsilon}|a^{\perp}\rangle)|\eta_a\rangle$$

(where $a^{\perp}$ is a state orthogonal to $|a\rangle$) for any (small) deserved ε. Then

$$||\ |a\rangle - \sqrt{1-\varepsilon}|a\rangle + \sqrt{\varepsilon}|a^{\perp}\rangle\ || = O(\sqrt{\varepsilon})$$

So

$$||\ U_{PE}^{-1}|a\rangle|\eta_a\rangle - |0\rangle|\eta_a\rangle\ || = O(\sqrt{\varepsilon})$$

(as unitaries preserve lengths). So we can approximate $QFT_{\mathcal{Q}}$ to any desired precision (omit details).

3 Amplitude Amplification

Note that this is a very good name – a fifth order literation (both starting with *Ampli*).
Apothesis of technique in Grover's algorithm.

Some background:
We'll make much use of *reflection operators*.

—Lecture 9—
A reminder that we don't have lecture next thursday.

Reflection operators:
• State $|\alpha\rangle$ in $\mathcal{H}_d \to$ 1-dimensional subspace L_α and $(d-1)$-dimensional orthogonal complement $L_\alpha^\perp$

$$I_{|\alpha\rangle} \overset{def}{=} I - 2|\alpha \times \alpha|$$

has $I_{|\alpha\rangle}|\alpha\rangle = -|\alpha\rangle$, $I_{|\alpha\rangle}|\beta\rangle = |\beta\rangle$ for any $|\beta\rangle \perp |\alpha\rangle$.
So $I_{|\alpha\rangle}$ is reflection in $(d-1)$-dimensional subspace $L_\alpha^\perp$.

Note that for any unitary U, $UI_{|\alpha\rangle}U^\dagger = I_{U|\alpha\rangle}$, since $U|\alpha \times \alpha|Y^\dagger = |\xi \times \xi|$ ofr $\xi = U|\alpha\rangle$ (basically a change of basis).

• Take k-dimensional subspace $A \subseteq \mathcal{H}_d$, and any orthonormal basis $|a_1\rangle, ..., |a_k\rangle$.
Then $P_A = \sum_{i=1}^k |a_i \times a_i|$ is projection operator into A.
Define $I_A = I - 2P_A$. Then we have $I_A|\xi\rangle = |\xi\rangle$ if $|\xi\rangle \in A^\perp$, and $I_A|\xi\rangle = -|\xi\rangle$ if $|\xi\rangle \in A$.
So I_A is reflection in $(d-k)$ dimensional mirror $A^\perp$.

Recap of Grover's algorithm (part II notes page 68-73):
• search for unique *good* item in unstructured database of $N = 2^n$ items formalised as: (write B_n to be the set of all n-bit strings, $N = 2^n$): Given oracle for $f : B_n \to B$, promised that there is unique $x_0 \in B_n$ with $f(x_0) = 1$, and we wish to find x_0.
This is closely related to class NP and Boolean satisfiability problem (see part II notes p 67-68).
Using one query to $(n+1)$-qubit $\mathcal{U}_f$, we can implement reflection operator $I_{|x_0\rangle} : |x\rangle \to |x\rangle$ if $x \neq x_0$, and to $-|x\rangle$ if $x = x_0$.
(viz. apply $\mathcal{U}_f$ to $|x\rangle(\frac{|0\rangle-|1\rangle}{\sqrt{2}})$ and discard the last qubit.)
Then consider *Grover iteration operator* on n qubits:

$$Q \overset{def}{=} -H_n I_{|0...0\rangle} H_n I_{|x_0\rangle} = -I_{|\psi_0\rangle} I_{|x_0\rangle}$$

here $H_n = H \otimes H \otimes ... \otimes H = H_n^\dagger$, and $|\psi_0\rangle = H^n|0...0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in B_n} |x\rangle$.
So one application of Q uses 1 query to $\mathcal{U}_f$.

Theorem. (Grover, 1996)
In 2-dimensional span of $|\psi_0\rangle$ and (unknown) $|x_0\rangle$, the action of Q is rotation by angle 2α where $\sin\alpha = \frac{1}{\sqrt{N}}$.

Hence (Grover's algorithm) to find x_0 given U_f:
1. Make $|\psi_0\rangle$;
2. Apply Q m times where $m = \frac{\arccos(\frac{1}{\sqrt{N}})}{2\arctan(frac1\sqrt{N})}$ to rotate $|\psi_0\rangle$ very close to $|x_0\rangle$.
3. Measure to see x_0 with high probability $\sim 1 - \frac{1}{N}$.

For large N, $\arccos(\frac{1}{\sqrt{N}}) \approx \pi/2$, $\arcsin(\frac{1}{\sqrt{N}}) \approx \frac{1}{\sqrt{N}}$ so $m = \frac{\pi}{4}\sqrt{N}$ iterations/queries to U_f suffice.
Classically we need $O(N)$ queries to see x_0 with any *constant* probability (independent of N), so get *square-root* speed up quantumly.

Amplitude Amplification:
Let G be any subspace (*good subspace*) of state space $\mathcal{H}$, and $G^\perp$ is orthogonal complement (*bad subspace*) $\mathcal{J} = G \oplus G^\perp$.
Given any $|\psi\rangle \in \mathcal{H}$, we have unique decompoisiton with *real positive* coefficients

$$|\psi\rangle = \sin\theta|g\rangle + \cos\theta|b\rangle$$

where $|g\rangle \in G$, $|b\rangle \in G^\perp$ normalised. Introduce reflections: flip $|\psi\rangle$ and good vectors: $I_{|\psi\rangle} = I - 2|\psi \times \psi|$, $I_G = I - 2P_G$ (projection into G), so $\sin\theta = ||P_G|\psi\rangle||$ is the length of good projection.
Introduce $Q \overset{def}{=} -I_{|\psi\rangle}I_G$.

Theorem. (Amplitude Amplification)
In the 2-dimensional subspace spanned by $|g\rangle$ and $|\psi\rangle$ (or equivalently by orthonormal vectors $|g\rangle$ and $|b\rangle$), Q is rotation by 2θ where $\sin\theta$ is the length of good projection of $|\psi\rangle$.

Proof. We have $I_G|g\rangle = -|g\rangle$, $I_G|b\rangle = |b\rangle$. So $Q|g\rangle = +I_{|\psi\rangle}|g\rangle$, $Q|b\rangle = -I_{|\psi\rangle}|b\rangle$. Now

$$\begin{aligned} I_{|\psi\rangle} &= I - 2(\sin\theta|g\rangle + \cos\theta|b\rangle)(\sin\theta\langle g| + \cos\theta\langle b|) \\ &= I - 2[\sin^2\theta|g \times g| + \sin\theta\cos\theta|g \times b| + \sin\theta\cos\theta|b \times g| + \cos^2\theta|b \times b|]|b\rangle \end{aligned}$$

And direct calculation (using $\langle g|b\rangle = 0$, $\langle g|g\rangle = \langle b|b\rangle = 1$) gives

$$\begin{aligned} Q|b\rangle &= I_{|\psi\rangle}|b\rangle \\ &= 2\sin\theta\cos\theta|g\rangle - (1 - 2\cos^2\theta)|b\rangle \\ &= \cos 2\theta|b\rangle + \sin 2\theta|g\rangle \end{aligned}$$

and $Q|g\rangle = +I_{|\psi\rangle}|g\rangle = -\sin 2\theta|b\rangle + \cos 2\theta|g\rangle$.
So in $\{|b\rangle, |g\rangle\}$ basis, matrix of Q is exactly the matrix of rotation by 2θ. □

—Lecture 10—

Let's continue on Amplitude Amplification.

Last time we showed that $Q = -I_{|\psi\rangle}I_g$ is the rotation through 2θ in the plane of $|\psi\rangle$ and $|g\rangle$, i.e. in $|b\rangle$ and $|g\rangle$ (orthonormal).

So $Q^n|\psi\rangle = \sin(2n+1)\theta|g\rangle + \cos(2n+1)\theta|b\rangle$, and if we measure $Q^n|\psi\rangle$ for good vs bad, we get $prob(good) = \sin^2(2n+1)\theta$.
We want to maximize this: it is maximised when $(2n+1)\theta = \pi/2$, i.e. $n = \frac{\pi^2}{4\theta} - \frac{1}{2}$.

Example. *If* we had $\theta = 4/6$, then $n = \frac{\pi}{4\theta} - \frac{1}{2} = 1$ is an exact integer. So Q^1 rotates $|\psi\rangle$ *exactly* onto $|g\rangle$, so we see good result with certainty!

Generally, for given θ, n is not an integer. So we use n to be the nearest integer to $(\frac{\pi}{4\theta} - \frac{1}{2}) \approx \frac{\pi}{4\theta}$ (for small θ), which equals $O(\frac{1}{\theta} = O(\frac{1}{\sin\theta}) = O(\frac{1}{||good\ proj\ of\ |\psi||}))$, and $Q^n|\psi\rangle$ will be within angle $\pm\theta$ of $|g\rangle$, so probability of good result is at least $\cos^2\theta \approx 1 - O(\theta^2)$.

All this can be implemented oif $I_{|\psi\rangle}$ and I_G can be implemented. See example sheet – for I_G, suffices for G to be spanned by computational basis states $|x\rangle$'s, and indicator function $f(x) = 1$ for x good and 0 for x bad efficiently computable. For $I_{|\psi\rangle}$, usally have $|\psi\rangle = H_n|00...0\rangle$ (H is the Hadamard gate). Then $I_{|\psi\rangle}$ can be implemented in linear $O(n)$ time.

Notes:
(1) In AA process, relative amplitudes of good labels in $|g\rangle$ stay *same* as they were in $|\psi\rangle = \sin\theta|g\rangle + \cos\theta|b\rangle$.
(2) Final state is generally not exactly $|g\rangle$, but *if* $\sin\theta$ is *known*, then we can modify AA process to make it exact, i.e. giving $|g\rangle$ state exactly (see example sheet).

Applications of AA:
(1) *Grover Search* with one or *more* (k) good items in N:

$$\begin{aligned}|\psi\rangle = |\psi_0\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x\in B_n} |x\rangle \\ &= \sqrt{\frac{k}{N}}\left(\frac{1}{\sqrt{k}} \sum_{good\ |g\rangle} |x\rangle\right) + \sqrt{\frac{N-k}{N}}\left(\frac{1}{\sqrt{N-k}} \sum_{x\ bad} |x\rangle\right)\end{aligned}$$

G spanned by good x's, $\sin\theta = \frac{k}{n}$ so Q is rotation through 2θ, $\theta = \arcsin\sqrt{k/N} \approx \sqrt{k/N}$, where $k \ll N$; and we only need $O(\sqrt{N/k})$ queries.

Note: for 2-bit case, $N = 4$ with $k = 1$ good item; we have $\theta = \arcsin(1/2) = \pi/6$, so one application of Q rotates $|\psi_0\rangle$ exactly onto $|x_{good}\rangle$, i.e. a *single* query suffices to find a unique good item in four, *with certainty*!

(2) Square-root sppedup of general quantum algorithms:
Let A be a quantum algorithm/circuit (sequence of unitary gates). on input, say $|0...0\rangle$. So final state is $A|0...0\rangle$.
Good labels = desired computational outcomes

$$A|0.000\rangle = \alpha|a\rangle + \beta|b\rangle, \alpha = \sin\theta$$

where $|a\rangle$ is normalised, genrally unequal superposition $\sum_{good\ x} c_x|x\rangle$.
So $Prob$(success in 1 run) $= |\alpha|^2$, so $O(\frac{1}{|\alpha|^2})$ repetitions of A needed to succeed with any *constant* high probability $1 - \varepsilon$.

Instead use AA: assumed we can check if answer is good or bad (e.g. factoring). So we can then implement $I_G : |x\rangle \to -|x\rangle$ if x is good, and $\to |x\rangle$ if x is bad.
Consider $|\psi\rangle = A|0...0\rangle$ and $Q = -I_{A|0...0\rangle} I_G = -(AI_{|0...0\rangle}A^\dagger)I_G$. All parts are implementable (A is the algorithm, $A^\dagger$ is inverse gate in reverse order, and $I_{|0...0\rangle}$ see example sheet).
By AA theorem, Q is rotation through 2θ, where $\sin\theta = |\alpha|$. So after $n \approx \frac{\pi}{4\theta} = O(\frac{1}{\theta}) = O(\frac{1}{\sin\theta}) = O(\frac{1}{|\alpha|})$ (for small $|\alpha|$).
Repetitions $A|0...0\rangle$ will be rotate very near to $|g\rangle$, and final measurement will succeed with high probability.
Each application of $\mathcal{Q}$ needs one A and one $A^\dagger$; $A^\dagger$ is the *inverse gate in reverse order*, i.e. the time complexity is the same, i.e. $O(\frac{1}{|\alpha|})$ repetition of Q gives square root time speed up over direct method.

Also, if success probability of A (i.e. $|\alpha|^2$) is known, then *improved* modification of the AA process that is *exact* can be applied; we convert probabilistic algorithm A into *deterministic* one, giving a good outcome with certainty.

4 Quantum Counting

Given $f : B_n \to B$ a boolean function with an unknown number k good x's, we want ot *estimate* k (rather than just find some good x).

Recall that Grover operator Q_G for f is rotaion through 2θ in 2-dimensional space of $|\psi_0 = \frac{1}{\sqrt{2^n}} \sum_{x \in B_n} |x\rangle$ and its good projection $|g\rangle = \frac{1}{\sqrt{k}} \sum_{good\ x} |x\rangle$, with $\sin\theta = \sqrt{k/N} \approx \theta$ for $k \ll N$.

5 Example Class 1

5.1 Question 1

Basic representation theory exercises.
For details see Part II Representation Theory.

5.2 Question 2

Let $G = (\mathbb{Z}_2^n, \oplus)$ where $\oplus$ is componentwise-addition. Subgroup K generated by $a_1...a_k$, $K = \{b_1a_1 \oplus ... \oplus b_ka_k : b_1, ..., b_k \in \mathbb{Z}_2\}$. Note that K has size 2^k if a_i's are LI, and so does any coset of K.
Then $f(x) = f(x \oplus a_i)$ for all a_i's $\equiv f$ constant on cosets of K. f is 2^k-to-1: a_k's all linearly independent and f different on different cosets.

Shift invariant states: For $\mathbb{Z}_2$ irreps are $\chi_a(x) = (-1)^{ax)}$, $a, x \in \mathbb{Z}_2$ (-1 is the 2nd root of unity). So irreps on $(\mathbb{Z}_2)^n$ are $\chi_a(x) = (-1)^{a_1x_1}...(-1)^{a_nx_n}$ where $a = a_1...a_n$, $x = x_1...x_n$ are in $\mathbb{Z}_2^n$.
We also introduce a dot product $a \cdot x = a_1x_1 \oplus ... \oplus a_nx_n \in \mathbb{Z}_2$ where $\oplus$ here is $+$ modulo 2.
So shift invariant states

$$|\chi\rangle = \frac{1}{\sqrt{|G|}} \sum_g \overline{\chi(g)} |g\rangle$$

So

$$|\chi_a\rangle = \frac{1}{\sqrt{2^n}} \sum_{b \in \mathbb{Z}_2^n} (-1)^{a \cdot b} |b\rangle$$

in n qubits. So

$$(QFT)_{alr(?)} = \frac{1}{\sqrt{2^n}} (-1)^{a \cdot b} = \frac{1}{\sqrt{2^n}} (-1)^{a_1b_1}...(-1)^{a_nb_n}$$

so $QFT = H \otimes H \otimes ... \otimes H$ where H is the Hadamard gate.
For second part, it's just calculation:
Probability that first string is LI is $1 - 2^{-m}$ (just exclude 0...0);
Probability that first 2 strings are LI given the first is is $1 - 2/2^m$ (i.e. as 1st string x_1 spans 2 strings, namely 0...0 and x_1).
...Probability that first j strings are LI given the first $j - 1$ are is $1 - 2^{j-1}/2^m$.
So by Bayes rule, probability that all of them are LI is $(1 - 1/2^m)...(1 - 2^{m-2}/2^m)(1 - 2^{m-1}/2^m)$. Use the hint given we get that is at least $\frac{1}{2}(1 - 1/2) = 1/4$.

Stanrdard HSP algorithm:
1. query to f, get random coset state $|y \oplus K\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in K} |y \oplus x\rangle, y \in \mathbb{Z}_2^n$.
Apply $QFT = H^{\otimes n}$ and measure; our theory assures that output is then uniformly random $c \in (\mathbb{Z}_2)^n$ s.t. irrep χ_c of G restricted to K is *trivial irrep* of K, i.e .$\chi_c(a) = 1$ for all $a \in K$.

So $(-1)^{a \cdot c} = 1$ for all $a \in K$, i.e. $c \cdot a = 0 \pmod 2$ for all $a = K$, i.e. $c_1 a_1 \oplus ... \oplus c_n a_n = 0$ for $a = a_1 ... a_n$.
We know K viewed as subspace of $(\mathbb{Z}_2)^n$ (n dimensional vector space over field $\mathbb{Z}_2$) has dimension k. So $(n-k)$ LI c_i's with $c_1 \cdot a = 0$ suffice to determine K as null spaceof linear system of $(n-k)$ equations.
So run HSP algorithm $(n-k)$ times: by (b) we'll get $(n-k)$ LI $c's$ with probability at least 1/4, and we can solve for elements of K.