

# Category Theory

October 19, 2018

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## 0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

# 1 Definitions and examples

## Definition. (1.1)

A category  $\mathcal{C}$  consists of:

- (a) a collection  $\text{ob } \mathcal{C}$  of *objects*  $A, B, C$ ;
- (b) a collection  $\text{mor } \mathcal{C}$  of *morphisms*  $f, g, h$ ;
- (c) two operations domain, codomain assigning to each  $f \in \text{mor } \mathcal{C}$  a pair of objects, its *domain* and *codomain*; we write  $A \xrightarrow{f} B$  to mean  $f$  is a morphism and  $\text{dom } f = A, \text{cod } f = B$ ;
- (d) an operation assigning to each  $A \in \text{ob } \mathcal{C}$  a morphism  $A \xrightarrow{1_A} A$ ;
- (e) a partial binary operation  $(f, g) \rightarrow fg$  on morphisms, such that  $fg$  is defined iff  $\text{dom } f = \text{cod } g$ , and  $\text{dom}(fg) = \text{dom } g$ ,  $\text{cod}(fg) = \text{cod}(f)$  if  $fg$  is defined, satisfying:
- (f)  $f1_A = f = 1_B f$  for any  $A \xrightarrow{f} B$ ;
- (g)  $(fg)h = f(gh)$  whenever  $fg$  and  $gh$  are defined.

## Remark. (1.2)

- (a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call  $\mathcal{C}$  *small* if  $\text{ob } \mathcal{C}$  and  $\text{mor } \mathcal{C}$  are sets.
- (b) Some texts say  $fg$  means  $f$  followed by  $g$ , i.e.  $fg$  is defined iff  $\text{cod } f = \text{dom } g$ .
- (c) Note that a morphism  $f$  is an identity iff  $fg = g$  and  $hf = h$  whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

## Example. (1.3)

- (a) The category **Set** has all sets as objects, and all functions between sets as morphisms.

Strictly speaking, morphisms  $A \rightarrow B$  are pairs  $(f, B)$  where  $f$  is a set-theoretic function. (See part II logic and sets)

- (b) The category **Gp** has all groups as objects, group homomorphisms as morphisms.

Similarly, **Ring** is the category of rings, **Mod<sub>R</sub>** is the category of  $R$ -modules.

- (c) The category **Top** has all topological spaces as objects, and continuous functions as morphisms.

Similarly, **Unif** has all uniform spaces and uniformly continuous functions as morphisms, **Mf** has all manifolds and smooth maps correspondingly.

- (d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given  $\mathcal{C}$ , we call an equivalence relation  $\simeq$  on  $\text{mor } \mathcal{C}$  a *congruence* if  $f \simeq g \implies \text{dom } f = \text{dom } g$  and  $\text{cod } f = \text{cod } g$ , and  $f \simeq g \implies fh \simeq gh$  and  $kf \simeq kg$  whenever the composites are defined. Then we have a category  $\mathcal{C}/\simeq$  with the same objects as  $\mathcal{C}$ , but congruence classes as morphisms instead.

- (e) Given  $\mathcal{C}$ , the *opposite category*  $\mathcal{C}^{op}$  has the same objects and morphisms as  $\mathcal{C}$ , but  $\text{dom}$  and  $\text{cod}$  are interchanged, and  $fg$  in  $\mathcal{C}^{op}$  is  $gf$  in  $\mathcal{C}$ .

This leads to the *duality principle*: if  $P$  is a true statement about categories, so is the statement  $P^*$  obtained from  $P$  by reversing all arrows.

- (f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat ( $\boxtimes$ ) with one object in which every morphism is an isomorphism (i.e. for all  $f, \exists g$  s.t.  $fg$  and  $gf$  are identities).

(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space  $X$ , the *fundamental groupoid*  $\pi(x)$  has all points of  $X$  as objects, and morphisms  $x \rightarrow y$  are homotopy classes  $rel\{0, 1\}$  of paths  $u : [0, 1] \rightarrow X$  with  $u(0) = x$ ,  $u(1) = y$  (if you know how to prove that the fundamental group is a group, you can prove that  $\pi(x)$  is a groupoid).

(h) A *discrete cat* is one whose only morphism are identities.

A *preorder* is a cat  $\mathcal{C}$  in which, for any pair  $(A, B)$ ,  $\exists$  at most 1 morphism  $A \rightarrow B$ .

A small preorder is a set equipped with a binary relation which is reflexive and transitive.

In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category **Rel** has the same objects as *set*, but morphisms  $A \rightarrow B$  are arbitrary relations  $R \subseteq A \times B$ . Given  $R$  and  $S \subseteq B \times C$ , we define  $S \cdot R = \{(a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R, (b, c) \in S)\}$ .

The identity  $1_A : A \rightarrow A$  is  $\{(a, a) \mid a \in A\}$ .

Similarly, the category **Part** are for sets and partial functions (i.e. relations s.t.  $(a, b) \in R$  and  $(a, b') \in R \implies b = b'$ ).

(j) Let  $K$  be a field. The category **Mat<sub>K</sub>** has natural numbers as objects, and morphism  $n \rightarrow p$  are  $(p \times n)$  matrices with entries from  $K$ . Composition is matrix multiplication.

(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

#### Definition. (1.4)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

(a) a mapping  $A \rightarrow FA$  from  $\text{ob } \mathcal{C}$  to  $\text{ob } \mathcal{D}$ ;

(b) a mapping  $f \rightarrow Ff$  from  $\text{mor } \mathcal{C}$  to  $\text{mor } \mathcal{D}$ ,

such that  $\text{dom}(Ff) = F(\text{dom } f)$ ,  $\text{cod}(Ff) = F(\text{cod } f)$ ,  $1_{FA} = F(1_A)$ , and  $(Ff)(Fg) = F(fg)$  whenever  $fg$  is defined.

#### Example. (1.5)

(a) We have *forgetful functors*  $U : \mathbf{Gp} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{Set}, \mathbf{Top} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{AbGp}$  (forget  $\times$ ),  $\mathbf{Ring} \rightarrow \mathbf{Mon}$  (Category of all monoids) (forget  $+$ ).

(b) Given a set  $A$ , the free group  $FA$  has the property:

Given any group  $G$  and any function  $A \xrightarrow{f} UG$  (?), there's a unique homomorphism  $FA \xrightarrow{\tilde{f}} G$  extending  $f$ . Here  $F$  is a functor  $\mathbf{Set} \rightarrow \mathbf{Gp}$ : given  $A \xrightarrow{f} B$ , we define  $Ff$  to be the unique homomorphism extending  $A \xrightarrow{f} B \leftrightarrow UFB$ .

**Functoriality** follows from uniqueness given  $B \xrightarrow{f} C$ .  $F(gf)$  and  $(Fg)(Ff)$  are both homomorphisms extending  $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow UFC$ .

(c) Given a set  $A$ , we write  $PA$  for the set of all subsets of  $A$ .

We can make  $P$  into a functor  $\mathbf{Set} \rightarrow \mathbf{Set}$ , given  $A \xrightarrow{f} B$ , we defined  $Pf(A') = \{f(a) \mid a \in A'\}$  for  $A' \subseteq A$ .

But we also have a functor  $P^* : \mathbf{Set} \rightarrow \mathbf{Set}^{op}$  defined on objects by  $P$ , but  $P^*f(B') = \{a \in A \mid f(a) \in B'\}$  for  $B' \subseteq B$ .

By a *contravariant* functor  $\mathcal{C} \rightarrow \mathcal{D}$ , we mean a functor  $\mathcal{C} \rightarrow \mathcal{D}^{op}$  (or  $\mathcal{C}^{op} \rightarrow \mathcal{D}$ ).

A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

- (d) Let  $K$  be a field. We have a functor  $*$  :  $\mathbf{Mod}_K \rightarrow \mathbf{Mod}_K^{op}$  defined by  $V^* = \{ \text{linear maps } V \rightarrow K \}$ , and if  $V \xrightarrow{f} W$ ,  $f^*(\theta : W \rightarrow K) = \theta f$ .
- (e) We have a functor  $op$  :  $\mathbf{Cat} \rightarrow \mathbf{Cat}$ , which is the identity on morphisms (note that this is a covariant).
- (f) A functor between monoids is a monoid homomorphism.
- (g) A functor between posets is an order-preserving map.
- (h) Let  $G$  be a group. A functor  $F \circ G \rightarrow \mathbf{Set}$  consists of a set  $A = F*$  together with an action of  $G$  on  $A$ , i.e. a *permutation representation* of  $G$ . Similarly, a functor  $G \rightarrow \mathbf{Mod}_K$  is a  $K$ -linear representation of  $G$ .
- (i) The construction of the fundamental group  $\pi(X, X)$  of a space  $X$  with basepoint  $X$  is a functor  $\mathbf{Top}_* \rightarrow \mathbf{Gp}$  where  $\mathbf{Top}_*$  is the category of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor  $\mathbf{Top} \rightarrow \mathbf{Gpd}$ , where  $\mathbf{Gpd}$  is the category of groupoids and functors between them.

**Definition.** (1.6)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$  (why two arrows?) two functors. A *natural transformation*  $\alpha : F \rightarrow G$  consists of an assignment  $A \rightarrow \alpha_A$  from  $\text{ob } \mathcal{C}$  to  $\text{mor } \mathcal{D}$  (think about this), such that  $\text{dom}_{\alpha_A} = FA$  and  $\text{cod}_{\alpha_A} = GA$  for all  $A$ , and for all  $A \xrightarrow{f} B$  in  $\mathcal{C}$ , the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e.  $\alpha_B(Ff) = (Gf)_{\alpha_A}$ ).

(1.3) (l) Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , we write  $[\mathcal{C}, \mathcal{D}]$  for the category whose objects are functors  $\mathcal{C} \rightarrow \mathcal{D}$  and whose morphisms are natural transformations.

**Example.** (1.7)

- (a) Let  $K$  be a field,  $V$  a vector space over  $K$ . There is a linear map  $\alpha_V : V \rightarrow V^{**}$  given by  $\alpha_V(v)\theta = \theta(v)$  for  $\theta \in V^*$ . This is the  $V$ -component of a natural transformation  $1_{\mathbf{Mod}_K} \rightarrow ** : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K$ .
- (b) For any set  $A$ , we have a mapping  $\sigma_A : A \rightarrow PA$  sending  $a$  to  $\{a\}$ . If  $f : A \rightarrow B$ , then  $Pf\{a\} = \{f(a)\}$ . So  $\sigma$  is a natural transformation  $1_{\mathbf{Set}} \rightarrow P$ .
- (c) Let  $F : \mathbf{Set} \rightarrow \mathbf{Gp}$  be the free group functor (1.5(b)), and  $U : \mathbf{Gp} \rightarrow \mathbf{Set}$  the forgetful functor. The inclusions  $A \rightarrow UFA$  form a natural transformation  $1_{\mathbf{Set}} \rightarrow UF$ .
- (d) Let  $G, H$  be groups and  $f, g : G \rightrightarrows H$  be two homomorphisms. A natural transformation  $\alpha : f \rightarrow g$  corresponds to an element  $h = \alpha_*$  of  $H$ , s.t.  $hf(x) \rightarrow g(x)h$  for all  $x \in G$  or equivalently  $f(x) = h^{-1}g(x)h$ , i.e.  $f$  and  $g$  are conjugate group homomorphisms.
- (e) Let  $A$  and  $B$  be two  $G$ -sets, regarded as functors:  $G \rightrightarrows \mathbf{Set}$ . A natural transformation  $A \rightarrow B$  is a function  $f$  satisfying  $f(g \cdot a) = g \cdot f(a)$  for all  $a \in A$ , i.e. a  $G$ -equivariant map.

**Lemma.** (1.8)

Let  $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$  be two functors, and  $\alpha : F \rightarrow G$  a natural transformation. Then  $\alpha$  is an isomorphism in  $[\mathcal{C}, \mathcal{D}]$  iff each  $\alpha_A$  is an isomorphism in  $\mathcal{D}$ .

*Proof.* Forward is trivial (ok, I'll check this later). For backward, suppose each  $\alpha_A$  has an inverse  $\beta_A$ . Given  $f : A \rightarrow B$  in  $\mathcal{C}$ , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

□

commutes. But as  $\alpha$  is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

So  $\beta$  is a natural transformation as well.

**Definition.** (1.9)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. By an *equivalence* between  $\mathcal{C}$  and  $\mathcal{D}$ , we mean a pair of functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  together with natural isomorphisms  $\alpha : 1_{\mathcal{C}} \rightarrow GF$  and  $\beta : FG \rightarrow 1_{\mathcal{D}}$ .

We write  $\mathcal{C} \cong \mathcal{D}$  if  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent.

We say a property  $P$  of categories is a *categorical property* if whenever  $\mathcal{C}$  has  $P$  and  $\mathcal{C} \cong \mathcal{D}$ , then  $\mathcal{D}$  has  $P$ .

For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

**Example.** (1.10)

(a) The category **Part** is equivalent to the category **Set**<sub>\*</sub> of pointed sets (and basepoint preserving functions (as morphisms)):

- We define  $F : \mathbf{Set}_* \rightarrow \mathbf{Part}$  by  $F(A, a) = A \setminus \{a\}$ , and if  $f : (A, a) \rightarrow (B, b)$ , then  $Ff(x) = f(x)$  if  $f(x) \neq b$ , and undefined otherwise;
- and  $G : \mathbf{Part} \rightarrow \mathbf{Set}_*$  by  $G(A) = A^+ = (A \cup \{A\}, A)$ , and if  $f : A \rightarrow B$  is a partial function, we define  $Gf : A^+ \rightarrow B^+$  by  $Gf(x) = f(x)$  if  $x \in A$  and  $f(x)$  defined, and equals  $B$  otherwise.

The composite  $FG$  is the identity on **Part**, but  $GF$  is not the identity. However, there is an isomorphism  $(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$  sending  $a$  to  $A \setminus \{a\}$  and everything else to itself and this is natural.

Note that there can be no isomorphism from **Set**<sub>\*</sub> to **Part**, since **Part** has a 1-element isomorphism class  $\{\phi\}$  but **Set**<sub>\*</sub> doesn't.

(So we see that equivalent categories can be non-isomorphic. According to a [post](#) on SO, this usually happens when there are multiple copies of the *same* thing in one but not the other. However, we can't generally *discard obsolete copies* in one as that generally requires AC and is not a very useful thing to do anyway – In short, *identifying isomorphic objects is often an extremely bad idea.*)

(b) The category **fdMod**<sub>K</sub> of finite-dimensional vector spaces over  $K$  is equivalent to **fdMod**<sub>K</sub><sup>op</sup>, the functors in both directions are  $*$  (the dual operator) and both isomorphisms are the natural transformations of 1.7(a) (double dual).

(c) **fdMod**<sub>K</sub> is also equivalent to **Mat**<sub>K</sub> (1.3(j)):

We define  $F : \mathbf{Mat}_K \rightarrow \mathbf{fdMod}_K$  by  $F(n) = K^n$ , and  $F(A)$  is the linear map represented by  $A$  w.r.t. the standard bases of  $K^n$  and  $K^p$ .

To define  $G : \mathbf{fdMod}_K \rightarrow \mathbf{Mat}_K$ , choose a basis for each finite dimensional vector

space, and define  $G(V) = \dim V$ ,  $G(V \xrightarrow{f} W)$  to be the matrix representing  $f$  w.r.t. chosen bases.  $GF$  is the identity, provided we choose the standard bases for the spaces  $K^n$ ;  $FG \neq 1$ , but the chosen bases give isomorphisms  $FG(V) = K^{\dim V} \rightarrow V$  for each  $V$ , which form a natural isomorphism.

—Lecture 4—

**Definition.** (1.11)

Let  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  be a functor.

(a) We say  $F$  is *faithful* if, given  $f, f' \in \text{mor } \mathcal{C}$  with  $\text{dom } f = \text{dom } f'$ ,  $\text{cod } f = \text{cod } f'$ , and  $Ff = Ff'$ , then  $f = f'$  (injectivity on morphisms. The name comes more from representation theory);

(b) We say  $F$  is *full* if, given  $FA \xrightarrow{g} FB$  in  $\mathcal{D}$ , there exists  $A \xrightarrow{f} B$  in  $\mathcal{C}$  with  $Ff = g$ . (this is something like surjectivity on morphisms, but see below);

(c) We say  $F$  is *essentially surjective* if, for every  $B \in \text{ob } \mathcal{D}$ , there exists  $A \in \text{ob } \mathcal{C}$  and isomorphism  $FA \rightarrow B$  in  $\mathcal{D}$ .

We say a subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is *full* if the inclusion  $\mathcal{C}' \rightarrow \mathcal{C}$  is a full functor (basically, if the objects are kept, any morphism between them must be kept). For example, **Gp** is a full subcategory of **Mon** (the category of all monoids), but **Mon** is not a full subcategory of the category **SGp** of semigroups (consider e.g. the homomorphism that sends everything in  $(\mathbb{Z}, \cdot)$  to  $(0, \cdot)$  (which is also a semigroup); but this doesn't preserve 1 so is not a morphism in **Mon**).

**Lemma.** (1.12)

Assuming the axiom of choice, a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is part of an equivalence  $\mathcal{C} \simeq \mathcal{D}$  if it's full, faithful, and essentially surjective.

*Proof.*  $\Rightarrow$ : Suppose given  $G, \alpha, \beta$  as in (1.9). Then for each  $B \in \text{ob } \mathcal{D}$ ,  $\beta_B$  is an isomorphism  $FGB \rightarrow B$ , so  $F$  is essentially surjective.

Given  $A \xrightarrow{f} B$  in  $\mathcal{C}$ , we can recover  $f$  from  $Ff$  as composite  $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B$ . Hence if  $A \xrightarrow{f'} B$  satisfies  $Ff = Ff'$ , then  $f = f'$ . So  $F$  is faithful;

Lastly, for fullness, given  $FA \xrightarrow{g} FB$ , define  $f$  to be the composite  $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$ . Then  $GFf = \alpha_B f \alpha_A^{-1}$ , which by construction is just  $Gg$ . But  $G$  is faithful for the same reason as  $f$ , so  $Ff = g$ .

$\Leftarrow$ : (need to find suitable  $G, \alpha, \beta$  for  $F$ .) For each  $B \in \text{ob } \mathcal{D}$ , choose  $GB \in \text{ob } \mathcal{C}$  and an isomorphism  $\beta_B : FGB \rightarrow B$  in  $\mathcal{D}$ . Given  $B \xrightarrow{g} B'$ , define  $Gg : GB \rightarrow GB'$  to be the unique morphism whose image under  $F$  is  $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$ .

Uniqueness implies functoriality: given  $B' \xrightarrow{g'} B''$ ,  $(Gg')(Gg)$  and  $G(g'g)$  have the same image under  $F$ , so they are equal.

By construction,  $\beta$  is a natural transformation  $FG \rightarrow 1_{\mathcal{D}}$ .

Given  $A \in \text{ob } \mathcal{C}$ , define  $\alpha_A : A \rightarrow GFA$  to be the unique morphism whose image under  $F$  is  $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$ .  $\alpha_A$  is an isomorphism, since  $\beta_{FA}$  also has a unique pre-image under  $F$ . And  $\alpha$  is a natural transformation, since any naturality



square for  $\alpha$  (the commutative square when we defined natural transformation) is mapped by  $F$  to a commutative square, and  $F$  is faithful.  $\square$

**Definition.** (1.13)

By a *skeleton* of a category, we mean a full subcategory  $\mathcal{C}_0$  containing one object from each isomorphism class. We say  $\mathcal{C}$  is *skeletal* if it's a skeleton of itself.

For example,  $\mathbf{Mat}_{\mathbf{K}}$  is a skeletal, and the image of  $F : \mathbf{Mat}_{\mathbf{K}} \rightarrow \mathbf{fdMod}_{\mathbf{K}}$  of 1.10(c) is a skeleton of  $\mathbf{fdMod}_{\mathbf{K}}$ .

(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

**Definition.** (1.14)

Let  $A \xrightarrow{f} B$  be a morphism in  $\mathcal{C}$ .

(a) We say  $f$  is a *monomorphism* (or  $f$  is *monic*) if, given any pair  $C \xrightleftharpoons[h]{g} A$ ,  $fg = fh$  implies  $g = h$ .

(b) We say  $f$  is an *epimorphism* (or *epic*) if it's a monomorphism in  $\mathcal{C}^{op}$ , i.e. if  $gf = hf$  implies  $g = h$ .

We denote monomorphisms by  $A \xrightarrow{f} B$ , and epimorphisms by  $A \xrightarrow{f} B$ .

Any isomorphism is monic and epic: more generally, if  $f$  has a left inverse (i.e.  $\exists g$  s.t.  $gf$  is an identity), then it's monic. We call such monomorphisms *split*.

We say  $\mathcal{C}$  is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

**Example.** (1.15)

(a) As usual we consider **Set** first. In **Set**, monomorphisms correspond to injections ( $\Leftarrow$  is easy (ok); for  $\Rightarrow$ , take  $C = 1 = \{*\}$ ), and epimorphisms correspond to surjections ( $\Leftarrow$  is easy; for  $\Rightarrow$ , use morphisms  $B \rightrightarrows 2 = \{0, 1\}$ ). So **Set** is balanced.

(b) In **Gp**, monomorphisms again correspond to injections (for  $\Rightarrow$  use homomorphisms  $\mathbb{Z} \rightarrow A$ ); epimorphisms again correspond to surjections ( $\Rightarrow$  use free products with amalgamation – this is a non-trivial fact about groups, read more if free). So **Gp** is also balanced.

(c) In **Rng** (obvious notation), monomorphisms correspond to injections (proof is much like for **Gp**). However, not all epimorphisms are surjective. For example

the inclusion  $\mathbb{Z} \rightarrow \mathbb{Q}$  is an epimorphism, since if  $\mathbb{Q} \xrightleftharpoons[g]{f} R$  agree on all integers, they agree everywhere. So **Rng** is not balanced.

(d) One final example is **Top**. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to **Set** (check). However, **Top** is not balanced since a continuous bijection need not have continuous inverse.

## 2 The Yoneda Lemma

Let's not start on the content this lecture. Why are we talking about one single lemma in a chapter? Well it's not really a lemma. There's some story behind this, check here for an [obituary](#) which probably has the story that lecture was talking about in class.

—Lecture 5—

### Definition. (2.1)

We say a category  $\mathcal{C}$  is *locally small* if, for any two objects  $A, B$ , the morphisms  $A \rightarrow B$  in  $\mathcal{C}$  form a set  $\mathcal{C}(A, B)$ .

If we fix  $A$  and let  $B$  vary, the assignment  $B \rightarrow \mathcal{C}(A, B)$  becomes a functor  $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ : given  $B \xrightarrow{f} C$ ,  $\mathcal{C}(A, f)$  is the mapping  $g \rightarrow fg$ . Similarly,  $A \rightarrow \mathcal{C}(A, B)$  defines a functor  $\mathcal{C}(-, B) : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ .

### Lemma. (2.2)

- (i) Let  $\mathcal{C}$  be a locally small category,  $A \in \text{ob } \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathbf{Set}$  a functor. Then natural transformations  $\mathcal{C}(A, -) \rightarrow F$  are in bijection with elements of  $FA$ ;
- (ii) Moreover, this bijection is natural in both  $A$  and  $F$ .

*Proof.* (i) Given  $\alpha : \mathcal{C}(A, -) \rightarrow F$ , we define  $\Phi(x) = \alpha_A(1_A) \in FA$ . Given  $x \in FA$ , we define  $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$  by  $\Psi(x)_B(A \xrightarrow{f} B) = Ff(x) \in FB$ .  $\Psi(x)$  is natural: given  $g : B \rightarrow C$ , we have

$$\begin{aligned}\Psi(x)_C \mathcal{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x), \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x), \\ \Phi\Psi(x) &= \Psi(X)_A(1_A) = F(1_A)(x) = x\end{aligned}$$

Given  $\alpha$ ,

$$\begin{aligned}\Psi\Phi(\alpha)_B(f)\Psi(\alpha_A(1_A))_B(f) &= Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathcal{C}(A, f)(1_A) = \alpha_B(f)\end{aligned}$$

So  $\Psi\Phi(\alpha) = \alpha$ . □

### Corollary. (2.3)

The assignment  $A \rightarrow \mathcal{C}(A, -)$  defines a full and faithful functor  $\mathcal{C}^{op} \rightarrow [\mathcal{C}, \mathbf{Set}]$ .

*Proof.* Put  $F = \mathcal{C}(B, -)$  in 2.2(i): we get a bijection between  $\mathcal{C}(B, A)$  and morphisms  $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$  in  $[\mathcal{C}, \mathbf{Set}]$ . We need to verify this is functorial: but it sends  $f : B \rightarrow A$  to the natural transformation  $g \rightarrow gf$ . So functoriality follows from associativity. □

We call this functor (or the functor  $\mathcal{C} \rightarrow [\mathcal{C}^{op}, \mathbf{Set}]$  sending  $A$  to  $\mathcal{C}(-, A)$ ) the *Yoneda embedding* of  $\mathcal{C}$ , and denote it by  $Y$ .

Now let's go back to prove 2.2(ii):

*Proof.* (ii) Suppose for the moment that  $\mathcal{C}$  is small, so that  $[\mathcal{C}, \mathbf{Set}]$  is locally small. Then we have two functors  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$ : one sends  $(A, F)$  to  $FA$ , and the other is the composite:  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{op} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-, -)} \mathbf{Set}$ . 2.2(ii) says that these are naturally isomorphic. We can translate this into an elementary statement, making sense even when  $\mathcal{C}$  isn't small. Given  $A \xrightarrow{f} B$  and  $F \xrightarrow{\alpha} G$ , the two ways of producing an element of  $GB$  from a natural transformation  $\beta : \mathcal{C}(A, -) \rightarrow F$  give the same result, namely

$$\alpha_B(Ff)\beta_A(1_A) = (Gf)\alpha_A\beta_A(1_A)$$

which is equal to  $\alpha_B\beta_B(f)$ .  $\square$

**Definition.** (2.4)

We say a functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$  is *representable* if it's isomorphic to  $\mathcal{C}(A, -)$  for some  $A$ . By a representation of  $F$ , we mean a pair  $(A, x)$  where  $x \in FA$  is such that  $\Psi(x)$  is an isomorphism.

We also call  $x$  a *universal element* of  $F$ .

**Corollary.** (2.5)

If  $(A, x)$  and  $(B, y)$  are both representations of  $F$ , then there's a unique isomorphism  $f : A \rightarrow B$  such that  $(Ff)(x) = y$ .

*Proof.* Consider the composite  $\mathcal{C}(B, -) \xrightarrow{\Psi(y)^{-1}} F \xrightarrow{\Psi(x)} \mathcal{C}(A, -)$ . By (2.3) this is of the form  $Y(f)$  for a unique isomorphism  $f : A \rightarrow B$ , and the diagram

$$\begin{array}{ccc} \mathcal{C}(B, -) & \xrightarrow{Y(f)} & \mathcal{C}(A, -) \\ & \searrow & \swarrow \\ \Psi(y) & & \Psi(x) \\ & \searrow & \swarrow \\ & F & \end{array}$$

commutes iff  $(Ff)(x) = y$ .  $\square$

**Example.** (2.6)

(a) The forgetful functor  $\mathbf{Gp} \rightarrow \mathbf{Set}$  is representable by  $(\mathbb{Z}, 1)$ ,  $\mathbf{Rng} \rightarrow \mathbf{Set}$  by  $(\mathbb{Z}[X], X)$ , and  $\mathbf{Top} \rightarrow \mathbf{Set}$  by  $(\{*\}, *)$ .

(b) The functor  $P^* : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$  is representable by  $(\{0, 1\}, \{1\})$ : this is the bijection between subsets and characteristic functions.

(c) Let  $G$  be a group. The unique (up to isomorphism) representable functor  $G(*, -) : G \rightarrow \mathbf{Set}$  is the *Cayley representation* of  $G$ , i.e. the set  $\cup G$  with  $G$  acting by left multiplication.

(d) Let  $A$  and  $B$  be two objects of a small category  $\mathcal{C}$ . We have a functor  $\mathcal{C}^{op} \rightarrow \mathbf{Set}$  sending  $C$  to  $\mathcal{C}(C, A) \times \mathcal{C}(C, B)$ . A representation of this, if it exists, is called a (categorical) *product* of  $A$  and  $B$ , and denoted  $(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B))$ .

This pair has the property that, for any pair  $(C \xrightarrow{f} A, C \xrightarrow{g} B)$ , there's a unique  $C \xrightarrow{h} A \times B$  with  $\pi_1 h = f$  and  $\pi_2 h = g$ .

Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top**, ..., they are *just* cartesian products, in posets they are binary meets (see sheet 1 Q1).

Dually, we have the notion of *coproduct*  $(A + B, A \xrightarrow{\mu_1} A + B, B \xrightarrow{\mu_2} A + B)$ . These also exist in many categories of interest.

—Lecture 6—

(f) (Lecturer didn't like (e) so jumped to (f) directly) Let  $A \rightrightarrows B$  be morphisms in locally small category  $\mathcal{C}$ . We have a functor  $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$  defined by

$$F(C) = \{h \in \mathcal{C}(C, A) \mid fh = gh\}$$

A representation (see (2.4)) of  $F$ , if it exists, is called an *equalizer* of  $(f, g)$ : It consists of an object  $E$  and a morphism  $E \xrightarrow{e} A$  s.t.  $fe = ge$ , and every  $h$  with  $fh = gh$  factors uniquely (see proof of 2.9(i) which gives an insight of what this means) through  $e$ .

In **Set**, we take  $E = \{x \in A \mid f(x) = g(x)\}$  and  $e = \text{inclusion}$ . Similar constructions work in **Gp**, **Rng**, **Top**, ...

Dually, we have the notion of *coequalizer*.

**Remark.** (2.7)

If  $e$  occurs as an equalizer, then it is a monomorphism, since any  $h$  factors through it in at most one way. We say a monomorphism is *regular* if it occurs as an equalizer.

Split monomorphisms are regular (cf sheet1 Q6(i)).

Note that regular epic monomorphisms are isomorphisms: if the equalizer  $e$  of  $(f, g)$  is epic, then  $f = g$ , so  $e \cong 1_{\text{cod } e}$ .

**Definition.** (2.8)

Let  $\mathcal{C}$  be a category,  $\mathcal{G}$  a class of objects of  $\mathcal{C}$ .

(a) We say  $\mathcal{G}$  is a *separating family* for  $\mathcal{C}$ , if given  $A \rightrightarrows B$  such that  $fh = gh$  for

all  $G \xrightarrow{h} A$  with  $G \in \mathcal{G}$ , then  $f = g$ .

(i.e. the functors  $\mathcal{C}(G, -)$ ,  $G \in \mathcal{G}$ , are collectively faithful.)

(b) We say  $\mathcal{G}$  is a *detecting family* if, given  $A \xrightarrow{f} B$  such that every  $G \xrightarrow{h} B$  with  $G \in \mathcal{G}$  factors uniquely through  $f$ , then  $f$  is an isomorphism.

If  $\mathcal{G} = \{G\}$ , we call  $G$  a *separator/detector*.

**Lemma.** (2.9)

(i) If  $\mathcal{C}$  is a balanced category, then any separating family is detecting.

(ii) If  $\mathcal{C}$  has equalizers, then any detecting family is separating.

*Proof.* (i) Suppose  $\mathcal{G}$  is separating and  $A \xrightarrow{f} B$  satisfies the condition of 2.8(b). If  $B \xrightarrow[g]{h} C$  satisfy  $gf = hf$ , then  $gx = hx$  for every  $G \xrightarrow{x} B$ , so  $g = h$ , i.e.  $f$  is epic.

Similarly if  $D \xrightarrow[l]{k} A$  satisfy  $fk = fl$ , then  $ky = ly$  for any  $G \xrightarrow{y} D$ , since both are factorizations of  $fky$  through  $f$ . So  $k = l$ , i.e.  $f$  is monic.

But  $\mathcal{C}$  is balanced. So  $f$  is an isomorphism.

(ii) Suppose  $\mathcal{G}$  is detecting and  $A \xrightarrow[g]{f} B$  satisfies the condition of 2.8(a). Then the equalizer  $E \xrightarrow{e} A$  of  $(f, g)$  is isomorphism, so  $f = g$ .  $\square$

**Example.** (2.10)

(a) In  $[\mathcal{C}, \mathbf{Set}]$ , the family  $\{\mathcal{C}(A, -) | A \in \text{ob } \mathcal{C}\}$  is both separating and detecting (just a restatement of Yoneda Lemma).

(b) In  $\mathbf{Set}$ ,  $1 = \{*\}$  (any one element set) is both a separator and a detector, since it represents the identity functor  $\mathbf{Set} \rightarrow \mathbf{Set}$ .

Similarly,  $\mathbb{Z}$  is both in  $\mathbf{Gp}$ , since it represents the forgetful functor  $\mathbf{Gp} \rightarrow \mathbf{Set}$ . Also,  $2 = \{0, 1\}$  is a coseparator and a codetector in  $\mathbf{Set}$ , since it represents  $P^* : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$ .

(c) In  $\mathbf{Top}$ ,  $1 = \{*\}$  is a separator since it represents the forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$ , but not a detector.

In fact,  $\mathbf{Top}$  has no detecting *set* of objects (note that this doesn't mean it has no detecting family).

For any infinite cardinal  $\kappa$ , let  $X$  be a discrete space of cardinality  $\kappa$ , and  $Y$  the same set with *co*- $< \kappa$  topology, i.e.  $F \subseteq Y$  is closed iff  $F = Y$  or  $\text{Card } F < \kappa$  (think about, e.g. cocountable topology, then this name makes sense).

The identity  $X \rightarrow Y$  is continuous, but not a homeomorphism (topologically). So if  $\{G_i | i \in I\}$  is any set of spaces, taking  $\kappa > \text{Card } G_i$  for all  $i$  yields an example to show that the set is not detecting.

(d) (some Algebraic Topology stuff) Let  $\mathcal{C}$  be the category of pointed connected *CW*-complexes and homotopy classes of (basepoint-preserving) continuous mappings.

JHC Whitehead proved that  $X \xrightarrow{f} Y$  in this category induces isomorphisms  $\pi_n(X) \rightarrow \pi_n(Y)$  for all  $n$ , then it's an isomorphism in  $\mathcal{C}$ .

This says that  $\{S^n | n \geq 1\}$  is a detecting set of  $\mathcal{C}$ .

But PJ Freyd showed there is no faithful functor  $\mathcal{C} \rightarrow \mathbf{Set}$ , so no separating *set*: if  $\{G_i | i \in I\}$  were separating, then  $x \rightarrow \coprod \mathcal{C}(G_i, x)$  (disjoint unions?) would be faithful.

Note that any functor of the form  $\mathcal{C}(A, -)$  preserves monomorphisms, but they don't normally preserve epimorphisms.

**Definition.** (2.11)

We say an object  $P$  is *Projective* if, given

$$\begin{array}{c} P \\ \downarrow f \\ A \xrightarrow{e} B \end{array}$$

(recall the two head right arrow means epimorphisms) there exists  $P \xrightarrow{g} A$  with

$eg = f$ .

(If  $\mathcal{C}$  is locally small, this says  $\mathcal{C}(P, -)$  preserves epimorphisms).

Dually, an *injective* object of  $\mathcal{C}$  is a projective object of  $\mathcal{C}^{op}$ .

Given a class  $\mathcal{E}$  of epimorphisms, we say  $P$  is  $\mathcal{E}$ -projective if it satisfies the condition for all  $e \in \mathcal{E}$ .

**Lemma.** (2.12)

Representable functors are (pointwise)(?) projective in  $[\mathcal{C}, \mathbf{Set}]$ .

*Proof.* Suppose given

$$\begin{array}{c} \mathcal{C}(A, -) \\ \downarrow \beta \\ F \xrightarrow{\alpha} G \end{array}$$

where  $\alpha$  is pointwise surjective. By Yoneda,  $\beta$  corresponds to some  $y \in GA$ , and we can find  $x \in FA$  with  $\alpha_A(x) = y$ . Now if  $\gamma : \mathcal{C}(A, -) \rightarrow F$  corresponds to  $x$ , then naturality of the Yoneda bijection yields  $\alpha\gamma = \beta$ .  $\square$

—Lecture 7— First example class: Friday 26th October, 2pm MR3.

Lecture is happy to mark any question we hand in!

### 3 Adjunctions

#### Definition. (3.1)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories and  $\mathcal{C} \xrightarrow{F} \mathcal{D}$ ,  $\mathcal{D} \xrightarrow{G} \mathcal{C}$  two functors.

By an *adjunction* between  $F$  and  $G$  we mean a bijection between morphisms

$FA \xrightarrow{f} B$  in  $\mathcal{D}$  and morphisms  $A \xrightarrow{g} GB$  in  $\mathcal{C}$ , which is natural in  $A$  and  $B$ , i.e. given  $A' \xrightarrow{g} A$  and  $B \xrightarrow{h} B'$ , we have  $h\hat{f}(Fg) = (\widehat{Gh})fg : FA' \rightarrow B'$ .

We say  $F$  is *left adjoint* to  $G$ , and write  $(F \dashv G)$ .

#### Example. (3.2)

(a) The free functor  $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$  is left adjoint to the forgetful functor  $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$ , since any function  $f : A \rightarrow UB$  extends uniquely to a homomorphism  $\hat{f} : FA \rightarrow B$ .

Naturality in  $B$  is *easy* (lecturer says so), naturality in  $A$  follows from the definition of  $F$  as a functor.

(b) The forgetful functor  $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$  has a left adjoint  $D$  which equips any set with the discrete topology, and also a right adjoint  $I$  which equips a set  $A$  with the discrete (lecturer had *indiscrete* here?) topology  $\{\phi, A\}$ .

(c) The functor  $\text{ob} : \mathbf{Cat} \rightarrow \mathbf{Set}$  (recall  $\mathbf{Cat}$  is the category of small categories) has a left adjoint  $D$  sending  $A$  to the *discrete* category with  $\text{ob}(DA) = A$  and only identity morphisms, and a right adjoint  $I$  sending  $A$  to the category with  $\text{ob}(IA) = A$  and one morphism  $x \rightarrow y$  for each  $(x, y) \in A \times A$ . In this case  $D$  in turn has a left adjoint  $\pi_0$  sending a small category  $\mathcal{C}$  to its set of *connected components*, i.e. the quotient of  $\text{ob}\mathcal{C}$  by the smallest equivalence relation identifying  $\text{dom } f$  with  $\text{cod } f$  for all  $f \in \text{mor } \mathcal{C}$ .

(d) Let  $M$  be the monoid  $\{1, e\}$  with  $e^2 = e$ . An object of  $[M, \mathbf{Set}]$  is a pair  $(A, e)$  (the images of the functor?), where  $e : A \rightarrow A$  satisfies  $e^2 = e$ .

We have a functor  $G : [M, \mathbf{Set}] \rightarrow \mathbf{Set}$  sending  $(A, e)$  to  $\{x \in A \mid e(x) = x\} = \{e(x) \mid x \in A\}$  and a functor  $F : \mathbf{Set} \rightarrow [M, \mathbf{Set}]$  sending  $A$  to  $(A, 1_A)$ .

I claim  $(F \dashv G \dashv F)$ : given  $f : (A, 1_A) \rightarrow (B, e)$ , it must take values in  $G(B, e)$ , and any  $g : (B, e) \rightarrow (A, 1_A)$  is determined by its values on the image of  $e$ .

(e) Let  $\mathbf{1}$  be the discrete category with one object  $*$ . For any  $\mathcal{C}$ , there's a unique functor  $\mathcal{C} \rightarrow \mathbf{1}$ : a left adjoint for this picks out an *initial* object of  $\mathcal{C}$ , i.e. an object  $I$  s.t. there exists a unique  $I \rightarrow A$  for each  $A \in \text{ob } \mathcal{C}$ .

Dually, a right adjoint for  $\mathcal{C} \rightarrow \mathbf{1}$  corresponds to a *terminal* object of  $\mathcal{C}$  (think about what this means).

(f) Let  $A \xrightarrow{f} B$  be a morphism in  $\mathbf{Set}$ . We can regard  $PA$  and  $PB$  as posets, and we have functors  $PA \xrightleftharpoons[P^*f]{Pf} PB$ .

I claim  $(PF \dashv P^*f)$ : we have  $Pf(A') \subseteq B' \iff f(x) \in B'$  for all  $x \in A' \iff A' \subseteq P^*f(B')$ .

(g) (*Galois Connection*) Suppose given sets  $A, B$  and a relation  $R \subseteq A \times B$ . We define mappings  $(-)^l, (-)^r$  between  $PA$  and  $PB$  by

$$S^r = \{y \in B \mid (\forall x \in S)((x, y) \in R)\} \text{ for } S \subseteq A$$

$$T^l = \{x \in A \mid (\forall y \in T)((x, y) \in R)\} \text{ for } T \subseteq B$$

The mappings are order-reserving (i.e. contravariant functors), and  $T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l$ .

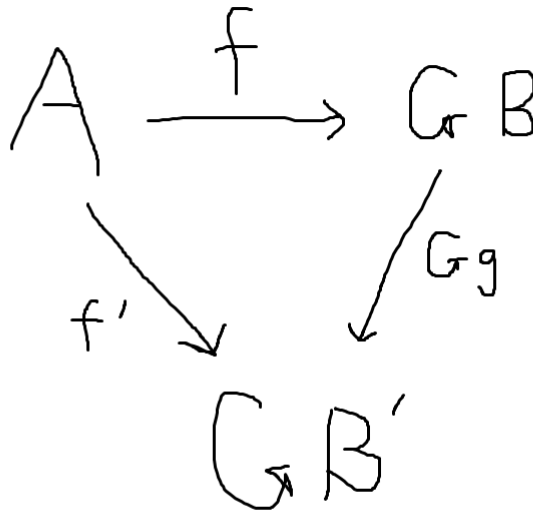
We say  $()^r$  and  $()^l$  are *adjoint on the right*.

(h) Let's now consider, as a functor,  $P^* : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$  is self-adjoint on the right, since functions  $A \rightarrow PB$  correspond bijectively to subsets of  $A \times B$ , and hence to functions  $B \rightarrow PA$ .

**Theorem.** (3.3)

(sorry I forgot to charge the other laptop today, the diagrams don't look very nice)

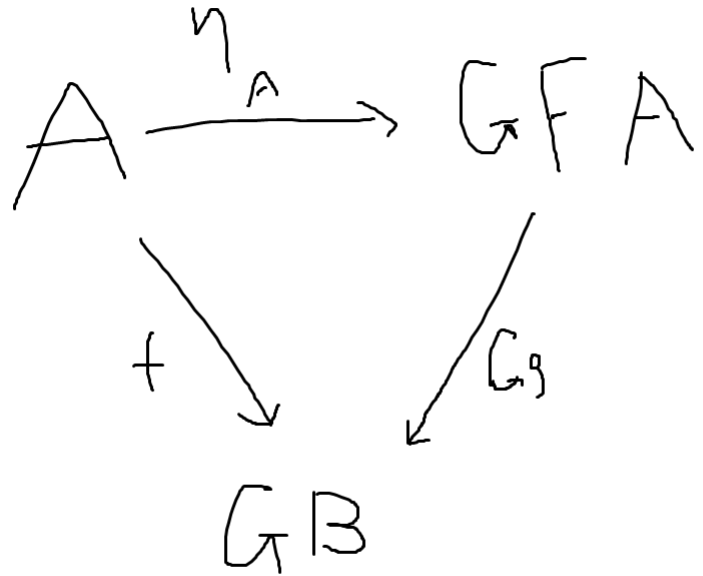
Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor. Then specifying a left adjoint for  $G$  is equivalent to specifying an initial object of  $(A \downarrow G)$  for each  $A \in \text{ob } \mathcal{C}$ , where  $(A \downarrow G)$  has objects pairs  $(B, f)$  with  $A \xrightarrow{f} GB$ , and morphisms  $(B, f) \rightarrow (B', f')$  are morphisms



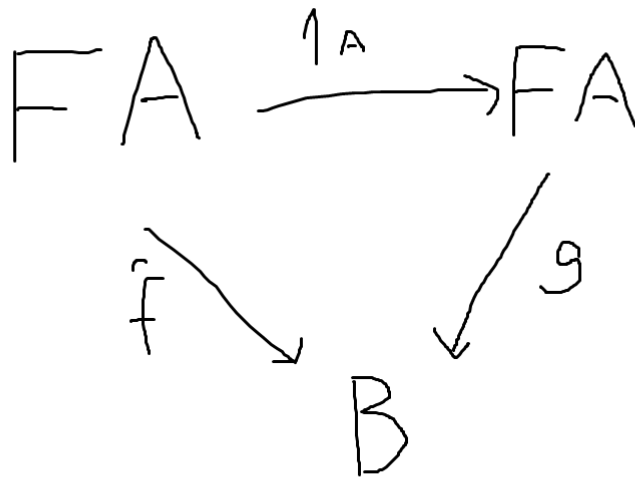
$B \xrightarrow{g} B'$  such that commutes.

*Proof.* Suppose given  $(F \dashv G)$ . Consider the morphism  $\eta_A : A \rightarrow GFA$  correspond to  $FA \xrightarrow{\eta} FA$ . Then  $(FA, \eta_A)$  is an object of  $(A \downarrow G)$ . Moreover, given  $g :$





$FA \rightarrow B$  and  $f : A \rightarrow GB$ , the diagram



commutes iff  
commutes, i.e.  $g = \hat{f}$ .

So  $(FA, \eta_A)$  is initial in  $(A \downarrow G)$ .

Conversely, suppose given an initial object  $(FA, \eta_A)$  for each  $(A \downarrow G)$ . Given

$A \xrightarrow{f} A'$ , we define  $Ff : FA \rightarrow FA'$  to be the unique morphism making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute.

Functoriality follows from uniqueness: given  $f' : A' \rightarrow A''$ ,  $F(f'f)$  and  $(Ff')(Ff)$  are both morphisms  $(FA, \eta_A) \rightarrow (FA'', \eta_{A''} F'f)$  in  $(A \downarrow G)$ .

Note that we haven't finished: we still have to verify natural adjunctions. We'll finish off this next monday.  $\square$