

Topics in Set Theory

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0 Introduction

—Lecture 1—

Example classes: 4th Feb, 18th Feb, 4th Mar 330-5pm MR5; fourth class undecided (probably on 15th).

Although the name of this course is *Topics* in Set theory, for all of its history only one topic is discussed. So maybe this course should be called *One Topic in Set Theory*, or probably just *the Continuum Hypothesis*: in this course we'll just solve one problem: the continuum problem, which we've known in the end that the problem is independent from ZFC.

Let's have some background stories first. In the second ICM congress (1900, Paris), Hilbert posed the famous 23 Hilbert questions, with the first one being the Continuum Hypothesis (a hypothesis at that time). The original formulation of CH was:

Any infinite subset of real numbers is either equinumerous to the set of natural numbers, or to the set of real numbers.

We could definitely formulate it better, but that is less important. More modern version of CH would be a short equation

$$2^{\aleph_0} = \aleph_1$$

which seemingly has nothing to do with the previous problem. However, in ZFC these two statements are equivalent:

- if $2^{\aleph_0} > \aleph_1$, in particular, $2^{\aleph_0} \geq \aleph_2$. Since $2^{\aleph_0} \sim \mathbb{R}$, we get an injection $i : \aleph_2 \rightarrow \mathbb{R}$. Consider $X := i[\aleph_1] \subseteq \mathbb{R}$. Clearly, $i \upharpoonright \aleph_1$ (i restricted to \aleph_1) is a bijection between \aleph_1 and X , so $X \sim \aleph_1$; but \aleph_1 , being uncountable, is not in bijection with natural numbers, and is not in bijection with real numbers. Thus X refutes CH.

If $2^{\aleph_0} = \aleph_1$, let $X \subseteq \mathbb{R}$. Consider $b : 2^{\aleph_0} \rightarrow \mathbb{R}$ a bijection. If X is infinite, then $b^{-1}[X] \subseteq 2^{\aleph_0}$. Thus the cardinality of X is either \aleph_0 or \aleph_1 (which $\sim \mathbb{N}$ and \mathbb{R} respectively). So $2^{\aleph_0} = \aleph_1 \implies \text{CH}$.

In 1938, Gödel proved that ZFC does not prove $\neg\text{CH}$, and in 1961 Cohen proved that ZFC does not prove CH, by methods of *inner models* and *forcing* (sometimes also called *outer models*, which is not incorrect) respectively. The latter has become the most important method in Set theory since then.

From logic (see Part II Logic and Set Theory) we have Gödel's Completeness Theorem: a theory T is consistent iff it has a model. So from the above two statements, it seems that we're going to prove that there are models for ZFC+CH and ZFC+ $\neg\text{CH}$; but this is obviously not possible because of the incompleteness phenomenon: we know we can't prove the consistency of ZFC (as a result, we can't even prove there is a model of ZFC)! So instead we could only prove the following:

$$\text{Cons}(\text{ZFC}) \rightarrow \text{Cons}(\text{ZFC} + \text{CH})$$

or equivalently, if $M \models ZFC$, then there is $N \models ZFC + CH$ (and similar for the other half).

1 Model theory of set theory

1.1 Absoluteness

For a moment, we will assume that we have a model $(M, \in) \models ZFC$. Unfortunately this first assumption doesn't make much sense, because model theory is based on set theory and we don't have anything if ZFC is inconsistent. We refer to the canonical objects in M by the usual symbols, e.g. $0, 1, 2, 3, 4, \dots, \omega, \omega + 1, \dots$.

What would an *inner model* be? Take $A \subseteq M$, and consider (A, \in) . It is a substructure of (M, \in) , because there are no function symbols or constant symbols in the language of set theory. This might be counterintuitive, because we're using symbols like ϕ and $\{\cdot\}$ all the time! However, these are technically not part of language of set theory as they can all be defined without any use of function symbols, i.e. they are just abbreviations. For example, $X = \phi$ abbreviates $\forall w(\neg w \in X)$; $X = \{Y\}$ abbreviates $\forall w(w \in X \iff w = Y)$, and similarly for \cup and \mathcal{P} ; and also for relation symbols such as \subseteq , which abbreviates $\forall w(w \in X \rightarrow w \in Y)$. Note that $X = \phi$ is NOT the formula that it looks like; in particular, it is not quantifier free (because it abbreviates $\forall w(\neg w \in X)$)! So we need to take extra care when we do things in this course.

Definition. If φ is a formula in n free variables, we say φ is *upwards absolute* between A and M if for all $a_1, \dots, a_n \in A$,

$$(A, \in) \models \varphi(a_1, \dots, a_n) \implies (M, \in) \models \varphi(a_1, \dots, a_n)$$

and we say φ is *downwards absolute* between A and M if for all $a_1, \dots, a_n \in A$,

$$(M, \in) \models \varphi(a_1, \dots, a_n) \implies (A, \in) \models \varphi(a_1, \dots, a_n)$$

and φ is *absolute* between A and M if it is both upwards and downwards absolute.

Observation:

(a) If φ is *quantifier-free*, then φ is absolute between A and M . But this doesn't really help much, because almost nothing is quantifier-free: without quantifiers we can only say things like $A \in B$ and $A = B$, and conjunctions of those; that's pretty much all.

(b) We say that a formula is Σ_1 if it is of the form

$$\exists x_1 \dots \exists x_n \varphi(x_1, \dots, x_n)$$

where φ is q.f.;

we say a formula is Π_1 if it is of the form

$$\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)$$

where φ is q.f..

(c) If φ is Π_1 , it is downward absolute; if it's Σ_1 then it is upwards absolute. So in particular, note that $X = \phi$ is downward absolute.

—Lecture 2—

As an example, write $0, 1, 2, 3, \dots$ for the ordinals in M , and let $A := M \setminus \{1\}$. In A , we have $0, 2$, but no 1 ; we also have $\{1\}$. If we use $\Phi_0(x)$ to denote the formula $\forall w(\neg w \in x) \iff x = \phi$. Clearly $(M, \in) \models \Phi_0(0)$, so by π_1 -downwards absoluteness, $(A, \in) \models \Phi_0(0)$.

Now, how many elements does $2 = \{0, 1\}$ have? In M we obviously know 2 has 2 elements; but in A , 2 only has one element 0 , and $\{1\}$ has no element: $(A, \in) \models \Phi_0(\{1\})$! Clearly $(M, \in) \not\models \Phi_0(\{1\})$, so Φ_0 is not absolute between A and M . As a corollary, we get $(A, \in) \not\models$ extensionality (we can uniquely specify sets by specifying their elements).

Remark. We could go on, defining formulas $\Phi_1(x), \Phi_2(x)$, etc to analyse which of the elements correspond to the natural numbers in A .

Reminder (from Part II Logic and Set Theory): we say A is *transitive in M* if for all $a \in A$ and $x \in M$ s.t. $(M, \in) \models x \in a$, we have $x \in A$. The problem for the above A is that it is not transitive. As long as that is fixed, we have the following:

Proposition. If A is transitive, then Φ_0 is absolute between A and M .

Proof. Since Φ_0 is Π_1 , we only need to show upwards absoluteness. Suppose $a \in A$ s.t. $(A, \in) \models \Phi_0(a)$, and suppose for contradiction that $a \neq 0$. Then there is some $x \in a$. By transitivity, $x \in A$. But then $\Phi_0(a) : \forall w(w \notin a)$ is not true in (A, \in) . \square

Similarly, if Φ_n is the formula describing the natural number n , and there is $a \in A$ s.t. $(A, \in) \models \Phi_n(a)$, and A is transitive, then $a = n$.

Proposition. If A is transitive in M , then $(A, \in) \models$ extensionality.

Proof. Take $a, b \in A$ with $a \neq b$. So by extensionality in (M, \in) , find, WLOG some $c \in a \setminus b$. Since $c \in a \in A$, by transitivity $c \in A$. Note that all of these quantifier-free formulas are absolute, so (A, \in) also models them; in particular, $(A, \in) \models c \in a, c \notin b$. So a, b do not satisfy the assumptions of extensionality. \square

Consider now $A = \omega + 2 = \{0, 1, 2, \dots, \omega, \omega + 1\} \subseteq M$. This is clearly transitive subset of M because it's an ordinal. So $(A, \in) \models$ extensionality, but clearly it isn't anything like a model of set theory as it is too thin. Consider the formula $x = \mathcal{P}(y)$. Unfortunately, this is not a formula, as \mathcal{P} is undefined. We have to expand it properly:

$$\begin{aligned} x &= \mathcal{P}(y) \\ \iff x &= \{z; z \subseteq y\} \\ \iff \forall w(w \in x \leftrightarrow w \subseteq y) \\ \iff \forall w(w \in x \leftrightarrow (\forall v(v \in w \rightarrow v \in y))) \end{aligned}$$

In A , what is $\mathcal{P}(\omega)$? We have $(A, \in) \models \omega + 1 = \mathcal{P}(\omega)$, which is obviously not what we want for $\mathcal{P}(\omega)$ to be.

Definition. (Bounded quantification)

We first define the notations $\exists v \in w \varphi$ to be $\exists v(v \in w \wedge \varphi)$, and $\forall v \in w \varphi$ to be $\forall v(v \in w \rightarrow \varphi)$, and we call these quantifiers *bounded*.

Now we say a formula φ is Δ_0 if it is in the smallest set S of formulas with the following properties:

1. All q-f formulas are in S ;
2. If $\varphi, \psi \in S$, then so are:

- 2a. $\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \varphi \leftrightarrow \psi$;
- 2b. $\neg\varphi$;
- 2c. $\exists x \in w \varphi, \forall v \in w \varphi$.

Theorem. If φ is Δ_0 and A is transitive, then φ is absolute between A and M .

Proof. We already know that quantifier free formulas are absolute, and absoluteness is obviously preserved under propositional connectives. The only case left is (2c).

Let's just do $\varphi \rightarrow \exists v \in w \varphi = \exists v(v \in w \wedge \varphi)$. So suppose φ is absolute. We need to deal with downwards absoluteness: we have $(M, \in) \models \exists v \in a \varphi(v, a)$ for some $a \in A$, i.e. $(M, \in) \models \exists v(v \in a \wedge (\varphi(v, a)))$.

Let's find $m \in M$ s.t. $(M, \in) \models m \in a \wedge \varphi(m, a)$.

Now $m \in a \in A$, so $m \in A$. By absoluteness of φ , we get $(A, \in) \models m \in a \wedge \varphi(m, a) \implies (A, \in) \models \exists v \in a \varphi(v, a)$. \square

Let T be any *set theory*. Then we say that φ is Δ_0^T if there is a Δ_0 formula ψ s.t. $T \vdash \varphi \leftrightarrow \psi$. So we get, as a corollary:

Corollary. If A is transitive in M , and both (M, \in) and (A, \in) are models of T , then Δ_0^T formulas are absolute between A and M .

We may also define Σ_1^T formulas to be the formulas that are T -equivalent to $\exists v_1 \dots \exists v_n \psi$ where ψ is Δ_0 , and similarly for Π_1^T formulas. So Σ_1^T (Π_1^T) formulas are upwards (downwards) absolute between A and M respectively.

On Wednesday we will look at what formulas are actually in these classes.

—Lecture 3—

Last time we fixed some *set theory* T , and defined formula classes Δ_0^T , Σ_0^T and Π_1^T . We showed that Δ_0^T formulas are absolute between A, M if A is transitive and $A, M \models T$, and also Σ_1^T and Π_1^T upwards and downwards respectively.

Even if you haven't paid attention you would have realize that we have some 0 and some 1 as subscripts here. So what is Δ_1^T ?

Definition. A formula is Δ_1^T if it is both Σ_1^T and Π_1^T .

Note that this definition is only possible upon taking equivalence classes on T , else no formula could be both Σ_1 and Π_1 .

Corollary. If A is transitive, $A, M \models T$, and ϕ is Δ_1^T , then ϕ is absolute between A and M .

Now we have to think of what a *set theory* is. We have to think of which axioms we're using. Preferably we would have extensionality, and then let's have pairing, union, power set, separation.

We denote this by FST_0 (finite set theory), with the 0 denoting that we don't have foundation yet. We use FST to denote FST_0 +foundation(regularity).

Now if we add infinity in, we reach the original version of Zermelo set theory Z_0 . However, nowadays we often call $Z = Z_0$ +foundation the Zermelo set theory.

For ordinary people these are enough (or far more than enough). But set-theorists realized later that they need replacement; we call this ZF_0 (of course ZF for the version with foundation). And lastly if we add choice in we get ZFC_0 (with foundation we get ZFC).

1.2 Long List of Δ_0^T formulas

Now we find a long list of Δ_0^T formulas. We start with the more trivial ones:

1. $x \in y$;
2. $x = y$;

These two are Δ_0 without T needed.

3. $x \subseteq y$. Apparently this is not a formula: we think it means $\forall w(w \in x \rightarrow w \in y)$, which we might abbreviate it as $\forall w \in x(w \in y)$, which is exactly the (2c) in definition of Δ_0 . So this is Δ_0 without T as well.

4. $\Phi_0(x) : \forall w(w \notin x) : \iff \forall w(\neg w \in x)$. If you took part II Logic and Set theory, you'll disagree that this is a formula, because \neg is not a thing; but let's not be so parsimonious on the syntax, but write it as $\forall w(w \in x \rightarrow \neg x = x)$, so this is also Δ_0 in predicate logic.

We say that an operation $x_1, \dots, x_n \rightarrow F(x_1, \dots, x_n)$ is defined by a formula in class Γ (where Γ is any class of formulas) in the theory T if there is a formula $\Psi \in \Gamma$ s.t.

- (1) $T \vdash \forall x_1 \dots \forall x_n \exists y \Psi(x_1, \dots, x_n, y)$;
- (2) $T \vdash \forall x_1 \dots \forall x_n \forall y, z (\Psi(x_1, \dots, x_n, y) \wedge \Psi(x_1, \dots, x_n, z) \rightarrow y = z)$;
- (3) $\Psi(x_1, \dots, x_n, y)$ iff $y = F(x_1, \dots, x_n)$.

Note that the first two are formal requirements, but the last one is an informal requirement as we haven't defined what F is.

Examples: $x \rightarrow \{x\}$, $x, y \rightarrow \{x, y\}$ (these are operations in FST_0). Note that these are informal because notations like $\{\cdot\}$ are undefined.

Let's now continue our lists:

5. $x \rightarrow \{x\}$. We need a formula $\Psi(x, z) \leftrightarrow 'z = \{x\}' \leftrightarrow \forall w(w \in z \leftrightarrow w = x)$.

This is not Δ_0 yet because we have a \leftrightarrow here. We rewrite it as

$\forall w((w \in z \rightarrow w = x) \wedge (w = x \rightarrow w \in z))$, but the second part is not Δ_0 . So we again rewrite it as

$\exists w \in z(w = x) \wedge \forall w \in z(w \in z \rightarrow w = x)$. So this is Δ_0 , with some very weak set theory being sufficient.

Similar to 5, we also have

6. $x, y \rightarrow \{x, y\}$;
 7. $x, y \rightarrow x \cup y$;
 8. $x, y \rightarrow x \cap y$;
 9. $x, y \rightarrow x \setminus y$;
 10. $x, y \rightarrow (x, y)$, the ordered pair, where we define it as $\{\{x\}, \{x, y\}\}$. Note that we could apply 5 and 6 (twice) to get this one.
- The last one gives us the motivation that if two operations f, g_1, \dots, g_k are defined by Δ_0^T -formulas, then so is the operation

$$x_1, \dots, x_n \rightarrow f(g_1(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n))$$

We then naturally have

11. $x \rightarrow x \cup \{x\} := S(x)$ (by the previous fact from 5 and 7).
12. $x \rightarrow \cup x$; (obvious if we write this fully out)
13. the formula φ describing " x is transitive".
14. the formula describing x is an ordered pair. At first look it looks like this is unbounded, but that's not the case: the quantifiers for the two components of x are bounded by $\cup x$.
15. $a, b \rightarrow a \times b$;
16. the formula " x is a binary relation";
17. $x \rightarrow \text{dom}(x) := \{y : \exists p \in x (p \text{ is an ordered pair, } p = (v, w), y = v)\}$;
18. $x \rightarrow \text{range}(x) := \{y : \exists p \in x (p \text{ is an ordered pair and } p = (v, y))\}$;
19. the formula ' x is a function';
20. the formula ' x is injective';
21. the formula ' x is a function from A to B ';
22. the formula ' x is a surjection from A to B ';
23. the formula ' x is a bijection from A to B '.

Note that we've only used some very few axioms: union, pairing and some finite version of separation, and nothing more.

Let's also agree on the definition of an ordinal: α is an ordinal if α is a transitive set well-ordered by \in . Of course we have to also agree on what being well-ordered means: it's totally ordered + well-founded, i.e. $\forall X (X \subseteq \alpha \rightarrow X \text{ has a } \in\text{-least element})$.

Being totally ordered is Δ_0 formula (check); however, the sentence (X, R) being well-founded is not obviously absolute, since the bound for the $\forall Z (Z \subseteq X \dots)$ quantifier is the power set. We'll talk about absoluteness of well-foundedness on Friday.

However, we don't actually need the general well-foundedness; we only need well-foundedness by \in , but that is given by axiom of foundation! So in models with the axiom of foundation, α is an ordinal iff α is transitive and totally ordered by \in .

—Lecture 4—

We're still on our list of things that are absolute for transitive models. We ended

with ordinals last time, where we defined that x is an ordinal iff x is transitive and (x, \in) is a well-order. We went into an issue there, because that consists of (x, \in) is a total order, which is fine; but then it also needs \in is a well-founded relation on x , which is only Π_1 . The good side is that if T contains the axiom of foundation, then $\Phi_{ord}(x)$ is equivalent to x is transitive and (x, \in) is a total order (as the last part is guaranteed), which is Δ_0^T . Therefore we can expand our list:

- 24. ' x is an ordinal' is Δ_0^T (for the right choice of T);
This is not as harmless as before, because we actually need T to include the axiom of foundation.
- 25. ' x is a successor ordinal', which is equivalent to ' x is an ordinal' and $\exists y \in x (y$ is the \in -largest element of x);
- 26. ' x is a limit ordinal';
- 27. $x = \omega$ (the smallest limit ordinal; similarly, $x = \omega + \omega$, $x = \omega + 1$, $x = \omega + \omega + 1$, $x = \omega^2$, $x = \omega^3$, $x = \omega^\omega$, ...)

1.3 Absoluteness of well-foundedness

If (X, R) is well-founded, we can define a rank function $rk : X \rightarrow \alpha$, where α is some ordinal, s.t. rk is order-preserving between (X, R) and (α, \in) . This theorem is proved using the right instances of Replacement. In particular, ZF proves:

(X, R) is well-founded $\iff \exists \alpha \exists f$ α is an ordinal, and f is an order-preserving function from (X, R) to (α, \in) . RHS is Σ_1^{ZF} , and LHS is Π_1^{ZF} . Thus for sufficiently strong T , (X, R) is well-founded is Δ_1^T and hence absolute for transitive models of T .

We generalize this to concepts defined by transfinite recursion. But first let's recall what it is: let (X, R) be well-founded, let F be 'functional', so for every x there is a unique y s.t. $x = F(y)$. Then there is a unique f with domain X and for all $x \in X$, $f(x) = F(f|IS_R(x))$, where $IS_R(x) = \{z \in x : zRx\}$.

Proposition. Let T be a set theory that is strong enough to prove the transfinite recursion theorem for F . Let F be absolute for transfinite models of T . Let (X, R) be in A . Then f defined by transfinite recursion is absolute between A and M .

Example. Let L be any first-order language whose symbols are all in A . Then the set of L -formulas and the set of L -sentences are in A .

The relation $S \models \varphi$ (note that this is not q-f, although it is bounded by S) is defined by recursion, and thus is absolute between A and M .

So: if S is an L -structure, $S \in A$,

$(A, \in) \models S \models \varphi \iff (M, \in) \models S \models \varphi$.

Gödel's incompleteness theorem roughly says that, if T is a theory whose set of axioms are recursive enumerable, and its axioms are strong enough to do some arithmetics, then $T \not\models Cons(T)$ (which is a sentence in L). Examples for T : PA,

Z , ZF , ZFC , $ZFC + \varphi$ any one additional formula.

In particular, $ZFC^* := ZFC + Cons(ZFC) \not\models Cons(ZFC + Cons(ZFC))$.

By completeness theorem, $Cons(T) \iff \exists M(M \models T)$. Note that LHS is Π_1 and RHS is Σ_1 , so completeness theorem tells us that this is a Δ_1 concept. Note also that LHS is a bounded concept, since all quantifiers are bounded (LHS is Δ_0^Z).

Now write β for 'there is a transitive set A s.t. $(A, \in) \models ZFC$ '. Note that this is stronger than the consistency of ZFC as it also specifies that there is a particular model for it. In particular, we can't prove it even in ZFC^* :

Theorem. If ZFC^* is consistent, then $ZFC^* \not\models \beta$.

Proof. Let $(M, \in) \models ZFC^*$. Suppose $ZFC^* \vdash \beta$. So $(M, \in) \models \beta$. So we found a transitive set A in M s.t. (A, \in) is a model of ZFC . By assumption, $(M, \in) \models Cons(ZFC)$, so $(A, \in) \models Cons(ZFC)$ since $Cons(ZFC)$ is absolute between transitive models. So $(A, \in) \models ZFC^*$. So we proved $Cons(ZFC^*)$, contradicting Gödel's incompleteness theorem. \square

That means that assuming β is not an obvious assumption, so we need to study under which (natural) assumptions β is true.

1.4 Concrete transitive models of ZFC

So let's investigate transitive models A inside M . The two most basic constructions:

- (1) von Neumann hierarchy (cumulative hierarchy);
- (2) hereditarily small sets.

(1) is defined as follows by transfinite recursion: $V_0 := \phi$, $V_{\alpha+1} := \mathcal{P}(V_\alpha)$, $V_\lambda := \cup_{\alpha < \lambda} V_\alpha$ for λ being limit ordinals.

Proposition. $\forall \alpha V_\alpha$ is transitive (see Part II Logic and Set Theory). [Induction, with key lemma: if X is transitive, then $\mathcal{P}(X)$ is transitive]

If λ is a limit ordinal, then $V_\lambda \models FST$.

If $\lambda > \omega$ and a limit ordinal, then $V_\lambda \models Z$.

For (2): let κ be a cardinal. We say X is hereditarily smaller than κ if $|tcl(X)| < \kappa$ (transitive closure).

Let $H_\kappa := \{X : X \text{ is hereditarily smaller than } \kappa\}$. This is obviously transitive.

We'll continue next Monday.

—Lecture 5—

We'll start with V_α today.

We know (if you don't know then I invite you to learn it): if λ is a limit then $V_\lambda \models FST$; if further $\lambda > \omega$, then $V_\lambda \models Z$. (On example sheet 1. I haven't printed them today, so if you want to try them before Wednesday, find an envelope next to my office at C0.10 from this afternoon).

The critical axiom here is *Replacement*. The test case is $\lambda = \omega + \omega$, which we expect replacement fails. Remember replacement says that if we have a function $F : V_{\omega+\omega} \rightarrow V_{\omega+\omega}$ that is definable in $V_{\omega+\omega}$, and $x \in V_{\omega+\omega}$, then $\{F(y) : y \in x\} \in V_{\omega+\omega}$, i.e. image of elements of a set under a function is a set.

The idea is to take $x = \omega$, and define F to be the function that takes $n \rightarrow \omega + n$, and $y \rightarrow 0$ if $y \notin \omega$. Remember that definable just means that we need a formula that uniquely specify the image (see previous notes). Let $Y = \{F(n) : n \in \omega\}$, which needs to be a set if replacement holds. Then Y is a subset of $V_{\omega+\omega}$, so $Y \in V_{\omega+\omega+1}$; but it is not bounded, so it is not in $V_{\omega+\omega}$. This example shows concretely that $V_{\omega+\omega} \models \neg \text{Replacement}$.

What we needed in this particular example is a function that takes a bounded sequence to an unbounded one. Similarly, if α is any ordinal s.t. there is a definable function $f : \omega \rightarrow \alpha$, s.t. the range of f is unbounded in α , then $V_\alpha \models \neg \text{Replacement}$.

Even more general, if $\beta < \alpha$, and a definable function $f : \beta \rightarrow \alpha$ with unbounded range, then $V_\alpha \models \neg \text{Replacement}$.

Reminder: we call a cardinal κ *regular* if there is no partition $\kappa = \cup_{i \in I} A_i$ s.t. $|I|, |A_i| < \kappa$ for all $i \in I$.

Equivalently, for every $\alpha < \kappa$, there is no unbounded function $f : \alpha \rightarrow \kappa$.

We know, for example, that \aleph_1 is regular. Moreover, for any α , $\aleph_{\alpha+1}$ is regular. So this gives us the next candidate, $\alpha = \aleph_1$, which replacement cannot fail in the above way. So how does it fail? Note that $\mathcal{P}(\omega) \in V_{\omega+2} \subseteq V_{\omega_1}$. Clearly, there is a surjection $s : \mathcal{P}(\omega) \rightarrow \omega_1$. But that means its range is unbounded in ω_1 . Thus $V_{\omega_1} \models \neg \text{Replacement}$ either.

In general, if κ is regular and there is $\lambda < \kappa$ with $|\mathcal{P}(\lambda)| \geq \kappa$, then the same argument shows $V_\kappa \models \neg \text{Replacement}$.

Definition. A cardinal κ is called *inaccessible* if

- (a) κ is regular;
- (b) $\forall \lambda < \kappa, |\mathcal{P}(\lambda)| < \kappa$.

Side note: Related to the question, 'are there regular limit cardinals'? Under GCH: $\forall \kappa, 2^\kappa = \kappa^+$, we have that κ is inaccessible iff κ is regular limit. This suggests that the above question cannot be answered that easily (we usually call regular limit cardinals *weakly inaccessible*).

Let's assume that $\kappa > \omega$ is inaccessible (because ω is actually inaccessible, which kind of make sense, because *infinity* is the ultimate thing that is not accessible from everything smaller than it).

Lemma. $\forall \lambda < \kappa, |V_\lambda| < \kappa$.

Proof. Clearly $|V_\omega| = \aleph_0$, so $|V_\omega| < \kappa$.

By induction, suppose $|V_\lambda| < \kappa$. Then $V_{\lambda+1} = \mathcal{P}(V_\lambda)$. Therefore $|V_{\lambda+1}| = |\mathcal{P}(V_\lambda)| < \kappa$ by (b).

Now let $\lambda < \kappa$ be a limit ordinal. Then $V_\lambda = \cup_{\alpha < \lambda} V_\alpha$. So suppose by contradiction that $|V_\lambda| = \kappa$. But $|V_\alpha| < \kappa$ for all $\alpha < \lambda$, so we can write κ as a union of λ many things of smaller cardinals, contradicting regularity. \square

Theorem. If κ is inaccessible, then $V_\kappa \models \text{Replacement}$.

Proof. We're actually going to something slightly stronger: take any function $F : V_\kappa \rightarrow V_\kappa$ (without caring of its definability), and any $x \in V_\kappa$. Note that $V_\kappa = \cup_{\alpha < \kappa} V_\alpha$. So we can find $\alpha \in \kappa$ s.t. $x \in V_\alpha$. Since V_α is transitive, $x \subseteq V_\alpha$. But that means $|x| \leq |V_\alpha| < \kappa$ (by lemma).

te that this is exactly what went wrong in V_{ω_1} .

Now consider $\{F(y) : y \in x\}$. \square