Model Theory

October 17, 2018

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1 Langauges and structures

Definition. (1.1) A language L consists of:
•(i) a set $\mathcal{F}$ of function symbols, and for each $f \in \mathcal{F}$, a positive integer n_f, the arity of f;
•(ii) a set $\mathcal{R}$ of relation symbols, and for each $R \in \mathcal{R}$, a positive integer n_R, the arity of R;
•(iii) a set $\mathcal{C}$ of constant symbols.
Note that each of the above three sets can be empty.

Example. $L = \{\{\cdot, -1\}, \{1\}\}$ where $\cdot$ is a binary function, -1 is a unary function, and 1 is a constant. We call this L_{gp} (language of groups);
$L_{lo} = \{<\}$, where $<$ is a binary relation (linear order).

Definition. (1.2) Given a language L, say, an L-structure consists of:
(i) a set M, the *domain*;
(ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \to M$;
(iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$;
(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the *interpretation* of f, R, c respectively.

Notation. (1.3)
We often fail to distinguish between the symbols in the language L and their interpretations in a L-structure, if the context allows.

We may write $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$.

Example. (1.4)
(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, -1\}, 1 \rangle$ is an L_{gp}-structure.
$\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is also an L_{gp}-structure (here $+$ is a binary and $-$ is the unary negation function).
$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo} structure ($<$ is the interpretation of relation).

Definition. (1.5)
Let L be a language, let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.
An *embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an injection $\alpha : M \to N$ that preserves the structure:
(i) For all $f \in \mathcal{F}$, and $a_1, ..., a_{n_f} \in M$,

$$\alpha(f^M(a_1, ..., a_{n_f})) = f^N(\alpha(a_1), ..., \alpha(a_{n_f}))$$

(ii) For all $R \in \mathcal{R}$, and $a_1, ..., a_{n_R} \in M$,

$$(a_1, ..., a_{n_R}) \in R^M \iff (\alpha(a_1), ..., \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if. (iii) For all $c \in \mathcal{C}$, we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding $\mathcal{M} \to \mathcal{N}$ is also called an *isomorphism* of $\mathcal{M}$ onto $\mathcal{N}$.

(1.6) Exercise. Let G_1, G_2 be groups, regarded as L_{gp}-structures.
Check that $G_1 \cong G_2$ in the usual algebra sense, if and only if there is an isomprhism $\alpha : G_1 \to G_2$ in the sense of above definition 1.5.

2 Terms, formulae, and their interpretations

In addition to the symbols of L, we also have:
(i) infinitely many variables, $\{x_i\}_{i \in I}$;
(ii) logical connectives, $\wedge, \neg$ (also express $\vee, \rightarrow, \leftrightarrow$);
(iii) quantifier $\exists$ (also express $\forall$);
(iv) punctuations (,).

Definition. (2.1)
L-terms are defined recursively as follows:
• any variable x_i is a term;
• any constant symbol is a term;
• for any $f \in \mathcal{F}$,

$$f(t_1, ..., t_{n_f})$$

for any terms $t_1, ..., t_{n_f}$ is a term;
• nothing else is a term.

Notation: we write $t(x_1, ..., x_n)$ to mean that the variables appearing in t are among $x_1, .., x_n$.

Example. In $\mathcal{R} =< \mathbb{R}, \cdot, -1, 1 >$,
• $(\cdot(x_1, x_2), x_3)$ is a term $(x_1 \cdot x_2) \cdot x_3)$;
• $(\cdot(1, x_1))^{-1}$ is a term $(1 \cdot x)^{-1}$.

Definition. (2.2)
If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, ..., x_k)$ we assign a function

$$t^M : M^k \to M$$

defined as follows:
(i) If $t = x_i, t^M[a_1, ..., a_k] = a_i$;
(ii) If $t = c$ is a constant, $t^M[a_1, ..., a_k] = c^m$;
(iii) If $t = f(t_1(x_1, ..., x_k), ..., t_{n_f}(x_1, ..., x_k))$,

$$t^M(a_1, ..., a_k) = f^M(t_1^M(a_1, ..., a_k), ..., t_{n_f}^M(a_1, ..., a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on $http$: $users.mct.open.ac.uk/sb27627/MT.html$ (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

Remark. (The lecture forgot about this last time) Any language L includes an equality symbol $=$.

Last time we assigned a function t^m. In L_{gp}, the term $x_2 \cdot x_3$ can be described as, say $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), ...$
Then the term $x_2 \cdot x_3$ can be assigned to functions $t_1^M : M^3 \to M : (a_1, a_2, a_3) \to (a_2 \cdot a_3)$, or $t_2^M : M^4 \to M : (a_1, a_2, a_3, a_4) \to (a_2 \cdot a_3)$. These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term t to be the number of symbols of L occuring in t.

Fact (2.3): Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any L-term $t(x_1, ..., x_k)$ and $a_1, ..., a_k \in M$, we have

$$\alpha(t^M(a_1, ..., a_k)) = t^N(\alpha(a_1), ..., \alpha(a_k))$$

Proof. Prove by induction on complexity of t.
Let $\bar{a} = (a_1, ..., a_k)$ and $\bar{x} = (x_1, ..., x_l)$. Then:
(i) if $t = x_i$a a variable, then $t^M(\bar{a}) = a_i$, and $t^N(\alpha(a_1), ..., \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds;
(ii) if $t = c$ is a constant, then $t^M(\bar{a}) = c^M$, and $t^N(\alpha(\bar{a})) = c^N$ by definition of a term. The key here is that, since α is an embedding we have $\alpha(c^M) = c^N$;
(iii) if $t = f(t_1(\bar{x}, ..., t_{n_f}(\bar{x})))$, then

$$\alpha(f^M(t_1^M(\bar{a}), ..., t_{n_f}(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), ..., \alpha(t_{n_f}^M(\bar{a})))$$

as α is an embedding. But $t_1(\bar{x}), ..., t_{n_f}(\bar{x})$ have lower complexity than t, so the inductive hypothesis applies. □

Exercise (2.4): conclude the proof of the above fact.
(Actually is it not done?)

Definition. (2.5)
The set of *atmoic formulas* of L is defined as follows:
(i) if t_1, t_2 are L-terms, then $t_1 = t_2$ is an atomic formula;
(ii) if R is a relation symbol, and $t_1, ..., t_{n_R}$ are L-terms, then $R(t_1, ..., t_{n_R})$ is an atomic formula;
(iii) nothing else is an atomic formula.

Definition. (2.6)
The set of L-formulas is defined as follows:
(i) any atomic formula is an L-formula;
(ii) if ϕ is an L-formula, then so is $\neg\phi$;
(iii) if ϕ and ψ are L-formulas, then so is $\phi \wedge \psi$;
(iv) if ϕ is an L-formula, for any $i \geq 1$, $\exists x_i \phi$ is a formula;
(v) nothing else is a formula (note that $\forall$ can be constructed by $\neg$ and $\exists$).

Example. In L_{gp}, $x_1 \cdot x_1 = x_2$, or $x_1 \cdot x_2 = 1$ are both atomic formulas; $\exists x_1(x_1 \cdot x_2) = 1$ is an L-formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier $\exists$. We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example, $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp}-sentence.

Notation: $\phi(x_1, ..., x_k)$ means that the free variables in ϕ are among $x_1, ..., x_k$.

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

Definition. (2.7)
Let $\phi(x_1, ..., x_k)$ be an L-formula, let $\mathcal{M}$ be an L-structure, and let $\bar{a} = a_1, ..., a_k$ be elements of $\mathcal{M}$.
We define $\mathcal{M} \vDash \phi(\bar{a})$ (syntactic implication, read as *M models* $\phi(\bar{a})$) as follows:
(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$;
(ii) if ϕ is $R(t_1, ..., t_{n_R})$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff

$$\left(t_1^M(\bar{a}), ..., t_{n_R}^M(\bar{a})\right) \in R^M$$

(iii) if ϕ is a conjunction, say $\psi \wedge \chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \chi(\bar{a})$;
(iv) if ϕ is $\exists x_j \chi(x_1, ..., x_k, x_j)$ (where we'll assume that x_j is not one of the free variables $x_1, ..., x_k$), then $\mathcal{M} \vDash \phi(\bar{a})$ iff there exists $b \in \mathcal{M}$ s.t. $\mathcal{M} \vDash \chi(a_1, ..., a_k, b)$;
(v) (lecture forgets this, this should probably be more in front rather than in the end) if ϕ is $\neg\psi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \psi(\bar{a})$.

Example. Consider $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1\rangle$, the multiplicative group of non-negative reals, and suppose we have $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$, then $\mathcal{R} \vDash \phi(1)$, but $\mathcal{R} \nvDash \phi(-1)$.

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):
• $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$;
• $\phi \to \psi$ for $\neg\phi \vee \psi$;
• $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$;
• $\forall x_i \phi$ for $\neg\exists x_i(\neg\phi)$.

Proposition. (2.9)
Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding.
Let $\phi(\bar{x})$ be an atomic(!) formula, and $\bar{a} \in M^{|\bar{x}|}$, here $|\bar{x}|$ means the length of the tuple $\bar{x}$ (from now on, when we write a tuple like $\bar{a}$, we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a}))$$

Question: if ϕ is an L-formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.
Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.
In definition 2.7 we defined what it means for $\mathcal{M} \models \phi(\bar{a})$, in particular we defined: if $\phi \equiv \neg\chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \chi(\bar{a})$. Here by $\mathcal{M} \models \phi(\bar{a})$ we mean $\mathcal{M} \models \neg\chi(\bar{a})$, and $\chi(\bar{a})$ is *shorter* than $\phi(\bar{a})$, so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

Proof. There are two cases:
• $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. Use Fact (2.3). (exercise on example sheet)
• $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), ..., t_{n_R}(\bar{x}))$. Then $\mathcal{M} \models R(t_1(\bar{a}), ..., t_{n_R}(\bar{a}))$ if and only if ... (lecturer says work this out by yourself. Basically the induction step). □

Proposition. (2.10)
Exercise: show that prop (2.9) holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).
(I guess that also suggests when does it not hold for general formulas – see below).

Example. (2.11, Do embeddings preserve all formulas? No.)
Let $\mathcal{Z} = (\mathbb{Z}, <)$ an $L_{lo}-$structure, $\mathcal{Q} = (\mathbb{Q}, <)$ also an $L_{lo}-$structure. Then

$$\begin{aligned} \alpha :& \mathbb{Z} \to \mathbb{Q} \\ & n \to n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3(x_1 < x_3 \wedge x_3 < x_2)$$

Now $\mathcal{Q} \models \phi(1, 2)$ but $\mathcal{Z} \not\models \phi(1, 2)$.

Fact (2.12) (From now on we'll stop saying that $\mathcal{M}, \mathcal{N}$ are L-structures etc to save time) Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L-formula, and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).

3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let L be a language, $\mathcal{M}, \mathcal{N}$ be L-structures.

Definition. (3.1)
An *$\mathcal{L}$-theory* T is a set of L-sentences.
$\mathcal{M}$ is a *model* of T if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$.
The class of all the models of T is written $Mod(T)$.
The *theory of* $\mathcal{M}$ is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L-\text{structure and } \mathcal{M} \vDash \sigma\}$$

Example. (3.2)
Let T_{gp} be the set of L_{gp}-sentences:
(i) $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$;
(ii) $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$;
(iii) $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$.
Clearly, for a group G, $G \vDash T_{gp}$ (as they are just the group axioms). However, for a specific group G, clearly the theory of it, $Th(G)$ is lartger than T_{gp}.

Definition. (3.3)
$\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if $Th(\mathcal{M}) = Th(\mathcal{N})$.
We write $\mathcal{M} \equiv \mathcal{N}$.
Clearly, if $\mathcal{M} \simeq \mathcal{N}$ ($\simeq$ means isomorphism), then $\mathcal{M} \equiv \mathcal{N}$.
But if $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!
We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as $L_{lo}-$structures(!).

Definition. (3.4)
(i) An embedding $\beta : \mathcal{M} \to \mathcal{N}$ is *elementary* if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\beta(\bar{a}))$$

(ii) If $M \subseteq N$, and $id : \mathcal{N} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.
(iii) If $M \subseteq N$ and $id : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then $\mathcal{M}$ is said to be an *elementary substructure* of $\mathcal{N}$, written as $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example. (3.5)
Let $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an $L_{lo}-$structure where $<$ is the usual order;
Let $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$, also an $L_{lo}-$structure with the same $<$.
Then $\mathcal{M} \simeq \mathcal{N}$ as $L_{lo}-$structures. So $\mathcal{M} \equiv \mathcal{N}$ (since they are isomoprhic).
Also, $\mathcal{M} \subseteq \mathcal{N}$ (read as *is a substructure of*), since the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$. *However*, $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if we pick the formula $\phi(x) \equiv \exists y (x < y)$, then $\mathcal{N} \vDash \phi(1)$, but $\mathcal{M} \nvDash \phi(1)$.

Definition. (3.6)
Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where c_a are constant symbols). An interpretation of $\mathcal{M}$ as an L-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way, i.e. $c_a^M = a$.
In this context, the elements of A are called *parameters*.
If $\mathcal{M}$ and $\mathcal{N}$ are two structures, and $A \subseteq M \cap N$, then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$ structures.