Representation Theory

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0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces.*

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over $\mathbb{C}$. Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise: • F is a field, usually $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or $\bar{F}_p$ (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
• V is a vector space over F, always finite dimensional;
$GL(V) = \{\theta : V \to V, \theta$ linear, invertible$\}$, i.e. $\det \theta \neq 0$.

Recall from Linear Algebra:
If $\dim_F V = n < \infty$, choose basis $e_1, ..., e_n$ over F, so we can identify it with F^n. Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \to A_\theta$ ($A_{\theta_1\theta_2} = A_{\theta_1} A_{\theta_2}$ and bijection). Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = XA_1X^{-1}$.

Recall that $\mathrm{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A.

(1.3) $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$, hence we can define $\mathrm{tr}(\theta) = \mathrm{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V in f.d. over $\mathbb{C}$, with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $EndV$ is the set of all ilnear maps $V \to V$, e.g. $End(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over $\mathbb{C}$, $\alpha \in End(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1}(X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a $\mathbb{C}$-vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:
(1.6) The symmetric group, $S_n = Sym(X)$ on the set $X = \{1, ..., n\}$ is the set of all permutations of X. $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m: $C_m =< x : x^m = 1 >$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in $\mathbb{C}$ (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of $\mathbb{C}$). We also have the group of rotaitons, centre O of regular $m-$gon in $\mathbb{R}^2$ (inside $GL_2(\mathbb{R})$).

(1.8) Dihedrap groups D_{2m} of order $2m =< x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} >$. Think of this as the set of rotations and reflections preserving a regular m-gon.

(1.9) Quaternino group, $Q_8 =< x, y|x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} >$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \to X$ by $(g, x) \to g \cdot x$ for $g \in G$, $x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X, we obtain a homomorphism $\theta : G \to Sym(X)$, called the *permutation representation* of G.

Proof. For $g \in G$, the function $\theta_g : X \to X$ by $x \to gx$ is a permutation on X, with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G$, $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$ since $(g_1 g_2)x = g_1(g_2 x)$ for $x \in X$. $\square$