

# Category Theory

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## 0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

# 1 Definitions and examples

## Definition. (1.1)

A category  $\mathcal{C}$  consists of:

- (a) a collection  $\text{ob } \mathcal{C}$  of *objects*  $A, B, C$ ;
- (b) a collection  $\text{mor } \mathcal{C}$  of *morphisms*  $f, g, h$ ;
- (c) two operations *domain*, *codomain* assigning to each  $f \in \text{mor } \mathcal{C}$  a pair of objects, its *domain* and *codomain*; we write  $A \xrightarrow{f} B$  to mean  $f$  is a *morphism* and  $\text{dom } f = A, \text{cod } f = B$ ;
- (d) an operation assigning to each  $A \in \text{ob } \mathcal{C}$  a morphism  $A \xrightarrow{1_A} A$ ;
- (e) a partial binary operation  $(f, g) \rightarrow fg$  on morphisms, such that  $fg$  is defined iff  $\text{dom } f = \text{cod } g$ , and  $\text{dom}(fg) = \text{dom } g$ ,  $\text{cod}(fg) = \text{cod}(f)$  if  $fg$  is defined, satisfying:
  - (f)  $f1_A = f = 1_B f$  for any  $A \xrightarrow{f} B$ ;
  - (g)  $(fg)h = f(gh)$  whenever  $fg$  and  $gh$  are defined.

## Remark. (1.2)

- (a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call  $\mathcal{C}$  *small* (nice name?) if  $\text{ob } \mathcal{C}$  and  $\text{mor } \mathcal{C}$  are sets.
- (b) Some texts say  $fg$  means  $f$  followed by  $g$ , i.e.  $fg$  is defined iff  $\text{cod } f = \text{dom } g$ .
- (c) Note that a morphism  $f$  is an identity iff  $fg = g$  and  $hf = h$  whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

## Example. (1.3)

- (a) The category *set* has all sets as objects, and all functions between sets as morphisms.

Strictly speaking, morphisms  $A \rightarrow B$  are pairs  $(f, B)$  where  $f$  is a set-theoretic function. (?)

- (b) The category *Grp* has all groups as objects, group homomorphisms as morphisms.

Similarly, *Ring* is the category of rings, *Mod<sub>R</sub>* is the category of  $R$ -modules.

- (c) The category *Top* has all topological spaces as objects, and continuous functions as morphisms.

Similarly, *Unif* has all uniform spaces and uniformly continuous functions as morphisms, *Mf* has all manifolds and smooth maps correspondingly.

- (d) The category *Htpy* has the same objects as *Top*, but morphisms are homotopy classes of continuous functions. More generally, given  $\mathcal{C}$ , we call an equivalence relation  $\simeq$  on  $\text{mor } \mathcal{C}$  a *congruence* if  $f \simeq g \implies \text{dom } f = \text{dom } g$  and  $\text{cod } f = \text{cod } g$ , and  $f \simeq g \implies fh \simeq gh$  and  $kf \simeq kg$  whenever the composites are defined. Then we have a category  $\mathcal{C}/\simeq$  with the same objects as  $\mathcal{C}$ , but congruence classes as morphisms instead.

- (e) Given  $\mathcal{C}$ , the *opposite category*  $\mathcal{C}^{op}$  has the same objects and morphisms as  $\mathcal{C}$ , but  $\text{dom}$  and  $\text{cod}$  are interchanged, and  $fg$  in  $\mathcal{C}^{op}$  is  $gf$  in  $\mathcal{C}$ .

This leads to the *duality principle*: if  $P$  is a true statement about categories, so is the statement  $P^*$  obtained from  $P$  by reversing all arrows.

- (f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (nice abbreviation) with one object in which

every morphism is an isomorphism (i.e. for all  $f, \exists g$  s.t.  $fg$  and  $gf$  are identities).  
 (g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space  $X$ , the *fundamental groupoid*  $\pi(x)$  has all points of  $X$  as objects, and morphisms  $x \rightarrow y$  are homotopy classes  $rel\{0, 1\}$  of paths  $u : [0, 1] \rightarrow X$  with  $u(0) = x, u(1) = y$  (if you know how to prove that the fundamental group is a group, you can prove that  $\pi(x)$  is a groupoid).

(h) A *discrete cat* is one whose only morphism are identities.

A *preorder* is a cat  $\mathcal{C}$  in which, for any pair  $(A, B)$ ,  $\exists$  at most 1 morphism  $A \rightarrow B$ .

A small preorder is a set equipped with a binary relation which is reflexive and transitive.

In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category *Rel* has the same objects as *set*, but morphisms  $A \rightarrow B$  are arbitrary relations  $R \subseteq A \times B$ . Given  $R$  and  $S \subseteq B \times C$ , we define  $S \cdot R = \{(a, c) \in A \times C | (\exists b \in B)((a, b) \in R, (b, c) \in S)\}$ .

The identity  $1_A : A \rightarrow A$  is  $\{(a, a) | a \in A\}$ .

Similarly, the category *Part* are for sets and partial functions (i.e. relations s.t.  $(a, b) \in R$  and  $(a, b') \in R \implies b = b'$ ).

(j) Let  $K$  be a field. The category *Mat<sub>K</sub>* has natural numbers as objects, and morphism  $n \rightarrow p$  are  $(p \times n)$  matrices with entries from  $K$ . Composition is matrix multiplication.

(k) We write *Cat* for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

**Definition.** (1.4)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of:

(a) a mapping  $A \rightarrow FA$  from  $\text{ob } \mathcal{C}$  to  $\text{ob } \mathcal{D}$ ;

(b) a mapping  $f \rightarrow Ff$  from  $\text{mor } \mathcal{C}$  to  $\text{mor } \mathcal{D}$ ,

such that  $\text{dom}(Ff) = F(\text{dom } f)$ ,  $\text{cod}(Ff) = F(\text{cod } f)$ ,  $1_{FA} = F(1_A)$ , and  $(Ff)(Fg) = F(fg)$  whenever  $fg$  is defined.

**Example.** (1.5)

(a) We have *forgetful functors*  $U : Gp \rightarrow Set, Ring \rightarrow Set, Top \rightarrow Set, Ring \rightarrow AbGp$  (forget  $\times$ ),  $Ring \rightarrow Mon$  (Category of all monoids) (forget  $+$ ).

(b) Given a set  $A$ , the free group  $F_A$  has the property:

Given any group  $G$  and any function  $A \xrightarrow{f} UG$  (?), there's a unique homomorphism  $F_A \xrightarrow{Ff} G$  extending  $f$ .

$F$  is a functor  $Set \rightarrow Gp$ : given  $A \xrightarrow{f} B$ , we define  $Ff$  to be the unique homomorphism extending  $A \xrightarrow{f} B \hookrightarrow UFB$ . Functoriality follows from uniqueness given  $B \xrightarrow{f} C$ .  $F(gf)$  and  $(Fg)(Ff)$  are both homomorphisms extending  $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow UFC$ .

(c) Given a set  $A$ , we write  $PA$  for the set of all subsets of  $A$ .

We can make  $P$  into a functor  $Set \rightarrow Set$ , given  $A \xrightarrow{f} B$ , we defined  $Pf(A') = \{f(a) | a \in A'\}$  for  $A' \subseteq A$ .

But we also have a functor  $P^* : Set \rightarrow Set^{op}$  defined on objects by  $P$ , but  $P^*f(B') = \{a \in A | f(a) \in B'\}$  for  $B' \subseteq B$ .

By a *contravariant* functor  $\mathcal{C} \rightarrow \mathcal{D}$ , we mean a functor  $\mathcal{C} \rightarrow \mathcal{D}^{op}$  (or  $\mathcal{C}^{op} \rightarrow \mathcal{D}$ ). A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

(d) Let  $K$  be a field. We have a functor  $*$  :  $Mod_K \rightarrow Mod_K^{op}$  defined by  $V^* = \{ \text{linear maps } V \rightarrow K \}$ , and if  $V \xrightarrow{f} W$ ,  $f^*(\theta : W \rightarrow K) = \theta f$ .

(e) We have a functor  $op : Cat \rightarrow Cat$ , which is the identity on morphisms (note that this is a covariant).

(f) A functor between monoids is a monoid homomorphism.

(g) A functor between posets is an order-preserving map.

(h) Let  $G$  be a group. A functor  $F \circ G \rightarrow Set$  consists of a set  $A = F*$  together with an action of  $G$  on  $A$ , i.e. a *permutation representation* of  $G$ .

Similarly, a functor  $G \rightarrow Mod_K$  is a  $K$ -linear representation of  $G$ . (i) The construction of the fundamental group  $\pi(X, X)$  of a space  $X$  with basepoint  $X$  is a functor  $Top_* \rightarrow Gp$  where  $Top_*$  is the category of spaces with a chosen basepoint.

Similarly, the fundamental groupoid is a functor  $Top \rightarrow Gpd$ , where  $Gpd$  is the category of groupoids and functors between them.

**Definition.** (1.6)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$  (why two arrows?) two functors.

A *natural transformation*  $\alpha : F \rightarrow G$  consists of an assignment  $A \mapsto \alpha_A$  from  $ob \mathcal{C}$  to  $mor \mathcal{D}$ , such that  $dom_{\alpha_A} = FA$  and  $cod_{\alpha_A} = GA$  for all  $A$ , and for all

$A \xrightarrow{f} B$  in  $\mathcal{C}$ , the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e.  $\alpha_B(Ff) = (Gf)_{\alpha_A}$ ).

(1.3) (l) Given categories  $\mathcal{C}$  and  $\mathcal{D}$ , we write  $[\mathcal{C}, \mathcal{D}]$  for the category whose objects are functors  $\mathcal{C} \rightarrow \mathcal{D}$  and whose morphisms are natural transformations.

**Example.** (1.7)

(a) Let  $K$  be a field,  $V$  a vector space over  $K$ . There is a linear map  $\alpha_V : V \rightarrow V^{**}$  given by  $\alpha_V(v)\theta = \theta(v)$  for  $\theta \in V^*$ .

This is the  $V$ -component of a natural transformation  $1_{Mod_K} \rightarrow ** : Mod_K \rightarrow Mod_K$ .

(b) For any set  $A$ , we have a mapping  $\sigma_A : A \rightarrow PA$  sending  $a$  to  $\{a\}$ . If  $f : A \rightarrow B$ , then  $Pf\{a\} = \{f(a)\}$ . So  $\sigma$  is a natural transformation  $1_{Set} \rightarrow P$ .

(c) Let  $F : Set \rightarrow Gp$  be the free group functor (1.5(b)), and  $U : Gp \rightarrow Set$  the forgetful functor. The inclusions  $A \rightarrow UFA$  form a natural transformation  $1_{Set} \rightarrow UF$ .

(d) Let  $G, H$  be groups and  $f, g : G \rightrightarrows H$  be two homomorphisms. A natural transformation  $\alpha : f \rightarrow g$  corresponds to an element  $h = \alpha_*$  of  $H$ , s.t.  $hf(x) \rightarrow g(x)h$  for all  $x \in G$  or equivalently  $f(x) = h^{-1}g(x)h$ , i.e.  $f$  and  $g$  are conjugate group homomorphisms.

(e) Let  $A$  and  $B$  be two  $G$ -sets, regarded as functors:  $G \rightrightarrows Set$ . A natural transformation  $A \rightarrow B$  is a function  $f$  satisfying  $f(g \cdot a) = g \cdot f(a)$  for all  $a \in A$ , i.e. a  $G$ -equivariant map.

**Lemma.** (1.8)

Let  $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$  be two functors, and  $\alpha : F \rightarrow G$  a natural transforms. Then  $\alpha$  is an isomorphism in  $[\mathcal{C}, \mathcal{D}]$  iff each  $\alpha_A$  is an isomorphism in  $\mathcal{D}$ .

*Proof.* Forward is trivial (ok, I'll check this later). For backward, suppose each  $\alpha_A$  has an inverse  $\beta_A$ . Given  $f : A \rightarrow B$  in  $\mathcal{C}$ , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

□

commutes. But as  $\alpha$  is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

**Definition.** (1.9)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. By an *equivalence* between  $\mathcal{C}$  and  $\mathcal{D}$ , we mean a pair of functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  together with natural isomorphisms  $\alpha : 1_{\mathcal{C}} \rightarrow GF$  and  $\beta : FG \rightarrow 1_{\mathcal{D}}$ .

We write  $\mathcal{C} \cong \mathcal{D}$  if  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent.

We say a property  $P$  of categories is a *categorical property* if whenever  $\mathcal{C}$  has  $P$  and  $\mathcal{C} \cong \mathcal{D}$ , then  $\mathcal{D}$  has  $P$ .

For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

**Example.** (1.10)

(a) The category *Part* is equivalent to the category *Set*<sub>\*</sub> of pointed sets (and basepoint, pre(orders?) as functions):

- We define  $F : \text{Set}_* \rightarrow \text{Part}$  by  $F(A, a) = A \setminus \{a\}$ , and if  $f : (A, a) \rightarrow (B, b)$ , then  $Ff(x) = f(x)$  if  $f(x) \neq b$ , and undefined otherwise;
- and  $G : \text{Part} \rightarrow \text{Set}_*$  by  $G(A) = A^+ = A \cup \{A\}$ , and if  $f : A \rightarrow B$  is a partial function, we define  $Gf : A^+ \rightarrow B^+$  by  $Gf(x) = f(x)$  if  $x \in A$  and  $f(x)$  defined, and equals  $B$  otherwise.

The composite  $FG$  is the identity on *Part*, but  $GF$  is not the identity. However, there is an isomorphism  $(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$  sending  $a$  to  $A \setminus \{a\}$  and everything else to itself and this is natural.

Note that there can be no isomorphism from *Set*<sub>\*</sub> to *Part*, since *Part* has a 1-element isomorphism class  $\{\phi\}$  but *Set*<sub>\*</sub> doesn't.

(b) The category *fdMod*<sub>K</sub> of finite-dimensional vector spaces over  $K$  is equivalent to *fdMod*<sub>K</sub><sup>op</sup>, the functors in both directions are  $(-1)^*$  (???) and both isomorphisms are the natural transformations of 1.7(a).

(c) *fdMod*<sub>K</sub> is also equivalent to *Mat*<sub>K</sub> (1.3(j)).

We define  $F : \text{Mat}_K \rightarrow \text{fdMod}_K$  by  $F(n) = K^n$ , and  $F(A)$  is the linear map represented by  $A$  w.r.t. the standard bases of  $K^n$  and  $K^p$ .

To define  $G : \text{fdMod}_K \rightarrow \text{Mat}_K$ , choose a basis for each finite dimensional vector space, and define  $G(V) = \dim V$ ,  $G(V \xrightarrow{f} W)$  to be the matrix representing  $f$  w.r.t. chosen bases.  $GF$  is the identity, provided we choose the standard

bases for the spaces  $K^n$ ;  $FG \neq 1$ , but the chosen bases give isomorphisms  $FG(V) = K^{\dim V} \rightarrow V$  for each  $V$ , which form a natural isomorphism.

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**Definition.** (1.11)

Let  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  be a functor.

(a) We say  $F$  is *faithful* if, given  $f, f' \in \text{mor } \mathcal{C}$  with  $\text{dom } f = \text{dom } f'$ ,  $\text{cod } f = \text{cod } f'$ , and  $Ff = Ff'$ , then  $f = f'$  (injectivity on morphisms. The name comes more from representation theory);

(b) We say  $F$  is *full* if, given  $FA \xrightarrow{g} FB$  in  $\mathcal{D}$ , there exists  $A \xrightarrow{f} B$  in  $\mathcal{C}$  with  $Ff = g$ . (this is something like surjective, but see below);

(c) We say  $F$  is *essentially surjective* if, for every  $B \in \text{ob } \mathcal{D}$ , there exists  $A \in \text{ob } \mathcal{C}$  and isomorphism  $FA \rightarrow B$  in  $\mathcal{D}$ .

We say a subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is full if the inclusion  $\mathcal{C}' \rightarrow \mathcal{C}$  is a full functor. For example,  $Gp$  is a full subcategory of  $Mon$  (the category of all monoids), but  $Mon$  is not a full subcategory of the category  $SGp$  of semigroups.

**Lemma.** (1.12)

Assuming the axiom of choice, a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is part of an equivalence  $\mathcal{C} \simeq \mathcal{D}$  (see (1.9) for what an equivalence is. I think here it means one of the functors  $F, G$ ) if it's full, faithful, and essentially surjective.

*Proof.*  $\Rightarrow$ : Suppose given  $G, \alpha, \beta$  as in (1.9). Then for each  $B \in \text{ob } \mathcal{D}$ ,  $\beta_B$  is an isomorphism  $FGB \rightarrow B$ , so  $F$  is essentially surjective.

Given  $A \xrightarrow{f} B$  in  $\mathcal{C}$ , we can recover  $f$  from  $Ff$  as composite  $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B$ . Hence if  $A \xrightarrow{f'} B$  satisfies  $Ff = Ff'$ , then  $f = f'$ . So  $F$  is faithful;

Lastly, for fullness, given  $FA \xrightarrow{g} FB$ , define  $f$  to be the composite  $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$ . Then  $GFf = \alpha_B f \alpha_A^{-1}$ , which by construction is just  $Gg$ . But  $G$  is faithful for the same reason as  $f$ , so  $Ff = g$ .

$\Leftarrow$ : (need to find suitable  $G, \alpha, \beta$  for  $F$ .) For each  $B \in \text{ob } \mathcal{D}$ , choose  $GB \in \text{ob } \mathcal{C}$  and an isomorphism  $\beta_B : FGB \rightarrow B$  in  $\mathcal{D}$ . Given  $B \xrightarrow{g} B'$ , define  $Gg : GB \rightarrow GB'$  to be the unique morphism whose image under  $F$  is  $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$ .

Uniqueness implies functoriality (check what this means – think it appeared somewhere before): given  $B' \xrightarrow{g'} B''$ ,  $(Gg')(Gg)$  and  $G(g'g)$  have the same image under  $F$ , so they are equal.

By construction,  $\beta$  is a natural transformation  $FG \rightarrow 1_{\mathcal{D}}$ .

Given  $A \in \text{ob } \mathcal{C}$ , define  $\alpha_A : A \rightarrow GFA$  to be the unique morphism whose image under  $F$  is  $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$ .  $\alpha_A$  is an isomorphism, since  $\beta_{FA}$  also has a unique pre-image under  $F$ . And  $\alpha$  is a natural transformation, since any naturality square for  $\alpha$  (oh I think the lecture means the commutative square when we defined natural transformation. check) is mapped by  $F$  to a commutative square, and  $F$  is faithful.  $\square$



**Definition.** (1.13)

By a *skeleton* of a category, we mean a full subcategory  $\mathcal{C}_0$  containing one object from each isomorphism class. We say  $\mathcal{C}$  is *skeletal* if it's a skeleton of itself.

For example,  $Mat_K$  is a skeletal, and the image of  $F : Mat_K \rightarrow fdMod_K$  of 1.10(c) is a skeleton of  $fdMod_K$ .

(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

**Definition.** (1.14)

Let  $A \xrightarrow{f} B$  be a morphism in  $\mathcal{C}$ .

(a) We say  $f$  is a *monomorphism* (or  $f$  is *monic*) if, given any pair  $C \xrightleftharpoons[h]{g} A$ ,

$fg = fh$  implies  $g = h$ .

(b) We say  $f$  is an *epimorphism* (or *epic*) if it's a monomorphism in  $\mathcal{C}^{op}$ , i.e. if  $gf = hf$  implies  $g = h$ .

We denote monomorphisms by  $A \xrightarrow{f} B$ , and epimorphisms by  $A \xrightarrow{f} B$ .

Any isomorphism is monic and epic: more generally, if  $f$  has a left inverse (i.e.  $\exists g$  s.t.  $gf$  is an identity), then it's monic. We call such monomorphisms *split*.

We say  $\mathcal{C}$  is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

**Example.** (1.15)

(a) As usual we consider *Set* first. In *Set*, monomorphisms correspond to injections ( $\Leftarrow$  is easy (ok); for  $\Rightarrow$ , take  $C = 1 = \{*\}$ ), and epimorphisms correspond to surjections ( $\Leftarrow$  is easy; for  $\Rightarrow$ , use morphisms  $B \rightrightarrows 2 = \{0, 1\}$ ). So *Set* is balanced.

(b) In *Gp*, monomorphisms again correspond to injections (for  $\Rightarrow$  use homomorphisms  $\mathbb{Z} \rightarrow A$ ); epimorphisms again correspond to surjections ( $\Rightarrow$  use free products with amalgamation – this is a non-trivial fact about groups, read more if free). So *Gp* is also balanced.

(c) In *Rng* (obvious notation), monomorphisms correspond to injections (proof is much like for *Gp*). However, not all epimorphisms are surjective. For example

the inclusion  $\mathbb{Z} \rightarrow \mathbb{Q}$  is an epimorphism, since if  $\mathbb{Q} \xrightleftharpoons[g]{f} R$  agree on all integers,

they agree everywhere. So *Rng* is not balanced.

(d) One final example is *Top*. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to *Set* (check). However, *Top* is not balanced since a continuous bijection need not have continuous inverse.

## 2 The Yoneda Lemma

Let's not start on the content this lecture. Why are we talking about one single lemma in a chapter? Well it's not really a lemma, even worse we're not even sure it should be credited to Yoneda. (wiki seems to have a more complete story on this)