Stochastic Financial Models

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0 Motivation

An investor needs a certain quantity of a share (or currency, good, etc), however, not right now ($t = 0$) but at a later time ($t = 1$). The price of the share $S(w)$ at time $t = 1$ is random, but already today one has to make calculation with it so there is risk. For example, 500USD $\approx$ 370 GBP today. What about in one year?

Possible solution: purchase a financial derivative such as:
• forward contract: right and obligation to buy a share at time $t = 1$ for a strike price K specified at time $t = 0$. Its value at time $t = 1$ should be $H(w) = S(w) - K$ is positive if $S(w) > K$, and negative i f $S(w) < K$;
• call-option: the right, but no obligation, to do the same thing as above. Its value at $t = 1$ should be $H(w) = (S(w) - K)^+$, i.e. $S(w) - K$ if that is positive, and 0 otherwise (no obligation to exercise the option).

One question: what is the fair price for such a derivative?

1) Classical approach: Regard payoff $H(w)$ as lottery, modelled by a random variable on $(\Omega, F, \mathbb{P})$, where $\mathbb{P}$ is the 'objective probability measure'.
a) very classical: fair price = expected discounted payoff = $\mathbb{E}[\frac{H}{1+r}]$, where r is the interest rate for funds/loans from $t = 0$ to $t = 1$.
Assumption: both interest rates are the same for large investors.
b) classical: subjective assessment of the risk (by the seller of H) by utility functions.

2) More modern approach: suppose the primary risk (share) can only be traded in $t = 0$ and $t = 1$.
Hedging strategy: θ^1 = number of shares held between $t = 0$ and $t = 1$;
θ^0 = balance on bank account with interest rate r.
Here we allow θ^1 to be either positive or negative (i.e. allow short-selling).

Price at $t = 0$: $\theta^0 + \theta^1\pi^1 = V_0$, where π^1 is the price of one share at $t = 0$.
The value of this portfolio at $t = 1$: $\theta^0(1 + r) + \theta^1 S(w) = V(w)$.

Requirement: value of derivative = value of strategy, $H(w) = V(w)$ for all $w \in \Omega$.

For example, for forward contract: $S(w) - K = V(w) = \theta^0(1 + r) + \theta^1 S(w)$, we should choose $\theta^1 = 1, \theta^0 = \frac{-K}{1+r}$, so $V_0 = \pi^1 - \frac{K}{1+r}$. The seller of H has no risk if he uses this strategy(?).

Even more, $\pi(H) = V_0$ is the unique fair price for the forward contract. Any other price $\tilde{\pi} \neq V_0$ would lead to *arbitrage*: a riskless opportunity to make profit, which should be excluded). For example, if $\tilde{\pi} > V_0$, at $t = 0$ sell forward for $\tilde{\pi}$ and buy the strategy for V_0. Then in $t = 1$ deliver share and repay the loan. We gain a pure profit at $t = 1$: $(\tilde{\pi} - V_0)(1 + r) > 0$, i.e. arbitrage.

Questions: how to characterize arbitrage-free market? How to determine fair prices of options and derivatives?

1 Utility and mean variance

The market is interaction of agents trading goods. Individual agents have preferences over different contingent(?) claims (=specified random payment). Agents' preferences are expressed by an expected utility representation. Y is preferred to X means $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$ with utility function $U : \mathbb{R} \to [-\infty, \infty)$ which is non-decreasing. We assume U to be concave, in the sense that we expect agents to dislike risks.

Definition. (1.1) A function $U : \mathbb{R} \to [-\infty, \infty)$ is *concave* if $\forall p \in [0,1]$, $pU(x) + (1-p)U(y) \leq U(px + (1-p)y)$. Let $P(U) = \{x : U(x) > -\infty\}$.