

Category Theory

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0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

1 Definitions and examples

Definition. (1.1)

A category \mathcal{C} consists of:

- (a) a collection $\text{ob } \mathcal{C}$ of *objects* A, B, C ;
- (b) a collection $\text{mor } \mathcal{C}$ of *morphisms* f, g, h ;
- (c) two operations domain, codomain assigning to each $f \in \text{mor } \mathcal{C}$ a pair of objects, its *domain* and *codomain*; we write $A \xrightarrow{f} B$ to mean f is a morphism and $\text{dom } f = A, \text{cod } f = B$;
- (d) an operation assigning to each $A \in \text{ob } \mathcal{C}$ a morphism $A \xrightarrow{1_A} A$;
- (e) a partial binary operation $(f, g) \rightarrow fg$ on morphisms, such that fg is defined iff $\text{dom } f = \text{cod } g$, and $\text{dom}(fg) = \text{dom } g$, $\text{cod}(fg) = \text{cod}(f)$ if fg is defined, satisfying:
- (f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$;
- (g) $(fg)h = f(gh)$ whenever fg and gh are defined.

Remark. (1.2)

- (a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call \mathcal{C} *small* if $\text{ob } \mathcal{C}$ and $\text{mor } \mathcal{C}$ are sets.
- (b) Some texts say fg means f followed by g , i.e. fg is defined iff $\text{cod } f = \text{dom } g$.
- (c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

Example. (1.3)

- (a) The category **Set** has all sets as objects, and all functions between sets as morphisms.

Strictly speaking, morphisms $A \rightarrow B$ are pairs (f, B) where f is a set-theoretic function. (See part II logic and sets)

- (b) The category **Gp** has all groups as objects, group homomorphisms as morphisms.

Similarly, **Ring** is the category of rings, **Mod_R** is the category of R -modules.

- (c) The category **Top** has all topological spaces as objects, and continuous functions as morphisms.

Similarly, **Unif** has all uniform spaces and uniformly continuous functions as morphisms, **Mf** has all manifolds and smooth maps correspondingly.

- (d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given \mathcal{C} , we call an equivalence relation \simeq on $\text{mor } \mathcal{C}$ a *congruence* if $f \simeq g \implies \text{dom } f = \text{dom } g$ and $\text{cod } f = \text{cod } g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category \mathcal{C}/\simeq with the same objects as \mathcal{C} , but congruence classes as morphisms instead.

- (e) Given \mathcal{C} , the *opposite category* \mathcal{C}^{op} has the same objects and morphisms as \mathcal{C} , but dom and cod are interchanged, and fg in \mathcal{C}^{op} is gf in \mathcal{C} .

This leads to the *duality principle*: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.

- (f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (\boxtimes) with one object in which every morphism is an isomorphism (i.e. for all $f, \exists g$ s.t. fg and gf are identities).

(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space X , the *fundamental groupoid* $\pi(x)$ has all points of X as objects, and morphisms $x \rightarrow y$ are homotopy classes $rel\{0, 1\}$ of paths $u : [0, 1] \rightarrow X$ with $u(0) = x$, $u(1) = y$ (if you know how to prove that the fundamental group is a group, you can prove that $\pi(x)$ is a groupoid).

(h) A *discrete cat* is one whose only morphism are identities.

A *preorder* is a cat \mathcal{C} in which, for any pair (A, B) , \exists at most 1 morphism $A \rightarrow B$.

A small preorder is a set equipped with a binary relation which is reflexive and transitive.

In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category **Rel** has the same objects as *set*, but morphisms $A \rightarrow B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define $S \cdot R = \{(a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R, (b, c) \in S)\}$.

The identity $1_A : A \rightarrow A$ is $\{(a, a) \mid a \in A\}$.

Similarly, the category **Part** are for sets and partial functions (i.e. relations s.t. $(a, b) \in R$ and $(a, b') \in R \implies b = b'$).

(j) Let K be a field. The category **Mat_K** has natural numbers as objects, and morphism $n \rightarrow p$ are $(p \times n)$ matrices with entries from K . Composition is matrix multiplication.

(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

Definition. (1.4)

Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

(a) a mapping $A \mapsto FA$ from $\text{ob } \mathcal{C}$ to $\text{ob } \mathcal{D}$;

(b) a mapping $f \mapsto Ff$ from $\text{mor } \mathcal{C}$ to $\text{mor } \mathcal{D}$,

such that $\text{dom}(Ff) = F(\text{dom } f)$, $\text{cod}(Ff) = F(\text{cod } f)$, $1_{FA} = F(1_A)$, and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Example. (1.5)

(a) We have *forgetful functors* $U : \mathbf{Gp} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{Set}, \mathbf{Top} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{AbGp}$ (forget \times), $\mathbf{Ring} \rightarrow \mathbf{Mon}$ (Category of all monoids) (forget $+$).

(b) Given a set A , the free group FA has the property:

Given any group G and any function $A \xrightarrow{f} UG$ (?), there's a unique homomorphism $FA \xrightarrow{\tilde{f}} G$ extending f . Here F is a functor $\mathbf{Set} \rightarrow \mathbf{Gp}$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow UFB$.

Functoriality follows from uniqueness given $B \xrightarrow{f} C$. $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC$.

(c) Given a set A , we write PA for the set of all subsets of A .

We can make P into a functor $\mathbf{Set} \rightarrow \mathbf{Set}$, given $A \xrightarrow{f} B$, we defined $Pf(A') = \{f(a) \mid a \in A'\}$ for $A' \subseteq A$.

But we also have a functor $P^* : \mathbf{Set} \rightarrow \mathbf{Set}^{op}$ defined on objects by P , but $P^*f(B') = \{a \in A \mid f(a) \in B'\}$ for $B' \subseteq B$.

By a *contravariant* functor $\mathcal{C} \rightarrow \mathcal{D}$, we mean a functor $\mathcal{C} \rightarrow \mathcal{D}^{op}$ (or $\mathcal{C}^{op} \rightarrow \mathcal{D}$).

A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

- (d) Let K be a field. We have a functor $*$: $\mathbf{Mod}_K \rightarrow \mathbf{Mod}_K^{op}$ defined by $V^* = \{ \text{linear maps } V \rightarrow K \}$, and if $V \xrightarrow{f} W$, $f^*(\theta : W \rightarrow K) = \theta f$.
- (e) We have a functor op : $\mathbf{Cat} \rightarrow \mathbf{Cat}$, which is the identity on morphisms (note that this is a covariant).
- (f) A functor between monoids is a monoid homomorphism.
- (g) A functor between posets is an order-preserving map.
- (h) Let G be a group. A functor $F \circ G \rightarrow \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A , i.e. a *permutation representation* of G . Similarly, a functor $G \rightarrow \mathbf{Mod}_K$ is a K -linear representation of G .
- (i) The construction of the fundamental group $\pi(X, X)$ of a space X with basepoint X is a functor $\mathbf{Top}_* \rightarrow \mathbf{Gp}$ where \mathbf{Top}_* is the category of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor $\mathbf{Top} \rightarrow \mathbf{Gpd}$, where \mathbf{Gpd} is the category of groupoids and functors between them.

Definition. (1.6)

Let \mathcal{C} and \mathcal{D} be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ (why two arrows?) two functors. A *natural transformation* $\alpha : F \rightarrow G$ consists of an assignment $A \rightarrow \alpha_A$ from $\text{ob } \mathcal{C}$ to $\text{mor } \mathcal{D}$ (think about this), such that $\text{dom}_{\alpha_A} = FA$ and $\text{cod}_{\alpha_A} = GA$ for all A , and for all $A \xrightarrow{f} B$ in \mathcal{C} , the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)_{\alpha_A}$).

(1.3) (l) Given categories \mathcal{C} and \mathcal{D} , we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \rightarrow \mathcal{D}$ and whose morphisms are natural transformations.

Example. (1.7)

- (a) Let K be a field, V a vector space over K . There is a linear map $\alpha_V : V \rightarrow V^{**}$ given by $\alpha_V(v)\theta = \theta(v)$ for $\theta \in V^*$. This is the V -component of a natural transformation $1_{\mathbf{Mod}_K} \rightarrow ** : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K$.
- (b) For any set A , we have a mapping $\sigma_A : A \rightarrow PA$ sending a to $\{a\}$. If $f : A \rightarrow B$, then $Pf\{a\} = \{f(a)\}$. So σ is a natural transformation $1_{\mathbf{Set}} \rightarrow P$.
- (c) Let $F : \mathbf{Set} \rightarrow \mathbf{Gp}$ be the free group functor (1.5(b)), and $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ the forgetful functor. The inclusions $A \rightarrow UFA$ form a natural transformation $1_{\mathbf{Set}} \rightarrow UF$.
- (d) Let G, H be groups and $f, g : G \rightrightarrows H$ be two homomorphisms. A natural transformation $\alpha : f \rightarrow g$ corresponds to an element $h = \alpha_*$ of H , s.t. $hf(x) \rightarrow g(x)h$ for all $x \in G$ or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
- (e) Let A and B be two G -sets, regarded as functors: $G \rightrightarrows \mathbf{Set}$. A natural transformation $A \rightarrow B$ is a function f satisfying $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$, i.e. a G -equivariant map.

Lemma. (1.8)

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors, and $\alpha : F \rightarrow G$ a natural transformation. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each α_A is an isomorphism in \mathcal{D} .

Proof. Forward is trivial (ok, I'll check this later). For backward, suppose each α_A has an inverse β_A . Given $f : A \rightarrow B$ in \mathcal{C} , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

□

commutes. But as α is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

So β is a natural transformation as well.

Definition. (1.9)

Let \mathcal{C} and \mathcal{D} be categories. By an *equivalence* between \mathcal{C} and \mathcal{D} , we mean a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$ and $\beta : FG \rightarrow 1_{\mathcal{D}}$.

We write $\mathcal{C} \cong \mathcal{D}$ if \mathcal{C} and \mathcal{D} are equivalent.

We say a property P of categories is a *categorical property* if whenever \mathcal{C} has P and $\mathcal{C} \cong \mathcal{D}$, then \mathcal{D} has P .

For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

Example. (1.10)

(a) The category **Part** is equivalent to the category **Set**_{*} of pointed sets (and basepoint preserving functions (as morphisms)):

- We define $F : \mathbf{Set}_* \rightarrow \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$, and if $f : (A, a) \rightarrow (B, b)$, then $Ff(x) = f(x)$ if $f(x) \neq b$, and undefined otherwise;
- and $G : \mathbf{Part} \rightarrow \mathbf{Set}_*$ by $G(A) = A^+ = (A \cup \{A\}, A)$, and if $f : A \rightarrow B$ is a partial function, we define $Gf : A^+ \rightarrow B^+$ by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ defined, and equals B otherwise.

The composite FG is the identity on **Part**, but GF is not the identity. However, there is an isomorphism $(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$ sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.

Note that there can be no isomorphism from **Set**_{*} to **Part**, since **Part** has a 1-element isomorphism class $\{\phi\}$ but **Set**_{*} doesn't.

(So we see that equivalent categories can be non-isomorphic. According to a [post](#) on SO, this usually happens when there are multiple copies of the *same* thing in one but not the other. However, we can't generally *discard obsolete copies* in one as that generally requires AC and is not a very useful thing to do anyway – In short, *identifying isomorphic objects is often an extremely bad idea.*)

(b) The category **fdMod**_K of finite-dimensional vector spaces over K is equivalent to **fdMod**_K^{op}, the functors in both directions are $*$ (the dual operator) and both isomorphisms are the natural transformations of 1.7(a) (double dual).

(c) **fdMod**_K is also equivalent to **Mat**_K (1.3(j)):

We define $F : \mathbf{Mat}_K \rightarrow \mathbf{fdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A w.r.t. the standard bases of K^n and K^p .

To define $G : \mathbf{fdMod}_K \rightarrow \mathbf{Mat}_K$, choose a basis for each finite dimensional vector

space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ to be the matrix representing f w.r.t. chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n ; $FG \neq 1$, but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \rightarrow V$ for each V , which form a natural isomorphism.

—Lecture 4—

Definition. (1.11)

Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor.

(a) We say F is *faithful* if, given $f, f' \in \text{mor } \mathcal{C}$ with $\text{dom } f = \text{dom } f'$, $\text{cod } f = \text{cod } f'$, and $Ff = Ff'$, then $f = f'$ (injectivity on morphisms. The name comes more from representation theory);

(b) We say F is *full* if, given $FA \xrightarrow{g} FB$ in \mathcal{D} , there exists $A \xrightarrow{f} B$ in \mathcal{C} with $Ff = g$. (this is something like surjectivity on morphisms, but see below);

(c) We say F is *essentially surjective* if, for every $B \in \text{ob } \mathcal{D}$, there exists $A \in \text{ob } \mathcal{C}$ and isomorphism $FA \rightarrow B$ in \mathcal{D} .

We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *full* if the inclusion $\mathcal{C}' \rightarrow \mathcal{C}$ is a full functor (basically, if the objects are kept, any morphism between them must be kept). For example, **Gp** is a full subcategory of **Mon** (the category of all monoids), but **Mon** is not a full subcategory of the category **SGp** of semigroups (consider e.g. the homomorphism that sends everything in (\mathbb{Z}, \cdot) to $(0, \cdot)$ (which is also a semigroup); but this doesn't preserve 1 so is not a morphism in **Mon**).

Lemma. (1.12)

Assuming the axiom of choice, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ if it's full, faithful, and essentially surjective.

Proof. \Rightarrow : Suppose given G, α, β as in (1.9). Then for each $B \in \text{ob } \mathcal{D}$, β_B is an isomorphism $FGB \rightarrow B$, so F is essentially surjective.

Given $A \xrightarrow{f} B$ in \mathcal{C} , we can recover f from Ff as composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B$. Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$. So F is faithful;

Lastly, for fullness, given $FA \xrightarrow{g} FB$, define f to be the composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$. Then $GFf = \alpha_B f \alpha_A^{-1}$, which by construction is just Gg . But G is faithful for the same reason as f , so $Ff = g$.

\Leftarrow : (need to find suitable G, α, β for F .) For each $B \in \text{ob } \mathcal{D}$, choose $GB \in \text{ob } \mathcal{C}$ and an isomorphism $\beta_B : FGB \rightarrow B$ in \mathcal{D} . Given $B \xrightarrow{g} B'$, define $Gg : GB \rightarrow GB'$ to be the unique morphism whose image under F is $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$.

Uniqueness implies functoriality: given $B' \xrightarrow{g'} B''$, $(Gg')(Gg)$ and $G(g'g)$ have the same image under F , so they are equal.

By construction, β is a natural transformation $FG \rightarrow 1_{\mathcal{D}}$.

Given $A \in \text{ob } \mathcal{C}$, define $\alpha_A : A \rightarrow GFA$ to be the unique morphism whose image under F is $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$. α_A is an isomorphism, since β_{FA} also has a unique pre-image under F . And α is a natural transformation, since any naturality

square for α (the commutative square when we defined natural transformation) is mapped by F to a commutative square, and F is faithful. \square

Definition. (1.13)

By a *skeleton* of a category, we mean a full subcategory \mathcal{C}_0 containing one object from each isomorphism class. We say \mathcal{C} is *skeletal* if it's a skeleton of itself.

For example, $\mathbf{Mat}_{\mathbf{K}}$ is a skeletal, and the image of $F : \mathbf{Mat}_{\mathbf{K}} \rightarrow \mathbf{fdMod}_{\mathbf{K}}$ of 1.10(c) is a skeleton of $\mathbf{fdMod}_{\mathbf{K}}$.

(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

Definition. (1.14)

Let $A \xrightarrow{f} B$ be a morphism in \mathcal{C} .

(a) We say f is a *monomorphism* (or f is *monic*) if, given any pair $C \rightrightarrows_h^g A$, $fg = fh$ implies $g = h$.

(b) We say f is an *epimorphism* (or *epic*) if it's a monomorphism in \mathcal{C}^{op} , i.e. if $gf = hf$ implies $g = h$.

We denote monomorphisms by $A \xrightarrow{f} B$, and epimorphisms by $A \xrightarrow{f} B$.

Any isomorphism is monic and epic: more generally, if f has a left inverse (i.e. $\exists g$ s.t. gf is an identity), then it's monic. We call such monomorphisms *split*.

We say \mathcal{C} is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

Example. (1.15)

(a) As usual we consider **Set** first. In **Set**, monomorphisms correspond to injections (\Leftarrow is easy (ok); for \Rightarrow , take $C \rightrightarrows 1 = \{*\}$), and epimorphisms correspond to surjections (\Leftarrow is easy; for \Rightarrow , use morphisms $B \rightrightarrows 2 = \{0, 1\}$). So **Set** is balanced.

(b) In **Gp**, monomorphisms again correspond to injections (for \Rightarrow use homomorphisms $\mathbb{Z} \rightarrow A$); epimorphisms again correspond to surjections (\Rightarrow use **free products with amalgamation** – this is a non-trivial fact about groups, read more if free). So **Gp** is also balanced.

(c) In **Rng** (obvious notation), monomorphisms correspond to injections (proof is much like for **Gp**). However, not all epimorphisms are surjective. For example

the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \rightrightarrows_g^f R$ (any ring) agree on all integers, they agree everywhere. So **Rng** is not balanced.

(d) One final example is **Top**. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to **Set** (check). However, **Top** is not balanced since a continuous bijection need not have continuous inverse.

2 The Yoneda Lemma

—Lecture 5—

Definition. (2.1)

We say a category \mathcal{C} is *locally small* if, for any two objects A, B , the morphisms $A \rightarrow B$ in \mathcal{C} form a set $\mathcal{C}(A, B)$.

If we fix A and let B vary, the assignment $B \rightarrow \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathcal{C}(A, f)$ is the mapping $g \rightarrow fg$ for all $g \in \mathcal{C}(B, C)$. Similarly, $A \rightarrow \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ (for $A \xrightarrow{f} C \in \text{mor } \mathcal{C}^{op}$, maps $g \rightarrow gf$).

Lemma. (2.2)

- (i) Let \mathcal{C} be a locally small category, $A \in \text{ob } \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathbf{Set}$ a functor. Then natural transformations $\mathcal{C}(A, -) \rightarrow F$ are in bijection with elements of FA ;
- (ii) Moreover, this bijection is natural in both A and F .

Proof. (i) Given $\alpha : \mathcal{C}(A, -) \rightarrow F$, we define $\Phi(x) = \alpha_A(1_A) \in FA$. Given $x \in FA$, we define $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$ by $\Psi(x)_B(A \xrightarrow{f} B) = Ff(x) \in FB$. $\Psi(x)$ is natural: given $g : B \rightarrow C$, we have

$$\begin{aligned} \Psi(x)_C \mathcal{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x), \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x), \\ \Phi\Psi(x) &= \Psi(x)_A(1_A) = F(1_A)(x) = x \end{aligned}$$

Given α ,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f)\Psi(\alpha_A(1_A))_B(f) &= Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathcal{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

So $\Psi\Phi(\alpha) = \alpha$. □

Corollary. (2.3)

The assignment $A \rightarrow \mathcal{C}(A, -)$ defines a full and faithful functor $\mathcal{C}^{op} \rightarrow [\mathcal{C}, \mathbf{Set}]$.

Proof. Put $F = \mathcal{C}(B, -)$ in 2.2(i): we get a bijection between $\mathcal{C}(B, A)$ and morphisms $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$ in $[\mathcal{C}, \mathbf{Set}]$. We need to verify this is functorial: but it sends $f : B \rightarrow A$ to the natural transformation $g \rightarrow gf$. So functoriality follows from associativity. □

We call this functor (or the functor $\mathcal{C} \rightarrow [\mathcal{C}^{op}, \mathbf{Set}]$ sending A to $\mathcal{C}(-, A)$) the *Yoneda embedding* of \mathcal{C} , and denote it by Y .

Now let's go back to prove 2.2(ii):

Proof. (ii) Suppose for the moment that \mathcal{C} is small, so that $[\mathcal{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$: one sends (A, F) to FA , and the

other is the composite: $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{op} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-, -)} \mathbf{Set}$. 2.2(ii) says that these are naturally isomorphic. We can translate this into an elementary statement, making sense even when \mathcal{C} isn't small. Given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, the two ways of producing an element of GB from a natural transformation $\beta : \mathcal{C}(A, -) \rightarrow F$ give the same result, namely

$$\alpha_B(Ff)\beta_A(1_A) = (Gf)\alpha_A\beta_A(1_A)$$

which is equal to $\alpha_B\beta_B(f)$. \square

Definition. (2.4)

We say a functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is *representable* if it's isomorphic to $\mathcal{C}(A, -)$ for some A . By a representation of F , we mean a pair (A, x) where $x \in FA$ is such that $\Psi(x)$ is an isomorphism.

We also call x a *universal element* of F .

Corollary. (2.5)

If (A, x) and (B, y) are both representations of F , then there's a unique isomorphism $f : A \rightarrow B$ such that $(Ff)(x) = y$.

Proof. Consider the composite $\mathcal{C}(B, -) \xrightarrow{\Psi(y)^{-1}} F \xrightarrow{\Psi(x)} \mathcal{C}(A, -)$. By (2.3) this is of the form $Y(f)$ for a unique isomorphism $f : A \rightarrow B$, and the diagram

$$\begin{array}{ccc} \mathcal{C}(B, -) & \xrightarrow{Y(f)} & \mathcal{C}(A, -) \\ & \searrow & \swarrow \\ \Psi(y) & & \Psi(x) \end{array}$$

F

commutes iff $(Ff)(x) = y$. \square

Example. (2.6)

(a) The forgetful functor $\mathbf{Gp} \rightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$, $\mathbf{Rng} \rightarrow \mathbf{Set}$ by $(\mathbb{Z}[X], X)$, and $\mathbf{Top} \rightarrow \mathbf{Set}$ by $(\{*\}, *)$.

(b) The functor $P^* : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$ is representable by $(\{0, 1\}, \{1\})$: this is the bijection between subsets and characteristic functions.

(c) Let G be a group. The unique (up to isomorphism) representable functor $G(*, -) : G \rightarrow \mathbf{Set}$ is the *Cayley representation* of G , i.e. the set $\cup G$ with G

acting by left multiplication.

(d) Let A and B be two objects of a small category \mathcal{C} . We have a functor $\mathcal{C}^{op} \rightarrow \mathbf{Set}$ sending C to $\mathcal{C}(C, A) \times \mathcal{C}(C, B)$. A representation of this, if it exists, is called a (categorical) *product* of A and B , and denoted $(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B))$.

This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$, there's a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.

Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top**, ..., they are *just* cartesian products, in posets they are binary meets (see sheet 1 Q1).

Dually, we have the notion of *coproduct* $(A + B, A \xrightarrow{\mu_1} A + B, B \xrightarrow{\mu_2} A + B)$. These also exist in many categories of interest.

—Lecture 6—

(f) (Lecturer didn't like (e) so jumped to (f) directly) Let $A \xrightarrow{f} B$ be morphisms in locally small category \mathcal{C} . We have a functor $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ defined by

$$F(C) = \{h \in \mathcal{C}(C, A) \mid fh = gh\}$$

A representation (see (2.4)) of F , if it exists, is called an *equalizer* of (f, g) : It consists of an object E and a morphism $E \xrightarrow{e} A$ s.t. $fe = ge$, and every h with $fh = gh$ factors uniquely (see proof of 2.9(i) which gives an insight of what this means) through e .

In **Set**, we take $E = \{x \in A \mid f(x) = g(x)\}$ and $e = \text{inclusion}$. Similar constructions work in **Gp**, **Rng**, **Top**, ...

Dually, we have the notion of *coequalizer*.

Remark. (2.7)

If e occurs as an equalizer, then it is a monomorphism, since any h factors through it in at most one way. We say a monomorphism is *regular* if it occurs as an equalizer.

Split monomorphisms are regular (cf sheet1 Q6(i)).

Note that regular epic monomorphisms are isomorphisms: if the equalizer e of (f, g) is epic, then $f = g$, so $e \cong 1_{\text{cod } e}$.

Definition. (2.8)

Let \mathcal{C} be a category, \mathcal{G} a class of objects of \mathcal{C} .

(a) We say \mathcal{G} is a *separating family* for \mathcal{C} , if given $A \xrightarrow{f} B$ such that $fh = gh$ for

all $G \xrightarrow{h} A$ with $G \in \mathcal{G}$, then $f = g$.

(i.e. the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$, are collectively faithful.)

(b) We say \mathcal{G} is a *detecting family* if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} B$ with $G \in \mathcal{G}$ factors uniquely through f , then f is an isomorphism.

If $\mathcal{G} = \{G\}$, we call G a *separator/detector*.

Lemma. (2.9)

(i) If \mathcal{C} is a balanced category, then any separating family is detecting.

(ii) If \mathcal{C} has equalizers, then any detecting family is separating.

Proof. (i) Suppose \mathcal{G} is separating and $A \xrightarrow{f} B$ satisfies the condition of 2.8(b). If $B \xrightarrow{g} C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is

epic.

Similarly if $D \begin{smallmatrix} k \\ \rightrightarrows \\ l \end{smallmatrix} A$ satisfy $fk = fl$, then $ky = ly$ for any $G \xrightarrow{y} D$, since both are factorizations of fky through f . So $k = l$, i.e. f is monic.

But \mathcal{C} is balanced. So f is an isomorphism.

(ii) Suppose \mathcal{G} is detecting and $A \begin{smallmatrix} f \\ \rightrightarrows \\ g \end{smallmatrix} B$ satisfies the condition of 2.8(a). Then the equalizer $E \xrightarrow{e} A$ of (f, g) is isomorphism, so $f = g$. \square

Example. (2.10)

(a) In $[\mathcal{C}, \mathbf{Set}]$, the family $\{\mathcal{C}(A, -) | A \in \text{ob } \mathcal{C}\}$ is both separating and detecting (just a restatement of Yoneda Lemma).

(b) In \mathbf{Set} , $1 = \{*\}$ (any one element set) is both a separator and a detector, since it represents the identity functor $\mathbf{Set} \rightarrow \mathbf{Set}$.

Similarly, \mathbb{Z} is both in \mathbf{Gp} , since it represents the forgetful functor $\mathbf{Gp} \rightarrow \mathbf{Set}$. Also, $2 = \{0, 1\}$ is a coseparator and a codetector in \mathbf{Set} , since it represents $P^* : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$.

(c) In \mathbf{Top} , $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$, but not a detector.

In fact, \mathbf{Top} has no detecting *set* of objects (note that this doesn't mean it has no detecting family).

For any infinite cardinal κ , let X be a discrete space of cardinality κ , and Y the same set with *co- $< \kappa$* topology, i.e. $F \subseteq Y$ is closed iff $F = Y$ or $\text{Card } F < \kappa$ (think about, e.g. cocountable topology, then this name makes sense).

The identity $X \rightarrow Y$ is continuous, but not a homeomorphism (topologically). So if $\{G_i | i \in I\}$ is any set of spaces, taking $\kappa > \text{Card } G_i$ for all i yields an example to show that the set is not detecting.

(d) (some Algebraic Topology stuff) Let \mathcal{C} be the category of pointed connected *CW*-complexes and homotopy classes of (basepoint-preserving) continuous mappings.

JHC Whitehead proved that $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \rightarrow \pi_n(Y)$ for all n , then it's an isomorphism in \mathcal{C} .

This says that $\{S^n | n \geq 1\}$ is a detecting set of \mathcal{C} .

But PJ Freyd showed there is no faithful functor $\mathcal{C} \rightarrow \mathbf{Set}$, so no separating *set*: if $\{G_i | i \in I\}$ were separating, then $x \rightarrow \coprod \mathcal{C}(G_i, x)$ (disjoint unions?) would be faithful.

Note that any functor of the form $\mathcal{C}(A, -)$ preserves monomorphisms, but they don't normally preserve epimorphisms.

Definition. (2.11)

We say an object P is *Projective* if, given

$$\begin{array}{c} P \\ \downarrow f \\ A \xrightarrow{e} B \end{array}$$

(recall the two head right arrow means epimorphisms) there exists $P \xrightarrow{g} A$ with $eg = f$.

(If \mathcal{C} is locally small, this says $\mathcal{C}(P, -)$ preserves epimorphisms).

Dually, an *injective* object of \mathcal{C} is a projective object of \mathcal{C}^{op} .

Given a class \mathcal{E} of epimorphisms, we say P is \mathcal{E} -projective if it satisfies the condition for all $e \in \mathcal{E}$.

Lemma. (2.12)

Representable functors are (pointwise)(?) projective in $[\mathcal{C}, \mathbf{Set}]$.

Proof. Suppose given

$$\begin{array}{c} \mathcal{C}(A, -) \\ \downarrow \beta \\ F \xrightarrow{\alpha} G \end{array}$$

where α is pointwise surjective. By Yoneda, β corresponds to some $y \in GA$, and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathcal{C}(A, -) \rightarrow F$ corresponds to x , then naturality of the Yoneda bijection yields $\alpha\gamma = \beta$. \square

—Lecture 7— First example class: Friday 26th October, 2pm MR3.

Lecture is happy to mark any question we hand in!

3 Adjunctions

Definition. (3.1)

Let \mathcal{C} and \mathcal{D} be two categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ two functors.

By an *adjunction* between F and G we mean a bijection between morphisms

$FA \xrightarrow{\hat{f}} B$ in \mathcal{D} and morphisms $A \xrightarrow{f} GB$ in \mathcal{C} , which is natural in A and B , i.e. given $A' \xrightarrow{g} A$ and $B \xrightarrow{h} B'$, we have $h\hat{f}(Fg) = \widehat{(Gh)}fg : FA' \rightarrow B'$.

We say F is *left adjoint* to G , and write $(F \dashv G)$.

Example. (3.2)

(a) The free functor $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$ is left adjoint to the forgetful functor $\mathbf{Gp} \xrightarrow{U} \mathbf{Set}$, since any function $f : A \rightarrow UB$ extends uniquely to a homomorphism $\hat{f} : FA \rightarrow B$.

Naturality in B is *easy* (lecturer says so), naturality in A follows from the definition of F as a functor.

(b) The forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ has a left adjoint D which equips any set with the discrete topology, *and* also a right adjoint I which equips a set A with the discrete (lecturer had *indiscrete* here?) topology $\{\phi, A\}$.

(c) The functor $\text{ob} : \mathbf{Cat} \rightarrow \mathbf{Set}$ (recall \mathbf{Cat} is the category of small categories) has a left adjoint D sending A to the *discrete* category with $\text{ob}(DA) = A$ and only identity morphisms, and a right adjoint I sending A to the category with $\text{ob}(IA) = A$ and one morphism $x \rightarrow y$ for each $(x, y) \in A \times A$. In this case D in turn has a left adjoint π_0 sending a small category \mathcal{C} to its set of *connected components*, i.e. the quotient of $\text{ob}\mathcal{C}$ by the smallest equivalence relation identifying $\text{dom } f$ with $\text{cod } f$ for all $f \in \text{mor } \mathcal{C}$.

(d) Let M be the monoid $\{1, e\}$ with $e^2 = e$. An object of $[M, \mathbf{Set}]$ is a pair (A, e) (the images of the functor?), where $e : A \rightarrow A$ satisfies $e^2 = e$.

We have a functor $G : [M, \mathbf{Set}] \rightarrow \mathbf{Set}$ sending (A, e) to $\{x \in A \mid e(x) = x\} = \{e(x) \mid x \in A\}$ and a functor $F : \mathbf{Set} \rightarrow [M, \mathbf{Set}]$ sending A to $(A, 1_A)$.

I claim $(F \dashv G \dashv F)$: given $f : (A, 1_A) \rightarrow (B, e)$, it must take values in $G(B, e)$, and any $g : (B, e) \rightarrow (A, 1_A)$ is determined by its values on the image of e .

(e) Let $\mathbf{1}$ be the discrete category with one object $*$. For any \mathcal{C} , there's a unique functor $\mathcal{C} \rightarrow \mathbf{1}$: a left adjoint for this picks out an *initial* object of \mathcal{C} , i.e. an object I s.t. there exists a unique $I \rightarrow A$ for each $A \in \text{ob } \mathcal{C}$.

Dually, a right adjoint for $\mathcal{C} \rightarrow \mathbf{1}$ corresponds to a *terminal* object of \mathcal{C} (think about what this means).

(f) Let $A \xrightarrow{f} B$ be a morphism in \mathbf{Set} . We can regard PA and PB as posets, and we have functors $PA \xrightleftharpoons[P^*f]{Pf} PB$.

I claim $(PF \dashv P^*f)$: we have $Pf(A') \subseteq B' \iff f(x) \in B'$ for all $x \in A' \iff A' \subseteq P^*f(B')$.

(g) (*Galois Connection*) Suppose given sets A, B and a relation $R \subseteq A \times B$. We define mappings $(-)^l, (-)^r$ between PA and PB by

$$S^r = \{y \in B \mid (\forall x \in S)((x, y) \in R)\} \text{ for } S \subseteq A$$

$$T^l = \{x \in A \mid (\forall y \in T)((x, y) \in R)\} \text{ for } T \subseteq B$$

The mappings are order-reserving (i.e. contravariant functors), and $T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l$.

We say $()^r$ and $()^l$ are *adjoint on the right*.

(h) Let's now consider, as a functor, $P^* : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$ is self-adjoint on the right, since functions $A \rightarrow PB$ correspond bijectively to subsets of $A \times B$, and hence to functions $B \rightarrow PA$.

Theorem. (3.3)

Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each $A \in \text{ob } \mathcal{C}$, where $(A \downarrow G)$ has objects pairs (B, f) with $A \xrightarrow{f} GB$, and morphisms $(B, f) \rightarrow (B', f')$ are morphisms $B \xrightarrow{g} B'$ such that

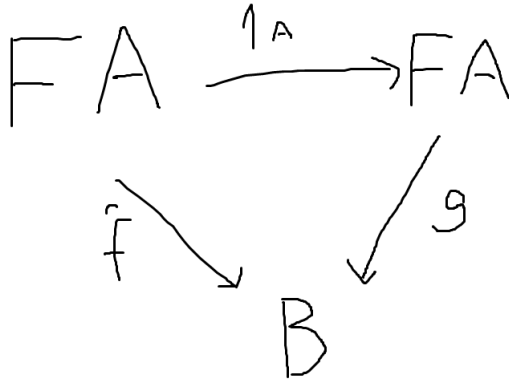
$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow f' & \swarrow Gg \\ & & GB' \end{array}$$

commutes.

Proof. Suppose given $(F \dashv G)$. Consider the morphism $\eta_A : A \rightarrow GFA$ correspond to $FA \xrightarrow{\eta} FA$. Then (FA, η_A) is an object of $(A \downarrow G)$. Moreover, given $g : FA \rightarrow B$ and $f : A \rightarrow GB$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow f & \swarrow Gg \\ & & GB \end{array}$$

commutes iff



commutes, i.e. $g = \hat{f}$.

So (FA, η_A) is initial in $(A \downarrow G)$.

Conversely, suppose given an initial object (FA, η_A) for each $(A \downarrow G)$. Given $A \xrightarrow{f} A'$, we define $Ff : FA \rightarrow FA'$ to be the unique morphism making

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & GFA \\
 \downarrow f & & \downarrow GFf \\
 A' & \xrightarrow{\eta_{A'}} & GFA'
 \end{array}$$

commute.

Functoriality follows from uniqueness: given $f' : A' \rightarrow A''$, $F(f'f)$ and $(Ff')(Ff)$ are both morphisms $(FA, \eta_A) \rightarrow (FA'', \eta_{A''} F'f)$ in $(A \downarrow G)$.

Note that we haven't finished: we still have to verify natural adjunctions. We'll finish off this next monday. \square