

Category Theory

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0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

1 Definitions and examples

Definition. (1.1)

A category \mathcal{C} consists of:

- (a) a collection $\text{ob } \mathcal{C}$ of *objects* A, B, C ;
- (b) a collection $\text{mor } \mathcal{C}$ of *morphisms* f, g, h ;
- (c) two operations domain, codomain assigning to each $f \in \text{mor } \mathcal{C}$ a pair of objects, its *domain* and *codomain*; we write $A \xrightarrow{f} B$ to mean f is a morphism and $\text{dom } f = A, \text{cod } f = B$;
- (d) an operation assigning to each $A \in \text{ob } \mathcal{C}$ a morphism $A \xrightarrow{1_A} A$;
- (e) a partial binary operation $(f, g) \rightarrow fg$ on morphisms, such that fg is defined iff $\text{dom } f = \text{cod } g$, and $\text{dom}(fg) = \text{dom } g$, $\text{cod}(fg) = \text{cod}(f)$ if fg is defined, satisfying:
- (f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$;
- (g) $(fg)h = f(gh)$ whenever fg and gh are defined.

Remark. (1.2)

- (a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call \mathcal{C} *small* if $\text{ob } \mathcal{C}$ and $\text{mor } \mathcal{C}$ are sets.
- (b) Some texts say fg means f followed by g , i.e. fg is defined iff $\text{cod } f = \text{dom } g$.
- (c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

Example. (1.3)

- (a) The category **Set** has all sets as objects, and all functions between sets as morphisms.

Strictly speaking, morphisms $A \rightarrow B$ are pairs (f, B) where f is a set-theoretic function. (See part II logic and sets)

- (b) The category **Gp** has all groups as objects, group homomorphisms as morphisms.

Similarly, **Ring** is the category of rings, **Mod_R** is the category of R -modules.

- (c) The category **Top** has all topological spaces as objects, and continuous functions as morphisms.

Similarly, **Unif** has all uniform spaces and uniformly continuous functions as morphisms, **Mf** has all manifolds and smooth maps correspondingly.

- (d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classes of continuous functions. More generally, given \mathcal{C} , we call an equivalence relation \simeq on $\text{mor } \mathcal{C}$ a *congruence* if $f \simeq g \implies \text{dom } f = \text{dom } g$ and $\text{cod } f = \text{cod } g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category \mathcal{C}/\simeq with the same objects as \mathcal{C} , but congruence classes as morphisms instead.

- (e) Given \mathcal{C} , the *opposite category* \mathcal{C}^{op} has the same objects and morphisms as \mathcal{C} , but dom and cod are interchanged, and fg in \mathcal{C}^{op} is gf in \mathcal{C} .

This leads to the *duality principle*: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.

- (f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (\boxtimes) with one object in which every morphism is an isomorphism (i.e. for all $f, \exists g$ s.t. fg and gf are identities).

(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space X , the *fundamental groupoid* $\pi(x)$ has all points of X as objects, and morphisms $x \rightarrow y$ are homotopy classes $rel\{0, 1\}$ of paths $u : [0, 1] \rightarrow X$ with $u(0) = x$, $u(1) = y$ (if you know how to prove that the fundamental group is a group, you can prove that $\pi(x)$ is a groupoid).

(h) A *discrete cat* is one whose only morphism are identities.

A *preorder* is a cat \mathcal{C} in which, for any pair (A, B) , \exists at most 1 morphism $A \rightarrow B$.

A small preorder is a set equipped with a binary relation which is reflexive and transitive.

In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.

(i) The category **Rel** has the same objects as *set*, but morphisms $A \rightarrow B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define $S \cdot R = \{(a, c) \in A \times C \mid (\exists b \in B)((a, b) \in R, (b, c) \in S)\}$.

The identity $1_A : A \rightarrow A$ is $\{(a, a) \mid a \in A\}$.

Similarly, the category **Part** are for sets and partial functions (i.e. relations s.t. $(a, b) \in R$ and $(a, b') \in R \implies b = b'$).

(j) Let K be a field. The category **Mat_K** has natural numbers as objects, and morphism $n \rightarrow p$ are $(p \times n)$ matrices with entries from K . Composition is matrix multiplication.

(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

Definition. (1.4)

Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

(a) a mapping $A \rightarrow FA$ from $\text{ob } \mathcal{C}$ to $\text{ob } \mathcal{D}$;

(b) a mapping $f \rightarrow Ff$ from $\text{mor } \mathcal{C}$ to $\text{mor } \mathcal{D}$,

such that $\text{dom}(Ff) = F(\text{dom } f)$, $\text{cod}(Ff) = F(\text{cod } f)$, $1_{FA} = F(1_A)$, and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Example. (1.5)

(a) We have *forgetful functors* $U : \mathbf{Gp} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{Set}, \mathbf{Top} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{AbGp}$ (forget \times), $\mathbf{Ring} \rightarrow \mathbf{Mon}$ (Category of all monoids) (forget $+$).

(b) Given a set A , the free group FA has the property:

Given any group G and any function $A \xrightarrow{f} UG$ (?), there's a unique homomorphism $FA \xrightarrow{\tilde{f}} G$ extending f . Here F is a functor $\mathbf{Set} \rightarrow \mathbf{Gp}$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \leftrightarrow UFB$. Functoriality follows from uniqueness given $B \xrightarrow{f} C$. $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow UFC$.

(c) Given a set A , we write PA for the set of all subsets of A .

We can make P into a functor $\mathbf{Set} \rightarrow \mathbf{Set}$, given $A \xrightarrow{f} B$, we defined $Pf(A') = \{f(a) \mid a \in A'\}$ for $A' \subseteq A$.

But we also have a functor $P^* : \mathbf{Set} \rightarrow \mathbf{Set}^{op}$ defined on objects by P , but $P^*f(B') = \{a \in A \mid f(a) \in B'\}$ for $B' \subseteq B$.

By a *contravariant* functor $\mathcal{C} \rightarrow \mathcal{D}$, we mean a functor $\mathcal{C} \rightarrow \mathcal{D}^{op}$ (or $\mathcal{C}^{op} \rightarrow \mathcal{D}$).

A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

- (d) Let K be a field. We have a functor $*$: $\mathbf{Mod}_K \rightarrow \mathbf{Mod}_K^{op}$ defined by $V^* = \{ \text{linear maps } V \rightarrow K \}$, and if $V \xrightarrow{f} W$, $f^*(\theta : W \rightarrow K) = \theta f$.
- (e) We have a functor op : $\mathbf{Cat} \rightarrow \mathbf{Cat}$, which is the identity on morphisms (note that this is a covariant).
- (f) A functor between monoids is a monoid homomorphism.
- (g) A functor between posets is an order-preserving map.
- (h) Let G be a group. A functor $F \circ G \rightarrow \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A , i.e. a *permutation representation* of G . Similarly, a functor $G \rightarrow \mathbf{Mod}_K$ is a K -linear representation of G .
- (i) The construction of the fundamental group $\pi(X, X)$ of a space X with basepoint X is a functor $\mathbf{Top}_* \rightarrow \mathbf{Gp}$ where \mathbf{Top}_* is the category of spaces with a chosen basepoint. Similarly, the fundamental groupoid is a functor $\mathbf{Top} \rightarrow \mathbf{Gpd}$, where \mathbf{Gpd} is the category of groupoids and functors between them.

Definition. (1.6)

Let \mathcal{C} and \mathcal{D} be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ (why two arrows?) two functors. A *natural transformation* $\alpha : F \rightarrow G$ consists of an assignment $A \rightarrow \alpha_A$ from $\text{ob } \mathcal{C}$ to $\text{mor } \mathcal{D}$ (think about this), such that $\text{dom}_{\alpha_A} = FA$ and $\text{cod}_{\alpha_A} = GA$ for all A , and for all $A \xrightarrow{f} B$ in \mathcal{C} , the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)_{\alpha_A}$).

(1.3) (l) Given categories \mathcal{C} and \mathcal{D} , we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \rightarrow \mathcal{D}$ and whose morphisms are natural transformations.

Example. (1.7)

- (a) Let K be a field, V a vector space over K . There is a linear map $\alpha_V : V \rightarrow V^{**}$ given by $\alpha_V(v)\theta = \theta(v)$ for $\theta \in V^*$. This is the V -component of a natural transformation $1_{\mathbf{Mod}_K} \rightarrow ** : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K$.
- (b) For any set A , we have a mapping $\sigma_A : A \rightarrow PA$ sending a to $\{a\}$. If $f : A \rightarrow B$, then $Pf\{a\} = \{f(a)\}$. So σ is a natural transformation $1_{\mathbf{Set}} \rightarrow P$.
- (c) Let $F : \mathbf{Set} \rightarrow \mathbf{Gp}$ be the free group functor (1.5(b)), and $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ the forgetful functor. The inclusions $A \rightarrow UFA$ form a natural transformation $1_{\mathbf{Set}} \rightarrow UF$.
- (d) Let G, H be groups and $f, g : G \rightrightarrows H$ be two homomorphisms. A natural transformation $\alpha : f \rightarrow g$ corresponds to an element $h = \alpha_*$ of H , s.t. $hf(x) \rightarrow g(x)h$ for all $x \in G$ or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
- (e) Let A and B be two G -sets, regarded as functors: $G \rightrightarrows \mathbf{Set}$. A natural transformation $A \rightarrow B$ is a function f satisfying $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$, i.e. a G -equivariant map.

Lemma. (1.8)

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors, and $\alpha : F \rightarrow G$ a natural transformation. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each α_A is an isomorphism in \mathcal{D} .

Proof. Forward is trivial (ok, I'll check this later). For backward, suppose each α_A has an inverse β_A . Given $f : A \rightarrow B$ in \mathcal{C} , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

□

commutes. But as α is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

Definition. (1.9)

Let \mathcal{C} and \mathcal{D} be categories. By an *equivalence* between \mathcal{C} and \mathcal{D} , we mean a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$ and $\beta : FG \rightarrow 1_{\mathcal{D}}$.

We write $\mathcal{C} \cong \mathcal{D}$ if \mathcal{C} and \mathcal{D} are equivalent.

We say a property P of categories is a *categorical property* if whenever \mathcal{C} has P and $\mathcal{C} \cong \mathcal{D}$, then \mathcal{D} has P .

For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

Example. (1.10)

(a) The category **Part** is equivalent to the category **Set**_{*} of pointed sets (and basepoint preserving functions (as morphisms)):

- We define $F : \mathbf{Set}_* \rightarrow \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$, and if $f : (A, a) \rightarrow (B, b)$, then $Ff(x) = f(x)$ if $f(x) \neq b$, and undefined otherwise;
- and $G : \mathbf{Part} \rightarrow \mathbf{Set}_*$ by $G(A) = A^+ = (A \cup \{A\}, A)$, and if $f : A \rightarrow B$ is a partial function, we define $Gf : A^+ \rightarrow B^+$ by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ defined, and equals B otherwise.

The composite FG is the identity on **Part**, but GF is not the identity. However, there is an isomorphism $(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$ sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.

Note that there can be no isomorphism from **Set**_{*} to **Part**, since **Part** has a 1-element isomorphism class $\{\phi\}$ but **Set**_{*} doesn't.

(So we see that equivalent categories can be non-isomorphic. According to a **post** on SO, this usually happens when there are multiple copies of the *same* thing in one but not the other. However, we can't generally *discard obsolete copies* in one as that generally requires AC and is not a very useful thing to do anyway – In short, *identifying isomorphic objects is often an extremely bad idea.*)

(b) The category **fdMod**_K of finite-dimensional vector spaces over K is equivalent to **fdMod**_K^{op}, the functors in both directions are $*$ (the dual operator) and both isomorphisms are the natural transformations of 1.7(a) (double dual).

(c) **fdMod**_K is also equivalent to **Mat**_K (1.3(j)):

We define $F : \mathbf{Mat}_K \rightarrow \mathbf{fdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A w.r.t. the standard bases of K^n and K^p .

To define $G : \mathbf{fdMod}_K \rightarrow \mathbf{Mat}_K$, choose a basis for each finite dimensional vector space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ to be the matrix representing

f w.r.t. chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n ; $FG \neq 1$, but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \rightarrow V$ for each V , which form a natural isomorphism.

—Lecture 4—

Definition. (1.11)

Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor.

(a) We say F is *faithful* if, given $f, f' \in \text{mor } \mathcal{C}$ with $\text{dom } f = \text{dom } f'$, $\text{cod } f = \text{cod } f'$, and $Ff = Ff'$, then $f = f'$ (injectivity on morphisms. The name comes more from representation theory);

(b) We say F is *full* if, given $FA \xrightarrow{g} FB$ in \mathcal{D} , there exists $A \xrightarrow{f} B$ in \mathcal{C} with $Ff = g$. (this is something like surjective, but see below);

(c) We say F is *essentially surjective* if, for every $B \in \text{ob } \mathcal{D}$, there exists $A \in \text{ob } \mathcal{C}$ and isomorphism $FA \rightarrow B$ in \mathcal{D} .

We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is full if the inclusion $\mathcal{C}' \rightarrow \mathcal{C}$ is a full functor. For example, **Gp** is a full subcategory of **Mon** (the category of all monoids), but **Mon** is not a full subcategory of the category **SGp** of semigroups.

Lemma. (1.12)

Assuming the axiom of choice, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ (see (1.9) for what an equivalence is. I think here it means one of the functors F, G) if it's full, faithful, and essentially surjective.

Proof. \Rightarrow : Suppose given G, α, β as in (1.9). Then for each $B \in \text{ob } \mathcal{D}$, β_B is an isomorphism $FGB \rightarrow B$, so F is essentially surjective.

Given $A \xrightarrow{f} B$ in \mathcal{C} , we can recover f from Ff as composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B$. Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$. So F is faithful;

Lastly, for fullness, given $FA \xrightarrow{g} FB$, define f to be the composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$. Then $GFf = \alpha_B f \alpha_A^{-1}$, which by construction is just Gg . But G is faithful for the same reason as f , so $Ff = g$.

\Leftarrow : (need to find suitable G, α, β for F .) For each $B \in \text{ob } \mathcal{D}$, choose $GB \in \text{ob } \mathcal{C}$ and an isomorphism $\beta_B : FGB \rightarrow B$ in \mathcal{D} . Given $B \xrightarrow{g} B'$, define $Gg : GB \rightarrow GB'$ to be the unique morphism whose image under F is $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$.

Uniqueness implies functoriality (check what this means – think it appeared somewhere before): given $B' \xrightarrow{g'} B''$, $(Gg')(Gg)$ and $G(g'g)$ have the same image under F , so they are equal.

By construction, β is a natural transformation $FG \rightarrow 1_{\mathcal{D}}$.

Given $A \in \text{ob } \mathcal{C}$, define $\alpha_A : A \rightarrow GFA$ to be the unique morphism whose image under F is $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$. α_A is an isomorphism, since β_{FA} also has a unique pre-image under F . And α is a natural transformation, since any naturality square for α (the commutative square when we defined natural transformation. check) is mapped by F to a commutative square, and F is faithful. \square

Definition. (1.13)

By a *skeleton* of a category, we mean a full subcategory \mathcal{C}_0 containing one object from each isomorphism class. We say \mathcal{C} is *skeletal* if it's a skeleton of itself.

For example, $\mathbf{Mat}_{\mathbf{K}}$ is a skeletal, and the image of $F : \mathbf{Mat}_{\mathbf{K}} \rightarrow \mathbf{fdMod}_{\mathbf{K}}$ of 1.10(c) is a skeleton of $\mathbf{fdMod}_{\mathbf{K}}$.

(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

Definition. (1.14)

Let $A \xrightarrow{f} B$ be a morphism in \mathcal{C} .

(a) We say f is a *monomorphism* (or f is *monic*) if, given any pair $C \xrightleftharpoons[h]{g} A$,

$fg = fh$ implies $g = h$.

(b) We say f is an *epimorphism* (or *epic*) if it's a monomorphism in \mathcal{C}^{op} , i.e. if $gf = hf$ implies $g = h$.

We denote monomorphisms by $A \xrightarrow{f} B$, and epimorphisms by $A \xrightarrow{f} B$.

Any isomorphism is monic and epic: more generally, if f has a left inverse (i.e. $\exists g$ s.t. gf is an identity), then it's monic. We call such monomorphisms *split*.

We say \mathcal{C} is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

Example. (1.15)

(a) As usual we consider **Set** first. In **Set**, monomorphisms correspond to injections (\Leftarrow is easy (ok); for \Rightarrow , take $C = 1 = \{*\}$), and epimorphisms correspond to surjections (\Leftarrow is easy; for \Rightarrow , use morphisms $B \rightrightarrows 2 = \{0, 1\}$). So **Set** is balanced.

(b) In **Gp**, monomorphisms again correspond to injections (for \Rightarrow use homomorphisms $\mathbb{Z} \rightarrow A$); epimorphisms again correspond to surjections (\Rightarrow use free products with amalgamation – this is a non-trivial fact about groups, read more if free). So **Gp** is also balanced.

(c) In **Rng** (obvious notation), monomorphisms correspond to injections (proof is much like for **Gp**). However, not all epimorphisms are surjective. For example

the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \xrightleftharpoons[g]{f} R$ agree on all integers,

they agree everywhere. So **Rng** is not balanced.

(d) One final example is **Top**. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to **Set** (check). However, **Top** is not balanced since a continuous bijection need not have continuous inverse.

2 The Yoneda Lemma

Let's not start on the content this lecture. Why are we talking about one single lemma in a chapter? Well it's not really a lemma. There's some story behind this, check here for an [obituary](#) which probably has the story that lecture was talking about in class.