Applied Probability

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0 Miscellaneous

Some speech

Google lecture's name to find his homepage and example sheets or probably some notice of a change of room

1 Poisson process

Suppose we have a Geiger counter. We model the "click process" as a family $\{N(t) : t \geq 0\}$, where $N(t)$ denotes the total number of ticks up to time t. Now note that $N(t) \in \{0, 1, ...\}$, $N(s) \leq N(t)$ if $s \leq t$, N increases by unit jumps, and $N(0) = 0$. We also assert that N is right-continuous, i.e. $\lim_{x \to t^+} N(x) = N(t)$.

Definition. (infinitesimal definition)
A *Poisson process* with intensity λ is a process $N = (N(t) : t \geq 0)$ which takes values in $S = \{0, 1, 2, ...\}$, s.t.:
(a) $N(0) = 0$, $N(s) \leq N(t)$ if $s \leq t$;
(b)

$$\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda h + o(h) & m = 1 \\ o(h) & m > 1 \\ 1 - \lambda h & m = 0 \end{cases}$$

Recall that $g(h) = o(h)$ means that $\frac{g(h)}{h} \to 0$ as $h \to 0$;
(c) if $s < t$, then $N(t) - N(s)$ is independent of all arrivals prior to s.

Theorem. $N(t)$ has the Poisson distribution with parameter λt.

Proof. Study $N(t+h)$ given $N(t)$. We have

$$\begin{aligned} \mathbb{P}(N(t+h) = j) &= \sum_{i \leq j} \mathbb{P}(N(t+h) \\ &= j | N(t) = i) \mathbb{P}(N(t) = i) \\ &= (1 - \lambda h)\mathbb{P}(N(t) = j) + \lambda h \mathbb{P}(N(t) = h - 1) + o(h) \end{aligned}$$

So

$$\frac{\mathbb{P}(N(t+h) = j) - \mathbb{P}(N(t) = j)}{h} = -\lambda \mathbb{P}(N(t) = j) + \lambda \mathbb{P}(N(t) = j - 1) + \frac{o(h)}{h}$$

write $p_n(t) = \mathbb{P}(N(t) = n)$, then let $h \to 0^+$ we get

$$\begin{aligned} p'j(t) = -\lambda p_j(t) + \lambda p_{j-1}(t) j \geq 1 \\ p'0_(t) = -\lambda p_0(t) \end{aligned}$$

with boundary condition $p_0(0) = 1$.
We solve p_0 to get $p_0(t) = e^{-\lambda(t)}$. Then we can use this to inductively solve $p_1, p_2, ...$ to get the desired result. □

An alternative derivation from the differential equations:
Let $G(s, t) = \sum_j s^j p_j(t)$. Now we take the set of differential equation, multiplying each one by s^j, then we get

$$\frac{\partial G}{\partial t} = \lambda(s - 1)G$$

Then we have

$$G(s, t) = A(s) e^{\lambda(s-1)t}$$

We also have $G(s, 0) = 1$ so we should be able to plug in a suitable value of s to get the desired result (I probably missed that).