Advanced Probability

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0 Reviews

0.1 Measure spaces

Let E be a set. Let $\mathcal{E}$ be a set of subsets of E. We say that $\mathcal{E}$ is a σ-algebra on E if:
• $\phi \in \mathcal{E}$;
• $\mathcal{E}$ is closed under countable unions and complements.
In that case, $(E, \mathcal{E})$ is called a *measurable space*.

We call the elements of $\mathcal{E}$ *measurable sets*.

Let μ be a function $\mathcal{E} \to [0, \infty]$. We say μ is a measure if:
• $\mu(\phi) = 0$; • μ is countably additive: for all sequences (A_n) of disjoint elements of $\mathcal{E}$, then

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$$

In that case, the triple $(E, \mathcal{E}, \mu)$ is called a *measure space*.

Given a topological space E, there is a smallest σ-algebra containing all the open sets in E. This is the *Borel σ-algebra of $\mathcal{E}$*, denoted $\mathcal{B}(E)$.

In particular, for the real line $\mathbb{R}$, we will just write $\mathcal{B} = \mathcal{B}(\mathbb{R})$ for simplicity.

0.2 Integration of measurable functions

Let $(E, \mathcal{E})$ and $(E', \mathcal{E}')$ be measurable spaces. A function $f : E \to E'$ is *measurable* if $f^{-1}(A) = \{x \in E : f(x) \in A\} \in \mathcal{E} \forall A \in \mathcal{E}'$.

If we refer to a measurable function f without specifying range, the default is $(\mathbb{R}, \mathcal{B})$.

Similarly, if we refer to f as a non-negative measurable function, then we mean $E' = [0, \infty]$, $\mathcal{E}' = \mathcal{B}([0, \infty])$.

It is worth notice that under this set of definitions, a non-negative measurable function might not be $\mathbb{R}$-measurable (since we allowed ∞).

We write $m\mathcal{E}^+$ for set of non-negative measurable functions.

Theorem. Let $(E, \mathcal{E}, \mu)$ be a measure space. There exists a unique map $\tilde{\mu} : m\mathcal{E}^+ \to [0, \infty]$ such that:
•(a) $\tilde{\mu}(1_A) = \mu(A)$ for all $A \in \mathcal{E}$, where 1_A is the indicator function;
•(b) $\tilde{\mu}(\alpha f + \beta g) = \alpha\tilde{\mu}(f) + \beta\tilde{\mu}(g)$ for all $\alpha, \beta \in [0, \infty)$, $f, g \in m\mathcal{E}^+$ (linearity);
•(c) $\tilde{\mu}(f) = \lim_{n\to\infty} \tilde{\mu}(f_n)$ for any non-decreasing sequence $(f_n : n \in \mathbb{N})$ in $m\mathcal{E}^+$ such that $f_n(x) \to f(x)$ for all $x \in E$ (monotone-convergence).

We'll only prove uniqueness. For existence, see II Probability and Measure notes.

From now on, write μ for $\tilde{\mu}$.
We'll call $\mu(f)$ the *integral* of f w.r.t. μ.
We also write $\int_E f d\mu = \int Ef(x)\mu(dx)$.

A *simple function* is a finite linear combination of indicator functions of measurable sets with positive coefficients, i.e. f is simple if

$$f = \sum_{k=1}^{n} \alpha_k 1_{A_k}$$

for some $n \geq 0$, $\alpha_k \in (0, \infty)$, $A_k \in \mathcal{E} \forall k = 1, ..., n$.

From (a) and (b), for f simple,

$$\mu(f) = \ sum_{k=1}^{n} \alpha_k \mu(A_k)$$

Also, if $f, g \in m\mathcal{E}^+$ with $f \leq g$, then $f + h = g$ where $h = g - f \cdot 1_{f<\infty} \in m\mathcal{E}^+$. Then since $\mu(h) \geq 0$, (b) implies $\mu(f) \leq \mu(g)$.

Take $f \in m\mathcal{E}^+$. Define for $x \in E$, $n \in \mathbb{N}$,

$$f_n(x) = \left(2^{-n} \lfloor 2^n f(x) \rfloor\right) \wedge n$$

where $\wedge$ means taking the minimum. Note that (f_n) is a non-decreasing sequence of simple functions that converges to f pointwise everywhere on E. Then by (c),

$$\mu(f) = \lim_{n \to \infty} \mu(f_n)$$

So we have shown uniqueness: μ is uniquely determined by the measure (provided that it exists, which we're not going to show).

When is $\mu(f)$ zero (for $f \in m\mathcal{E}^+$)? For measurable functions f, g, we say $f = g$ *almost everywhere* if

$$\mu(\{x \in E : f(x) \neq g(x)\}) = 0$$

i.e. they only disagree on a measure-zero set.

We can show, for $f \in m\mathcal{E}^+$, that $\mu(f) = 0$ if and only if $f = 0$ almost everywhere.

Let f be a measurable function. We say that f is *integrable* if $\mu(|f|) < \infty$.

Write $L^1 = L^1(E, \mathcal{E}, \mu)$ for the set of all integrable functions. We extend the integral to L^1 by setting $\mu(f) = \mu(f^+) - \mu(f^-)$, where

$$f^{\pm}(x) = 0 \vee (\pm f(x))$$

where $\vee$ means the maximum (so $f = f^+ - f^-$). Note that now f^+, f^- are both non-negative, with disjoint support. Then we can show that L^1 is a vector space, and $\mu : L^1 \to \mathbb{R}$ is linear.

Lemma. (Fatou's lemma)
Let $(f_n : n \in \mathbb{N})$ be any sequence in $m\mathcal{E}^+$. Then

$$\mu(\liminf_{n \to \infty} f_n) \leq \liminf_{n \to \infty} \mu(f_n)$$

The proof is a straight forward application of monotone convergence.
The only hard part is to remember which way the inequality is (consider a sliding block function to the right).