Topics in Set Theory Sheet 4

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34.

Fix $A \in M[G]$, a formula $\varphi(x, y, x_1, ..., x_n)$ which specifies the function that we want to use for replacement, and fix parameters $a_1, ..., a_n \in M[G]$. We want a name for

$$B := \{y : M[G] \vDash \exists x \in A\varphi(x, y, a_1, ..., a_n)\}$$

Take a name σ for A and some names $\tau_1, ..., \tau_n$ for $a_1, ..., a_n$. Define a formula

$$\psi(x, y, x_1, ..., x_n, A) := x \in A \wedge \varphi(x, y, x_1, ..., x_n)$$

and now consider the name

$$\rho := \{(\pi, p) : p \Vdash^* \exists x\psi(x, \pi, \tau_1, ..., \tau_n, \sigma)\}$$

We claim that $val(\rho, G) = B$.
$\subseteq$: suppose $y \in val(\rho, G)$. So there is $(\pi, p) \in \rho$ with $p \in G$ and $val(\pi, G) = y$. So $p \Vdash^* \pi \in \exists x\psi(x, \pi, \tau_1, ..., \tau_n, \sigma)$. By definition, this means that the set

$$D := \{r : \exists \mu \in M^{\mathbb{P}}(r \Vdash^* \psi(\mu, \pi, \tau_1, ..., \tau_n, \sigma))\}$$

is dense below p. Also $p \in G$, so $G \cap D \neq \phi$,[1] so take some $q \in G \cap D$. $q \in D$ means there exists a name μ s.t.

$$q \Vdash^* \psi(\mu, \pi, \tau_1, ..., \tau_n, \sigma)$$

But $q \in G$ as well, so by FT,

$$M[G] \vDash \psi(\mu, \pi, \tau_1, ..., \tau_n, \sigma)$$

which translates to: there exists x ($:= val(\mu, G)$) that

$$M[G] \vDash \psi(x, y, a_1, ..., a_n, A)$$
$$\iff M[G] \vDash x \in A \wedge \varphi(x, y, x_1, ..., x_n)$$

we can rewrite this as

$$M[G] \vDash \exists x(x \in A \wedge \varphi(x, y, x_1, ..., x_n))$$

i.e. $y \in B$.
$\supseteq$: suppose $y \in B$. So $M[G] \vDash \exists x\psi(x, y, a_1, ...a_n, A)$. Now take a name π for y; so by FT, there exists $p \in G$ s.t. $p \Vdash^* \exists x\psi(x, \pi, \tau_1, ..., \tau_n, \sigma)$. But this is exactly the requirement for $(\pi, p) \in \rho$. So $y \in val(\rho, G)$.

[1] On lecture 19 this was just stated as a fact without proof, but I can't see why it's trivial. My proof of this claim: suppose otherwise, that $G \cap D = \phi$. Add all elements q s.t. there's no $r \in D$ s.t. $r \leq q$ to D to form a new set D'. Then D' is dense, so $G \cap D' \neq \phi$, so pick $r \in G \cap D'$. Now if $r \notin D$, then since G is a filter, pick $q \in G$ s.t. $q \leq p, r$. Since D is dense below p and $q \leq p$, we can pick $s \in D$ s.t. $s \in q$. But then $s \leq q \leq r$, and $s \in D$, $r \in D' \setminus D$, contradicting our definition of D'.

35.

Fix $A \in M[G]$, and $\phi \notin A$. We want to prove that there exists $f \in M[G]$ s.t. f is a function from $A \to \cup A$, and $x \in A \implies f(x) \in x$.

Take a name σ of A. We'd like a name π that satisfies the following:
(1) $val(\pi, G)$ is a function $A \to \cup A$;
(2) if $(\tau_1, p_1), (\tau_2, p_2) \in \pi$, $p_1, p_2 \in G$, and

36.

I'm assuming every instance of '$\mathbb{P}$-generic over M' actually means '$\mathbb{P}$-generic filter over M' (and similarly for later questions).

G is a $\mathbb{P}$-generic filter:
• Let $p \in G$, and $p \leq p'$. Then $i(p) \leq i(p')$. But H is a filter, and $i(p) \in H$, so $i(p') \in H$. So $p' \in G$.
• Let $p, p' \in G$; we want to prove that p, p' have a witness of compatibility in G. Suppose otherwise, so if $p'' \leq p, p'$ then $i(p'') \notin H$. But $i(p)$ and $i(p')$ still need to have a witness of compatibility in H, say $q \in H$; so this q cannot be in the image $i(\mathbb{P})$.
• We've proved that G is a filter; now we prove that G is $\mathbb{P}$-generic over M. In fact we prove that it's $\mathbb{P}$-antichain generic over M. Let $A \subset \mathbb{P}$ be a maximal antichain, we want to prove that $\exists a \in A(i(a) \in H)$.
Suppose otherwise. Write $i(A)$ for the set containing images of element of A under i. Then $i(A)$ cannot be an maximal antichain (as H is $\mathbb{Q}$-antichain generic, so every maximal antichain has non-empty intersection with H). Obviously by (b), $i(A)$ is still an antichain. So (as $M[H] \vDash AC$) we extend it to a maximal antichain A'. Now $A' \cap H \neq \phi$, so take $q \in A' \cap H$. By assumption, $q \notin i(A)$. Now apply (c), we get a $p \in \mathbb{P}$ s.t. $p' \leq p \implies i(p')$ and q are compatible. But A is a maximal antichain in $\mathbb{P}$, so p has to be compatible with some element of A, say p_a. So let $p_c \in \mathbb{P}$ witness that p and p_a are compatible; in particular, $p_c \leq p$. By (c), $i(p_c)$ and q are compatible. But $p_c \leq p_a$ and hence $i(p_c) \leq i(p_a)$, so $i(p_a)$ and q are compatible; but both $i(p_a)$ and q are in the antichain A' of $\mathbb{Q}$. Contradiction (see diagram below).