

# Representation Theory

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## 0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example,  $SU(n)$  and  $O(n)$ . The vector spaces we conside are finite dimensional, and usually over  $\mathbb{C}$ . Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

## 1 Group actions

Throughout this course, if not specified otherwise:

- $F$  is a field, usually  $\mathbb{C}$ ,  $\mathbb{R}$  or  $\mathbb{Q}$ . When the field is one of these, we are discussing *ordinary representation theory*. Sometimes  $F = F_p$  or  $\bar{F}_p$  (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
- $V$  is a vector space over  $F$ , always finite dimensional;  
 $GL(V) = \{\theta : V \rightarrow V, \theta \text{ linear, invertible}\}$ , i.e.  $\det \theta \neq 0$ .

Recall from Linear Algebra:

If  $\dim_F V = n < \infty$ , choose basis  $e_1, \dots, e_n$  over  $F$ , so we can identify it with  $F^n$ . Then  $\theta \in GL(V)$  corresponds to an  $n \times n$  matrix  $A_\theta = (a_{ij})$ , where  $\theta(e_j) = \sum_i a_{ij} e_i$ . In fact, we have  $A_\theta \in GL_n(F)$ , the general linear group.

(1.1)  $GL(V) \cong GL_n(F)$  as groups by  $\theta \rightarrow A_\theta$  ( $A_{\theta_1 \theta_2} = A_{\theta_1} A_{\theta_2}$  and bijection).  
 Choosing different basis gives different isomorphism to  $GL_n(F)$ , but:

(1.2) Matrices  $A_1, A_2$  represent the same element of  $GL(V)$  w.r.t different bases iff they are conjugate (similar), i.e.  $\exists X \in GL_n(F)$  s.t.  $A_2 = X A_1 X^{-1}$ .

Recall that  $\text{tr}(A) = \sum_i a_{ii}$  where  $A = (a_{ij})$ , the *trace* of  $A$ .

(1.3)  $\text{tr}(X A X^{-1}) = \text{tr}(A)$ , hence we can define  $\text{tr}(\theta) = \text{tr}(A_{\theta_1})$  independent of basis.

(1.4) Let  $\alpha \in GL(V)$  where  $V$  in f.d. over  $\mathbb{C}$ , with  $\alpha^m = \iota$  for some  $m$  (here  $\iota$  is the identity map). Then  $\alpha$  is diagonalisable.

Recall  $\text{End} V$  is the set of all linear maps  $V \rightarrow V$ , e.g.  $\text{End}(F^n) = M_n(F)$  some  $n \times n$  matrices.

(1.5) *Proposition.* Take  $V$  f.d. over  $\mathbb{C}$ ,  $\alpha \in \text{End}(V)$ . Then  $\alpha$  is diagonalisable iff there exists a polynomial  $f$  with distinct linear factors with  $f(\alpha) = 0$ . For example, in (1.4), where  $\alpha^m = \iota$ , we take  $f = X^m - 1 = \prod_{j=0}^{m-1} (X - \omega^j)$  where  $\omega = e^{2\pi i/m}$  is the  $(m^{\text{th}})$  root of unity. In fact we have:

(1.4)\* A finite family of commuting separately diagonalisable automorphisms of a  $\mathbb{C}$ -vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:

(1.6) The symmetric group,  $S_n = \text{Sym}(X)$  on the set  $X = \{1, \dots, n\}$  is the set of all permutations of  $X$ .  $|S_n| = n!$ . The alternating group  $A_n$  on  $X$  is the set of products of an even number of transpositions (2-cycles).  $|A_n| = \frac{n!}{2}$ .

(1.7) Cyclic groups of order  $m$ :  $C_m = \langle x : x^m = 1 \rangle$ . For example,  $(\mathbb{Z}/m\mathbb{Z}, +)$ ; also, the group of  $m^{\text{th}}$  roots of unity in  $\mathbb{C}$  (inside  $GL_1(\mathbb{C}) = \mathbb{C}^*$ , the multiplicative group of  $\mathbb{C}$ ). We also have the group of rotations, centre  $O$  of regular  $m$ -gon in  $\mathbb{R}^2$  (inside  $GL_2(\mathbb{R})$ ).

(1.8) Dihedral groups  $D_{2m}$  of order  $2m = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ . Think of this as the set of rotations and reflections preserving a regular  $m$ -gon.

(1.9) Quaternion group,  $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$  of order 8. For example, in  $GL_2(\mathbb{C})$ , put  $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ , then  $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$ .

(1.10) The conjugacy class (ccls) of  $g \in G$  is  $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$ . Then  $|\mathcal{C}_G(g)| = |G : C_G(g)|$ , where  $C_G(g) = \{x \in G : xg = gx\}$ , the centraliser of  $g \in G$ .

(1.11) Let  $G$  be a group,  $X$  be a set.  $G$  acts on  $X$  if there exists a map  $\cdot : G \times X \rightarrow X$  by  $(g, x) \rightarrow g \cdot x$  for  $g \in G, x \in X$ , s.t.  $1 \cdot x = x$  for all  $x \in X$ ,  $(gh) \cdot x = g \cdot (h \cdot x)$  for all  $g, h \in G, x \in X$ .

(1.12) Given an action of  $G$  on  $X$ , we obtain a homomorphism  $\theta : G \rightarrow \text{Sym}(X)$ , called the *permutation representation* of  $G$ .

*Proof.* For  $g \in G$ , the function  $\theta_g : X \rightarrow X$  by  $x \rightarrow gx$  is a permutation on  $X$ , with inverse  $\theta_{g^{-1}}$ . Moreover,  $\forall g_1, g_2 \in G, \theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2}$  since  $(g_1 g_2)x = g_1(g_2 x)$  for  $x \in X$ .  $\square$

## 2 Basic Definitions

### 2.1 Representations

Let  $G$  be finite,  $F$  be a field, usually  $\mathbb{C}$ .

**Definition.** (2.1)

Let  $V$  be a f.d. vector space over  $F$ . A (linear, in some books) *representation* of  $G$  on  $V$  is a group homomorphism

$$\rho = \rho_V : G \rightarrow GL(V)$$

Write  $\rho_g$  for the image  $\rho_V(g)$ ; so for each  $g \in G$ ,  $\rho_g \in GL(V)$ , and  $\rho_{g_1 g_2} = \rho_{g_1} \rho_{g_2}$ , and  $(\rho_g)^{-1} = \rho_{g^{-1}}$ .

The *dimension* (or *degree*) of  $\rho$  is  $\dim_F V$ .

(2.2) Recall  $\ker \rho \triangleleft G$  (kernel is a normal subgroup), and  $G/\ker \rho \cong \rho(G) \leq GL(V)$  (1st isomorphism theorem). We say  $\rho$  is *faithful* if  $\ker \rho = 1$ .

An alternative (and equivalent) approach is to observe that a representation of  $G$  on  $V$  is "the same as" a *linear action* of  $G$ :

**Definition.** (2.3)

$G$  *acts linearly* on  $V$  if there exists a *linear action*

$$\begin{aligned} G \times V &\rightarrow V \\ (g, v) &\rightarrow gv \end{aligned}$$

By linear action we mean: (action)  $(g_1 g_2)v = g_1(g_2 v)$ ,  $1v = v \ \forall g_1, g_2 \in G, v \in V$ , and (linear)  $g(v_1 + v_2) = gv_1 + gv_2$ ,  $g(\lambda v) = \lambda gv \ \forall g \in G, v_1, v_2 \in V, \lambda \in F$ .

Now if  $G$  acts linearly on  $V$ , the map

$$\begin{aligned} G &\rightarrow GL(V) \\ g &\rightarrow \rho_g \end{aligned}$$

with  $\rho_g : v \rightarrow gv$  is a representation of  $G$ . Conversely, given a representation  $\rho : G \rightarrow GL(V)$ , we have a linear action of  $G$  on  $V$  via  $g \cdot v := \rho(g)v \ \forall v \in V, g \in G$ .

(2.4) In (2.3) we also say that  $V$  is a  $G$ -space or that  $V$  is a  $G$ -module. In fact if we define the *group algebra*  $FG$ , or  $F[G]$ , to be  $\{\sum \alpha_j g : \alpha_j \in F\}$  with natural addition and multiplication, then  $V$  is actually a  $FG$ -module (in the sense from GRM).

(2.5)  $R$  is a *matrix representation* of  $G$  of degree  $n$  if  $R$  is a homomorphism  $G \rightarrow GL_n(F)$ . Given representation  $\rho : G \rightarrow GL(V)$  with  $\dim_F V = n$ , fix basis  $B$ ; we get matrix representation

$$\begin{aligned} G &\rightarrow GL_n(F) \\ g &\rightarrow [\rho(g)]_B \end{aligned}$$

Conversely, given matrix representation  $R : G \rightarrow GL_n(F)$ , we get representation

$$\begin{aligned}\rho : G &\rightarrow GL(F^n) \\ g &\rightarrow \rho_g\end{aligned}$$

via  $\rho_g(v) = R_g v$  where  $R_g$  is the matrix of  $g$ .

**Example.** (2.6)

Given any group  $G$ , take  $V = F$  the 1-dimensional space, and

$$\begin{aligned}\rho : G &\rightarrow GL(F) \\ g &\rightarrow (id : F \rightarrow F)\end{aligned}$$

is known as the trivial representation of  $G$ . So  $\deg \rho = 1$  ( $\dim_F F = 1$ ).

**Example.** (2.7)

Let  $G = C_4 = \langle x : x^4 = 1 \rangle$ . Let  $n = 2$ , and  $F = \mathbb{C}$ . Note that any  $R : x \rightarrow X$  will determine  $x^j \rightarrow X^j$  as it is a homomorphism, and also we need  $X^4 = I$ . So we can take  $X$  to be diagonal matrix – any such with diagonal entries a root to  $x^4 = 1$ , i.e.  $\{\pm 1, \pm i\}$ , or if  $X$  is not diagonal then it will be similar to a diagonal matrix by (1.4) ( $X^4 = I$ ).

## 2.2 Equivalent representations

**Definition.** (2.8)

Fix  $G, F$ . Let  $V, V'$  be  $F$ -spaces, and  $\rho : G \rightarrow GL(V)$ ,  $\rho' : G \rightarrow GL(V')$  which are representations of  $G$ . The linear map  $\phi : V \rightarrow V'$  is a  $G$ -homomorphism if

$$\phi \rho(g) = \rho'(g) \phi \forall g \in G(*)$$

We can understand this more by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \phi \downarrow & \searrow & \downarrow \phi \\ V' & \xrightarrow{\rho_{g'}} & V' \end{array}$$

We say  $\phi$  *intertwines*  $\rho, \rho'$ . Write  $\text{Hom}_G(V, V')$  for the  $F$ -space of all these.  $\phi$  is a  $G$ -isomorphism if it is also bijective; if such  $\phi$  exists,  $\rho, \rho'$  are isomorphic/equivalent representations. If  $\phi$  is a  $G$ -isomorphism, we can write (\*) as  $\rho' = \phi\rho\phi^{-1}$ .

**Lemma.** (2.9)

The relation "being isomorphic" is an equivalent relation on the set of all representations of  $G$  (over  $F$ ).

**Remark.** (2.10)

If  $\rho, \rho'$  are isomorphic representations, they have the same dimension.

The converse may be false:  $C_4$  has four non-isomorphic 1-dimensional representations: if  $\omega = e^{2\pi i/4}$  then they are  $\rho_j(x^i) = \omega^{ij}$  ( $0 \leq i \leq 3$ ).

**Remark.** (2.11)

Given  $G, V$  over  $F$  of dimension  $n$  and  $\rho : G \rightarrow GL(V)$ . Fix basis  $B$  for  $V$ : we get a linear isomorphism

$$\begin{aligned} \phi : V &\rightarrow F^n \\ v &\rightarrow [v]_B \end{aligned}$$

and we get a representation  $\rho' : G \rightarrow GL(F^n)$  isomorphic to  $\rho$ :

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \\ \downarrow \phi & & \downarrow \phi \\ F^n & \xrightarrow{\rho'} & F^n \end{array}$$

(2.12) In terms of matrix representations, we have

$$\begin{aligned} R : G &\rightarrow GL_n(F), \\ R' : G &\rightarrow GL_n(F) \end{aligned}$$

are  $(G)$ -isomorphic or equivalent if there exists a nonsingular matrix  $X \in GL_n(F)$  with  $R'(g) = XR(g)X^{-1} \forall g \in G$ .

In terms of linear  $G$ -actions, the actions of  $G$  on  $V, V'$  are  $G$ -isomorphic if there exists isomorphisms  $\phi : V \rightarrow V'$  such that  $g : \phi(v) = \phi(gv) \forall v \in V, g \in G$ .



### 2.3 Subrepresentations

**Definition.** (2.13)

Let  $\rho : G \rightarrow GL(V)$  be a representation of  $G$ . We say  $W \leq V$  is a  $G$ -subspace if it's a subspace and it is  $\rho(G)$ -invariant, i.e.  $\rho_g(W) \leq W \forall g \in G$ . Obviously  $\{0\}$  and  $V$  are  $G$ -subspaces, however.

$\rho$  is *irreducible/simple* representation if there are no proper  $G$ -subspaces.

**Example.** (2.14)

Any 1-dimensional representation of  $G$  is irreducible, but not conversely, e.g.  $D_8$  has 2-dimensional  $\mathbb{C}$ -irreducible representation.

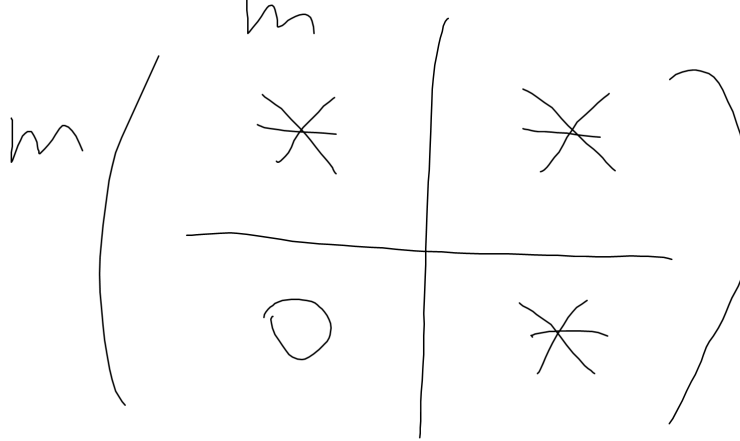
(2.15) In definition (2.13), if  $W$  is a  $G$ -subspace, then the corresponding map

$$\begin{aligned} G &\rightarrow GL(W) \\ g &\rightarrow \rho(g)|_W \end{aligned}$$

is a representation of  $G$ , a *subrepresentation* of  $\rho$ .

**Lemma.** (2.16)

In definition (2.13), given  $\rho : G \rightarrow GL(V)$ , if  $W$  is a  $G$ -subspace of  $V$  and if  $B = \{v_1, \dots, v_n\}$  is a basis containing basis  $B_1 = \{v_1, \dots, v_m\}$  of  $W$  ( $0 < m < n$ ) then the matrix of  $\rho(g)$  w.r.t.  $B$  has block upper triangular form as the graph below, for



each  $g \in G$ .

**Example.** (2.17)

(i) The irreducible representations of  $C_4 = \langle x : x^4 = 1 \rangle$  are all 1-dimensional and four of these are  $x \rightarrow i, x \rightarrow -1, x \rightarrow -i, x \rightarrow 1$ . In general,  $C_m = \langle x : x^m = 1 \rangle$  has precisely  $m$  irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)\* or see (4.4) below).

(ii)  $G = D_6$ : any irreducible  $C$ -representation has dimension  $\leq 2$ .

Let  $\rho : G \rightarrow GL(V)$  be irreducible  $G$ -representation. Let  $r, s$  be rotation and reflection in  $D_6$  respectively. Let  $V$  be eigenvector of  $\rho(r)$ . So  $\rho(r)v = \lambda v$

for some  $\lambda \neq 0$ . Let  $W = \text{span}\{v, \rho(s)v\} \leq V$ . Since  $\rho(s)\rho(s)v = v$  and  $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$ , both of which are in  $W$ ; so  $W$  is  $G$ -invariant, i.e. a  $G$ -subspace. Since  $V$  is irreducible,  $W = V$ .

**Definition.** (2.18)

We say that  $\rho : G \rightarrow GL(V)$  is *decomposable* if there are proper  $G$ -invariant subspaces  $U, W$  with  $V = U \oplus W$ . Say  $\rho$  is direct sum  $\rho_U \oplus \rho_W$ . If no such decomposition exists, we say that  $\rho$  is *indecomposable*.

**Lemma.** (2.19)

Suppose  $\rho : G \rightarrow GL(V)$  is decomposable with  $G$ -invariant decomposition  $V = U \oplus W$ . If  $B$  is a basis  $\{\underbrace{u_1, \dots, u_k}_{B_1}, \underbrace{w_1, \dots, w_l}_{B_2}\}$  of  $V$  consisting of basis of  $U$  and basis of  $W$ , then w.r.t.  $B$ ,  $\rho(g)_B$  is a block diagonal matrix  $\forall g \in G$  as

$$\rho(g)_B = \begin{pmatrix} [\rho_U(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

**Definition.** (2.20)

If  $\rho : G \rightarrow GL(V)$ ,  $\rho' : G \rightarrow GL(V')$ , the *direct sum* of  $\rho, \rho'$  is

$$\rho \oplus \rho' : G \rightarrow GL(V \oplus V')$$

where  $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$ , a *block diagonal action*. For matrix representations  $R : G \rightarrow GL_n(F)$ ,  $R' : G \rightarrow GL_{n'}(F)$ , define  $R \oplus R' : G \rightarrow GL_{n+n'}(F)$ :

$$g \rightarrow \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

### 3 Complete reducibility and Maschke's theorem

**Definition.** (3.1)

A representation  $\rho : G \rightarrow GL(V)$  is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

**Remark.** (3.2)

- (1) The converse is false;
- (2) See sheet 1 Q3:  $\mathbb{C}$ -representation of  $\mathbb{Z}$  is not completely reducible and also representation of  $C_p$  over  $\mathbb{F}_p$  is not c.r..

From now on, take  $G$  finite and  $\text{char } F = 0$ .

**Theorem.** (3.3)

Every f.d. representation  $V$  of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus \dots \oplus V_r$$

is a direct sum of representations, each  $V_i$  irreducible.

It is enough to prove:

**Theorem.** (3.4 Maschke's theorem, 1899)

Let  $G$  be finite,  $\rho : G \rightarrow GL(V)$  a f.d. representation,  $\text{char } F = 0$ . If  $W$  is a  $G$ -subspace of  $V$ , then there exists a  $G$ -subspace  $U$  of  $V$  s.t.  $V = W \oplus U$ , a direct sum of  $G$ -subspaces.

*Proof.* (1)

Let  $W'$  be any *vector subspace* complement of  $W$  in  $V$ , i.e.  $V = W \oplus W'$  as vector spaces, and  $W \cap W' = 0$ . Let  $q : V \rightarrow W$  be the projection of  $V$  onto  $W$  along  $W'$  ( $\ker q = W'$ ), i.e. if  $v = w + w'$  then  $q(v) = w$ . Define

$$\bar{q} : v \rightarrow \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of  $q$  over  $G$ . Note that in order for  $\frac{1}{|G|}$  to exist, we need  $\text{char } F = 0$ .

It still works if  $\text{char } F \nmid |G|$ .

Claim (1):  $\bar{q} : V \rightarrow W$ : For  $v \in V$ ,  $g(q(g^{-1}v)) \in W$  and  $gW \leq W$ ;

Claim (2):  $\bar{q}(w) = w$  for  $w \in W$ :

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w = w$$

So these two claims imply that  $\bar{q}$  projects  $V$  onto  $W$ .

Claim (3) If  $h \in G$  then  $h\bar{q}(v) = \bar{q}(hv)$  ( $v \in V$ ):

$$\begin{aligned}
 h\bar{q}(v) &= h \frac{1}{|G|} \sum_g g \cdot q(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g h g q(g^{-1}v) \\
 &= \frac{1}{|G|} \sum_g (hg) q((hg)^{-1}hv) \\
 &= \frac{1}{|G|} \sum_g g q(g^{-1}(hv)) \\
 &= \bar{q}(hv) \\
 &= \bar{q}(hv)
 \end{aligned}$$

We'll then show that the kernel of this map is  $G$ -invariant, so this gives a  $G$ -summand on Thursday.

Let's now show  $\ker \bar{q}$  is  $G$ -invariant. If  $v \in \ker \bar{q}$ , then  $h\bar{q}(v) = 0 = \bar{q}(hv)$ , so  $hv \in \ker \bar{q}$ . Thus  $V = \text{im } \bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$  is a  $G$ -subspace decomposition.

We can deduce (3.3) from (3.4) by induction on  $\dim V$ . If  $\dim V = 0$  or  $V$  is irreducible, then result is clear. Otherwise,  $V$  has non-trivial  $G$ -invariant subspace,  $W$ . Then by (3.4), there exists  $G$ -invariant complement  $U$  s.t.  $V = U \oplus W$  as representations of  $G$ . But  $\dim U, \dim W < \dim V$ . So by induction they can be broken up into direct sum of irreducible subrepresentations.  $\square$

The second proof uses inner products, hence we need to take  $F = \mathbb{C}$  and can be generalised to compact groups in section 15.

Recall, for  $V$  a  $\mathbb{C}$ -space,  $\langle, \rangle$  is a *Hermitian inner product* if

- (a)  $\langle w, v \rangle = \overline{\langle v, w \rangle} \ \forall v, w$  (Hermitian);
- (b) linear in RHS (sesquilinear);
- (c)  $\langle v, v \rangle > 0$  iff  $v \neq 0$  (positive definite).

Additionally,  $\langle, \rangle$  is  *$G$ -invariant* if

- (d)  $\langle gv, gw \rangle = \langle v, w \rangle \ \forall v, w \in V, g \in G$ .

Note if  $W$  is  $G$ -invariant subspace of  $V$ , with  $G$ -invariant inner product, then  $W^\perp$  is also  $G$ -invariant, and  $V \oplus W^\perp$ . For all  $v \in W^\perp, g \in G$ , we have to show that  $gv \in W^\perp$ . But  $v \in W^\perp \iff \langle v, w \rangle = 0 \ \forall w \in W$ . Thus by (d),  $\langle gv, gw \rangle = 0 \ \forall g \in G \ \forall w \in W$ . Hence  $\langle gv, w' \rangle = 0 \ \forall w' \in W$ . Since we can choose  $w = g^{-1}w' \in W$  by  $G$ -invariance of  $W$ . Thus  $gv \in W^\perp$  since  $g$  was arbitrary.

Hence if there is a  $G$ -invariant inner product on any  $G$ -space, we get another proof of Maschke's theorem:

(3.4\*) (Weyl's unitary trick)

Let  $\rho$  be a complex representation of the finite group  $G$  on the  $\mathbb{C}$ -space  $V$ . Then there is a  $G$ -invariant Hermitian inner product on  $V$ .

**Remark.** Recall the *unitary group*  $U(V)$  on  $V$ :  $\{f \in GL(V) : (fu, fv) = (u, v) \ \forall u, v \in V\} = \{A \in GL_n(\mathbb{C}) : A\bar{A}^T = I\} (= U(n))$  by choosing orthonormal

basis.

Sheet 1 Q.12: any finite subgroup of  $GL_n(\mathbb{C})$  is conjugate to a subgroup of  $U(n)$ .

*Proof.* (2)

There exist an inner product on  $V$ : take basis  $e_1, \dots, e_n$  and define  $(e_i, e_j) = \delta_{ij}$ , extended sesquilinearly. Now

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} (gv, gw)$$

we claim that  $\langle, \rangle$  is sesquilinear, positive definite and  $G$ -invariant: if  $h \in G$ , then

$$\begin{aligned} \langle hv, hw \rangle &= \frac{1}{|G|} \sum_{g \in G} ((gh)v, (gh)w) \\ &= \frac{1}{|G|} \sum_{g' \in G} (g'v, g'w) \\ &= \langle v, w \rangle \end{aligned}$$

for all  $v, w \in V$ . □

**Definition.** (3.5, the regular representation)

Recall *group algebra* of  $G$  is  $F$ -space  $FG = \text{span}\{e_g : g \in G\}$ . There is a linear  $G$ -action

$$h \in G, h \sum_{g \in G} a_g e_g = \sum_{g \in G} a_g e_{hg} (= \sum_{g' \in G} a_{h^{-1}g'} e_{g'})$$

$\rho_{reg}$  is the corresponding representation, the *regular representation* of  $G$ . This is faithful of  $\dim |G|$ .  $FG$  is the *regular module*.

**Proposition.** Let  $\rho$  be an irreducible representation of  $G$  over a field of characteristic 0. Then  $\rho$  is isomorphic to a subrepresentation of  $\rho_{reg}$ .

*Proof.* Take  $\rho : G \rightarrow GL(V)$  irreducible and let  $0 \neq v \in V$ . Let  $\theta : FG \rightarrow V$  by  $\sum a_g e_g \rightarrow \sum a_g gv$ . Check this is a  $G$ -homomorphism. Now  $V$  is irreducible so  $\text{im } \theta = V$  (since  $\text{im } \theta$  is a  $G$ -subspace).

Also  $\ker \theta$  is  $G$ -subspace of  $FG$ . Let  $W$  be  $G$ -complement of  $\ker \theta$  in  $FG$  (Maschke), so that  $W < FG$  is  $G$ -subspace and  $FG = \ker \theta \oplus W$ . Thus  $W \cong FG / \ker \theta \cong (G\text{-isomorphism}) \text{im } \theta \cong V$ . □

More generally,

**Definition.** (3.7)

Let  $F$  be a field. Let  $G$  act on set  $X$ . Let  $FX = \text{span}\{e_x : x \in X\}$  with  $G$ -action

$$g(\sum a_x e_x) = \sum a_x e_{gx}$$

The representation  $G \rightarrow GL(V)$  where  $V = FX$  is the corresponding *permutation representation*. See section 7.

## 4 Schur's lemma

It's really unfair that such an important result is only remembered by a lemma, so we shall call it a theorem.

**Theorem.** (4.1, Schur)

- (a) Assume  $V, W$  are irreducible  $G$ -spaces over field  $F$ . Then any  $G$ -homomorphism  $\theta : V \rightarrow W$  is either 0 or an isomorphism.
- (b) Assume  $F$  is algebraically closed, and let  $V$  be an irreducible  $G$ -space. Then any  $G$ -endomorphism  $V \rightarrow V$  is a scalar multiple of the identity map  $\iota_V$ .

*Proof.* (a) Let  $\theta : V \rightarrow W$  be a  $G$ -homomorphism. Then  $\ker \theta$  is  $G$  subspace of  $V$  and, since  $V$  is irreducible, we get  $\ker \theta = 0$  or  $\ker \theta = V$ .

And  $\text{im} \theta$  is  $G$ -subspace of  $W$ , so as  $W$  is irreducible,  $\text{im} \theta$  is either 0 or  $W$ . Hence, either  $\theta = 0$  or  $\theta$  is injective and surjective, hence isomorphism.

(b) Since  $F$  is algebraically closed,  $\theta$  has an eigenvalue,  $\lambda$ . Then  $\theta - \lambda \iota$  is singular  $G$ -endomorphism of  $V$ , but it cannot be an isomorphism, so it is 0 (by (a)). So  $\theta = \lambda \iota_V$ .  $\square$

Recall from (2.8), the  $F$ -space  $\text{Hom}_G(V, W)$  of all  $G$ -homomorphisms  $V \rightarrow W$ . Write  $\text{End}_G(V)$  for the  $G$ -endomorphisms of  $V$ .

**Corollary.** (4.2)

If  $V, W$  are irreducible complex  $G$ -spaces, then

$$\dim_{\mathbb{C}} \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* If  $V, W$  are not  $G$ -isomorphic then the only  $G$ -homomorphism  $V \rightarrow W$  is 0 by (4.1). Assume  $v \cong_G W$  and  $\theta_1, \theta_2 \in \text{Hom}_G(V, W)$ , both non-zero. Then  $\theta_2$  is invertible by (4.1), and  $\theta_2^{-1}\theta_1 \in \text{End}_G(V)$ , and non-zero, so  $\theta_2^{-1}\theta_1 = \lambda \iota_V$  for some  $\lambda \in \mathbb{C}$ . Hence  $\theta_1 = \lambda \theta_2$ .  $\square$

**Corollary.** (4.3)

If finite group  $G$  has a faithful complex irreducible representation, then  $Z(G)$ , the centre of the group, is cyclic.

Note that the converse is false (Sheet 1, Q10).

*Proof.* Let  $\rho : G \rightarrow GL(V)$  be faithful irreducible complex representation. Let  $z \in Z(G)$ , so  $zg = gz \forall g \in G$ , hence the map  $\phi_z : v \rightarrow z(v)$  ( $v \in V$ ) is  $G$ -endomorphism of  $V$ , hence is multiplication by scalar  $\mu_z$ , say.

By Schur's lemma,  $z(v) = \mu_z v \forall v$ . Then the map

$$\begin{aligned} Z(G) &\rightarrow \mathbb{C}^* \text{ (multiplicative group)} \\ z &\rightarrow \mu_z \end{aligned}$$

is a representation of  $Z$  and is faithful, since  $\rho$  is. Thus  $Z(G)$  is isomorphic to some finite subgroup of  $\mathbb{C}^*$ , so is cyclic.  $\square$

Let's now consider representation of finite abelian groups.

**Corollary.** (4.4)

The irreducible  $\mathbb{C}$ -representations of a finite abelian group are all 1-dimensional.

*Proof. Either:* use (1.4)\* to invoke simultaneous diagonalisation: if  $v$  is an eigenvector for each  $g \in G$ , and if  $V$  is irreducible, then  $V = \langle v \rangle$ .

*Or:* Let  $V$  be an irreducible  $\mathbb{C}$ -representation. For  $g \in G$ , the map

$$\begin{array}{ccc} \theta_g : V & \rightarrow & v \\ v & \rightarrow & gv \end{array}$$

is a  $G$ -endomorphism of  $V$ , and as  $V$  irreducible,  $\theta_g = \lambda_g \text{id}_V$  for some  $\lambda_g \in \mathbb{C}$ . Thus  $gv = \lambda_g v$  for any  $g \in G$  (so  $\langle v \rangle$  is a  $G$ -subspace of  $V$ ). Thus as  $0 \neq V$  is irreducible,  $V = \langle v \rangle$ , which is 1-dimensional.  $\square$

**Remark.** Schur's lemma fails over non-algebraically closed field, in particular, over  $\mathbb{R}$ . For example, let's consider the cyclic group  $C_3$ . It has 2 irreducible  $\mathbb{R}$ -representations, one of dimension 1 (maps everything to 1) and one of dimension 2 (imo consider  $\mathbb{C}$  as a dimension 2 space over  $\mathbb{R}$ , then map the generator to the 3rd root of unity?) (so 'contradicting' with Schur's lemma via the corollary above).

Recall that every finite abelian group  $G$  is isomorphic to a product of cyclic groups (see GRM). For example,  $C_6 = C_2 \times C_3$ . In fact, it can be written as a product of  $C_{p^\alpha}$  for various primes  $p$  and  $\alpha \geq 1$ , and the factors are uniquely determined up to reordering.

**Proposition.** (4.5)

The finite abelian group  $G = C_{n_1} \times \dots \times C_{n_r}$  has precisely  $|G|$  irreducible  $\mathbb{C}$ -representations, as described below:

*Proof.* Write  $G = \langle x_1 \rangle \times \dots \times \langle x_r \rangle$  where  $|x_j| = n_j$ . Suppose  $\rho$  is irreducible, so by (4.4), it's 1-dimensional:  $\rho : G \rightarrow \mathbb{C}^*$ .

Let  $\rho(1, \dots, x_j, \dots, 1)$  (all 1 apart from the  $j^{\text{th}}$  entry) be  $\lambda_j$ . Then  $\lambda_j^{n_j} = 1$ , so  $\lambda_j$  is a  $n_j$ -th root of unity. Now, the values  $(\lambda_1, \dots, \lambda_r)$  determine  $\rho$ :

$$\rho(x_1^{j_1}, \dots, x_r^{j_r}) = \lambda_1^{j_1} \dots \lambda_r^{j_r}$$

thus  $\rho \leftrightarrow (\lambda_1, \dots, \lambda_r)$  with  $\lambda_j^{n_j} = 1 \forall j$ ; we have  $n_1 \dots n_r$  such  $r$ -tuples, each giving 1-dimensional representation.  $\square$

**Example.** (4.6)

Consider  $G = C_4 = \langle x \rangle$ . We could have  $\rho_1(x) = 1, \rho_2(x) = i, \rho_3(x) = -1, \rho_4(x) = -i$ .

Warning: There is no "natural" 1-1 correspondence between the elements of  $G$  and the representations of  $G$  ( $G$ -finite abelian). If you choose an isomorphism  $G \cong C_{a_1} \times \dots \times C_{a_r}$ , then we can identify the two sets (elements of groups and representations of  $G$ ), but it depends on the choice of isomorphism.

Isotypical decomposition:

Recall any diagonalisable endomorphism  $\alpha : V \rightarrow V$  gives eigenspace decomposition of  $V \cong \oplus_\lambda V(\lambda)$ , where  $V(\lambda) = \{v : \alpha v = \lambda v\}$ . This is *caconical* (one of the three useless words: *arbitrary*(anything), *canonical*(only one choice), *uniform*(you can choose, but it doesn't really matter)), in the sense that it depends on  $\alpha$  alone (and nothing else).

There is no canonical eigenbasis of  $V$ : must choose basis in each  $V(\lambda)$ .

We know that in *char* 0 every representation  $V$  decomposes as  $\oplus n_i V_i$ ,  $V_i$  irreducible,  $n_i \geq 0$ . How unique is this?

We have this wishlist (4.7):

- (a) Uniqueness: for each  $V$  there is only one way to decompose  $V$  as above. However, this doesn't work obviously.
- (b) Isotypes: for each  $V$ , there exists a unique collection of subrepresentations  $U_1, \dots, U_k$  s.t.  $V = \oplus U_i$  and, if  $V_i \subseteq U_i$  and  $V_j' \subseteq U_j$  are irreducible subrepresentations, then  $V_i \cong V_j'$  iff  $i = j$ .
- (c) Uniqueness of factors: If  $\oplus_{i=1}^k V_i \cong \oplus_{i=1}^{k'} V_i'$  with  $V_i, V_i'$  irreducible, then  $k = k'$ , and  $\exists \pi \in S_k$  such that  $V_{\pi(i)}' \cong V_i$  (Krull-Schmidt theorem). For (b),(c) see Teleman section 5.

**Lemma.** (4.8)

Let  $V, V_1, V_2$  be  $G$ -spaces over  $F$ .

- (i)  $\text{Hom}_G(V, V_1 \oplus V_2) \cong \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2)$ ;
- (ii)  $\text{Hom}_G(V_1 \oplus V_2, V) \cong \text{Hom}_G(V_1, V) \oplus \text{Hom}_G(V_2, V)$ ;

*Proof.* (i) Let  $\pi_i : V_1 \oplus V_2 \rightarrow V_i$  be  $G$ -linear projections onto  $V_i$ , with kernel  $V_{3-i}$  ( $i = 1, 2$ ).

Consider

$$\begin{aligned} \text{Hom}_G(V, V_1 \oplus V_2) &\rightarrow \text{Hom}_G(V, V_1) \oplus \text{Hom}_G(V, V_2) \\ \phi &\rightarrow (\pi_1 \phi, \pi_2 \phi) \end{aligned}$$

This map has inverse  $(\psi_1, \psi_2) \rightarrow \psi_1 + \psi_2$ . Check details.

- (ii) The map  $\phi \rightarrow (\phi|_{V_1}, \phi|_{V_2})$  has inverse  $(\psi_1, \psi_2) \rightarrow \psi_1 \pi_1 + \psi_2 \pi_2$ . □

**Lemma.** Let  $F$  be algebraically closed,  $V = \oplus_1^n V_i$  a decomposition of  $G$ -space into irreducible summands. Then, for each irreducible representation  $S$  of  $G$ ,

$$\#\{j : V_j \cong S\} = \dim \text{Hom}_G(S, V)$$

where  $\#$  means 'number of times'. This is called the *multiplicity* of  $S$  in  $V$ .

*Proof.* Induction on  $n$ .  $n = 0, 1$  are trivial.

If  $n > 1$ ,  $V = \oplus_1^{n-1} V_i \oplus V_n$ . By (4.8) we have

$$\dim \text{Hom}_G(S, \oplus_1^{n-1} V_i \oplus V_n) = \dim \text{Hom}(S, \oplus_1^{n-1} V_i) + \underbrace{\dim \text{Hom}_G(S, V_n)}_{\text{Schur's lemma}}$$

□



**Definition.** (4.10)

A decomposition of  $V$  as  $\oplus W_j$  where each  $W_j \cong n_j$  copies of irreducible representations  $S_j$  (each non-isomorphic for each  $j$ ) is the *canonical decomposition* or the decomposition into *isotypical components*  $W_j$ . For  $F$  algebraically closed,  $n_j = \dim \operatorname{Hom}_G(S_j, V)$ .

## 5 Character theory

We want to attach invariants to representation  $\rho$  of a finite group  $G$  on  $V$ . Matrix coefficients of  $\rho(g)$  are basis dependent, so not true invariants.

Let's take  $F = \mathbb{C}$ ,  $G$  finite,  $\rho = \rho_V : G \rightarrow GL(V)$  be a representation of  $G$ .

**Definition.** (5.1)

The *character*  $\chi_\rho = \chi_V = \chi$  is defined as  $\chi(g) = \text{tr } \rho(g) = \text{tr } R(g)$  where  $R(g)$  is any matrix representation of  $\rho(g)$  w.r.t. any basis.

The degree of  $\chi_V$  is  $\dim_{\mathbb{C}} V$ .

Thus  $\chi$  is a function  $G \rightarrow \mathbb{C}$ .  $\chi$  is *linear* (not a universal name) if  $\dim V = 1$ , in which case  $\chi$  is a homomorphism  $G \rightarrow \mathbb{C}^*$  ( $= GL_1(\mathbb{C})$ ).

$\chi$  is irreducible if  $\rho$  is;  $\chi$  is faithful if  $\rho$  is; and,  $\chi$  is trivial, or principal, if  $\rho$  is the trivial representation (2.6). We write  $\chi = 1_G$  in that case.

$\chi$  is a complete invariant in the sense that it determines  $\rho$  up to isomorphism – see (5.7).

**Theorem.** (5.2, first properties)

- (i)  $\chi_V(1) = \dim_{\mathbb{C}} V$ ; (clear:  $\text{tr } I_n = n$ )
- (ii)  $\chi_V$  is a *class function*, via it is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$$

Thus  $\chi_V$  is constant on conjugacy classes.

- (iii)  $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ , the complex conjugate;

- (iv) For two representations  $V, W$ ,  $\chi_{V \oplus W} = \chi_V + \chi_W$ .

*Proof.* (ii)  $\chi(hgh^{-1}) = \text{tr}(R_h R_g R_h^{-1}) = \text{tr}(R_g) = \chi(g)$ .

(iii) Recall  $g \in G$  has finite order, so we can assume  $\rho(g)$  is represented by a diagonal matrix  $\text{Diag}(\lambda_1, \dots, \lambda_n)$ . Then  $\chi(g) = \sum \lambda_i$ . Now  $g^{-1}$  is represented by the matrix  $\text{Diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$ , and hence  $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\chi(g)}$  (since  $\lambda_i$ 's are roots of unity – since  $g^k = 1$  for some  $k$ ! (I mean an exclamation mark here to express surprise) and by homomorphism we know that).

(iv) Suppose  $V = V_1 \oplus V_2$ ,  $\rho_i : G \rightarrow GL(V_i)$ ,  $\rho : G \rightarrow GL(V)$ . Take basis  $B = B_1 \cup B_2$  of  $V$  w.r.t.  $B$ ,  $\rho(g)$  has matrix of block form  $\text{Diag}([\rho_1(g)]_{B_1}, [\rho_2(g)]_{B_2})$  and as  $\chi(g)$  is the trace of the above matrix, it is equal to  $\text{tr } \rho_1(g) + \text{tr } \rho_2(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$ .  $\square$

**Remark.** We see later that  $\chi_1, \chi_2$  character of  $G$  implies that  $\chi_1 \chi_2$  is also a character of  $G$ : uses tensor products, see (9.6).

**Lemma.** (5.3)

Let  $\rho : G \rightarrow GL(V)$  be a complex representation *affording* the character  $\chi$  (i.e.  $\chi$  is a character of  $\rho$ ). Then  $|\chi(g)| \leq \chi(1)$ , with equality iff  $\rho(g) = \lambda I$  for some  $\lambda \in \mathbb{C}$ , a root of unity. Moreover,  $\chi(g) = \chi(1)$  iff  $g \in \ker \rho$ .

*Proof.* Fix  $g$ . W.r.t. basis of  $V$  of eigenvalues  $\rho(g)$ , the matrix of  $\rho(g)$  is  $\text{Diag}(\lambda_1, \dots, \lambda_n)$ . Hence  $|\chi(g)| = |\sum \lambda_j| \leq \sum |\lambda_j| = \sum 1 = \dim V = \chi(1)$ . Equality holds iff all  $\lambda_j$  are equal (to  $\lambda$ , say).

If  $\chi(g) = \chi(1)$ , then  $\rho(g) = \lambda I$  has  $\chi(g) = \lambda \chi(1)$ .  $\square$

**Lemma.** (5.4)

- (a) If  $\chi$  is a complex irreducible character of  $G$ , so is  $\bar{\chi}$ ;
- (b) Under the same assumption, so is  $\varepsilon\chi$  for any linear character  $\varepsilon$  of  $G$ .

*Proof.* If  $R : G \rightarrow GL_n(\mathbb{C})$  is a complex irreducible representation then so is  $\bar{R} : G \rightarrow GL_n(\mathbb{C})$  by  $g \rightarrow \bar{R}(g)$ . Similarly for  $R' : g \rightarrow \varepsilon(g)R(g)$  for  $g \in G$ . Check the details.  $\square$

**Definition.** (5.5)

$\mathcal{C}(G) = \{f : G \rightarrow \mathbb{C} : f(ghg^{-1}) = f(g) \forall h, g \in G\}$ , the  $\mathbb{C}$ -space of class functions (we call it a space since  $f_1 + f_2 : g \rightarrow f_1(g) + f_2(g)$ ,  $\lambda f : g \rightarrow \lambda f(g)$  are still in  $\mathcal{C}(G)$ ), so this is a vector space.

Let  $k = k(G)$  be the number of ccls of  $G$ . List the ccls  $\mathcal{C}_1, \dots, \mathcal{C}_k$ . Conventionally we choose  $g_1 = 1, g_2, \dots, g_k$ , representatives of the ccls (hence  $\mathcal{C}_1 = \{1\}$ ). Note that  $\dim_{\mathbb{C}} \mathcal{C}(G) = k$  (the characteristic functions  $\delta_j$  of each ccl which maps any element in the ccl to 1 and others to 0 form a basis).

We define Hermitian inner product on  $\mathcal{C}(G)$ :

$$\begin{aligned} \langle f, f' \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} f'(g) \\ &= \frac{1}{|G|} \sum_{j=1}^k |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) \\ &= \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \overline{f(g_j)} f'(g_j) \end{aligned}$$

using  $|\mathcal{C}_x| = |G : C_G(x)|$ , where  $\mathcal{C}_x$  is the ccl of  $x$ ,  $C_G(x)$  is the centraliser of  $x$ . For characters

$$\langle \chi, \chi' \rangle = \sum_{j=1}^k \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

is a real symmetric form (in fact,  $\langle \chi, \chi' \rangle \in \mathbb{Z}$  – see later).

**Theorem.** (5.6)

The  $\mathbb{C}$ -irreducible characters of  $G$  form an orthonormal basis of  $\mathcal{C}(G)$ . Moreover,

- (a) If  $\rho : G \rightarrow GL(V), \rho' : G \rightarrow GL(V')$  are irreducible representations of  $G$  affording characters  $\chi, \chi'$  respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho, \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

we call this 'row orthogonality'.

- (b) Each class function of  $G$  can be expressed as a linear combination of  $G$ . This will be proved later in section 6.

**Corollary.** (5.7)

Complex representations of *finite* groups are characterised by their characters. We emphasise on finiteness here: for example,  $G = \mathbb{Z}$ , consider  $1 \rightarrow I_2, 1 \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  are non-isomorphic but have same character.

*Proof.* Let  $\rho : G \rightarrow GL(V)$  be representation affording  $\chi$  ( $G$  finite over  $\mathbb{C}$ ). (3.3) says

$$\rho = m_1 \rho_1 \oplus \dots \oplus m_k \rho_k$$

where  $\rho_1, \dots, \rho_k$  are irreducible, and  $m_j \geq 0$ . Then  $m_j = \langle \chi, \chi_j \rangle$  where  $\chi_j$  is afforded by  $\rho_j$ : we have  $\chi = m_1 \chi_1 + \dots + m_k \chi_k$ , but the  $\rho_i$ 's are orthonormal.  $\square$

**Corollary.** (5.8, irreducibility criterion)

If  $\rho$  is  $\mathbb{C}$ -representation of  $G$  affording  $\chi$ , then  $\rho$  irreducible  $\iff \langle \chi, \chi \rangle = 1$ .

*Proof.* Forward is just the statement of orthonormality. Conversely, assume  $\langle \chi, \chi \rangle = 1$ . Now take a (complete) decomposition of  $\rho$  and take characters of it we get  $\chi = \sum m_j \chi_j$  with  $\chi_j$  irreducible and  $m_j \geq 0$ . Then  $\sum m_j^2 = 1$ . Hence  $\chi = \chi_j$  for some  $j$  (since the  $m_j$ 's are obviously integers), so is irreducible.  $\square$

**Corollary.** (5.9)

If the irreducible  $\mathbb{C}$ -representations of  $G$  are  $\rho_1, \dots, \rho_k$  have dimensions  $n_1, \dots, n_k$ , then

$$|G| = \sum_{i=1}^k n_i^2$$

*Proof.* Recall from (3.5),  $\rho_{reg} : G \rightarrow GL(\mathbb{C}G)$ , the regular representation  $G$  of dimension  $|G|$  (where  $\mathbb{C}G$  is just a  $G$ -space with basis  $\{e_g : g \in G\}$  and any  $h \in G$  permutes the  $e_g$ :  $e_g \rightarrow e_{hg}$ ).

Let  $\pi_{reg}$  be its character, the *regular character* of  $G$ .

Claim 1:  $\pi_{reg}(1) = |G|$ ,  $\pi_{reg}(h) = 0$  if  $h \neq 1$ .

This is clear: take  $h \in G, h \neq 1$ , then we always have 0 down the diagonal since  $h$  permutes things around, so the trace is 0; if  $h = 1$  then we have an identity matrix so trace is  $\dim \rho = |G|$ .

Claim 2:  $\pi_{reg} = \sum n_j \chi_j$  with  $n_j = \chi_j(1)$ .

This is because

$$\begin{aligned} n_j &= \langle \pi_{reg}, \chi_j \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\pi_{reg}(g)} \chi_j(g) \\ &= \frac{1}{|G|} \cdot |G| \chi_j(1) = \chi_j(1) \end{aligned}$$

(all the other  $\pi_{reg}(g)$  are zero by claim 1).

Our corollary is then obvious by just calculating  $|G| = \pi_{reg}(1)$ .  $\square$

**Corollary.** (5.10)

Number of irreducible characters of  $G$  (up to equivalence) =  $k$  (=number of ccls).

**Corollary.** (5.11)

Elements  $g_1, g_2 \in G$  are conjugate iff  $\chi(g_1) = \chi(g_2)$  for all irreducible characters of  $G$ .

*Proof.* Forward: characters are class functions;

Backward: Let  $\delta$  be the characteristic function of the class of  $g_1$ . In particular,  $\delta$  is a class function, so can be written as a linear combination of the irreducible characters of  $G$ . Hence  $\delta(g_2) = \delta(g_1) = 1$ , so  $g_2 \in \mathcal{C}_G(g_1)$ .  $\square$

In the end let's introduce a good friend which will be around for the next few lectures:

Recall from (5.5), the inner product on  $\mathcal{C}(G)$  and the real symmetric form  $\langle, \rangle$  on characters:

**Definition.** The *character table* of  $G$  is the  $k \times k$  matrix (where  $k$  is the number of ccls)  $X = [\chi_i(g_j)]$ , the  $i^{th}$  character on the  $j^{th}$  class, where we let  $\chi_1 = 1_G, \chi_2, \dots, \chi_k$  are the irreducible characters of  $G$ , and  $\mathcal{C}_1 = \{1\}, \dots, \mathcal{C}_k$  are the ccls with  $g_j \in \mathcal{C}_j$  (as we defined in 5.5).

So the  $(i, j)^{th}$  entry of  $X$  is just  $\chi_i(g_j)$ .

**Example.** (5.13)

(a)  $C_3 = \langle x : x^3 = 1 \rangle$ . The character table is

	1	$x$	$x^2$
$\chi_1$	1	1	1
$\chi_2$	1	$\omega$	$\omega^2$
$\chi_3$	1	$\omega^2$	$\omega$

where  $\omega = e^{2\pi i/3}$ .

(b)  $G = D_6 \cong S_3 = \langle r, s : r^3 = s^2 = 1, sr^{-1} = r^{-1} \rangle$ .

ccls of  $G$ :  $\mathcal{C}_1 = \{1\}$ ,  $\mathcal{C}_2 = \{r, r^{-1}\}$ ,  $\mathcal{C}_3 = \{s, sr, sr^2\}$ . We have 3 irreducible representations over  $\mathbb{C}$ :  $1_G$  (trivial);  $\mathcal{S}$  (sign):  $x \rightarrow 1$  for  $x$  even,  $x \rightarrow -1$  for  $x$  odd; and  $W$  (2-dimensional):  $sr^i$  acts by matrix with eigenvalues  $\pm 1$ ;  $r^k$  acts by the matrix

$$\begin{pmatrix} \cos 2k\pi/3 & -\sin 2k\pi/3 \\ \sin 2k\pi/3 & \cos 2k\pi/3 \end{pmatrix}$$

so  $\chi_w(sr^i) = 0 \ \forall j$ ,  $\chi_w(r^k) = 2 \cos 2k\pi/3 = -1 \ \forall k$ . So the charactable is:

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$1_G$	1	1	1
$\chi_s$	1	-1	1
$\chi_w$	2	0	-1

## 6 Proofs and orthogonality

We want to prove(5.6): irreducible characters form orthonormal basis for the space of  $\mathbb{C}$ -class functions.

*Proof.* (of 5.6 (a))

Fix bases of  $V$  and  $V'$ . Write  $R(g)$ ,  $R'(g)$  for matrices of  $\rho(g)$ ,  $\rho'(g)$  w.r.t. these bases, respectively. Then

$$\begin{aligned}\langle \chi', \chi \rangle &= \frac{1}{|G|} \chi'(g^{-1}) \chi(g) \\ &= \frac{1}{|G|} \sum_{g \in G, i, j \text{ s.t. } 1 \leq i \leq n', 1 \leq j \leq n} R'(g^{-1})_{ii} R(g)_{jj}\end{aligned}$$

the trick is to define something that annihilates almost the whole thing. Let  $\phi : V \rightarrow V'$  be linear and define

$$\begin{aligned}\tilde{\phi} : V &\rightarrow V' \\ v &\rightarrow \frac{1}{|G|} \sum_{g \in G} \rho'(g^{-1}) \phi \rho(g) v\end{aligned}$$

We claim that this is a  $G$ -homomorphism: if  $h \in G$ , let's calculate

$$\begin{aligned}\rho'(h^{-1}) \tilde{\phi} \rho(h)(v) &= \frac{1}{|G|} \sum_{g \in G} \rho'(gh)^{-1} \phi \rho(gh)(v) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho'(g'^{-1}) \phi \rho(g')(v) \\ &= \tilde{\phi}(v)\end{aligned}$$

(when  $g$  runs through  $G$ ,  $gh$  runs through  $G$  as well). So (2.8) is satisfied, i.e.  $\phi$  is a  $G$ -homomorphism.

Case 1:  $\rho, \rho'$  are not isomorphic. Schur's lemma says  $\tilde{\phi} = 0$  for any given linear  $\phi : V \rightarrow V'$ . Take  $\phi = \varepsilon_{\alpha\beta}$ , having matrix  $E_{\alpha\beta}$  (w.r.t our basis). This is 0 everywhere except 1 in the  $(\alpha, \beta)$ -position. Then  $\varepsilon_{\alpha\beta} = 0$ . So  $\frac{1}{|G|} \sum_{g \in G} (R'(g^{-1}) E_{\alpha\beta} R(g))_{ij} = 0$ . So  $\frac{1}{|G|} \sum R'(G^{-1})_{i\alpha} R(g)_{\beta j} = 0 \forall i, j$ , with  $\alpha = i, \beta = j$ . Now  $\frac{1}{|G|} \sum_{g \in G} R'(g^{-1})_{ii} R(g)_{jj} = 0$  sum over  $i, j$ . Then  $\langle \chi', \chi \rangle = 0$ . Case 2:  $\rho, \rho'$  isomorphic. So  $\chi = \chi'$ ; take  $V = V'$ ,  $\rho = \rho'$ . If  $\phi : V \rightarrow V$  is linear endomorphism, we claim  $\text{tr } \phi = \text{tr } \tilde{\phi}$ :

$$\text{tr } \tilde{\phi} = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\rho(g)^{-1} \phi \rho(g)) = \frac{1}{|G|} \sum_{g \in G} \text{tr } \phi = \text{tr } \phi$$

By Schur's lemma,  $\tilde{\phi} = \lambda \iota_V$  for some  $\lambda \in \mathbb{C}$  (depending on  $\phi$ ). Then  $\lambda = \frac{1}{n} \text{tr } \phi$ . Let  $\phi = \varepsilon_{\alpha\beta}$ . So  $\text{tr } \phi = \delta_{\alpha\beta}$ . Hence  $\varepsilon_{\alpha\beta} = \frac{1}{n} \delta_{\alpha\beta} \iota_v = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \varepsilon_{\alpha\beta} \rho(g)$ . In terms of matrices, take  $(i, j)$ -entry:  $\frac{1}{|G|} \sum_j R(g^{-1})_{i\alpha} R(g)_{\beta j} = \frac{1}{n} \delta_{\alpha\beta} \delta_{ij} \forall i, j$ . Put  $\alpha = i, \beta = j$  to get  $\frac{1}{|G|} \sum_g R(g^{-1})_{ii} R(g)_{jj} = \frac{1}{n} \delta_{ij}$ . Finally sum over  $i, j$  to get  $\langle \chi, \chi \rangle = 1$ .  $\square$

Before proving (b), let's prove column orthogonality:

**Theorem.** (6.1, column orthogonality relations)

$$\sum_{i=1}^k \overline{\chi_i(g_j)} \chi_i(g_l) = \delta_{jl} |C_G(g_j)|$$

having an easy corollary

**Corollary.** (6.2)

$$|G| = \sum_{i=1}^k \chi_i^2(1).$$

*Proof.* (of (6.1))

$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum \overline{\chi_i(g_l)} \chi_j(g_l) / |C_G(g_l)|$ . Consider the character table  $X = (\chi_i(g_j))$ . Then  $\bar{X} D^{-1} X^T = I_{k \times k}$  where  $D = \text{Diag}(|C_G(g_1)|, \dots, |C_G(g_k)|)$ .

Since  $X$  is square, it follows that  $D^{-1} \bar{X}^T$  is the inverse of  $X$ , so  $\bar{X}^T X = D$ .  $\square$

*Proof.* (of (5.6(b)))

The  $\chi_i$  generate  $\mathcal{C}_G$ . Let all the irreducible characters  $\chi_1, \dots, \chi_l$  of  $G$ : claim these generate  $\mathcal{C}_G$ , the  $\mathbb{C}$ -space of class functions on  $G$ . It's enough to show that the orthogonal complement to  $\text{span}\{\chi_1, \dots, \chi_l\}$  in  $\mathcal{C}_G$  is  $\{0\}$ . To see this, assume  $f \in \mathcal{C}_G$  with  $\langle f, \chi_j \rangle = 0 \forall j$ . Let  $\rho : G \rightarrow GL(V)$  be irreducible representation affording  $\chi \in \{\chi_1, \dots, \chi_l\}$ . Then  $\langle f, \chi \rangle = 0$ .

Consider

$$\frac{1}{|G|} \sum_G \overline{f(g)} \rho(g) : V \rightarrow V$$

This is a  $G$ -homomorphism, so as  $\rho$  is irreducible, it must be  $\lambda_i$  for some  $\lambda \in \mathbb{C}$ .

Now

$$\begin{aligned} n\lambda &= \text{tr} \frac{1}{|G|} \sum_g \overline{f(g)} \rho(g) \\ &= \frac{1}{|G|} \sum_g \overline{f(g)} \chi(g) = 0 = \langle f, \chi \rangle \end{aligned}$$

So  $\lambda = 0$ . Hence  $\sum \overline{f(g)} \rho(g) = 0$ , the zero endomorphism on  $V$  for all representations  $\rho$  (complete reducibility).

Take  $\rho = \rho_{\text{reg}}$  where  $\rho_{\text{reg}}(g) : e_1 \rightarrow e_g$  ( $g \in G$ ). So

$$\sum_g \overline{f(g)} \rho_{\text{reg}}(g) : e_1 \rightarrow \sum_g \overline{f(g)} e_g$$

So it follows  $\sum_g \overline{f(g)} e_g = 0$ . So  $\overline{f(g)} = 0 \forall g \in G$ , so  $f \equiv 0$ .  $\square$

Various corollaries now follow:

- The number of irreducible representations of  $G$  = number of ccls; (5.10)
- Column orthogonality (6.1);
- $|G| = \sum n_i^2$  (6.2);
- $g_1 \sim g_2 \iff \chi(g_1) = \chi(g_2)$  for all irreducible  $\chi$  (5.11);
- If  $g \in G$ ,  $g \sim g^{-1} \iff \chi(g) \in \mathbb{R}$  for all irreducible  $\chi$ .

## 7 Permutation representations

Preview was given in (3.7). Recall: •  $G$  finite group acting on finite set  $X = \{x_1, \dots, x_n\}$ ;

•  $\mathbb{C}X = \mathbb{C}$ -space, with basis  $\{e_{x_1}, \dots, e_{x_n}\}$  of dimension  $|X|$ , so is  $\{\sum_j a_j e_{x_j} : a_j \in \mathbb{C}\}$ ;

• corresponding permutation representation  $\rho_X : G \rightarrow GL(\mathbb{C}X)$  by  $g \rightarrow \rho(g)$ , where  $\rho(g)$  sends  $e_{x_j} \rightarrow e_{gx_j}$ , extending linearly.

•  $\rho_X$  is the *permutation representation* corresponding to the action of  $G$  on  $X$ .

• matrices representing  $\rho_X(g)$  w.r.t. basis  $\{e_x\}_{x \in X}$  are permutation matrices: 0 except for one 1 in each row and column, and  $(\rho(g))_{ij} = 1$  iff  $gx_j = x_i$ . Consider its character:

(7.1) Permutation character,  $\pi_X$ , is

$$\pi_X(g) = |\text{Fix}_X(g)| = |\{x \in X : gx = x\}|.$$

(7.2)  $\rho_X$  always contains  $1_G$ :  $\text{span}\{e_{x_1} + \dots + e_{x_n}\}$  is a trivial  $G$ -subspace of  $\mathbb{C}X$  with  $G$ -invariant complement  $\text{span}\{\sum a_x e_x : \sum a_x = 0\}$ .

**Lemma.** (7.3, Burnside's lemma, after Cauchy, Frobenius)  $\langle \pi_X, 1 \rangle =$  number of orbits of  $G$  on  $X$ .

*Proof.* If  $X = X_1 \cup \dots \cup X_l$  disjoint union of orbits, then  $\pi_X = \pi_{X_1} + \dots + \pi_{X_l}$ , with  $\pi_{X_j}$  permutation character of  $G$  on  $X_j$ , so to prove the claim it's enough to show that if  $G$  is transitive on  $X$  then  $\langle \pi_X, 1 \rangle = 1$ . Assume  $G$  is transitive on  $X$ . Now

$$\begin{aligned} \langle \pi_X, 1 \rangle &= \frac{1}{|G|} \sum_g \pi_X(g) = \frac{1}{|G|} |\{(g, x) \in G \times X : gx = x\}| \\ &= \frac{1}{|G|} \sum_{x \in X} |G_x| = \frac{1}{|G|} |X| |G_x| = \frac{1}{|G|} |G| = 1 \end{aligned}$$

(Note the use of orbit-stabilizer theorem). □

**Lemma.** (7.4)

Let  $G$  act on the sets  $X_1, X_2$ . Then  $G$  acts on  $X_1 \times X_2$  via  $g(x_1, x_2) = (gx_1, gx_2)$ . The character  $\pi_{X_1 \times X_2} = \pi_{X_1} \pi_{X_2}$  and so  $\langle \pi_{X_1}, \pi_{X_2} \rangle =$  number of orbits of  $G$  on  $X_1 \times X_2$ .

*Proof.* If  $g \in G$  then  $\pi_{X_1 \times X_2}(g) = \pi_{X_1}(g) \pi_{X_2}(g)$ . And we have

$$\langle \pi_{X_1}, \pi_{X_2} \rangle = \langle \pi_{X_1} \pi_{X_2}, 1 \rangle = \langle \pi_{X_1 \times X_2}, 1 \rangle = (7.3) \text{ number of orbits of } G \text{ on } X_1 \times X_2.$$

□

**Definition.** (7.5)

Let  $G$  act on  $X$ ,  $|X| > 2$ . Then  $G$  is *2-transitive* on  $X$  if  $G$  has precisely two orbits on  $X \times X : \{(x, x) : x \in X\}$  and  $\{(x_1, x_2) : x_i \in X, x_1 \neq x_2\}$ .



**Lemma.** (7.6)

Let  $G$  act on  $X$ ,  $|X| > 2$ . Then  $\pi_X = 1 + \chi$  with  $\chi$  irreducible  $\iff G$  is 2-transitive on  $X$ .

*Proof.*  $\pi_X = m_1 1 + m_2 \chi_2 + \dots + m_l \chi_l$  with  $1, \chi_2, \dots, \chi_l$  distinct irreducible characters and  $m_i \in \mathbb{N}$ . Then

$$\langle \pi_X, \pi_X \rangle = \sum_{i=1}^l m_i^2$$

hence  $G$  is 2-transitive on  $X \iff l = 2, m_1 = m_2 = 1$ .  $\square$

**Example.** (7.7)

Consider  $S_n$  acting on  $X = \{1, \dots, n\}$  which is 2-transitive. Hence  $\pi_X = 1 + \chi$  with  $\chi$  irreducible of degree  $n - 1$ . Similarly for  $A_n$  ( $n > 3$ ).

**Example.** (7.8)

$\subset \subset 1$	1	3	8	6	
rep	1	$(1, 2)(3, 4)$	$(1, 2, 3)$	$(1, 2, 3, 4)$	
$\chi_1$	1	1	1	1	
$\chi_2$	1	1	1	-1	
$\chi_3$	3	-1	0	-1	
$\chi_4$	3	-1	0	1	
$\chi_5$	2	...	...	...	

Consider  $G = S_4$ .