Introduction to Discrete Analysis

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0 Introduction

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1 The discrete Fourier transform

Let N be a fixed positive integer. Write ω for $e^{2\pi i/N}$, and $\mathbb{Z}_N$ for $\mathbb{Z}/n\mathbb{Z}$. Let $f : \mathbb{Z}_N \to \mathbb{C}$. Given $r \in \mathbb{Z}_N$, define $\hat{f}(r)$ to be

$$\frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x)\omega^{-rx}$$

From now on we use the notation $\mathbb{E}_{x\in\mathbb{Z}_N}$ for $\frac{1}{N}\sum_{x\in\mathbb{Z}_N}$, so $\hat{f}(r) = \mathbb{E}_x f(x)e^{-\frac{2\pi i r x}{N}}$.

If we write ω_r for the function $x \to \omega^{rx}$, and $\langle f, g\rangle$ for $\mathbb{E}_x f(x)\overline{g(x)}$, then $\hat{f}(r) = \langle f, \omega_r\rangle$. So the discrete fourier transforn is basically expanding the function f in the set of orthonormal basis ω_r.

Let us write $||f||_p$ for $\mathbb{E}_x|f(x)|^p)^{1/p}$ (the L_p-norm), and call the resulting space $L_p(\mathbb{Z}_n)$.

Important convention: we use *averages* for the 'original functions' in 'physical spaces', and *sums* for their Fourier transforms in 'frequency space' (referring to $\mathbb{E}$: $\langle , \rangle$ is average in the original space but just $\sum$ in frequency space, i.e. for $\hat{f}, \hat{g}$ etc.)

Lemma. (Parseval's identity)
If $f, g : \mathbb{Z}_n \to \mathbb{C}$, then $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$.

Proof.

$$\begin{aligned}
\langle \hat{f}, \hat{g}\rangle &= \sum_r \hat{f}(r)\overline{\hat{g}(r)} \\
&= \sum_r (\mathbb{E}_x f(x)\omega^{-rx})(\overline{\mathbb{E}_y g(y)\omega^{-ry}}) \\
&= \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)}\sum_r \omega^{-r(x-y)} \\
&= \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)} n\delta_{xy} \\
&= \langle f, g\rangle
\end{aligned}$$

□

Lemma. (Convolution identity)

$$\widehat{f * g}(r) = \hat{f}(r)\hat{g}(r)$$

where

$$(f * g)(x) = \mathbb{E}_{y+z=x} f(y)g(z) = \mathbb{E}_y f(y)g(x-y)$$

Proof.

$$\begin{aligned}\widehat{f*g}(r) &= \mathbb{E}_x f*g(x)\omega^{-rx}\\ &= \mathbb{E}_x\mathbb{E}_{y+z=x} f(y)g(z)\omega^{-rx}\\ &= \mathbb{E}_x\mathbb{E}_{y+z=x} f(y)g(z)\omega^{-ry}\omega^{-rz}\\ &= \mathbb{E}_y\mathbb{E}_z f(y)\omega^{-ry}g(z)\omega^{-rz}\\ &= \hat{f}(r)\hat{g}(r)\end{aligned}$$

□

Lemma. (Inversion formula)

$$f(x) = \sum_r \hat{f}(r)\omega^{rx}$$

(note the sign of ω^{rx}).

Proof.

$$\begin{aligned}\sum_r \hat{f}(r)\omega^{rx} &= \sum_r \mathbb{E}_y f(y)\omega^{r(x-y)}\\ &= \mathbb{E}_y f(y)\sum_r \omega^{r(x-y)}\\ &= \mathbb{E}_y f(y) n\delta_{xy}\\ &= f(x)\end{aligned}$$

This is really just the statement that we get the original vector back when we sum up its components. □

Further observations: If f is real-valued, then $\hat{f}(-r) = \mathbb{E}_x f(x)\omega^{rx} = \overline{\mathbb{E}_x f(x)\omega^{-rx}} = \overline{\hat{f}(r)}$.

If $A \subset \mathbb{Z}_n$, write A (instead of $1_A, \chi_A$) for the characteristic function of A. Then $\hat{A}(0) = \mathbb{E}_x A(x) = \frac{|A|}{N}$, the *density* of A.

Also, $||\hat{A}||_2^2 = \langle \hat{A}, \hat{A}\rangle = \langle A, A\rangle = \mathbb{E}_x A(x)^2 = \mathbb{E}_x A(x) = \frac{|A|}{N}$, again the density.

Let $f : \mathbb{Z}_n \to \mathbb{C}$. Given $\mu \in \mathbb{Z}_n$, define $f_\mu(x)$ to be $f(\mu^{-1}x)$ (so we need $(\mu, N) = 1$). Then

$$\begin{aligned}\hat{f}_\mu(r) &= \mathbb{E}_x f_\mu(x)\omega^{-rx}\\ &= \mathbb{E}_x f(x/\mu)\omega^{-rx}\\ &= \mathbb{E}_x f(x)\omega^{-r\mu x}\\ &= \hat{f}(\mu r)\end{aligned}$$

1.1 Roth's theorem

Theorem. For every $\delta > 0$, $\exists N$ s.t. if $A \subset \{1, ..., N\}$ is a set of size at least δN, then A must contain an arithmetic progression of length 3.

This is also true for 4,5,..., but the proof is much harder – Szemeredi's theorem. Basic strategy of proof: show that if A has density δ and no AP of length 3 (3AP), then there's a long AP in $P \subset \{1, 2, ..., n\}$ s.t.

$$|A \cap P| \geq (\delta + c(\delta))|p|$$

where $c(\delta)$ is some positive number. But then we can continue this argument to expand $A \cap P$ to infinity (note that $|A \cap P|$ is an integer, so each time increase by 1 at least).

The best known relationship between δ and the N required is around $\delta \sim \frac{c}{\log \log N}$ for some constant c.