Representation Theory

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0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over $\mathbb{C}$. Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise:
• F is a field, usually $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or $\bar{F}_p$ (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
• V is a vector space over F, always finite dimensional;
$GL(V) = \{\theta : V \to V, \theta \text{ linear, invertible}\}$, i.e. $\det \theta \neq 0$.

Recall from Linear Algebra:
If $\dim_F V = n < \infty$, choose basis $e_1, ..., e_n$ over F, so we can identify it with F^n. Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \to A_\theta$ ($A_{\theta_1\theta_2} = A_{\theta_1} A_{\theta_2}$ and bijection). Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = X A_1 X^{-1}$.

Recall that $\mathrm{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A.

(1.3) $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$, hence we can define $\mathrm{tr}(\theta) = \mathrm{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V in f.d. over $\mathbb{C}$, with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $EndV$ is the set of all ilnear maps $V \to V$, e.g. $End(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over $\mathbb{C}$, $\alpha \in End(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1}(X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a $\mathbb{C}$-vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:
(1.6) The symmetric group, $S_n = Sym(X)$ on the set $X = \{1, ..., n\}$ is the set of all permutations of X. $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m: $C_m =< x : x^m = 1 >$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in $\mathbb{C}$ (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of $\mathbb{C}$). We also have the group of rotations, centre O of regular $m-$gon in $\mathbb{R}^2$ (inside $GL_2(\mathbb{R})$).

(1.8) Dihedral groups D_{2m} of order $2m =< x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} >$. Think of this as the set of rotations and reflections preserving a regular m-gon.

(1.9) Quaternion group, $Q_8 =< x, y|x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} >$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right), j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), k = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \to X$ by $(g, x) \to g \cdot x$ for $g \in G$, $x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X, we obtain a homomorphism $\theta : G \to Sym(X)$, called the *permutation representation* of G.

Proof. For $g \in G$, the function $\theta_g : X \to X$ by $x \to gx$ is a permutation on X, with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G$, $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$ since $(g_1g_2)x = g_1(g_2x)$ for $x \in X$. $\square$

2 Basic Definitions

2.1 Representations

Let G be finite, F be a field, usually $\mathbb{C}$.

Definition. (2.1)
Let V be a f.d. vector space over F. A (linear, in some books) *representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \quad \to GL(V)$$

Write ρ_g for the image $\rho_V(g)$; so for each $g \in G$, $\rho_g \in GL(V)$, and $\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}$, and $(\rho_g)^{-1} = \rho_{g^{-1}}$.
The *dimension* (or *degree*) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \lhd G$ (kernel is a normal subgroup), and $G/\ker\rho \cong \rho(G) \leq GL(V)$ (1st isomorphism theorem). We say ρ is *faithful* if $\ker\rho = 1$.

An alternative (and equivalent) approach is to observe that a representation of G on V is "the same as" a *linear action* of G:

Definition. (2.3)
G *acts linearly* on V if there exists a *linear action*

$$\begin{aligned} G \times V &\to V \\ (g, v) &\to gv \end{aligned}$$

By linear action we mean: (action) $(g_1 g_2)v = g_1(g_2 v)$, $1v = v$ $\forall g_1, g_2 \in G, v \in V$, and (linear) $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv$ $\forall g \in G, v_1, v_2 \in V, \lambda \in F$.
Now if G acts linearly on V, the map

$$\begin{aligned} G &\to GL(V) \\ g &\to \rho_g \end{aligned}$$

with $\rho_g : v \to gv$ is a representation of G. Conversely, given a representaiton $\rho : G \to GL(V)$, we have a linear action of G on V via $g \cdot v := \rho(g)v$ $\forall v \in V, g \in G$.

(2.4) In (2.3) we also say that V is a G-space or that V is a G-module. In fact if we define the *group algebra* FG, or $F[G]$, to be $\{\sum \alpha_j g : \alpha_j \in F\}$ with natural addition and multiplication, then V is actually a FG-module (in the sense from GRM).

(2.5) R is a *matrix representation* of G of degree n if R is a homomorphism $G \to GL_n(F)$. Given representation $\rho : G \to GL(V)$ with $\dim_F V = n$, fix basis B; we get matrix representation

$$\begin{aligned} G &\to GL_n(F) \\ g &\to [\rho(g)]_B \end{aligned}$$

Conversely, given matrix representation $R : G \to GL_n(F)$, we get representation

$$\begin{aligned} \rho : G &\to GL(F^n) \\ g &\to \rho_g \end{aligned}$$

via $\rho_g(v) = R_g v$ where R_g is the matrix of g.

Example. (2.6)
Given any group G, take $V = F$ the 1-dimensional space, and

$$\begin{aligned} \rho : G &\to GL(F) \\ g &\to (id : F \to F) \end{aligned}$$

is known as the trivial representation of G. So $\deg \rho = 1$ ($\dim_F F = 1$).

Example. (2.7)
Let $G = C_4 =< x : x^4 = 1 >$. Let $n = 2$, and $F = \mathbb{C}$. Note that any $R : x \to X$ will determine $x^j \to X^j$ as it is a homomorphism, and also we need $X^4 = I$. So we can take X to be diagonal matrix – any such with diagonal entries a root to $x^4 = 1$, i.e. $\{\pm 1, \pm i\}$, or if X is not diagonal then it will be similar to a diagonal matrix by (1.4) ($X^4 = I$).

2.2 Equivalent representations

Definition. (2.8)
Fix G, F. Let V, V' be F-spaces, and $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ which are representations of G. The linear map $\phi : V \to V'$ is a G-homomorphism if

$$\phi\rho(g) = \rho'(g)\phi \forall g \in G (*)$$

We can understand this more by the following diagram:

$$\begin{tikzcd} V \arrow[r, "\rho_g"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\curvearrowright"] & V \arrow[d, "\varphi"] \\ V' \arrow[r, "\rho_g'"] & V' \end{tikzcd}$$

We say ϕ *intertwines* ρ, ρ'. Write $Hom_G(V, V')$ for the F-space of all these. ϕ is a G-isomorphism if it is also bijective; if such ϕ exists, ρ, ρ' are isomorphic/equivalent representations. If ϕ is a G-isomorphism, we can write $(*)$ as $\rho' = \phi\rho\phi^{-1}$.

Lemma. (2.9)
The relation "being isomorphic" is an equivalent relation on the set of all representations of G (over F).

Remark. (2.10)
If ρ, ρ' are isomorphic representations, they have the same dimension.

The converse may be false: C_4 has four non-isomorphic 1-dimensional representations: if $\omega = e^{2\pi i/4}$ then they are $\rho_j(x^i) = \omega^{ij}$ $(0 \leq i \leq 3)$.

Remark. (2.11)
Given G, V over F of dimension n and $\rho : G \to GL(V)$. Fix basis B for V: we get a linear isomorphism

$$\begin{aligned} \phi : V &\to F^n \\ v &\to [v]_B \end{aligned}$$

and we get a representation $\rho' : G \to GL(F^n)$ isomorphic to ρ:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\varphi"] & V \arrow[d, "\varphi"] \\ F^n \arrow[r, "\rho'"'] & F^n \end{tikzcd}$$

(2.12) In terms of matrix representations, we have

$$\begin{aligned} R : G &\to GL_n(F), \\ R' : G &\to GL_n(F) \end{aligned}$$

are (G)-isomorphic or equivalent if there exists a nonsingular matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1}$ $\forall g \in G$.

In terms of linear G-actions, the actions of G on V,V' are G-isomorphic if there exists isomorphisms $\phi : V \to V'$ such that $g : \phi(v) = \phi(gv)$ $\forall v \in V, g \in G$.

2.3 Subrepresentations

Definition. (2.13)
Let $\rho : G \to GL(V)$ be a representation of G. We say $W \leq V$ is a G-subspace if it's a subspace and it is $\rho(G)$-invariant, i.e. $\rho_g(W) \leq W \forall g \in G$. Obviously $\{0\}$ and V are G-subspaces, however.
ρ is *irreducible/simple* representation if there are no proper G-subspaces.

Example. (2.14)
Any 1-dimensional representaiton of G is irreducible, but not conversely, e.g. D_8 has 2-dimensional $\mathbb{C}$-irreducible representation.

(2.15) In definition (2.13), if W is a G-subspace, then the corresponding map

$$\begin{aligned} G &\to GL(W) \\ g &\to \rho(g)|_W \end{aligned}$$

is a representation of G, a *subrepresentation* of ρ.

Lemma. (2.16)
In definition (2.13), given $\rho : G \to GL(V)$, if W is a G-subspace of V and if $B = \{v_1, ..., v_n\}$ is a basis containing basis $B_1 = \{v_1, ..., v_m\}$ of W $(0 < m < n)$ then the matrix of $\rho(g)$ w.r.t. B has block upper triangular form as the graph below, for

$$\begin{array}{cc} & \begin{array}{cc} m & \end{array} \\ m & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

each $g \in G$.

Example. (2.17)
(i) The irreducible representaitons of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1-dimensional and four of these are $x \to i, x \to -1, x \to -i, x \to 1$. In general, $C_m = \langle x : x^m = 1 \rangle$ has precisely m irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)* or see (4.4) below).
(ii) $G = D_6$: any irreducible C-representation has dimension ≤ 2.
Let $\rho : G \to GL(V)$ be irreducible G-representation. Let r, s be rotation and reflection in D_6 respectively. Let V be eigenvector of $\rho(r)$. So $\rho(r)v = \lambda v$

for some $\lambda \neq 0$. Let $W = span\{v, \rho(s)v\} \leq V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, both of which are in W; so W is G-invariant, i.e. a G-subspace. Since V is irreducible, $W = V$.

Definition. (2.18)
We say at $\rho : G \to GL(V)$ is *decomposable* if there are proper G-invariant subspaces U, W with $V = U \oplus W$. Say ρ is direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists, we say that ρ is *indecomposable.*

Lemma. (2.19)
Suppose $\rho : G \to GL(V)$ is decomposable with G-invariant decomposition $V = U \oplus W$. If B is a basis $\{\underbrace{u_1, ..., u_k}_{B_1}, \underbrace{w_1, ..., w_l}_{B_2}\}$ of V consisting of basis of U and basis of W, then w.r.t. B, $\rho(g)_B$ is a block diagonal matrix $\forall g \in G$ as

$$\rho(g)_B = \begin{pmatrix} [\rho_W(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

Definition. (2.20)
If $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$, the *direct sum* of ρ, ρ' is

$$\rho \oplus \rho' : G \to GL(V \oplus V')$$

where $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$, a *block diagonal action.* For matrix representations $R : G \to GL_n(F)$, $R' : G \to GL_{n'}(F)$, define $R \oplus R' : G \to GL_{n+n'}(F)$:

$$g \to \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Definition. (3.1)
A representation $\rho : G \to GL(V)$ is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

Remark. (3.2)
(1) The converse is false;
(2) See sheet 1 Q3: $\mathbb{C}$-representaiton of $\mathbb{Z}$ is not completely reducible and also representaiton of C_p over $\mathbb{F}_p$ is not c.r..

Fron now on, take G finite and *char* $F = 0$.

Theorem. (3.3)
Every f.d. representation V of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus ... \oplus V_r$$

is a direct sum of representations, each V_i irreducible.

It is enough to prove:

Theorem. (3.4 Maschke's theorem, 1899)
Let G be finite, $\rho : G \to GL(V)$ a f.d. representation, *char* $F = 0$. If W is a G-subspace of V, then there exists a G-subspace U of V s.t. $V = W \oplus U$, a direct sum of G-subspaces.

Proof. (1)
Let W' be any *vector subspace* complement of W in V, i.e. $V = W \oplus W'$ as vector spaces, and $W \cap W' = 0$. Let $q : V \to W$ be th projection of V onto W along W' ($\ker q = W'$), i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\bar{q} : v \to \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of q over G. Note that in order for $\frac{1}{|G|}$ to exists, we need *char* $F = 0$. It still works if *char* $F \nmid |G|$.
Claim (1): $\bar{q} : V \to W$: For $v \in V$, $g(q^{-1}v) \in W$ and $gW \leq W$;
Claim (2): $\bar{q}(w) = w$ for $w \in W$:

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum g(g^{-1}w) = \frac{1}{|G|} \sum w = w$$

So these two claims imply that $\bar{q}$ projects V onto W.

Claim (3) If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ $(v \in V)$:

$$\begin{aligned} h\bar{q}(v) &= h\frac{1}{|G|}\sum_g g \cdot q(g^{-1}v) \\ &= \frac{1}{|G|}\sum_g hgq(g^{-1}v) \\ &= \frac{1}{|G|}\sum (hg)q((hg)^{-1}hv) \\ &= \frac{1}{|G|}\sum_g gq(g^{-1}(hv)) \\ &= \bar{q}(hv \\ &= \bar{q}(hv)) \end{aligned}$$

We'll then show that the kernel of this map is G-invariant, so this gives a G-summand on Thursday. $\square$