Advanced Financial Models

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0 Introduction

www.staslab.cam.ac.uk/ mike/AFM/ for course material. However lecture notes only come after lectures, so taking notes is still necessary..

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Assumptions for this course:
No dividends, zero tick size (continuous), no transaction costs, no short-selling constraints, infinitely divisible assets, no bid-ask spread, infinite market depth, agents have preferences for expected utility.

1 Discrete time models

We'll assume there are n assets with price P_t^i at time t for asset i. Apparently P_t^i is a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We'll use the notation $P = (P_t^1, ..., P_t^n)_{t\geq 0}$ which is a n-dimensional stochastic process.

Information available at time t is modelled by a σ-algebra $\mathcal{F}_t \subseteq \mathcal{F}$.

The assumption will be $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$ (in other words, $\mathcal{F}$ is a filtration):

Definition. (Filtration)
A *Filtration* is a collection of σ-algebra $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

We'll assume that $\mathcal{F}_0$ is trivial, i.e. if A is $\mathcal{F}_0$ measure then $\mathbb{P} = 0$ or 1. As a result $P_0^1, ..., P_0^n$ are constants.

We assume that $(P_t)_{t\geq 0}$ is adapted to the filtration, i.e. P_t is $\mathcal{F}_t$-measurable for all $t \geq 0$. Usually we assume the filtration is generated by P itself (so all the information is in the price).

Let c_t be the amount consumed at time t which will be a $\mathcal{F}_t$-measurable scalar, $H_t = (H_t^1, ..., H_t^n)$ be the vector of portfolio weights, so H_t^i is the number of shares held at $(t-1, t]$ (remember we are in discrete time), which will be $\mathcal{F}_{t-1}$-measurable. $(c_t)_{t\geq 0}$ is adapted ($\mathcal{F}_t$-measurable), $(H_t)_{t\geq 1}$ is predictable/previsible ($\mathcal{F}_{t-1}$-measurable, n-dimensional.

Definition. The process (c, H) is self-financing if $H_t \cdot P_t = c_t + H_{t+1} \cdot P_t$ for all $t \geq 1$.

For example, X_0 be the initial wealth, $X_0 - c_0$ is the post-consumption wealth $(= H_1 \cdot P_0)$, $X_1 = H_1 \cdot P_1$ is the pre-consumption wealth at time 1, $X_1 - c_1$ is the post consumption wealth (which is $H_2 \cdot P_1$. In other words, at any specific time, consumption takes place and then price updates.

Background assumption on behaviour of an agent:
Say c^2 is preffered to c^1 iff $\mathbb{E}U(c_0^1, c_1^1, ...) < \mathbb{E}U(c_0^2, c_1^2, ...)$ where U is some investor utility function, which is increasing in all c_i, concave (so risk-avert)

Definition. An arbitrage is a self-financing investment-consumption strategy (c, H) such that there exists a non-random time $T > 0$ s.t. $c_0 = -H_1 \cdot P_0$, $c_t = (H_t - H_{t+1}) \cdot P_t$ for $1 \leq t \leq T-1$, and $c_T = H_T \cdot P_T$ (in words, with initial wealth $X_0 = 0$ and post-consumption wealth at T $X_T - C_T = 0$), that $\mathbb{P}(c_t \geq 0)$ for all $0 \leq t \leq T) = 1$, and $\mathbb{P}(c_t > 0$ for some $0 \leq t \leq T) > 0$.

Suppose (c^1, H^1) is self-financing with iniital wealth X_0, $(c^2, H^2) = (c^1, H^1) + (c, H)$, where (c, H) is an arbitrage. Then c^2 is preferred to c^1.

The inverstor who believes there is an arbitrage would have no optimal investment-consumption policy.

Even further background assumption: the market is in equillibrium (supply = demand).

Definition. Given the market model, a martingale deflator is positive adapted process $(Y_t)_{t\geq 0}$ such that $(Y_t \cdot P_t)_{t\geq 0}$ is a martingale.

Theorem. (First fundamental theorem of asset pricing)
The market has no arbitrage if and only if there exists a martingale deflator.

Definition. Given an $(\Omega, \mathcal{F}, \mathbb{P})$-integrable X ($\mathbb{E}|X| < \infty$), and $\mathcal{G} \subseteq \mathcal{F}$) a sub-$\sigma$-algebra of $\mathcal{F}$, a conditional expectation of X given $\mathcal{G}$ is an integrable Y that is $\mathcal{G}$-measurable and such that $\mathbb{E}(X1_G) = \mathbb{E}(Y1_G)$ for all $G \in \mathcal{G}$.

Theorem. The conditional expectations exist and are unique in the sense that, if Y^1, Y^2 are both conditional expecations, then $Y^1 = Y^2$ almost surely.

We'll use the notation $Y = \mathbb{E}(X|\mathcal{G})$.

Example. Let $(G_n)_n$ be a partition of Ω, $\mathcal{G} = \sigma(G_n)$. Then $\mathbb{E}(X|\mathcal{G})(\omega) = \frac{\mathbb{E}(X1_{G_n})}{\mathbb{P}(G_n)} = \mathbb{E}(X|G_n)$ if $\omega \in G_n$ and $\mathbb{P}(G_n) > 0$, or anything else if $\mathbb{P}(G_n) = 0$ is a valid conditional expectation.

Theorem. Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra, and assume all conditional expectations we need exist, i.e. all random variables are integrable.
• 1. Linearity: $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})\ \forall a, b \in \mathbb{R}$;
• 2. Positivity: If $X \geq 0$ a.s., then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.; furthermore, if $\mathbb{E}(X|\mathcal{G}) = 0$ a.s., then $X = 0$ a.s..
• 3. Jensen's inequality: If f is convex, then $\mathbb{E}(f(X)|\mathcal{G}) \geq f(\mathbb{E}(X|\mathcal{G}))$ a.s. (here we obviously assume f is integrable).
• 4. Monotone convergence: If $X_n \geq 0\ \forall n$ a.s. and $X_n \nearrow X$ a.s., then $\mathbb{E}(X_n|\mathcal{G}) \nearrow \mathbb{E}(X|\mathcal{G})$.
• 5. Fatou's lemma: If $X_n \geq 0$, then $\liminf \mathbb{E}(X_n|\mathcal{G}) \geq \mathbb{E}(\liminf X_n|\mathcal{G})$.
• 6. Dominated convergence: If $\mathbb{E}(\sup|X_n|) < \infty$ and $X_n \to X$ a.s., then $\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G})$ a.s..
The derivation of the above are similar to that of normal expectations. Now let's introduce something that's only for conditional expectations:
• 7. If X and $\mathcal{G}$ are *independent*, i.e. $\{X \leq x\}$ and G are independent for all $x \in \mathbb{R}$, $G \in \mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.
• 8. Slot property: If X and XY are integrable, Y is $\mathcal{G}$-measurable, then $\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G})$. In particular, $\mathbb{E}(Y|\mathcal{G}) = Y$.
• 9. Tower property: If $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, then

$$\mathbb{E}\left(E(X|\mathcal{G})|\mathcal{H}\right) = \mathbb{E}\left(\mathbb{E}(X|\mathcal{H})|\mathcal{G}\right) = \mathbb{E}(X|\mathcal{H})$$

Definition. Given $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $(\mathcal{F}_t)_{t\geq 0}$, a martingale $\mathcal{M} = (\mathcal{M}_t)_{t\geq 0}$ is an adapted process (i.e. $\mathcal{M}_t is \mathcal{F}_t$-measurable), and integrable process (i.e $\mathbb{E}(|\mathcal{M}_t) < \infty \forall t)$, and $\mathbb{E}(\mathcal{M}_t|\mathcal{F}_f) = \mathcal{M}_s$ for $0 \leq s \leq t$.

Remark. This definition is the same in both discrete and continuous time.

Proposition. In discrete time, an integrable adapted process $(\mathcal{M}_t)_{t\geq 0}$ is a martingale iff $\mathbb{E}(\mathcal{M}_{t+1}|\mathcal{F}_t) = \mathcal{M}_t$ for all $t \geq 0$.

Proof. If $\mathcal{M}$ is a martingale then this condition holds trivially by definition. Now suppose $\mathbb{E}(\mathcal{M}_{t+1}|\mathcal{F}_t) = \mathcal{M}_t \ \forall t \geq 0$, we claim $\mathbb{E}(\mathcal{M}_T|\mathcal{F}_t) = \mathcal{M}_t$ for all $T \geq t$. Prove by induction:
This is true for $T = t+1$. Now $\mathbb{E}(\mathcal{M}_{T+1}|\mathcal{F}_t) = \mathbb{E}(\mathbb{E}(\mathcal{M}_{T+1}|\mathcal{F}_T)|\mathcal{F}_t) = \mathbb{E}(\mathcal{M}_T|\mathcal{F}_t)$ by tower property, so done. □

Example. 1. (Forward construction) Given $\xi_1, \xi_2, ...$ independent and integrable, $\mathbb{E}(\xi_n) = 0 \forall n$, $S_0 = 0$ and $S_t = \xi_1 + ... + \xi_t$ for $t \geq 1$. Then $(S_t)_{t \geq 0}$ is a martingale (relative to the filtration generated by ξ_i ,i.e. $\mathcal{F} = \sigma(\xi_1, ...\xi_t)$).

Proof. Adaptedness and integrability are easy (measurability and integrability are preserved by additions);
Now
$$\begin{aligned}\mathbb{E}(S_{t+1}|\mathcal{F}_t) &= \mathbb{E}(S_t + \xi_{t+1}|\mathcal{F}_t) \\ &= \mathbb{E}(S_t|\mathcal{F}_t) + \mathbb{E}(\xi_{t+1}|\mathcal{F}_t) \\ &= S_t + \mathbb{E}(\xi_{t+1}) = S_t\end{aligned}$$
we've used that S_t is $\mathcal{F}_t$-measurable and the independence of ξ_i. □

2. (Backward construction) Given X integrable, and a filtration $(\mathcal{F}_t)_{t \geq 0}$, let
$$M_t = \mathbb{E}(X|\mathcal{F}_t)$$
then $(M_t)_{t \geq 0}$ is a martingale.

Proof. As before we don'e have to do adaptedness and integrability as conditional expectations are integrable and adapted. Now
$$\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(X|\mathcal{F}_s) = M_s \forall t > s$$
by tower property. □