Introduction to Discrete Analysis

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0 Introduction

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1 The discrete Fourier transform

Let N be a fixed positive integer. Write ω for $e^{2\pi i/N}$, and $\mathbb{Z}_N$ for $\mathbb{Z}/n\mathbb{Z}$. Let $f : \mathbb{Z}_N \to \mathbb{C}$. Given $r \in \mathbb{Z}_N$, define $\hat{f}(r)$ to be

$$\frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x)\omega^{-rx}$$

From now on we use the notation $\mathbb{E}_{x\in\mathbb{Z}_N}$ for $\frac{1}{N}\sum_{x\in\mathbb{Z}_N}$, so $\hat{f}(r) = \mathbb{E}_x f(x)e^{-\frac{2\pi i r x}{N}}$.

If we write ω_r for the function $x \to \omega^{rx}$, and $\langle f, g\rangle$ for $\mathbb{E}_x f(x)\overline{g(x)}$, then $\hat{f}(r) = \langle f, \omega_r\rangle$. So the discrete fourier transforn is basically expanding the function f in the set of orthonormal basis ω_r.

Let us write $||f||_p$ for $\mathbb{E}_x|f(x)|^p)^{1/p}$ (the L_p-norm), and call the resulting space $L_p(\mathbb{Z}_n)$.

Important convention: we use *averages* for the 'original functions' in 'physical spaces', and *sums* for their Fourier transforms in 'frequency space' (referring to $\mathbb{E}$: $\langle,\rangle$ is average in the original space but just $\sum$ in frequency space, i.e. for $\hat{f}, \hat{g}$ etc.)

Lemma. (1, Parseval's identity)
If $f, g : \mathbb{Z}_n \to \mathbb{C}$, then $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$.

Proof.

$$\begin{aligned}
\langle \hat{f}, \hat{g}\rangle &= \sum_r \hat{f}(r)\overline{\hat{g}(r)} \\
&= \sum_r (\mathbb{E}_x f(x)\omega^{-rx})(\overline{\mathbb{E}_y g(y)\omega^{-ry}}) \\
&= \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)} \sum_r \omega^{-r(x-y)} \\
&= \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)} n\delta_{xy} \\
&= \langle f, g\rangle
\end{aligned}$$

$\square$

Lemma. (2, Convolution identity)

$$\widehat{f * g}(r) = \hat{f}(r)\hat{g}(r)$$

where

$$(f * g)(x) = \mathbb{E}_{y+z=x} f(y)g(z) = \mathbb{E}_y f(y)g(x-y)$$

Proof.

$$\begin{aligned}\widehat{f*g}(r) &= \mathbb{E}_x f*g(x)\omega^{-rx} \\ &= \mathbb{E}_x\mathbb{E}_{y+z=x} f(y)g(z)\omega^{-rx} \\ &= \mathbb{E}_x\mathbb{E}_{y+z=x} f(y)g(z)\omega^{-ry}\omega^{-rz} \\ &= \mathbb{E}_y\mathbb{E}_z f(y)\omega^{-ry} g(z)\omega^{-rz} \\ &= \hat{f}(r)\hat{g}(r)\end{aligned}$$

□

Lemma. (3, Inversion formula)

$$f(x) = \sum_r \hat{f}(r)\omega^{rx}$$

(note the sign of ω^{rx}).

Proof.

$$\begin{aligned}\sum_r \hat{f}(r)\omega^{rx} &= \sum_r \mathbb{E}_y f(y)\omega^{r(x-y)} \\ &= \mathbb{E}_y f(y)\sum_r \omega^{r(x-y)} \\ &= \mathbb{E}_y f(y) n\delta_{xy} \\ &= f(x)\end{aligned}$$

This is really just the statement that we get the original vector back when we sum up its components. □

Further observations: If f is real-valued, then $\hat{f}(-r) = \mathbb{E}_x f(x)\omega^{rx} = \overline{\mathbb{E}_x f(x)\omega^{-rx}} = \overline{\hat{f}(r)}$.

If $A \subset \mathbb{Z}_n$, write A (instead of $1_A, \chi_A$) for the characteristic function of A. Then $\hat{A}(0) = \mathbb{E}_x A(x) = \frac{|A|}{N}$, the *density* of A.

Also, $||\hat{A}||_2^2 = \langle \hat{A}, \hat{A}\rangle = \langle A, A\rangle = \mathbb{E}_x A(x)^2 = \mathbb{E}_x A(x) = \frac{|A|}{N}$, again the density.

Let $f : \mathbb{Z}_n \to \mathbb{C}$. Given $\mu \in \mathbb{Z}_n$, define $f_\mu(x)$ to be $f(\mu^{-1}x)$ (so we need $(\mu, N) = 1$). Then

$$\begin{aligned}\hat{f}_\mu(r) &= \mathbb{E}_x f_\mu(x)\omega^{-rx} \\ &= \mathbb{E}_x f(x/\mu)\omega^{-rx} \\ &= \mathbb{E}_x f(x)\omega^{-r\mu x} \\ &= \hat{f}(\mu r)\end{aligned}$$

1.1 4, Roth's theorem

Theorem. For every $\delta > 0$, $\exists N$ s.t. if $A \subset \{1, ..., N\}$ is a set of size at least δN, then A must contain an arithmetic progression of length 3.

This is also true for 4,5,..., but the proof is much harder − Szemeredi's theorem. Basic strategy of proof: show that if A has density δ and no AP of length 3 (3AP), then there's a long AP in $P \subset \{1, 2, ..., n\}$ s.t.

$$|A \cap P| \geq (\delta + c(\delta))|p|$$

where $c(\delta)$ is some positive number. But then we can continue this argument to expand $A \cap P$ to infinity (note that $|A \cap P|$ is an integer, so each time increase by 1 at least).

The best known relationship between δ and the N required is around $\delta \sim \frac{c}{\log \log N}$ for some constant c.

—Lecture 2—

Lemma. (5)
Let N be odd, $A, B, C \subset \mathbb{Z}_N$ have densties α, β, γ.
If $\max_{r \neq 0} |\hat{A}(r)| \leq \frac{\alpha(\beta\gamma)^{1/2}}{2}$ and $\frac{\alpha\beta\gamma}{2} > \frac{1}{N}$, then there exists $x, d \in \mathbb{Z}_N$ with $d \neq 0$ s.t. $(x, x+d, x+2d) \in A \times B \times C$.

Proof.

$$\begin{aligned}
\mathbb{E}_{x,d} A(x)B(x+d)C(x+2d) &= \mathbb{E}_{x+z=2y} A(x)B(y)C(z) \\
&= \mathbb{E}_u (\mathbb{E}_{x+z=u} A(x)C(z)) \mathbb{E}_{2y=u} B(y) \\
&= \mathbb{E}_u A * C(u) B_2(u) \\
&= \langle A * C, B_2 \rangle \\
&= \langle \widehat{A * C}, \hat{B}_2 \rangle \\
&= \langle \hat{A}\hat{C}, \hat{B}_2 \rangle \\
&= \sum_r \hat{A}(r)\hat{C}(r)\hat{B}(-2r) \\
&= \alpha\beta\gamma + \sum_{r \neq 0} \hat{A}(r)\hat{C}(r)\hat{B}(-2r)
\end{aligned}$$

Recall here the notation is $B_2(u) = B(u/2)$. now

$$\begin{aligned}
|\sum_{r \neq 0} \hat{A}(r)\hat{B}(-2r)\hat{C}(r)| &\leq \frac{\alpha(\beta\gamma)^{1/2}}{2} \sum_{r \neq 0} |\hat{B}(-2r)||\hat{C}(r)| \\
&\leq \frac{\alpha(\beta\gamma)^{1/2}}{2} \left(\sum_r |\hat{B}(-2r)^2 \right)^{1/2} \left(\sum_r |\hat{C}(r)|^2 \right)^{1/2} \quad \text{By Cauchy-Schwarz} \\
&= \frac{\alpha(\beta\gamma)^{1/2}}{2} ||\hat{B}||_2 ||\hat{C}||_2 \\
&= \frac{\alpha(\beta\gamma)^{1/2}}{2} ||B||_2 ||C||_2 \\
&= \frac{\alpha\beta\gamma}{2}
\end{aligned}$$

The contribution to $\mathbb{E}_{x,d} A(x)B(x+d)C(x+2d)$ from $d = 0$ is at most $\frac{1}{N}$, so if $\frac{\alpha\beta\gamma}{2} > \frac{1}{N}$, we are done. □

Now let A be a subset of $\{1, ..., N\}$ with density $\geq \delta$ and let $B = C = A \cap [\frac{N}{3}, \frac{2N}{3})$. If B has density $< \frac{\delta}{5}$ (??), then either $A \cap [1, \frac{N}{3}]$ or $A \cap [\frac{2N}{3}, N]$ has density at least $\frac{2\delta}{5}$. In that case we find an AP P of length about $N/3$ such that $|A \cap P|/|P| \geq \frac{6\delta}{5}$.
Otherwise, ew find that if $\max_{r \neq 0} |\hat{A}(r)| \leq \frac{\delta}{10}$ and $\frac{\delta^3}{50} > \frac{1}{N}$, then $A \times B \times C$ contains a 3AP, so A contains a 3AP.
So if A does not contain a 3AP, then either we find P of length about $N/3$ with $|A \cap P|/|P| \geq \frac{6\delta}{5}$, or ther exists $r \neq 0$ s.t. $|\hat{A}(r)| \geq \frac{\delta}{10}$.

Definition. If X is a finite set and $f : X \to \mathbb{C}$, $Y \subset X$, write $osc(f|_Y)$ to mean $\max_{y_1, y_2 \in Y} |f(y_1) - f(y_2)|$.

Lemma. (6)
Let $r \in \mathbb{Z}_n$ and let $\varepsilon > 0$. Then there is a partition of $\{1, 2, ..., N\}$ into arithmetic progressions P_i of length at least $c(\varepsilon)\sqrt{N}$ such that

$$osc(\omega_r|_{P_i}) \leq \varepsilon$$

for each i.

Proof. Let $t = \lfloor \sqrt{N} \rfloor$. Of the numbers $1, \omega^r, ..., \omega^t r$, there must be two that differ by at most $\frac{2\pi}{t}$.
If $|\omega^{ar} - \omega^{br}| \leq \frac{2\pi}{t}$ with $a < b$, then $|1 - \omega^{dr}| \leq \frac{2\pi}{t}$ where $d = b - a$. Then $|\omega^{urd} - \omega^{vrd}| \leq |\omega^{urd} - \omega^{(u+1)rd}| + ... + |\omega^{(v-1)rd} - \omega^{vrd}| \leq \frac{2\pi}{t}(v - u)$.
So if P is a progression with common differnece d and length l, then $osc(\omega_r|_P) \leq \frac{2\pi l}{t}$. So divide up $\{1, ..., N\}$ into residue classes mod d, and partition each residue class into parts of length between $\frac{\varepsilon t}{4\pi}$ and $\frac{\varepsilon t}{2\pi}$ (possible, since $d \leq t \leq \sqrt{N}$).
We are done, with $c(\varepsilon) = \frac{\varepsilon}{16}$ (a casual choice). □

Now let us use the information that $r \neq 0$ and $|\hat{A}(r)| \geq \frac{\delta^2}{10}$.
Define the *balanced function* f of A by $f(x) = A(x) = \frac{|A|}{N}$ for each x.
Note that $\hat{f}(0) = 0$ and $\hat{f}(r) = \hat{A}(r)$ for all $r \neq 0$.

Now let $P_1, ..., P_m$ be given by Lemma 6 with $\varepsilon = \delta^2/20$. Then

$$\begin{aligned}
\frac{\delta^2}{10} &\leq |\hat{f}(r)| \\
&= \frac{1}{N} |\sum_x f(x)\omega^{-rx}| \\
&\leq \frac{1}{N} \sum_{i=1}^{m} |\sum_{x \in P_i} f(x)\omega^{-rx}| \\
&\leq \frac{1}{N} \sum_{i=1}^{N} \left[\left| \sum_{x \in P_i} f(x)\omega^{-rx_i} \right| + \left| \sum_{x in P_i} f(x)(\omega^{-rx} - \omega^{-rx_i}) \right| \right] \quad x_i \in P_i \text{ arbitrary} \\
&\leq \frac{1}{N} \sum_{i=1}^{m} |\sum_{x \in P_i} f(x)| + \frac{\delta^2}{20}
\end{aligned}$$

Therefore $\sum_{i=1}^{m} |\sum_{x \in P_i} f(x)| \geq \frac{\delta^2 N}{20}$.

We also have $\sum_{i=1}^{m} \sum_{x \in P_i} f(x) = 0$, so

$$\sum_{i=1}^{m} \left(\left| \sum_{x \in P_i} f(x) \right| + \sum_{x \in P_i} f(x) \right) \geq \frac{\delta^2}{20} \sum_{i=1}^{m} |P_i|$$

Therefore,

$$\begin{aligned} &| \sum_{x \in P_i} f(x)| + \sum_{x \in P_i} f(x) \geq \frac{\delta^2}{20} |P_i| \\ &\implies \sum_{x \in P_i} f(x) \geq \frac{\delta^2}{40} |P_i| \\ &\implies |A \cap P_i| \geq (\delta + \frac{\delta^2}{40}) |P_i| \end{aligned}$$