

# Model Theory

November 2, 2018

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# 1 Langauges and structures

**Definition.** (1.1) A language  $L$  consists of:

- (i) a set  $\mathcal{F}$  of function symbols, and for each  $f \in \mathcal{F}$ , a positive integer  $n_f$ , the arity of  $f$ ;
- (ii) a set  $\mathcal{R}$  of relation symbols, and for each  $R \in \mathcal{R}$ , a positive integer  $n_R$ , the arity of  $R$ ;
- (iii) a set  $\mathcal{C}$  of constant symbols.

Note that each of the above three sets can be empty.

**Example.**  $L = \{\{\cdot, -1\}, \{1\}\}$  where  $\cdot$  is a binary function,  $-1$  is a unary function, and  $1$  is a constant. We call this  $L_{gp}$  (language of groups);  $L_{lo} = \{<\}$ , where  $<$  is a binary relation (linear order).

**Definition.** (1.2)

Given a language  $L$ , say, an  $L$ -structure consists of:

- (i) a set  $M$ , the *domain*;
- (ii) for each  $f \in \mathcal{F}$ , a function  $f^M : M^{n_f} \rightarrow M$ ;
- (iii) for each  $R \in \mathcal{R}$ , a relation  $R^M \subseteq M^{n_R}$ ;
- (iv) for each  $c \in \mathcal{C}$ , an element  $c^M \in M$ .

$f^M, R^M, c^M$  are called the *interpretation* of  $f, R, c$  respectively.

**Notation.** (1.3)

We often fail to distinguish between the symbols in the language  $L$  and their interpretations in a  $L$ -structure, if the context allows.

We may write  $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$ .

**Example.** (1.4)

(a)  $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, -1\}, 1 \rangle$  is an  $L_{gp}$ -structure.

$\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$  is also an  $L_{gp}$ -structure (here  $+$  is a binary and  $-$  is the unary negation function).

$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$  is an  $L_{lo}$  structure ( $<$  is the interpretation of relation).

**Definition.** (1.5)

Let  $L$  be a language, let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures.

An *embedding* of  $\mathcal{M}$  into  $\mathcal{N}$  is an injection  $\alpha : M \rightarrow N$  that preserves the structure:

- (i) For all  $f \in \mathcal{F}$ , and  $a_1, \dots, a_{n_f} \in M$ ,

$$\alpha(f^M(a_1, \dots, a_{n_f})) = f^N(\alpha(a_1), \dots, \alpha(a_{n_f}))$$

- (ii) For all  $R \in \mathcal{R}$ , and  $a_1, \dots, a_{n_R} \in M$ ,

$$(a_1, \dots, a_{n_R}) \in R^M \iff (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if.

- (iii) For all  $c \in \mathcal{C}$ , we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding  $\mathcal{M} \rightarrow \mathcal{N}$  is also called an *isomorphism* of  $\mathcal{M}$  onto  $\mathcal{N}$ .

(1.6) Exercise. Let  $G_1, G_2$  be groups, regarded as  $L_{gp}$ -structures. Check that  $G_1 \cong G_2$  in the usual algebra sense, if and only if there is an isomorphism  $\alpha : G_1 \rightarrow G_2$  in the sense of above definition 1.5.

## 2 Terms, formulae, and their interpretations

In addition to the symbols of  $L$ , we also have:

- (i) infinitely many variables,  $\{x_i\}_{i \in I}$ ;
- (ii) logical connectives,  $\wedge, \neg$  (also express  $\vee, \rightarrow, \leftrightarrow$ );
- (iii) quantifier  $\exists$  (also express  $\forall$ );
- (iv) punctuations  $(, )$ .

**Definition.** (2.1)

$L$ -terms are defined recursively as follows:

- any variable  $x_i$  is a term;
- any constant symbol is a term;
- for any  $f \in \mathcal{F}$ ,

$$f(t_1, \dots, t_{n_f})$$

for any terms  $t_1, \dots, t_{n_f}$  is a term;

- nothing else is a term.

Notation: we write  $t(x_1, \dots, x_n)$  to mean that the variables appearing in  $t$  are among  $x_1, \dots, x_n$ .

**Example.** In  $\mathcal{R} = \langle \mathbb{R}, \cdot, -1, 1 \rangle$ ,

- $(\cdot(x_1, x_2), x_3)$  is a term  $(x_1 \cdot x_2) \cdot x_3$ ;
- $(\cdot(1, x_1))^{-1}$  is a term  $(1 \cdot x)^{-1}$ .

**Definition.** (2.2)

If  $\mathcal{M}$  is an  $L$ -structure, to each  $L$ -term  $t(x_1, \dots, x_k)$  we assign a function

$$t^M : M^k \rightarrow M$$

defined as follows:

- (i) If  $t = x_i$ ,  $t^M[a_1, \dots, a_k] = a_i$ ;
- (ii) If  $t = c$  is a constant,  $t^M[a_1, \dots, a_k] = c^M$ ;
- (iii) If  $t = f(t_1(x_1, \dots, x_k), \dots, t_{n_f}(x_1, \dots, x_k))$ ,

$$t^M(a_1, \dots, a_k) = f^M(t_1^M(a_1, \dots, a_k), \dots, t_{n_f}^M(a_1, \dots, a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on [http](http://users.mct.open.ac.uk/sb27627/MT.html) :

[users.mct.open.ac.uk/sb27627/MT.html](http://users.mct.open.ac.uk/sb27627/MT.html) (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

**Remark.** (The lecture forgot about this last time) Any language  $L$  includes an equality symbol  $=$ .

Last time we assigned a function  $t^m$ . In  $L_{gp}$ , the term  $x_2 \cdot x_3$  can be described as, say  $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), \dots$

Then the term  $x_2 \cdot x_3$  can be assigned to functions  $t_1^M : M^3 \rightarrow M : (a_1, a_2, a_3) \rightarrow (a_2 \cdot a_3)$ , or  $t_2^M : M^4 \rightarrow M : (a_1, a_2, a_3, a_4) \rightarrow (a_2 \cdot a_3)$ . These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term  $t$  to be the number of symbols of  $L$  occurring in  $t$ .

Fact (2.3): Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures, and let  $\alpha : \mathcal{M} \rightarrow \mathcal{N}$  be an embedding. For any  $L$ -term  $t(x_1, \dots, x_k)$  and  $a_1, \dots, a_k \in M$ , we have

$$\alpha(t^M(a_1, \dots, a_k)) = t^N(\alpha(a_1), \dots, \alpha(a_k))$$

*Proof.* Prove by induction on complexity of  $t$ .

Let  $\bar{a} = (a_1, \dots, a_k)$  and  $\bar{x} = (x_1, \dots, x_l)$ . Then:

- (i) if  $t = x_i$  a variable, then  $t^M(\bar{a}) = a_i$ , and  $t^N(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$ , so the conclusion holds;
- (ii) if  $t = c$  is a constant, then  $t^M(\bar{a}) = c^M$ , and  $t^N(\alpha(\bar{a})) = c^N$  by definition of a term. The key here is that, since  $\alpha$  is an embedding we have  $\alpha(c^M) = c^N$ ;
- (iii) if  $t = f(t_1(\bar{x}, \dots, t_{n_f}(\bar{x})))$ , then

$$\alpha(f^M(t_1^M(\bar{a}), \dots, t_{n_f}^M(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), \dots, \alpha(t_{n_f}^M(\bar{a})))$$

as  $\alpha$  is an embedding. But  $t_1(\bar{x}), \dots, t_{n_f}(\bar{x})$  have lower complexity than  $t$ , so the inductive hypothesis applies.  $\square$

Exercise (2.4): conclude the proof of the above fact.  
(Actually is it not done?)

**Definition.** (2.5)

The set of *atomic formulas* of  $L$  is defined as follows:

- (i) if  $t_1, t_2$  are  $L$ -terms, then  $t_1 = t_2$  is an atomic formula;
- (ii) if  $R$  is a relation symbol, and  $t_1, \dots, t_{n_R}$  are  $L$ -terms, then  $R(t_1, \dots, t_{n_R})$  is an atomic formula;
- (iii) nothing else is an atomic formula.

**Definition.** (2.6)

The set of  $L$ -formulas is defined as follows:

- (i) any atomic formula is an  $L$ -formula;
- (ii) if  $\phi$  is an  $L$ -formula, then so is  $\neg\phi$ ;
- (iii) if  $\phi$  and  $\psi$  are  $L$ -formulas, then so is  $\phi \wedge \psi$ ;
- (iv) if  $\phi$  is an  $L$ -formula, for any  $i \geq 1$ ,  $\exists x_i \phi$  is a formula;
- (v) nothing else is a formula (note that  $\forall$  can be constructed by  $\neg$  and  $\exists$ ).

**Example.** In  $L_{gp}$ ,  $x_1 \cdot x_1 = x_2$ , or  $x_1 \cdot x_2 = 1$  are both atomic formulas;  $\exists x_1(x_1 \cdot x_2) = 1$  is an  $L$ -formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier  $\exists$ . We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example,  $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$  is an  $L_{gp}$ -sentence.

Notation:  $\phi(x_1, \dots, x_k)$  means that the free variables in  $\phi$  are among  $x_1, \dots, x_k$ .

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

**Definition.** (2.7)

Let  $\phi(x_1, \dots, x_k)$  be an  $L$ -formula, let  $\mathcal{M}$  be an  $L$ -structure, and let  $\bar{a} = a_1, \dots, a_k$  be elements of  $\mathcal{M}$ .

We define  $\mathcal{M} \models \phi(\bar{a})$  (syntactic implication, read as  $\mathcal{M}$  models  $\phi(\bar{a})$ ) as follows:

- (i) if  $\phi$  is  $t_1 = t_2$ , then  $\mathcal{M} \models \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$ ;
- (ii) if  $\phi$  is  $R(t_1, \dots, t_{n_R})$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff

$$(t_1^M(\bar{a}), \dots, t_{n_R}^M(\bar{a})) \in R^M$$

- (iii) if  $\phi$  is a conjunction, say  $\psi \wedge \chi$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{M} \models \psi(\bar{a})$  and  $\mathcal{M} \models \chi(\bar{a})$ ;
- (iv) if  $\phi$  is  $\exists x_j \chi(x_1, \dots, x_k, x_j)$  (where we'll assume that  $x_j$  is not one of the free variables  $x_1, \dots, x_k$ ), then  $\mathcal{M} \models \phi(\bar{a})$  iff there exists  $b \in \mathcal{M}$  s.t.  $\mathcal{M} \models \chi(a_1, \dots, a_k, b)$ ;
- (v) (lecture forgets this, this should probably be more in front rather than in the end) if  $\phi$  is  $\neg\psi$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{M} \not\models \psi(\bar{a})$ .

**Example.** Consider  $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1 \rangle$ , the multiplicative group of non-negative reals, and suppose we have  $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$ , then  $\mathcal{R} \models \phi(1)$ , but  $\mathcal{R} \not\models \phi(-1)$ .

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):

- $\phi \vee \psi$  for  $\neg(\neg\phi \wedge \neg\psi)$ ;
- $\phi \rightarrow \psi$  for  $\neg\phi \vee \psi$ ;
- $\phi \leftrightarrow \psi$  for  $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ ;
- $\forall x_i \phi$  for  $\neg \exists x_i (\neg\phi)$ .

**Proposition.** (2.9)

Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures, and let  $\alpha : \mathcal{M} \rightarrow \mathcal{N}$  be an embedding.

Let  $\phi(\bar{x})$  be an atomic(!) formula, and  $\bar{a} \in M^{|\bar{x}|}$ , here  $|\bar{x}|$  means the length of the tuple  $\bar{x}$  (from now on, when we write a tuple like  $\bar{a}$ , we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

Question: if  $\phi$  is an  $L$ -formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.

Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.

In definition 2.7 we defined what it means for  $\mathcal{M} \models \phi(\bar{a})$ , in particular we defined: if  $\phi \equiv \neg\chi$ , then  $\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{M} \not\models \chi(\bar{a})$ . Here by  $\mathcal{M} \models \phi(\bar{a})$  we mean  $\mathcal{M} \models \neg\chi(\bar{a})$ , and  $\chi(\bar{a})$  is *shorter* than  $\phi(\bar{a})$ , so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

*Proof.* There are two cases:

- $\phi(\bar{x})$  is of the form  $t_1(\bar{x}) = t_2(\bar{x})$  where  $t_1, t_2$  are terms. Use Fact (2.3). (exercise on example sheet)
- $\phi(\bar{x})$  is of the form  $R(t_1(\bar{x}), \dots, t_{n_R}(\bar{x}))$ . Then  $\mathcal{M} \models R(t_1(\bar{a}), \dots, t_{n_R}(\bar{a}))$  if and only if ... (lecturer says work this out by yourself. Basically the induction step).  $\square$

**Proposition.** (2.10)

Exercise: show that prop (2.9) holds if  $\phi(\bar{x})$  is a formula without quantifiers (a quantifier-free formula).

(I guess that also suggests when does it not hold for general formulas – see below).

**Example.** (2.11, Do embeddings preserve all formulas? No.)

Let  $\mathcal{Z} = (\mathbb{Z}, <)$  an  $L_{lo}$ -structure,  $\mathcal{Q} = (\mathbb{Q}, <)$  also an  $L_{lo}$ -structure. Then

$$\begin{aligned} \alpha : \mathbb{Z} &\rightarrow \mathbb{Q} \\ n &\rightarrow n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3 (x_1 < x_3 \wedge x_3 < x_2)$$

Now  $\mathcal{Q} \models \phi(1, 2)$  but  $\mathcal{Z} \not\models \phi(1, 2)$ .

Fact (2.12) (From now on we'll stop saying that  $\mathcal{M}, \mathcal{N}$  are  $L$ -structures etc to save time) Let  $\alpha : \mathcal{M} \rightarrow \mathcal{N}$  be an isomorphism. Then if  $\phi(\bar{x})$  is an  $L$ -formula, and  $\bar{a} \in M^{|\bar{x}|}$ , then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).



### 3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let  $L$  be a language,  $\mathcal{M}, \mathcal{N}$  be  $L$ -structures.

**Definition.** (3.1)

An  $L$ -theory  $T$  is a set of  $L$ -sentences.

$\mathcal{M}$  is a *model* of  $T$  if  $\mathcal{M} \models \sigma$  for all  $\sigma \in T$ . We write  $\mathcal{M} \models T$ .

The class of all the models of  $T$  is written  $Mod(T)$ .

The *theory of*  $\mathcal{M}$  is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L\text{-sentence and } \mathcal{M} \models \sigma\}$$

**Example.** (3.2)

Let  $T_{gp}$  be the set of  $L_{gp}$ -sentences:

(i)  $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$ ;

(ii)  $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$ ;

(iii)  $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$ .

Clearly, for a group  $G$ ,  $G \models T_{gp}$  (as they are just the group axioms). However, for a specific group  $G$ , clearly the theory of it,  $Th(G)$  is larger than  $T_{gp}$ .

**Definition.** (3.3)

$\mathcal{M}$  and  $\mathcal{N}$  are *elementarily equivalent* if  $Th(\mathcal{M}) = Th(\mathcal{N})$ .

We write  $\mathcal{M} \equiv \mathcal{N}$ .

Clearly, if  $\mathcal{M} \simeq \mathcal{N}$  ( $\simeq$  means isomorphism), then  $\mathcal{M} \equiv \mathcal{N}$ .

But if  $\mathcal{M}$  and  $\mathcal{N}$  are not isomorphic, establishing whether  $\mathcal{M} \equiv \mathcal{N}$  can be highly non-trivial!

We'll see  $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$  as  $L_{lo}$ -structures(!).

**Definition.** (3.4)

(i) An embedding  $\beta : \mathcal{M} \rightarrow \mathcal{N}$  is *elementary* if for all formulas  $\phi(\bar{x})$  and  $\bar{a} \in M^{|\bar{x}|}$ ,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\beta(\bar{a}))$$

(ii) If  $M \subseteq N$ , and  $id : \mathcal{M} \rightarrow \mathcal{N}$  is an embedding, then  $\mathcal{M}$  is a *substructure* of  $\mathcal{N}$ .

(iii) If  $M \subseteq N$  and  $id : \mathcal{M} \rightarrow \mathcal{N}$  is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then  $\mathcal{M}$  is said to be an *elementary substructure* of  $\mathcal{N}$ , written as  $\mathcal{M} \preceq \mathcal{N}$ .

**Example.** (3.5)

Let  $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$ , an  $L_{lo}$ -structure where  $<$  is the usual order;

Let  $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$ , also an  $L_{lo}$ -structure with the same  $<$ .

Then  $\mathcal{M} \simeq \mathcal{N}$  as  $L_{lo}$ -structures. So  $\mathcal{M} \equiv \mathcal{N}$  (since they are isomorphic).

Also,  $\mathcal{M} \subseteq \mathcal{N}$  (read as *is a substructure of*), since the ordering  $<$  coincides on  $\mathcal{M}$  and  $\mathcal{N}$ . However,  $\mathcal{M} \not\preceq \mathcal{N}$ , since if we pick the formula  $\phi(x) \equiv \exists y (x < y)$ , then  $\mathcal{N} \models \phi(1)$ , but  $\mathcal{M} \not\models \phi(1)$ .

**Definition.** (3.6)

Let  $\mathcal{M}$  be an  $L$ -structure,  $A \subseteq M$ , then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where  $c_a$  are constant symbols). An interpretation of  $\mathcal{M}$  as an  $L$ -structure extends to an interpretation of  $\mathcal{M}$  as an  $L(A)$ -structure in the obvious way, i.e.  $c_a^M = a$ .

In this context, the elements of  $A$  are called *parameters*.

If  $\mathcal{M}$  and  $\mathcal{N}$  are two structures, and  $A \subseteq M \cap N$ , then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean  $\mathcal{M}, \mathcal{N}$  satisfy exactly the same  $L(A)$  sentences.

—Lecture 4—

Reminder: we have a lecture next Monday (22nd Oct)!

**Proposition.** It turns out that,  $\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$  (where  $M$  is the domain of  $\mathcal{M}$ ).

**Lemma.** (3.8, Tarski-Vaught test)

Let  $\mathcal{N}$  be an  $L$ -structure, let  $A \subseteq N$ . The following are equivalent:

- (i)  $A$  is the domain of a substructure  $\mathcal{M}$  s.t.  $\mathcal{M} \preceq \mathcal{N}$ ;
- (ii) if  $\phi(x) \in L(A)$  (with an abuse of notations  $\phi(x, c_{a_1}, \dots, c_{a_n}) = \phi(x, a_1, \dots, a_n)$ ), if  $\mathcal{N} \models \exists x \phi(x)$ , then  $\mathcal{N} \models \phi(b)$  for some  $b \in A$ .

*Proof.* (i)  $\implies$  (ii): Suppose  $\mathcal{N} \models \exists x \phi(x)$ . Then by elementarity,  $\mathcal{M} \models \exists x \phi(x)$ , and so  $\mathcal{M} \models \phi(b)$  for  $b \in M$ . So (again by elementarity),  $\mathcal{N} \models \phi(b)$ .

(ii)  $\implies$  (i): This is the harder direction. First we prove that  $A$  is the domain of a substructure  $\mathcal{M} \subseteq \mathcal{N}$ .

By Sheet 1 Q4, it suffices to check:

- (a) For each constant  $c$ ,  $c^N \in A$ ;
- (b) For each function symbol  $f$ ,  $f^N(\bar{a}) \in A$  (for all  $\bar{a} \in A^{n_R}$ );

For (a), use property (ii) with  $\exists x(x = c)$ .

For (b), use property (ii) with the formula  $\exists x((\bar{a} = x))$ .

So we now have  $\mathcal{M} \subseteq \mathcal{N}$ , and domain of  $\mathcal{M}$  is  $A$ . But we actually want to prove that  $\mathcal{M} \preceq \mathcal{N}$ . Now let  $\chi(\bar{x})$  be an  $L$ -formula.

We want to show that for  $\bar{a} \in A^{|\bar{x}|}$   $\mathcal{M} \models \chi(\bar{a}) \iff \mathcal{N} \models \chi(\bar{a})$  (\*).

By induction on the complexity of  $\chi(\bar{x})$ :

- if  $\chi(\bar{x})$  is atomic, (\*) follows from  $\mathcal{M} \subseteq \mathcal{N}$  (since  $\mathcal{M}$  is a substructure!);
- if  $\chi(\bar{x})$  is  $\neg\psi(\bar{x})$  or  $\chi(\bar{x})$  is  $\psi(\bar{x}) \wedge \xi(\bar{x})$ , it's a straightforward induction;
- (interesting case) if  $\chi(\bar{x}) = \exists y \psi(\bar{x}, y)$  where  $\psi(\bar{x}, y)$  is an  $L$ -formula, suppose that  $\mathcal{M} \models \chi(\bar{a})$ , then  $\mathcal{M} \models \exists y \psi(\bar{a}, y)$ , hence  $\mathcal{M} \models \psi(\bar{a}, b)$  for some  $b \in A = \text{dom}(\mathcal{M})$  (this is the definition of truth).

But then  $\mathcal{N} \models \psi(\bar{a}, b)$  by inductive hypothesis, so  $\mathcal{N} \models \chi(\bar{a})$ .

Now let  $\mathcal{N} \models \chi(\bar{a})$ , i.e.  $\mathcal{N} \models \exists y \psi(\bar{a}, y)$  (we find a *witness* for it). By property (ii),  $\mathcal{N} \models \psi(\bar{a}, b)$  for some  $b \in A = \text{dom}(\mathcal{M})$ .

Again by inductive hypothesis, we have  $\mathcal{M} \models \psi(\bar{a}, b)$ , and so in particular,  $\mathcal{M} \models \chi(\bar{a})$  as it has got a witness there.  $\square$

**Remark.** (3.9)

Even more assumptions: let's assume that the set of variables is countably infinite. Then:

- the cardinality of the set of  $L$ -formulas is  $|L| + \omega$  (where by  $|L|$  we mean

the number of symbols. For example,  $|L_{gp}| = 3$ ,  $|L_{lo}| = 1$ ), where we abuse another notation that we use  $\omega$  as cardinals (rather than ordinals) (note that the formulas are just strings of finite length);

- if  $A$  is a set of parameters in some structure, the cardinality of the set  $L(A)$  is  $|A| + |L| + \omega$ , where by  $+$  here we merely mean  $\max\{|L|, |A|, \omega\}$  (instead of addition), and same for the  $+$  above.

**Definition.** (3.10)

Let  $\lambda$  be an ordinal. Then a *chain of length  $\lambda$*  of sets is a sequence  $\langle M_i : i < \lambda \rangle$ , where  $M_i \subseteq M_j$  for all  $i \leq j < \lambda$ .

A chain of  $L$ -structures is a sequence:  $\langle \mathcal{M}_i : i < \lambda \rangle$  s.t.  $\mathcal{M}_i \subseteq \mathcal{M}_j$  (note that it's substructure here) for  $i \leq j < \lambda$ .

The *union* of this chain is the  $L$ -structure  $\mathcal{M}$  defined as follows:

- the domain is  $\bigcup_{i < \lambda} M_i$  (when you think of this, you can always start with the case  $\lambda = \omega$ );
- for constants  $c$ ,  $c^{\mathcal{M}} = c^{\mathcal{M}_i}$  for any  $i < \lambda$  (this is well defined, because of the substructure condition above);
- if  $f$  is a function symbol,  $\bar{a} \in M^{|\bar{a}|}$  (why the mod sign here),  $f^{\mathcal{M}}\bar{a} = f^{\mathcal{M}_i}\bar{a}$  where  $i$  is s.t.  $\bar{a} \in M_i^{|\bar{a}|}$ ;
- if  $R$  is a relation symbol, then  $R^{\mathcal{M}} = \bigcup_{i < \lambda} R^{\mathcal{M}_i}$ .

**Theorem.** (3.11, Downward Löwenheim-Skolem theorem)

(Recall that in part II Logic and Set Theory we had the countable version of this)

Let  $\mathcal{N}$  be an  $L$ -structure, and  $|\mathcal{N}| \geq |L| + \omega$ . Let  $A \subseteq N$ . Then for every cardinal  $\lambda$  s.t.  $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$ , there is  $\mathcal{M} \preceq \mathcal{N}$  s.t.

- $A \subseteq M$ ;
- $|\mathcal{M}| = \lambda$ .



(It helps to think about the case  $|A| = \omega$  and  $|N|$  is uncountable.)

A quick example how this could be useful (we'll go very sloppy here): think of  $(\mathbb{C}, +, \cdot, -, \cdot^{-1}, 0, 1)$  as a field. Consider  $\mathbb{Q} \subseteq \mathbb{C}$  (both as subset and substructure). Note that algebraic closeness is a property of  $\mathbb{C}$ . By downward Löwenheim-

Skolem, there is a substructure in  $\mathcal{C}$  that contains  $\mathbb{Q}$  that is also algebraically closed (apparently, the set of algebraic numbers).

*Proof.* We build a chain  $\langle A_i : i < \lambda \rangle$ , with  $A_i \subseteq N$ , s.t.  $|A_i| = \lambda$ .

(our goal: define an elementary substructure with domain  $M = \bigcup_{i < \omega} A_i$ ).

Base case: Let  $A_0 \subseteq N$  be such that  $A \subseteq A_0$  and  $|A_0| = \lambda$ .

Successors: At stage  $i + 1$ , assume  $A_i$  has been built, with  $|A_i| = \lambda$ .

Let  $\langle \phi_k(x) : k < \lambda \rangle$  be an enumeration of those  $L(A_i)$ -formulas such that  $\mathcal{N} \models \exists x \phi_k(x)$ . Let  $a_k$  be such that  $\mathcal{N} \models \phi_k(a_k)$ , and let  $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$  (basically, with those witnesses added). Then  $|A_{i+1}| = \lambda$  (note that we haven't increased the size).

Now let  $M = \bigcup_{i < \omega} A_i$  (note the subscript range). We use lemma (3.8) to show that  $M$  is the domain of  $\mathcal{M} \preceq \mathcal{N}$ , and  $|M| = \lambda$ . We're running out of time, so we'll continue next Monday.

—Lecture 5—

Solutions to worksheet 1: either take along to lecture on Friday, or email them to [silvia.barbina@open.ac.uk](mailto:silvia.barbina@open.ac.uk).

Let's continue with the proof:



Start with  $A_0 \subset N$ ,  $A \subseteq A_0$ ,  $|A_0| = \lambda$ . The idea is to define  $\langle A_i : i < \omega \rangle$  so that  $M = \bigcup_{i < \omega} A_i$  satisfies (ii) via the TV test (3.8).

List all formulas  $\phi(x, \bar{a})$  ( $\bar{a}$  is a tuple in  $A_0$ ), and  $\mathcal{N} \models \phi(b, \bar{a})$  for some  $b$ .

Add each such  $b$  to  $A_0$  (one for each such  $\phi$ ).

Let  $A_1 = A_0 \cup \{ \text{all these } b\text{'s} \}$ .

Repeat for formulas  $\phi(x, \bar{a})$  where  $\bar{a}$  is in  $A_1, \dots$

Eventually,  $\langle A_i : i < \omega \rangle$  is such that  $M = \bigcup_{i < \omega} A_i$  is as required (i.e.  $M$  is the domain of some elementary substructure of  $\mathcal{N}$  that we need).

We claim that  $M$  satisfies condition (ii) in Lemma (3.8): Let  $\mathcal{N} \models \exists x \psi(x, \bar{a})$ , where  $\bar{a}$  is a tuple in  $M$ . Then  $\bar{a}$  is a *finite* tuple, so there is an  $i$  s.t.  $\bar{a}$  is in  $A_i$ .

Then  $A_{i+1}$ , by construction, contains  $b$  s.t.  $\mathcal{N} \models \phi(b, \bar{a})$ . But  $A_{i+1} \subseteq M, b \in M$ . Then apply (3.8) we're done.  $\square$

## 4 Two relational structures

**Definition.** (4.1, dense linear orders)

A *linear order* is an  $L_{lo} = \{<\}$ -structure such that:

- (i)  $\forall x \neg(x < x)$ ;
- (ii)  $\forall xyz((x < y \wedge y < z) \rightarrow x < z)$ ;
- (iii)  $\forall xy((x < y) \vee (y < x) \vee x = y)$  (total).

A linear order is *dense* if, in addition, it also satisfies:

- (iv)  $\exists xy(x < y)$ ;
- (v)  $\forall xy, (x < y \rightarrow \exists z(x < z \wedge z < y))$  (density).

A linear order has no endpoints if, in addition,

- (vi)  $\forall x(\exists y(x < y) \wedge \exists z(z < x))$ .

We use  $T_{dlo}$  to denote the theory that includes all axioms (i) to (vi), and  $T_{lo}$  is the theory that includes axioms (i) to (iii) only.

**Remark.** (iv) and (v) imply that if  $\mathcal{M} \models T_{dlo}$ , then  $|\mathcal{M}| \geq \omega$ .

**Definition.** (4.2)

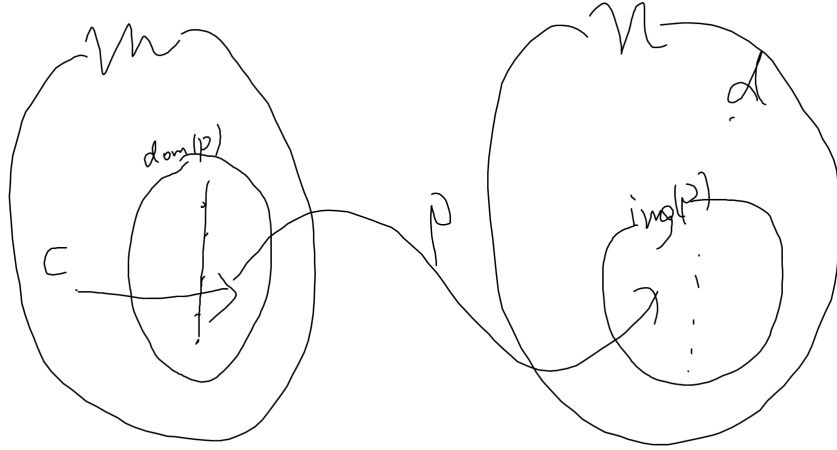
If  $\mathcal{M}, \mathcal{N} \models T_{lo}$ , then an *injective* map  $p : A \subseteq M \rightarrow N$  is a *partial embedding* if  $\mathcal{M} \models a < b \implies \mathcal{N} \models p(a) < p(b)$ .

In particular, if  $|\text{dom}(p)| < \omega$ , then  $p$  is a *finite* partial embedding.

**Lemma.** (4.3, extension lemma)

Take a linear order  $\mathcal{M} \models T_{lo}$ , and a dense linear endpoints  $\mathcal{N} \models T_{dlo}$ , and let  $p : M \rightarrow N$  be a finite partial embedding. Then if  $c \in \mathcal{M}$ , there is a finite partial embedding  $\hat{p}$  s.t.  $p \subseteq \hat{p}$  and  $c \in \text{dom}(\hat{p})$ .

(we can always add one extra element in our embedding.)



*Proof.*

Case 1:  $c$  is greater than all elements in  $\text{dom}(p)$ . In that case, pick an element  $d \in \mathcal{N}$  s.t.  $d > b$  for all  $b \in \text{img}(p)$ ;

Case 2:  $a_i < c < a_{i+1}$  where  $a_i, a_{i+1} \in \text{dom}(p)$ . Then we choose  $\mathcal{N} \models p(a_i) < d < p(a_{i+1})$ , where  $d$  is chosen appropriately by density (here's the case why we

need  $\mathcal{N}$  to be dense;

Case 3:  $c$  is less than all elements in  $\text{dom}(p)$ . This is similar to case 1.

Note that the ability to extend by one point allows us to embed any finite linear order into a dense linear order without endpoints.  $\square$

**Theorem.** (4.4)

Let  $\mathcal{M}, \mathcal{N} \models T_{dlo}$  s.t.  $|\mathcal{M}| = |\mathcal{N}| = \omega$ . Let  $p : A \subseteq M \rightarrow N$  be a finite partial embedding.

Then there is an isomorphism  $\pi : \mathcal{M} \rightarrow \mathcal{N}$  s.t.  $p \subseteq \pi$ .

*Proof.* Enumerate  $M, N$ , say  $M = \langle a_i : i < \omega \rangle$ ,  $N = \langle b_i : i < \omega \rangle$  (sequences of elements).

We define, inductively, a chain of finite partial embedding  $\langle p_i : i < \omega \rangle$  (idea:  $\pi = \bigcup_{i < \omega} p_i$ ).

Let's start with  $p_0 = p$ . At stage  $i + 1$ , suppose we are given  $p_i$ . We want to include  $a_i$  in  $\text{dom } p_{i+1}$ , and  $b_i$  in the  $\text{img}(p_{i+1})$ .

(Lecturer calls this a *back and forth* method) Forth step: By lemma 4.3, we can extend  $p_i$  to  $p_{i+\frac{1}{2}}$  such that  $a_i \in \text{dom}(p_{i+\frac{1}{2}})$ ;

Back step: By lemma 4.3 again applied to  $(p_{i+\frac{1}{2}})^{-1}$  to include  $b_i \in \text{dom}(p_{i+1}^{-1})$  (i.e. in the range of  $p_{i+1}$ ).

We claim that  $p_{i+1}$  extends  $p_i$  as required.

Let  $\pi = \bigcup_{i < \omega} p_i$ . Then (check)  $\pi$  is an isomorphism (i.e. order-preserving bijection).  $\square$

**Definition.** (4.5)

An  $L$ -theory is *consistent* if there is  $L$ -structure  $\mathcal{M}$  s.t.  $\mathcal{M} \models T$ .

If  $T$  is a theory in  $L$  and  $\phi$  is an  $L$ -sentence, then  $T \vdash \phi$  (read as  $T$  entails  $\phi$ , note that this has nothing to do with syntactic implication) if for all  $\mathcal{M}$  such that  $\mathcal{M} \models T$ , we have  $\mathcal{M} \models \phi$ .

Finally, an  $L$ -theory  $T$  is *complete* if for all  $L$ -sentences  $\phi$ , either  $T \vdash \phi$  or  $T \vdash \neg\phi$  (see part II Logic and Set Theory).

For example,  $T_{dlo}$  is complete.

—Lecture 6—

**Definition.** (4.6)

A theory  $T$  in a countable language with a (infinitely) countable model is  *$\omega$ -categorical* if any two countable models of  $T$  are isomorphic.

**Corollary.** (4.7 of theorem (4.4))

$T_{dlo}$  is  $\omega$ -categorical.

*Proof.* If  $\mathcal{M}, \mathcal{N} \models T_{dlo}$ ,  $|\mathcal{M}| = |\mathcal{N}| = \omega$ , then  $\phi$  (the empty map) is a finite partial embedding. But by theorem (4.4) we get  $\mathcal{M} \simeq \mathcal{N}$ .

(We can also use any  $\{\langle a, b \rangle\}$  where  $a \in \mathcal{M}$  and  $b \in \mathcal{M}$  as initial finite partial embedding).  $\square$

**Theorem.** (4.8)

(erratum 26th Oct 2018: lecturer wants to add a condition  $T$  has no finite models.

Then the problem with (4.11) is fixed.)

If  $T$  is an  $\omega$ -categorical theory in a countable language, then  $T$  is complete.

*Proof.* Let  $\mathcal{M} \models T$  and  $\phi$  be an  $L$ -sentence.

If  $\mathcal{M} \models \phi$ , suppose  $\mathcal{N} \models T$ . Then by theorem (3.11) (Downward Lowenheim-Skolem), there are  $\mathcal{M}' \preceq \mathcal{M}$ ,  $\mathcal{N}' \preceq \mathcal{N}$  s.t.  $|\mathcal{M}'| = |\mathcal{N}'| = \omega$ .

But  $\mathcal{M}' \simeq \mathcal{N}'$  (by  $\omega$ -categoricity), so in particular  $\mathcal{M}' \equiv \mathcal{N}'$ , and so  $\mathcal{N}' \models \phi$ . By elementarity,  $\mathcal{N} \models \phi$ .

The case  $\mathcal{M} \models \neg\phi$  is similar.

(Think about if  $T$  could have a finite model.) □

**Corollary.** (4.9)

$T_{dlo}$  is complete.

**Definition.** (4.10)

If  $\mathcal{M}, \mathcal{N}$  are  $L$ -structures, a map  $f$  such that  $\text{dom}(f) \subseteq M$  (the domain of  $\mathcal{M}$ ), and  $\text{img}(f) \subseteq N$  is a (partial) *elementary map* if for all  $L$ -formulas  $\phi(\bar{x})$  and  $\bar{a} \in (\text{dom}(f))^{|\bar{x}|}$ , then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(f(\bar{a}))$$

**Remark.** (4.11)

A map  $f$  is elementary iff every finite restriction of  $f$  is elementary.

(Why? For forward, if  $f_0 \subseteq f$  is a finite restriction that is not elementary, then for some formula  $\phi(\bar{x})$ ,  $\bar{a} \in \text{dom}(f_0)$ , the above equivalence doesn't hold; but then that equivalence doesn't hold for  $f$  either; contradiction; for backward, if  $f$  is not elementary, then the above equivalence fails on a finite tuple, so the above equivalence fails on some finite restriction.)

**Proposition.** (4.12)

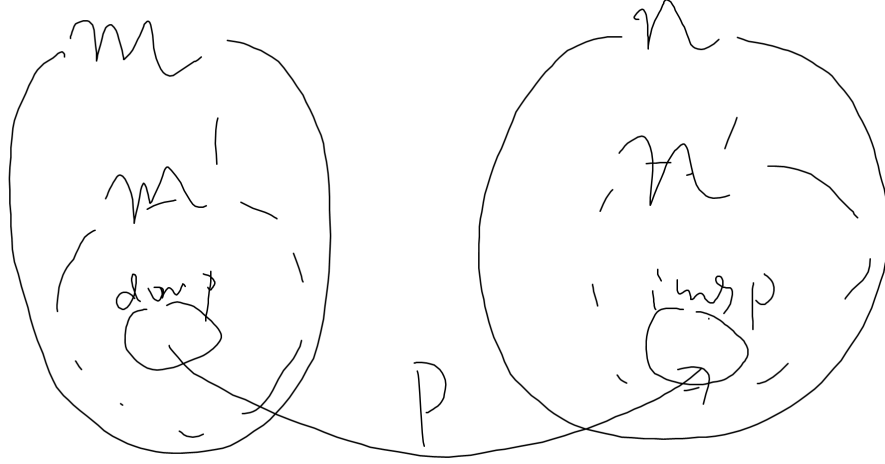
Let  $\mathcal{M}, \mathcal{N} \models T_{dlo}$ , and let  $p : A \subseteq M \rightarrow N$  be a partial embedding. Then  $p$  is elementary.

*Proof.* By remark (4.11), it suffices to consider  $p$  finite.

By Downward L-S theoem (3.11), we choose  $\mathcal{M}', \mathcal{N}'$  such that

- (i)  $|\mathcal{M}'| = |\mathcal{N}'| = \omega$ ;
- (ii)  $\mathcal{M}' \preceq \mathcal{M}$ ,  $\mathcal{N}' \preceq \mathcal{N}$ ;
- (iii)  $\text{dom}(p) \subseteq M'$ ,  $\text{img}(p) \subseteq N'$ .





Now  $p$  is a finite partial embedding between countable models, so  $p$  extends to an isomorphism  $\pi : \mathcal{M}' \rightarrow \mathcal{N}'$ .

In particular,  $\pi$  is an elementary map between  $\mathcal{M}$  and  $\mathcal{N}$ . □

**Corollary.** (4.13)

$(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$ .

*Proof.* Use proposition (4.12) with  $id : \mathbb{Q} \rightarrow \mathbb{R}$ . □

**Definition.** (4.14)

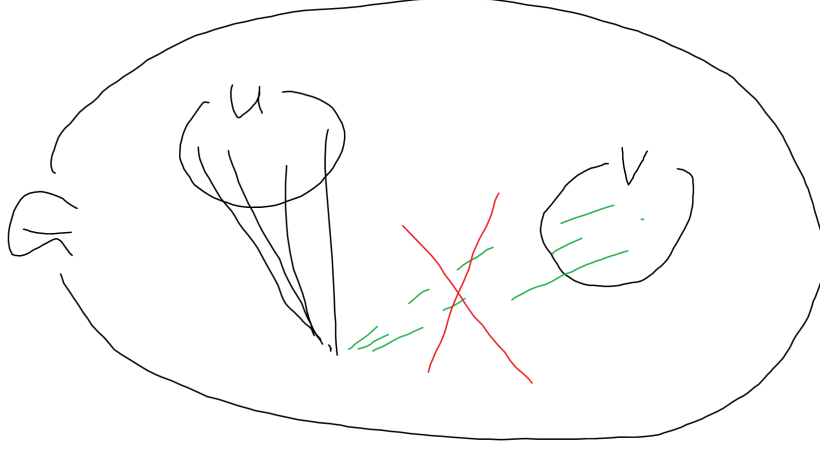
(See Part II Logic and Set Theory)

Let  $L_{gph} = \{R\}$ , where  $R$  is a binary relation symbol.

An  $L_{gph}$ -structure is a graph if

- (i)  $\forall x \neg R(x, x)$ ;
- (ii)  $\forall xy (R(x, y) \leftrightarrow R(y, x))$ .

An  $L_{gph}$ -structure is a **random graph** if it is a graph such that the following axiom-schema  $(r_n)$  hold:



$$\forall x_0 \dots x_n, y_0 \dots y_n, \left( \bigwedge_{i,j=0}^n x_i \neq y_j \rightarrow \exists z \left( \bigwedge_{i=0}^n (z \neq x_i) \wedge (z \neq y_i) \wedge R(z, x_i) \wedge \neg R(z, y_i) \right) \right)$$

(iii)  $\exists xy (x \neq y)$ .

**Remark.** (similar to what is mentioned in the link above)

A random graph is infinite. Given a finite subset, we can always find a vertex that is connected to every vertex in the subset (likewise for not connected).

**Fact.** (4.15)

There is a random graph.

*Proof.* Let the domain be  $\omega$ , let  $i, j \in \omega$  such that  $i < j$ . Write  $j$  as a sum of distinct powers of 2. Then  $\{i, j\}$  is an edge iff  $2^i$  appears in the sum.

As an exercise, prove that  $\omega$  with this definition of  $R$  is indeed a random graph.  $\square$

**Definition.** (4.16, or more precisely just notations)

$T_{gph} = \{\text{axioms (i), (ii)}\}$ ,  $T_{rg} = T_{gph} \cup \{(iii), (r_n) : n \in \omega\}$ .

If  $\mathcal{M}, \mathcal{N} \models T_{gph}$ , a partial embedding is an injection  $p : A \subseteq M \rightarrow N$  s.t.  $\mathcal{M} \models R(p(a), p(b)) \iff \mathcal{N} \models R(a, b)$  for all  $a, b$  in the domain.

**Lemma.** (4.17)

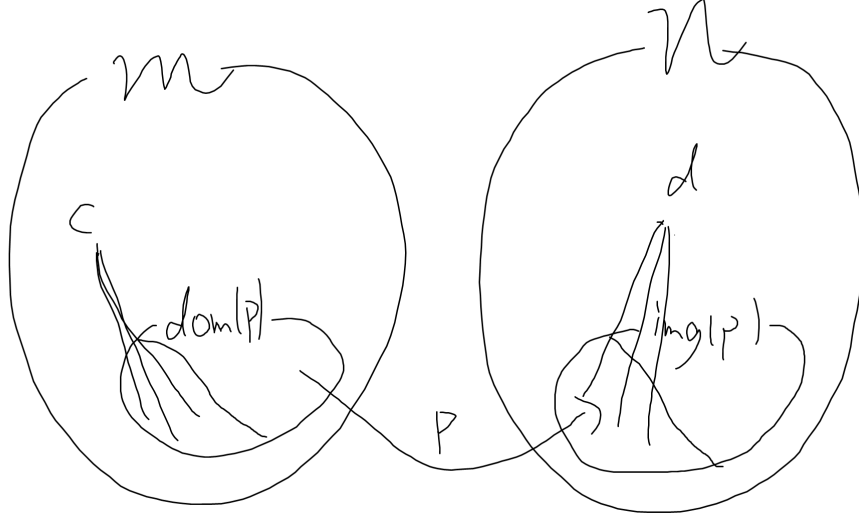
Let  $\mathcal{M} \models T_{gph}, \mathcal{N} \models T_{rg}$ , let  $p : A \subseteq M \rightarrow N$  be a finite partial embedding, and let  $c \in M$ .

Then there is a map  $\hat{p} : \hat{A} \subseteq M \rightarrow N$  such that  $\hat{p}$  is a partial embedding,  $c \in \text{dom } \hat{p}$ ,  $p \subseteq \hat{p}$ .

(This is like another extension lemma.)

We'll prove this next time.

Last time we defined what a random graph is (in this course). We also defined what is a partial embedding in this theory (just preserves all edges). Let's continue with the proof of the lemma now. Let  $c \in M$ ,  $c \notin \text{dom}(p)$ .



Find  $d \in N$  such that  $\mathcal{N} \models R(d, p(a)) \iff \mathcal{M} \models R(c, a)$ .

**Theorem.** (4.18)

Let  $\mathcal{M}, \mathcal{N} \models T_{rg}$  and  $|\mathcal{M}| = |\mathcal{N}| = \omega$ , and  $P : A \subseteq M \rightarrow N$  is a finite partial embedding.

Then  $\mathcal{M} \simeq \mathcal{N}$ , by an isomorphism that extends  $p$ .

*Proof.* Same as proof of Theorem (4.4) (there is only one model of  $T_{dlo}$  up to isomorphism), but with lemma (4.17) instead of lemma (4.3).  $\square$

**Corollary.** (4.19)

$T_{rg}$  is  $\omega$ -categorical (see definition (4.6) – this is just a restatement of the theorem above) and complete.

In particular, every finite partial embedding between models of  $T_{rg}$  is an elementary map.

**Remark.** The unique (up to isomorphism) model of  $T_{rg}$  is *the* countable random graph, or the *Rado* graph.

It is universal w.r.t. finite and countable graphs (i.e. it embeds all).

Another nice property (which you are not required to see this immediately – it is far from trivial) *ultrahomogeneous*, i.e. every isomorphism between finite substructures extends to an automorphism of the whole graph.

Google *David Marker's* book, or *Tent-Ziegler*. Warning: both of them contain a lot of typos.

## 5 Compactness

### Definition. (5.1)

Suppose we have a  $L$ -theory  $T$ .

- (i)  $T$  is *finitely satisfiable* if every finite subset of sentences in  $T$  has a model.
- (ii)  $T$  is *maximal* if for all  $L$ -sentences  $\sigma$ , either  $\sigma \in T$  or  $\neg\sigma \in T$ .
- (iii)  $T$  has the *witness property* (WP): if for all  $\phi(x)$  ( $L$ -formula with 1 free variable), there is a constant  $c \in \mathcal{C}$  s.t.

$$(\exists x \phi(x) \rightarrow \phi(c)) \in T$$

### Lemma. (5.2)

If  $T$  is maximal and finitely satisfiable (we'll sometimes use *f.s.* from now onwards), and  $\phi$  is an  $L$ -sentence, and  $\Delta \stackrel{fin}{\subseteq} T$  and  $\Delta \vdash \phi$ , then  $\phi \in T$ .  
(Prove it by yourself)

### Lemma. (5.3)

Let  $T$  be a maximal, f.s. theory with WP. Then  $T$  has a model.

Moreover, if  $\lambda$  is a cardinal and  $|\mathcal{C}| \leq \lambda$  ( $\mathcal{C}$  is the set of constants in  $L$ ), then  $T$  has a model of size at most  $\lambda$ .

*Proof.* Let  $\mathcal{C}$  be the constants of  $L$ . Let  $c, d \in \mathcal{C}$ , define  $c \sim d$  iff  $c = d \in T$ .

We claim that  $\sim$  is an equivalence relation: reflexivity and symmetry are trivial; for transitivity, let  $c \sim d$  and  $d \sim e$ . Then  $c = d \in T$  and  $d = e \in T$ . Then by the lemma  $c = e \in T$  as it is implied by the two sentences. So  $c \sim e$ .

Notation: we'll use  $c / \sim = c^*$  to denote the equivalence class of  $c$ .

Now define a structure  $\mathcal{M}$  whose domain is  $\mathcal{C} / \sim = M$ . Clearly,  $|M| \leq \lambda$  if  $|\mathcal{C}| \leq \lambda$ .

We must define interpretations in  $\mathcal{M}$  for symbols for  $L$ :

- If  $c \in \mathcal{C}$ , then  $c^m = c^* (= c / \sim)$ ;
- If  $R \in \mathcal{R}$  is a relation symbol, we define  $R^{\mathcal{M}} = \{(c_1^*, \dots, c_{n_R}^*) : R(c_1, \dots, c_{n_R}) \in T\}$ .

We have to check that  $R^{\mathcal{M}}$  is well-defined: suppose  $\bar{c}, \bar{d} \in \mathcal{C}^{n_R}$  and suppose  $c_i \sim d_i$  for each  $i$ , i.e.  $c_i = d_i \in T$  for every  $i = 1, \dots, n_R$ . However, now

$$R(\bar{c}) \in T \iff R(\bar{d}) \in T$$

by maximality of  $T$  and the previous lemma. So that  $R^{\mathcal{M}}$  is well defined.

- If  $f \in \mathcal{F}$  is a function symbol, then  $f\bar{c} = d \in T$  for some  $d \in \mathcal{C}$ : this is because  $\exists x(f(\bar{c}) = x) \in T$  by maximality and f.s..

Then define  $f^{\mathcal{M}}(\bar{c}^*) = \bar{d}^*$  (obvious notation).

We also have to check this is well-defined. Lecturer decides to left this to us.

Now we claim that the terms behave nicely as what the theory says, i.e. if  $t(x_1, \dots, x_n)$  is an  $L$ -term and  $c_1, \dots, c_n, d \in \mathcal{C}$ , then  $t(c_1, \dots, c_n) = d \in T \iff t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$ .

- $\implies$ : by induction on the complexity of  $T$  (lecturer decided to leave this as another exercise).
- $\Leftarrow$ : Assume  $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$ . Then  $t(c_1, \dots, c_n) = e \in T$  for some constant  $e$  (why? As our theory is maximal, it has to say what the result is when we apply

$t$  to these terms). We then use  $\implies$  to get that  $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$ .

But then  $d^* = e^*$ , i.e.  $d = e \in T$ . So by lemma (5.2), the sentence implied by these two sentences,  $t(c_1, \dots, c_n) = d \in T$ .

The last massive claim: for all  $L$ -formulas  $\phi(\bar{x})$  and  $\bar{c} \in \mathcal{C}^{|\bar{x}|}$ , we have

$$\mathcal{M} \models \phi(\bar{c}) \iff \phi(\bar{c}) \in T$$

The proof is by induction on complexity of  $\phi(\bar{x})$  (The lecturer decided to leave yet another proof to us – lots of work to be done here. Lecturer is speeding up!).  $\square$

—Lecture 8—

**Lemma.** (5.4)

Let  $T$  be a f.s.  $L$ -theory. Then there are  $L^* \supseteq L$  and a f.s.  $T^* \supseteq T$  such that:

- (i)  $|L^*| = |L| + \omega$ ;
- (ii) any  $L^*$ -theory extending  $T^*$  has WP.

*Proof.* Define  $\langle L_i : i < \omega \rangle$  a chain of languages containing  $L$  and s.t.  $|L_i| = |L| + \omega$ , and  $\langle T_i : i < \omega \rangle$  of f.s. theories s.t.  $\forall i$   $T_i$  is an  $L_i$ -theory, and  $T_i \supseteq T$ .

• Base step:  $L_0 = L$  and  $T_0 = T$ .

• At stage  $i + 1$ ,  $L_i$  and  $T_i$  are given. List all  $L_i$ -formulas  $\phi(x)$  (one free variable  $x$ ) and let  $L_{i+1} = L_i \cup \{c_\phi : \phi(x) \text{ is an } L_i\text{-formula}\}$ .

For all  $\phi(x)$  ( $L_i$ -formula), let  $\Phi_\phi$  be the  $L_{i+1}$ -sentence  $\exists x \phi(x) \rightarrow \phi(c_\phi)$ .

Then  $T_{i+1} := T_i \cup \{\Phi_\phi : \phi(x) \text{ is } L_i\text{-formula}\}$ .

Claim:  $T_{i+1}$  is f.s.. Why is that? Let's take a finite subset  $\Delta \stackrel{fin}{\subseteq} T_{i+1}$ . Then  $\Delta = \Delta_0 \cup \{\Phi_{\phi_2}, \dots, \Phi_{\phi_n}\}$  where  $\Delta_0 \subseteq T_i$ .

Let  $\mathcal{M} \models \Delta_0$  ( $\mathcal{M}$  is an  $L_i$ -structure; it exists because  $T_i$  is f.s.).

We define an  $L_{i+1}$ -structure  $\mathcal{M}'$  with domain  $M$  (of  $\mathcal{M}$ ). Define the interpretation of new constants as follows: if  $\mathcal{M} \models \exists x \phi(x)$ , then  $c_\phi^{\mathcal{M}'} = a$  for any  $a \in \mathcal{M}$  s.t.  $\mathcal{M} \models \phi(a)$ .

Otherwise,  $c_\phi^{\mathcal{M}'}$  is arbitrary.

Then  $\mathcal{M}' \models \Delta$ .

Now let  $L^* = \bigcup_{i < \omega} L_i$ ,  $T^* = \bigcup_{i < \omega} T_i$ .

By our construction, any extension of  $T^*$  has WP (check), and  $T^*$  is f.s. ( $\Delta \stackrel{fin}{\subseteq} T^*$ , then  $\Delta \subseteq T_i$  for some  $i$ ).  $\square$

**Lemma.** (5.5)

If  $T$  is f.s., there exists a maximal f.s.  $T' \supseteq T$  (of underlying language  $L$ ).

*Proof.* Let  $I := \{S : S \text{ is a f.s. } L\text{-theory s.t. } T \subseteq S\}$ .

$I$  is partially ordered by inclusion.

If  $\langle C_i : i < \lambda \rangle$  is a chain in  $I$ , then  $\bigcup_{i < \lambda} C_i$  is an upper bound for the chain: it is f.s. ( $\Delta \subseteq \bigcup C_i$  is s.t.  $\Delta \subseteq C_i$ ).

Then by Zorn's lemma,  $I$  has a maximal element (w.r.t.  $\subseteq$ ).

We claim that the maximal element  $T'$  of  $I$  is the required extension of  $T$  (check that  $\forall L$ -sentences  $\sigma$ ,  $\sigma \in T'$  or  $\neg \sigma \in T'$ ).  $\square$

**Theorem.** (5.6, Compactness)

If  $T$  is a f.s.  $L$ -theory, and  $\lambda \geq |L| + \omega$ , then there is  $\mathcal{M} \models T$  s.t.  $|\mathcal{M}| \leq \lambda$ .  
(See part II Logic and Set Theory for the finite case).

*Proof.* (sketch)

Extend  $T$  to  $T^*$ , an  $L^*$ -theory that is f.s., and s.t. any  $S \supseteq T^*$  has WP (by lemma (5.4)).

By lemma (5.5), we can extend this  $T^*$  to a maximal f.s. theory  $T'$ ; and by the lemma  $T'$  would have WP since it is an extension of  $T^*$ .

Now  $T'$  is maximal and f.s., so we can use lemma (5.3) to show that there is a model  $\mathcal{M} \models T'$ . Then in particular,  $\mathcal{M} \models T$  (check  $\mathcal{M} \leq \lambda$ ).  $\square$

**Definition.** (5.7)

Let  $L$  be a language. Then an  $L$ -type  $p(\bar{x})$  is a set of  $L$ -formulas whose free variables are in  $\bar{x}$  (and  $\bar{x} = \langle x_i : i < \lambda \rangle$ ).

- An  $L$ -type is *satisfiable* if there is an  $L$ -structure  $\mathcal{M}$  and an assignment  $\bar{a} \in \mathcal{M}^{|\bar{x}|}$  s.t.  $\mathcal{M} \models \phi(\bar{a})$  for all  $\phi(\bar{x}) \in p(\bar{x})$  (we also say  $p(\bar{x})$  is *consistent*, and  $\bar{a}$  *realizes*  $p(\bar{x})$  in  $\mathcal{M}$ ).

We write  $\mathcal{M} \models p(\bar{a})$ , or  $\mathcal{M}, \bar{a} \models p(\bar{x})$ .

We also say that  $p(\bar{x})$  is *satisfied in*  $\mathcal{M}$ .

- A type  $p(\bar{x})$  is *finitely satisfiable* if every finite subset of  $p(\bar{x})$  is satisfiable (we may say  $p(\bar{x})$  is *finitely consistent*).

**Remark.** An  $L$ -type may be finitely satisfiable in a model  $\mathcal{M}$  (by this we mean every finite subset is satisfiable in  $\mathcal{M}$ ), but not satisfiable in  $\mathcal{M}$ .

**Example.** Let  $\mathcal{M} = (\mathbb{N}, <)$ . Let  $\phi_n(x)$  say *there are at least  $n$  elements less than  $x$* , and  $p(x) := \{\phi_n(x) : n < \omega\}$ .

Is  $p(x)$  finitely satisfiable in  $\mathcal{M}$ ? Yes; but obviously  $p(x)$  is not satisfiable in  $\mathcal{M}$ .

**Theorem.** (5.8)

Every finitely satisfiable  $L$ -type  $p(\bar{x})$  is satisfiable (not necessarily in the original model, of course).

*Proof.* Let  $\bar{x} = \langle x_i : i < \lambda \rangle$ , let  $\langle c_i : i < \lambda \rangle$  be new constants (not in  $L$ ).

Expand  $L$  to  $L' = L \cup \{c_i : i < \lambda\}$ . Then  $p(\bar{c})$  is an  $L'$ -theory, and theorem (5.6) applied to  $p(\bar{c})$  gives the desired conclusion (think).  $\square$

—Lecture 9—

Example class 2: 19th November, Monday.

**Lemma.** Let  $\mathcal{M}$  be a structure, let  $\bar{a} = \langle a_i : i < \lambda \rangle$  enumerate  $\mathcal{M}$ . Let

$$q(\bar{x}) = \{\phi(\bar{x}) : \mathcal{M} \models \phi(\bar{a})\}$$

where  $|\text{bar } x| = \lambda$ .

Then  $q(\bar{x})$  is satisfiable in  $\mathcal{N}$  iff there is  $\beta : \mathcal{M} \rightarrow \mathcal{N}$  that is an elementary embedding.

*Proof.* If  $q(\bar{x})$  is satisfiable in  $\mathcal{N}$ , there is  $\bar{b} \in N^{|\bar{x}|}$  s.t.  $\mathcal{N} \models \phi(\bar{b})$  for all  $\phi(\bar{x}) \in q(\bar{x})$ .

Then  $\beta : a_i \rightarrow b_i$  ( $i < \lambda$ ) is an elementary embedding ( $\beta$  preserves, for example, atomic formulas of the form  $f(a_{i_1}, \dots, a_{i_n}) = a_{i_{n+1}}$ , so it is an embedding.)

More generally, for any  $\phi(\bar{x})$  ( $L$ -formula),

$\mathcal{M} \models \phi(\bar{a})$  iff  $\mathcal{N} \models \phi(\bar{b})$ , but  $\beta(\bar{a}) = \bar{b}$ .

Conversely, if  $\beta : \mathcal{M} \rightarrow \mathcal{N}$  is elementary, then  $\beta(\bar{a})$  satisfies  $q(\bar{x})$  in  $\mathcal{N}$ .  $\square$

**Remark.** The above is usually stated as (diagram lemma): Let  $Th(\mathcal{M}_M)$  be a theory in  $L(M)$ ,  $\mathcal{N} \models Th(\mathcal{M}_M)$ , then  $\mathcal{M}$  embeds elementarily in  $\mathcal{N}$ .

**Remark.** (5.10)

We can consider types in  $L(A)$ , where  $A \subseteq \mathcal{M}$ .

In particular, we can consider  $A = M$  the whole domain of  $\mathcal{M}$ .

Types of this kind are said to *have parameters in A* (or to be *over A*).

If  $p(\bar{x})$  is a type over  $M$ , then there is a  $\bar{a}$ , an enumeration of  $M$ , and a type  $p'(\bar{x}, \bar{z})$  where the  $\bar{z}$  are *new constants*,  $|\bar{z}| = |\bar{a}|$ , and  $p(\bar{x}) = p'(\bar{x}, \bar{a})$ .

**Theorem.** (5.11)

If  $\mathcal{M}$  is a structure, and  $p(x)$  is a type in  $L(M)$  that is finitely satisfiable in  $\mathcal{M}$ , then  $p(\bar{x})$  is satisfiable in some  $\mathcal{M}$  such that  $\mathcal{M} \preceq \mathcal{N}$ .

**Example.** Consider  $\mathcal{M} = (\mathbb{Q}, <)$ , and let  $\langle a_i : i < \omega \rangle$  a sequence in  $\mathbb{Q}$  that converges to  $\sqrt{2}$  from below, and let  $\langle b_i : i < \omega \rangle \subseteq \mathbb{Q}$  tend to  $\sqrt{2}$  from above.

Let  $\phi_n(x) \equiv a_n < x < b_n$ . Then let  $p(x) = \{\phi_n(x) \text{ s.t. } n < \omega\}$ .

Then  $p(x)$  is an  $L(\mathbb{Q})$ -type which is finitely satisfiable in  $\mathbb{Q}$ . But  $p(x)$  is *not* satisfiable in  $\mathcal{M}$ .

It is, however, satisfiable in  $(\mathbb{R}, <)$  which contains  $(\mathbb{Q}, <)$  as an elementary substructure.

(we can actually also use the example about natural numbers in last lecture – just need to add an  $\infty$ .)

*Proof.* (of 5.11)

Let  $\langle a_i : i < \lambda \rangle$  enumerate  $\mathcal{M}$ , and let  $q(\bar{x}) := \{\phi(\bar{z}) : \mathcal{M} \models \phi(\bar{a})\}$ , where  $|\bar{z}| = \lambda$  and the  $z_i$  are new variables (so not among the  $\bar{x}$ ).

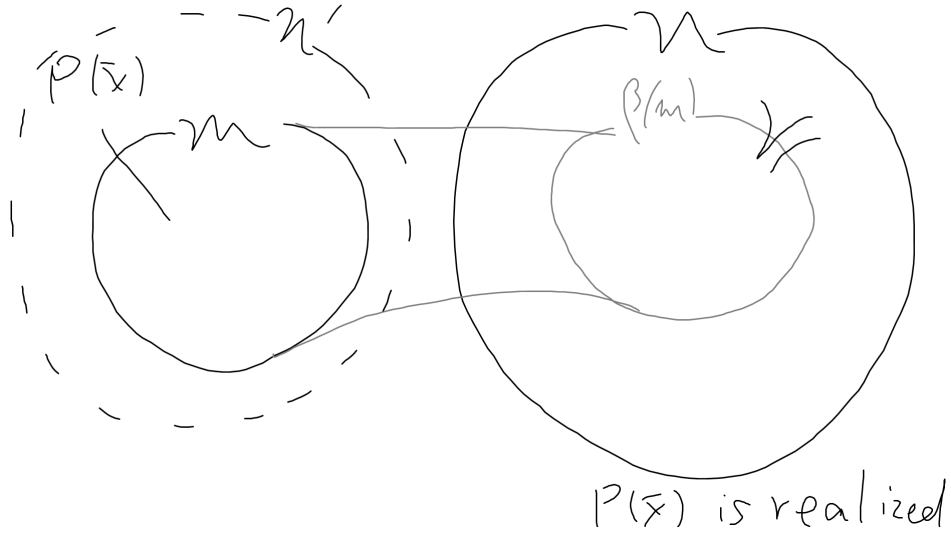
Write  $p(\bar{x})$  as  $p'(\bar{x}, \bar{a})$  for some  $p'(\bar{x}, \bar{z})$  (an  $L$ -type).

We claim that  $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$  is finitely satisfiable in  $\mathcal{M}$ : this is because  $p'(\bar{x}, \bar{a})$  is finitely satisfiable by hypothesis and  $q(\bar{z})$  is realized by  $\bar{a}$ .

Then, by theorem (5.8) (compactness for types),  $p'(\bar{x}, \bar{z}) \cup q(\bar{z})$  is satisfiable.

That is, there is  $\mathcal{N}$  and  $\bar{b} \in \mathcal{N}^{|\bar{z}|}$  and  $\bar{c} \in \mathcal{N}^{|\bar{x}|}$  such that  $\mathcal{N} \models p'(\bar{c}, \bar{b}) \cup q(\bar{b})$ .

In particular,  $\mathcal{N} \models q(\bar{b})$ . Then by lemma (5.9),  $\beta : \mathcal{M} \rightarrow \mathcal{N}$  by  $a_i \rightarrow b_i$  is an elementary embedding.



(Get  $\mathcal{N}' \simeq \mathcal{N}$  s.t.  $\mathcal{M} \preccurlyeq \mathcal{N}'$ ,  $\mathcal{M} \simeq \beta(\mathcal{M})$ .) □

**Theorem.** (5.12, Upward Löwenheim-Skolem theorem)

(See part II Logic and Set theory for the (countable?) case)

Let  $\mathcal{M}$  be s.t.  $|\mathcal{M}| \geq \omega$ . Then for any  $\lambda \geq |\mathcal{M}| + |L|$ , there is  $\mathcal{N}$  s.t.  $\mathcal{M} \preccurlyeq \mathcal{N}$  with  $|\mathcal{N}| = \lambda$ .

*Proof.* Let  $\bar{x} = \langle x_i : i < \lambda \rangle$  a tuple of distinct variables.

Let  $p(\bar{x}) = \{x_i \neq x_j : i < j < \lambda\}$ . Then  $p(\bar{x})$  is finitely consistent in  $\mathcal{M}$ . So by theorem (5.11),  $p(\bar{x})$  is realized in some  $\mathcal{N}$  that is an elementary extension of  $\mathcal{M}$ , and  $|\mathcal{N}| \geq \lambda$ .

Then by downward L-S theorem (3.11) we could have  $|\mathcal{N}| = \lambda$ . □



## 6 Saturation

**Definition.** (6.1)

Let  $\lambda$  be an infinite cardinal, let  $|\mathcal{M}| \geq \omega$ . Then  $\mathcal{M}$  is  $\lambda$ -saturated if  $\mathcal{M}$  realizes every type  $p(x)$  (with one free variable) such that:

- (i)  $p(x)$  has parameters in  $A \subseteq \mathcal{M}$  and  $|A| < \lambda$ ;
- (ii)  $p(x)$  is finitely consistent in  $\mathcal{M}$ .

We say  $\mathcal{M}$  is *saturated* if it is  $|\mathcal{M}|$ -saturated.

Question: can  $\mathcal{M}$  be  $\lambda$ -saturated if  $\lambda > |\mathcal{M}|$ ? If so,  $\mathcal{M}$  would satisfy f.s. types in  $L(M)$ .

(Hint: we could use  $p(x) = \{x \neq a_i : i < |\mathcal{M}|\}$  where  $\langle a_i : i < |\mathcal{M}| \rangle$  enumerates  $\mathcal{M}$ .)

## 7 Example class 1

### 7.1 Question 3

- (i) if  $\phi(\bar{x})$  is universal (i.e.  $\forall \bar{y} \psi(\bar{x}, \bar{y})$  where  $\psi(\bar{x}, \bar{y})$  is quantifier-free), then  $\mathcal{N} \models \phi(\beta(\bar{a})) \implies \mathcal{M} \models \phi(\bar{a})$ .
- (ii) Consider  $\mathcal{N} = (\{\frac{n}{2} : n \in \mathbb{Z}\}, +)$ . Assume for a  $X$  that  $\phi(x)$  exists. Then  $\mathcal{M} \models \phi(1)$  as 1 is odd. Now consider  $\beta : M \rightarrow N$  defined by  $n \rightarrow \frac{n}{2}$ ; it's an isomorphism between  $\mathcal{M}$  and  $\mathcal{N}$ ; Also,  $id : \mathcal{M} \rightarrow \mathcal{N}$  is an embedding. In particular, if  $\mathcal{M} \models \phi(1)$ , then  $\mathcal{N} \models \phi(1)$ ; but  $\mathcal{M} \equiv \mathcal{N}$  by the isomorphism, so  $\mathcal{M} \models \forall x(\phi(x) \leftrightarrow \forall y(y + y \neq x))$ , however that is not true in  $\mathcal{N}$ .

### 7.2 Question 4

- (i)  $\implies$  (ii):  $id : A \rightarrow N$  is an embedding.
- (ii)  $\implies$  (i): Define structure on  $A$ :  $c^{\mathcal{M}} = c^{\mathcal{N}}$ ,  $R^{\mathcal{M}} = R^{\mathcal{N}} \cap A^{n^R}$ , and similar for functions.

### 7.3 Question 5

Prove that  $\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ , where  $L(M) = L \cup \{c_a : a \in M\}$ .  
It should be clear that everything follows from the fact that,  $\mathcal{M} \models \phi(\bar{a})$  as an  $L$ -structure iff  $\mathcal{M} \models \phi(c_{a_1}, \dots, c_{a_n})$  as an  $L(M)$ -structure.

### 7.4 Question 6

We have  $[0, 1] \equiv [0, 2]$  (seen in lectures, as they are isomorphic as  $L_{lo}$ -structures. However,  $[0, 1] \not\equiv [0, 2]$  as 1 has a property in LHS which it doesn't have in RHS. Instead,  $(0, 1) \preceq (0, 2)$ . It's best to use part (i): If  $\mathcal{M}, \mathcal{N}$  are s.t.  $A \subseteq M \cap N$ ,

then  $\mathcal{M} \equiv_A \mathcal{N} \iff \mathcal{M} \equiv_B \mathcal{N}$  for every  $B \stackrel{fin}{\subseteq} A$ .

Select finite  $B \subseteq (0, 1) \cap (0, 2)$ , and show  $(0, 1) \equiv_B (0, 2)$ .

Induction on complexity of formulas are not strictly necessary; instead if we can find an isomorphism that fixes the elements of  $B$  then we're done.

Say  $B = \{b_0, \dots, b_n\}$ , where  $b_m \geq b_i \forall b_i \in B$ . We can just send every  $x \leq b_n$  in  $(0, 1)$  to the same  $x$  in  $(0, 2)$ , and rescale the interval  $(x, 1)$  to match  $(x, 2)$ , which gives an isomorphism.

### 7.5 Question 7

Let  $\phi(\bar{x})$  be a formula and  $\bar{a} \in M_i^{|\bar{x}|}$ .

It's required to prove that  $\mathcal{M}_i \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$ . By induction on  $\phi(\bar{x})$ :

if  $\phi(\bar{x})$  is atomic, claim follows from  $\mathcal{M}_i \subseteq \mathcal{N}$  (substructure);  $\wedge$  and  $\neg$  are also easy induction;

For existential formula  $\phi(\bar{x}) \equiv \exists y \psi(\bar{x}, y)$ , if  $\mathcal{M} \models \exists y \psi(\bar{a}, y)$ , then  $\mathcal{M} \models \psi(\bar{a}, b)$  for  $b$  in some  $\mathcal{M}_i$ , so  $\mathcal{N} \models \psi(\bar{a}, b)$ ;

Conversely, if  $\mathcal{N} \models \exists y \psi(\bar{a}, y)$ , then  $\mathcal{N} \models \psi(\bar{a}, b)$  for  $b \in \mathcal{N}$ . But then  $\bar{a}, b$  are in some  $\mathcal{M}_j$  for some finite  $j$ ; as a result, by inductive hypothesis  $\mathcal{M}_j \models \exists y \psi(\bar{a}, y)$ . But then  $\mathcal{M}_i \models \exists y \psi(\bar{a}, y)$  by elementarity.

Alternatively, use T-V test.

## 7.6 Question 8

(Lecturer is assuming CH here.....)

First show that  $(\mathbb{Q} + \mathbb{R}) \not\cong (\mathbb{R}, <)$ . Then for any  $U \models T_{dlo}$  with  $|U| = \lambda > 2^{\aleph_0}$ . Then prove  $(\mathbb{Q} + \mathbb{R} + U) \not\cong \mathbb{R} + U$ .