

Model Theory

October 17, 2018

<i>CONTENTS</i>	2
-----------------	---

Contents

1	Languages and structures	3
2	Terms, formulae, and their interpretations	4
3	Theories and Elementarity	8

1 Langauges and structures

Definition. (1.1) A language L consists of:

- (i) a set \mathcal{F} of function symbols, and for each $f \in \mathcal{F}$, a positive integer n_f , the arity of f ;
- (ii) a set \mathcal{R} of relation symbols, and for each $R \in \mathcal{R}$, a positive integer n_R , the arity of R ;
- (iii) a set \mathcal{C} of constant symbols.

Note that each of the above three sets can be empty.

Example. $L = \{\{ \cdot, -1 \}, \{1\}\}$ where \cdot is a binary function, -1 is a unary function, and 1 is a constant. We call this L_{gp} (language of groups); $L_{lo} = \{<\}$, where $<$ is a binary relation (linear order).

Definition. (1.2) Given a language L , say, an L -structure consists of:

- (i) a set M , the *domain*;
- (ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \rightarrow M$;
- (iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$;
- (iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the *interpretation* of f, R, c respectively.

Notation. (1.3)

We often fail to distinguish between the symbols in the language L and their interpretations in a L -structure, if the context allows.

We may write $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$.

Example. (1.4)

(a) $\mathcal{R} = \langle \mathbb{R}^+, \{ \cdot, -1 \}, 1 \rangle$ is an L_{gp} -structure.

$\mathcal{Z} = \langle \mathbb{Z}, \{ +, - \}, 0 \rangle$ is also an L_{gp} -structure (here $+$ is a binary and $-$ is the unary negation function).

$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo} structure ($<$ is the interpretation of relation).

Definition. (1.5)

Let L be a language, let \mathcal{M} and \mathcal{N} be L -structures.

An *embedding* of \mathcal{M} into \mathcal{N} is an injection $\alpha : M \rightarrow N$ that preserves the structure:

- (i) For all $f \in \mathcal{F}$, and $a_1, \dots, a_{n_f} \in M$,

$$\alpha(f^M(a_1, \dots, a_{n_f})) = f^N(\alpha(a_1), \dots, \alpha(a_{n_f}))$$

- (ii) For all $R \in \mathcal{R}$, and $a_1, \dots, a_{n_R} \in M$,

$$(a_1, \dots, a_{n_R}) \in R^M \iff (\alpha(a_1), \dots, \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if. (iii) For all $c \in \mathcal{C}$, we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding $\mathcal{M} \rightarrow \mathcal{N}$ is also called an *isomorphism* of \mathcal{M} onto \mathcal{N} .

(1.6) Exercise. Let G_1, G_2 be groups, regarded as L_{gp} -structures.

Check that $G_1 \cong G_2$ in the usual algebra sense, if and only if there is an isomorphism $\alpha : G_1 \rightarrow G_2$ in the sense of above definition 1.5.

2 Terms, formulae, and their interpretations

In addition to the symbols of L , we also have:

- (i) infinitely many variables, $\{x_i\}_{i \in I}$;
- (ii) logical connectives, \wedge, \neg (also express $\vee, \rightarrow, \leftrightarrow$);
- (iii) quantifier \exists (also express \forall);
- (iv) punctuations $(,)$.

Definition. (2.1)

L -terms are defined recursively as follows:

- any variable x_i is a term;
- any constant symbol is a term;
- for any $f \in \mathcal{F}$,

$$f(t_1, \dots, t_{n_f})$$

for any terms t_1, \dots, t_{n_f} is a term;

- nothing else is a term.

Notation: we write $t(x_1, \dots, x_n)$ to mean that the variables appearing in t are among x_1, \dots, x_n .

Example. In $\mathcal{R} = \langle \mathbb{R}, \cdot, -1, 1 \rangle$,

- $(\cdot(x_1, x_2), x_3)$ is a term $(x_1 \cdot x_2) \cdot x_3$;
- $(\cdot(1, x_1))^{-1}$ is a term $(1 \cdot x)^{-1}$.

Definition. (2.2)

If \mathcal{M} is an L -structure, to each L -term $t(x_1, \dots, x_k)$ we assign a function

$$t^M : M^k \rightarrow M$$

defined as follows:

- (i) If $t = x_i$, $t^M[a_1, \dots, a_k] = a_i$;
- (ii) If $t = c$ is a constant, $t^M[a_1, \dots, a_k] = c^M$;
- (iii) If $t = f(t_1(x_1, \dots, x_k), \dots, t_{n_f}(x_1, \dots, x_k))$,

$$t^M(a_1, \dots, a_k) = f^M(t_1^M(a_1, \dots, a_k), \dots, t_{n_f}^M(a_1, \dots, a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on [http](http://users.mct.open.ac.uk/sb27627/MT.html) :

users.mct.open.ac.uk/sb27627/MT.html (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

Remark. (The lecture forgot about this last time) Any language L includes an equality symbol $=$.

Last time we assigned a function t^m . In L_{gp} , the term $x_2 \cdot x_3$ can be described as, say $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), \dots$

Then the term $x_2 \cdot x_3$ can be assigned to functions $t_1^M : M^3 \rightarrow M : (a_1, a_2, a_3) \rightarrow (a_2 \cdot a_3)$, or $t_2^M : M^4 \rightarrow M : (a_1, a_2, a_3, a_4) \rightarrow (a_2 \cdot a_3)$. These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term t to be the number of symbols of L occurring in t .

Fact (2.3): Let \mathcal{M} and \mathcal{N} be L -structures, and let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an embedding. For any L -term $t(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in M$, we have

$$\alpha(t^M(a_1, \dots, a_k)) = t^N(\alpha(a_1), \dots, \alpha(a_k))$$

Proof. Prove by induction on complexity of t .

Let $\bar{a} = (a_1, \dots, a_k)$ and $\bar{x} = (x_1, \dots, x_l)$. Then:

- (i) if $t = x_i$ a variable, then $t^M(\bar{a}) = a_i$, and $t^N(\alpha(a_1), \dots, \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds;
- (ii) if $t = c$ is a constant, then $t^M(\bar{a}) = c^M$, and $t^N(\alpha(\bar{a})) = c^N$ by definition of a term. The key here is that, since α is an embedding we have $\alpha(c^M) = c^N$;
- (iii) if $t = f(t_1(\bar{x}, \dots, t_{n_f}(\bar{x})))$, then

$$\alpha(f^M(t_1^M(\bar{a}), \dots, t_{n_f}^M(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), \dots, \alpha(t_{n_f}^M(\bar{a})))$$

as α is an embedding. But $t_1(\bar{x}), \dots, t_{n_f}(\bar{x})$ have lower complexity than t , so the inductive hypothesis applies. \square

Exercise (2.4): conclude the proof of the above fact.
(Actually is it not done?)

Definition. (2.5)

The set of *atomic formulas* of L is defined as follows:

- (i) if t_1, t_2 are L -terms, then $t_1 = t_2$ is an atomic formula;
- (ii) if R is a relation symbol, and t_1, \dots, t_{n_R} are L -terms, then $R(t_1, \dots, t_{n_R})$ is an atomic formula;
- (iii) nothing else is an atomic formula.

Definition. (2.6)

The set of L -formulas is defined as follows:

- (i) any atomic formula is an L -formula;
- (ii) if ϕ is an L -formula, then so is $\neg\phi$;
- (iii) if ϕ and ψ are L -formulas, then so is $\phi \wedge \psi$;
- (iv) if ϕ is an L -formula, for any $i \geq 1$, $\exists x_i \phi$ is a formula;
- (v) nothing else is a formula (note that \forall can be constructed by \neg and \exists).

Example. In L_{gp} , $x_1 \cdot x_1 = x_2$, or $x_1 \cdot x_2 = 1$ are both atomic formulas; $\exists x_1(x_1 \cdot x_2) = 1$ is an L -formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier \exists . We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example, $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp} -sentence.

Notation: $\phi(x_1, \dots, x_k)$ means that the free variables in ϕ are among x_1, \dots, x_k .

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

Definition. (2.7)

Let $\phi(x_1, \dots, x_k)$ be an L -formula, let \mathcal{M} be an L -structure, and let $\bar{a} = a_1, \dots, a_k$ be elements of \mathcal{M} .

We define $\mathcal{M} \models \phi(\bar{a})$ (syntactic implication, read as \mathcal{M} models $\phi(\bar{a})$) as follows:

- (i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$;
- (ii) if ϕ is $R(t_1, \dots, t_{n_R})$, then $\mathcal{M} \models \phi(\bar{a})$ iff

$$(t_1^M(\bar{a}), \dots, t_{n_R}^M(\bar{a})) \in R^M$$

- (iii) if ϕ is a conjunction, say $\psi \wedge \chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \chi(\bar{a})$;
- (iv) if ϕ is $\exists x_j \chi(x_1, \dots, x_k, x_j)$ (where we'll assume that x_j is not one of the free variables x_1, \dots, x_k), then $\mathcal{M} \models \phi(\bar{a})$ iff there exists $b \in \mathcal{M}$ s.t. $\mathcal{M} \models \chi(a_1, \dots, a_k, b)$;
- (v) (lecture forgets this, this should probably be more in front rather than in the end) if ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \psi(\bar{a})$.

Example. Consider $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1 \rangle$, the multiplicative group of non-negative reals, and suppose we have $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$, then $\mathcal{R} \models \phi(1)$, but $\mathcal{R} \not\models \phi(-1)$.

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):

- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$;
- $\phi \rightarrow \psi$ for $\neg\phi \vee \psi$;
- $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$;
- $\forall x_i \phi$ for $\neg \exists x_i (\neg\phi)$.

Proposition. (2.9)

Let \mathcal{M} and \mathcal{N} be L -structures, and let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an embedding.

Let $\phi(\bar{x})$ be an atomic(!) formula, and $\bar{a} \in M^{|\bar{x}|}$, here $|\bar{x}|$ means the length of the tuple \bar{x} (from now on, when we write a tuple like \bar{a} , we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

Question: if ϕ is an L -formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.

Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.

In definition 2.7 we defined what it means for $\mathcal{M} \models \phi(\bar{a})$, in particular we defined: if $\phi \equiv \neg\chi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \chi(\bar{a})$. Here by $\mathcal{M} \models \phi(\bar{a})$ we mean $\mathcal{M} \models \neg\chi(\bar{a})$, and $\chi(\bar{a})$ is *shorter* than $\phi(\bar{a})$, so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

Proof. There are two cases:

- $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. Use Fact (2.3). (exercise on example sheet)
- $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), \dots, t_{n_R}(\bar{x}))$. Then $\mathcal{M} \models R(t_1(\bar{a}), \dots, t_{n_R}(\bar{a}))$ if and only if ... (lecturer says work this out by yourself. Basically the induction step). \square

Proposition. (2.10)

Exercise: show that prop (2.9) holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).

(I guess that also suggests when does it not hold for general formulas – see below).

Example. (2.11, Do embeddings preserve all formulas? No.)

Let $\mathcal{Z} = (\mathbb{Z}, <)$ an L_{lo} -structure, $\mathcal{Q} = (\mathbb{Q}, <)$ also an L_{lo} -structure. Then

$$\begin{aligned} \alpha : \mathbb{Z} &\rightarrow \mathbb{Q} \\ n &\rightarrow n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3 (x_1 < x_3 \wedge x_3 < x_2)$$

Now $\mathcal{Q} \models \phi(1, 2)$ but $\mathcal{Z} \not\models \phi(1, 2)$.

Fact (2.12) (From now on we'll stop saying that \mathcal{M}, \mathcal{N} are L -structures etc to save time) Let $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L -formula, and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).

3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let L be a language, \mathcal{M}, \mathcal{N} be L -structures.

Definition. (3.1)

An \mathcal{L} -theory T is a set of L -sentences.

\mathcal{M} is a *model* of T if $\mathcal{M} \models \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \models T$.

The class of all the models of T is written $Mod(T)$.

The *theory* of \mathcal{M} is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L\text{-structure and } \mathcal{M} \models \sigma\}$$

Example. (3.2)

Let T_{gp} be the set of L_{gp} -sentences:

(i) $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$;

(ii) $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$;

(iii) $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$.

Clearly, for a group G , $G \models T_{gp}$ (as they are just the group axioms). However, for a specific group G , clearly the theory of it, $Th(G)$ is larger than T_{gp} .

Definition. (3.3)

\mathcal{M} and \mathcal{N} are *elementarily equivalent* if $Th(\mathcal{M}) = Th(\mathcal{N})$.

We write $\mathcal{M} \equiv \mathcal{N}$.

Clearly, if $\mathcal{M} \simeq \mathcal{N}$ (\simeq means isomorphism), then $\mathcal{M} \equiv \mathcal{N}$.

But if \mathcal{M} and \mathcal{N} are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!

We'll see $(\mathcal{Q}, <) \equiv (\mathcal{R}, <)$ as L_{lo} -structures(!).

Definition. (3.4)

(i) An embedding $\beta : \mathcal{M} \rightarrow \mathcal{N}$ is *elementary* if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\beta(\bar{a}))$$

(ii) If $M \subseteq N$, and $id : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding, then \mathcal{M} is a *substructure* of \mathcal{N} .

(iii) If $M \subseteq N$ and $id : \mathcal{M} \rightarrow \mathcal{N}$ is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then \mathcal{M} is said to be an *elementary substructure* of \mathcal{N} , written as $\mathcal{M} \preceq \mathcal{N}$.

Example. (3.5)

Let $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an L_{lo} -structure where $<$ is the usual order;

Let $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$, also an L_{lo} -structure with the same $<$.

Then $\mathcal{M} \simeq \mathcal{N}$ as L_{lo} -structures. So $\mathcal{M} \equiv \mathcal{N}$ (since they are isomorphic).

Also, $\mathcal{M} \subseteq \mathcal{N}$ (read as *is a substructure of*), since the ordering $<$ coincides on \mathcal{M} and \mathcal{N} . However, $\mathcal{M} \not\preceq \mathcal{N}$, since if we pick the formula $\phi(x) \equiv \exists y (x < y)$, then $\mathcal{N} \models \phi(1)$, but $\mathcal{M} \not\models \phi(1)$.

Definition. (3.6)

Let \mathcal{M} be an L -structure, $A \subseteq M$, then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where c_a are constant symbols). An interpretation of \mathcal{M} as an L -structure extends to an interpretation of \mathcal{M} as an $L(A)$ -structure in the obvious way, i.e. $c_a^{\mathcal{M}} = a$.

In this context, the elements of A are called *parameters*.

If \mathcal{M} and \mathcal{N} are two structures, and $A \subseteq M \cap N$, then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean \mathcal{M}, \mathcal{N} satisfy exactly the same $L(A)$ structures.