# Category Theory

October 14, 2018

# Contents

# 0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

# 1 Definitions and examples

**Definition.** (1.1)
A category $\mathcal{C}$ consists of:
(a) a collection $\operatorname{ob}\mathcal{C}$ of *objects* $A, B, C$;
(b) a collection $\operatorname{mor}\mathcal{C}$ of *morphisms* $f, g, h$;
(c) two operations domain, codomain assigining to each $f \in \operatorname{mor}\mathcal{C}$ a pair of objects, its *domain* and *codomain*; we write $A \xrightarrow{f} B$ to mean *$f$ is a morphism and* $\operatorname{dom} f = A, \operatorname{cod} f = B$;
(d) an operation assigning to each $A \in \operatorname{ob}\mathcal{C}$ a morphism $A \xrightarrow{1_A} A$;
(e) a partial binary operation $(f, g) \to fg$ on morphisms, such that $fg$ is defined iff $\operatorname{dom} f = \operatorname{cod} g$, and $\operatorname{dom}(fg) = \operatorname{dom} g$, $\operatorname{cod}(fg) = \operatorname{cod}(f)$ if $fg$ is defined, satisfying:
(f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$;
(g) $(fg)h = f(gh)$ whenever $fg$ and $gh$ are defined.

**Remark.** (1.2)
(a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call $\mathcal{C}$ *small* if $\operatorname{ob}\mathcal{C}$ and $\operatorname{mor}\mathcal{C}$ are sets.
(b) Some texts say $fg$ means $f$ followed by $g$, i.e. $fg$ is defined iff $\operatorname{cod} f = \operatorname{dom} g$.
(c) Note that a morphism $f$ is an identity iff $fg = g$ and $hf = h$ whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

**Example.** (1.3)
(a) The category **Set** has all sets as objects, and all functions between sets as morphisms.
Strictly speaking, morphisms $A \to B$ are pairs $(f, B)$ where $f$ is a set-theoretic function. (See part II logic and sets)
(b) The category **Gp** has all groups as objects, group homomorphisms as morphisms.
Similarly, **Ring** is the category of rings, $\mathbf{Mod_R}$ is the category of $R$-modules.
(c) The category **Top** has all topological spaces as objects, and continuous functions as morphisms.
Similarly, **Unif** has all uniform spaces and uniformly continuous functions as morphisms, **Mf** has all manifolds and smooth maps correspondingly.
(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classess of continuous functions. More generally, given $\mathcal{C}$, we call an equivalence relation $\simeq$ on $\operatorname{mor}\mathcal{C}$ a *congruence* if $f \simeq g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathcal{C}/\simeq$ with the same objects as $\mathcal{C}$, but congruence classes as morphisms instead.
(e) Given $\mathcal{C}$, the *opposite category* $C^{op}$ has the same objects and morphisms as $\mathcal{C}$, but dom and cod are interchanged, and $fg$ in $\mathcal{C}^{op}$ is $gf$ in $\mathcal{C}$.
This leads to the *duality principle*: if $P$ is a true statement about categories, so is the statement $P^*$ obtained from $P$ by reversing all arrows.
(f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (🐱) with one object in which every morphism is an isomorphism (i.e. for all $f, \exists g$ s.t. $fg$ and $gf$ are identities).

(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space $X$, the *fundamental groupoid* $\pi(x)$ has all points of $X$ as objects, and morphisms $x \to y$ are homotopy classes $rel\{0,1\}$ of paths $u : [0,1] \to X$ with $u(0) = x$, $u(1) = y$ (if you know how to prove that the fundamental group is a group, you can prove that $\pi(x)$ is a groupoid).
(h) A *discrete* cat is one whose only morphism are identities.
A *preorder* is a cat $\mathcal{C}$ in which, for any pair $(A, B)$, $\exists$ at most 1 morphism $A \to B$.
A small preorder is a set equipped with a binary relation which is reflexive and transitive.
In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.
(i) The category **Rel** has the same objects as *set*, but morphisms $A \to B$ are arbitrary relations $R \subseteq A \times B$. Given $R$ and $S \subseteq B \times C$, we define $S \cdot R = \{(a,c) \in A \times C | (\exists b \in B)((a,b) \in R, (b,c) \in S)\}$.
The identity $1_A : A \to A$ is $\{(a,a) | a \in A\}$.
Similarly, the category **Part** are for sets and partial functions (i.e. relations s.t. $(a,b) \in R$ and $(a,b') \in R \implies b = b'$).
(j) Let $K$ be a field. The cateogry $\mathbf{Mat_K}$ has natural numbers as objects, and morphism $n \to p$ are $(p \times n)$ matrices with entries from $K$. Composition is matrix multiplication.
(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

**Definition.** (1.4)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:
(a) a mapping $A \to FA$ from $\mathrm{ob}\,\mathcal{C}$ to $\mathrm{ob}\,\mathcal{D}$;
(b) a mapping $f \to Ff$ from $\mathrm{mor}\,\mathcal{C}$ to $\mathrm{mor}\,\mathcal{D}$,
such that $\mathrm{dom}(Ff) = F(\mathrm{dom}\, f)$, $\mathrm{cod}(Ff) = F(\mathrm{cod}\, f)$, $1_{FA} = F(1_A)$, and $(Ff)(Fg) = F(fg)$ whenever $fg$ is defined.

**Example.** (1.5)
(a) We have *forgetful functors* $U$: **Gp** $\to$ **Set**, **Ring** $\to$ **Set**, **Top** $\to$ **Set**, **Ring** $\to$ **AbGp** (forget $\times$), **Ring** $\to$ **Mon** (Category of all monoids) (forget $+$).
(b) Given a set $A$, the free group $FA$ has the property:
Given any group $G$ and any function $A \xrightarrow{f} UG$ (?), there's a unique homomorphism $FA \xrightarrow{\bar{f}} G$ extending $f$. Here $F$ is a functor **Set** $\to$ **Gp**: given $A \xrightarrow{f} B$, we define $Ff$ to be the unique homomorphism extending $A \xrightarrow{f} B \leftrightarrow UFB$. Functorality follows from uniqueness given $B \xrightarrow{f} C$. $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \to UFC$.
(c) Given a set $A$, we write $PA$ for the set of all subsets of $A$.
We can make $P$ into a functor **Set** $\to$ **Set**, given $A \xrightarrow{f} B$, we defined $Pf(A') = \{f(a) | a \in A'\}$ for $A' \subseteq A$.
But we also have a functor $P^* :$ **Set** $\to$ $\mathbf{Set}^{op}$ defined on objects by $P$, but $P^* f(B') = \{a \in A | f(a) \in B'\}$ for $B' \subseteq B$.
By a *contravariant* functor $\mathcal{C} \to \mathcal{D}$, we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or $\mathcal{C}^{op} \to \mathcal{D}$).
A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

(d) Let $K$ be a field. We have a functor $* : \mathbf{Mod_K} \to \mathbf{Mod_K}^{op}$ defined by $V^* = \{$ linear maps $V \to K\}$, and if $V \xrightarrow{f} W$, $f^*(\theta : W \to K) = \theta f$.
(e) We have a functor $op : \mathbf{Cat} \to \mathbf{Cat}$, which is the identity on morphisms (note that this is a covariant).
(f) A functor between monoids is a monoid homomorphism.
(g) A functor between posets is an order-preserving map.
(h) Let $G$ be a group. A functor $F \circ G \to \mathbf{Set}$ consists of a set $A = F*$ together with an action of $G$ on $A$, i.e. a *permutation representation* of $G$.
Similarly, a functor $G \to \mathbf{Mod_K}$ is a $K$-linear representation of $G$.
(i) The construction of the fundamental group $\pi(X, X)$ of a space $X$ with basepoint $X$ is a functor $\mathbf{Top}* \to \mathbf{Gp}$ where $\mathbf{Top}*$ is the category of spaces with a chosen basepoint.
Similarly, the fundamental groupoid is a functor $\mathbf{Top} \to \mathbf{Gpd}$, where $\mathbf{Gpd}$ is the category of groupoids and functors between them.

**Definition.** (1.6)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ (why two arrows?) two functors.
A *natural transformation* $\alpha : F \to G$ consists of an assignment $A \to \alpha_A$ from $\mathrm{ob}\,\mathcal{C}$ to $\mathrm{mor}\,\mathcal{D}$ (think about this), such that $\mathrm{dom}_{\alpha_A} = FA$ and $\mathrm{cod}_{\alpha A} = GA$ for all $A$, and for all $A \xrightarrow{f} B$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)_{\alpha A}$).

(1.3) (l) Given categories $\mathcal{C}$ and $\mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

**Example.** (1.7)
(a) Let $K$ be a field, $V$ a vector space over $K$. There is a linear map $\alpha_V : V \to V^{**}$ given by $\alpha_V(v)\theta = \theta(v)$ for $\theta \in V^*$.
This is the $V$-component of a natural transformation $1_{\mathbf{Mod_K}} \to ** : \mathbf{Mod_K} \to \mathbf{Mod_K}$.
(b) For any set $A$, we have a mapping $\sigma_A : A \to PA$ sending $a$ to $\{a\}$. If $f : A \to B$, then $Pf\{a\} = \{f(a)\}$. So $\sigma$ is a natural transformation $1_{\mathbf{Set}} \to P$.
(c) Let $F$:$\mathbf{Set} \to \mathbf{Gp}$ be the free group functor (1.5(b)), and $U : \mathbf{Gp} \to \mathbf{Set}$ the forgetful functor. The inclusions $A \to UFA$ form a natural transformation $1_{\mathbf{Set}} \to UF$.
(d) Let $G, H$ be groups and $f, g : G \rightrightarrows H$ be two homomorphisms. A natural transformation $\alpha : f \to g$ corresponds to an element $h = \alpha_*$ of $H$, s.t. $hf(x) \to g(x)h$ for all $x \in G$ or equivalently $f(x) = h^{-1}g(x)h$, i.e. $f$ and $g$ are conjugate group homomorphisms.
(e) Let $A$ and $B$ be two $G$-sets, regarded as functors: $G \rightrightarrows \mathbf{Set}$. A natural transformation $A \to B$ is a function $f$ satisfying $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$, i.e. a $G$-equivariant map.

**Lemma.** (1.8)
Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors, and $\alpha : F \to G$ a natural transformation. Then $\alpha$ is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each $\alpha_A$ is an isomorphism in $\mathcal{D}$.

*Proof.* Forward is trivial (ok, I'll check this later). For backward, suppose each $\alpha_A$ has an inverse $\beta_A$. Given $f : A \to B$ in $\mathcal{C}$, we need to show that

$$\begin{array}{ll} GA & \xrightarrow{Gf} GB \\ \downarrow \beta_A & \quad \downarrow \beta_B \\ FA & \xrightarrow{Ff} FB \end{array}$$

□

commutes. But as $\alpha$ is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

**Definition.** (1.9)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. By an *equivalence* between $\mathcal{C}$ and $\mathcal{D}$, we mean a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$.
We write $\mathcal{C} \cong \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent.
We say a property $P$ of categories is a *categorical property* if whenever $\mathcal{C}$ has $P$ and $\mathcal{C} \cong \mathcal{D}$, then $\mathcal{D}$ has $P$.
For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

**Example.** (1.10)
(a) The category **Part** is equivalent to the category $\mathbf{Set}_*$ of pointed sets (and basepoint preserving functions (as morphisms)):
• We define $F : \mathbf{Set}_* \to \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$, and if $f : (A, a) \to (B, b)$, then $Ff(x) = f(x)$ if $f(x) \neq b$, and undefined otherwise;
• and $G : \mathbf{Part} \to \mathbf{Set}_*$ by $G(A) = A^+ = (A \cup \{A\}, A)$, and if $f : A \to B$ is a partial function, we define $Gf : A^+ \to B^+$ by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ defined, and equals $B$ otherwise.
The composite $FG$ is the identity on **Part**, but $GF$ is not the identity. However, there is an isomoprhism $(A, a) \to ((A \setminus \{a\})^+, A \setminus \{a\})$ sending $a$ to $A \setminus \{a\}$ and everything else to itself and this is natural.
Note that there can be no isomoprhism from $\mathbf{Set}_*$ to **Part**, since **Part** has a 1-element isomorphism class $\{\phi\}$ but $\mathbf{Set}_*$ doesn't.
(So we see that equivalent categories can be non-isomorphic. According to a post on SO, this usually happens when there are multiple copies of the 'same' thing in one but not the other. However, we can't generally 'discard obsolete copies' in one as that generally requires AC and is not a very useful thing to do anyway – In short, *identifying isomorphic objects is often an extremely bad idea.*
(b) The category $\mathbf{fdMod_K}$ of finite-dimensional vector spaces over $K$ is equivalent to $\mathbf{fdMod_K}^{op}$, the functors in both directions are $*$ (the dual operator) and both isomorphisms are the natural transformations of 1.7(a) (double dual).
(c) $\mathbf{fdMod_K}$ is also equivalent to $\mathbf{Mat}_K$ (1.3(j)):
We define $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by $A$ w.r.t. the standard bases of $K^n$ and $K^p$.
To define $G : \mathbf{fdMod_K} \to \mathbf{Mat_K}$, choose a basis for each finite dimensional vector space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ to be the matrix representing

$f$ w.r.t. chosen bases. $GF$ is the identity, provided we choose the standard bases for the spaces $K^n$; $FG \neq 1$, but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \to V$ for each $V$, which form a natural isomorphism.

—Lecture 4—

**Definition.** (1.11)
Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor.
(a) We say $F$ is *faithful* if, given $f, f' \in \operatorname{mor} \mathcal{C}$ with $\operatorname{dom} f = \operatorname{dom} f'$, $\operatorname{cod} f = \operatorname{cod} f'$, and $Ff = Ff'$, then $f = f'$ (injectivity on morphisms. The name comes more from representation theory);
(b) We say $F$ is *full* if, given $FA \xrightarrow{g} FB$ in $\mathcal{D}$, ther exists $A \xrightarrow{f} B$ in $\mathcal{C}$ with $Ff = g$. (this is something like surjective, but see below);
(c) We say $F$ is *essentially surjective* if, forevery $B \in \operatorname{ob} \mathcal{D}$, there exists $A \in \operatorname{ob} \mathcal{C}$ and isomorphism $FA \to B$ in $\mathcal{D}$.
We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is full if the inclusion $\mathcal{C}' \to \mathcal{C}$ is a full functor. For example, **Gp** is a full subcategory of **Mon** (the category of all monoids), but **Mon** is not a full subcategory of the category **SGp** of semigroups.

**Lemma.** (1.12)
Assuming the axiom of choice, a functor $F \cdot \mathcal{C} \to \mathcal{D}$ is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ (see (1.9) for what an equivalence is. I think here it means one of the functors $F, G$) if it's full, faithful, and essentially surjective.

*Proof.* $\Rightarrow$: Suppose given $G, \alpha, \beta$ as in (1.9). Then for each $B \in \operatorname{ob} \mathcal{D}$, $\beta_B$ is an isomorphism $FGB \to B$, so $F$ is essentially surjective.
Given $A \xrightarrow{f} B$ in $\mathcal{C}$, we can recover $f$ from $Ff$ as composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_b^{-1}} B$. Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$. So $F$ is faithful;
Lastly, for fullness, given $FA \xrightarrow{g} FB$, define $f$ to be the composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$. Then $GFf = \alpha_B f \alpha_A^{-1}$, which by construction is just $Gg$. But $G$ is faithful for the same reason as $f$, so $Ff = g$.

$\Leftarrow$: (need to find suitable $G, \alpha, \beta$ for $F$.) For each $B \in \operatorname{ob} \mathcal{D}$, choose $GB \in \operatorname{ob} \mathcal{C}$ and an isomorphism $\beta_B : FGB \to B$ in $\mathcal{D}$. Given $B \xrightarrow{g} B'$, define $Gg : GB \to GB'$ to be the unique morphism whose image under $F$ is $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$.
Uniqueness implies functoriality (check what this means – think it appeared somewhere before): given $B' \xrightarrow{g'} B''$, $(Gg')(Gg)$ and $G(g'g)$ have the same image under $F$, so they are equal.
By construction, $\beta$ is a natural trnasformation $FG \to 1_{\mathcal{D}}$.
Given $A \in \operatorname{ob} \mathcal{C}$, define $\alpha_A : A \to GFA$ to be the unique morphism whose image under $F$ is $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$. $\alpha_A$ is an isomorphism, since $\beta_{FA}$ also has a unique pre-image under $F$. And $\alpha$ is a natural transformation, since any naturality square for $\alpha$ (the commutative square when we defined natural transformation. check) is mapped by $F$ to a commutative square, and $F$ is faithful. $\square$

**Definition.** (1.13)
By a *skeleton* of a category, we mean a full subcategory $\mathcal{C}_0$ containing one object from each isomorphism class. We say $\mathcal{C}$ is *skeletal* if it's a skeleton of itself.
For example, $\mathbf{Mat_K}$ is a skeletal, and the image of $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ of 1.10(c) is a skeleton of $\mathbf{fdMod_K}$.
(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

**Definition.** (1.14)
Let $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$.
(a) We say $f$ is a *monomorphism* (or $f$ is *monic*) if, given any pair $C \underset{h}{\overset{g}{\rightrightarrows}} A$, $fg = fh$ implies $g = h$.
(b) We say $f$ is an *epimorphism* (or *epic*) if it's a monomorphism in $\mathcal{C}^{op}$, i.e. if $gf = hf$ implies $g = h$.
We denote monomorphisms by $A \overset{f}{\rightarrowtail} B$, and epimorphisms by $A \overset{f}{\twoheadrightarrow} B$.
Any isomorphism is monic and epic: more generally, if $f$ has a left inverse (i.e. $\exists g$ s.t. $gf$ is an identity), then it's monic. We call such monomorphisms *split*.
We say $\mathcal{C}$ is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

**Example.** (1.15)
(a) As usual we consider **Set** first. In **Set**, monomorphisms correspond to injections ($\Leftarrow$ is easy (ok); for $\Rightarrow$, take $C = 1 = \{*\}$), and epimorphsims correspond to surjections ($\Leftarrow$ is easy; for $\Rightarrow$, use morphisms $B \rightrightarrows 2 = \{0, 1\}$). So **Set** is balanced.
(b) In **Gp**, monomorphisms again correspond to injections (for $\Rightarrow$ use homomorphisms $\mathbb{Z} \to A$); epimorphisms again correspond to surjections ($\Rightarrow$ use free products with amalgamation – this is a non-trivial fact about groups, read more if free). So **Gp** is also balanced.
(c) In **Rng** (obvious notation), monomorphisms correspond to injections (proof is much like for **Gp**). However, not all epimorphisms are surjective. For example the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \underset{g}{\overset{f}{\rightrightarrows}} R$ agree on all integers, they agree everywhere. So **Rng** is not balanced.
(d) One final example is **Top**. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to **Set** (check). However, **Top** is not balanced since a continuous bijection need not have continuous inverse.

# 2 The Yoneda Lemma

Let's not start on the content this lecture. Why are we talking about one single lemma in a chapter? Well it's not really a lemma. There's some story behind this, check here for an obituary which probably has the story that lecture was talking about in class.