Connection between Model theory and Combinatorics

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Contents

0 Introduction

—Lecture 1—

Several things we'll look into in this course:
- Intro to stability;
- stable Ramsey/Erdös-Hajnal;
- stable regularity lemma;
- independence property;
- dividing lines in unstable theories.

1 Introduction to stability

1.1 History

• (definition of first-order language, some examples including L_{gp}, L_{agp}, L_{lo} and L_{gr})
• (definition of L-structures, also some examples)
• (definition of L-formulas, with some examples)
• (definition of L-sentences)
• (definition of L-theory)
• (definition of models of an L-theory)

We use the notation $I_T(\kappa)$ to mean the number of models of T of size κ up to isomorphism.

Result by Morley: let T be a conutable theory. If $I_T(\kappa) = 1$ for some uncountable κ, then $I_T(\kappa) = 1 \forall \kappa$. Examples include the theory of vector spaces over a fixed field, and the theory of algebraically closed fields (ACF).

1.2 The order property

Definition. Let T be a theory, $\mathcal{M} \vDash T$, $k \geq 1$ an integer. A formula $\phi(x, y)$ (for the time being, let it have two free variables) is said to have the k-order property (k-OP) if there are sequences $a_i, b_j, i, j = 1, ..., k$ s.t. $\mathcal{M} \vDash \phi(a_i, b_j)$ iff $i \leq j$.
A formula $\phi(x, y)$ is said to be k-stable if it does not have k-OP.

Example. • Consider Th_{gr} and a model $G = \langle V, E \rangle$. Then the formula $E(x, y)$ is k-stable if G does not contain a half-graph of height k (from the definition it's obvious what it means) as an induced bipartite subgraph. We'll sometimes say the graph G is k-stable.
• Consider Th_{agp} and a model $\langle G, +, -, A \rangle$, where A is a unary relation (so basically specifies a subset of G). The formula $\phi(x, y) = A(x + y)$ (i.e. $x + y \in A$) is k-stable if G does not contain sequences a_i, b_j of length k s.t. $a_i + b_j \in A$ iff $i \leq j$. We'll sometimes say the set A is k-stable.

Lemma. Let G be an abelian group. If $H \leq G$, then H is 2-stable.

Proof. We want to show that H can't have 2-OP. Suppose there are $a_1, a_2, b_1, b_2 \in G$ s.t. $a_i + b_j \in H$ for $1 \leq i \leq j \leq 2$, i.e. $a_1 + b_1, a_1 + b_2, a_2 + b_2 \in H$, but $a_2 + b_1$ not in H. But that is not possible because $a_2 + b_1 = (a_1 + b_1) - (a_1 + b_2) + (a_2 + b_2)$. $\square$

Lemma. Let G be an abelian group, $H \leq G$, and U a union of k cosets of H. Then U is $(k + 1)$-stable.

Proof. Suppose we had $a_1, ..., a_{k+1}, b_1, ..., b_{k+1} \in G$ witnessing $(k+1)$-OP. Then by pigeonhole principle, there exists $1 \leq i < j \leq k+1$ s.t. $a_1 + b_i$ and $a_1 + b_j$ lie in the same coset of H. Then $b_i + H = b_j + H$, and then $a_j + b_i = \underbrace{(a_j + b_j)}_{\in U} + \underbrace{(b_i - b_j)}_{\in H} \in U$. $\square$

Exercise. Let $A \subseteq G$ be a Sidon set, i.e. it contains no non-trivial solutions to $x + y = z + w$. Show that A is 3-stable. Are all 3-stable sets Sidon sets?