Model Theory

October 23, 2018

Contents

1 Langauges and structures

Definition. (1.1) A language L consists of:
•(i) a set $\mathcal{F}$ of function symbols, and for each $f \in \mathcal{F}$, a positive integer n_f, the arity of f;
•(ii) a set $\mathcal{R}$ of relation symbols, and for each $R \in \mathcal{R}$, a positive integer n_R, the arity of R;
•(iii) a set $\mathcal{C}$ of constant symbols.
Note that each of the above three sets can be empty.

Example. $L = \{\{\cdot, -1\}, \{1\}\}$ where $\cdot$ is a binary function, -1 is a unary function, and 1 is a constant. We call this L_{gp} (language of groups);
$L_{lo} = \{<\}$, where $<$ is a binary relation (linear order).

Definition. (1.2)
Given a language L, say, an L-structure consists of:
(i) a set M, the *domain*;
(ii) for each $f \in \mathcal{F}$, a function $f^M : M^{n_f} \to M$;
(iii) for each $R \in \mathcal{R}$, a relation $R^M \subseteq M^{n_R}$;
(iv) for each $c \in \mathcal{C}$, an element $c^M \in M$.

f^M, R^M, c^M are called the *interpretation* of f, R, c respectively.

Notation. (1.3)
We often fail to distinguish between the symbols in the language L and their interpretations in a L-structure, if the context allows.

We may write $\mathcal{M} = \langle M, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$.

Example. (1.4)
(a) $\mathcal{R} = \langle \mathbb{R}^+, \{\cdot, -1\}, 1 \rangle$ is an L_{gp}-structure.
$\mathcal{Z} = \langle \mathbb{Z}, \{+, -\}, 0 \rangle$ is also an L_{gp}-structure (here $+$ is a binary and $-$ is the unary negation function).
$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$ is an L_{lo} structure ($<$ is the interpretation of relation).

Definition. (1.5)
Let L be a language, let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.
An *embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an injection $\alpha : M \to N$ that preserves the structure:
(i) For all $f \in \mathcal{F}$, and $a_1, ..., a_{n_f} \in M$,

$$\alpha(f^M(a_1, ..., a_{n_f})) = f^N(\alpha(a_1), ..., \alpha(a_{n_f}))$$

(ii) For all $R \in \mathcal{R}$, and $a_1, ..., a_{n_R} \in M$,

$$(a_1, ..., a_{n_R}) \in R^M \iff (\alpha(a_1), ..., \alpha(a_{n_R})) \in R^N$$

Note that this is an if and only if.
(iii) For all $c \in \mathcal{C}$, we need

$$\alpha(c^M) = c^N$$

As anyone could expect, a surjective embedding $\mathcal{M} \to \mathcal{N}$ is also called an *isomorphism* of $\mathcal{M}$ onto $\mathcal{N}$.

(1.6) Exercise. Let G_1, G_2 be groups, regarded as L_{gp}-structures.
Check that $G_1 \cong G_2$ in the usual algebra sense, if and only if there is an isomprhism $\alpha : G_1 \ \to G_2$ in the sense of above definition 1.5.

2 Terms, formulae, and their interpretations

In addition to the symbols of L, we also have:
(i) infinitely many variables, $\{x_i\}_{i\in I}$;
(ii) logical connectives, $\wedge, \neg$ (also express $\vee, \rightarrow, \leftrightarrow$);
(iii) quantifier $\exists$ (also express $\forall$);
(iv) punctuations (,).

Definition. (2.1)
L-terms are defined recursively as follows:
• any variable x_i is a term;
• any constant symbol is a term;
• for any $f \in \mathcal{F}$,

$$f(t_1, ..., t_{n_f})$$

for any terms $t_1, ..., t_{n_f}$ is a term;
• nothing else is a term.

Notation: we write $t(x_1, ..., x_n)$ to mean that the variables appearing in t are among $x_1, .., x_n$.

Example. In $\mathcal{R} =< \mathbb{R}, \cdot, -1, 1 >$,
• $(\cdot(x_1, x_2), x_3)$ is a term $(x_1 \cdot x_2) \cdot x_3)$;
• $(\cdot(1, x_1))^{-1}$ is a term $(1 \cdot x)^{-1}$.

Definition. (2.2)
If $\mathcal{M}$ is an L-structure, to each L-term $t(x_1, ..., x_k)$ we assign a function

$$t^M : M^k \to M$$

defined as follows:
(i) If $t = x_i, t^M[a_1, ..., a_k] = a_i$;
(ii) If $t = c$ is a constant, $t^M[a_1, ..., a_k] = c^m$;
(iii) If $t = f(t_1(x_1, ..., x_k), ..., t_{n_f}(x_1, ..., x_k))$,

$$t^M(a_1, ..., a_k) = f^M(t_1^M(a_1, ..., a_k), ..., t_{n_f}^M(a_1, ..., a_k))$$

—Lecture 2—

No lecture this friday (12th Oct)! Will have an extra one on Monday 22 Oct at 12 (MR12).

First example class: Monday 29th Oct at 12.

Info on course and notes on $http:$ $users.mct.open.ac.uk/sb27627/MT.html$ (it seems that it only comes after lecture, and is hand-written, so this notes still continues), or google *Silvia Barbina MCT* and follow link *Part III Model Theory* on lecturer's homepage.

Remark. (The lecture forgot about this last time) Any language L includes an equality symbol $=$.

Last time we assigned a function t^m. In L_{gp}, the term $x_2 \cdot x_3$ can be described as, say $t_1(x_1, x_2, x_3), t_2(x_1, x_2, x_3, x_4), ...$
Then the term $x_2 \cdot x_3$ can be assigned to functions $t_1^M : M^3 \to M : (a_1, a_2, a_3) \to (a_2 \cdot a_3)$, or $t_2^M : M^4 \to M : (a_1, a_2, a_3, a_4) \to (a_2 \cdot a_3)$. These syntactic things are not really important – we just have to know that there is a corresponding action for each term.

We now define the *complexity* of a term t to be the number of symbols of L occuring in t.

Fact (2.3): Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding. For any L-term $t(x_1, ..., x_k)$ and $a_1, ..., a_k \in M$, we have

$$\alpha(t^M(a_1, ..., a_k)) = t^N(\alpha(a_1), ..., \alpha(a_k))$$

Proof. Prove by induction on complexity of t.
Let $\bar{a} = (a_1, ..., a_k)$ and $\bar{x} = (x_1, ..., x_l)$. Then:
(i) if $t = x_i$a a variable, then $t^M(\bar{a}) = a_i$, and $t^N(\alpha(a_1), ..., \alpha(a_k)) = \alpha(a_i)$, so the conclusion holds;
(ii) if $t = c$ is a constant, then $t^M(\bar{a}) = c^M$, and $t^N(\alpha(\bar{a})) = c^N$ by definition of a term. The key here is that, since α is an embedding we have $\alpha(c^M) = c^N$;
(iii) if $t = f(t_1(\bar{x}, ..., t_{n_f}(\bar{x})))$, then

$$\alpha(f^M(t_1^M(\bar{a}), ..., t_{n_f}(\bar{a}))) = f^N(\alpha(t_1^M(\bar{a})), ..., \alpha(t_{n_f}^M(\bar{a})))$$

as α is an embedding. But $t_1(\bar{x}), ..., t_{n_f}(\bar{x})$ have lower complexity than t, so the inductive hypothesis applies. □

Exercise (2.4): conclude the proof of the above fact.
(Actually is it not done?)

Definition. (2.5)
The set of *atmoic formulas* of L is defined as follows:
(i) if t_1, t_2 are L-terms, then $t_1 = t_2$ is an atomic formula;
(ii) if R is a relation symbol, and $t_1, ..., t_{n_R}$ are L-terms, then $R(t_1, ..., t_{n_R})$ is an atomic formula;
(iii) nothing else is an atomic formula.

Definition. (2.6)
The set of L-formulas is defined as follows:
(i) any atomic formula is an L-formula;
(ii) if ϕ is an L-formula, then so is $\neg\phi$;
(iii) if ϕ and ψ are L-formulas, then so is $\phi \wedge \psi$;
(iv) if ϕ is an L-formula, for any $i \geq 1$, $\exists x_i \phi$ is a formula;
(v) nothing else is a formula (note that $\forall$ can be constructed by $\neg$ and $\exists$).

Example. In L_{gp}, $x_1 \cdot x_1 = x_2$, or $x_1 \cdot x_2 = 1$ are both atomic formulas;
$\exists x_1 (x_1 \cdot x_2) = 1$ is an L-formula, but (obviously) not atomic.

A variable occurs *freely* in a formula if it does not occur within the scope of a quantifier $\exists$. We sometimes also say that the variable is *free* (from Part II Logic and Sets). Otherwise we say the variable is *bound*.

We'll use the convention that no variable occurs both freely and as a bound variable in the same formula.

A *sentence* is a formula with no free variables. For example, $\exists x_1 \exists x_2 (x_1 \cdot x_2 = 1)$ is an L_{gp}-sentence.

Notation: $\phi(x_1, ..., x_k)$ means that the free variables in ϕ are among $x_1, ..., x_k$.

Now we introduce a long and inductive (and also in logic and sets) definition for which sentences are *true*:

Definition. (2.7)
Let $\phi(x_1, ..., x_k)$ be an L-formula, let $\mathcal{M}$ be an L-structure, and let $\bar{a} = a_1, ..., a_k$ be elements of $\mathcal{M}$.
We define $\mathcal{M} \vDash \phi(\bar{a})$ (syntactic implication, read as *M models* $\phi(\bar{a})$) as follows:
(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a}) \iff t_1^M(\bar{a}) = t_2^M(\bar{a})$;
(ii) if ϕ is $R(t_1, ..., t_{n_R})$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff

$$\left(t_1^M(\bar{a}), ..., t_{n_R}^M(\bar{a})\right) \in R^M$$

(iii) if ϕ is a conjunction, say $\psi \wedge \chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \chi(\bar{a})$;
(iv) if ϕ is $\exists x_j \chi(x_1, ..., x_k, x_j)$ (where we'll assume that x_j is not one of the free variables $x_1, ..., x_k$), then $\mathcal{M} \vDash \phi(\bar{a})$ iff there exists $b \in \mathcal{M}$ s.t. $\mathcal{M} \vDash \chi(a_1, ..., a_k, b)$;
(v) (lecture forgets this, this should probably be more in front rather than in the end) if ϕ is $\neg\psi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \psi(\bar{a})$.

Example. Consider $\mathcal{R} = \langle \mathbb{R}^*, \cdot, -1, 1 \rangle$, the multiplicative group of non-negative reals, and suppose we have $\phi(x_1) = \exists x_2 (x_2 \cdot x_2 = x_1)$, then $\mathcal{R} \vDash \phi(1)$, but $\mathcal{R} \nvDash \phi(-1)$.

Notation (2.8) (useful abbreviations, closer to real life. The precise formulas are not that important – the abbreviations mean what we expect in real life):
- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$;
- $\phi \to \psi$ for $\neg\phi \vee \psi$;
- $\phi \leftrightarrow \psi$ for $(\phi \to \psi) \wedge (\psi \to \phi)$;
- $\forall x_i \phi$ for $\neg\exists x_i(\neg\phi)$.

Proposition. (2.9)
Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures, and let $\alpha : \mathcal{M} \to \mathcal{N}$ be an embedding.
Let $\phi(\bar{x})$ be an atomic(!) formula, and $\bar{a} \in M^{|\bar{x}|}$, here $|\bar{x}|$ means the length of the tuple $\bar{x}$ (from now on, when we write a tuple like $\bar{a}$, we will assume that it has the correct length without explicitly stating that), then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a}))$$

Question: if ϕ is an L-formula, not necessarily atomic, does (2.9) still hold? (the answer is no!)

—Lecture 3—

Lecturer wants to reiterate that her email address is *silvia.barbina@open.ac.uk*. Just bring the work along. Unfortunately lecturer doesn't have an office here, so

no pigeonhole.
Check website for example sheet 1!

Additional assumption: assume the set of variables in a language are indexed by a linearly ordered set.
In definition 2.7 we defined what it means for $\mathcal{M} \vDash \phi(\bar{a})$, in particular we defined: if $\phi \equiv \neg\chi$, then $\mathcal{M} \vDash \phi(\bar{a})$ iff $\mathcal{M} \nvDash \chi(\bar{a})$. Here by $\mathcal{M} \vDash \phi(\bar{a})$ we mean $\mathcal{M} \vDash \neg\chi(\bar{a})$, and $\chi(\bar{a})$ is *shorter* than $\phi(\bar{a})$, so this definition by induction works.

Now let's go back to a sketch proof of (2.9).

Proof. There are two cases:
• $\phi(\bar{x})$ is of the form $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are terms. Use Fact (2.3). (exercise on example sheet)
• $\phi(\bar{x})$ is of the form $R(t_1(\bar{x}), ..., t_{n_R}(\bar{x}))$. Then $\mathcal{M} \vDash R(t_1(\bar{a}), ..., t_{n_R}(\bar{a}))$ if and only if ... (lecturer says work this out by yourself. Basically the induction step). □

Proposition. (2.10)
Exercise: show that prop (2.9) holds if $\phi(\bar{x})$ is a formula without quantifiers (a quantifier-free formula).
(I guess that also suggests when does it not hold for general formulas – see below).

Example. (2.11, Do embeddings preserve all formulas? No.)
Let $\mathcal{Z} = (\mathbb{Z}, <)$ an $L_{lo}-$structure, $\mathcal{Q} = (\mathbb{Q}, <)$ also an $L_{lo}-$structure. Then

$$\begin{aligned} \alpha :&\mathbb{Z} \to \mathbb{Q} \\ &n \to n \end{aligned}$$

is an embedding (check). But:

$$\phi(x_1, x_2) \equiv \exists x_3 (x_1 < x_3 \wedge x_3 < x_2)$$

Now $\mathcal{Q} \vDash \phi(1, 2)$ but $\mathcal{Z} \nvDash \phi(1, 2)$.

Fact (2.12) (From now on we'll stop saying that $\mathcal{M}, \mathcal{N}$ are L-structures etc to save time) Let $\alpha : \mathcal{M} \to \mathcal{N}$ be an isomorphism. Then if $\phi(\bar{x})$ is an L-formula, and $\bar{a} \in M^{|\bar{x}|}$, then

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\alpha(\bar{a}))$$

The proof is left as an exercise (another one).

3 Theories and Elementarity

This is where the core materials begin.

Throughout this chapter, let L be a language, $\mathcal{M}, \mathcal{N}$ be L-structures.

Definition. (3.1)
An L-*theory* T is a set of L-sentences.
$\mathcal{M}$ is a *model* of T if $\mathcal{M} \vDash \sigma$ for all $\sigma \in T$. We write $\mathcal{M} \vDash T$.
The class of all the models of T is written $Mod(T)$.
The *theory of* $\mathcal{M}$ is the set

$$Th(\mathcal{M}) = \{\sigma : \sigma \text{ is an } L - \text{sentence and } \mathcal{M} \vDash \sigma\}$$

Example. (3.2)
Let T_{gp} be the set of L_{gp}-sentences:
(i) $\forall x_1 x_2 x_3 (x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3)$;
(ii) $\forall x_1 (x_1 \cdot 1 = 1 \cdot x_1 = x_1)$;
(iii) $\forall x_1 (x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1)$.
Clearly, for a group G, $G \vDash T_{gp}$ (as they are just the group axioms). However, for a specific group G, clearly the theory of it, $Th(G)$ is lartger than T_{gp}.

Definition. (3.3)
$\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if $Th(\mathcal{M}) = Th(\mathcal{N})$.
We write $\mathcal{M} \equiv \mathcal{N}$.
Clearly, if $\mathcal{M} \simeq \mathcal{N}$ ($\simeq$ means isomorphism), then $\mathcal{M} \equiv \mathcal{N}$.
But if $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, establishing whether $\mathcal{M} \equiv \mathcal{N}$ can be highly non-trivial!
We'll see $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as $L_{lo}-$structures(!).

Definition. (3.4)
(i) An embedding $\beta : \mathcal{M} \to \mathcal{N}$ is *elementary* if for all formulas $\phi(\bar{x})$ and $\bar{a} \in M^{|\bar{x}|}$,

$$\mathcal{M} \vDash \phi(\bar{a}) \iff \mathcal{N} \vDash \phi(\beta(\bar{a}))$$

(ii) If $M \subseteq N$, and $id : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.
(iii) If $M \subseteq N$ and $id : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* (just accept it without thinking of what it actually means in reality), then $\mathcal{M}$ is said to be an *elementary substructure* of $\mathcal{N}$, written as $\mathcal{M} \preccurlyeq \mathcal{N}$.

Example. (3.5)
Let $\mathcal{M} = [0, 1] \subseteq \mathbb{R}$, an $L_{lo}-$structure where $<$ is the usual order;
Let $\mathcal{N} = [0, 2] \subseteq \mathbb{R}$, also an $L_{lo}-$structure with the same $<$.
Then $\mathcal{M} \simeq \mathcal{N}$ as $L_{lo}-$structures. So $\mathcal{M} \equiv \mathcal{N}$ (since they are isomoprhic).
Also, $\mathcal{M} \subseteq \mathcal{N}$ (read as *is a substructure of*), since the ordering $<$ coincides on $\mathcal{M}$ and $\mathcal{N}$. *However*, $\mathcal{M} \not\preccurlyeq \mathcal{N}$, since if we pick the formula $\phi(x) \equiv \exists y (x < y)$, then $\mathcal{N} \vDash \phi(1)$, but $\mathcal{M} \nvDash \phi(1)$.

Definition. (3.6)
Let $\mathcal{M}$ be an L-structure, $A \subseteq M$, then

$$L(A) = L \cup \{c_a : a \in A\}$$

(where c_a are constant symbols). An interpretation of $\mathcal{M}$ as an L-structure extends to an interpretation of $\mathcal{M}$ as an $L(A)$-structure in the obvious way, i.e. $c_a^M = a$.
In this context, the elements of A are called *parameters*.
If $\mathcal{M}$ and $\mathcal{N}$ are two structures, and $A \subseteq M \cap N$, then

$$\mathcal{M} \equiv_A \mathcal{N}$$

where we mean $\mathcal{M}, \mathcal{N}$ satisfy exactly the same $L(A)$ structures.

—Lecture 4—

Reminder: we have a lecture next Monday (22nd Oct)!

Proposition. It turns out that, $\mathcal{M} \preccurlyeq \mathcal{N} \iff \mathcal{M} \equiv_M \mathcal{N}$ (where M is the domain of $\mathcal{M}$).

Lemma. (3.8, Tarski-Vaught test)
Let $\mathcal{N}$ be an L-structure, let $A \subseteq N$. The follwing are equivalent:
(i) A is the domain of a structure $\mathcal{M}$ s.t. $\mathcal{M} \preccurlyeq \mathcal{N}$;
(ii) if $\phi(x) \in L(A)$ (with an abuse of notations $\phi(x, c_{a_1}, ..., c_{a_n}) = \phi(x, a_1, ..., a_n)$), if $\mathcal{N} \vDash \exists x \phi(x)$, then $\mathcal{N} \vDash \phi(b)$ for some $b \in A$.

Proof. (i) $\implies$ (ii): Suppose $\mathcal{N} \vDash \exists x \phi(x)$. Then by elementarity, $\mathcal{M} \vDash \exists x \phi(x)$, and so $\mathcal{M} \vDash \phi(b)$ for $b \in \mathcal{M}$. So (again by elementarity), $\mathcal{N} \vDash \phi(b)$.
(ii) $\implies$ (i): This is the harder direction. First we prove that A is the domain of a substructure $\mathcal{M} \subseteq \mathcal{N}$.
By Sheet 1 Q4, it suffices to check:
(a) For each constant c, $c^N \in A$;
(b) For each function symbol f, $f^N(\bar{a}) \in A$ (for all $\bar{a} \in A^{n_R}$);
For (a), use property (ii) with $\exists x(x = c)$.
For (b), use property (ii) with the formula $\exists x((\bar{a}) = x)$.
So we now have $\mathcal{M} \subseteq \mathcal{N}$, and domain of $\mathcal{M}$ is A. But we actually want to prove that $\mathcal{M} \preccurlyeq \mathcal{N}$. Now let $\chi(\bar{x})$ be an L-formula.
We want to show that for $\bar{a} \in A^{|\bar{x}|}$ $\mathcal{M} \vDash \chi(\bar{a}) \iff \mathcal{N} \vDash \chi(\bar{a})$ (*).
By induction on the complexity of $\chi(\bar{x})$:
- if $\chi(\bar{x})$ is atomic, (*) follows from $\mathcal{M} \subseteq \mathcal{N}$ (since $\mathcal{M}$ is a substructure!);
- if $\chi(\bar{x})$ is $\neg\psi(\bar{x})$ or $\chi(\bar{x})$ is $\psi(\bar{x}) \wedge \xi(\bar{x})$, it's a straightforward induction;
- (interesting case) if $\chi(\bar{x}) = \exists y \psi(\bar{x}, y)$ where $\psi(\bar{x}, y)$ is an L-formula, suppose that $\mathcal{M} \vDash \chi(\bar{a})$, then $\mathcal{M} \vDash \exists y \psi(\bar{a}, y)$, hence $\mathcal{M} \vDash \psi(\bar{a}, b)$ for some $b \in A = \mathrm{dom}(\mathcal{M})$ (this is the definition of truth).

But then $\mathcal{N} \vDash \psi(\bar{a}, b)$ by inductive hypothesis, so $\mathcal{N} \vDash \chi(\bar{a})$.
Now let $\mathcal{N} \vDash \chi(\bar{a})$, i.e. $\mathcal{N} \vDash \exists y \psi(\bar{a}, y)$ (we find a *witness* for it). By property (ii), $\mathcal{N} \vDash \psi(\bar{a}, b)$ for some $b \in A = \mathrm{dom}(\mathcal{M})$.
Again by inductive hypothesis, we have $\mathcal{M} \vDash \psi(\bar{a}, b)$, and so in particular, $\mathcal{M} \vDash \chi(\bar{a})$ as it has got a witness there. ☐

Remark. (3.9)
Even more assumptions: let's assume that the set of variables is countably infinite. Then:
- the cardinality of the set of L-formulas is $|L| + \omega$ (where by $|L|$ we mean

the number of symbols. For example, $|L_{gp}| = 3$, $|L_{lo}| = 1$), where we abuse another notation that we use ω as cardinals (rather than ordinals) (note that the formulas are just strings of finite length);
• if A is a set of parameters in some structure, the cardinality of the set $L(A)$ is $|A| + |L| + \omega$, where by + here we merely mean $\max\{|L|, |A|, \omega\}$ (instead of addition), and same for the + above.

Definition. (3.10)
Let λ be an ordinal. Then *a chain of length* λ of sets is a sequence $\langle M_i : i < \lambda \rangle$, where $M_i \subseteq M_j$ for all $i \leq j < \lambda$.
A chain of L-structures is a seequence: $\langle \mathcal{M}_i : i < \lambda \rangle$ s.t. $\mathcal{M}_i \subseteq \mathcal{M}_j$ (note that it's substructure here) for $i \leq j < \lambda$.
The *union* of this chain is the L-structure $\mathcal{M}$ defined as follows:
• the domain is $\bigcup_{i<\lambda} M_i$ (when you think of this, you can always start with the case $\lambda = \omega$);
• for constants c, $c^M = c^{M_i}$ for any $i < \lambda$ (this is well defined, because of the substructure condition above);
• if f is a function symbol, $\bar{a} \in M^{|n_f|}$ (why the mod sign here), $f^M \bar{a} = f^{M_i} \bar{a}$ where i is s.t. $\bar{a} \in M_i^{|n_f|}$;
• if R is a relation synbol, then $R^M = \bigcup_{i<\lambda} R^{M_i}$.

Theorem. (3.11, Downward Löwenheim-Skolem theorem)
(Recall that in part II Logic and Set Theory we had the countable version of this)
Let $\mathcal{N}$ be an L-structure, and $|\mathcal{N}| \geq |L| + \omega$. Let $A \subseteq N$. Then for every cardinal λ s.t. $|L| + |A| + \omega \leq \lambda \leq |\mathcal{N}|$, there is $\mathcal{M} \preccurlyeq \mathcal{N}$ s.t.
(i) $A \subseteq M$;
(ii) $|\mathcal{M}| = \lambda$.

(It helps to think about the case $|A| = \omega$ and $|N|$ is uncountable.)
A quick example how this could be useful (we'll go very sloppy here): think of $(\mathbb{C}, +, \cdot, -, \cdot^{-1}, 0, 1)$ as a field. Consider $\mathbb{Q} \subseteq \mathbb{C}$ (both as subset and substructure). Note that algebraic closeness is a property of $\mathbb{C}$. By downward Löwenheim-

Skolem, there is a substructure in $\mathcal{C}$ that contains $\mathbb{Q}$ that is also algebraically closed (apparently, the set of algebraic numbers).

Proof. We build a chain $\langle A_i : i < \lambda \rangle$, with $A_i \subseteq N$, s.t. $|A_i| = \lambda$.
(our goal: define an elementary substructure with domain $M = \bigcup_{i<\omega} A_i$).
Base case: Let $A_0 \subseteq N$ be such that $A \subseteq A_0$ and $|A_0| = \lambda$.
Successors: At stage $i + 1$, assume A_i has been built, with $|A_i| = \lambda$.
Let $\langle \phi_k(x) : k < \lambda \rangle$ be an enumeration of those $L(A_i)$-formulas such that $\mathcal{N} \vDash \exists x \phi_k(x)$. Let a_k be such that $\mathcal{N} \vDash \phi_k(a_k)$, and let $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$ (basically, with those witnesses added). Then $|A_{i+1}| = \lambda$ (note that we haven't increased the size).
Now let $M = \bigcup_{i<\omega} A_i$ (note the subscript range). We use lemma (3.8) to show that M is the domain of $\mathcal{M} \preccurlyeq \mathcal{N}$, and $|M| = \lambda$. We're running out of time, so we'll continue next Monday.

—Lecture 5—
Solutions to worksheet 1: either take along to lecture on Friday, or email them to *silvia.barbina@open.ac.uk*.

Let's continue with the proof:

Start with $A_0 \subset N$, $A \subseteq A_0$, $|A_0| = l$. The idea is to define $\langle A_i : i < \omega \rangle$ so that $M = \cup_{i<\omega} A_i$ satisfies (ii) via the TV test (3.8).
List all formulas $\phi(x, \bar{a})$ ($\bar{a}$ is a tuple in A_0), and $\mathcal{N} \vDash \phi(b, \bar{a})$ for some b.
Add each such b to A_0 (one for each such ϕ).
Let $A_1 = A_0 \cup \{$ all thes b's$\}$.
Repeat for formulas $\phi(x, \bar{a})$ where $\bar{a}$ is in A_1,...
Eventually, $\langle A_i : i < \omega \rangle$ is such that $M = \cup_{i<\omega} A_i$ is as required (i.e. M is the domain of some elementary substrucutre of $\mathcal{N}$ that we need).
We claim that M satisfies condition (ii) in Lemma (3.8): Let $\mathcal{N} \vDash \exists x \psi(x, \bar{a})$, where $\bar{a}$ is a tuple in M. Then $\bar{a}$ is a *finite* tuple, so there is an i s.t. $\bar{a}$ is in A_i.

Then A_{i+1}, by construction, contains b s.t. $\mathcal{N} \models \phi(b, \bar{a})$. But $A_{i+1} \subseteq M, b \in M$. Then apply (3.8) we're done. $\square$

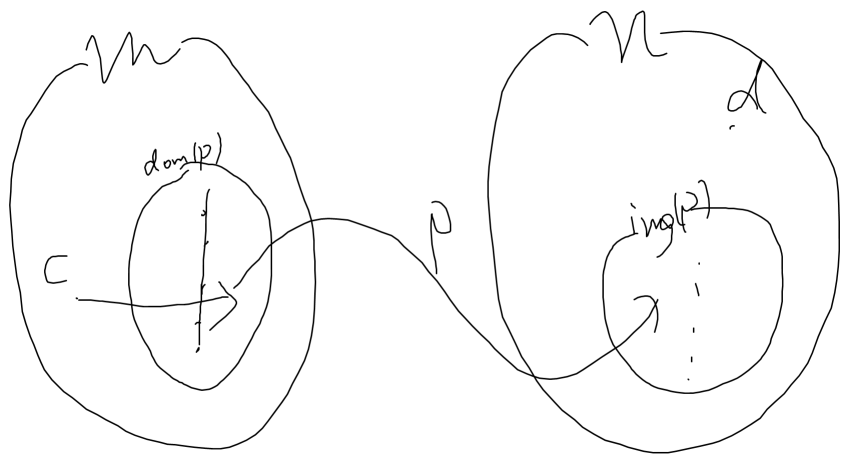
4 Two relational structures

Definition. (4.1, dense linear orders)
A *linear order* is an $L_{lo} = \{<\}$-structure such that:
(i) $\forall x \neg(x < x)$;
(ii) $\forall xyz((x < y \wedge y < z) \rightarrow x < z)$;
(iii) $\forall xy((x < y) \vee (y < x) \vee x = y)$ (total).
A linear order is *dense* if, in addition, it also satisfies:
(iv) $\exists xy(x < y)$;
(v) $\forall xy, (x < y \rightarrow \exists z(x < z \wedge z < y))$ (density).
A linear order has no endpoints if, in addition,
(vi) $\forall x(\exists y(x < y) \wedge \exists z(z < x))$.
We use T_{dlo} to denote the theory that includes all axioms (i) to (vi), and T_{lo} is the theory that includes axioms (i) to (iii) only.

Remark. (iv) and (v) imply that if $\mathcal{M} \models T_{dlo}$, then $|\mathcal{M}| \geq \omega$.

Definition. (4.2)
If $\mathcal{M}, \mathcal{N} \models T_{lo}$, then an *injective* map $p : A \subseteq M \rightarrow N$ is a *partial embedding* if $\mathcal{M} \models a < b \implies \mathcal{N} \models p(a) < p(b)$.
In particular, if $|\operatorname{dom}(p)| < \omega$, then p is a *finite* partial embedding.

Lemma. (4.3, extension lemma)
Take a linear order $\mathcal{M} \models T_{lo}$, and a dense linear endpoints $\mathcal{N} \models T_{dlo}$, and let $p : M \rightarrow N$ be a finite partial embedding. Then if $c \in \mathcal{M}$, there is a finite partial embedding $\hat{p}$ s.t. $p \subseteq \hat{p}$ and $c \in \operatorname{dom}(\hat{p})$.
(*we can always add one extra element in our embedding.*)



Proof.

Case 1: c is greater than all elements in $\operatorname{dom}(p)$. In that case, pick an element $d \in \mathcal{N}$ s.t. $d > b$ for all $b \in img(p)$;
Case 2: $a_i < c < a_{i+1}$ where $a_i, a_{i+1} \in \operatorname{dom}(p)$. Then we choose $\mathcal{N} \models p(a_i) < d < p(a_{i+1})$, where d is chosen appropriately by density (here's the case why we

need $\mathcal{N}$ to be dense;
Case 3: c is less than all elements in $\mathrm{dom}(p)$. This is similar to case 1.

Note that the ability to extend by one point allows us to embed any finite linear order into a dense linear order without endpoints. $\square$

Theorem. (4.4)
Let $\mathcal{M}, \mathcal{N} \vDash T_{dlo}$ s.t. $|\mathcal{M}| = |\mathcal{N}| = \omega$. Let $p : A \subseteq M \to N$ be a finite partial embedding.
Then there is an isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ s.t. $p \subseteq \pi$.

Proof. Enumerate M, N, say $M = \langle a_i : i < \omega \rangle$, $N = \langle b_i : i < \omega \rangle$ (sequences of elements).
We define, inductively, a chain of finite partial embedding $\langle p_i : i < \omega \rangle$ (idea: $\pi = \cup_{i<\omega} p_i$).
Let's start with $p_0 = p$. At stage $i+1$, suppose we are given p_i. We want to include a_i in $\mathrm{dom}\, p_{i+1}$, and b_i in the $img(p_{i+1})$.
(Lecturer calls this a *back and forth* method) Forth step: By lemma 4.3, we can extend p_i to $p_{i+\frac{1}{2}}$ such that $a_i \in \mathrm{dom}(p_{i+\frac{1}{2}})$;
Back step: By lemma 4.3 again applied to $(p_{i+\frac{1}{2}})^{-1}$ to include $b_i \in \mathrm{dom}(p_{i+1}^{-1})$ (i.e. in the range of p_{i+1}).
We claim that p_{i+1} extends p_i as required.
Let $\pi = \cup_{i<\omega} p_i$. Then (check) π is an isomorphism (i.e. order-preserving bijection). $\square$

Definition. (4.5)
An L-theory is *consistent* if it there is L-structure $\mathcal{M}$ s.t. $\mathcal{M} \vDash T$.
If T is a theory in L and ϕ is an L-sentence, then $T \vdash \phi$ (read as T *entails* ϕ, note that this has nothing to do with syntactic implication) if for all $\mathcal{M}$ such that $\mathcal{M} \vDash T$, we have $\mathcal{M} \vDash \phi$.
Finally, an L-theory T is *complete* if for all L-sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$ (see part II Logic and Set Theory).
For example, T_{dlo} is complete.