

Logic and Set Theory

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0 Miscellaneous

Some introductory speech

1 Propositional logic

Let P denote a set of *primitive proposition*, unless otherwise stated, $P = \{p_1, p_2, \dots\}$.

Definition. The *language* or *set of propositions* $L = L(P)$ is defined inductively by:

- (1) $p \in L \forall p \in P$;
- (2) $\perp \in L$, where \perp is read as 'false';
- (3) If $p, q \in L$, then $(p \implies q) \in L$. For example, $(p_1 \implies L)$, $((p_1 \implies p_2) \implies (p_1 \implies p_3))$.

Note that at this point, each proposition is only a finite string of symbols from the alphabet $(,), \implies, \perp, p_1, p_2, \dots$ and do not really mean anything (until we define so).

By *inductively define*, we mean more precisely that we set $L_1 = P \cup \{\perp\}$, and $L_{n+1} = L_n \cup \{(p \implies q) : p, q \in L_n\}$, and then put $L = L_1 \cup L_2 \cup \dots$

Each proposition is built up *uniquely* from 1) and 2) using 3). For example, $((p_1 \implies p_2) \implies (p_1 \implies p_3))$ came from $(p_1 \implies p_2)$ and $(p_1 \implies p_3)$. We often omit outer brackets or use different brackets for clarity.

Now we can define some useful things:

- $\neg p$ (not p), as an abbreviation for $p \implies \perp$;
- $p \vee q$ (p or q), as an abbreviation for $(\neg p) \implies q$;
- $p \wedge q$ (p and q), as an abbreviation for $(p \implies (\neg q))$.

These definitions 'make sense' in the way that we expect them to.

Definition. A *valuation* is a function $v : L \rightarrow \{0, 1\}$ s.t.

- (1) $v(\perp) = 0$; (2)

$$v(p \implies q) = \begin{cases} 0 & v(p) = 1, v(q) = 0 \\ 1 & \text{else} \end{cases} \quad \forall p, q \in L$$

Remark. On $\{0, 1\}$, we could define a constant \perp by $\perp = 0$, and an operation \implies by $a \implies b = 0$ if $a = 1, b = 0$ and 1 otherwise. Then a valuation is a function $L \rightarrow \{0, 1\}$ that preserves the structure $(\perp \text{ and } \implies)$, i.e. a homomorphism.

Proposition. (1) If v, v' are valuations with $v(p) = v'(p) \forall p \in P$, then $v = v'$ (on L).

(2) For any $w : P \rightarrow \{0, 1\}$, there exists a valuation v with $v(p) = w(p) \forall p \in P$. In short, a valuation is defined by its value on P , and any values will do.

Proof. (1) We have $v(p) = v'(p) \forall p \in L_1$. However, if $v(p) = v'(p)$ and $v(q) = v'(q)$ then $v(p \implies q) = v'(p \implies q)$, so $v = v'$ on L_2 . Continue inductively we have $v = v'$ on $L_n \forall n$.

(2) Set $v(p) = w(p) \forall p \in P$ and $v(\perp) = 0$: this defines v on L_1 . Having defined v on L_n , use the rules for valuation to inductively define v on L_{n+1} so we can extend v to L . \square

Definition. We say p is a *tautology*, written $\models p$, if $v(p) = 1 \forall$ valuations v .
Some examples:

(1) $p \implies (q \implies p)$: a true statement implies by anything. We can verify this by:

$v(p)$	$v(q)$	$v(q \implies p)$	$v(p \implies (q \implies p))$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

So we see that this is indeed a tautology;

(2) $(\neg\neg p) \implies p$, i.e. $((p \implies \perp) \implies \perp) \implies p$, called the "law of excluded middle";

(3) $[p \implies (q \implies r)] \implies [(p \implies q) \implies (p \implies r)]$.

Indeed, if not then we have some v with $v(p \implies (q \implies r)) = 1$, $v((p \implies q) \implies (p \implies r)) = 0$. So $v(p \implies q) = 1$, $v(p \implies r) = 0$. This happens when $v(p) = 1$, $v(r) = 0$, so also $v(q) = 1$. But then $v(q \implies r) = 0$, so $v(p \implies (q \implies r)) = 0$.

Definition. For $S \subset L$, $t \in L$, say S *entails* or *semantically implies* t , written $S \models t$ if $v(s) = 1 \forall s \in S \implies v(t) = 1$, each valuation v .

("Whenever all of S is true, t is true as well.")

For example, $\{p \implies q, q \implies r\} \models (p \implies r)$. To prove this, suppose not: so we have v with $v(p \implies q) = v(q \implies r) = 1$ but $v(p \implies r) = 0$. So $v(p) = 1$, $v(r) = 0$, so $v(q) = 0$, but then $v(p \implies q) = 0$.

If $v(t) = 1$ we say t is true in v or that v is a model of t .

For $S \subset L$, v is a model of S if $v(s) = 1 \forall s \in S$. So $S \models t$ says that every model of S is a model of t . For example, in fact $\models t$ is the same as $\emptyset \models t$.

2 Syntactic implication

For a notion of 'proof', we will need axioms and deduction rules. As axioms, we'll take:

1. $p \implies (q \implies p) \forall p, q \in L$;
2. $[p \implies (q \implies r)] \implies [(p \implies q) \implies (p \implies r)] \forall p, q, r \in L$;
3. $(\neg\neg p) \implies p \forall p \in L$.

Note: these are all tautologies. Sometimes we say they are 3 axiom-schemes, as all of these are infinite sets of axioms.

As deduction rules,