Representation Theory

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Contents

0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over $\mathbb{C}$. Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise:
• F is a field, usually $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or $\bar{F}_p$ (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
• V is a vector space over F, always finite dimensional;
$GL(V) = \{\theta : V \to V, \theta \text{ linear, invertible}\}$, i.e. $\det\theta \neq 0$.

Recall from Linear Algebra:
If $\dim_F V = n < \infty$, choose basis $e_1, ..., e_n$ over F, so we can identify it with F^n. Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \to A_\theta$ ($A_{\theta_1\theta_2} = A_{\theta_1}A_{\theta_2}$ and bijection). Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = XA_1X^{-1}$.

Recall that $\mathrm{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A.

(1.3) $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$, hence we can define $\mathrm{tr}(\theta) = \mathrm{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V in f.d. over $\mathbb{C}$, with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $EndV$ is the set of all ilnear maps $V \to V$, e.g. $End(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over $\mathbb{C}$, $\alpha \in End(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1}(X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a $\mathbb{C}$-vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:
(1.6) The symmetric group, $S_n = Sym(X)$ on the set $X = \{1, ..., n\}$ is the set of all permutations of X. $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m: $C_m = < x : x^m = 1 >$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in $\mathbb{C}$ (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of $\mathbb{C}$). We also have the group of rotations, centre O of regular $m-$gon in $\mathbb{R}^2$ (inside $GL_2(\mathbb{R})$).

(1.8) Dihedral groups D_{2m} of order $2m = < x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} >$. Think of this as the set of rotations and reflections preserving a regular m-gon.

(1.9) Quaternion group, $Q_8 =< x, y|x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} >$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \to X$ by $(g, x) \to g \cdot x$ for $g \in G$, $x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X, we obtain a homomorphism $\theta : G \to Sym(X)$, called the *permutation representation* of G.

Proof. For $g \in G$, the function $\theta_g : X \to X$ by $x \to gx$ is a permutation on X, with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G$, $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$ since $(g_1g_2)x = g_1(g_2x)$ for $x \in X$. $\square$

2 Basic Definitions

2.1 Representations

Let G be finite, F be a field, usually $\mathbb{C}$.

Definition. (2.1)
Let V be a f.d. vector space over F. A (linear, in some books) *representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \to GL(V)$$

Write ρ_g for the image $\rho_V(g)$; so for each $g \in G$, $\rho_g \in GL(V)$, and $\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}$, and $(\rho_g)^{-1} = \rho_{g^{-1}}$.
The *dimension* (or *degree*) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \lhd G$ (kernel is a normal subgroup), and $G/\ker \rho \cong \rho(G) \leq GL(V)$ (1st isomorphism theorem). We say ρ is *faithful* if $\ker \rho = 1$.

An alternative (and equivalent) approach is to observe that a representation of G on V is "the same as" a *linear action* of G:

Definition. (2.3)
G acts linearly on V if there exists a *linear action*

$$\begin{aligned} G \times V &\to V \\ (g, v) &\to gv \end{aligned}$$

By linear action we mean: (action) $(g_1 g_2)v = g_1(g_2 v)$, $1v = v \ \forall g_1, g_2 \in G, v \in V$, and (linear) $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv \ \forall g \in G, v_1, v_2 \in V, \lambda \in F$.
Now if G acts linearly on V, the map

$$\begin{aligned} G &\to GL(V) \\ g &\to \rho_g \end{aligned}$$

with $\rho_g : v \to gv$ is a representation of G. Conversely, given a representation $\rho : G \to GL(V)$, we have a linear action of G on V via $g \cdot v := \rho(g)v \ \forall v \in V, g \in G$.

(2.4) In (2.3) we also say that V is a G-space or that V is a G-module. In fact if we define the *group algebra* FG, or $F[G]$, to be $\{\sum \alpha_j g : \alpha_j \in F\}$ with natural addition and multiplication, then V is actually a FG-module (in the sense from GRM).

(2.5) R is a *matrix representation* of G of degree n if R is a homomorphism $G \to GL_n(F)$. Given representation $\rho : G \to GL(V)$ with $\dim_F V = n$, fix basis B; we get matrix representation

$$\begin{aligned} G &\to GL_n(F) \\ g &\to [\rho(g)]_B \end{aligned}$$

Conversely, given matrix representation $R : G \to GL_n(F)$, we get representation

$$\begin{aligned} \rho : G &\to GL(F^n) \\ g &\to \rho_g \end{aligned}$$

via $\rho_g(v) = R_g v$ where R_g is the matrix of g.

Example. (2.6)
Given any group G, take $V = F$ the 1-dimensional space, and

$$\begin{aligned} \rho : G &\to GL(F) \\ g &\to (id : F \to F) \end{aligned}$$

is known as the trivial representation of G. So $\deg \rho = 1$ ($\dim_F F = 1$).

Example. (2.7)
Let $G = C_4 =< x : x^4 = 1 >$. Let $n = 2$, and $F = \mathbb{C}$. Note that any $R : x \to X$ will determine $x^j \to X^j$ as it is a homomorphism, and also we need $X^4 = I$. So we can take X to be diagonal matrix – any such with diagonal entries a root to $x^4 = 1$, i.e. $\{\pm 1, \pm i\}$, or if X is not diagonal then it will be similar to a diagonal matrix by (1.4) ($X^4 = I$).

2.2 Equivalent representations

Definition. (2.8)
Fix G, F. Let V, V' be F-spaces, and $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ which are representations of G. The linear map $\phi : V \to V'$ is a G-homomorphism if

$$\phi\rho(g) = \rho'(g)\phi \forall g \in G (*)$$

We can understand this more by the following diagram:

$$\begin{tikzcd} V \arrow[r, "\rho_g"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ V' \arrow[r, "\rho_g'"] & V' \end{tikzcd}$$

We say ϕ *intertwines* ρ, ρ'. Write $Hom_G(V, V')$ for the F-space of all these. ϕ is a G-isomorphism if it is also bijective; if such ϕ exists, ρ, ρ' are isomorphic/equivalent representations. If ϕ is a G-isomorphism, we can write $(*)$ as $\rho' = \phi\rho\phi^{-1}$.

Lemma. (2.9)
The relation "being isomorphic" is an equivalent relation on the set of all representations of G (over F).

Remark. (2.10)
If ρ, ρ' are isomorphic representations, they have the same dimension.

The converse may be false: C_4 has four non-isomorphic 1-dimensional representations: if $\omega = e^{2\pi i/4}$ then they are $\rho_j(x^i) = \omega^{ij}$ $(0 \leq i \leq 3)$.

Remark. (2.11)
Given G, V over F of dimension n and $\rho : G \to GL(V)$. Fix basis B for V: we get a linear isomorphism

$$\begin{aligned} \phi : V &\to F^n \\ v &\to [v]_B \end{aligned}$$

and we get a representation $\rho' : G \to GL(F^n)$ isomorphic to ρ:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\varphi"] & V \arrow[d, "\varphi"] \\ F^n \arrow[r, "\rho'"'] & F^n \end{tikzcd}$$

(2.12) In terms of matrix representations, we have

$$\begin{aligned} R : G &\to GL_n(F), \\ R' : G &\to GL_n(F) \end{aligned}$$

are (G)-isomorphic or equivalent if there exists a nonsingular matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1}$ $\forall g \in G$.

In terms of linear G-actions, the actions of G on V,V' are G-isomorphic if there exists isomorphisms $\phi : V \to V'$ such that $g : \phi(v) = \phi(gv)$ $\forall v \in V, g \in G$.

2.3 Subrepresentations

Definition. (2.13)
Let $\rho : G \to GL(V)$ be a representation of G. We say $W \leq V$ is a G-subspace if it's a subspace and it is $\rho(G)$-invariant, i.e. $\rho_g(W) \leq W \forall g \in G$. Obviously $\{0\}$ and V are G-subspaces, however.
ρ is *irreducible/simple* representation if there are no proper G-subspaces.

Example. (2.14)
Any 1-dimensional representation of G is irreducible, but not conversely, e.g. D_8 has 2-dimensional $\mathbb{C}$-irreducible representation.

(2.15) In definition (2.13), if W is a G-subspace, then the corresponding map

$$\begin{aligned} G &\to GL(W) \\ g &\to \rho(g)|_W \end{aligned}$$

is a representation of G, a *subrepresentation* of ρ.

Lemma. (2.16)
In definition (2.13), given $\rho : G \to GL(V)$, if W is a G-subspace of V and if $B = \{v_1, ..., v_n\}$ is a basis containing basis $B_1 = \{v_1, ..., v_m\}$ of W $(0 < m < n)$ then the matrix of $\rho(g)$ w.r.t. B has block upper triangular form as the graph below, for

$$\begin{array}{cc} & \begin{array}{cc} m & \end{array} \\ m & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

each $g \in G$.

Example. (2.17)
(i) The irreducible representations of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1-dimensional and four of these are $x \to i, x \to -1, x \to -i, x \to 1$. In general, $C_m = \langle x : x^m = 1 \rangle$ has precisely m irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)* or see (4.4) below).
(ii) $G = D_6$: any irreducible C-representation has dimension ≤ 2.
Let $\rho : G \to GL(V)$ be irreducible G-representation. Let r, s be rotation and reflection in D_6 respectively. Let V be eigenvector of $\rho(r)$. So $\rho(r)v = \lambda v$

for some $\lambda \neq 0$. Let $W = span\{v, \rho(s)v\} \leq V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, both of which are in W; so W is G-invariant, i.e. a G-subspace. Since V is irreducible, $W = V$.

Definition. (2.18)
We say at $\rho : G \to GL(V)$ is *decomposable* if there are proper G-invariant subspaces U, W with $V = U \oplus W$. Say ρ is direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists, we say that ρ is *indecomposable.*

Lemma. (2.19)
Suppose $\rho : G \to GL(V)$ is decomposable with G-invariant decomposition $V = U \oplus W$. If B is a basis $\{\underbrace{u_1, ..., u_k}_{B_1}, \underbrace{w_1, ..., w_l}_{B_2}\}$ of V consisting of basis of U and basis of W, then w.r.t. B, $\rho(g)_B$ is a block diagonal matrix $\forall g \in G$ as

$$\rho(g)_B = \begin{pmatrix} [\rho_W(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

Definition. (2.20)
If $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$, the *direct sum* of ρ, ρ' is

$$\rho \oplus \rho' : G \to GL(V \oplus V')$$

where $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$, a *block diagonal action.* For matrix representations $R : G \to GL_n(F)$, $R' : G \to GL_{n'}(F)$, define $R \oplus R' : G \to GL_{n+n'}(F)$:

$$g \to \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Definition. (3.1)
A representation $\rho : G \to GL(V)$ is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

Remark. (3.2)
(1) The converse is false;
(2) See sheet 1 Q3: $\mathbb{C}$-representation of $\mathbb{Z}$ is not completely reducible and also representation of C_p over $\mathbb{F}_p$ is not c.r..

Fron now on, take G finite and $char\ F = 0$.

Theorem. (3.3)
Every f.d. representation V of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus ... \oplus V_r$$

is a direct sum of representations, each V_i irreducible.

It is enough to prove:

Theorem. (3.4 Maschke's theorem, 1899)
Let G be finite, $\rho : G \to GL(V)$ a f.d. representation, $char\ F = 0$. If W is a G-subspace of V, then there exists a G-subspace U of V s.t. $V = W \oplus U$, a direct sum of G-subspaces.

Proof. (1)
Let W' be any *vector subspace* complement of W in V, i.e. $V = W \oplus W'$ as vector spaces, and $W \cap W' = 0$. Let $q : V \to W$ be th projection of V onto W along W' ($\ker q = W'$), i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\bar{q} : v \to \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of q over G. Note that in order for $\frac{1}{|G|}$ to exists, we need $char\ F = 0$. It still works if $char\ F \nmid |G|$.
Claim (1): $\bar{q} : V \to W$: For $v \in V$, $g(q^{-1}v) \in W$ and $gW \leq W$;
Claim (2): $\bar{q}(w) = w$ for $w \in W$:

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum g(g^{-1}w) = \frac{1}{|G|} \sum w = w$$

So these two claims imply that $\bar{q}$ projects V onto W.

Claim (3) If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ $(v \in V)$:

$$\begin{aligned}
h\bar{q}(v) &= h\frac{1}{|G|}\sum_g g \cdot q(g^{-1}v) \\
&= \frac{1}{|G|}\sum_g hgq(g^{-1}v) \\
&= \frac{1}{|G|}\sum (hg)q((hg)^{-1}hv) \\
&= \frac{1}{|G|}\sum_g gq(g^{-1}(hv)) \\
&= \bar{q}(hv \\
&= \bar{q}(hv))
\end{aligned}$$

We'll then show that the kernel of this map is G-invariant, so this gives a G-summand on Thursday.

Let's now show $\ker \bar{q}$ is G-invariant. If $v \in \ker \bar{q}$, then $h\bar{q}(v) = 0 = \bar{q}(hv)$, so $hv \in \ker \bar{q}$. Thus $V = im\bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$ is a G-subspace decomposition.

We can deduce (3.3) from (3.4) by induction on $\dim V$. If $\dim V = 0$ or V is irreducible, then result is clear. Otherwise, V has non-trivial G-invariant subspace, W. Then by (3.4), there exists G-invariant complement U s.t. $V = U \oplus W$ as representations of G. But $\dim U, \dim W < \dim V$. So by induction they can be broken up into direct sum of irreducible subrepresentations. □

The second proof uses inner products, hence we need to take $F = \mathbb{C}$ and can be generalised to compact groups in section 15.
Recall, for V a $\mathbb{C}$-space, $\langle , \rangle$ is a *Hermitian inner product* if
(a) $\langle w, v\rangle = \overline{\langle v, w\rangle}$ $\forall v, w$ (Hermitian);
(b) linear in RHS (sesquilinear);
(c) $\langle v, v\rangle > 0$ iff $v \neq 0$ (positvie definite).

Additionally, $\langle , \rangle$ is *G-invariant* if
(d) $\langle gv, gw\rangle = \langle v, w\rangle$ $\forall v, w \in V, g \in G$.

Note if W is G-invariant subspace of V, with G-invariant inner product, then $W^\perp$ is also G-invariant, and $V \oplus W^\perp$. For all $v \in W^\perp$, $g \in G$, we have to show that $gv \in W^\perp$. But $v \in W^\perp \iff \langle v, w\rangle = 0 \forall w \in W$. Thus by (d), $\langle gv, gw\rangle = 0$ $\forall g \in G \forall w \in W$. Hence $\langle gv, w'\rangle = 0$ $\forall w' \in W$. Since we can choose $w = g^{-1}w' \in W$ by G-invariance of W. Thus $gv \in W^\perp$ since g was arbitrary.

Hence if there is a G-invariant inner product on any G-space, we get another proof of Maschke's theorem:

(3.4*) (Weyl's unitary trick)
Let ρ be a complex representation of the finite group G on the $\mathbb{C}$-space V. Then there is a G-invariant Hermitian inner product on V.

Remark. Recall the *unitary group* $U(V)$ on V: $\{f \in GL(V) : (fu, fv) = (u, v) \forall u, v \in V\} = \{A \in GL_n(\mathbb{C}) : A\bar{A}^T = I\} (= U(n))$ by choosing orthonormal

basis.
Sheet 1 Q.12: any finite subgroup of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Proof. (2)
There exist an inner product on V: take basis $e_1, ..., e_n$ and define $(e_i, e_j) = \delta_{ij}$, extended sesquilinearly. Now

$$\langle v, w\rangle := \frac{1}{|G|}\sum_{g\in G}(gv, gw)$$

we claim that $\langle , \rangle$ is sesquilinear, positive definite and G-invariant: if $h \in G$, then

$$\begin{aligned}\langle hv, hw\rangle = \frac{1}{|G|}\sum_{g\in G}((gh)v, (gh)w) \\ &= \frac{1}{|G|}\sum_{g'\in G}(g'v, g'w) \\ &= \langle v, w\rangle\end{aligned}$$

for all $v, w \in V$. □

Definition. (3.5, the regular representation)
Recall *group algebra* of G is F-space $FG = span\{e_g : g \in G\}$. There is a linear G-action

$$h \in G, h\sum_{g\in G} a_g e_g = \sum_{g\in G} a_g e_{hg} (= \sum_{g'\in G} a_{h^{-1}g'} e_{g'})$$

ρ_{reg} is the corresponding representation, the *regular representation* of G. This is faithful of $\dim |G|$. FG is the *regular module*.

Proposition. Let ρ be an irreducible representation of G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg}.

Proof. Take $\rho : G \in GL(V)$ irreducible and let $0 \neq v \in V$. Let $\theta : FG \to V$ by $\sum a_g e_g \to \sum a_g gv$. Check this is a G-homomorphism. Now V is irreducible so $im\theta = V$ (since $im\theta$ is a G-subspace).

Also $\ker\theta$ is G-subspace of FG. Let W be G-complement of $\ker\theta$ in FG (Maschke), so that $W < FG$ is G-subspace and $FG = \ker\theta \oplus W$. Thus $W \cong FG/\ker\theta \cong (G - isomorphism) im\theta \cong V$. □

More generally,

Definition. (3.7)
Let F be a field. Let G act on set X. Let $FX = span\{e_x : x \in X\}$ with G-action

$$g(\sum a_x e_x) = \sum a_x e_{gx}$$

The representation $G \to GL(V)$ where $V = FX$ is the corresponding *permutation representation*. See section 7.

4 Schur's lemma

It's really unfair that such an important result is only remembered by a lemma, so we shall call it a theorem.

Theorem. (4.1, Schur)
(a) Assume V, W are irreducible G-spaces over field F. Then any G-homomorphism $\theta : V \to W$ is either 0 or an isomorphism.
(b) Assume F is *algebraically closed*, and let V be an irreducible G-space. Then any G-endomorphism $V \to V$ is a scalar multiple of the identity map ι_V.

Proof. (a) Let $\theta : V \to W$ be a G-homomorphism. Then ker θ is G subspace of V and, since V is irreducible, we get $\ker \theta = 0$ or $\ker \theta = V$.
And $im\theta$ is G-subspace of W, so as W is irreducible, $im\theta$ is either 0 or W. Hence, either $\theta = 0$ or θ is injective and surjective, hence isomorphism.
(b) Since F is algebraically closed, θ has an eigenvalue, λ. Then $\theta - \lambda\iota$ is singular G-endomorphism of V, but it cannot be an isomorphism, so it is 0 (by (a)). So $\theta = \lambda\iota_V$. □

Recall from (2.8), the F-space $Hom_G(V, W)$ of all G-homomorphisms $V \to W$. Write $End_G(V)$ for the G-endomorphisms of V.

Corollary. (4.2)
If V, W are irreducible complex G-spaces, then

$$\dim_{\mathbb{C}} Hom_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G- \text{ isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V, W are not G-isomorphic then the only G-homomorphism $V \to W$ is 0 by (4.1). Assume $v \cong_G W$ and $\theta_1, \theta - 2 \in Hom_G(V, W)$, both non-zero. Then θ_2 is invertible by (4.1), and $\theta_2^{-1}\theta_1 \in End_G(V)$, and non-zero, so $\theta_2^{-1}\theta_1 = \lambda\iota_V$ for some $\lambda \in \mathbb{C}$. Hence $\theta_1 = \lambda\theta_2$. □

Corollary. (4.3)
If finite group G has a faithful complex irreducible representation, then $Z(G)$, the centre of the group, is cyclic.
Note that the converse is false (Sheet 1, Q10).

Proof. Let $\rho : G \to GL(V)$ be faithful irreducible complex representation. Let $z \in Z(G)$, so $zg = gz\ \forall g \in G$, hence the map $\phi_z : v \to z(v)$ $(v \in V)$ is G-endomorphism of V, hence is multiplication by scalar μ_z, say.
By Schur's lemma, $z(v) = \mu_z v\ \forall v$. Then the map

$$\begin{aligned} Z(G) &\to \mathbb{C}^* \text{ (multiplicative group)} \\ z &\to \mu_z \end{aligned}$$

is a representation of Z and is faithful, since ρ is. Thus $Z(G)$ is isomorphic to some finite subgroup of $\mathbb{C}^*$, so is cyclic. □

Let's now consider representation of finite abelian groups.

Corollary. (4.4)
The irreducible $\mathbb{C}$-representations of a finite abelian group are all 1-dimensional.

Proof. Either: use (1.4)* to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$, and if V is irreducible, then $V = \langle v \rangle$.
Or: Let V be an irreducible $\mathbb{C}$-representation. For $g \in G$, the map

$$\begin{aligned} \theta_g : V &\to v \\ v &\to gv \end{aligned}$$

is a G-endomorphism of V, and as V irreducible, $\theta_g = \lambda_g \iota_V$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any $g \in G$ (so $\langle v \rangle$ is a G-subspace of V). Thus as $0 \neq V$ is irreducible, $V = \langle v \rangle$, which is 1-dimensional. $\square$

Remark. Schur's lemma fails over non-algebraically closed field, in particular, over $\mathbb{R}$. For example, let's consider the cyclic group C_3. It has 2 irreducible $\mathbb{R}$-representations, one of dimension 1 (maps everything to 1) and one of dimension 2(imo consider $\mathbb{C}$ as a dimension 2 space over $\mathbb{R}$, then map the generator to the 3rd root of unity?) (so 'contradicting' with Schur's lemma via the corollary above).
Recall that every finite abelian group G is isomorphic to a product of cyclic groups (see GRM). For example, $C_6 = C_2 \times C_3$. In fact, it can be written as a product of C_{p^α} for various primes p and $\alpha \geq 1$, and the factors are uniquely determined up to reordering.

Proposition. (4.5)
The finite abelian group $G = C_{n_1} \times ... \times C_{n_r}$ has precisely $|G|$ irreducible $\mathbb{C}$-representations, as described below:

Proof. Write $G = \langle x_1 \rangle \times ... \langle x_r \rangle$ where $|x_j| = n_j$. Suppose ρ is irreducible, so by (4.4), it's 1-dimensional: $\rho : G \to \mathbb{C}^*$.
Let $\rho(1, ..., x_j, ..., 1)$ (all 1 apart from the j^{th} entry) be λ_j. Then $\lambda_j^{n_j} = 1$, so λ_j is a n_j-th root of unity. Now, the values $(\lambda_1, ..., \lambda_r)$ determine ρ:

$$\rho(x_1^{j_1}, ..., x_r^{j_r}) = \lambda_1^{j_1} ... \lambda_r^{j_r}$$

thus $\rho \leftrightarrow (\lambda_1, ..., \lambda_r)$ with $\lambda_j^{n_j} = 1 \ \forall j$; we have $n_1 ... n_r$ such r-tuples, each giving 1-dimensional representation. $\square$

Example. (4.6)
Consider $G = C_4 = \langle x \rangle$. We could have $\rho_1(x) = 1, \rho_2(x) = i, \rho_3(x) = -1, \rho_4(x) = -i$.

Warning: There is no "natural" 1-1 correspondence between the elements of G and the representations of G (G-finite abelian). If you choose an isomorphism $G \cong C_{a_1} \times ... \times C_{a_r}$, then we can identify the two sets (elements of groups and representations of G), but it depends on the choice of isomorphism.

Isotypical decomposition:

Recall any diagonalisable endomorphism $\alpha : V \to V$ gives eigenspace decomposition of $V \cong \oplus_\lambda V(\lambda)$, where $V(\lambda) = \{v : \alpha v = \lambda v\}$. This is *caconical* (one of the three useless words: *arbitrary*(anything), *canonical*(only one choice), *uniform*(you can choose, but it doesn't really matter)), in the sense that it depends on α alone (and nothing else).
There is no canonical eigenbasis of V: must choose basis in each $V(\lambda)$.

We know that in *char* 0 every representation V decomposes as $\oplus n_i V_i$, V_i irreducible, $n_i \geq 0$. How unique is this?

We have this wishlist (4.7):
(a) Uniqueness: for each V there is only one way to decompose V as above. However, this doesn't work obviously.
(b) Isotypes: for each V, there exists a unique collection of subrepresentations $U_1, ..., U_k$ s.t. $V = \oplus U_i$ and, if $V_i \subseteq U_i$ and $V_j' \subseteq U_j$ are irreducible subrepresentations, then $V_i \cong V_j'$ iff $i = j$.
(c) Uniqueness of factors: If $\oplus_{i=1}^k V_i \cong \oplus_{i=1}^k V_i'$ with V_i, V_i' irreducible, then $k = k'$, and $\exists \pi \in S_k$ such that $V'_{\pi(i)} \cong V_i$ (Krull-Schimdt theorem). For (b),(c) see Teleman section 5.

Lemma. (4.8)
Let V, V_1, V_2 be $G-$spaces over F.
(i) $Hom_G(V, V_1 \oplus V_2) \cong Hom_G(V, V_1) \oplus Hom_G(V, V_2)$;
(ii) $Hom_G(V_1 \oplus V_2, V) \cong Hom_G(V_1, V) \oplus Hom_G(V_2, V)$;

Proof. (i) Let $\pi_i : V_1 \oplus V_2 \to V_i$ be G-linear projections onto V_i, with kernel V_{3-i} $(i = 1, 2)$.
Consider

$$Hom_G(V, V_1 \oplus V_2) \to Hom_G(V, V_1) \oplus Hom_G(V, V_2)$$
$$\phi \to (\pi_1 \phi, \pi_2 \phi)$$

This map has inverse $(\psi_1, \psi_2) \to \psi_1 + \psi_2)$. Check details.
(ii) The map $\phi \to (\phi|_{V_1}, \phi|_{V_2})$ has inverse $(\psi_1, \psi_2) \to \psi_1 \pi_1 + \psi_2 \pi_2$. □

Lemma. Let F be algebraically closed, $V = \oplus_1^n V_i$ a decomposition of G-space into irreducible summands. Then, for each irreducible representation S of G,

$$\#\{j : V_j \cong S\} = \dim Hom_G(S, V)$$

where # means 'number of times'. This is called the *multiplicity* of S in V.

Proof. Indunction on n. $n = 0, 1$ are trivial.
If $n > 1$, $V = \oplus_1^{n-1} V_i \oplus V_n$. By (4.8) we have

$$\dim Hom_G(S, \oplus_1^{n-1} V_i \oplus V_n) = \dim Hom(S, \oplus_1^{n-1} V_i) + \underbrace{\dim Hom_G(S, V_n)}_{\text{Schur's lemma}}$$

□

Definition. (4.10)
A decomposition of V as $\oplus W_j$ where each $W_j \cong n_j$ copies of irreducible representations S_j (each non-isomorphic for each j) is the *canonical decomposition* or the decomposition into *isotypical components* W_j. For F algebraically closed, $n_j = \dim Hom_G(S_j, V)$.

5 Character theory

We want to attach invariants to representation ρ of a finite group G on V. Matrix coefficients of $\rho(g)$ are basis dependent, so not true invariants.
Let's take $F = \mathbb{C}$, G finite, $\rho = \rho_V : G \to GL(V)$ be a representation of G.

Definition. (5.1)
The *character* $\chi_\rho = \chi_V = \chi$ is defined as $\chi(g) = \operatorname{tr}\rho(g) = \operatorname{tr} R(g)$ where $R(g)$ is any matrix representation of $\rho(g)$ w.r.t. any basis.
The degree of χ_V is $\dim_{\mathbb{C}} V$.
Thus χ is a function $G \to \mathbb{C}$. χ is *linear* (not a universal name) if $\dim V = 1$, in which case χ is a homomorphism $G \to \mathbb{C}^*$ $(= GL_1(\mathbb{C}))$.
χ is irreducible if ρ is; χ is faithful if ρ is; and, χ is trivial, or principal, if ρ is the trivial representation (2.6). We write $\chi = 1_G$ in that case.
χ is a complete invariant in the sense that it determines ρ up to isomorphism – see (5.7).

Theorem. (5.2, first properties)
(i) $\chi_V(1) = \dim_{\mathbb{C}} V$; (clear: $\operatorname{tr} I_n = n$)
(ii) χ_V is a *class function*, via it is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$$

Thus χ_V is constant on conjugacy classes.
(iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, the complex conjugate;
(iv) For two representations V, W, $\chi_{V \oplus W} = \chi_V + \chi_W$.

Proof. (ii) $\chi(hgh^{-1}) = \operatorname{tr}(R_h R_g R_h^{-1}) = \operatorname{tr}(R_g) = \chi(g)$.
(iii) Recall $g \in G$ has finite order, so we can assume $\rho(g)$ is represented by a diagonal matrix $Diag(\lambda_1, ..., \lambda_n)$. Then $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by the matrix $Diag(\lambda_1^{-1}, ... \lambda_n^{-1})$, and hence $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda_i} = \overline{\chi(g)}$ (since λ_i's are roots of unity – since $g^k = 1$ for some k!(I mean an exclamation mark here to express surprise) and by homomorphism we know that).
(iv) Suppose $V = V_1 \oplus V_2$, $\rho_i : G \to GL(V_i)$, $\rho : G \to GL(V)$. Take basis $B = B_1 \cup B_2$ of V w.r.t B, $\rho(g)$ has matrix of block form $Diag([\rho_1(g)]_{B_1}, [\rho_2(g)]_{B_2})$ and as $\chi(g)$ is the trace of the above matrix, it is equal ot $\operatorname{tr}\rho_1(g) + \operatorname{tr}\rho_2(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$. □

Remark. We see later that χ_1, χ_2 character of G implies that $\chi_1\chi_2$ is also a character of G: uses tensor products, see (9.6).

Lemma. (5.3)
Let $\rho : G \to GL(V)$ be a copmlex representation *affording* the character χ (i.e. χ is a character of ρ). Then $|\chi(g)| \le \chi(1)$, with equality iff $\rho(g) = \lambda_I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ iff $g \in \ker\rho$.

Proof. Fix g. W.r.t. basis of V of eigenvalues $\rho(g)$, the matrix of $\rho(g)$ is $Diag(\lambda_1, ..., \lambda_n)$. Hence $|\chi(g)| = |\sum \lambda_j| \le \sum |\lambda_j| = \sum 1 = \dim V = \chi(1)$. Equality holds iff all λ_j are equal (to λ, say).
If $\chi(g) = \chi(1)$, then $\rho(g) = \lambda\iota$ has $\chi(g) = \lambda\chi(1)$. □

Lemma. (5.4)
(a) If χ is a complex irreducible character of G, so is $\bar{\chi}$;
(b) Under the same assumption, so is $\varepsilon\chi$ for any linear character ε of G.

Proof. If $R : G \to GL_n(\mathbb{C})$ is a complex irreducible representation then so is $\bar{R} : G \to GL_n(\mathbb{C})$ by $g \to \bar{R}(g)$. Similarly for $R' : g \to \varepsilon(g)R(g)$ for $g \in G$. Check the details. $\square$

Definition. (5.5)
$\mathcal{C}(G) = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g) \forall h, g \in G\}$, the $\mathbb{C}$-space of class functions (we call it a space since $f_1 + f_2 : g \to f_1(g) + f_2(g)$, $\lambda f : g \to \lambda f(g)$ are still in $\mathcal{C}(G)$), so this is a vector space.
Let $k = k(G)$ be the number of ccls of G. List the ccls $\mathcal{C}_1, ..., \mathcal{C}_k$. Conventionally we choose $g_1 = 1, g_2, ..., g_k$, representatives of the ccls (hence $\mathcal{C}_1 = \{1\}$). Note that $\dim_{\mathbb{C}} \mathcal{C}(G) = k$ (the characteristic functions δ_j of each ccl which maps any element in the ccl to 1 and others to 0 form a basis).
We define Hermitian inner product on $\mathcal{C}(G)$:

$$\begin{aligned}\langle f, f' \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{f(G)} f'(g) \\ &= \frac{1}{|G|} \sum_{j=1}^{k} |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) \\ &= \sum_{j=1}^{k} \frac{1}{|C_G(g_j)} \overline{f(g_j)} f'(g_j)\end{aligned}$$

using $|\mathcal{C}_x| = |G : C_g(x)|$, where $\mathcal{C}_x$ is the ccl of x, $C_G(x)$ is the centraliser of x. For characters

$$\langle \chi, \chi' \rangle = \sum \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

is a real symmetric form (in fact, $\langle \chi, \chi' \rangle \in \mathbb{Z}$ – see later).

Theorem. (5.6)
The $\mathbb{C}$-irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$. Moreover,
(a) If $\rho : G \to GL(V), \rho' : G \to GL(V')$ are irreducible representations of G affording characters χ, χ' respecitvely, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho, \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

we call this 'row orthogonality'.
(b) Each class function of G can be expressed as a linear combination of G.
This will be proved later in section 6.

Corollary. (5.7)
Complex representations of *finite* groups are characterised by their characters. We emphasise on finiteness here: for example, $G = \mathbb{Z}$, consider $1 \to I_2$, $1 \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are non-isomorphic but have same character.

Proof. Let $\rho : G \to GL(V)$ be representation affording χ (G finite over $\mathbb{C}$). (3.3) says

$$\rho = m_1\rho_1 \oplus ... \oplus m_k\rho_k$$

where $\rho_1, ..., \rho_k$ are irreducible, and $m_j \geq 0$. Then $m_j = \langle \chi, \chi_j \rangle$ where χ_j is afforded by ρ_j: we have $\chi = m_1\chi_1 + ... + m_k\chi_k$, but the ρ_i's are orthonormal. □

Corollary. (5.8, irreduciblility criterion)
If ρ is $\mathbb{C}$-representation of G affording χ, then ρ irreducible $\iff$ $\langle \chi, \chi \rangle = 1$.

Proof. Forward is just the statement of orthonormality. Conversely, assume $bra\chi, \chi\rangle = 1$. Now take a (complete) decomposition of ρ and take characters of it we get $\chi = \sum m_j\chi_j$ with χ_j irreducible and $m_j \geq 0$. Then $\sum m_j^2 = 1$. Hence $\chi = \chi_j$ for some j (since the m_j's are obviously integers), so is irreducible. □

Corollary. (5.9)
If the irreducible $\mathbb{C}$-representations of G are $\rho_1, ..., \rho_k$ have dimensions $n_1, ..., n_k$, then

$$|G| = \sum_{i=1}^{k} n_i^2$$

Proof. Recall from (3.5), $\rho_{reg}; G \to GL(\mathbb{C}G)$, the regular representation G of dimension $|G|$ (where $\mathbb{C}G$ is just a G-space with basis $\{e_g : g \in G\}$ and any $h \in G$ permutes the e_g: $e_g \to e_{hg}$).
Let π_{reg} be its charcter, the *regular character* of G.
Claim 1: $\pi_{reg}(1) = |G|$, $\pi_{reg}(h) = 0$ if $h \neq 1$.
This is clear: take $h \in G, h \neq 1$, then we always have 0 down the diagonal since h permutes things around, so the trace is 0; if $h = 1$ then we have an identity matrix so trace is $\dim \rho = |G|$.
Claim 2: $\pi_{reg} = \sum n_j\chi_j$ with $n_j = \chi_j(1)$.
This is because

$$\begin{aligned} n_j &= \langle \pi_{reg}, \chi_j \rangle \\ &= \frac{1}{|G|}\sum_{g \in G} \overline{\pi_{reg}(g)}\chi_j(g) \\ &= \frac{1}{|G|} \cdot |G|\chi_j(1) = \chi_j(1) \end{aligned}$$

(all the other $\pi_{reg}(g)$ are zero by claim 1).
Our corollary is then obvious by just calculating $|G| = \pi_{reg}(1)$. □

Corollary. (5.10)
Number of irreducible characters of G (up to equivalence) $= k$ (=number of ccls).

Corollary. (5.11)
Elements $g_1, g_2 \in G$ are conjugate iff $\chi(g_1) = \chi(g_2)$ for all irreducible characters of G.

Proof. Forward: characters are class functions;
Backward: Let δ be the characteristic function of the class of g_1. In particular, δ is a class function, so can be written as a linear combination of the irreducible characters of G. Hence $\delta(g_2) = \delta(g_1) = 1$, so $g_2 \in \mathcal{C}_G(g_1)$. □

In the end let's introduce a good friend which will be around for the next few lectures:
Recall from (5.5), the inner product on $\mathcal{C}(G)$ and the real symmetric form $\langle , \rangle$ on characters:

Definition. The *character table* of G is the $k \times k$ matrix (where k is the number of ccls) $X = [\chi_i(g_j)]$, the i^{th} character on the j^{th} class, where we let $\chi_1 = 1_G, \chi_2, ..., \chi_k$ are the irreducible characters of G, and $\mathcal{C}_1 = \{1\}, ..., \mathcal{C}_k$ are the ccls with $g_j \in \mathcal{C}_j$ (as we defined in 5.5).
So the $(i,j)^{th}$ entry of X is just $\chi_i(g_j)$.

Example. (5.13)
(a) $C_3 = \langle x : x^3 = 1 \rangle$. The character table is

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = e^{2\pi i/3}$.
(b) $G = D_6 \cong S_3 = \langle r, s : r^3 = s^2 = 1, sr^{-1} = r^{-1} \rangle$.
ccls of G: $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r, r^{-1}$, $\mathcal{C}_3 = \{s, sr, sr^2\}$. We have 3 irreducible representations over $\mathbb{C}$: 1_G (trivial); $\mathcal{S}$ (sign): $x \to 1$ for x even, $x \to -1$ for x odd; and W (2-dimensional): sr^i acts by matrix with eigenvalues ± 1; r^k acts by the matrix

$$\begin{matrix} \cos 2k\pi/3 & -\sin 2k\pi/3 \\ \sin 2k\pi/3 & \cos 2k\pi/3 \end{matrix}$$

so $\chi_w(sr^i) = 0 \ \forall j$, $\chi_w(r^k) = 2\cos 2k\pi/3 = -1 \ \forall k$. So the charactable is:

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
1_G	1	1	1
χ_s	1	-1	1
χ_w	2	0	-1

6 Proofs and orthogonality

We want to prove(5.6): irreducible characters form orthonormal basis for the space of $\mathbb{C}$-class functions.

Proof. (of 5.6 (a))
Fix bases of V and V'. Write $R(g)$, $R'(g)$ for matrices of $\rho(g), \rho'(g)$ w.r.t. these bases, respectively. Then

$$\begin{aligned}\langle \chi', \chi \rangle &= \frac{1}{|G|}\chi'(g^{-1})\chi(g) \\ &= \frac{1}{|G|}\sum_{g\in G, i,j \ s.t. 1\le i\le n', 1\le j\le n} R'(g^{-1})_{ii}R(g)_{jj}\end{aligned}$$

the trick is to define something that annhilates almost the whole thing. Let $\phi : V \to V'$ be linear and define

$$\begin{aligned}\tilde{\phi} : V \to & \quad V' \\ v \to & \quad \frac{1}{|G|}\sum_{g\in G}\rho'(g^{-1})\phi\rho(g)v\end{aligned}$$

We claim that this is a G-homomorphism: if $h \in G$, let's calculate

$$\begin{aligned}\rho'(h^{-1})\tilde{\phi}\rho(h)(v) &= \frac{1}{|G|}\sum_{g\in G}\rho'(gh)^{-1}\phi\rho(gh)(v) \\ &= \frac{1}{|G|}\sum_{g'\in G}\rho'(g'^{-1})\phi\rho(g')(v) \\ &= \tilde{\phi}(v)\end{aligned}$$

(when g runs through G, gh runs through G as well). So (2.8) is satisfied, i.e. ϕ is a G-homomorphism.

Case 1: ρ, ρ' are not isomorphic. Schur's lemma says $\tilde{\phi} = 0$ for any given linear $\phi : V \to V'$. Take $\phi - \varepsilon_{\alpha\beta}$, having matrix $E_{\alpha\beta}$ (w.r.t our basis). This is 0 everywhere except 1 in the (α, β)-position. Then $\tilde{\varepsilon_{\alpha\beta}} = 0$. So $\frac{1}{|G|}\sum_{g\in G}(R'(g^{-1})E_{\alpha\beta}R(g))_{ij} = 0$. So $\frac{1}{|G|}\sum R'(G^{-1})_{i\alpha}R(g)_{\beta j} = 0 \ \forall i, j$, with $\alpha = i, \beta = j$. Now $\frac{1}{|G|}\sum_{g\in G} R'(g^{-1})_{ii}R(g)_{jj} = 0$ sum over i, j. Then $\langle \chi', \chi\rangle = 0$.
Case 2: ρ, ρ' isomorphic. So $\chi = \chi'$; take $V = V'$, $\rho = \rho'$. If $\phi : V \to V$ is linear endomorphism, we claim $\operatorname{tr}\phi = \operatorname{tr}\tilde{\phi}$:

$$\operatorname{tr}\tilde{\phi} = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho(g)^{-1}\phi\rho(g)) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}\phi = \operatorname{tr}\phi$$

By Schur's lemma, $\tilde{\phi} = \lambda\iota_V$ for some $\lambda \in \mathbb{C}$ (depending on ϕ). Then $\lambda = \frac{1}{n}\operatorname{tr}\phi$. Let $\phi = \varepsilon_{\alpha\beta}$. So $\operatorname{tr}\phi = \delta_{\alpha\beta}$. Hence $\tilde{\varepsilon_{\alpha\beta}} = \frac{1}{n}\delta_{\alpha\beta}\iota_v = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})\varepsilon_{\alpha\beta}\rho(g)$. In terms of matrices, take (i, j)-entry: $\frac{1}{|G|}\sum_j R(g^{-1})_{i\alpha}R(g)_{\beta j} = \frac{1}{n}\delta_{\alpha\beta}\delta_{ij} \ \forall i, j$. Put $\alpha = i, \beta = j$ to get $\frac{1}{|G|}\sum_g R(g^{-1})_{ii}R(g)_{jj} = \frac{1}{n}\delta_{ij}$. Finally sum over i, j to get $\langle\chi, \chi\rangle = 1$. $\square$

Before proving (b), let's prove column orthogonality:

Theorem. (6.1, column orthogonality relations)

$$\sum_{i=1}^{k} \overline{\chi_i(g_j)}\chi_i(g_l) = \delta_{jl}|C_G(g_j)|$$

having an easy corollary

Corollary. (6.2)
$|G| = \sum_{i=1}^{k} \chi_i^2(1)$.

Proof. (of (6.1))
$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum \overline{\chi_i(g_l)}\chi_j(g_l)/|C_G(g_l)|$. Consider the character table $X = (\chi_i(g_j))$. Then $\bar{X}D^{-1}X^T = I_{k\times k}$ where $D = Diag(|C_G(g_1)|, ..., |C_G(g_k)|)$.
Since X is quare, it follows that $d6{-}1\bar{X}^T$ is the inverse of X, so $\bar{X}^T X = D$. □

Proof. (of (5.6(b)))
The χ_i generate $\mathcal{C}_G$. Let all the irreducible characters $\chi_1, ..., \chi_l$ of G: claim these generate $\mathcal{C}_G$, the $\mathbb{C}$-space of class functions on G. It's enough to show that the orthogonal complement to $span\{\chi_1, ..., \chi_l\}$ in $\mathcal{C}_G$ is $\{0\}$. To see this, assume $f \in \mathcal{C}_G$ with $\langle f, \chi_j \rangle = 0 \forall j$. Let $\rho : G \to GL(V)$ be irreducible representation affording $\chi \in \{\chi_1, ..., \chi_l\}$. Then $\langle f, \chi \rangle = 0$.
Consider

$$\frac{1}{|G|}\sum_G \overline{f(g)}\rho(g) : V \to V$$

This is a G-homomorphism, so as ρ is irreducible, it must be λ_ι for some $\lambda \in \mathbb{C}$.
Now

$$\begin{aligned} n\lambda &= \operatorname{tr} \frac{1}{|G|}\sum_g \overline{f(g)}\rho(g) \\ &= \frac{1}{|G|}\sum_g \overline{f(g)}\chi(g) = 0 = \langle f, \chi \rangle \end{aligned}$$

So $\lambda = 0$. Hence $\sum \overline{f(g)}\rho(g) = 0$, the zero endomorphism on V for all representations ρ (complete reducibility).

Take $\rho = \rho_{reg}$ where $\rho_{reg}(g) : e_1 \to e_g$ $(g \in G)$. So

$$\sum_g \overline{f(g)}\rho_{reg}(g) : e_1 \to \sum_g \overline{f(g)}e_g$$

So it follows $\sum_g \overline{f(g)}e_g = 0$. So $\overline{f(g)} = 0 \forall g \in G$, so $f \equiv 0$. □

Variuous corollaries now follow:
- The number of irreducible representations of G = number of ccls; (5.10)
- Column orthogonality (6.1);
- $|G| = \sum n_i^2$ (6.2);
- $g_1 \tilde{G} g_2 \iff \chi(g_1) = \chi(g_2)$ for all irreducible χ (5.11);
- If $g \in G$, $g \tilde{G} g^{-1} \iff \chi(g) \in \mathbb{R}$ for all irreducible χ.

7 Permutation representations

Preview was given in (3.7). Recall: • G finite group acting on finite set $X = \{x_1, ..., x_n\}$;
• $\mathbb{C}X = \mathbb{C}$-space, with basis $\{e_{x_1}, ..., e_{x_n}\}$ of dimension $|X|$, so is $\{\sum_j a_j e_{x_j} : a_j \in \mathbb{C}\}$;
• corresponding permutation representation $\rho_X : G \to GL(\mathbb{C}X)$ by $g \to \rho(g)$, where $\rho(g)$ sends $e_{x_j} \to e_{gx_j}$, extending linearly.
• ρ_X is the *permutation representation* corresponding to the action of G on X.
• matrices representing $\rho_X(g)$ w.r.t. basis $\{e_x\}_{x\in X}$ are permutation matrices: 0 except for one 1 in each row and column, and $(\rho(g))_{ij} = 1$ iff $gx_j = x_i$. Consider its character:

(7.1) Permutation character, π_X, is

$$\pi_X(g) = |Fix_X(g)| = |\{x \in X : gx = x\}|.$$

(7.2) ρ_X always contains 1_G: $span\{e_{x_1} + ... + e_{x_n}\}$ is a trivial G-subspace of $\mathbb{C}X$ with G-invariant complement $span\{\sum a_x e_x : \sum a_x = 0\}$.

Lemma. (7.3, Burnside's lemma, after Cauchy, Frobenius) $\langle \pi_X, 1\rangle$ = number of orbits of G on X.

Proof. If $X = X_1 \cup ... \cup X_l$ disjoint union of orbits, then $\pi_X = \pi_{X_1} + ... + \pi_{X_l}$, with π_{X_j} permutation character of G on X_j, so to prove the claim it's enough to show that if G is transitive on X then $\langle \pi_X, 1\rangle = 1$. Assume G is transitive on X. Now

$$\begin{aligned}\langle \pi_X, 1\rangle &= \frac{1}{|G|}\sum_g \pi_X(g) = \frac{1}{|G}|\{(g,x) \in G \times X : gx = x\}| \\ &= \frac{1}{|G|}\sum_{x\in X}|G_x| = \frac{1}{|G|}|X||G_x| = \frac{1}{|G|}|G| = 1\end{aligned}$$

(Note the use of orbit-stabilizer theorem). □

Lemma. (7.4)
Let G act on the sets X_1, X_2. Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1\times X_2} = \pi_{X_1}\pi_{X_2}$ and so $\langle \pi_{X_1}, \pi_{X_2}\rangle$ = number of orbits of G on $X_1 \times X_2$.

Proof. If $g \in G$ then $\pi_{X_1\times X_2}(g) = \pi_{X_1}(g)\pi_{X_2}(g)$. And we have

$\langle \pi_{X_1}, \pi_{X_2}\rangle = \langle \pi_{X_1}\pi_{X_2}, 1\rangle = \langle \pi_{X_1\times X_2}, 1\rangle =$ (7.3) number of orbits of G on $X_1 \times X_2$.

□

Definition. (7.5)
Let G act on X, $|X| > 2$. Then G is *2-transitive* on X if G has precisely two orbits on $X \times X : \{(x, x) : x \in X\}$ and $\{x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

Lemma. (7.6)
Let G act on X, $|X| > 2$. Then $\pi_X = 1 + \chi$ with χ irreducible $\iff$ G is 2-transitive on X.

Proof. $\pi_X = m_1 1 + m_2\chi_2 + ... + m_l\chi_l$ with $1, \chi_2, ..., \chi_l$ distinct irreducible characters and $m_i \in \mathbb{N}$. Then

$$\langle \pi_X, \pi_X \rangle = \sum_{i=1}^{l} m_i^2$$

hence G is 2-transitive on X $\iff$ $l = 2, m_1 = m_2 = 1$. $\square$

Example. (7.7)
Consider S_n acting on $X = \{1, ..., n\}$ which is 2-transitive. Hence $\pi_X = 1 + \chi$ with χ irreducible of degree $n - 1$. Similarly for A_n $(n > 3)$.

Example. (7.8)
Consider $G = S_4$.

ccl	1	3	8	6	6
rep	1	(12)(34)	(123)	(1234)	(12)
$1_G = \chi_1$	1	1	1	1	1
sign $= \chi_2$	1	1	1	-1	-1
$\pi_X - 1 = \chi_3$	3	-1	0	-1	1
$\chi_2\chi_3 = \chi_4$	3	-1	0	1	-1
χ_5	2	...	...	...	...

Get by orthog

Last lecture we were talking about using column orthogonality to find χ_5. Indeed we have

$$\chi_{reg} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5$$

So we can use this to find χ_5. Also, $S_4/V_4 \cong S_3$ by 'lifting' – see next chapter.