Analysis

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1 Differentiation

Theorem. (IFT) Assume I is an open interval, $f : I \to \mathbb{R}$ is differentiable on I, $f'(x) \neq 0 \forall x \in I$. Then $J = f(I)$ is an open interval, f is strictly monotonic, and hence bijection $I \to J$. Moreover, $f^{-1} : J \to I$ is differentiable, and

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

Proof. f is injective: if $x < y$ and $f(x) = f(y)$. Then by Rolle's theorem (f differentiable and hence continuous on I),$\exists c \in (x, y)$s.t. $f'(c) = 0$. Contradiction.
f is strictly monotonic: $\forall x < y$ in I, either $f(x) < f(y)$ or $f(x) > f(y)$. We next show that $\forall a < b < c$ in I either $f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$.
If not then $\exists a < b < c$ in I s.t.
either $f(a) < f(b), f(b) > f(c)$
or $f(a) > f(b), f(b) < f(c)$.
In the first case, fix w s.t. $\max(f(a), f(c)) < w < f(b)$. Then $f(a) < w < f(b)$ so by IVT, $\exists x \in (a, b)$ s.t. $f(x) = w$, and $f(b) > w > f(c)$ so by IVT $\exists y \in (b, c)$ s.t. $f(y) = w$. Contradicts with injectivity.
The other case is similar (apply the first case to $(-f)$).
Fix $a < b$ in I. We show that if $f(a) < f(b)$ then f is strictly increasing on I. The case $f(a) > f(b)$ will be similar and then f is strictly decreasing on I.
Let $x \in I$. If $x < a$ then considering the triple $x < a < b$ we obtain $f(x) < f(a)$.
If $a < x$ then
either $x < b$ and considering $a < x < b$, get $f(a) < f(x)$
or $x > b$ and considering $a < b < x$, get $f(a) < f(x)$
or $x = b$ and then $f(a) < f(b) = f(x)$.
So far we have $\forall x < y$ in I if $x = a$ or $y = a$ then $f(x) < f(y)$.
For arbitrary $x < y$ in I with $a \neq x$ and $a \neq y$ we have 3 cases: $a < x < y$,$x < a < y$,$x < y < a$, applying the previous claim we get $f(x) < f(y)$.
J is an interval: Let $x < y < z$ in $\mathbb{R}$ s.t. $x, z \in J$. We have $a, b \in I$ s.t. $x = f(a)$,$z = f(b)$. So by IVT, $\exists c$ between a, b s.t. $f(c) = y$, so $y \in J$.
J is an open interval: Given $y \in J$,$\exists b \in I$ s.t. $f(b) = y$.
I is an open interval, so $\exists a, c \in I$,$a < b < c$.
Then either $f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$.
So y is not an endpoint of J.

Now $f : I \to J$ is a strictly monotonic bijection, so $f^{-1} : J \to I$ is continuous by Theorem 3.6.
f^{-1} differentiable: Let $y \in J$. We consider

$$\frac{f^{-1}(y + k) - f^{-1}(y)}{k} \text{ as } k \to 0.$$

For given k, let $h = f^{-1}(y + k) - f^{-1}(y)$ and let $x = f^{-1}(y)$.
Then $f^{-1}(y + k) = h + f^{-1}(y) = x + h$,

$k = f(x+h) - f(x)$,

$$\frac{f^{-1}(y+k) - f^{-1}(y)}{k} = \frac{h}{f(x+h) - f(x)} = \frac{1}{\frac{f(x+h)-f(x)}{h}}$$

Here $h = h(k)$ depends on k, $h(k) \neq 0$ if $k \neq 0$ and $h(k) \to 0$ as $k \to 0$ since f^{-1} is continuous.
So
$$\frac{f^{-1}(y+k) - f^{-1}(y)}{k} \to \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

□

Example. Fix $n \in \mathbb{N}$, consider $f : (0, \infty) \to \mathbb{R}$,$f(x) = x^n$.
f is strictly increasing, onto $(0, \infty)$, differentiable, $f'(x) = nx^{n-1}$.
We have $f^{-1} : (0, \infty) \to (0, \infty)$,$f^{-1}(x) = x^{\frac{1}{n}}$ (definition).
The extra information from IFT is that f^{-1} is differentiable, and

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{n\left(y^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n} y^{\frac{1}{n}-1}.$$

For $\alpha = \frac{p}{q}$, $p, q \in \mathbb{N}$, $x^\alpha = \left(x^{\frac{1}{q}}\right)^p$ is differentiable by Chain Rule:
$\frac{d}{dx}(x^\alpha) = p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q} x^{\frac{1}{q}-1} = \alpha x^{\alpha-1}$.
For $\alpha \in \mathbb{Q}$,$\alpha < 0$,$x^\alpha = \frac{1}{x^{-\alpha}}$ is differentiable, and
$\frac{d}{dx}(x^\alpha) = -\frac{1}{(x^{-\alpha})^2} \cdot (-\alpha) x^{-\alpha-1} = \alpha x^{\alpha-1}$.
Need *exp*, *log* for $\alpha \in \mathbb{R}$.

1.1 Complex differentiation

Given $f : \mathbb{C} \to \mathbb{C}$, $a \in \mathbb{C}$, we say f is complex differentiable at a if

$$\lim_{h \to 0} \frac{f((a+h) - f(a)}{h}$$

exists and we denote the limit by $f'(a)$ and call it the *derivative of* f *at* a.
Say f is *complex differentiable* on $\mathbb{C}$ (or *holomorphic*) if it's complex differentiable at every $a \in \mathbb{C}$.
f is complex differentiable at $a \iff \exists \lambda \in \mathbb{C}\ f(a+) = f(a) + \lambda h + \epsilon(h) \cdot h$, where $\epsilon(h) \to 0$ as $h \to 0$. Then $\lambda = f'(a)$.

Proposition. f is complex differentiable at $a \implies f$ is continuous at a.

Property 2, Theorem 3 also hold.

For $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely and hence converges.
For $z = 0$ ok, for $z \neq 0$ we use the ratio test:

$$\frac{|\frac{z^{n+1}}{(n+1)!}|}{|\frac{z^n}{n!}|} = \frac{|z|}{n+1} \to 0 \text{ as } n \to \infty.$$

We define the *exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Theorem. (Properties of exp)
1) $\exp(z+n) = \exp(z)\exp(w)\ \forall z, w \in \mathbb{C}$;
2) $\underline{\exp(0)} = 1, \exp(z) \neq 0 \forall z \in \mathbb{C}$;
3) $\exp(z) = \exp(\overline{z})$;
4) $\exp(x) \in \mathbb{R} \forall x \in \mathbb{R}$;
5) $\exp(ix) \in T = \{z \in \mathbb{C} | |z| = 1\}\ \forall x \in \mathbb{R}$;
6) exp is complex differentiable at 0, $\exp'(0) = 1$;
7) exp is holomorphic, and $\exp'(z) = \exp(z)$.

Proof. 1) Let $a_k = \frac{z^n}{n'}, b_n = \frac{w^n}{n!}, c_n = \frac{(z+w)^n}{n!}$ for $n \geq 0$.
$c_n = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} z^j w^{n-j} = \sum_{j=0}^{n} \frac{1}{j!(n-j)!} z^j w^{n-j} = \sum_{j+k=n} a_j b_j$.

$$\begin{aligned}
&|\sum_{n=0}^{N} c_n - \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right)| \\
&\qquad = |\sum_{n=0}^{N}\sum_{j+k=n} a_j b_k - \sum_{j,k=0}^{N} a_j b_k| \\
&\qquad = |\sum_{j,k=0,j+k>N}^{N} a_j b_k| \\
&\qquad \leq \sum_{j,k=0,j+k>N}^{N} |a_j||b_k| \\
&\qquad \leq \sum_{j,k=0,j>\frac{N}{2} \text{ or } k>\frac{N}{2}}^{N} |a_j||b_k| \\
&\qquad \leq \sum_{\frac{N}{2}<j\leq N} |a_j| \cdot \sum_{k=0}^{N} |b_k| + \sum_{\frac{N}{2}<k\leq N} |b_k| \cdot \sum_{j=0}^{N} |a_j| \\
&\qquad \to 0 \text{ as } N \to \infty.
\end{aligned}$$

So $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

2) $\exp(0)$ by definition.
$1 = \exp(0) = \exp(z + (-z)) = \exp(z)\exp(-z)$.
So $\exp(z) \neq 0 \forall z \in \mathbb{C}$.

3) $\overline{\exp(z)} = \exp(\overline{z}) \forall z \in \mathbb{C}$.

$$\begin{aligned}\overline{\exp(z)} &= \overline{\left(\lim_{N\to\infty} \sum_{n=0}^{N} \frac{z^n}{n!}\right)} \\ &= \lim_{N\to\infty} \overline{\left(\sum_{n=0}^{N} \frac{z^n}{n!}\right)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(\overline{z})^n}{n!} \\ &= \exp(\overline{z}).\end{aligned}$$

4) $\exp(x) \in \mathbb{R} \forall x \in \mathbb{R}$ and $\exp(ix) \in T \forall x \in \mathbb{R}$.
$\overline{\exp(x)} = \exp(\overline{x}) = \exp(x)$, so $\exp(x) \in \mathbb{R}$.

$$\begin{aligned}|\exp(ix)|^2 = \exp(ix) \\ \overline{\exp(ix)} \\ &= \exp(ix)\exp(-ix) \\ &= \exp(0) \\ &= 1.\end{aligned}$$

5) $\exp'(0) = 1$.

$$\begin{aligned}\exp(h) &= \sum_{n=0}^{\infty} \frac{h^n}{n!} \\ &= 1 + h + \sum_{n=2}^{\infty} \frac{h^n}{n!} \\ &= \exp(0) + h + h\sum_{n=2}^{\infty} \frac{h^{n-1}}{n!}.\end{aligned}$$

Define $\epsilon(h) = \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!}$. Need $\epsilon(h) \to 0$ as $h \to 0$.
We have

$$\begin{aligned}|\epsilon(h)| &\leq \sum_{n=2}^{\infty} |\frac{h^{n-1}}{n!} \\ &= \sum_{n=2}^{\infty} \frac{|h|^{n-1}}{n!} \\ &\leq \sum_{n=2}^{\infty} |h|^{n-1} \text{ assume } |h| \leq 1 \\ &= \frac{|h|}{1-|h|}\end{aligned}$$

So $\epsilon(h) \to 0$ as $h \to 0$. Done.

6) $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\begin{aligned}\frac{\exp(z+h)-\exp(z)}{h} \\ &= \frac{\exp(z)\cdot\exp(h)-\exp(z)}{h} \\ &= \exp(z)\frac{\exp(h)-\exp(0)}{h} \\ &\to \exp(z)\end{aligned}$$

as $h \to 0$.
So $\exp'(z) = \exp(z)$. □

By 4) we have a real function $\exp : \mathbb{R} \to \mathbb{R}$.

Theorem. $\exp : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, differentiable bijection of $\mathbb{R}$ onto $\mathbb{R}^+$;
For $x \geq 0$, $\exp(x) \geq 1+x$ so $\exp(x) \to \infty$ as $x \to \infty$;
For $x \leq 0$, $\exp(x) = \frac{1}{\exp(-x)} \to 0$ as $x \to -\infty$.

Proof. For $x \geq 0$, $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq 1+x > 0$. So for $x \leq 0$,

$$\begin{aligned}1 &= \exp(x+(-x)) \\ &= \exp(x)\exp(-x)\end{aligned}$$

So $\exp(x) = \frac{1}{\exp(-x)} > 0$
since $\exp'(x) = \exp(x) > 0 \forall x \in \mathbb{R}$. □

By Corollary 6, $\exp : \mathbb{R} \to \mathbb{R}^+$ is strictly increasing.
Given $y \in \mathbb{R}^+$, choose $n \in \mathbb{N}$ s.t. $n > y > \frac{1}{n}$. So $\exp(n) \geq 1+n > y$, and $\exp(-n) = \frac{1}{\exp(n)} \leq \frac{1}{1+n} < \frac{1}{n} < y$.
By IVT, $\exists x \in (-n, n)$, $\exp(x) = y$.
Finally, $\exp(x) \geq 1+x \to \infty$ as $x \to \infty$ for $x \geq 0$.
For $x \leq 0$, $\exp(x) = \frac{1}{\exp(-x)} \to 0$ as $x \to -\infty$.

We define the *logarithm* to be the function $\log : \mathbb{R}^+ \to \mathbb{R}$ that is the inverse of $\exp : \mathbb{R} \to \mathbb{R}^+$.

Theorem. $\log : \mathbb{R}^+ \to \mathbb{R}$ is a strictly increasing, differentiable bijection. For $y > 0$, $\log'(y) = \frac{1}{y}$, $\log 1 = 0$,$\log(xy) = \log x + \log y \forall x, y > 0$, $\log x \to \infty$ as $x \to \infty$, $\log x \to \infty$ as $x \to 0$.

Proof. If $0 < x < y$ and $\log x \geq \log y$ then
$x = \exp(\log x) \geq \exp(\log y) = y$, contradiction.
Since $\exp'(x) = \exp(x) \neq 0 \forall x \in \mathbb{R}$, by IFT, log is differentiable, and

$$\log'(y) = \frac{1}{\exp(\log y)} = \frac{1}{y}.$$

$\log 1 = 0$ since $1 = \exp(0)$.
$\exp(\log x + \log y) = \exp(\log x)\exp(\log y) = xy$;
Apply log:
$\log x + \log y = \log(xy)$.
Since log, exp, are strictly increasing, $\log x > c \iff x > \exp c$, $\log x < c \iff x < \exp c$.
So it follows immediately that
$\log x \to \infty$ as $x \to \infty$,
$\log x \to -\infty$ as $x \to 0^+$. □

Definition. Define for $x > 0$, $\alpha \in \mathbb{R}$

$$x^\alpha = \exp(\alpha \log x).$$

Theorem. • 1) for $\alpha \in \mathbb{Q}$, x^α agrees with the previous definition.
• 2) for $\alpha > 0$, $x \to x^\alpha$ is a strictly increasing differentiable bijection: $\mathbb{R}^+ \to \mathbb{R}^+$; for $\alpha < 0$, $x \to x^\alpha$ is a strictly decreasing differentiable bijection: $\mathbb{R}^+ \to \mathbb{R}^+$; $\forall \alpha$, $f(x) = x^\alpha$, $f'(x) = \alpha x^{\alpha-1}$;

$\forall x, y > 0, \forall \alpha, \beta \in \mathbb{R}$:
• 3) $(xy)^\alpha = x^\alpha y^\alpha$,
$x^{\alpha+\beta} = x^\alpha x^\beta$, $(x^\alpha)^\beta = x^{\alpha\beta}$;
• 4) $\frac{x^\alpha}{\exp(x)} \to 0$ as $x \to \infty \forall x \in \mathbb{R}$,
$\frac{\log x}{x^\alpha} \to 0$ as $x \to \infty \forall x > 0$.

Proof. 1)

$$\begin{aligned} x^n &= \exp(n \log x) \\ &= \exp(\log x + \log x + ... + \log x) \text{ n times} \\ &= \exp(\log x)\exp(\log x)...\exp(\log x) \text{ n times} \\ &= x \cdot x \cdot ... \cdot x \text{ n times} \end{aligned}$$

which is the old definition of x^n.

$$\begin{aligned} \left(x^{\frac{1}{n}}\right)^n &= \left(\exp\left(\frac{1}{n}\log x\right)\right)^n \\ &= \exp\left(\frac{1}{n}\log x\right) \cdot \exp\left(\frac{1}{n}\log x\right) \cdot ... \cdot \exp\left(\frac{1}{n}\log x\right) \text{ n times} \\ &= \exp(\log x) \\ &= x. \end{aligned}$$

So the new $x^{\frac{1}{n}}$ is the unique $y > 0$ such that $y^n = x$, also same as the old definition.

So now for $\alpha \in \mathbb{Q}^+$, it follows that the two definitions coincide.
For $\alpha \in \mathbb{Q}^-$,

$$\begin{aligned} x^\alpha &= \exp(\alpha \log x) \\ &= \exp(-(-\alpha)\log x) \\ &= \frac{1}{\exp(-\alpha \log x)} \\ &= \frac{1}{x^{-\alpha}} \end{aligned}$$

also the old definition.

2) Immediate from properties of log and exp.
e.g. for $f(x) = x^\alpha = \exp(\alpha \log x)$, by chain rule,

$$\begin{aligned} f'(x) &= \exp(\alpha \log x)\,\alpha \frac{1}{x} \\ &= \alpha \frac{\exp(\alpha \log x)}{\exp(\log x)} \\ &= \alpha \exp(\alpha \log x - \log x) \\ &= \alpha \exp((\alpha - 1)\log x) \end{aligned}$$

$$= \alpha x^{\alpha-1}.$$

3) also immediate from properties of log and exp. (exercise)

4) For $x > 0$, $\exp(x) > \frac{x^n}{n!}$ for any $n \in \mathbb{N}$.
Given $\alpha \in \mathbb{R}$, choose $n \in \mathbb{N}, n > \alpha$, then

$$\begin{aligned} \frac{x^\alpha}{\exp(x)} &< \frac{x^\alpha}{x^{-n} n!} \\ &= (n!)\, x^{\alpha-n} \\ &= (n!) \exp((\alpha - n)\log x) \to 0 \end{aligned}$$

as $x \to \infty$.

Now let $y = \log(x\alpha) = \alpha \log x \to \infty$ as $x \to \infty$, so $\frac{\log x}{x^\alpha} = \frac{1}{\alpha}\frac{y}{\exp(y)} \to 0$ as $x \to \infty$. □

We can define $x^\alpha = \exp(\alpha \log x)$ for $x \in \mathbb{R}, x > 0$ and $\alpha \in \mathbb{C}$.
Exercise: define $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.
Show that $e = \exp(1)$, $e^z = \exp(z)$.
Need $e^z = e^w \iff z - w \in 2\pi\mathbb{Z}$ (what is π?)

1.2 Trigonometric and Hyperbolic functions

Definition. Define functions $\sin, \cos, \sinh, \cosh : \mathbb{C} \to \mathbb{C}$:

$$\begin{aligned}
\sin z &= \frac{e^iz - e^-iz}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{6} + \frac{z^5}{120} - ... \\
\cos z &= \frac{e^iz + e^-iz}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{24} - ... \\
\sinh z &= \frac{e^z - e^-z}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\
\cosh z &= \frac{e^z + e^-z}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}
\end{aligned}$$

Proposition. (Properties of trigonometric functions)
1) If f is any of these trigonometric functions, then $\overline{f(z)} > f(\overline{z})$.
Also $f(-z) = -f(z)$ for $f = \sin, \sinh$ (odd),
$f(-z) = f(z)$ for $f = \cos, \cosh$ (even),
$\sin(0) = \sinh(0) = 0$, $\cos(0) = \cosh(0) = 1$.

2)

$$\begin{aligned}
\sin(z+w) &= \sin z \cos w + \cos z \sin w, \sin(2z) = 2 \sin z \cos z \\
\cos(z+w) &= \cos z \cos w - \sin z \sin w, \cos(2z) = \cos^2 z - \sin^2 z \\
\sinh(z+w) &= \sinh z \cosh w + \cosh z \sinh w, \sinh(2z) = 2 \sinh z \cosh z \\
\cosh(z+w) &= \cosh z \cosh w + \sinh z \sinh w, \cosh(2z) = \cosh^2 z + \sinh^2 z.
\end{aligned}$$

3)

$$\begin{aligned}
1 &= \cos(0) = \cos^2 z + \sin^2 z \\
1 &= \cosh(0) = \cosh^2 z - \sinh^2 z
\end{aligned}$$

4)

$$\begin{aligned}
e^{iz} &= \cos z + i \sin z \\
e^z &= \cosh z + \sinh z
\end{aligned}$$

5) All the four functions are complex differentiable, with

$$\sin' z = \cos z, \cos' z = -\sin z, \sinh' z = \cosh z, \cosh' z = \sinh z.$$

Proof. Immediate from the previous theorem. □

This implies that $\sin x, \cos x \in \mathbb{R}$ for $x \in \mathbb{R}$.
Since $\cos^2 x + \sin^2 x = 1$, we have $\cos x, \sin x \in [-1, 1]$.
Have functions $\sin, \cos : \mathbb{R} \to [-1, 1]$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \left(-\frac{x^6}{6!} + \frac{x^8}{8!}\right) + \left(-\frac{-x^10}{10!} + \frac{x^12}{12!}\right) + ...$$

At $x = 2$, each term in the brackets is negative.
So
$$\cos(z) < 1 - \frac{2^2}{2} + \frac{2^4}{4!} = 1 - 2 + \frac{16}{84} < 0$$
Since $\cos(0) = 1 > 0$ and cos is continuous, by IVT
$$\exists z \in (0, 2)\, s.t. \cos(z) = 0.$$
So $A = \{z \geq 0 | \cos(z) = 0\} \neq \phi$ is bounded below by 0, so $\inf A$ exists.

Definition. $\pi = 2 \times \inf A$, i.e. $\frac{\pi}{2} = \inf A$.

Claim:
$$\cos\frac{\pi}{2} = 0$$
and so $\frac{\pi}{2} \geq 0$, and it is the least positive zero of cos.

Proof. $\forall n \in \mathbb{N}$, $\frac{\pi}{2} + \frac{1}{n} > \inf A$, so $\exists x_n \in A$ s.t. $\frac{\pi}{2} + \frac{1}{n} > x_n \geq \frac{\pi}{2}$.
So $x_n \to \frac{\pi}{2}$ and have $\cos(x_n) \to \cos\frac{\pi}{2}$.(cos is continuous). So $\cos\frac{\pi}{2} = 0$.
$\cos(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$. So $\sin'(x) = \cos x < 0$ and hence sin is strictly increasing on $[0, \frac{\pi}{2}]$. □

So $\sin^2\frac{\pi}{2} = 1 - \cos^2\frac{\pi}{2} = 1$, so $\sin\frac{\pi}{2} = 1$.
And $\sin x > 0$ on $\left(0, \frac{\pi}{2}\right)$ and hence cos is strictly decreasing on $[0, \frac{\pi}{2}]$.
$\sin\pi = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0$, $\cos\pi = \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2} = -1$.
$\sin(\pi - x) = \sin\pi\cos(-x) + \cos\pi\sin(-x) = \sin x$.
$x \to \frac{\pi}{2} + x$, $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} + x\right)$;
$x \to -x$, $\sin(\pi + x) = \sin(-x) = -\sin(x) = -\sin(\pi + x)$;
$\sin(2\pi + x) = -\sin(-x) = \sin x$.
So $\sin(2\pi n + x) = \sin x \forall x \in \mathbb{R} \forall n \in \mathbb{Z}$.
$\sin\left(x + \frac{\pi}{2}\right) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2} = \cos x \implies$ usual properties of cos (symmetry, periodicity).

Proposition. For $z \in \mathbb{C}, e^z = 1 \iff z \in 2\pi i\mathbb{Z}$.
So $e^z = e^w \iff z - w \in 2\pi i\mathbb{Z}$.

Proof. If $z = 2\pi in, n \in \mathbb{Z}$ then
$$\begin{aligned} e^z &= e^{(2\pi n)}i \\ &= \cos(2\pi n) + 2\sin(2\pi n) \\ &= \cos(0) + i\sin(0) \\ &= 1. \end{aligned}$$
Conversely, assume $e^z = 1$ and write $z = x + iy$, $x, y \in \mathbb{R}$,
$$1 = e^z = e^x e^{iy}$$

taking modulus,

$$1 = e^x$$

So$x = 0$.
So $1 = e^{iy} = \cos(y) + i\sin(y)$ and hence $\cos(y) = 1$.
Since cos is strictly decreasing on $[0, \pi]$, we have

$$\cos t = 1, t \in [0, \pi] \iff t = 0$$

Since cos is symmetric in $x = \pi$, for $t \in [0, 2\pi)$,

$$\cos t = 1 \iff t = 0$$

Now choose $n \in \mathbb{Z}$ s.t. $y - 2\pi n \in [0, 2\pi)$.
Then $\cos(y - 2\pi n) = \cos(y) = 1$ and so $y - 2\pi n = 0$. Hence $z \in 2\pi i\mathbb{Z}$. □

For $x \in \mathbb{R}$,

$$\sinh x = \frac{e^x - e^{-x}}{2} \in \mathbb{R}, \sinh : \mathbb{R} \to \mathbb{R}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \in \mathbb{R}, \cosh : \mathbb{R} \to \mathbb{R}$$

$\cosh x > 0$, $\sinh'(x) = \cosh(x)$. So sinh is strictly increasing.
$\cosh'(x) = \sinh(x) > 0$ for $x > 0$.
So cosh is strictly increasing on $[0, \infty)$.
$\cosh x > \sinh x$, and

$$\frac{\cosh x}{\sinh x} = \frac{1 + e^{-2x}}{1 - e^{-2x}} \to 1$$

as $x \to \infty$.
Define $\tan z = \frac{\sin z}{\cos z}$, $\tanh z = \frac{\sinh z}{\cosh z}$.

1.3 Derivative of higher orders

Definition. Let $A \subset \mathbb{R}, f : A \to \mathbb{R}$ be a function.
For $a \in A$, say f is twice differentiable at a if f is defined and differentiable on some open interval I containing a ($I \subset A$), and $f' : I \to \mathbb{R}$, $x \to f'(x)$ is differentiable at a. The second derivative of f at a is $(f')'(a)$.
We denote this by $f''(a)$ or $f^{(2)}(a)$ (also sometimes write $f^{(1)}$ for f',$f^{(0)}$ for f).
f is twice differentiable on A if f is twice differentiable at every $a \in A$, then the second derivative of f is the function $f'' : A \to \mathbb{R}, x \to f''(x)$.
In this case $\forall a \in A \exists r > 0, (a - r, a + r) \subset A$.
Typically $A = \mathbb{R}$ or some open interval or $\mathbb{R}\setminus\{0\}$.

In general for $n \geq 2$, f is n times differentiable at a if f is $n-1$ times differentiable on some open interval I containing a, and $f^{(n-1)} : I \to \mathbb{R}, x \to f^{(n-1)}(x)$ is differentiable at a. We write $f^{(n)}(a)$ for $\left(f^{(n-1)}\right)'(a)$ called the n^{th} derivative of f at a.
f is n times differentiable on A if f is n times differentiable at every $a \in A$.

Then the n^{th} derivative of f on A is the function: $f^{(n)} : A \to \mathbb{R}, x \to f^{(n)}(x)$.
f is infinitely differentiable on A (or C^{∞}) if f is n times differentiable on A $\forall n \in \mathbb{N}$.
f is n times continuously differentiable on A (or C^n) if f is n times differentiable on A, and $f^{(n)} : A \to \mathbb{R}$ is continuous.

Example. Let
$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$
be a polynomial. Then
$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + ... + n a_n x^{n-1}$$
is also a polynomial. So by induction, p is C^{∞}.

Theorem. Let $n \in \mathbb{N}$, $a \in \mathbb{R}$, let f be n times differentiable at a. Then $\exists \delta > 0$ and a function $R_n : (-\delta, \delta) \to \mathbb{R}$ s.t.
$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + ... + \frac{f^{(n)}(a)}{n!} + R_n(h) \tag{1}$$
for all $h \in (-\delta, \delta)$, and $R_n(h) = o(h^n)$, i.e.
$$\frac{R_n(h)}{h^n} \to 0$$
as $h \to 0$.

Proof. By definition, $\exists \delta > 0$ s.t. f is defined and is $n-1$ times differentiable on $(a-\delta, a+\delta)$, and
$$f^{(n)}(a) = \lim_{h \to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h}.$$
We define
$$R_n(n) = f(a+h) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k, |h| < \delta$$
So (1) holds.
$\bullet n = 1 :$
$$\frac{R_1(h)}{h} = \frac{f(a+h) - f(a)}{h} - f'(a) \to 0$$
$\bullet n \geq 2 : R_n(h)$ is $(n-1)$ times differentiable, and
$$\begin{aligned} R_n^{(k)}(h) &= f^{(k)}(a+h) \\ &= \sum_{l=k}^{n} \frac{f^{(l)}(a)}{l!} l(l-1)...(l-k+1)h^{l-k}, \\ R_n^{(k)}(0) &= 0 \ \forall k = 0, 1, ..., n-1. \end{aligned}$$
Now let
$$\begin{aligned} g(h) &= h^n, \\ g^{(k)}(h) &= n(n-1)...(n-k+1)h^{n-k}, \\ g^{(k)}(0) &= 0, \\ g^{(k)}(h) &\neq 0 \ \forall 0 < |h| < \delta. \end{aligned}$$

Then

$$\begin{aligned}\frac{R_n^{(n-1)}(h)}{g^{(n-1)}(h)} &= \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a) - hf^{(n)}(a)}{(n!)\,h} \\ &= \left(\frac{1}{n!}\right)\left[\frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h} - f^{(n)}(a)\right] \to 0\end{aligned}$$

as $h \to 0$.
So apply L' Hôpital's rule $(n-1)$ times, we obtain

$$\frac{R_n(h)}{h^n} \to 0$$

as $h \to 0$. $\square$

Suppose f is a C^∞ function. Then the previous theorem applies for all n. So does

$$\begin{aligned} f(a+h) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k + R_n(h) \\ \implies (?) f(a+h) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} h^k \end{aligned}$$

(called the *Taylor series*) on $(-\delta, \delta)$ on some $\delta > 0$?

Example. $f = \exp : \mathbb{R} \to \mathbb{R}$.
$f' = f$, so f is C^∞ and $f^{(n)} = f \ \forall n$.
Taylor series at 0:

$$\sum_{k=0}^{\infty} \frac{1}{k!} h^k = \exp(h)$$

for all $h \in \mathbb{R}$...

In general the answer is NO!

Problem. R_n depends on n. What if $R_n(h) = n^{n+1} h^{n+1}$?
For all n, $\frac{R_n(h)}{h^n} \to 0$ as $h \to 0$.
For fixed $h \neq 0$, $R_n(h) \not\to 0$ as $n \to \infty$.

Theorem. (Taylor's theorem with the Lagrange remainder)
Let $a \in \mathbb{R}$, $\delta > 0$, $n \in \mathbb{N}$. Assume $f : (a-\delta, a+\delta) \to \mathbb{R}$ is n times differentiable. Then $\forall h \in (-\delta, \delta)$, $\exists \theta \in (0,1)$ s.t.

$$f(a+h) = f(a) + \sum_{k=1}^{n-1} \frac{f^{(k)}(a)}{k!} h^k + \frac{f^{(n)}(a+\theta h)}{n!} h^n.$$

Remark. • when $n = 1$:

$$\begin{aligned} f(a+h) &= f(a) + f'(a+\theta h)\, h, \\ \frac{f(a+h) - f(a)}{h} &= f'(a+\theta h) \end{aligned}$$

while $a+\theta h$ is between a and $a+h$. So this is MVT!
• (1) says

$$f\left(a+h\right)=f\left(a\right)+\sum_{k=1}^{n}\frac{f^{(k)}\left(a\right)}{k!}h^{k}+R_{n}\left(h\right),$$
$$R_{n}\left(h\right)=\frac{h^{n}}{n!}\left(f^{(n)}\left(a+\theta h\right)-f^{(n)}\left(a\right)\right)$$

This is not obviously $o\left(h^{n}\right)$.

Proof. For $n=1$ the theorem is just MVT.
For $n\geq 2$, fix $h\in(-\delta,\delta)$, WLOG $h\neq 0$. Choose $A\in\mathbb{R}$ s.t.

$$f\left(a+h\right)=f\left(a\right)+\sum_{k=1}^{n-1}\frac{f^{(k)}\left(a\right)}{k!}h^{k}+\frac{Ah^{n}}{n!}$$

want to prove:

$$A=f^{(n)}\left(a+\theta h\right)$$

for some $\theta\in(0,1)$.
Define

$$g\left(t\right)=f\left(t\right)+\sum_{k=1}^{n-1}\frac{f^{(k)}\left(t\right)}{k!}\left(a+h-t\right)^{k}+\frac{A}{n!}\left(a+h-t\right)^{n}$$

For t in the closed interval between a and $a+b$.
g is continuous differentiable on the open interval between a and $a+h$.
Taylor expansion of f about t:

$$f\left(t+u\right)=f\left(t\right)=\sum_{k=1}^{n-1}\frac{f^{(k)}\left(t\right)}{k!}u^{k}+\text{ error}$$

when $u=a+h-t$,

$$f\left(a+h\right)=f\left(t\right)+\sum_{k=1}^{n-1}\frac{f^{(k)}\left(t\right)}{k!}\left(a+h-t\right)^{k}+\text{ error}$$

Now

$$g\left(a\right)=f\left(a+h\right),$$
$$g\left(a+h\right)=f\left(a+h\right)$$

By Rolle's theorem, $\exists\theta\in(0,1)$ s.t. $g'\left(a+\theta h\right)=0$.

$$\begin{aligned}g'\left(t\right)&=f'\left(t\right)+\sum_{k=1}^{n-1}\left[-\frac{f^{(k)}\left(t\right)}{(k-1)!}\left(a+h-t\right)^{k-1}+\frac{f^{(k+1)}\left(t\right)}{k!}\left(a+h-t\right)^{k}\right]-A\frac{\left(a+h-t\right)^{n-1}}{(n-1)!}\\&=\frac{f^{(n)}\left(t\right)}{(n-1)!}\left(a+h-t\right)^{n-1}-A\frac{\left(a+h-t\right)^{n-1}}{(n-1)!}\end{aligned}$$

So $g'\left(a+\theta h\right)=0$, which implies $A=f^{(n)}\left(a+\theta h\right)$. □

Example. Fix $\alpha \in \mathbb{R}$, $f : (-1, \infty) \to \mathbb{R}$.

$$\begin{aligned} f(x) &= (1+x)^{\alpha} = \exp(\alpha \log(1+x)) \\ f'(x) &= \alpha (1+x)^{\alpha-1} \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ f^{(n)}(x) &= \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n} \end{aligned}$$

So f is C^{∞} on $(-1, \infty)$ and its Taylor series at 0 is

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots$$

This converges to $f(x)$ $\forall x \in (-1, 1)$ (binomial theorem).

Remark. • $\alpha \in \mathbb{Z}, \alpha \geq 0$, then $\alpha^{\underline{n}} = 0 \forall n > \alpha$.(c.f. number and sets, falling power)
So

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = \sum_{n=0}^{\alpha} \binom{\alpha}{n} x^n = (1+x)^n$$

• $\alpha = -1$:

$$(1+x)^{-1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$
$$\alpha^{\underline{n}} = (-1)(-2)\dots(-n) = (-1)^n n!$$

Proof for general α deferred until Chapter 6(integration).
We an give a proof for $|x| < \frac{1}{2}$. From the previous theorem:

$$f(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + \binom{\alpha}{n+1} (1+\theta_n x)^{n+1} x^{n+1}$$

for some $\theta_n \in (0, 1)$.

$$|\frac{x^{n+1}}{(n+1)!} \alpha^{\underline{n+1}} (1+\theta_n x)^{\alpha-n-1}| \leq C \cdot n^m (2|x|)^n$$

(for $m <\in \mathbb{N}, m > |\alpha|$)

$$|\frac{x}{1+\theta_n x} \leq \frac{|x|}{1-|\theta_n x|} \leq 2|x|$$

2 Power series

Definition. A series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

is a power series about a. Here $(a_n)_{n=0}^{\infty}$ is a complex sequence, $a, z \in \mathbb{C}$. Think of $a, (a_n)$ as fixed and z as a variable.
Consider

$$D = \left\{ z \in \mathbb{C} | \sum_{n=0}^{\infty} a_n (z-a)^n \text{ converges} \right\}$$

and define $f : D \to \mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Example. 1) $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$. $a_n = \frac{1}{n!}$, $a = 0$.
Here $D = \mathbb{C}$ and $f = \exp$.
2) $\sum_{n=1}^{\infty} \frac{-1}{n} (1-z)^n$. $a = 1$, $a_n = \frac{(-1)^{n-1}}{n}$, $a_0 = 0$ (note $(1-z)^n = (-1)^n (z-1)^n$).
3) $\sum_{n=1}^{\infty} n^n z^n$. $a_n = n$, $n \geq 1$, $a_0 = 0$, $a = 0$.
D={0}: given $z \neq 0$, $\exists N$ s.t. $|Nz| > 1$. Then $\forall n \geq N$, $|(nz)^n| \geq 1$. So $(nz)^n \not\to 0$.

Notation. $D(a,r) = \{z \in \mathbb{C} || z-a| < r\}$ the open disc with centre a, radius r.
$\bar{D}(a,r) = \{z \in \mathbb{C} || z-a| < r\}$ the closed disc with centre a, radius r.

Note: $\{z | \sum a_n (z-a)^n \text{ converges }\} = \{z | \sum a_n z^n \text{ converges }\} + a$. So WLOG $a = 0$.

Theorem. Suppose $\sum_{n=0}^{\infty} a_n w^n$ converges. Then $\forall z \in \mathbb{C}$, if $|z| < |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

Proof. Since $\sum a_n w^n$ converges, $a_n w^n \to 0$ as $n \to \infty$.
So $\exists N \leq \mathbb{N}$, $\forall n \geq N$, $|a_n w^n| \leq 1$. Then $\forall n \geq \mathbb{N}$,

$$|a_n z^n| = |a_n w^n \left(\frac{z}{w}\right)^n| \leq |\frac{z}{w}|^n$$

This converges as $|\frac{z}{w}| < 1$ (geometric series). So by comparison test, $\sum a_n z^n$ converges absolutely. □

Convention. 1) Let $[0,\infty] = [0,\infty) \cup \{\infty\}$.
Extend $\leq$ to $[0,\infty]$ by $x \leq \infty$ $\forall x \in [0,\infty]$, so $x < \infty$ $\forall x \in [0,\infty]$.
2) So $|z| < \infty$ $\forall z < \mathbb{C}$, and $\nexists z \in \mathbb{C}, |z| > \infty$. Write $D(a,\infty) = \mathbb{C}$ by convention.
3) $A \subset [0,\infty)$, $a \neq \phi$. If a is not bounded above then $\forall C \geq 0$, $\exists a \in A$ s.t. $a > C$.
We define $\sup A = \infty$.

Theorem. For power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

There exists a unique $R \in [0, \infty]$ s.t. $\forall z \in \mathbb{C}$,

$$|z-a| < R \implies \sum_{n=0}^{\infty} a_n (z-a)^n \text{ converges absolutely,}$$
$$|z-a| > R \implies \sum_{n=0}^{\infty} a_n (z-a)^n \text{ diverges.}$$

Proof. WLOG let $a = 0$.
• uniqueness: Suppose $R < S$ both work. Then fix $z \in \mathbb{C}$ with $R < |z| < S$.
Definition of $R \implies \sum a_n z^n$ diverges;
Definition of $S \implies \sum a_n z^n$ converges absolutely. Contradiction.
• existence: Let

$$A = \left\{ |z| \mid \sum_{n=0}^{\infty} a_n z^n \text{ converges} \right\}$$

$A \neq \phi$ as $0 \in A$. Let $R = \sup A$ (recall that $\sup A = \infty$ when A is not bounded above).
Let $z \in \mathbb{C}$. If $|z| < R$, then $\exists w \in \mathbb{C}$ s.t. $|z| < |w|$ and $\sum_{n=0}^{\infty} a_n w^n$ converges.
Hence by the previous theorem, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.
If $|z| > R$, then $|z| \in A$, so $\sum_{n=0}^{\infty} a_n z^n$ diverges. □

Definition. R is called the *radius of convergence* of $\sum_{n=0}^{\infty} a_n (z-a)^n$.

Remark. This theorem says nothing about convergence or otherwise of the power series when $|z-a| = R$.

Example. • 1)

$$\sum_{n=0}^{\infty} z^n$$

converges if $|z| < 1$ (to $\frac{1}{1-z}$).
When $|z| \geq 1$, then $|z^n| \geq 1$ $\forall n$, so $z^n \not\to 0$ and $\sum z^n$ is divergent.
It follows that $R = 1$.
• 2)

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges if $z = -1$ by the alternating series test. So $R \geq 1$.
On the other hand, this diverges when $z = 1$ (harmonic series). So $R \leq 1$.
So $R = 1$.
(in fact the series converges $\forall z$ s.t. $|z| = 1, z \neq 1$).
• 3)

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

Here

$$|z| \leq 1 \implies |\frac{z^n}{n^2}| \leq \frac{1}{n^2}$$

So by the comparison test, the series converges absolutely. So $R \geq 1$.
When $|z| > 1$ then $|\frac{z^n}{n^2}| = \frac{|z^n|}{n^2} \to \infty$ as $n \to \infty$. So the series diverges. So $R \leq 1$.
So $R = 1$ (the series converges absolutely for all $|z| = 1$).

Theorem. Assume

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

has radius of convergence $R > 0$.
Let $f : D(a, R) \to \mathbb{C}$ ($D(a, R)$ is the disc with centre a and radius R) be defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Then

$$\sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

also has radius of convergence R.
Setting

$$g : D(a, R) \to \mathbb{C},$$
$$g(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

we have f is complex differentiable on $D(a, R)$, and

$$f'(z) = g(z) \, \forall z \in D(a, R).$$

Remark. So we can differentiate a power series term-by-term inside the radius of convergence. i.e.,

$$\frac{d}{dz} \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} \frac{d}{dz}$$

But this is dangerous in general.

Corollary. A power series is infinitely complex differentiable inside the radius of convergence.

Example. (non-examinable)
The previous theorem tells that exp is differentiable. We'll deduce that

$$\exp(z+w) = \exp z \cdot \exp w$$

Proof. Let

$$f(z) = \exp(z) \cdot \exp(-z).$$

So $f' \equiv 0(?)$, i.e. $f \equiv 1$, so $\exp(-z) = \frac{1}{\exp(z)}$.
Then fix w, $g(z) = \exp(z+w)\exp(-z)$. So $g' \equiv 0$, $g \equiv \exp w$.
So $\forall z$,

$$\exp(z+w)\exp(-z) = \exp w,$$
$$\exp(z+w) = \exp(z)\exp(w)$$

$\square$

Theorem. (non-examinable)
Suppose

$$f : D(a, R) \to \mathbb{C}$$

is complex differentiable. Then $\exists (a_n)_{n=0}^{\infty}$ in $\mathbb{C}$, s.t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \, \forall z \in D(a, R)$$

Proof. WLOG let $a = 0$. Fix $z \in D(0, R)$. Fix $\delta > 0$ s.t. $|z| + \delta < R$ for $0 < |h| < \delta$. Then

$$|\frac{f(z+h) - f(z)}{h} - g(z)| = \sum_{n=1}^{\infty} |a_n| | \left[\frac{(z+h)^n - z^n}{h} - nz^{n-1} \right] | \tag{2}$$

Then look at the second term, i.e. (let $\delta = |h|$)

$$\begin{aligned} |\frac{(z+h)^n - z^n - hnz^{n-1}}{h}| &\le \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} \delta^k \frac{|h|}{\delta^2} \\ &\le (|z| + \delta)^n \frac{|h|}{\delta^2} \end{aligned}$$

So (2) is at most

$$\left(\sum_{n=0}^{\infty} |a_n| (|z| + \delta)^n \right) \frac{|h|}{\delta^2} \to 0$$

as $h \to 0$. □

Corollary. (non-examinable)
If

$$f : D(a, R) \to \mathbb{C}$$

is complex differentiable, then it is infinitely complex differentiable (holomorphic).

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1) a_n (z-a)^{n-k}$$

So

$$f^{(k)}(a) = n! a_n$$

So

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

is the taylor series!

Corollary. (non-examinable)
Let

$$f, g : D(a, R) \to \mathbb{C}$$

be complex differentiable.
Suppose $\exists \delta > 0 \, (\delta < R)$ s.t.

$$f \equiv g \text{ on } D(a, \delta)$$

Then

$$f \equiv g \text{ on } D(a, R).$$

3 Integration

Suppose $[a,b]$ is a closed bounded interval with $a \leq b$, and

$$f : [a,b] \to \mathbb{R}$$

is a bounded function, i.e. $\exists C$ s.t. $|f(t)| \leq C\ \forall t \in [a,b]$.

A *dissection of* $[a,b]$ is a finite sequence

$$\mathcal{D} : a = x_0 < x_1 < x_2 < ... < x_n = b$$

The *lower sum of* f *w.r.t.* $\mathcal{D}$ is

$$\mathcal{S}_{\mathcal{D}}(f) = \sum_{k=1}^{n} (x_k - x_{k-1}) \inf f[x_{k-1}, x_k]$$

The *upper sum of* f *w.r.t.* $\mathcal{D}$ is

$$S_{\mathcal{D}}(f) = \sum_{k=1}^{n} (x_k - x_{k-1}) \sup f[x_{k-1}, x_k]$$

Notation. For $A \subset [a,b]$,

$$\sup_A f = \sup \{f(x) \,|x \in A\}$$

Note that $\mathcal{S}_{\mathcal{D}}(f) \leq S_{\mathcal{D}}(f)$.

Definition. $\mathcal{D}'$ is a *refinement* of $\mathcal{D}$ if it contains all the points in $\mathcal{D}$. Write $\mathcal{D} \leq \mathcal{D}'$.

Lemma. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $\mathcal{D}, \mathcal{D}'$ be dissections of $[a,b]$ with $\mathcal{D} \leq \mathcal{D}'$. Then

$$\mathcal{S}_{\mathcal{D}}(f) \leq \mathcal{S}_{\mathcal{D}'}(f) \leq S_{\mathcal{D}'}(f) \leq S_{\mathcal{D}}(f).$$

Proof. Say $\mathcal{D} : a = x_0 < x_1 < ... < x_n = b$.
We may assume that $\mathcal{D}'$ has only one extra point c, then the rest can be done by induction.
Choose k s.t. $x_{k-1} < c < x_k$. Then

$$\begin{aligned}\inf_{[x_{k-1},x_k]} f \cdot (x_k - x_{k-1}) &= (c - x_{k-1}) \cdot \inf_{[x_{k-1},x_k]} f + (x_k + c) \inf_{[x_{k-1},x_k]} f \\ &\leq (c - x_{k-1}) \inf_{[x_{k-1},c]} f + (x_k - c) \inf_{[c,x_k]} f\end{aligned}$$

We obtain

$$\mathcal{S}_{\mathcal{D}}(f) \leq \mathcal{S}_{\mathcal{D}'}(f).$$

Similarly,

$$S_{\mathcal{D}'}(f) \leq S_{\mathcal{D}}(f)$$

and we always have

$$\mathcal{S}_{\mathcal{D}'}(f) \leq S_{\mathcal{D}'}(f)$$

So done. □

Corollary. If $\mathcal{D}_1$, $\mathcal{D}_2$ are two dissections of $[a,b]$, then

$$\mathcal{S}_{\mathcal{D}_1}(f) \leq S_{\mathcal{D}_2}(f)$$

Here f is as in the previous lemma.

Proof. Let $D = \mathcal{D}_1 \cup \mathcal{D}_2$, the common refinement of $\mathcal{D}_1$ and $\mathcal{D}_2$, i.e. the union of the points of $\mathcal{D}_1$ and $\mathcal{D}_2$.
By the previous lemma,

$$\mathcal{S}_{\mathcal{D}_1}(f) \leq \mathcal{S}_{\mathcal{D}}(f) \leq S_{\mathcal{D}}(f) \leq S_{\mathcal{D}_2}(f).$$

□

Definition. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
The *upper (Riemann) integral* of f on $[a,b]$ is

$$\overline{\int_a^b} f = \inf_{\mathcal{D}} S_{\mathcal{D}}(f)$$

i.e. (the inf taken over all dissections $\mathcal{D}$ of $[a,b]$.
The *lower (Riemann) integral* of f on $[a,b]$ is

$$\underline{\int_a^b} f = \sup_{\mathcal{D}} \mathcal{S}_{\mathcal{D}}(f)$$

By the previous corollary, given a dissection $\mathcal{D}_1$, $\mathcal{S}_{\mathcal{D}_1}(f)$ is a lower bound of $\{S_{\mathcal{D}}(f) \,|\, \mathcal{D} \text{ any dissection of } [a,b]\}$. So the upper integral exists and is at least $\mathcal{S}_{\mathcal{D}_1}(f)$.
Since $\mathcal{D}_1$ was arbitrary, $\overline{\int_a^b} f$ is an upper bound of $\{\mathcal{S}_{\mathcal{D}}(f) \,|\, \mathcal{D} \text{ any dissection of } [a,b]\}$.
Hence $\underline{\int_a^b} f$ exists and

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Definition. (Integrability)
Say f is (Riemann) integrable on $[a,b]$ if it is bounded, and

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

We define the integral of f on $[a,b]$ to be the common value of them, and denote it by

$$\int_a^b f$$

or

$$\int_a^b f(t)\,dt.$$

Proposition. Suppose $f : [a,b] \to \mathbb{R}$ is integrable and let

$$m = \inf_{[a,b]} f$$
$$M = \sup_{[a,b]} f$$

Then

$$m(b-a) \le \int_a^b f \le M(b-a)$$

and

$$|\int_a^b f| \le (b-a) \sup_{[a,b]} |f|$$

Proof. Consider $\mathcal{D} : a < b$. Then

$$\mathcal{S}_{\mathcal{D}}(f) = m(b-a)$$
$$S_{\mathcal{D}}(f) = M(b-a)$$

In addition

$$M \le \sup_{[a,b]} |f|$$
$$m \ge -\sup_{[a,b]} |f|$$

So the result follows. $\square$

Example. • 1) If $f(x) = c \ \forall x \in [a,b]$ then f is integrable, and

$$\int_a^b f(t)\,dt = c(b-a)$$

Note that $m = M = c$. Then

$$c(b-a) = m(b-a) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le M(b-a) = c(b-a)$$

So the upper and lower integral are equal and the value is $c(b-a)$.

• 2) $f : [0,1] \to \mathbb{R}$, $f(x) = x$.
Consider

$$\mathcal{D}_n : 0 \le \frac{1}{n} \le \frac{2}{n} \le ... \le \frac{n}{n} = 1$$

Then

$$\mathcal{S}_{\mathcal{D}_n}(f) = \sum_{k=1}^{n} \frac{1}{n}\frac{k-1}{n} = \frac{(n-1)n}{2n^2} = \frac{n-1}{2n} \to \frac{1}{2}$$

So

$$\underline{\int_0^1} f \ge \sup_n \mathcal{S}_{\mathcal{D}_n}(f) = \frac{1}{2}$$

Similarly,

$$S_{\mathcal{D}_n}(f) = \sum_{k=1}^{n} \frac{1}{n}\frac{k}{n} = \frac{(n+1)n}{2n^2} = \frac{n+1}{2n} \to \frac{1}{2}$$

So

$$\bar{\int_0^1} f \leq \inf_n S_{\mathcal{D}_n}(f) = \frac{1}{2}$$

So it follows that f is integrable and is equal to $\frac{1}{2}$.

Theorem. A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if

$$\forall \epsilon > 0 \exists \mathcal{D} \text{ s.t. } S_{\mathcal{D}}(f) - \mathcal{S}_{\mathcal{D}}(f) < \epsilon$$

Proof. • Forward: Since

$$\int_a^b = \inf_{\mathcal{D}} S_{\mathcal{D}}(f) = \sup_{\mathcal{D}} \mathcal{S}_{\mathcal{D}}(f)$$

we know that $\exists \mathcal{D}_1, \mathcal{D}_2$ with

$$S_{\mathcal{D}_1}(f) < \int_a^b f + \frac{\epsilon}{2},$$
$$\mathcal{S}_{\mathcal{D}_2}(f) > \int_a^b f - \frac{\epsilon}{2}.$$

Set $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. Then

$$\int_a^b f - \frac{\epsilon}{2} < \mathcal{S}_{\mathcal{D}_2}(f) \leq \mathcal{S}_{\mathcal{D}}(f) \leq S_{\mathcal{D}}(f) \leq S_{\mathcal{D}_1}(f) < \int_a^b f + \frac{\epsilon}{2}$$

• Backward: Suppose

$$S_{\mathcal{D}}(f) - \mathcal{S}_{\mathcal{D}}(f) < \epsilon$$

Then

$$\bar{\int_a^b} f \leq S_{\mathcal{D}}(f) < \mathcal{S}_{\mathcal{D}}(f) + \epsilon \leq \underline{\int_a^b} f + \epsilon$$

So

$$\bar{\int_a^b} f \leq \underline{\int_a^b} f + \epsilon \forall \epsilon > 0$$

Hence

$$\bar{\int_a^b} f \leq \underline{\int_a^b} f$$

So f is integrable. $\square$

Corollary. A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if there exists a sequence $\mathcal{D}_n$ of dissections s.t.

$$S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f) \to 0$$

as $n \to \infty$. Then both $\mathcal{S}_{\mathcal{D}_n}(f)$ and $S_{\mathcal{D}_n}(f)$ converge to $\int_a^b f$.
Moreover, if

$$\mathcal{D}_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{m_n}^{(n)} = b$$

and
$$\xi_k^{(n)} \in \left[x_{k-1}^{(n)}, x_k^{(n)}\right]$$
For $k = 1, 2, ..., m_n$. Then
$$\sum_{k=1}^{m_n} f\left(\xi_k^{(n)}\right)\left(x_k^{(n)} - x_{k-1}^{(n)}\right) \to \int_a^b f$$
as $n \to \infty$.

Proof. The first part is immediate from the previous theorem.
For the second part,
$$\begin{aligned}\int_a^b f &\leq S_{\mathcal{D}_n}(f) \\ &= \mathcal{S}_{\mathcal{D}_n}(f) + \left(S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f)\right) \\ &\leq \int_a^b f + \left(S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f)\right)\end{aligned}$$
Hence
$$S_{\mathcal{D}_n}(f) \to \int_a^b f$$
as $n \to \infty$. Then
$$\mathcal{S}_{\mathcal{D}_n}(f) = S_{\mathcal{D}_n}(f) - \left(S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f)\right) \to \int_a^b f$$
□

Together with
$$\inf_{\left[x_{k-1}^{(n)}, x_k^{(n)}\right]} f \leq f\left(\xi_k^{(n)}\right) \leq \sup_{\left[x_{k-1}^{(n)}, x_k^{(n)}\right]} f$$
Hence
$$\mathcal{S}_{\mathcal{D}_n}(f) \leq \sum_{k=1}^{m_n} f\left(\xi_k^{(n)}\right)\left(x_k^{(n)} - x_{k-1}^{(n)}\right) \leq S_{\mathcal{D}_n}(f)$$

Remark. *Darboux*: if f is integrable and
$$\mathcal{D}_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{m_n}^{(n)} = b$$
is such that
$$|\mathcal{D}_n| = \max_{1 \leq k \leq m_n}\left(x_k^{(n)} - x_{k-1}^{(n)}\right) \to 0$$
Then
$$S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f) \to 0$$

Lemma. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. Assume there exists $k \geq 0$ such that
$$|f(x) - f(y)| \leq K|g(x) - g(y)| \forall x, y \in [a, b]$$
Then if g is integrable, then f is also integrable.

Proof. Given $\epsilon > 0$, there exists $\mathcal{D}$ such that

$$S_{\mathcal{D}}(g) - \mathcal{S}_{\mathcal{D}}(g) < \epsilon$$

Now let

$$\mathcal{D} : a = x_0 < x_1 < ... < x_n = b$$

And let $I = [x_{k-1}, x_k]$. Then

$$\begin{aligned} \sup_I f - \inf_I f &= \sup_{x,y \in I} |f(x) - f(y)| \\ &\leq K \sup_{x,y \in I} |g(x) - g(y)| \\ &= K \left(\sup_I g - \inf_I g \right) \end{aligned}$$

Multiply by $|I| = x_k - x_{k-1}$ and sum over k,

$$S_{\mathcal{D}}(f) - \mathcal{S}_{\mathcal{D}}(f) \leq K (S_{\mathcal{D}}(g) - \mathcal{S}_{\mathcal{D}}(g)) < k\epsilon$$

As ϵ is arbitrary, f is integrable. □

Theorem. Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then
1) $\lambda f + \mu g$ is integrable, and

$$\int_a^b (\lambda f + \mu y) = \lambda \int_a^b f + \mu \int_a^b g$$

2) If $f \leq g$, then

$$\int_a^b f \leq \int_a^b g$$

3) $|f|$ is integrable, and

$$|\int_a^b f| \leq \int_a^b |f|$$

4) $\max(f, g)$, $\min(f, g)$ are integrable;
5) $f \cdot g$ is integrable, and (Cauchy-Schwarz inequality)

$$|\int_a^b fg| \leq \left(\int_a^b f^2 \right)^{\frac{1}{2}} \left(\int_a^b g^2 \right)^{\frac{1}{2}}$$

Proof. 1) Enough to consider $f + g$, λf for $\lambda \geq 0$, and $-f$.
We know from a previous corollary that there exists a sequence $\mathcal{D}_n$ of dissections of $[a, b]$ s.t. $S_{\mathcal{D}_n}(f)$, $\mathcal{S}_{\mathcal{D}_n}(f)$ both converge to $\int_a^b f$, and$S_{\mathcal{D}_n}(g)$, $\mathcal{S}_{\mathcal{D}_n}(g)$ both converge to $\int_a^b g$.
For an interval $I \subset [a, b]$, we have

$$\begin{aligned} &\sup_I (f + g) \leq \sup_I f + \sup_I g \\ &\inf_I (f + g) \geq \inf_I f + \inf_I g \\ &\mathcal{S}_{\mathcal{D}_n}(f) + \mathcal{S}_{\mathcal{D}_n}(g) \leq \mathcal{S}_{\mathcal{D}_n}(f + g) \leq S_{\mathcal{D}_n}(f + g) \leq S_{\mathcal{D}_n}(f) + S_{\mathcal{D}_n}(g) \end{aligned}$$

As $n \to \infty$, LHS and RHS both tend to $\int_a^b f + \int_a^b g$. So $f+g$ is integrable and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Also

$$\sup_I (\lambda f) = \lambda \sup_I f$$
$$\inf_I (\lambda f) = \lambda \inf_I f$$

Hence

$$S_{\mathcal{D}_n}(\lambda f) = \lambda S_{\mathcal{D}_n}(f) \to \lambda \int_a^b f$$
$$\mathcal{S}_{\mathcal{D}_n}(\lambda f) = \lambda \mathcal{S}_{\mathcal{D}_n}(f) \to \lambda \int_a^b f$$

By the previous corollary, λf is integrable and

$$\int_a^b (\lambda f) = \lambda \int_a^b f$$

Finally,

$$\sup_I (-f) = -\inf_I f,$$
$$\inf_I (-f) = -\sup_I f$$

We get

$$S_{\mathcal{D}_n}(-f) = -\mathcal{S}_{\mathcal{D}_n}(f) \to -\int_a^b f,$$
$$\mathcal{S}_{\mathcal{D}_n}(-f) = -S_{\mathcal{D}_n}(f) \to -\int_a^b f$$

Hence $-f$ is integrable and

$$\int_a^b (-f) = -\int_a^b f.$$

2) If $f \leq g$ then $g - f \geq 0$. Hence

$$\int_a^b g - \int_a^b f = \int_a^b (g-f) \geq (b-a) \inf_{[a,b]} (g-f) \geq 0$$

3)Note that

$$||f(x)| - f(y)|| \leq |f(x) - f(y)|$$

for all $x, y \in [a,b]$. So by the previous lemma, together with the assumption that f is integrable, we know that $|f|$ is integrable. Since

$$f \leq |f|, -f \leq |f|,$$

by 2) we know that

$$\int_a^b f \leq \int_a^b |f|,$$
$$-\int_a^b f = \int_a^b (-f) \leq \int_a^b |f|$$

Hence
$$|\int_a^b f| \leq \int_a^b |f|.$$
4) Given $s, t \in \mathbb{R}$,
$$\max(s,t) = \frac{s+t}{2} + \frac{|s-t|}{2}$$
So if $h = \max(f,g)$, then for all $x \in [a,b]$,
$$h(x) = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$$
Hence h is integrable by 1) and 3).
($\min(f,g)$ can be proved similarly).
5) Let
$$M = \sup_{[a,b]} |f|$$
This exists since f must be bounded on $[a,b]$. Then for all $x, y \in [a,b]$,
$$|f^2(x) - f^2(y)| = |f(x) - f(y)| \cdot |f(x) + f(y)| \leq 2M|f(x) - f(y)|$$
So by the previous lemma, f^2 is integrable. Hence by 1),
$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$
is integrable.
Next, we have
$$0 \leq \int_a^b (f - \lambda g)^2 = \int_a^b f^2 + \lambda^2 \int_a^b g^2 - 2\lambda \int_a^b fg$$
(using 2) and 1) respectively) for all $\lambda \in \mathbb{R}$.
Now put
$$\lambda = \frac{\int_a^b fg}{\int_a^b g^2}$$
After *some algebra*, we obtain the required inequality. $\square$

Proposition. 1) Assume $h : [a,b] \to \mathbb{R}$ satisfies that $h(x) = 0$ for all but finitely many x. Then h is integrable, and
$$\int_a^b h = 0$$
2) Assume $f : [a,b] \to \mathbb{R}$ is integrable, and $g(x) = f(x)$ for all but finitely many x. Then g is integrable, and
$$\int_a^b g = \int_a^b f$$

Proof. Choose
$$a = c_0 < c_1 < ... < c_n = b$$

s.t. $h(x) = 0\ \forall x \in [a,b] \setminus \{c_0, c_1, ..., c_n\}$ (it is not necessary for all of $f(c_i)$ to be non-zero).
If

$$M = \max_{0 \le i \le n} |h(c_i)|$$

Then $|h(x)| \le M\ \forall x$, so h is bounded.
Fix $\delta > 0$ s.t.

$$\delta < \frac{1}{2}(c_k - c_{k-1})$$

for all $1 \le k \le n$.
Now consider $\mathcal{D} : a, a+\delta, c_1 - \delta, c_1 + \delta, c_2 - \delta, c_2 + \delta, ..., c_n - \delta, c_n = b$. We have

$$\inf_{[c_{k-1}+\delta, c_k - \delta]} h = \sup_{[c_{k-1}+\delta, c_k - \delta]} h = 0$$

for all $1 \le k \le n$.
If $I = [c_k - \delta, c_k + \delta]$ $(1 \le k \le n-1)$ or $I = [a, a+\delta]$ or $I = [b-\delta, b]$, we have

$$\sup_I h \le M$$
$$\inf_I h \ge -M$$

Hence

$$\mathcal{S}_{\mathcal{D}}(h) \ge (n-1)\, 2\delta(-M) + 2\delta(-M) = -2Mn\delta,$$
$$S_{\mathcal{D}}(h) \le (n-1)\, 2\delta M + 2\delta M = 2Mn\delta$$

Hence

$$-2Mn\delta \le \underline{\int_a^b} h \le \overline{\int_a^b} h \le 2Mn\delta$$

But δ is arbitrary. So h is integrable and the integral is 0.
2) $g = f + (g - f)$. By 1) $g - f$ is integrable, and

$$\int_a^b (g-f) = 0$$

So

$$\int_a^b g = \int_a^b f$$

□

Theorem. Every continuous function is integrable.

Proof. Let $f : [a,b] \to \mathbb{R}$ be continuous. That means

$$\forall x \in [a,b]\, \forall \epsilon < 0 \exists \delta > 0 \forall y \in [a,b]\, |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

We claim that

$$\forall \epsilon < 0 \exists \delta > 0 \forall x, y \in [a,b]\, |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

(actually this is the definition of uniform convergence).
Proof of claim: Suppose otherwise. Then

$$\exists \epsilon > 0 \forall \delta > 0 \exists x, y \in [a,b]\, |y - x| < \delta, |f(y) - f(x)| \ge \epsilon$$

In particular,

$$\forall n \in \mathbb{N}, \exists x_n, y_n \in [a,b] \text{ s.t. } |x_n - y_n| < \frac{1}{n}, |f(x_n) - f(y_n)| \geq \epsilon$$

Now (x_n) is bounded. So by B-W theorem, $\exists k_1 < k_2 < k_3 < \ldots$ in $\mathbb{N}$ s.t. $(x_{k_n})_{n=1}^{\infty}$ converges to some $x \in \mathbb{R}$.
Since $a \leq x_{k_n} \leq b$ for all n, we have $x \in [a,b]$. Then since

$$|y_{k_n} - x_{k_n}| < \frac{1}{k_n} \leq \frac{1}{n} \to 0$$

as $n \to \infty$, so

$$y_{k_n} = x_{k_n} + (y_{k_n} - x_{k_n}) \to x$$

as $n \to \infty$.
As f is continuous,

$$\epsilon \leq |f(x_{k_n}) - f(y_{k_n})| \to |f(x) - f(x)| = 0$$

Contradiction.
Now back to the main proof.
Given $n \in \mathbb{N}$, choose $\delta_n > 0$ s.t.

$$\forall x, y, |x - y| < \delta_n \implies |f(x) - f(y)| < \frac{1}{n}$$

Then choose a dissection $\mathcal{D}_n$ s.t.

$$|\mathcal{D}_n| < \delta_n$$

$$(\text{ if } \mathcal{D} : a = x_0 < x_1 < \ldots < x_m = b, \text{ then } |\mathcal{D}| = \max_{1 \leq k \leq m} (x_k - x_{k-1}))$$

If I is an interval of $\mathcal{D}_n$ then

$$\sup_I f - \inf_I f \leq \frac{1}{n}$$

Hence

$$S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f) \leq \frac{1}{n}(b-a) \to 0$$

as $n \to \infty$. So f is integrable. □

Theorem. Monotonic functions are integrable.

Proof. Let $f : [a,b] \to \mathbb{R}$ be a monotonic function. WLOG let f be increasing (otherwise look at $-f$).
Let

$$\mathcal{D}_n : a + \frac{k}{n}(b-a), 0 \leq n (n \in \mathbb{N})$$

Then

$$\begin{aligned} S_{\mathcal{D}_n}(f) - \mathcal{S}_{\mathcal{D}_n}(f) &= \sum_{k=1}^{n} \frac{b-a}{n} \left(\sup_{\left[a+\frac{k-1}{n}(b-a), a+\frac{k}{n}(b-a)\right]} f - \inf_{\left[a+\frac{k-1}{n}(b-a), a+\frac{k}{n}(b-a)\right]} \right) \\ &= \frac{b-a}{n} \sum_{k=1}^{n} \left(f\left(a + \frac{k}{n}(b-a)\right) - f\left(a + \frac{k-1}{n}(b-a)\right)\right) \\ &= \frac{b-a}{n}(f(b) - f(a)) \to 0 \end{aligned}$$

as $n \to \infty$. So f is integrable by the previous corollary. □

(A note on the proof of the integral form of Cauchy-Schwarz inequality:

$$0 \leq \int_a^b (f - \lambda g)^2 = \int_a^b f^2 + \lambda^2 \int_a^b g^2 - s\lambda \int_a^b fg$$

for all $\lambda \in \mathbb{R}$.
Putting

$$\lambda = \frac{\int_a^b fg}{\int_a^b g^2}$$

yields the result, provided the denominator is not zero.
If $\int_a^b g^2 = 0$, then we get

$$2\lambda \int_a^b fg \leq \int_a^b f^2$$

for all $\lambda \in \mathbb{R}$.
Since λ is arbitrary, this forces

$$\int_a^b fg = 0$$

so the result still holds.)

Theorem. Let $a < b$, f a bounded function on $[a, b]$.
1) If $a < c < b$ and f is integrable on $[a, c]$ and $[c, b]$, then it's integrable on $[a, b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

2) If f is integrable on $[a, b]$, then f is integrable on $[c, d]$ whenever $a \leq c < d \leq b$.

Proof. 1) There exists sequences $\mathcal{D}'_n$ and $\mathcal{D}''_n$ of dissections of $[a, c]$ and $[c, b]$ respectively, such that

$$S_{\mathcal{D}'_n}(f), \mathcal{S}_{\mathcal{D}'_n}(f) \to \int_a^c f$$
$$S_{\mathcal{D}''_n}(f), \mathcal{S}_{\mathcal{D}''_n}(f) \to \int_c^b f$$

Then

$$\mathcal{D}_n = \mathcal{D}'_n \cup \mathcal{D}''_n$$

is a dissection of $[a, b]$, and

$$S_{\mathcal{D}_n}(f) = S_{\mathcal{D}'_n}(f) + S_{\mathcal{D}''_n}(f) \to \int_a^c f + \int_c^b f$$
$$\mathcal{S}_{\mathcal{D}_n}(f) = \mathcal{S}_{\mathcal{D}'_n}(f) + \mathcal{S}_{\mathcal{D}''_n}(f) \to \int_a^c f + \int_c^b f$$

So f is integrable on $[a,b]$ and tends to the expected value.
2) Given $\epsilon > 0$, there is a dissection $\mathcal{D}$ of $[a,b]$ s.t.

$$S_{\mathcal{D}}(f) - \mathcal{S}_{\mathcal{D}}(f) < \epsilon$$

WLOG we may assume that c, d are in $\mathcal{D}$ (otherwise add them into $\mathcal{D}$ which refines it, and will make the difference between the upper and lower integral even smaller).
So

$$\mathcal{D} : a = x_0 < x_1 < ... < x_n = b, c = x_{j-1}, d = x_k$$

for some $1 \leq j \leq k \leq n$.
Then

$$D' : x_{j-1} < x_j < ... < x_k$$

is a dissection of $[c,d]$. Then

$$\begin{aligned} S_{\mathcal{D}'}(f) - S_{D'}(f) &= \sum_{i=j}^{k} (x_i - x_{i-1}) \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) \\ &\leq \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) \\ &= S_{\mathcal{D}}(f) - \mathcal{S}_{\mathcal{D}}(f) \\ &< \epsilon \end{aligned}$$

So S is integrable on $[c,d]$. □

Corollary. Let a, b, f be as in the previous theorem. Consider

$$a = c_0 < c_1 < ... < c_k = b$$

Then f is integrable on $[a,b]$ if and only if f is integrable on $[c_{j-1}, c_j]$, for all $1 \leq j \leq k$, and then

$$\int_a^b f = \sum_{j=1}^{k} \int_{c_{j-1}}^{c_j} f$$

Corollary. Piecewise monotonic functions are integrable. $f : [a,b] \to \mathbb{R}$ is *piecewise monotonic* if there exists $a = c_0 < c_1 < ... < c_k = b$ such that f is monotonic on each $[c_{j-1}, c_j]$.

Proof. By the theorem that monotonic functions are integrable, and the previous corollary. □

Theorem. If $a < b$, f is a bounded function on $[a,b]$, continuous at all except finitely many points. Then f is integrable.

Proof. Choose

$$a = c_0 < c_1 < ... < c_k = b$$

such that f is continuous at x if $x \notin \{c_0, c_1, ..., c_k\}$.
Let $M = \sup_{[a,b]} |f|$. Choose $\delta > 0$ such that $\delta < \frac{1}{2}(c_j - c_{j-1})$ for all j and $4M\delta k < \frac{\epsilon}{2}$.
f is continuous on $[c_{j-1} + \delta, c_j - \delta]$ for $1 \le j \le k$. So it's integrable, so there exists a dissection $\mathcal{D}_j$ s.t.

$$S_{\mathcal{D}_j}(f) - \mathcal{S}_{\mathcal{D}_j}(f) < \frac{\epsilon}{2k}$$

Now consider

$$\mathcal{D} = \bigcup_{j=1}^{k} \mathcal{D}_j \bigcup \{a, b\}$$

which is a dissection of $[a, b]$. Then

$$\begin{aligned} S_{\mathcal{D}}(f) - \mathcal{S}_{\mathcal{D}}(f) &= \sum_{j=1}^{k} \left(S_{\mathcal{D}_j}(f) - \mathcal{S}_{\mathcal{D}_j}(f)\right) + \text{ contributions from the small segments around } c_j's \\ &\le k \cdot \frac{\epsilon}{2k} + 2\delta \cdot 2M \cdot (k-1) + \delta \cdot 2M \cdot 2 \\ &= \frac{\epsilon}{2} + 4M\delta k \\ &< \epsilon \end{aligned}$$

□

Example. (a function that is not integrable)
Let

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

for any interval $I \subset [0, 1]$ of positive length, we have

$$\sup_I f = 1, \inf_I f = 0$$

Then for all dissection $\mathcal{D}$, $S_{\mathcal{D}}(f) = 1$, $\mathcal{S}_{\mathcal{D}}(f) = 0$. So

$$\overline{\int_0^1} f = 1 \neq 0 = \underline{\int_0^1} f$$

So f is not integrable.

Definition. Given $a < b$, f integrable on $[a, b]$, we *define*

$$\int_b^a = -\int_a^b f$$

So if f is integrable on some closed, bounded interval containing a, b, c (in any order), then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

This comes from the theorem about integrals on union of intervals and this definitions (a few cases for signs).
(eg if $c < b < a$ then

$$\int_c^a f = \int_c^b f + \int_b^a f$$
$$-\int_a^c f = \int_c^b f - \int_a^b f$$

So consistent.)
Note:

$$|\int_a^b f| \leq |b-a| \sup_{[a,b]} |f|$$

Since if $a < b$, this holds by proposition 3;
if $b < a$ then

$$\begin{aligned} |\int_a^b f| = |-\int_b^a f| &= |\int_b^a f| \\ &\leq (a-b) \sup |f| = |b-a| \sup |f|. \end{aligned}$$

Definition. (indefinite integral)
Suppose $a < b$, f is integrable on $[a,b]$ and $c \in [a,b]$. The function

$$F(x) = \int_c^x f(t)\,dt, x \in [a,b]$$

is called <u>an</u> *indefinite integral of f on* $[a,b]$ (since this depends on c).
Note:

$$F(y) - F(x) = \int_x^y f(t)\,dt$$

This does not depend on c.

Theorem. If $a < b$, f integrable on $[a,b]$, F is an indefinite integral of f on $[a,b]$, then F is continuous. In fact, there exists some $k \geq 0$ such that

$$|F(y) - F(x)| \leq k|y-x|$$

Proof. Let $K = \sup_{[a,b]} |f|$. Then

$$\begin{aligned} |F(y) - F(x)| &= |\int_x^y f(t)\,dt| \\ &\leq |y-x| \cdot \sup_{[a,b]} |f| \\ &= K|y-x| \end{aligned}$$

So the second part is done.
Now given $x \in [a,b], \epsilon > 0$, letting $\delta = \frac{\epsilon}{k}$, we have

$$\forall y \in [a,b], |y-x| < \delta \implies |F(y) - F(x)| < \epsilon.$$

So F is continuous. $\square$

Theorem. (Fundamental theorem of Calculus)
Let a, b, f, F be as in the previous theorem. If $c \in [a, b]$ and f is continuous at c, then F is differentiable at c, and

$$F'(c) = f(c)$$

Note: if $c = a$,

$$F'(a) = \lim_{h \to 0^+} \frac{F(a+h) - F(a)}{h}$$

and if $c = b$,

$$F'(b) = \lim_{h \to 0^-} \frac{F(b+h) - F(b)}{h}.$$

Proof. Given $\epsilon > 0$, there is a $\delta > 0$ s.t.

$$\forall t \in [a, b] \, |t - c| < \delta \implies |f(t) - f(c)| < \epsilon$$

Now we have

$$\begin{aligned} \left|\frac{F(c+h) - F(c)}{h} - f(c)\right| &= \left|\frac{1}{h}\int_c^{c+h} f(t)\,dt - f(c)\right| \\ &= \left|\frac{1}{h}\int_c^{c+h} (f(t) - f(c))\,dt\right| \\ &\leq \frac{1}{|h|}|h| \cdot \sup\{|f(t) - f(c)| : c \leq t \leq c+h\} \\ &\leq \epsilon \end{aligned}$$

Whenever $0 < |h| < \delta$ (provided $c + h \in [a, b]$). □

Let $a < b$, f, F be functions on $[a, b]$. We say F is an *antiderivative of* f *on* $[a, b]$ if F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

Corollary. Let $a < b$, f a continuous function on $[a, b]$. Then f has an antiderivative F on $[a, b]$. Moreover, if G is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)\,dt = G(b) - g(a)$$

Proof. For the first part, just take F to be an indefinite integral of f on $[a, b]$.
For the second part, we have

$$(F - G)'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$$

for all x.
So (by mean value theorem) $F - G$ is a constant. Hence

$$G(b) - G(a) = F(b) - F(a) = \int_a^b f(t)\,dt$$

by definition of indefinite integrals. □

Remark. • this corollary shows that the differential equation

$$\frac{dy}{dt} = f$$

has a solution when f is continuous, and it is unique up to a constant.
So given $y_0 \in \mathbb{R}$, the initial value problem

$$\begin{cases} \frac{dy}{dt} = f \\ y(a) = y \end{cases}$$

has a unique solution.
• this corollary provides a way for computing

$$\int_a^b f(t)\, dt$$

when f is continuous.

Theorem. Let $a < b$, f integrable on $[a, b]$. Assume f has an antiderivative G. Then

$$\int_a^b f(t)\, dt = G(b) - G(a)$$

Proof. By a previous corollary(5) we know that there exists a sequence $\mathcal{D}_n$ of dissections of $[a, b]$ such that $S_{\mathcal{D}_n}(f)$ and $\mathcal{S}_{\mathcal{D}_n}(f)$ both converge to $\int_a^b f(t)\, dt$. Say

$$\mathcal{D}_n : a = x_0^{(n)} < x_1^{(n)} < \ldots < x_{m_n}^{(n)} = b$$

for some $m_n \in \mathbb{N}$.
Apply mean value theorem to G on $\left[x_{k-1}^{(n)}, x_k^{(n)}\right]$ we get, there exists $\xi_k^{(n)} \in \left(x_{k-1}^{(n)}, x_k^{(n)}\right)$ such that

$$\frac{G\left(x_k^{(n)}\right) - G\left(x_{k-1}^{(n)}\right)}{x_k^{(n) - x_{k-1}^{(n)}}} = G'\left(\xi_k^{(n)}\right) = f\left(\xi_k^{(n)}\right)$$

So

$$\sum_{k=1}^{m_n} f\left(\xi_k^{(n)}\right)\left(x_k^{(n)} - x_{k-1}^{(n)}\right) = \sum_{k=1}^{m_n}\left(G\left(x_k^{(n)}\right) - G\left(x_{k-1}^{(n)}\right)\right) = G(b) - G(a)$$

Then by corollary 5, $LHS \to \int_a^b f(t)\, dt$ as $n \to \infty$. $\square$

Remark. Let f, G be as in the previous theorem. Then

$$\int_a^x f(t)\, dt = G(x) - G(a)$$

for all $x \in [a, b]$.
So any indefinite integral of f must be differentiable.

Corollary. Let $a < b$, f, g be integrable functions on $[a, b]$. Assume F, G are antiderivatives of f, g respectively on $[a, b]$. (eg this happens if f, g are continuous) Then

$$\int_a^b fG = G(b)F(b) - G(a)F(a) - \int_a^b Fg$$

Proof. Let

$$H(x) = F(x)G(x)$$

for $x \in [a, b]$. By product rule, H is differentiable, and

$$H'(x) = f(x)G(x) + F(x)g(x)$$

RHS is integrable. Thus

$$H(b) - H(a) = \int_a^b (fG + Fg)$$

rearrange to get the desired result. □

Corollary. (Change of variable)
Let $a < b$, $\varphi : [a, b] \to \mathbb{R}$ be continuously differentiable. Let f be a continuous function on the closed bounded interval $\varphi([a, b])$. Then

$$\int_{\varphi(a)}^{\varphi(b)} = \int_a^b f(\varphi(t))\varphi'(t)\,dt$$

Remark. As φ is continuous, there exists $c, d \in [a, b]$ such that $\varphi(c) \le \varphi(x) \le \varphi(d)$ for all $x \in [a, b]$. Then by IVT,

$$\varphi([a, b]) = [\varphi(c), \varphi(d)]$$

We do not assume that $\varphi(a)$, $\varphi(b)$ are the end points of this interval.

Proof. Let F be an antiderivative of f on $\varphi([a, b])$ (this exists by a previous corollary(17)). By chain rule,

$$(F \circ \varphi)'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$$

ans is continuous. By corollary 17,

$$\int_a^b f(\varphi(t))\varphi'(t)\,dt = F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy$$

□

Note that this corollary remains true if φ is differentiable, φ' is integrable, f is integrable, and f has antiderivative.
(need: φ continuous, f integrable $\implies$ $f \circ \varphi$ integrable).

Theorem. (Taylor's theorem with the integral remainder)
Assume $a, \delta \in \mathbb{R}$, $\delta > 0$, $f : (a-\delta, a+\delta) \to \mathbb{R}$ is n times continuously differentiable. Then for all $h \in (-\delta, \delta)$,

$$f(a+h) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} h^k + \frac{1}{(n-1)!} \int_0^h (h-t)^{n-1} f^{(n)}(a+t)\, dt$$

Proof. Induction on n:
$n = 1$:

$$\begin{aligned} RHS &= f(a) + \int_0^h f'(a+t)\, dt \\ &= f(a) + f(a+h) - f(a) \\ &= LHS. \end{aligned}$$

$n \geq 1$: assume result for n. Then

$$\begin{aligned} &\frac{1}{n!} \int_0^h (h-t)^n f^{(n+1)}(a+t)\, dt \\ &= \left[\frac{(h-t)^n}{n!} f^{(n)}(a+t) \right]_0^h + \frac{1}{n!} \int_0^h n(h-t)^{n-1} \cdot f^{(n)}(a+t)\, dt \\ &= -\frac{h^n}{n!} f^{(n)}(a) + \frac{1}{(n-1)!} \int_0^h (h-t)^{n-1} f^{(n)}(a+t)\, dt \\ &= -\frac{h^n}{n!} f^{(n)}(a) + \left(f(a+h) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} h^k \right) \end{aligned}$$

rearrange to get the desired results. $\square$