Category Theory

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0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

1 Definitions an examples

Definition. (1.1)
A category $\mathcal{C}$ consists of:
• (a) a collection $\operatorname{ob}\mathcal{C}$ of *objects* A, B, C;
• (b) a collection $\operatorname{mor}\mathcal{C}$ of *morphisms* f, g, h;
• (c) two operations domain, codomain assigining to each $f \in \operatorname{mor}\mathcal{C}$, a pair of objects, it's *domain* and *codomain*m we write $A \xrightarrow{f} B$ to mean f *is a morphism and* $\operatorname{dom} f = A, \operatorname{cod} f = B$;
• (d) an operation assigning to each $A \in \operatorname{ob}\mathcal{C}$ a morphism $A \xrightarrow{1_A} A$;
• (e) a partial binary operation $(f, g) \to fg$ on morphisms, such that fg is defined iff $\operatorname{dom} f = \operatorname{cod} g$, and $\operatorname{dom}(fg) = \operatorname{dom} g$, $\operatorname{cod}(fg) = \operatorname{cod}(f)$ if fg is defined, satisfying:
• (f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$;
• (g) $(fg)h = f(gh)$ whenever fg and gh are defined.

Remark. (1.2)
(a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call $\mathcal{C}$ *small* (nice name?) if $\operatorname{ob}\mathcal{C}$ and $\operatorname{mor}\mathcal{C}$ are sets.
(b) Some texts say fg means f followed by g, i.e. fg is defined iff $\operatorname{cod} f = \operatorname{dom} g$.
(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

Example. (1.3)
(a) The category *set* has all sets as objects, and all functions between sets as morphisms.
Strictly speaking, morphisms $A \to B$ are pairs (f, B) where f is a set-theoretic function. (?)
(b) The category *Gp* has all groups as objects, group homomorphisms as morphisms.
Similarly, *Ring* is the category of rings, Mod_R is the category of R-modules.
(c) The cateogry Top has all topological spaces as objects, and continuous functions as morphisms.
Similarly, $Unif$ has all uniform spaces and uniformly continuous functions as morphisms, Mf has all manifolds and smooth maps correspondingly.
(d) The category *Htpy* has the same objects as Top, but morphisms are homotopy classess of continuous functions. More generally, given $\mathcal{C}$, we call an equivalence relation $\simeq$ on $\operatorname{mor}\mathcal{C}$ a *congruence* if $f \simeq g \implies \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathcal{C}/\simeq$ with the same objects as $\mathcal{C}$, but congruence classes as morphisms instead.
(e) Given $\mathcal{C}$, the *opposite category* C^{op} has the same objects and morphisms as $\mathcal{C}$, but dom and cod are interchanged, and fg in $\mathcal{C}^{op}$ is gf in $\mathcal{C}$.
This leads to the *duality principle*: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.
(f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (nice abbreviation) with one object in which

every morphism is an isomorphism (i.e. for all $f, \exists g$ s.t. fg and gf are identities).
(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space X, the *fundamental groupoid* $\pi(x)$ has all points of X as objects, and morphisms $x \to y$ are homotopy classes $rel\{0,1\}$ of paths $u : [0,1] \to X$ with $u(0) = x$, $u(1) = y$ (if you know how to prove that the fundamental group is a group, you can prove that $\pi(x)$ is a groupoid).
(h) A *discrete* cat is one whose only morphism are identities.
A *preorder* is a cat $\mathcal{C}$ in which, for any pair (A, B), $\exists$ at most 1 morphism $A \to B$.
A small preoder is a set equipped with a binary relation which is reflexive and transitive.
In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.
(i) The category *Rel* has the same objects a *set* (???), but morphisms $A \to B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define $S \cdot R = \{(a, c) \in A \times C | (\exists b \in B)((a, b) \in R, (b, c) \in S)\}$.
The identity $1_A : A \to A$ is $\{(a, a) | a \in A\}$.
Similarly, the category $Part$ are for sets and partial functions (i.e. relations s.t. $(a, b) \in R$ and $(a, b') \in R \implies b = b'$).
(j) Let K be a field. The cateogry Mat_K has natural numbers as objects, and morphism $n \to p$ are $(p \times n)$ matrices with entries from K. Composition is matrix multiplication.
(k) We write *Cat* for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

Definition. (1.4)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:
(a) a mapping $A \to FA$ from $\operatorname{ob}\mathcal{C}$ to $\operatorname{ob}\mathcal{D}$;
(b) a mapping $f \to Ff$ from $\operatorname{mor}\mathcal{C}$ to $\operatorname{mor}\mathcal{D}$
such that $\operatorname{dom}(Ff) = F(\operatorname{dom} f)$, $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$, $1_{FA} = F(1_A)$, and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Example. (1.5)
(a) We have *forgetful functors* $Gp \to Set$, $Ring \to Set$, $Top \to Set$, $Ring \to AbGp$ (forget $\times$), $Ring \to Monoid$ (forget $+$).
(b) Given a set A, the free group FA has the property:
Given any group G and any function $A \xrightarrow{f} UG$ (?), there's a unique homomorphism $FA \xrightarrow{F} G$ extending f.
F is a functor $Set \to Gp$: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \leftrightarrow UFB$ Functionality follows from uniqueness given $B \xrightarrow{f} C$. $F(gf)$ and $(Fg)(Ff)$ are both momoms extending $A \xrightarrow{f} B \xrightarrow{g} C \to UFC$.
(c) Given a set A, we write PA for the set of all subsets of A.
We can make P into a functor $Set \to Set$, given $A \xrightarrow{f} B$, we defined $Pf(A') = \{f(a) | a \in A'\}$ for $A' \subseteq A$.
But we also have a functor $P^* : Set \to Set^{op}$ defined on objects by P, but $P^*f(B') = \{a \in A | f(a) \in B'\}$ for $B' \subseteq B$.
By a *contravariant* functor $\mathcal{C} \to \mathcal{D}$, we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or $\mathcal{C}^{op} \to \mathcal{D}$).

A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).
(d) Let K be a field. We have a functor $* : Mod_K \to Mod_K^{op}$ defined by $V^* = \{$ linear maps $V \to K\}$, and if $V \xrightarrow{f} W$, $f^*(\theta : W \to K) = \theta f$.
(e) We have a functor $op : Cat \to Cat$, which is the identity on morphisms (note that this is a covariant).
(f) A functor between monoids is a monoid homomorphism.
(g) A functor between posets is an order-preserving map.
(h) Let G be a group. A functor $F \circ G \to Set$ consists of a set $A = F*$ together with an action of G on A, i.e. a *permutation representation* of G.
Similarly, a funcotr $G \to Mod_K$ is a K-linear representation of G. (i) The construction of the fundamental group $\pi(X, X)$ of a space X with basepoint X is a functor $Top* \to Gp$ where $Top*$ is the category of spaces with a chosen basepoint.
Similarly, the fundamental groupoid is a functor $Top \to Gpd$, where Gpd is the category of groupoids and functors between them.

Definition. (1.6)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ (why two arrows?) two functors. A *natural transformation* $\alpha : F \to G$ consists of an assignment $A \to \alpha_A$ from $\text{ob}\, C$ to $\text{mor}\, \mathcal{D}$, such that $\text{dom}_{\alpha_A} = FA$ and $\text{cod}_{\alpha A} = GA$ for all A, and for all $A \xrightarrow{f} B$ in $\mathcal{C}$, the square

$$\begin{array}{ll} FA & \xrightarrow{Ff} FB \\ \downarrow \alpha_A & \quad \downarrow \alpha_B \\ GA & \xrightarrow{Gf} GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)_{\alpha A}$).

(1.3) (l) Given categories $\mathcal{C}$ and $\mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

Example. (1.7)
(a) Let K be a field, V a vector space over K. There is a linear map $\alpha_V : V \to V^{**}$ given by $\alpha_V(v)\theta = \theta(v)$ for $\theta \in V^*$.
This is the V-component of a natural transformation $1_{Mod_K} \to ** : Mod_K \to Mod_K$.
(b) For any set A, we have a mapping $\sigma_A : A \to PA$ sending a to $\{a\}$. If $f : A \to B$, then $Pf\{a\} = \{f(a)\}$. So σ is a natural transformation $1_{Set} \to P$.
(c) Let F:$Set \to Gp$ be the free group functor (1.5(b)), and $U : Gp \to Set$ the forgetful functor. The inclusions $A \to UFA$ form a natural transformation $1_{Set} \to UF$.
(d) Let G, H be groups and $f, g : G \rightrightarrows H$ be two homomorphisms. A natural transformation $\alpha : f \to g$ corresponds to an element $h = \alpha_*$ of H, s.t. $hf(x) \to g(x)h$ for all $x \in G$ or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
(e) Let A and B be two G-sets, regarded as functors: $G \rightrightarrows Set$. A natural transformation $A \to B$ is a function f satisfying $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$, i.e. a G-equivariant map.

Lemma. (1.8)

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors, and $\alpha : F \to G$ a natural transforms. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each α_A is an isomorphism in $\mathcal{D}$.

Proof. Forward is trivial (ok, I'll check this later). For backward, suppose each α_A has an inverse β_A. Given $f : A \to B$ in $\mathcal{C}$, we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

□

commutes. But as α is natural,

$$(Ff)\beta_A = \beta_B \alpha_B (Ff) \beta_A = \beta_B (Gf) \alpha_A \beta_A = \beta_B (Gf)$$

Definition. (1.9)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. By an *equivalence* between $\mathcal{C}$ and $\mathcal{D}$, we mean a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together wish natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$.
We write $\mathcal{C} \cong \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent.
We say a property P of categories is a *categorical property* if whenever $\mathcal{C}$ has P and $\mathcal{C} \cong \mathcal{D}$, then $\mathcal{D}$ has P.
For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

Example. (1.10)
(a) The category $Part$ is equivalent to the category Set_* of pointed sets (and basepoint, pre(orders?) as functions):
• We define $F : Set_* \to Part$ by $F(A, a) = A \setminus \{a\}$, and if $f : (A, a) \to (B, b)$, then $Ff(x) = f(x)$ if $f(x) \neq b$, and undefined otherwise;
• and $G : Part \to Set_*$ by $G(A) = A^+ = A \cup \{A\}$, and if $f : A \to B$ is a partial function, we define $Gf : A^+ \to B^+$ by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ defined, and equals B otherwise.
The composite FG is the identity on $Part$, but GF is not the identity. However, there is an isomoprhism $(A, a) \to ((A \setminus \{a\})^+, A \setminus \{a\})$ sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.
Note that there can be no isomoprhism from Set_* to $Part$, since $Part$ has a 1-element isomorphism class $\{\phi\}$ but Set_* doesn't.
(b) The category $fdMod_K$ of finite-dimensional vector spaces over K is equivalent to $fdMod_K^{op}$, the functors in both directions are $(-1)^*$ (???) and both isomorphisms are the natural transformations of 1.7(a).
(c) $fdMod_K$ is also equivalent to Mat_K (1.3(j)).
We define $F : Mat_K \to fdMod_K$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A w.r.t. the standard bases of K^n and K^p.
To define $G : fdMod_K \to Mat_K$, choose a basis for each finite dimensional vector space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ to be the matrix representing f w.r.t. chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n; $FG \neq 1$, but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \to V$ for each V, which form a natural isomorphism.