Stochastic Financial Models

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0 Motivation

An investor needs a certain quantity of a share (or currency, good, etc), however, not right now ($t = 0$) but at a later time ($t = 1$). The price of the share $S(w)$ at time $t = 1$ is random, but already today one has to make calculation with it so there is risk. For example, 500USD $\approx$ 370 GBP today. What about in one year?

Possible solution: purchase a financial derivative such as:
• forward contract: right and obligation to buy a share at time $t = 1$ for a strike price K specified at time $t = 0$. Its value at time $t = 1$ should be $H(w) = S(w) - K$ is positive if $S(w) > K$, and negative i f $S(w) < K$;
• call-option: the right, but no obligation, to do the same thing as above. Its value at $t = 1$ should be $H(w) = (S(w) - K)^+$, i.e. $S(w) - K$ if that is positive, and 0 otherwise (no obligation to exercise the option).

One question: what is the fair price for such a derivative?

1) Classical approach: Regard payoff $H(w)$ as lottery, modelled by a random variable on $(\Omega, F, \mathbb{P})$, where $\mathbb{P}$ is the 'objective probability measure'.
a) very classical: fair price = expected discounted payoff = $\mathbb{E}[\frac{H}{1+r}]$, where r is the interest rate for funds/loans from $t = 0$ to $t = 1$.
Assumption: both interest rates are the same for large investors.
b) classical: subjective assessment of the risk (by the seller of H) by utility functions.

2) More modern approach: suppose the primary risk (share) can only be traded in $t = 0$ and $t = 1$.
Hedging strategy: θ^1 = number of shares held between $t = 0$ and $t = 1$;
θ^0 = balance on bank account with interest rate r.
Here we allow θ^1 to be either positive or negative (i.e. allow short-selling).

Price at $t = 0$: $\theta^0 + \theta^1\pi^1 = V_0$, where π^1 is the price of one share at $t = 0$.
The value of this portfolio at $t = 1$: $\theta^0(1 + r) + \theta^1 S(w) = V(w)$.

Requirement: value of derivative = value of strategy, $H(w) = V(w)$ for all $w \in \Omega$.

For example, for forward contract: $S(w) - K = V(w) = \theta^0(1 + r) + \theta^1 S(w)$, we should choose $\theta^1 = 1, \theta^0 = \frac{-K}{1+r}$, so $V_0 = \pi^1 - \frac{K}{1+r}$. The seller of H has no risk if he uses this strategy(?).

Even more, $\pi(H) = V_0$ is the unique fair price for the forward contract. Any other price $\tilde{\pi} \neq V_0$ would lead to *arbitrage*: a riskless opportunity to make profit, which should be excluded). For example, if $\tilde{\pi} > V_0$, at $t = 0$ sell forward for $\tilde{\pi}$ and buy the strategy for V_0. Then in $t = 1$ deliver share and repay the loan. We gain a pure profit at $t = 1$: $(\tilde{\pi} - V_0)(1 + r) > 0$, i.e. arbitrage.

Questions: how to characterize arbitrage-free market? How to determine fair prices of options and derivatives?

1 Utility and mean variance

The market is interaction of agents trading goods. Individual agents have preferences over different contingent(?) claims (=specified random payment). Agents' preferences are expressed by an expected utility representation. Y is preferred to X means $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$ with utility function $U : \mathbb{R} \to [-\infty, \infty)$ which is non-decreasing. We assume U to be concave, in the sense that we expect agents to dislike risks.

Definition. (1.1) A function $U : \mathbb{R} \to [-\infty, \infty)$ is *concave* if $\forall p \in [0,1]$, $pU(x) + (1-p)U(y) \leq U(px + (1-p)y)$. Let $P(U) = \{x : U(x) > -\infty\}$.

Remark. (1.2)
(a) If U is concave, then $-U$ is convex;
(b) Jensen's inequality: $\mathbb{E}[U(X)] \leq U(\mathbb{E}[X])$. Note that this means if an agent is offered a contingent claim X (a random variable) and a certain payment $\mathbb{E}[X]$, then the agent prefers the certain payment. This shows that concave utility functions implies risk-aversion.
If U is linear, the agent is risk neutral as it doesn't matter between the two forms of payment; if U is convex then the agent will be risk friendly, since X is now preferred to $\mathbb{E}[X]$.
(c) If $U(x) = -\infty$ then the outcome x is completely unacceptable.

Example. (1.3)
Some example of utility functions:
(1) $U(x) = -e^{-\gamma x}$, $\gamma > 0$ is the constant absolute risk aversion (CARA) utility.
(2)

$$U(x) = \begin{cases} \frac{x^{1-R}}{1-R} & x \geq 0 \\ -\infty & x < 0 \end{cases}$$

where $R > 0$, $r \neq 1$ is the constant relative risk aversion (CRRA) utility.
(3)

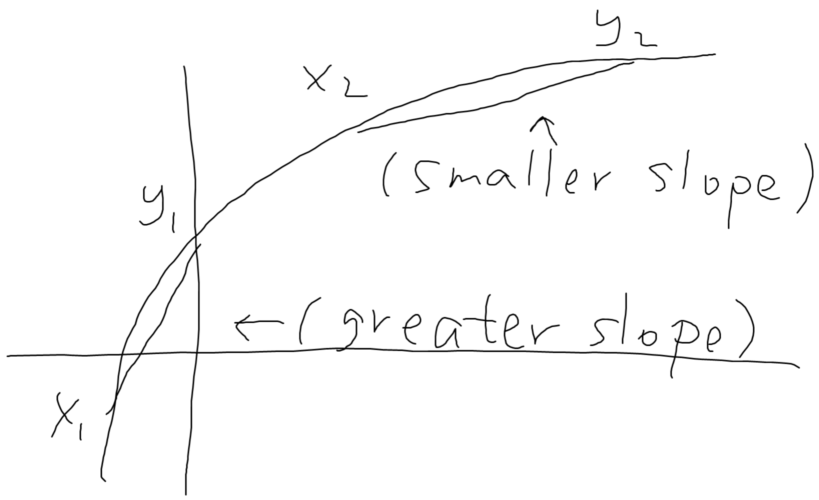
$$U(x) = \begin{cases} \log x & x > 0 \\ -\infty & x \leq 0 \end{cases}$$

is the logarithmic utility.
(4) $U(x) = \min(x, \alpha x)$, $\alpha \in [0, 1)$.
(5) $U(x) = -\frac{1}{2}x^2 + \alpha x$, $\alpha \geq 0$. This is concave, but not increasing.
(6) U_1, U_2 are utilities, $\alpha_1, \alpha_2 \geq 0$, then $\alpha_1 U_1 + \alpha_2 U_2$ is also a utility.
(7) If $\{U_\lambda, \lambda \in \Lambda\}$ is a family of utilities, then $U(x) = \inf_{\lambda \in \Lambda} U_\lambda(x)$ is a utility.

Proposition. (1.4)
$U : \mathbb{R} \to [-\infty, \infty)$ is concave if and only if

$$\frac{U(y_1 - U(x_1)}{y_1 - x_1} \geq \frac{U(y_2) - U(x_2)}{y_2 - x_2} \forall x_1 < y_1 \leq x_2 < y_2 \quad (1.1)$$

. See the graph below for intuition:



Proof. Let U be concave. It's enough to show the above equation with $y_1 = x_2$ (then apply it twice), so it suffices to show that

$$\begin{aligned}
&\frac{U(z) - U(x)}{z - x} \geq \frac{U(y) - U(z)}{y - z} \forall x < z < y \\
\iff &(y - x)U(z) \geq (z - x)U(y) + (y - z)U(x) \\
\iff &\underbrace{(U(z)}_{\geq pU(y)+(1-p)U(x)} \geq \underbrace{\frac{z - x}{y - x}}_{=p\in[0,1]} U(y) + \underbrace{\frac{y - z}{y - x}}_{=1-p} U(x) \ (1.2)
\end{aligned}$$

which holds by concavity.

Conversely, if (1.1) holds then (1.2) also holds which implies concavity. □

Corollary. (1.5)
(1) for concave U and $z \in intD(u)$, the interior of regions that U is differentiable, the left and right hand derivatives

$$U'_{-}(z) = \lim_{x\to z-} \frac{U(z) - U(x)}{z - x}$$

and

$$U'_{+}(z) = \lim_{y\to z+} \frac{U(y) - U(z)}{y - z}$$

exist. Both U'_{+} and U'_{-} are decreasing functions and they satisfy $U'_{-} \geq U'_{+}$.
(2) If $U \in C^2(\mathbb{R})$, then $U''(x) \leq 0 \forall x$ if and only if U is concave.

Assume now onward that all utilities are strictly increasing.

Now an agent with wealth w and C^2 utility U will accept a contingent claim X provided that

$$\mathbb{E}[U(w + X)] > U(w)$$

i.e. his expected utility increases after accepting the contingent claim. If X is asmall, by Taylor expansion we have approximately $U(w) + U'(w)\mathbb{E}[X] + \frac{1}{2}U''(w)\mathbb{E}[X^2] > U(w)$. Recall that $U'(w) > 0$ and $U''(w) < 0$, so the agent benefits from a positive $\mathbb{E}[X]$, and gets a disadvantage from $\mathbb{E}[X^2]$, sort of the variance. This corresponds to the real-life scenario where agents like larger expected gain but dislike larger risk.

The above is balanced if

$$\frac{2\mathbb{E}[X]}{\mathbb{E}[X^2]} = -\frac{U''(w)}{U'(w)}$$

, the *Arrow-Pratt coefficient(measure) of absolute risk-aversion.*

Similarly, if the proposed gamble is multiplicative instead of additive, i.e. wealth become $w(1+X)$ upon accepting the contingent claim, then the agent will accept it if $\mathbb{E}[U(w(1+X))] > U(w)$. By Taylor expansion we get, if $w > 0$, the agent should accept if

$$\frac{2\mathbb{E}[X]}{\mathbb{E}[X^2]} \geq -\frac{wU''(w)}{U'(w)}$$

, the *Arrow-Pratt coefficient of relative risk-aversion.*

Check by substitution how these relate to the examples (1) and (2) in (1.3), which explains the name of them (CARA and CRRA).

1.1 Reservation and marginal prices

If an agent can choose claims from an admissible set A, he will try to achieve $\sup_{X\in A} \mathbb{E}[U(X)]$.

Suppose the sup is attained in $X^* \in A$ and suppose that A is an affine space of the form $A = X^* + V$ where V is a vector space. Then, $\forall \xi \in V$, $t \in \mathbb{R}$, $\mathbb{E}[U(X^*)] \geq \mathbb{E}[U(X^* + t\xi)]$ and differentiating w.r.t. t gives

$$\mathbb{E}[U'(X^*)\xi] = 0 \ (1.3)$$

for all $\xi \in V$.