Topics in Set Theory

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Contents

0 Introduction

—Lecture 1—

Example classes: 4th Feb, 18th Feb, 4th Mar 330-5pm MR5; fourth class undecided (probably on 15th).

Although the name of this course is *Topics* in Set theory, for all of its history only one topic is discussed. So maybe this course should be called *One Topic in Set Theory*, or probably just *the Continuum Hypothesis*: in this course we'll just solve one problem: the continuum problem, which we've known in the end that the problem is independent from ZFC.

Let's have some background stories first. In the second ICM congress (1900, Paris), Hilbert posed the famous 23 Hilbert questions, with the first one being the Continuum Hypothesis (a hypothesis at that time). The original formulation of CH was:

Any infinite subset of real numbers is either equinumerous to the set of natural numbers, or to the set of real numbers.

We could definitely formulate it better, but that is less important. More modern version of CH would be a short equation

$$2^{\aleph_0} = \aleph_1$$

which seemingly has nothing to do with the previous problem. However, in ZFC these two statements are equivalent:
• if $2^{\aleph_0} > \aleph_1$, in particular, $2^{\aleph_0} \geq \aleph_2$. Since $2^{\aleph_0} \sim \mathbb{R}$, we get an injection $i : \aleph_2 \to \mathbb{R}$. Consider $X := i[\aleph_1] \subseteq \mathbb{R}$. Clearly, $i \upharpoonright \aleph_1$ (i restricted to $\aleph_1$) is a bijection between $\aleph_1$ and X, so $X \sim \aleph_1$; but $\aleph_1$, being uncountable, is not in bijection with natural numbers, and is not in bijection with real numbers. Thus X refutes CH.

If $2^{\aleph_0} = \aleph_1$, let $X \subseteq \mathbb{R}$. Consider $b : 2^{\aleph_0} \to \mathbb{R}$ a bijection. If X is infinite, then $b^{-1}[X] \subseteq 2^{\aleph_0}$. Thus the cardinality of X is either $\aleph_0$ or $\aleph_1$ (which $\sim \mathbb{N}$ and $\mathbb{R}$ respectively). So $2^{\aleph_0} = \aleph_1 \implies$ CH.

In 1938, Gödel proved that ZFC does not prove ¬CH, and in 1961 Cohen proved that ZFC does not prove CH, by methods of *inner models* and *forcing* (sometimes also called *outer models*, which is not incorrect) respectively. The latter has become the most important method in Set theory since then.

From logic (see Part II Logic and Set Theory) we have Gödel's Completeness Theorem: a theory T is consistent iff it has a model. So from the above two statements, it seems that we're going to prove that there are models for ZFC+CH and ZFC+¬CH; but this is obviously not possible because of the incompleteness phenomenon: we know we can't prove the consistency of ZFC (as a result, we can't even prove there is a model of ZFC)! So instead we could only prove the following:

$$Cons(ZFC) \to Cons(ZFC + CH)$$

or equivalently, if $M \vDash ZFC$, then there is $N \vDash ZFC + CH$ (and similar for the other half).

1 Model theory of set theory

For a moment, we will assume that we have a model $(M, \in) \models ZFC$. Unfortunately this first assumption doesn't make much sense, because model theory is based on set theory and we don't have anything if ZFC is inconsistent. We refer to the canonical objects in M by the usual symbols, e.g. $0, 1, 2, 3, 4, ..., \omega, \omega+1, ...$.

What would an *inner model* be? Take $A \subseteq M$, and consider $(A, \in)$. It is a substructure of $(M, \in)$, because there are no function symbols or constant symbols in the language of set theory. This might be counterintuitive, because we're using symbols like ϕ and $\{\cdot\}$ all the time! However, these are technically not part of language of set theory as they can all be defined without any use of function symbols, i.e. they are just abbreviations. For example, $X = \phi$ abbreviates $\forall w(\neg w \in X)$; $X = \{Y\}$ abbreviates $\forall w(w \in X \iff w = Y)$, and similarly for $\cup$ and $\mathcal{P}$; and also for relation symbols such as $\subseteq$, which abbreviates $\forall w(w \in X \to w \in Y)$. Note that $X = \phi$ is NOT the formula that it looks like; in particular, it is not quantifier free (because it abbreviates $\forall w(\neg w \in X)$)! So we need to take extra care when we do things in this course.

Definition. If φ is a formula in n free variables, we say φ is *upwards absolute* between A and M if for all $a_1, ..., a_n \in A$,

$$(A, \in) \models \varphi(a_1, ..., a_n) \implies (M, \in) \models \varphi(a_1, ..., a_n)$$

and we say φ is *downwards absolute* between A and M if for all $a_1, ..., a_n \in A$,

$$(M, \in) \models \varphi(a_1, ..., a_n) \implies (A, \in) \models \varphi(a_1, ..., a_n)$$

and φ is *absolute* between A and M if it is both upwards and downwards absolute.

Observation:
(a) If φ is *quantifier-free*, then φ is absolute between A and M. But this doesn't really help much, because almost nothing is quantifier-free: without quantifiers we can only say things like $A \in B$ and $A = B$, and conjunctions of those; that's pretty much all.
(b) We say that a formula is Σ_1 if it is of the form

$$\exists x_1 ... \exists x_n \varphi(x_1, ..., x_n)$$

where φ is q.f.;
we say a formula is Π_1 if it is of the form

$$\forall x_1 ... \forall x_n \varphi(x_1, ..., x_n)$$

where φ is q.f..
(c) If φ is Π_1, it is downward absolute; if it's Σ_1 then it is upwards absolute. So in particular, note that $X = \phi$ is downward absolute.

—Lecture 2—

As an example, write $0, 1, 2, 3, ...$ for the ordinals in M, and let $A := M \setminus \{1\}$. In A, we have $0, 2$, but no 1; we also have $\{1\}$. If we use $\Phi_0(x)$ to denote the

formula $\forall w(\neg w \in x) \iff x = \phi$. Clearly $(M, \in) \vDash \Phi_0(0)$, so by π_1-downwards absoluteness, $(A, \in) \vDash \Phi_0(0)$.
Now, how many elements does $2 = \{0, 1\}$ have? In M we obviously know 2 has 2 elements; but in A, 2 only has one element 0, and $\{1\}$ has no element: $(A, \in) \vDash \Phi_0(\{1\})$! Clearly $(M, \in) \nvDash \Phi_0(\{1\})$, so Φ_0 is not absolute between A and M. As a corollary, we get $(A, \in) \nvDash$ extensionality (we can uniquely specify sets by specifying their elements).

Remark. We could go on, defining formulas $\Phi_1(x), \Phi_2(x)$, etc to analyse which of the elements correspond to the natural numbers in A.

Reminder (from Part II Logic and Set Theory): we say A is *transitive in M* if for all $a \in A$ and $x \in M$ s.t. $(M, \in) \vDash x \in a$, we have $x \in A$. The problem for the above A is that it is not transitive. As long as that is fixed, we have the following:

Proposition. If A is transitive, then Φ_0 is absolute between A and M.

Proof. Since Φ_0 is Π_1, we only need to show upwards absoluteness. Suppose $a \in A$ s.t. $(A, \in) \vDash \Phi_0(a)$, and suppose for contradiction that $a \neq 0$. Then there is some $x \in a$. By transitivity, $x \in A$. But then $\Phi_0(a) : \forall w(w \notin a)$ is not true in $(A, \in)$. $\square$

Similarly, if Φ_n is the formula describing the natural number n, and there is $a \in A$ s.t. $(A, \in) \vDash \Phi_n(a)$, and A is transitive, then $a = n$.

Proposition. If A is transitive in M, then $(A, \in) \vDash$ extensionality.

Proof. Take $a, b \in A$ with $a \neq b$. So by extensionality in $(M, \in)$, find, WLOG some $c \in a \setminus b$. Since $c \in a \in A$, by transitivity $c \in A$. Note that all of these quantifier-free formulas are absolute, so $(A, \in)$ also models them; in particular, $(A, \in) \vDash c \in a, c \notin b$. So a, b do not satisfy the assumptions of extensionality. $\square$

Consider now $A = \omega + 2 = \{0, 1, 2, ..., \omega, \omega + 1\} \subseteq M$. This is clearly transitive subset of M because it's an ordinal. So $(A, \in) \vDash$ extensionality, but clearly it isn't anything like a model of set theory as it is too thin. Consider the formula $x = \mathcal{P}(y)$. Unfortunately, this is not a formula, as $\mathcal{P}$ is undefined. We have to expand it properly:

$$
\begin{aligned}
& x = \mathcal{P}(y) \\
\iff & x = \{z; z \subseteq y\} \\
\iff & \forall w(w \in x \leftrightarrow w \subseteq y) \\
\iff & \forall w(w \in x \leftrightarrow (\forall v(v \in w \to v \in y)))
\end{aligned}
$$

In A, what is $\mathcal{P}(\omega)$? We have $(A, \in) \vDash \omega + 1 = \mathcal{P}(\omega)$, which is obviously not what we want for $\mathcal{P}(\omega)$ to be.

Definition. (Bounded quantification)
We first define the notations $\exists v \in w\ \varphi$ to be $\exists v(v \in w \wedge \phi)$, and $\forall v \in w\ \varphi$ to be $\forall v(v \in w \to \phi)$, and we call these quantifiers *bounded.*
Now we say a formula φ is $\triangle_0$ if it is in the smallest set S of formulas with the following properties:
1. All q-f formulas are in S;
2. If $\varphi, \psi \in S$, then so are:

> 2a. $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$;
> 2b. $\neg\varphi$;
> 2c. $\exists x \in w\ \varphi$, $\forall v \in w\ \varphi$.

Theorem. If φ is $\triangle_0$ and A is transitive, then φ is absolute between A and M.

Proof. We already know that quantifier free formulas are absolute, and absoluteness is obviously preserved under propositional connectives. The only case left is (2c).
Let's just do $\varphi \to \exists v \in w\ \varphi = \exists v(v \in w \wedge \varphi)$. So suppose φ is absolute. We need to deal with downwards absoluteness: we have $(M, \in) \vDash \exists v \in a\varphi(v, a)$ for some $a \in A$, i.e. $(M, \in) \vDash \exists v(v \in a \wedge (\varphi(v, a))$.
Let's find $m \in M$ s.t. $(M, \in) \vDash m \in a \wedge \varphi(m, a)$.
Now $m \in a \in A$, so $m \in A$. By absoluteness of φ, we get $(A, \in) \vDash m \in a \wedge \varphi(m, a) \implies (A, \in) \vDash \exists v \in a\varphi(v, a)$. $\square$

Let T be any *set theory.* Then we say that φ is $\triangle_0^T$ if there is a $\triangle_0$ formula ψ s.t. $T \vdash \varphi \leftrightarrow \psi$. So we get, as a corollary:

Corollary. If A is transitive in M, and both $(M, \in)$ and $(A, \in)$ are models of T, then $\triangle_0^T$ formulas are absolute between A and M.
We may also define Σ_1^T formulas to be the formulas that are T-equivalent to $\exists v_1 ... \exists v_n \psi$ where ψ is $\triangle_0$, and similarly for Π_1^T formulas. So $\Sigma_1^T(\Pi_1^T)$ formulas are upwards(downwards) absolute between A and M respectively.

On Wednesday we will look at what formulas are actually in these classes.