Category Theory

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0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

1 Definitions and examples

Definition. (1.1)
A category $\mathcal{C}$ consists of:
(a) a collection $\mathrm{ob}\,\mathcal{C}$ of *objects* A, B, C;
(b) a collection $\mathrm{mor}\,\mathcal{C}$ of *morphisms* f, g, h;
(c) two operations domain, codomain assigining to each $f \in \mathrm{mor}\,\mathcal{C}$ a pair of objects, its *domain* and *codomain*; we write $A \xrightarrow{f} B$ to mean f *is a morphism and* $\mathrm{dom}\, f = A, \mathrm{cod}\, f = B$;
(d) an operation assigning to each $A \in \mathrm{ob}\,\mathcal{C}$ a morphism $A \xrightarrow{1_A} A$;
(e) a partial binary operation $(f, g) \to fg$ on morphisms, such that fg is defined iff $\mathrm{dom}\, f = \mathrm{cod}\, g$, and $\mathrm{dom}(fg) = \mathrm{dom}\, g$, $\mathrm{cod}(fg) = \mathrm{cod}(f)$ if fg is defined, satisfying:
(f) $f 1_A = f = 1_B f$ for any $A \xrightarrow{f} B$;
(g) $(fg)h = f(gh)$ whenever fg and gh are defined.

Remark. (1.2)
(a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call $\mathcal{C}$ *small* if $\mathrm{ob}\,\mathcal{C}$ and $\mathrm{mor}\,\mathcal{C}$ are sets.
(b) Some texts say fg means f followed by g, i.e. fg is defined iff $\mathrm{cod}\, f = \mathrm{dom}\, g$.
(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

Example. (1.3)
(a) The category **Set** has all sets as objects, and all functions between sets as morphisms.
Strictly speaking, morphisms $A \to B$ are pairs (f, B) where f is a set-theoretic function. (See part II logic and sets)
(b) The category **Gp** has all groups as objects, group homomorphisms as morphisms.
Similarly, **Ring** is the category of rings, $\mathbf{Mod_R}$ is the category of R-modules.
(c) The category **Top** has all topological spaces as objects, and continuous functions as morphisms.
Similarly, **Unif** has all uniform spaces and uniformly continuous functions as morphisms, **Mf** has all manifolds and smooth maps correspondingly.
(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classess of continuous functions. More generally, given $\mathcal{C}$, we call an equivalence relation $\simeq$ on $\mathrm{mor}\,\mathcal{C}$ a *congruence* if $f \simeq g \implies \mathrm{dom}\, f = \mathrm{dom}\, g$ and $\mathrm{cod}\, f = \mathrm{cod}\, g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathcal{C}/\simeq$ with the same objects as $\mathcal{C}$, but congruence classes as morphisms instead.
(e) Given $\mathcal{C}$, the *opposite category* C^{op} has the same objects and morphisms as $\mathcal{C}$, but dom and cod are interchanged, and fg in $\mathcal{C}^{op}$ is gf in $\mathcal{C}$.
This leads to the *duality principle*: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.
(f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (😸) with one object in which every morphism is an isomorphism (i.e. for all $f, \exists g$ s.t. fg and gf are identities).

(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space X, the *fundamental groupoid* $\pi(x)$ has all points of X as objects, and morphisms $x \to y$ are homotopy classes $rel\{0,1\}$ of paths $u : [0,1] \to X$ with $u(0) = x$, $u(1) = y$ (if you know how to prove that the fundamental group is a group, you can prove that $\pi(x)$ is a groupoid).
(h) A *discrete* cat is one whose only morphism are identities.
A *preorder* is a cat $\mathcal{C}$ in which, for any pair (A, B), $\exists$ at most 1 morphism $A \to B$.
A small preorder is a set equipped with a binary relation which is reflexive and transitive.
In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.
(i) The category **Rel** has the same objects as *set*, but morphisms $A \to B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define $S \cdot R = \{(a,c) \in A \times C | (\exists b \in B)((a,b) \in R, (b,c) \in S)\}$.
The identity $1_A : A \to A$ is $\{(a,a) | a \in A\}$.
Similarly, the category **Part** are for sets and partial functions (i.e. relations s.t. $(a,b) \in R$ and $(a,b') \in R \implies b = b'$).
(j) Let K be a field. The cateogry $\mathbf{Mat_K}$ has natural numbers as objects, and morphism $n \to p$ are $(p \times n)$ matrices with entries from K. Composition is matrix multiplication.
(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

Definition. (1.4)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:
(a) a mapping $A \to FA$ from $\operatorname{ob}\mathcal{C}$ to $\operatorname{ob}\mathcal{D}$;
(b) a mapping $f \to Ff$ from $\operatorname{mor}\mathcal{C}$ to $\operatorname{mor}\mathcal{D}$,
such that $\operatorname{dom}(Ff) = F(\operatorname{dom} f)$, $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$, $1_{FA} = F(1_A)$, and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Example. (1.5)
(a) We have *forgetful functors* U: **Gp** $\to$ **Set**, **Ring** $\to$ **Set**, **Top** $\to$ **Set**, **Ring** $\to$ **AbGp** (forget $\times$), **Ring** $\to$ **Mon** (Category of all monoids) (forget $+$).
(b) Given a set A, the free group FA has the property:
Given any group G and any function $A \xrightarrow{f} UG$ (?), there's a unique homomorphism $FA \xrightarrow{\bar{f}} G$ extending f. Here F is a functor **Set** $\to$ **Gp**: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \leftrightarrow UFB$. Functoriality follows from uniqueness given $B \xrightarrow{f} C$. $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \to UFC$.
(c) Given a set A, we write PA for the set of all subsets of A.
We can make P into a functor **Set** $\to$ **Set**, given $A \xrightarrow{f} B$, we defined $Pf(A') = \{f(a) | a \in A'\}$ for $A' \subseteq A$.
But we also have a functor $P^* :$ **Set** $\to$ $\mathbf{Set}^{op}$ defined on objects by P, but $P^* f(B') = \{a \in A | f(a) \in B'\}$ for $B' \subseteq B$.
By a *contravariant* functor $\mathcal{C} \to \mathcal{D}$, we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or $\mathcal{C}^{op} \to \mathcal{D}$).
A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

(d) Let K be a field. We have a functor $* : \mathbf{Mod_K} \to \mathbf{Mod_K}^{op}$ defined by $V^* = \{$ linear maps $V \to K\}$, and if $V \xrightarrow{f} W$, $f^*(\theta : W \to K) = \theta f$.
(e) We have a functor $op : \mathbf{Cat} \to \mathbf{Cat}$, which is the identity on morphisms (note that this is a covariant).
(f) A functor between monoids is a monoid homomorphism.
(g) A functor between posets is an order-preserving map.
(h) Let G be a group. A functor $F \circ G \to \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A, i.e. a *permutation representation* of G.
Similarly, a functor $G \to \mathbf{Mod_K}$ is a K-linear representation of G.
(i) The construction of the fundamental group $\pi(X, X)$ of a space X with basepoint X is a functor $\mathbf{Top}* \to \mathbf{Gp}$ where $\mathbf{Top}*$ is the category of spaces with a chosen basepoint.
Similarly, the fundamental groupoid is a functor $\mathbf{Top} \to \mathbf{Gpd}$, where $\mathbf{Gpd}$ is the category of groupoids and functors between them.

Definition. (1.6)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ (why two arrows?) two functors.
A *natural transformation* $\alpha : F \to G$ consists of an assignment $A \to \alpha_A$ from $\mathrm{ob}\,\mathcal{C}$ to $\mathrm{mor}\,\mathcal{D}$ (think about this), such that $\mathrm{dom}_{\alpha_A} = FA$ and $\mathrm{cod}_{\alpha A} = GA$ for all A, and for all $A \xrightarrow{f} B$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)_{\alpha A}$).

(1.3) (l) Given categories $\mathcal{C}$ and $\mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

Example. (1.7)
(a) Let K be a field, V a vector space over K. There is a linear map $\alpha_V : V \to V^{**}$ given by $\alpha_V(v)\theta = \theta(v)$ for $\theta \in V^*$.
This is the V-component of a natural transformation $1_{\mathbf{Mod_K}} \to ** : \mathbf{Mod_K} \to \mathbf{Mod_K}$.
(b) For any set A, we have a mapping $\sigma_A : A \to PA$ sending a to $\{a\}$. If $f : A \to B$, then $Pf\{a\} = \{f(a)\}$. So σ is a natural transformation $1_{\mathbf{Set}} \to P$.
(c) Let F:$\mathbf{Set} \to \mathbf{Gp}$ be the free group functor (1.5(b)), and $U : \mathbf{Gp} \to \mathbf{Set}$ the forgetful functor. The inclusions $A \to UFA$ form a natural transformation $1_{\mathbf{Set}} \to UF$.
(d) Let G, H be groups and $f, g : G \rightrightarrows H$ be two homomorphisms. A natural transformation $\alpha : f \to g$ corresponds to an element $h = \alpha_*$ of H, s.t. $hf(x) \to g(x)h$ for all $x \in G$ or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
(e) Let A and B be two G-sets, regarded as functors: $G \rightrightarrows \mathbf{Set}$. A natural transformation $A \to B$ is a function f satisfying $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$, i.e. a G-equivariant map.

Lemma. (1.8)
Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors, and $\alpha : F \to G$ a natural transformation. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each α_A is an isomorphism in $\mathcal{D}$.

Proof. Forward is trivial. For backward, suppose each α_A has an inverse β_A. Given $f : A \to B$ in $\mathcal{C}$, we need to show that

$$\begin{array}{ll} GA & \xrightarrow{Gf} GB \\ \downarrow \beta_A & \quad \downarrow \beta_B \\ FA & \xrightarrow{Ff} FB \end{array}$$

□

commutes. But as α is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

So β is a natural transformation as well.

Definition. (1.9)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. By an *equivalence* between $\mathcal{C}$ and $\mathcal{D}$, we mean a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_\mathcal{C} \to GF$ and $\beta : FG \to 1_\mathcal{D}$.
We write $\mathcal{C} \cong \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent.
We say a property P of categories is a *categorical property* if whenever $\mathcal{C}$ has P and $\mathcal{C} \cong \mathcal{D}$, then $\mathcal{D}$ has P.
For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

Example. (1.10)
(a) The category **Part** is equivalent to the category $\mathbf{Set}_*$ of pointed sets (and basepoint preserving functions (as morphisms)):
• We define $F : \mathbf{Set}_* \to \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$, and if $f : (A, a) \to (B, b)$, then $Ff(x) = f(x)$ if $f(x) \neq b$, and undefined otherwise;
• and $G : \mathbf{Part} \to \mathbf{Set}_*$ by $G(A) = A^+ = (A \cup \{A\}, A)$, and if $f : A \to B$ is a partial function, we define $Gf : A^+ \to B^+$ by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ defined, and equals B otherwise.
The composite FG is the identity on **Part**, but GF is not the identity. However, there is an isomoprhism $(A, a) \to ((A \setminus \{a\})^+, A \setminus \{a\})$ sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.
Note that there can be no isomoprhism from $\mathbf{Set}_*$ to **Part**, since **Part** has a 1-element isomorphism class $\{\phi\}$ but $\mathbf{Set}_*$ doesn't.
(So we see that equivalent categories can be non-isomorphic. According to a post on SO, this usually happens when there are multiple copies of the *same* thing in one but not the other. However, we can't generally *discard obsolete copies* in one as that generally requires AC and is not a very useful thing to do anyway – In short, *identifying isomorphic objects is often an extremely bad idea.*)
(b) The category $\mathbf{fdMod_K}$ of finite-dimensional vector spaces over K is equivalent to $\mathbf{fdMod_K}^{op}$, the functors in both directions are $*$ (the dual operator) and both isomorphisms are the natural transformations of 1.7(a) (double dual).
(c) $\mathbf{fdMod_K}$ is also equivalent to $\mathbf{Mat}_K$ (1.3(j)):
We define $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A w.r.t. the standard bases of K^n and K^p.
To define $G : \mathbf{fdMod_K} \to \mathbf{Mat_K}$, choose a basis for each finite dimensional vector

space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ to be the matrix representing f w.r.t. chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n; $FG \neq 1$, but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \to V$ for each V, which form a natural isomorphism.

—Lecture 4—

Definition. (1.11)
Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor.
(a) We say F is *faithful* if, given $f, f' \in \operatorname{mor}\mathcal{C}$ with $\operatorname{dom} f = \operatorname{dom} f'$, $\operatorname{cod} f = \operatorname{cod} f'$, and $Ff = Ff'$, then $f = f'$ (injectivity on morphisms. The name comes more from representation theory);
(b) We say F is *full* if, given $FA \xrightarrow{g} FB$ in $\mathcal{D}$, there exists $A \xrightarrow{f} B$ in $\mathcal{C}$ with $Ff = g$. (this is something like surjectivivity on morphisms, but see below);
(c) We say F is *essentially surjective* if, for every $B \in \operatorname{ob}\mathcal{D}$, there exists $A \in \operatorname{ob}\mathcal{C}$ and isomorphism $FA \to B$ in $\mathcal{D}$.
We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is full if the inclusion $\mathcal{C}' \to \mathcal{C}$ is a full functor (basically, if the objects are kept, any morphism between them must be kept). For example, **Gp** is a full subcategory of **Mon** (the category of all monoids), but **Mon** is not a full subcategory of the category **SGp** of semigroups (consider e.g. the homomorphism that sends everything in $(\mathbb{Z}, \cdot)$ to $(0, \cdot)$ (which is also a semigroup); but this doesn't preserve 1 so is not a morphism in **Mon**).

Lemma. (1.12)
Assuming the axiom of choice, a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ if it's full, faithful, and essentially surjective.

Proof. $\Rightarrow$: Suppose given G, α, β as in (1.9). Then for each $B \in \operatorname{ob}\mathcal{D}$, β_B is an isomorphism $FGB \to B$, so F is essentially surjective.
Given $A \xrightarrow{f} B$ in $\mathcal{C}$, we can recover f from Ff as composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_b^{-1}} B$. Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$. So F is faithful;
Lastly, for fullness, given $FA \xrightarrow{g} FB$, define f to be the composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$. Then $GFf = \alpha_B f \alpha_A^{-1}$, which by construction is just Gg. But G is faithful for the same reason as f, so $Ff = g$.

$\Leftarrow$: (need to find suitable G, α, β for F.) For each $B \in \operatorname{ob}\mathcal{D}$, choose $GB \in \operatorname{ob}\mathcal{C}$ and an isomorphism $\beta_B : FGB \to B$ in $\mathcal{D}$. Given $B \xrightarrow{g} B'$, define $Gg : GB \to GB'$ to be the unique morphism whose image under F is $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$.
Uniqueness implies functoriality: given $B' \xrightarrow{g'} B''$, $(Gg')(Gg)$ and $G(g'g)$ have the same image under F, so they are equal.
By construction, β is a natural transformation $FG \to 1_{\mathcal{D}}$.
Given $A \in \operatorname{ob}\mathcal{C}$, define $\alpha_A : A \to GFA$ to be the unique morphism whose image under F is $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$. α_A is an isomorphism, since β_{FA} also has a unique pre-image under F. And α is a natural transformation, since any naturality

square for α (the commutative square when we defined natural transformation) is mapped by F to a commutative square, and F is faithful. $\square$

Definition. (1.13)
By a *skeleton* of a category, we mean a full subcategory $\mathcal{C}_0$ containing one object from each isomorphism class. We say $\mathcal{C}$ is *skeletal* if it's a skeleton of itself.
For example, $\mathbf{Mat_K}$ is a skeletal, and the image of $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ of 1.10(c) is a skeleton of $\mathbf{fdMod_K}$.
(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

Definition. (1.14)
Let $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$.
(a) We say f is a *monomorphism* (or f is *monic*) if, given any pair $C \overset{g}{\underset{h}{\rightrightarrows}} A$, $fg = fh$ implies $g = h$.
(b) We say f is an *epimorphism* (or *epic*) if it's a monomorphism in $\mathcal{C}^{op}$, i.e. if $gf = hf$ implies $g = h$.
We denote monomorphisms by $A \overset{f}{\rightarrowtail} B$, and epimorphisms by $A \overset{f}{\twoheadrightarrow} B$.
Any isomorphism is monic and epic: more generally, if f has a left inverse (i.e. $\exists g$ s.t. gf is an identity), then it's monic. We call such monomorphisms *split*.
We say $\mathcal{C}$ is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

Example. (1.15)
(a) As usual we consider **Set** first. In **Set**, monomorphisms correspond to injections ($\Leftarrow$ is easy (ok); for $\Rightarrow$, take $C \rightrightarrows 1 = \{*\}$), and epimorphsims correspond to surjections ($\Leftarrow$ is easy; for $\Rightarrow$, use morphisms $B \rightrightarrows 2 = \{0, 1\}$). So **Set** is balanced.
(b) In **Gp**, monomorphisms again correspond to injections (for $\Rightarrow$ use homomorphisms $\mathbb{Z} \to A$); epimorphisms again correspond to surjections ($\Rightarrow$ use free products with amalgamation – this is a non-trivial fact about groups, read more if free). So **Gp** is also balanced.
(c) In **Rng** (obvious notation), monomorphisms correspond to injections (proof is much like for **Gp**). However, not all epimorphisms are surjective. For example the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \overset{f}{\underset{g}{\rightrightarrows}} R$ (any ring) agree on all integers, they agree everywhere. So **Rng** is not balanced.
(d) One final example is **Top**. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to **Set** (check). However, **Top** is not balanced since a continuous bijection need not have continuous inverse.

2 The Yoneda Lemma

—Lecture 5—

Definition. (2.1)
We say a category $\mathcal{C}$ is *locally small* if, for any two objects A, B, the morphisms $A \to B$ in $\mathcal{C}$ form a set $\mathcal{C}(A, B)$.
If we fix A and let B vary, the assignment $B \to \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathcal{C}(A, f)$ is the mapping $g \to fg$ for all $g \in \mathcal{C}(B, C)$. Similarly, $A \to \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : \mathcal{C}^{op} \to \mathbf{Set}$ (for $A \xrightarrow{f} C \in \operatorname{mor} \mathcal{C}^{op}$, maps $g \to gf$).

Lemma. (2.2)
(i) Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob} \mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then natural transformations $\mathcal{C}(A, -) \to F$ are in bijection with elements of FA;
(ii) Moreover, this bijection is natural in A and F.

Proof. (i) Given α:$\mathcal{C}(A, -) \to F$, we define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.[1]
Conversely, given $x \in FA$, we define $\Psi(x) : \mathcal{C}(A, -) \to F$ by $\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$.
$\Psi(x)$ is natural: given $g : B \to C$, we have

$$\begin{aligned} \Psi(x)_C \mathcal{C}(A, g)(f) = \Psi(x)_C(gf) = F(gf)(x), \\ (Fg)\Psi(x)_B(f) = (Fg)(Ff)(x) = F(gf)(x) \end{aligned}$$

Now given $x \in FA$, $\Phi\Psi(x) = \Psi(X)_A(1_A) = F(1_A)(x) = x$; given α,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f)\Psi(\alpha_A(1_A))_B(f) &= Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathcal{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

So $\Psi\Phi(\alpha) = \alpha$. So $\Psi\Phi$ and $\Phi\Psi$ are both identities on their respective domain (so we have a bijection). □

Corollary. (2.3)
The assignment $A \to \mathcal{C}(A, -)$ defines a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \mathbf{Set}]$.

Proof. Put $F = \mathcal{C}(B, -)$ in 2.2(i): we get a bijection between $\mathcal{C}(B, A)$ and morphisms $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ in $[\mathcal{C}, \mathbf{Set}]$[2]. We need to verify this is functorial: but it sends $f : B \to A$ to the natural transformation $g \to gf$. So functoriality follows from associativity. □

[1]Note $1_A \in \mathcal{C}(A, A)$, and $\alpha_A \in \operatorname{mor} \mathbf{Set}$ but $\operatorname{mor} \mathbf{Set}$ are just functions between sets, so this makes sense.

[2]Think very carefully about this... Given a morphism in $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$, the above gives us a way to identify it uniquely with an element in $\mathcal{C}(B, A)$ which is in $\operatorname{mor} \mathcal{C}^{op}$. But that alone is not enough; we also need the above functor to take that morphism *directly* to the original morphism. Luckily this is the case by the proof of 2.2(i), which is also explained in the later half of the sentence above.

We call this functor (or the functor $\mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ sending A to $\mathcal{C}(-, A)$) the *Yoneda embedding* of $\mathcal{C}$, and denote it by Y.

Now let's go back to prove 2.2(ii):

Proof. (ii) Suppose for the moment that $\mathcal{C}$ is small, so that $[\mathcal{C}, \mathbf{Set}]$ is locally small.[3] Then we have two functors $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$: one sends (A, F) to FA, and the other is the composite: $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{op} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-;-)} \mathbf{Set}$.[4]
2.2(ii) says that these are naturally isomorphic. We can translate this into an elementary statement, making sense even when $\mathcal{C}$ isn't small. Given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, the two ways of producing an element of GB from a natural transformation $\beta : \mathcal{C}(A, -) \to F$ give the same result, namely

$$\alpha_B (Ff) \beta_A (1_A) = (Gf) \alpha_A \beta_A (1_A)$$

which is equal to $\alpha_B \beta_B (f)$. □

Definition. (2.4)
We say a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if it's isomorphic to $\mathcal{C}(A, -)$ for some A. By a representation of F, we mean a pair (A, x) where $x \in FA$ is such that $\Psi(x)$ is an isomorphism.
We also call x a *universal element* of F.

Corollary. (2.5)
If (A, x) and (B, y) are both representations of F, then there's a unique isomorphism $f : A \to B$ such that $(Ff)(x) = y$.

Proof. Consider the composite $\mathcal{C}(B, -) \xrightarrow{\Psi(y)^{-1}} F \xrightarrow{\Psi(x)} \mathcal{C}(A, -)$. By (2.3) this is of the form $Y(f)$ for a unique isomorphism $f : A \to B$, and the diagram

$$\begin{tikzcd} \mathcal{C}(B,-) \arrow[rr, "Y(f)"] \arrow[dr, "\Psi(y)"'] & & \mathcal{C}(A,-) \arrow[dl, "\Psi(x)"] \\ & F & \end{tikzcd}$$

commutes iff $(Ff)(x) = y$. □

Example. (2.6)
(a) The forgetful functor $\mathbf{Gp} \to \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$, $\mathbf{Rng} \to \mathbf{Set}$ by $(\mathbb{Z}[X], X])$, and $\mathbf{Top} \to \mathbf{Set}$ by $(\{*\}, *)$.
(b) The functor $P^* : \mathbf{Set}^{op} \to \mathbf{Set}$ is representable by $(\{0, 1\}, \{\mathbf{1}\})$: this is the bijection between subsets and characteristic functions.
(c) Let G be a group. The unique (up to isomorphism) representable functor $G(*, -) : G \to \mathbf{Set}$ is the *Cayley representation* of G, i.e. the set $\cup G$ with G acting by left multiplication.

[3] Elements in mor$[\mathcal{C}, \mathbf{Set}]$ correspond to those in mor $\mathcal{C}^{op}$ by Yoneda.
[4] The second operator maps two functors two the set of natural transformations between them?

(d) Let A and B be two objects of a small category $\mathcal{C}$. We have a functor $\mathcal{C}^{op} \to \mathbf{Set}$ sending C to $\mathcal{C}(C,A) \times \mathcal{C}(C,B)$. A representation of this, if it exists, is called a (categorical) *product* of A and B, and denoted $(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B))$.
This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$, there's a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.
Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top**,..., they are *just* cartesian products, in posets they are binary meets (see sheet 1 Q1).
Dually, we have the notion of *coproduct* $(A + B, A \xrightarrow{\mu_1} A + B, B \xrightarrow{\mu_2} A + B)$.
These also exist in many categories of interest.
—Lecture 6—
(f) (Lecturer didn't like (e) so jumped to (f) directly) Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be morphisms in locally small category $\mathcal{C}$. We have a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by

$$F(C) = \{h \in \mathcal{C}(C,A) | fh = gh\}$$

A representation (see (2.4)) of F, if it exists, is called an *equalizer* of (f, g): It consists of an object E and a morphism $E \xrightarrow{e} A$ s.t. $fe = ge$, and every h with $fh = gh$ factors uniquely (see proof of 2.9(i) which gives an insight of what this means) through e.
In **Set**, we take $E = \{x \in A | f(x) = g(x)\}$ and e =inclusion. Similar constructions work in $\mathbf{Gp}, \mathbf{Rng}, \mathbf{Top}$,...
Dually, we have the notion of *coequalizer*.

Remark. (2.7)
If e occurs as an equalizer, then it is a monomorphism, since any h factors through it in at most one way. We say a monomorphism is *regular* if it occurs as an equalizer.
Split monomorphisms are regular (cf sheet1 Q6(i)).
Note that regular epic monomorphisms are isomorphisms: if the equalizer e of (f, g) is epic, then $f = g$, so $e \cong 1_{\operatorname{cod} e}$.

Definition. (2.8)
Let $\mathcal{C}$ be a category, $\mathcal{G}$ a class of objects of $\mathcal{C}$.
(a) We say $\mathcal{G}$ is a *separating family* for $\mathcal{C}$, if given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that $fh = gh$ for all $G \xrightarrow{h} A$ with $G \in \mathcal{G}$, then $f = g$.
(i.e. the functors $\mathcal{C}(G, -), G \in \mathcal{G}$, are collectively faithful.)
(b) We say $\mathcal{G}$ is a *detecting family* if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} B$ with $G \in \mathcal{G}$ factors uniquely through f, then f is an isomorphism.
If $\mathcal{G} = \{G\}$, we call G a *separator/detector*.

Lemma. (2.9)
(i) If $\mathcal{C}$ is a balanced category, then any saparating fmamily is detecting.
(ii) If $\mathcal{C}$ has equalizers, then any detecting family is separating.

Proof. (i) Suppose $\mathcal{G}$ is separating and $A \xrightarrow{f} B$ satisfies the condition of 2.8(b). If $B \underset{h}{\overset{g}{\rightrightarrows}} C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is

epic.
Similarly if $D \underset{l}{\overset{k}{\rightrightarrows}} A$ satisfy $fk = fl$, then $ky = ly$ for any $G \xrightarrow{y} D$, since both are factorizations of fky through f. So $k = l$, i.e. f is monic.
But $\mathcal{C}$ is balanced. So f is an isomorphism.
(ii) Suppose $\mathcal{G}$ is detecting and $A \underset{g}{\overset{f}{\rightrightarrows}} B$ satisfies the condition of 2.8(a). Then the equalizer $E \xrightarrow{e} A$ of (f, g) is isomorphism, so $f = g$. $\square$

Example. (2.10)
(a) In $[\mathcal{C}, \mathbf{Set}]$, the family $\{\mathcal{C}(A, -) | A \in \mathrm{ob}\,\mathcal{C}\}$ is both separating and detecting (just a restatement of Yoneda Lemma).
(b) In **Set**. $1 = \{*\}$ (any one element set) is both a separator and a detector, since it represents the identity functor $\mathbf{Set} \to \mathbf{Set}$.
Similarly, $\mathbb{Z}$ is both in **Gp**, since it represents the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$.
Also, $2 = \{0, 1\}$ is a coseparator and a codetector in **Set**, since it represents $P^* : \mathbf{Set}^{op} \to \mathbf{Set}$.
(c) In **Top**, $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \to \mathbf{Set}$, but not a detector.
In fact, **Top** has no detecting *set* of objects (note that this doesn't mean it has no detecting family).
For any infinite cardinal κ, let X be a discrete space of cardinality κ, and Y the same set with *co-*$<\kappa$ topology, i.e. $F \subseteq Y$ is closed iff $F = Y$ or $\mathrm{Card}\, F < \kappa$ (think about, e.g. cocountable topology, then this name makes sense).
The identity $X \to Y$ is continuous, but not a homeomorphism (topologically).
So if $\{G_i | i \in I\}$ is any set of spaces, taking $\kappa > \mathrm{Card}\, G_i$ for all i yields an example to show that the set is not detecting.
(d) (some Algebraic Topology stuff) Let $\mathcal{C}$ be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous mappings.
JHC Whitehead proved that $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all n, then it's an isomorphism in $\mathcal{C}$.
This says that $\{S^n | n \geq 1\}$ is a detecting set of $\mathcal{C}$.
But PJ Freyd showed there is no faithful functor $\mathcal{C} \to \mathbf{Set}$, so no separating *set*: if $\{G_i | i \in I\}$ were separating, then $x \to \coprod \mathcal{C}(G_i, x)$ (disjoint unions?) would be faithful.
Note that any functor of the form $\mathcal{C}(A, -)$ preserves monomorphisms, but they don't normally preserves epimorphisms.

Definition. (2.11)
We say an object P is *Projective* if, given

$$\begin{array}{c} P \\ \downarrow f \\ A \overset{e}{\twoheadrightarrow} B \end{array}$$

(recall the two head right arrow means epimorphisms) there exists $P \xrightarrow{g} A$ with $eg = f$.
(If $\mathcal{C}$ is locally small, this says $\mathcal{C}(P, -)$ preserves epimorphisms).
Dually, an *injective* object of $\mathcal{C}$ is a projective object of $\mathcal{C}^{op}$.

Given a class $\mathcal{E}$ of epimorphisms, we say P is *$\mathcal{E}$-projective* if it satisfies the condition for all $e \in \mathcal{E}$.

Lemma. (2.12)
Representable functors are (pointwise)(?) projective in $[\mathcal{C}, \mathbf{Set}]$.

Proof. Suppose given

$$\begin{array}{c} \mathcal{C}(A,-) \\ \downarrow \beta \\ F \overset{\alpha}{\twoheadrightarrow} G \end{array}$$

where α is pointwise surjective. By Yoneda, β corresponds to some $y \in GA$, and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathcal{C}(A,-) \to F$ corresponds to x, then naturality of the Yoneda bijection yields $\alpha\gamma = \beta$. $\square$

—Leture 7—
First example class: Friday 26th October, 2pm MR3.

Lecture is happy to mark any question we hand in!

3 Adjunctions

Definition. (3.1)
Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ two functors.
By an *adjunction* between F and G we mean a bijection between morphisms $FA \xrightarrow{\hat{f}} B$ in $\mathcal{D}$ and morphisms $A \xrightarrow{f} GB$ in $\mathcal{C}$, which is natural in A and B, i.e. given $A' \xrightarrow{g} A$ and $B \xrightarrow{h} B'$, we have $h\hat{f}(Fg) = \widehat{(Gh)fg} : FA' \to B'$.
We say F is *left adjoint* to G, and write $(F \dashv G)$.

Example. (3.2)
(a) The free functor $\mathbf{Set} \xrightarrow{F} \mathbf{Gp}$ is left adjoint to the forgetful functor $\mathbf{Gp} \xrightarrow{U}$ $\mathbf{Set}$, since any function $f : A \to UB$ extends uniquely to a homomorphisms $\hat{f} : FA \to B$.
Naturality in B is *easy* (lecturer says so), naturality in A follows from the definition of F as a functor.
(b) The forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ has a left adjoint D which equips any set with the discrete topology, *and* also a right adjoint I which equips a set A with the discrete (lecturer had *indiscrete* here?) topology $\{\phi, A\}$.
(c) The functor $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$ (recall $\mathbf{Cat}$ is the category of small categories) has a left adjoint D sending A to the *discrete* category with $\mathrm{ob}(DA) = A$ and only identity morphisms, and a right adjoint I sending A to the category with $\mathrm{ob}(IA) = A$ and one morphism $x \to y$ for each $(x, y) \in A \times A$. In this case D in turn has a left adjoint π_0 sending a small category $\mathcal{C}$ to its set of *connected components*, i.e. the quotient of $\mathrm{ob}\,\mathcal{C}$ by the smallest equivalence relation identifying $\mathrm{dom}\, f$ with $\mathrm{cod}\, f$ for all $f \in \mathrm{mor}\,\mathcal{C}$.
(d) Let M be the monoid $\{1, e\}$ with $e^2 = e$. An object of $[M, \mathbf{Set}]$ is a pair (A, e) (the images of the functor?), where $e : A \to A$ satisfies $e^2 = e$.
We have a functor $G : [M, \mathbf{Set}] \to \mathbf{Set}$ sending (A, e) to $\{x \in A | e(x) = x\} = \{e(x) | x \in A\}$ and a functor $F : \mathbf{Set} \to [M, \mathbf{Set}]$ sending A to $(A, 1_A)$.
I claim $(F \dashv G \dashv F)$: given $f : (A, 1_A) \to (B, e)$, it must take values in $G(B, e)$, and any $g : (B, e) \to (A, 1_A)$ is determined by its values on the image of e.
(e) Let $\mathbf{1}$ be the discrete category with one object $*$. For any $\mathcal{C}$, there's a unique functor $\mathcal{C} \to \mathbf{1}$: a left adjoint for this picks out an *initial* object of $\mathcal{C}$, i.e. an object I s.t. there exists a unique $I \to A$ for each $A \in \mathrm{ob}\,\mathcal{C}$.
Dually, a right adjoint for $\mathcal{C} \to \mathbf{1}$ corresponds to a *terminal* object of $\mathcal{C}$ (think about what this means).
(f) Let $A \xrightarrow{f} B$ be a morphism in $\mathbf{Set}$. We can regard PA and PB as posets, and we have functors $PA \underset{P^*f}{\overset{Pf}{\rightleftarrows}} PB$.
I claim $(PF \dashv P^*f)$: we have $Pf(A') \subseteq B' \iff f(x) \in B'$ for all $x \in A' \iff A' \subseteq P^*f(B')$.
(g) (*Galois Connection*) Suppose given sets A, B and a relation $R \subseteq A \times B$. We define mappings $(-)^l$,$(-)^r$ between PA and PB by

$$S^r = \{y \in B | (\forall x \in S)((x, y) \in R)\} \text{ for } S \subseteq A$$
$$T^l = \{x \in A | (\forall y \in T)((x, y) \in R)\} \text{ for } T \subseteq B$$

The mappings are order-reserving (i.e. contravariant functors), and $T \subseteq S^r \iff S \times T \subseteq R \iff S \subseteq T^l$.

We say $()^r$ and $()^l$ are *adjoint on the right.*
(h) Let's now consider, as a functor, $P^* : \mathbf{Set}^{op} \to \mathbf{Set}$ is self-adjoint on the right, since functions $A \to PB$ correspond bijectively to subsets of $A \times B$, and hence to functions $B \to PA$.

Theorem. (3.3)
Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each $A \in \mathrm{ob}\,\mathcal{C}$, where $(A \downarrow G)$ has objects pairs (B, f) with $A \xrightarrow{f} GB$, and morphisms $(B, f) \to (B', f')$ are morphisms $B \xrightarrow{g} B'$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f'"'] & & GB \arrow[dl, "Gg"] \\ & GB' & \end{tikzcd}$$

commutes.

Proof. Suppose given $(F \dashv G)$. Consider the morphism $\eta_A : A \to GFA$ correspond to $FA \xrightarrow{\eta} FA$. Then (FA, η_A) is an object of $(A \downarrow G)$. Moreover, given $g : FA \to B$ and $f : A \to GB$, the diagram

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & GFA \arrow[dl, "Gg"] \\ & GB & \end{tikzcd}$$

commutes iff

$$\begin{tikzcd} FA \arrow[rr, "1_A"] \arrow[dr, "\hat{f}"'] & & FA \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

commutes, i.e. $g = \hat{f}$.
So (FA, η_A) is initial in $(A \downarrow G)$.
Conversely, suppose given an initial object (FA, η_A) for each $(A \downarrow G)$. Given $A \xrightarrow{f} A'$, we define $Ff : FA \to FA'$ to be the unique morphism making

$$\begin{array}{ccc} A \xrightarrow{\eta_A} & & GFA \\ \downarrow f & & \downarrow GFf \\ A' \xrightarrow{\eta_{A'}} & & GFA' \end{array}$$

commute.
Functoriality follows from uniqueness: given $f' : A' \to A''$, $F(f'f)$ and $(Ff')(Ff)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''}F'f)$ in $(A \downarrow G)$.
Note that we haven't finished: we still have to verify natural adjunctions. We'll finish off this next monday.

—Lecture 8—
It's next monday now! Let's finish the proof:

To show $F \dashv G$: given $A \xrightarrow{f} GB$, we define $\hat{f} : FA \to B$ to be the unique morphism $(FA, \eta_A) \to (B, f)$ in $(a \downarrow G)$. This is a bijection with inverse $(FA \xrightarrow{g} B) \to (A \xrightarrow{\eta_a} GFA \xrightarrow{Gg} GB)$. The latter mapping is natural in B, as G is a functor; and also in A, since by construction, η is a natural transformation $1_\mathcal{C} \to GF$. □

Given an adjunction $(F \dashv G)$, the natural transformation $\eta : 1_\mathcal{C} \to GF$ emergin in the above proof (3.3) is called the *unit* of the adjunction.
Dually, we have a natural transformation traditionally denoted $\varepsilon : FG \to 1_\mathcal{D}$ s.t. $\varepsilon_B : FGB \to B$ corresponds to $GB \xrightarrow{1_{GB}} GB$, is called the *counit*.

Corollary. (3.4)
If F and F' are both left adjoint to $G : \mathcal{D} \to \mathcal{C}$, then they are naturally isomorphic.

Proof. For any A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \to (F'A, \eta'_A)$.
In any naturality square for α, the two ways round are both morphisms in $(A \downarrow G)$ whose domain is initial, so they are equal. So α is not only just an isomorphism (but also natural). □

Lemma. (3.5)
Given $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$, with $(F \dashv G)$ and $(H \dashv K)$, we have $(HF \dashv GK)$.

Proof. We have bijections between morphisms $A \to GKC$, morphisms $FA \to KC$ and morphisms $HFA \to C$, which are both natural in A and C. □

Corollary. (3.6)
Given a commutative square

$$\begin{array}{cc} \mathcal{C} \to & \mathcal{D} \\ \downarrow & \downarrow \\ \mathcal{E} \to & \mathcal{F} \end{array}$$

of categories and functors, if the functors all have left adjoints, then the diagram of left adjoinst commutes up to natural isomorphisms (?).

Proof. By (3.5), both ways round the diagram of left adjoinst are left adjoint to the composite $\mathcal{C} \to \mathcal{F}$, so by (3.4) they are isomorphic. □

Theorem. (3.7)
Given functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, specifyig an adjunction $(F \dashv G)$ is equivalent to specifying natural transformations $\eta : 1_\mathcal{C} \to GF$, $\varepsilon : FG \to 1_\mathcal{D}$ satisfying the commutative diagrams,

$$\begin{tikzcd} F \arrow[r, "F_\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G_\varepsilon"] \\ & G \end{tikzcd}$$

which are sometimes called the *triangular identities* (for obvious reason).

Proof. First suppose we are given $(F \dashv G)$. Define η and ε as in (3.3) and its dual; now consider the composite

$$FA \xrightarrow{F_{\eta_A}} FGFA \xrightarrow{\varepsilon_{FA}} FA$$

under the adjunction, this corresponds to

$$A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$$

But this also corresponds to 1_{FA}, so $\varepsilon_{FA} \cdot F_{\eta_A} = 1_F A$.
The other identity is dual to this one (check).
Conversely, suppose we are given η and ε satisfying the trianglular identities. Given $A \xrightarrow{f} GB$, let $\Phi(f)$ be the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\varepsilon_B} B$; and given $FA \xrightarrow{g} B$, let $\Psi(g)$ be $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$. Then Φ and Ψ are both natural; we now need to show they are inverse to each other. Let's do $\Psi\Phi$, say: now

$$\begin{aligned}\Psi\Phi(A \xrightarrow{f} GB) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\varepsilon_B} GB \\ &= A \xrightarrow{f} FB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G_{\varepsilon_B}} GB \\ &= f\end{aligned}$$

and dually, $\Phi\Psi(g) = g$. □

Lemma. (3.8)
Suppose given $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. Then there are isomorphisms $\alpha' : 1_{\mathcal{C}} \to GF$, $\beta' : FG \to 1_{\mathcal{D}}$ which satisfy the triangular identities. So $(F \dashv G)$ (and $(G \dashv F)$).

Proof. We define $\alpha' = \alpha$ and, in attempt to fix β', define β' to be the composite

$$FG \xrightarrow{(FGB)^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that $FGB = \beta_{FG}$, since

$$\begin{array}{ccc} FGFG & \xrightarrow{FGB} & FG \\ \downarrow \beta_{FG} & & \downarrow B \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$$

commutes by naturality of β, and β is monic. So it doesn't matter which way we choose above.
Now $(\beta'_F)(F\alpha')$ is the composite

$$\begin{aligned}& F \xrightarrow{F\alpha} FGF \xrightarrow{(\beta_{FGF})^{-1}} FGFGF \xrightarrow{(F\alpha_{GF})^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{(\beta_F)^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{(F\alpha_{GF})^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{(\beta_F)^{-1}} FGF \xrightarrow{\beta_F} F \\ &= 1_F\end{aligned}$$

Since $GF\alpha = \alpha_{GF}$ (similar reasoning as previous).
Now similarly $(G\beta')(\alpha' G)$ is

$$\begin{aligned}
&G \xrightarrow{\alpha_G} GFG \xrightarrow{(GFG\beta)^{-1}} GFGFG \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\
&= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\
&= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{G\beta} G \\
&= 1_G
\end{aligned}$$

□

Lemma. (3.9)
Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint F with counit $\varepsilon : FG \to 1_{\mathcal{D}}$, then:
(i) G is faithful iff ε is pointwise epic;
(ii) G is full and faithful iff ε is an isomorphism.
(and of course the dual results for unit – change epic to monic).

Proof. (i) Given $B \xrightarrow{g} B'$, Gg corresponds, under the adjunction, to the composite $FGB \xrightarrow{\varepsilon_B} B \xrightarrow{g} B'$. Hence the mapping $g \to Gg$ is injective on morphisms with domain B (and specified codomain) iff $g \to g\varepsilon_B$ is injecctive, i.e. iff ε_B is an epimorphism.
(ii) The proof of this is actually very similar: G is full and faithful iff $g \to g\varepsilon_B$ is bijective, but that forces ε to be an isomorphism: if $\alpha : B \to FGB$ is such that $\alpha\varepsilon_B = 1_{FGB}$, then this must be a two sided inverse as $\varepsilon_B\alpha\varepsilon_B = \varepsilon_B$, whence $\varepsilon_B\alpha = 1_B$. So ε_B is an isomorphism, for all B. □