Category Theory

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Contents

0 Introduction

I didn't go to the first 3 lectures, so no intro – sorry. I have no idea on what this course is about, let's see

1 Definitions and examples

Definition. (1.1)
A category $\mathcal{C}$ consists of:
(a) a collection $\mathrm{ob}\,\mathcal{C}$ of *objects* A, B, C;
(b) a collection $\mathrm{mor}\,\mathcal{C}$ of *morphisms* f, g, h;
(c) two operations domain, codomain assigining to each $f \in \mathrm{mor}\,\mathcal{C}$ a pair of objects, its *domain* and *codomain*; we write $A \xrightarrow{f} B$ to mean f *is a morphism and* $\mathrm{dom}\, f = A, \mathrm{cod}\, f = B$;
(d) an operation assigning to each $A \in \mathrm{ob}\,\mathcal{C}$ a morphism $A \xrightarrow{1_A} A$;
(e) a partial binary operation $(f, g) \to fg$ on morphisms, such that fg is defined iff $\mathrm{dom}\, f = \mathrm{cod}\, g$, and $\mathrm{dom}(fg) = \mathrm{dom}\, g$, $\mathrm{cod}(fg) = \mathrm{cod}(f)$ if fg is defined, satisfying:
(f) $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$;
(g) $(fg)h = f(gh)$ whenever fg and gh are defined.

Remark. (1.2)
(a) This definition is independent of any model of set theory. If we're given a particular model of set theory, we call $\mathcal{C}$ *small* if $\mathrm{ob}\,\mathcal{C}$ and $\mathrm{mor}\,\mathcal{C}$ are sets.
(b) Some texts say fg means f followed by g, i.e. fg is defined iff $\mathrm{cod}\, f = \mathrm{dom}\, g$.
(c) Note that a morphism f is an identity iff $fg = g$ and $hf = h$ whenever the composites are defined. So we could formulate the definition entirely in terms of morphisms.

Example. (1.3)
(a) The category **Set** has all sets as objects, and all functions between sets as morphisms.
Strictly speaking, morphisms $A \to B$ are pairs (f, B) where f is a set-theoretic function. (See part II logic and sets)
(b) The category **Gp** has all groups as objects, group homomorphisms as morphisms.
Similarly, **Ring** is the category of rings, $\mathbf{Mod_R}$ is the category of R-modules.
(c) The category **Top** has all topological spaces as objects, and continuous functions as morphisms.
Similarly, **Unif** has all uniform spaces and uniformly continuous functions as morphisms, **Mf** has all manifolds and smooth maps correspondingly.
(d) The category **Htpy** has the same objects as **Top**, but morphisms are homotopy classess of continuous functions. More generally, given $\mathcal{C}$, we call an equivalence relation $\simeq$ on $\mathrm{mor}\,\mathcal{C}$ a *congruence* if $f \simeq g \implies \mathrm{dom}\, f = \mathrm{dom}\, g$ and $\mathrm{cod}\, f = \mathrm{cod}\, g$, and $f \simeq g \implies fh \simeq gh$ and $kf \simeq kg$ whenever the composites are defined. Then we have a category $\mathcal{C}/\simeq$ with the same objects as $\mathcal{C}$, but congruence classes as morphisms instead.
(e) Given $\mathcal{C}$, the *opposite category* C^{op} has the same objects and morphisms as $\mathcal{C}$, but dom and cod are interchanged, and fg in $\mathcal{C}^{op}$ is gf in $\mathcal{C}$.
This leads to the *duality principle*: if P is a true statement about categories, so is the statement P^* obtained from P by reversing all arrows.
(f) A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small cat (🐱) with one object in which every morphism is an isomorphism (i.e. for all $f, \exists g$ s.t. fg and gf are identities).

(g) A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space X, the *fundamental groupoid* $\pi(x)$ has all points of X as objects, and morphisms $x \to y$ are homotopy classes $rel\{0,1\}$ of paths $u : [0,1] \to X$ with $u(0) = x$, $u(1) = y$ (if you know how to prove that the fundamental group is a group, you can prove that $\pi(x)$ is a groupoid).
(h) A *discrete* cat is one whose only morphism are identities.
A *preorder* is a cat $\mathcal{C}$ in which, for any pair (A, B), $\exists$ at most 1 morphism $A \to B$.
A small preorder is a set equipped with a binary relation which is reflexive and transitive.
In particular, a partially ordered set is a small preorder in which the only isomorphisms are identities.
(i) The category **Rel** has the same objects as *set*, but morphisms $A \to B$ are arbitrary relations $R \subseteq A \times B$. Given R and $S \subseteq B \times C$, we define $S \cdot R = \{(a,c) \in A \times C | (\exists b \in B)((a,b) \in R, (b,c) \in S)\}$.
The identity $1_A : A \to A$ is $\{(a,a) | a \in A\}$.
Similarly, the category **Part** are for sets and partial functions (i.e. relations s.t. $(a,b) \in R$ and $(a,b') \in R \implies b = b'$).
(j) Let K be a field. The cateogry $\mathbf{Mat_K}$ has natural numbers as objects, and morphism $n \to p$ are $(p \times n)$ matrices with entries from K. Composition is matrix multiplication.
(k) We write **Cat** for the category whose objects are all small categories, and whose morphisms are functors between them. (see below for definition of functors)

Definition. (1.4)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:
(a) a mapping $A \to FA$ from $\operatorname{ob}\mathcal{C}$ to $\operatorname{ob}\mathcal{D}$;
(b) a mapping $f \to Ff$ from $\operatorname{mor}\mathcal{C}$ to $\operatorname{mor}\mathcal{D}$,
such that $\operatorname{dom}(Ff) = F(\operatorname{dom} f)$, $\operatorname{cod}(Ff) = F(\operatorname{cod} f)$, $1_{FA} = F(1_A)$, and $(Ff)(Fg) = F(fg)$ whenever fg is defined.

Example. (1.5)
(a) We have *forgetful functors* U: **Gp** $\to$ **Set**, **Ring** $\to$ **Set**, **Top** $\to$ **Set**, **Ring** $\to$ **AbGp** (forget $\times$), **Ring** $\to$ **Mon** (Category of all monoids) (forget $+$).
(b) Given a set A, the free group FA has the property:
Given any group G and any function $A \xrightarrow{f} UG$ (?), there's a unique homomorphism $FA \xrightarrow{\bar{f}} G$ extending f. Here F is a functor **Set** $\to$ **Gp**: given $A \xrightarrow{f} B$, we define Ff to be the unique homomorphism extending $A \xrightarrow{f} B \leftrightarrow UFB$. Functorality follows from uniqueness given $B \xrightarrow{f} C$. $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending $A \xrightarrow{f} B \xrightarrow{g} C \to UFC$.
(c) Given a set A, we write PA for the set of all subsets of A.
We can make P into a functor **Set** $\to$ **Set**, given $A \xrightarrow{f} B$, we defined $Pf(A') = \{f(a) | a \in A'\}$ for $A' \subseteq A$.
But we also have a functor $P^* : \mathbf{Set} \to \mathbf{Set}^{op}$ defined on objects by P, but $P^*f(B') = \{a \in A | f(a) \in B'\}$ for $B' \subseteq B$.
By a *contravariant* functor $\mathcal{C} \to \mathcal{D}$, we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or $\mathcal{C}^{op} \to \mathcal{D}$). A *covariant* functor is one that doesn't reverse arrows (in *op* I guess?).

(d) Let K be a field. We have a functor $* : \mathbf{Mod_K} \to \mathbf{Mod_K}^{op}$ defined by $V^* = \{$ linear maps $V \to K\}$, and if $V \xrightarrow{f} W$, $f^*(\theta : W \to K) = \theta f$.
(e) We have a functor $op : \mathbf{Cat} \to \mathbf{Cat}$, which is the identity on morphisms (note that this is a covariant).
(f) A functor between monoids is a monoid homomorphism.
(g) A functor between posets is an order-preserving map.
(h) Let G be a group. A functor $F \circ G \to \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A, i.e. a *permutation representation* of G.
Similarly, a functor $G \to \mathbf{Mod_K}$ is a K-linear representation of G.
(i) The construction of the fundamental group $\pi(X, X)$ of a space X with base-point X is a functor $\mathbf{Top}* \to \mathbf{Gp}$ where $\mathbf{Top}*$ is the category of spaces with a chosen basepoint.
Similarly, the fundamental groupoid is a functor $\mathbf{Top} \to \mathbf{Gpd}$, where $\mathbf{Gpd}$ is the category of groupoids and functors between them.

Definition. (1.6)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ (why two arrows?) two functors.
A *natural transformation* $\alpha : F \to G$ consists of an assignment $A \to \alpha_A$ from $\mathrm{ob}\,\mathcal{C}$ to $\mathrm{mor}\,\mathcal{D}$ (think about this), such that $\mathrm{dom}_{\alpha_A} = FA$ and $\mathrm{cod}_{\alpha A} = GA$ for all A, and for all $A \xrightarrow{f} B$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes (i.e. $\alpha_B(Ff) = (Gf)_{\alpha A}$).

(1.3) (l) Given categories $\mathcal{C}$ and $\mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

Example. (1.7)
(a) Let K be a field, V a vector space over K. There is a linear map $\alpha_V : V \to V^{**}$ given by $\alpha_V(v)\theta = \theta(v)$ for $\theta \in V^*$.
This is the V-component of a natural transformation $1_{\mathbf{Mod_K}} \to ** : \mathbf{Mod_K} \to \mathbf{Mod_K}$.
(b) For any set A, we have a mapping $\sigma_A : A \to PA$ sending a to $\{a\}$. If $f : A \to B$, then $Pf\{a\} = \{f(a)\}$. So σ is a natural transformation $1_{\mathbf{Set}} \to P$.
(c) Let F:$\mathbf{Set} \to \mathbf{Gp}$ be the free group functor (1.5(b)), and $U : \mathbf{Gp} \to \mathbf{Set}$ the forgetful functor. The inclusions $A \to UFA$ form a natural transformation $1_{\mathbf{Set}} \to UF$.
(d) Let G, H be groups and $f, g : G \rightrightarrows H$ be two homomorphisms. A natural transformation $\alpha : f \to g$ corresponds to an element $h = \alpha_*$ of H, s.t. $hf(x) \to g(x)h$ for all $x \in G$ or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
(e) Let A and B be two G-sets, regarded as functors: $G \rightrightarrows \mathbf{Set}$. A natural transformation $A \to B$ is a function f satisfying $f(g \cdot a) = g \cdot f(a)$ for all $a \in A$, i.e. a G-equivariant map.

Lemma. (1.8)
Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors, and $\alpha : F \to G$ a natural transformation. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ iff each α_A is an isomorphism in $\mathcal{D}$.

Proof. Forward is trivial (ok, I'll check this later). For backward, suppose each α_A has an inverse β_A. Given $f : A \to B$ in $\mathcal{C}$, we need to show that

$$
\begin{array}{ll}
GA & \xrightarrow{Gf} GB \\
\downarrow \beta_A & \quad \downarrow \beta_B \\
FA & \xrightarrow{Ff} FB
\end{array}
$$

□

commutes. But as α is natural,

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

Definition. (1.9)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. By an *equivalence* between $\mathcal{C}$ and $\mathcal{D}$, we mean a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$.
We write $\mathcal{C} \cong \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent.
We say a property P of categories is a *categorical property* if whenever $\mathcal{C}$ has P and $\mathcal{C} \cong \mathcal{D}$, then $\mathcal{D}$ has P.
For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

Example. (1.10)
(a) The category **Part** is equivalent to the category $\mathbf{Set}_*$ of pointed sets (and basepoint preserving functions (as morphisms)):
• We define $F : \mathbf{Set}_* \to \mathbf{Part}$ by $F(A, a) = A \setminus \{a\}$, and if $f : (A, a) \to (B, b)$, then $Ff(x) = f(x)$ if $f(x) \neq b$, and undefined otherwise;
• and $G : \mathbf{Part} \to \mathbf{Set}_*$ by $G(A) = A^+ = (A \cup \{A\}, A)$, and if $f : A \to B$ is a partial function, we define $Gf : A^+ \to B^+$ by $Gf(x) = f(x)$ if $x \in A$ and $f(x)$ defined, and equals B otherwise.
The composite FG is the identity on **Part**, but GF is not the identity. However, there is an isomoprhism $(A, a) \to ((A \setminus \{a\})^+, A \setminus \{a\})$ sending a to $A \setminus \{a\}$ and everything else to itself and this is natural.
Note that there can be no isomoprhism from $\mathbf{Set}_*$ to **Part**, since **Part** has a 1-element isomorphism class $\{\phi\}$ but $\mathbf{Set}_*$ doesn't.
(So we see that equivalent categories can be non-isomorphic. According to a post on SO, this usually happens when there are multiple copies of the *same* thing in one but not the other. However, we can't generally *discard obsolete copies* in one as that generally requires AC and is not a very useful thing to do anyway – In short, *identifying isomorphic objects is often an extremely bad idea.*)
(b) The category $\mathbf{fdMod_K}$ of finite-dimensional vector spaces over K is equivalent to $\mathbf{fdMod_K}^{op}$, the functors in both directions are $*$ (the dual operator) and both isomorphisms are the natural transformations of 1.7(a) (double dual).
(c) $\mathbf{fdMod_K}$ is also equivalent to $\mathbf{Mat}_K$ (1.3(j)):
We define $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ by $F(n) = K^n$, and $F(A)$ is the linear map represented by A w.r.t. the standard bases of K^n and K^p.
To define $G : \mathbf{fdMod_K} \to \mathbf{Mat_K}$, choose a basis for each finite dimensional vector space, and define $G(V) = \dim V$, $G(V \xrightarrow{f} W)$ to be the matrix representing

f w.r.t. chosen bases. GF is the identity, provided we choose the standard bases for the spaces K^n; $FG \neq 1$, but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \to V$ for each V, which form a natural isomorphism.

—Lecture 4—

Definition. (1.11)
Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor.
(a) We say F is *faithful* if, given $f, f' \in \operatorname{mor} \mathcal{C}$ with $\operatorname{dom} f = \operatorname{dom} f'$, $\operatorname{cod} f = \operatorname{cod} f'$, and $Ff = Ff'$, then $f = f'$ (injectivity on morphisms. The name comes more from representation theory);
(b) We say F is *full* if, given $FA \xrightarrow{g} FB$ in $\mathcal{D}$, there exists $A \xrightarrow{f} B$ in $\mathcal{C}$ with $Ff = g$. (this is something like surjective, but see below);
(c) We say F is *essentially surjective* if, forevery $B \in \operatorname{ob} \mathcal{D}$, there exists $A \in \operatorname{ob} \mathcal{C}$ and isomorphism $FA \to B$ in $\mathcal{D}$.
We say a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is full if the inclusion $\mathcal{C}' \to \mathcal{C}$ is a full functor. For example, **Gp** is a full subcategory of **Mon** (the category of all monoids), but **Mon** is not a full subcategory of the category **SGp** of semigroups.

Lemma. (1.12)
Assuming the axiom of choice, a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ if it's full, faithful, and essentially surjective.

Proof. $\Rightarrow$: Suppose given G, α, β as in (1.9). Then for each $B \in \operatorname{ob} \mathcal{D}$, β_B is an isomorphism $FGB \to B$, so F is essentially surjective.
Given $A \xrightarrow{f} B$ in $\mathcal{C}$, we can recover f from Ff as composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_b^{-1}} B$. Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$, then $f = f'$. So F is faithful;
Lastly, for fullness, given $FA \xrightarrow{g} FB$, define f to be the composite $A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$. Then $GFf = \alpha_B f \alpha_A^{-1}$, which by construction is just Gg. But G is faithful for the same reason as f, so $Ff = g$.

$\Leftarrow$: (need to find suitable G, α, β for F.) For each $B \in \operatorname{ob} \mathcal{D}$, choose $GB \in \operatorname{ob} \mathcal{C}$ and an isomorphism $\beta_B : FGB \to B$ in $\mathcal{D}$. Given $B \xrightarrow{g} B'$, define $Gg : GB \to GB'$ to be the unique morphism whose image under F is $FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$.
Uniqueness implies functoriality (check what this means – think it appeared somewhere before): given $B' \xrightarrow{g'} B''$, $(Gg')(Gg)$ and $G(g'g)$ have the same image under F, so they are equal.
By construction, β is a natural trnasformation $FG \to 1_{\mathcal{D}}$.
Given $A \in \operatorname{ob} \mathcal{C}$, define $\alpha_A : A \to GFA$ to be the unique morphism whose image under F is $FA \xrightarrow{\beta_{FA}^{-1}} FGFA$. α_A is an isomorphism, since β_{FA} also has a unique pre-image under F. And α is a natural transformation, since any naturality square for α (the commutative square when we defined natural transformation. check) is mapped by F to a commutative square, and F is faithful. $\square$

Definition. (1.13)

By a *skeleton* of a category, we mean a full subcategory $\mathcal{C}_0$ containing one object from each isomorphism class. We say $\mathcal{C}$ is *skeletal* if it's a skeleton of itself.
For example, $\mathbf{Mat_K}$ is a skeletal, and the image of $F : \mathbf{Mat_K} \to \mathbf{fdMod_K}$ of 1.10(c) is a skeleton of $\mathbf{fdMod_K}$.
(there are some examples on wikipedia)

Warning: almost any assertion about skeletons is equivalent to axiom of choice (see q2 on example sheet 1).

Definition. (1.14)
Let $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$.
(a) We say f is a *monomorphism* (or f is *monic*) if, given any pair $C \overset{g}{\underset{h}{\rightrightarrows}} A$, $fg = fh$ implies $g = h$.
(b) We say f is an *epimorphism* (or *epic*) if it's a monomorphism in $\mathcal{C}^{op}$, i.e. if $gf = hf$ implies $g = h$.
We denote monomorphisms by $A \overset{f}{\rightarrowtail} B$, and epimorphisms by $A \overset{f}{\twoheadrightarrow} B$.
Any isomorphism is monic and epic: more generally, if f has a left inverse (i.e. $\exists g$ s.t. gf is an identity), then it's monic. We call such monomorphisms *split*.
We say $\mathcal{C}$ is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

Example. (1.15)
(a) As usual we consider **Set** first. In **Set**, monomorphisms correspond to injections ($\Leftarrow$ is easy (ok); for $\Rightarrow$, take $C = 1 = \{*\}$), and epimorphsims correspond to surjections ($\Leftarrow$ is easy; for $\Rightarrow$, use morphisms $B \rightrightarrows 2 = \{0, 1\}$). So **Set** is balanced.
(b) In **Gp**, monomorphisms again correspond to injections (for $\Rightarrow$ use homomorphisms $\mathbb{Z} \to A$); epimorphisms again correspond to surjections ($\Rightarrow$ use free products with amalgamation – this is a non-trivial fact about groups, read more if free). So **Gp** is also balanced.
(c) In **Rng** (obvious notation), monomorphisms correspond to injections (proof is much like for **Gp**). However, not all epimorphisms are surjective. For example the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism, since if $\mathbb{Q} \overset{f}{\underset{g}{\rightrightarrows}} R$ agree on all integers, they agree everywhere. So **Rng** is not balanced.
(d) One final example is **Top**. Again, monomorphisms are injections and epimorphisms are surjections (and vice versa): proof is similar to **Set** (check). However, **Top** is not balanced since a continuous bijection need not have continuous inverse.

2 The Yoneda Lemma

Let's not start on the content this lecture. Why are we talking about one single lemma in a chapter? Well it's not really a lemma. There's some story behind this, check here for an obituary which probably has the story that lecture was talking about in class.

—Lecture 5—

Definition. (2.1)
We say a category $\mathcal{C}$ is *locally small* if, for any two objects A, B, the morphisms $A \to B$ in $\mathcal{C}$ form a set $\mathcal{C}(A, B)$.
If we fix A and let B vary, the assignment $B \to \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathcal{C}(A, f)$ is the mapping $g \to fg$. Similarly, $A \to \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : C^{op} \to \mathbf{Set}$.

Lemma. (2.2)
(i) Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob} \mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then natural transformations $\mathcal{C}(A, -) \to F$ are in bijection with elements of FA;
(ii) Moreover, this bijection is natural in both A and F.

Proof. (i) Given α:$\mathcal{C}(A, -) \to F$, we define $\Phi(x) = \alpha_A(1_A) \in FA$. Given $x \in FA$, we define $\Psi(x) : \mathcal{C}(A, -) \to F$ by $\Psi(x)_B(A \xrightarrow{f} B) = Ff(x) \in FB$.
$\Psi(x)$ is natural: given $g : B \to C$, we have

$$\begin{aligned} \Psi(x)_C \mathcal{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x), \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x), \\ \Phi\Psi(x) &= \Psi(X)_A(1_A) = F(1_A)(x) = x \end{aligned}$$

Given α,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f)\Psi(\alpha_A(1_A))_B(f) &= Ff(\alpha_A(1_A)) \\ &= \alpha_B \mathcal{C}(A, f)(1_A) = \alpha_B(f) \end{aligned}$$

So $\Psi\Phi(\alpha) = \alpha$. $\square$

Corollary. (2.3)
The assignment $A \to \mathcal{C}(A, -)$ defines a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \mathbf{Set}]$.

Proof. Put $F = \mathcal{C}(B, -)$ in 2.2(i): we get a bijection between $\mathcal{C}(B, A)$ and morphisms $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ in $[\mathcal{C}, \mathbf{Set}]$. We need to verify this is functorial: but it sends $f : B \to A$ to the natural transformation $g \to gf$. So functoriality follows from associativity. $\square$

We call this functor (or the functor $\mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ sending A to $\mathcal{C}(-, A)$) the *Yoneda embedding* of $\mathcal{C}$, and denote it by Y.

Now let's go back to prove 2.2(ii):

Proof. (ii) Suppose for the moment that $\mathcal{C}$ is small, so that $[\mathcal{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$: one sends (A, F) to FA, and the other is the composite: $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{op} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-;-)} \mathbf{Set}$. 2.2(ii) says that these are naturally isomorphic. We can translate this into an elementary statement, making sense even when $\mathcal{C}$ isn't small. Given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, the two ways of producing an element of GB from a natural transformation $\beta : \mathcal{C}(A, -) \to F$ give the same result, namely

$$\alpha_B (Ff) \beta_A (1_A) = (Gf) \alpha_A \beta_A (1_A)$$

which is equal to $\alpha_B \beta_B (f)$. □

Definition. (2.4)
We say a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if it's isomorphic to $\mathcal{C}(A, -)$ for some A. By a representation of F, we mean a pair (A, x) where $x \in FA$ is such that $\Psi(x)$ is an isomorphism.
We also call x a *universal element* of F.

Corollary. (2.5)
If (A, x) and (B, y) are both representations of F, then there's a unique isomorphism $f : A \to B$ such that $(Ff)(x) = y$.

Proof. Consider the composite $\mathcal{C}(B, -) \xrightarrow{\Psi(y)^{-1}} F \xrightarrow{\Psi(x)} \mathcal{C}(A, -)$. By (2.3) this is of the form $Y(f)$ for a unique isomorphism $f : A \to B$, and the diagram

$$\begin{tikzcd}
\mathcal{C}(B,-) \arrow[rr, "Y(f)"] \arrow[dr, "\Psi(y)"'] & & \mathcal{C}(A,-) \arrow[dl, "\Psi(x)"] \\
& F &
\end{tikzcd}$$

commutes iff $(Ff)(x) = y$. □

Example. (2.6)
(a) The forgetful functor $\mathbf{Gp} \to \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$, $\mathbf{Rng} \to \mathbf{Set}$ by $(\mathbb{Z}[X], X])$, and $\mathbf{Top} \to \mathbf{Set}$ by $(\{*\}, *)$.
(b) The functor $P^* : \mathbf{Set}^{op} \to \mathbf{Set}$ is representable by $(\{0, 1\}, \{1\})$: this is the bijection between subsets and characteristic functions.

(c) Let G be a group. The unique (up to isomorphism) representable functor $G(*,-) : G \to \mathbf{Set}$ is the *Cayley representation* of G, i.e. the set $\cup G$ with G acting by left multiplication.
(d) Let A and B be two objects of a small category $\mathcal{C}$. We have a functor $\mathcal{C}^{op} \to \mathbf{Set}$ sending C to $\mathcal{C}(C,A) \times \mathcal{C}(C,B)$. A representation of this, if it exists, is called a (categorical) *product* of A and B, and denoted $(A \times B, (A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B))$.
This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$, there's a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.
Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top**,..., they are *just* cartesian products, in posets they are binary meets (see sheet 1 Q1).
Dually, we have the notion of *coproduct* $(A + B, A \xrightarrow{\mu_1} A + B, B \xrightarrow{\mu_2} A + B)$.
These also exist in many categories of interest.
—Lecture 6—
(f) (Did I miss (e)?) Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be morphisms in locally small category $\mathcal{C}$. We have a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by

$$F(C) = \{h \in \mathcal{C}(C,A) | fh = gh\}$$

A representation (see (2.4)) of F, if it exists, is called an *equalizer* of (f,g): It consists of an object E and a morphism $E \xrightarrow{e} A$ s.t. $fe = ge$, and every h with $fh = gh$ factors uniquely (see proof of 2.9(i) which gives an insight of what this means) through e.
In **Set**, we take $E = \{x \in A | f(x) = g(x)\}$ and $e =$inclusion. Similar constructions work in $\mathbf{Gp}, \mathbf{Rng}, \mathbf{Top}$,...
Dually, we have the notion of *coequalizer*.

Remark. (2.7)
If e occurs as an equalizer, then it is a monomorphism, since any h factors through it in at most one way. We say a monomorphism is *regular* if it occurs as an equalizer.
Split monomorphisms are regular (cf sheet1 Q6(i)).
Note that regular epic monomorphisms are isomorphisms: if the equalizer e of (f,g) is epic, then $f = g$, so $e \cong 1_{\operatorname{cod} e}$.

Definition. (2.8)
Let $\mathcal{C}$ be a category, $\mathcal{G}$ a class of objects of $\mathcal{C}$.
(a) We say $\mathcal{G}$ is a *separating family* for $\mathcal{C}$, if given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that $fh = gh$ for all $G \xrightarrow{h} A$ with $G \in \mathcal{G}$, then $f = g$.
(i.e. the functors $\mathcal{C}(G,-), G \in \mathcal{G}$, are collectively faithful.)
(b) We say $\mathcal{G}$ is a *detecting family* if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} B$ with $G \in \mathcal{G}$ factors uniquely through f, then f is an isomorphism.
If $\mathcal{G} = \{G\}$, we call G a *separator/detector*.

Lemma. (2.9)
(i) If $\mathcal{C}$ is a balanced category, then any saparating fmamily is detecting.
(ii) If $\mathcal{C}$ has equalizers, then any detecting family is separating.

Proof. (i) Suppose $\mathcal{G}$ is separating and $A \xrightarrow{f} B$ satisfies the condition of 2.8(b). If $B \underset{h}{\overset{g}{\rightrightarrows}} C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is epic.
Similarly if $D \underset{l}{\overset{k}{\rightrightarrows}} A$ satisfy $fk = fl$, then $ky = ly$ for any $G \xrightarrow{y} D$, since both are factorizations of fky through f. So $k = l$, i.e. f is monic.
But $\mathcal{C}$ is balanced. So f is an isomorphism.
(ii) Suppose $\mathcal{G}$ is detecting and $A \underset{g}{\overset{f}{\rightrightarrows}} B$ satisfies the condition of 2.8(a). Then the equalizer $E \xrightarrow{e} A$ of (f, g) is isomorphism, so $f = g$. □

Example. (2.10)
(a) In $[\mathcal{C}, \mathbf{Set}]$, the family $\{\mathcal{C}(A, -) | A \in \mathrm{ob}\, \mathcal{C}\}$ is both separating and detecting (just a restatement of Yoneda Lemma).
(b) In **Set**. $1 = \{*\}$ (any one element set) is both a separator and a detector, since it represents the identity functor $\mathbf{Set} \to \mathbf{Set}$.
Similarly, $\mathbb{Z}$ is both in **Gp**, since it represents the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$.
Also, $2 = \{0, 1\}$ is a coseparator and a codetector in **Set**, since it represents $P^* : \mathbf{Set}^{op} \to \mathbf{Set}$.
(c) In **Top**, $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \to \mathbf{Set}$, but not a detector.
In fact, **Top** has no detecting *set* of objects (note that this doesn't mean it has no detecting family).
For any infinite cardinal κ, let X be a discrete space of cardinality κ, and Y the same set with *co-*$< \kappa$ topology, i.e. $F \subseteq Y$ is closed iff $F = Y$ or $\mathrm{Card}\, F < \kappa$ (think about, e.g. cocountable topology, this name then makes sense).
The identity $X \to Y$ is continuous, but not a homeomorphism (topologically).
So if $\{G_i | i \in I\}$ is any set of spaces, taking $\kappa > \mathrm{Card}\, G_i$ for all i yields an example to show that the set is not detecting.
(d) (some Algebraic Topology stuff) Let $\mathcal{C}$ be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous mappings.
JHC Whitehead proved that $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all n, then it's an isomorphism in $\mathcal{C}$.
This says that $\{S^n | n \geq 1\}$ is a detecting set of $\mathcal{C}$.
But PJ Freyd showed there is no faithful functor $\mathcal{C} \to \mathbf{Set}$, so no separating *set*: if $\{G_i | i \in I\}$ were separating, then $x \to \coprod \mathcal{C}(G_i, x)$ (disjoint unions?) would be faithful.
Note that any functor of the form $\mathcal{C}(A, -)$ preserves monomorphisms, but they don't normally preserves epimorphisms.

Definition. (2.11)
We say an object P is *Projective* if, given

$$\begin{array}{cc} & P \\ & \downarrow f \\ A \overset{e}{\twoheadrightarrow} & B \end{array}$$

(recall the two head right arrow means epimorphisms) there exists $P \xrightarrow{g} A$ with

$eg = f$.
(If $\mathcal{C}$ is locally small, this says $\mathcal{C}(P, -)$ preserves epimorphisms).
Dually, an *injective* object of $\mathcal{C}$ is a projective object of $\mathcal{C}^{op}$.
Given a class $\mathcal{E}$ of epimorphisms, we say P is *$\mathcal{E}$-projective* if it satisfies the condition for all $e \in \mathcal{E}$.

Lemma. (2.12)
Representable functors are (pointwise)(?) projective in $[\mathcal{C}, \mathbf{Set}]$.

Proof. Suppose given

$$\begin{array}{ccc} & & \mathcal{C}(A,-) \\ & & \downarrow \beta \\ F & \overset{\alpha}{\twoheadrightarrow} & G \end{array}$$

where α is pointwise surjective. By Yoneda, β corresponds to some $y \in GA$, and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathcal{C}(A, -) \to F$ corresponds to x, then naturality of the Yoneda bijection yields $\alpha\gamma = \beta$. $\square$