Analysis II

October 31, 2016

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1 Vector spaces

1.1 Vector spaces

If $a_n \in \mathbb{R}$, $(a_n) \to a$ if for every $\epsilon > 0$, $\exists N$ such that $|a_n - a| < \epsilon$ whenever $n > N$.
Now consider a general vector space:

Definition. Let V be a real vector space. A *norm* on V is a function $||\cdot|| : V \to \mathbb{R}$ satisfying:
• $||\mathbf{v}|| \geq 0 \ \forall \mathbf{v} \in V$, and $||\mathbf{v}|| = 0 \iff \mathbf{v} = \mathbf{0}$;
•$||\lambda \mathbf{v}|| = |\lambda| \cdot ||\mathbf{v}||$, $\forall \lambda \in \mathbb{R}$ and $\mathbf{v} \in V$;
•$||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$, $\forall \mathbf{v}, \mathbf{w} \in V$ (triangle inequality).

Example. $||\mathbf{v}||_2 = \left(\sum v_i^2\right)^{\frac{1}{2}}$, the Euclidean norm;
$||\mathbf{v}||_1 = \sum |v_i|$;
$||\mathbf{v}||_\infty = \max \{|v_1|, ..., |v_n|\}$.

Example. Let $V = C[0,1] = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous}\}$. Then we can have the following norms:
• $||f||_1 = \int_0^1 |f(x)| dx$;
• $||f||_2 = \left(\int_0^1 f(x)^2 dx\right)^{\frac{1}{2}}$;
• $||f||_\infty = \max_{x \in [0,1]} |f(x)|$.

Notation. If $||\cdot||$ is a norm on V, we say the pair $(V, ||\cdot||)$ is a *normed space*.

Definition. Suppose $(V, ||\cdot||)$ is a normed vector space, and $(\mathbf{v}_n)$ is a sequence in V. We say $(\mathbf{v}_n)$ converges to $\mathbf{v} \in V$ if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, $||\mathbf{v}_n - \mathbf{v}|| < \varepsilon$.
Equivalently, $(\mathbf{v}_n) \to \mathbf{v}$ if and only if $||\mathbf{v}_n - \mathbf{v}|| \to 0$ in $\mathbb{R}$.

Example. Let $V = \mathbb{R}^n$, $\mathbf{v}_k = (v_{k,1}, ..., v_{k,n})$.
(a) $(\mathbf{v}_k) \to \mathbf{v}$ with respect to $||\cdot||_\infty$
$\iff ||\mathbf{v}_k - \mathbf{v}||_\infty \to 0$
$\iff \max \{|v_{k,i} - v_i|\} \to 0$
$\iff |v_{k,i} - v_i| \to 0$ for all $1 \leq i \leq n$
$\iff v_{k,i} \to v_i$.

So sequence converges if and only if every component converges.

(b) $(\mathbf{v}_k) \to \mathbf{v}$ with respect to $||\cdot||_1$
$\iff \sum_{i=1}^n |v_{k,i} - v_i| \to 0$
$\iff |v_{k,i} - v_i| \to 0$ for all $1 \leq i \leq n$
$\iff v_{k,i} \to v_i$.

Note the two different norms in (a) and (b) give the same notion of convergence.

We set a convention that, when talking about convergence in $\mathbb{R}^n$ without mentioning a norm, then it's with respect to $||\cdot||_1$ (or $||\cdot||_\infty$ or $||\cdot||_2$) (these all give the same notion of convergence).

Example. Let $V = C[0,1]$,

$$f_n(x) = \begin{cases} 1 - nx & x \in \left[0, \frac{1}{n}\right) \\ 0 & x \in \left[\frac{1}{n}, 1\right] \end{cases}$$

So

$$||f_n||_1 = \int_0^1 |f_n(x)|dx = \frac{1}{2n} \to 0$$

as $n \to \infty$. So $(f_n) \to 0$ with respect to $||\cdot||_1$.

On the other hand, $||f_n||_\infty = 1 \not\to 0$, so $(f_n) \not\to 0$ with respect to $||\cdot||_\infty$. Here the two different norms give two different notions of convergence.

1.2 Continuity

Let $(V, ||\cdot||)$ be a normed vector space.

Recall: If $\mathbf{v}_n \in V$ and $\mathbf{v} \in V$, the sequence $(\mathbf{v}_n) \to \mathbf{v}$ if for every $\varepsilon > 0$, there exists n such that $||\mathbf{v}_n - \mathbf{v}|| < \varepsilon$ when $n > N$.

Definition. Suppose V and W are normed spaces, and $f : V \to W$. We say f is *continuous* if the sequence $(f(\mathbf{v}_n)) \to f(\mathbf{v})$ in W whenever $(\mathbf{v}_n) \to \mathbf{v}$ in V.

Example. (1)$f : V \to \mathbb{R}^n$, $f(\mathbf{v}) = (f_1(\mathbf{v}), ..., f_n(\mathbf{v}))$. Then f is continuous if and only if $f_1, ..., f_n$ are all continuous.
(2) $p_i : \mathbb{R}^n \to \mathbb{R}$ by $p_i(\mathbf{v}) = v_i$. Then p_i is continuous.
(3) $V = C[0,1]$, $x \in [0,1]$, $p_x : C[0,1] \to \mathbb{R}$ by $p_x(f) = f(x)$ (linear map). Then p_x is continuous with respect to the uniform norm on $C[0,1]$:

$$\begin{aligned} & (f_n) \to f \text{ wrt } ||\cdot||_\infty \\ \iff & \max_{y \in [0,1]} |f_n(x) - f(x)| \to 0 \\ \implies & |f_n(x) - f(x)| \to 0 \\ \implies & (f_n(x)) \to f(x) \end{aligned}$$

However, p_x is not continuous with respect to $||\cdot||_1$ on $C[0,1]$. See examples in M&T.

So linear maps may not be continuous.
(4) If $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are continuous, so is $g \circ f : V_1 \to V_3$.
(5) $||\cdot|| : V \to \mathbb{R}$ is continuous.

Lemma. If $\mathbf{v}, \mathbf{w} \in V$, then $||\mathbf{w} - \mathbf{v}|| \geq |||\mathbf{w}|| - ||\mathbf{v}|||$.

Proof. Since $||\mathbf{v}|| + ||\mathbf{w} - \mathbf{v}|| \geq ||\mathbf{w}||$,
$||\mathbf{w} - \mathbf{v}|| \geq ||\mathbf{w}|| - ||\mathbf{v}||$.
Similarly, $||\mathbf{w} - \mathbf{v}|| = ||\mathbf{v} - \mathbf{w}|| \geq ||\mathbf{v}|| - ||\mathbf{w}||$. So $||\mathbf{w} - \mathbf{v}|| \geq |||\mathbf{w}|| - ||\mathbf{v}|||$. □

Now we can prove the 5^{th} example above:

Proof. Let $f(\mathbf{v}) = ||\mathbf{v}||$. Then if $(\mathbf{v}_n) \to \mathbf{v}$, $(||\mathbf{v}_n - \mathbf{v}||) \to 0$. But $||\mathbf{v}_n - \mathbf{v}|| \geq |||\mathbf{v}_n|| - ||\mathbf{v}||| = |f(\mathbf{v}_n) - f(\mathbf{v})| \geq 0$.
So by squeeze rule, $(|f(\mathbf{v}_n) - f(\mathbf{v})|) \to 0$, i.e. $f(\mathbf{v}_n) \to f(\mathbf{v})$. □

Proposition. $f : V \to W$ is continuous if and only if for every $\mathbf{v} \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||f(\mathbf{w}) - f(\mathbf{v})||_W < \varepsilon$$

whenever $||\mathbf{w} - \mathbf{v}||_V < \delta$.

Proof. Suppose the $\varepsilon - \delta$ condition hold. We'll show that f is continuous, i.e. if $(\mathbf{v}_n) \to \mathbf{v}$, then $(f(\mathbf{v}_n)) \to f(\mathbf{v})$.
Given $(\mathbf{v}_n) \to \mathbf{v}$ and $\varepsilon > 0$, pick $\delta > 0$ such that $||f(\mathbf{w}) - f(\mathbf{v})|| < \varepsilon$ whenever $||\mathbf{w} - \mathbf{v}|| < \delta$. Since $(\mathbf{v}_n) \to \mathbf{v}$, there exists N such that $||\mathbf{v}_n - \mathbf{v}|| < \delta$ whenever $n > N$, i.e. $||f(\mathbf{v}_n) - f(\mathbf{v})|| < \varepsilon$ when $n > N$. So $(f(\mathbf{v}_n)) \to f(\mathbf{v})$. So f is continuous.
If the $\varepsilon - \delta$ condition does not hold, then there exists $\mathbf{v} \in V$ and $\varepsilon > 0$ such that for every $n > 0$, there exists $\mathbf{v}_n$ with

$$||\mathbf{v} - \mathbf{v}_n|| < \frac{1}{n}$$

but

$$||f(\mathbf{v}) - f(\mathbf{v}_n)|| > \varepsilon$$

(Otherwise, take $\delta = \frac{1}{n}$ and we get a contradiction). Then $(\mathbf{v}_n) \to \mathbf{v}$, but $(f(\mathbf{v}_n)) \not\to f(\mathbf{v})$. So f is not continuous. □

1.2.1 Addendum

Suppose V, W are normed spaces and U_α is an open subset of V for all $\alpha \in A$. Let $U = \cup_{\alpha \in A} U_\alpha$.

Proposition. Suppose $f : U \to W$ and f is continuous on all U_α. Then f is continuous on U. It's important that U_α's are all open. For example, any $f : V \to W$ is continuous on $\{\mathbf{v}\}$, but may not be continuous on $\cup_{\mathbf{v} \in V} \{\mathbf{v}\} = V$.

Proof. Must show that given $\mathbf{v} \in U$ and $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$f(B_\delta(\mathbf{v}) \cap U) \subset B_\varepsilon(f(\mathbf{v}))$$

$\mathbf{v} \in \cup_{\alpha \in A} U_\alpha$, so $\mathbf{v} \in U_{\alpha_0}$ for some $\alpha_0 \in A$. f is continuous on U_{α_0}, so $\exists \delta_1 > 0$ s.t.

$$f(B_{\delta_1}(\mathbf{v}) \cap U_{\alpha_0}) \subset B_\varepsilon(f(\mathbf{v}))$$

U_{α_0} is open, so $\exists \delta_2 > 0$ s.t. $B_{\delta_2}(\mathbf{v}) \subset U_{\alpha_0}$.
Let $\delta = \min(\delta_1, \delta_2)$. Then $B_\delta(\mathbf{v}) \subset B_{\delta_1}(\mathbf{v})$ and $B_\delta(\mathbf{v}) \subset B_{\delta_2}(\mathbf{v}) \subset U_{\alpha_0}$.

So $B_\delta(\mathbf{v}) \subset B_{\delta_1}(\mathbf{v}) \cap U_{\alpha_0}$.
Thus

$$f(B_\delta(\mathbf{v}) \cap U) = f(B_\delta(\mathbf{v})) \subset f(B_{\delta_1}(\mathbf{v}) \cap U_{\alpha_0}) \subset B_\varepsilon(f(\mathbf{v}))$$

□

1.3 Open and Closed Subsets

Definition. If $\mathbf{v} \in V$ and $r > 0$,

$$B_r(\mathbf{v}) = \{\mathbf{w} \in V \| ||\mathbf{v} - \mathbf{w}|| < r\}$$

is the *open ball* of radius r centered at $\mathbf{v}$,

$$B_r(\mathbf{v}) = \{\mathbf{w} \in V \| ||\mathbf{v} - \mathbf{w}|| \leq r\}$$

is the *closed ball* of radius r centered at $\mathbf{v}$.

Now we can get an alternative definition of continuous:
• f is continuous if and only if for every $\mathbf{v} \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(\mathbf{v})) \subset B_\varepsilon(f(\mathbf{v}))$.

Definition. $U \subset V$ is an *open subset* of V if for every $\mathbf{u} \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{u}) \subset U$.

Proposition. If $f : V \to W$ is continuous and $U \subset W$ is open, then $f^{-1}(U)$ is open in V.

Proof. Suppose $\mathbf{v} \in f^{-1}(U)$, i.e. $f(\mathbf{v}) \in U$.
U is open, so there exists $\varepsilon > 0$ such that $B_\varepsilon(f(\mathbf{v})) \subset U$.
f is continuous, so $\exists \delta > 0$ such that $f(B_\delta(\mathbf{v})) \subset B_\varepsilon(f(\mathbf{v})) \subset U$, i.e. $B_\delta(\mathbf{v}) \subset f^{-1}(U)$ so $f^{-1}(U)$ is open.
The converse is also true(see M&T). □

Definition. (Open subsets) Recall $U \subset V$ is *open* in V if for every $\mathbf{u} \in U$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(\mathbf{u}) \subset U$.

Proposition. If $f : V \to W$ is continuous and $U \subset W$ is open, then $f^{-1}(U)$ is open in V.

Example. Given $\mathbf{v} \in V$, define

$$f_\mathbf{v} : V \to \mathbb{R}$$
$$f_\mathbf{v}(\mathbf{w}) = ||\mathbf{v} - \mathbf{w}||$$

Then $f_\mathbf{v}$ is continuous, so

$$B_r(\mathbf{v}) = f_\mathbf{v}^{-1}((-r, r))$$

is open in V, i.e. open balls are open.

Definition. (Closed subsets) Recall if $C \subset V$, $V - C = \{\mathbf{v} \in V | \mathbf{v} \notin C\}$ is the *complement* of C. $C \subset V$ is *closed* if $V - C$ is an open subset of V.

Corollary. If $f : V \to W$ is continuous and C is closed in W, then $f^{-1}(C)$ is closed in V.

Example. Let

$$C = \{(x, f(x)) \, | x \in \mathbb{R}\}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then C is closed in $\mathbb{R}^2$.

Proof. Let $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x) - y$ which is continuous.
Then $C = F^{-1}(\{0\})$ is closed, since $\{0\}$ is closed in $\mathbb{R}$. □

Example.

$$\overline{B}_r(\mathbf{v}) = f_{\mathbf{v}}^{-1}([0, r])$$

is closed in any normed space V.

Example. $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

Example. $V \subset V$, $\phi \subset V$ are both open and closed.

Proposition. C is closed in V if and only if for every sequence $(\mathbf{v}_n) \to \mathbf{v} \in V$ which satisfies $\mathbf{v}_n \in C$ for all n, we have $\mathbf{v} \in C$ as well.

Proof. Suppose C is closed in V, and $(\mathbf{v}_n) \to \mathbf{v}$ with $\mathbf{v} \notin C$.
Now $V - C$ is open, and $\mathbf{v} \in V - C$. So $\exists \varepsilon > 0$ s.t. $B_\varepsilon(\mathbf{v}) \subset V - C$.
Since $(\mathbf{v}_n) \to \mathbf{v}$, there exists N s.t. $\mathbf{v}_n \in B_\varepsilon(\mathbf{v}) \subset V - C$ for all $n > N$. So $\mathbf{v}_n \notin C$. Contradiction.

Conversely, suppose that C is not closed. Then $V - C$ is not open. So there exists $\mathbf{u} \in V - C$ such that for every $\varepsilon > 0$, $B_\varepsilon(\mathbf{v}) \not\subset V - C$, i.e. $B_\varepsilon(\mathbf{v}) \cap C \neq \phi$. Now pick $\mathbf{v}_n$ s.t. $\mathbf{v}_n \in B_{1/n}(\mathbf{v}) \cap C$. Then $||\mathbf{v}_n - \mathbf{v}|| < \frac{1}{n} \to 0$, so $(\mathbf{v}_n) \to \mathbf{v}$ for all $\mathbf{v}n \in C$, but $\mathbf{v} \notin C$. Contradiction. □

1.4 Lipschitz equivalence

We've seen in the first lecture that $|| \cdot ||_1$,$|| \cdot ||_2$,$|| \cdot ||_\infty$ all induce the same notion of convergence on $\mathbb{R}^n$. So $f : \mathbb{R}^n \to V$ is continuous with respect to $|| \cdot ||$ if and only if it's continuous with respect to $|| \cdot ||_\infty$.

Proposition. Suppose $|| \cdot ||$,$|| \cdot ||'$ are two norms on V. The map $id : (V, || \cdot ||) \to (V, || \cdot ||')$ by $id(\mathbf{v}) = \mathbf{v}$ is continuous if and only if there exists some constants $C > 0$ such that

$$||\mathbf{v}||' \leq C||\mathbf{v}||$$

for all $\mathbf{v} \in V$.

Proof. Suppose $||\mathbf{v}||' \leq C||\mathbf{v}||$ for all $\mathbf{v} \in V$.
If $(\mathbf{v}_n) \to \mathbf{v}$ with respect to $||\cdot||$, then $(||\mathbf{v} - \mathbf{v}_n||) \to 0$. But then

$$0 \leq ||\mathbf{v} - \mathbf{v}_n||' \leq C||\mathbf{v} - \mathbf{v}_n||$$

By the squeeze law, $||\mathbf{v} - \mathbf{v}_n||' \to 0$ as well. So $(\mathbf{v}_n) \to \mathbf{v}$ with respect to $||\cdot||'$. This means $id : (V, ||\cdot||) \to (V, ||\cdot||')$ is continuous.

Conversely, suppose $id : (V, ||\cdot||) \to (V, ||\cdot||')$ is continuous. Then there exists $\delta > 0$ s.t. $B_\delta(\mathbf{0}, ||\cdot||) \subset B_1(\mathbf{0}, ||\cdot||')$.
For any $\mathbf{v} \in V, \mathbf{v} \neq 0$, there exists k s.t. $||k\mathbf{v}|| = \frac{\delta}{2}$. So $k\mathbf{v} \in B_\delta(\mathbf{0}, ||\cdot||)$, so $k\mathbf{v} \in B_1(\mathbf{0}, ||\cdot||')$, i.e. $||k\mathbf{v}||' < 1 = \frac{2}{\delta}||k\mathbf{v}||$. Divide by $|k|$ we get

$$||\mathbf{v}||' \leq \frac{2}{\delta}||\mathbf{v}||$$

for all $\mathbf{v} \neq \mathbf{0}$. So we can take $C = \frac{2}{\delta}$. The case $\mathbf{v} = \mathbf{0}$ is trivial. $\square$

Definition. If $||\cdot||$ and $||\cdot||'$ are two norms on V, we say they are *Lipschitz equivalent* if there exists $C > 0$ s.t.

$$\frac{1}{C}||\mathbf{v}|| \leq ||\mathbf{v}||' \leq C||\mathbf{v}||$$

for all $\mathbf{v} \in V$, or say there exists C_1, C_2 such that

$$||\mathbf{v}|| \leq C_1||\mathbf{v}||'$$

and

$$||\mathbf{v}||' \leq C_2||\mathbf{v}||$$

That is also equivalent to

$$id : (V, ||\cdot||) \to (V, ||\cdot||')$$

and

$$id : (V, ||\cdot||') \to (V, ||\cdot||)$$

being both continuous.

Corollary. If $||\cdot||$ and $||\cdot||'$ are Lipschitz equivalent, then:
(a) $(\mathbf{v}_n) \to \mathbf{v}$ with respect to $||\cdot||$ if and only if $(\mathbf{v}_n) \to \mathbf{v}$ with respect to $||\cdot||'$.
(b) $f : V \to W$ is continuous with respect to $||\cdot||$ if and only if $f : V \to W$ is continuous with respect to $||\cdot||'$.
(c) $g : W \to V$ is continuous with respect to $||\cdot||$ if and only if $g : W \to V$ is continuous with respect to $||\cdot||'$.

Example. $||\mathbf{v}||_\infty \leq ||\mathbf{v}||_2 \leq ||\mathbf{v}||_1 \leq n||\mathbf{v}||_\infty$ for all $\mathbf{v} \in \mathbb{R}^n$. So $||\cdot||_\infty$, $||\cdot||_2$, $||\cdot||_1$ are all Lipschitz equivalent.

Problem. Can we find a norm on $\mathbb{R}^n$ that is not Lipschitz equivalent to these?

2 Uniform Convergence

2.1 Notions of Convergence

Let $A \subset \mathbb{R}$, $f, f_n : A \to \mathbb{R}$.

We've known the definition of continuous and boundedness from Analysis I. Now define $C(A)$ to be the set of continuous functions $f : A \to \mathbb{R}$, and $B(A)$ to be the set of bounded functions $F : A \to \mathbb{R}$. Both of these are vector spaces.

We have $C[0,1] \subset B[0,1]$ by maximum value theorem, while $C(0,1) \not\subset B(0,1)$ (take $f(x) = \frac{1}{x}$).

Definition. If $f, f_n : A \to \mathbb{R}$, we say $(f_n) \to f$ *pointwise* if $(f_n(x)) \to f(x)$ for every $x \in A$.

Definition. The *uniform norm* $||\cdot||_\infty$ on $B(A)$ is given by

$$||f||_\infty = \sup_{x \in A} |f(x)|$$

If $f, f_n : A \to \mathbb{R}$, we say $(f_n) \to f$ *uniformly* if $||f - f_n||_\infty \to 0$.

Equivalently, if $(f_n) \to f$ pointwise, then for every $x \in A$ and $\epsilon > 0$, $\exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon$ whenever $n > N$.
If $(f_n) \to f$ uniformly, given ε, we need to find some N that works for all $x \in A$.

Example. Let $A = \mathbb{R}$, $f_n(x) = x + \frac{1}{n}$, $f(x) = x$. Then $(f_n) \to f$ pointwise and uniformly.

Example. Let $A = \mathbb{R}$, $g_n(x) = \left(x + \frac{1}{n}\right)^2$, $g(x) = x^2$. Then $g(n) \to g$ pointwise, but $g_n - g = \frac{2x}{n} + \frac{1}{n^2}$ is not even bounded. So (g_n) does not converge to g uniformly. Nevertheless, $(g_n) \to g$ uniformly on $[a, b]$ for any $a, b \in \mathbb{R}$) (since convergence and uniform convergence is the same on compact sets).

Example. If $(f_n) \to f$ uniformly, then $(f_n) \to f$ pointwise (Immediate from definition).

Theorem. Suppose $f_n \in C(A)$ and $(f_n) \to f$ uniformly on A. Then $f \in C(A)$.

Proof. Given $x \in A$ and $\varepsilon > 0$, we need to find $\delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon$$

whenever $|x - y| < \delta$ and $y \in A$.
Since $(f_n) \to f$ uniformly, $\exists N$ s.t.

$$|f_n(y) - f(y)| < \frac{\varepsilon}{4}$$

whenever $n \geq N$ and $y \in A$.
Since f_N is continuous, $\exists \delta > 0$ s.t.

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{2}$$

whenever $|x-y|<\delta$ and $y \in A$. Then for $|x-y|<\delta$ and $y \in A$,

$$\begin{aligned} |f(x)-f(y)| &\leq |f(x)-f_N(x)|+|f_N(x)-f_N(y)|+|f_N(y)-f(y)| \\ &< \frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4}=\varepsilon \end{aligned}$$

which is what we wanted to prove. □

Corollary. $C[a,b]$ is a closed subset of $B[a,b]$ with respect to $||\cdot||_\infty$.

Proof. Recall that C is closed if $c \in C$ whenever $(c_n) \to c$ and $c_n \in C$. □

Example. Let $A=[0,1]$, $f_n(x)=x^n$, $f(x)=\begin{cases} 0 & x \in [0,1) \\ 1 & x=1 \end{cases}$.
Then $(f_n) \to f$ pointwise but not uniformly, since $f_n \in C[0,1]$, but $f \notin C[0,1]$.

Example. Let $f_n(x)=(1-x)x^n$. Then $(f_n) \to 0$ pointwise. In fact $(f_n) \to 0$ uniformly.

Proof. Given $\varepsilon>0$, we must find N s.t. $|f_n(x)|<\varepsilon$ for all $x \in [0,1]$ whenever $n>N$.

We know $1-\varepsilon<1$, so $(1-\varepsilon)^n \to 0$. Pick N s.t. $(1-\varepsilon)^n<\varepsilon$ whenever $n>N$. Then for $n>N$,

$$|(1-x)x^n|<1\cdot(1-\varepsilon)^n<\varepsilon$$

for $x \in [0,1-\varepsilon]$, and

$$|(1-x)x^n|<\varepsilon\cdot 1^n=\varepsilon$$

for $x \in (1-\varepsilon,1]$. □

Everything so far in this chapter works for $f: A \to W$, where $A \subset V$ and V, W are both normed spaces. (exercise)

Recall that if $f, f_n \in C[a,b]$ with $a,b \in \mathbb{R}$, then $(f_n) \to f$ in L^1 (with respect to $||\cdot||_1$) if

$$||f_n-f||_1=\int_a^b |f_n(x)-f(x)| \to 0$$

Lemma. If $(f_n) \to f$ uniformly on $[a,b]$ and $f_n \in C[a,b]$, then $(f_n) \to f$ in L^1 on $[a,b]$.

Proof. $(f_n) \to f$ uniformly implies that $f \in C[a,b]$.
Given $\varepsilon>0$, pick N s.t.

$$|f_n(x)-f(x)|<\frac{\varepsilon}{(b-a)}$$

for $n>N$ and $x \in [a,b]$. Then

$$||f_n-f||_1=\int_a^b |f_n(x)-f(x)|\,dx<\int_a^b \frac{\varepsilon}{b-a}dx=\varepsilon$$

So $(f_n) \to f$ in L^1. □

Example. Let $A = [0,1]$,

$$f_n(x) = \begin{cases} nx & x \in \left[0, \frac{1}{n}\right] \\ 2 - nx & x \in \left[\frac{1}{n}, \frac{2}{n}\right] \\ 0 & x \in \left[\frac{2}{n}, 1\right] \end{cases}$$

Then $(f_n) \to 0$ pointwise, and in L^1, but not uniformly.

Example. Let $A = [0,1]$,

$$f_n(x) = \begin{cases} n^2 x & x \in \left[0, \frac{1}{n}\right] \\ 2n - n^2 x & x \in \left[\frac{1}{n}, \frac{2}{n}\right] \\ 0 & x \in \left[\frac{2}{n}, 1\right] \end{cases}$$

Then $(f_n) \to f$ pointwise, but not in L^1, nor uniformly.

We woud like to say that a sequence of bounded integrable functions on $[0,1]$ that converges pointwise converges in L^1. But for this to be true, we need a better definition of $\int$ (in measure and probability).

2.2 Power series

Recall some facts about series of complex numbers from Analysis I, for $\sum_{i=0}^{\infty} c_i$, $c_i \in \mathbb{C}$:
1) $\sum_{i=0}^{\infty} c_i = c$ means $\left(\sum_{i=0}^{n} c_i\right) \to c$;
2) $\sum_{i=0}^{\infty} c_i$ converges if and only if $\sum_{i=k}^{\infty} c_i$ converges;
3) $\sum_{i=k}^{\infty} \alpha^i = \frac{\alpha^k}{1-\alpha}$ if $|\alpha| < 1$;
4) If $\sum_{i=0}^{\infty} c_i$ converges, then $(c_n) \to 0$;
5) If $0 < a_i < b_i$ for all i (here $a_i, b_i \in \mathbb{R}$), and $\sum_{i=0}^{\infty} b_i$ converges, then $\sum_{i=0}^{\infty} a_i$ converges as well;
6) If $\sum_{i=0}^{\infty} |c_i|$ converges, then $\sum_{i=0}^{\infty} c_i$ converges.

Corollary. If $|c_i| < b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges, then $\sum_{i=0}^{\infty} c_i$ converges.

Proof. Follows from (5) and (6). □

Definition. A *power series* is

$$\sum_{i=0}^{\infty} a_i \left(z_i\right)^i$$

where $a_i, c, z \in \mathbb{C}$. Call c the *center* of the series.

Proposition. Suppose $\sum_{i=0}^{\infty} a_i \left(z_0 - c\right)^i$ converges for some $z_0 \in \mathbb{C}$. Then the series $\sum_{i=0}^{\infty} a_i \left(z_0 - c\right)^i$ converges for all z with $|z - c| < |z_0 - c|$.

Proof. By (4), $\left(a_i \left(z_0 - c\right)^i\right) \to 0$. Pick N such that $\left|a_i \left(z_0 - c\right)^i\right| < 1$ for all $i \geq N$.

By (2), suffices to show that $\sum_{i=N}^{\infty} a_i (z-c)^i$ converges. Now

$$|a_i (z-c)^i| = |a_i (z_0 - c)^i| \cdot \left|\frac{z-c}{z_0-c}\right|^i \leq 1 \cdot \alpha^i$$

(call this 'Key Estimate', to be used later) for $i \geq N$ where $\alpha = \left|\frac{z-c}{z_0-c}\right|$.
For $|z-c| < |z_0 - c|$, $\alpha < 1$, so $\sum_{i=N}^{\infty} \alpha^i$ converges.
By corollary, it follows that $\sum_{i=0}^{\infty} a_i (z-c)^i$ converges. □

Definition.

$$R = \sup\left\{ |z-c| \middle| \sum_{i=0}^{\infty} a_i (z-c)^i \text{ converges} \right\}$$

is the *radius of convergence* of this series.

The above proposition says that $\sum_{i=0}^{\infty} a_i (z-c)^i$ converges for all $z \in B_R(c) = \{z \in \mathbb{C} | |z-c| < R\}$.

We can define $f : B_R(c) \to \mathbb{C}$ by

$$f(z) = \sum_{i=0}^{\infty} a_i (z-c)^i$$

Let

$$p_n(z) = a_i (z-c)^i$$

Then $(p_n) \to f$ pointwise on $B_R(c)$.

Theorem. With notation as above, $(p_n) \to f$ uniformly on $\bar{B}_r(c) = \{z \in \mathbb{C} | |z-c| \leq r\}$ for any $r < R$.

Proof. Fix $z_0 \in \mathbb{C}$ with $r < |z_0 - c| < R$. Then $\sum_{i=0}^{\infty} a_i (z_0 - c)^i$ converges. Let

$$E_n(z) = f(z) - p_n(z) = \sum_{i=n+1}^{\infty} a_i (z-c)^i$$

We want to show that given $\varepsilon > 0$, $\exists N$ s.t. $|E_n(z)| < \varepsilon$ for all $n > N$ and $z \in \bar{B}_r(c)$.

Pick N_0 with $|a_i (z_0 - c)^i| < 1$ for all $i \geq N_0$ as in the proof of the previous proposition.

Now for $n > N_0$, Key Estimate says that

$$\begin{aligned} |E_n(z)| &= \left| \sum_{i=m}^{\infty} a_i (z-c)^i \right| \\ &\leq \sum_{i=n+1}^{\infty} |a_i (z-c)^i| \\ &\leq \sum_{i=n+1}^{\infty} \alpha (z)^i \end{aligned}$$

where $\alpha(z) = \frac{|z-c|}{|z_0-c|}$.
If $z \in \bar{B}_r(c)$, $\alpha(z) \le \alpha_0 = \frac{r}{|z_0-c|} < 1$. So

$$|E_n(z)| \le \sum_{i=1}^{\infty} \alpha^i = \frac{\alpha_0^{n+1}}{1-\alpha_0}$$

Now $\alpha_0 < 1$, so $\frac{\alpha_0^{n+1}}{1-\alpha_0} \to 0$ as $n \to \infty$. Pick $N > N_0$ s.t. $\frac{\alpha_0^{n+1}}{1-\alpha_0} < \varepsilon$ for $n > N$. Then $|E_n(z)| < \varepsilon$ for all $n > N$ and $z \in \bar{B}_r(c)$ which is what we wanted. □

Remark. (p_n) may not converge uniformly on $B_R(c)$. For example, $\sum_{i=0}^{\infty} x^i$ has $R = 1$, and equals $f(x) = \frac{1}{1-x}$ on $B_1(0)$, but p_n is a polynomial, so bounded on $\bar{B}_1(0)$, so $f(x) - p_n(x)$ is not even a bounded function on $B_1(0)$.

Corollary.

$$f(z) = \sum_{i=0}^{\infty} a_i (z-c)^i$$

is a continuous map $f : B_R(c) \to \mathbb{C}$.

Proof. $p_n = \sum_{i=0}^{n} a_I (z-c)^i$ is a polynomial, so is continuous as a map $\mathbb{C} \to \mathbb{C}$. $(p_n) \to f$ uniformly on $\bar{B}_r(c)$ for any $r < R$, so $f : \bar{B}_r(c) \to \mathbb{C}$ is continuous for any $r < R$.
Given $z \in B_R(c)$, pick r with $z \in B_r(c)$. Then f is continuous at z. So f is continuous at all $z \in B_R(c)$, i.e. $f : B_R(c) \to \mathbb{C}$ is continuous. □

We can now construct lots of continuous functions using power series.

Example.

$$\exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

has $R = \infty$, so is a well defined, continuous function on $\mathbb{C}$.

Let $f(x) = \exp(x)$ for $x \in \mathbb{R}$. We want to show that $f'(x) = f(x)$:

$$\frac{d}{dx}\left(\sum_{i=0}^{\infty} \frac{x_i}{i!}\right) = \sum_{i=0}^{\infty} \frac{ix^{i-1}}{i!} = \sum_{i=1}^{\infty} \frac{x^{i-1}}{(i-1)!} = \exp(x)$$

this looks easy, but why does the first equality hold?

Example. Suppose

$$\sum_{i=0}^{\infty} a_i (z-c)$$

has radius of convergence R. Then if $p_n = \sum_{i=0}^{\infty} a_i (z-c)^i$, $(p_n) \to f(z) = \sum_{i=0}^{\infty} a_i (z-c)^i$ uniformly on $\bar{B}_r(c)$ for all $r < R \implies f$ is continuous on $\bar{B}_r(c)$ for $r \in R$.
Take $U_r = B_r(c)$, so f is continuous on U_r for $r < R$. U_r is open. So f is continuous on $\cup_{r<R} U_r = B_R(c)$.

2.3 Integration and Differentiation

Recall from Analysis I:

Theorem. (Fundamental Theorem of Calculus) If $f \in C[a,b]$, then

$$F(x) = \int_{x_0}^{x} f(y)\, dy$$

exists, and

$$F'(x) = f(x).$$

Some properties of integral:
Suppose $f, g \in C[a,b]$.
(1)

$$\int_{x_0}^{x} f(y) + \lambda g(y)\, dy = \int_{x_0}^{x} f(y)\, dy + \lambda \int_{x_0}^{x} g(y)\, dy$$

(2) If $f(y) \leq g(y)$ for all $y \in [a,b]$, then

$$\int_{x_0}^{x} f(y)\, dy \leq \int_{x_0}^{x} g(y)\, dy$$

(3)

$$\left| \int_{x}^{x_0} f(y)\, dy \right| \leq \left| \int_{x}^{x_0} |f(y)|\, dy \right|$$

Suppose $f_n \in C[a,b]$ and $(f_n) \to f$ uniformly on $[a,b]$. So $f \in C[a,b]$. Thus

$$F(x) = \int_{x_0}^{x} f_n(y)\, dy$$

and

$$F(x) = \int_{x_0}^{x} f(y)\, dy$$

are defined.

Proposition. $(F_n) \to F$ uniformly on $[a,b]$.

Proof. $(f_n) \to f$ uniformly, so given $\varepsilon > 0$, $\exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon$$

for all $n > N$ and $x \in [a,b]$. Choose N s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$$

for all $n > N$ and $x \in [a,b]$. Then for $x \in [a,b]$,

$$\begin{aligned}
|F_n(x) - F(x)| &= \left| \int_{x_0}^{x} (f_n(y) - f(y))\, dy \right| \\
&\leq \left| \int_{x_0}^{x} |f_n(y) - f(y)|\, dy \right| \\
&\leq \left| \int_{x_0}^{x} \frac{\varepsilon}{b-a} dy \right| dy \\
&= \frac{\varepsilon |x - x_0|}{|b-a|} \\
&\leq \varepsilon
\end{aligned}$$

So $(F_n) \to F$ uniformly on $[a,b]$. □

Note that $(f_n) \in C(\mathbb{R})$, $(f_n) \to f$ uniformly does not imply $(F_n) \to F$ uniformly on $\mathbb{R}$. (But does on $[a,b]$ for $a, b \in \mathbb{R}$).

Let

$$f(y) = \sum_{i=0}^{\infty} a_i (y-c)^i$$

be a real power series ($a_i, c, y \in \mathbb{R}$) with radius of convergence R. Then if the partial sum $p_n(y) = \sum_{i=0}^{n} a_i (y-c)^i$, then $(p_n) \to f$ uniformly on $[c-r, c+r]$ for any $r < R$.

Corollary.

$$\int_c^x f(y)\, dy = \sum_{i=0}^{\infty} \frac{a_i}{i+1} (x-c)^{i+1}$$

for all $x \in (c-R, c+R)$.

Proof. Given $x \in (c-R, c+R)$, pick r with $|x-c| < r < R$. Then $(p_n) \to f$ uniformly on $[c-r, c+r]$, so by proposition

$$(P_n) \to \int_c^x f(y)\, dy$$

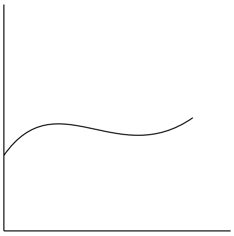
where

$$P_n = \int_c^x p_n(y)\, dy = \sum_{i=0}^{n} \frac{a_i}{i+1} (x-c)^{i+1}$$

□

Q: If $(f_n) \to f$ uniformly, what can I say about (f_n)?
A: Nothing, because:

Example. Take $f_n(x) = \frac{1}{n} \sin nx$, $x \in [0, \pi]$. Then $(f_n) \to 0$ uniformly on $[0, \pi]$, but $f_n'(x) = \cos nx$ doesn't converge for any $x \in (0, \pi)$.



Proposition. If

$$f(y) = \sum_{i=0}^{\infty} a_i (y-c)^i$$

converges on $(c-R, c+R)$, then

$$f(y) = \sum_{i=0}^{\infty} i a_i (y-c)^{i-1}$$

on $(c-R, c+R)$.

Proof.

Lemma.

$$\sum_{i=0}^{\infty} i a_i (y-c)^{i-1}$$

converges for all $y \in (c-R, c+R)$.

Pick y_0 with $|y-c| < |y_0 - c| < R$.
$\sum_{i=0}^{\infty} a_i (y-c)^i$ converges, so by 'Key Estimate', $\exists N$ s.t.

$$|a_i (y-c)|^i < \alpha^i$$

for all $i \geq N$, where $\alpha = \left|\frac{y-c}{y_0-c}\right| < 1$.

If $y = c$, $\sum i a_i (y-c)^{i-1}$ obviously converges. If not, estimate

$$\left|i a_i (y-c)^{i-1}\right| < \frac{i}{|y-c|}\alpha^i$$

Now $\sum_{i=0}^{\infty} \frac{i}{|y-c|}\alpha^i$ converges by Ratio Test. So $\sum_{i=0}^{\infty} i a_i (y-c)^{i-1}$ converges as well. $\square$

Now begin the proof of proposition:

$$g(y) = \sum_{i=0}^{\infty} i a_i (y-c)^{i-1}$$

is continuous on $(c-R, c+R)$. So by corollary,

$$\int_c^x g(y)\, dy = \sum_{i=1}^{\infty} a_i (x-c)^i = f(x) - f(c)$$

By Fundamental Theorem of Calculus, $f'(x) = g(x)$. $\square$

Application: Power series solutions of ODEs are legit (as long as we check the radius of convergence).

3 Compactness

3.1 Compact subsets of $\mathbb{R}^n$

Let V be a normed space. Then if $(\mathbf{v}_n) \to \mathbf{v} \in V$ and $(\mathbf{v_{n_j}})$ is a subsequence of $(\mathbf{v}_n)$, then $(\mathbf{v_{n_j}}) \to \mathbf{v}$. We leave this as an exercise.

Definition. $A \subset V$ is bounded if $\exists M \in \mathbb{R}$ s.t. $||\mathbf{v}|| \leq M$ for all $\mathbf{v} \in A$.

If $|| \cdot ||$ and $|| \cdot ||'$ are Lipschitz equivalent, then boundedness with respect to the two norms are equivalent.

Corollary. (Bolzano-Weierstrass in $\mathbb{R}^n$) If $(\mathbf{v}_k)$ is a bounded sequence in $\mathbb{R}^n$, it has a converging subsequence.

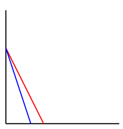
Proof. To prove this, simply pick a subsequence with the first coordinate convergent, then pick a subsequence of that subsequence with the second coordinate convergent, etc..

Let $\mathbf{v}_k = (v_{1,k}, ..., v_{n,k})$.
$(\mathbf{v}_k)$ is bounded, so $(v_{i,k})$ is bounded for all $1 \leq i \leq n$. By B-W theorem, there exists a convergent subsequence $\left(v_{1,k_j^1}\right)$ of $(v_{1,k})$. Now the sequence $\left(v_{2,k_j^1}\right)$ is bounded. So by B-W, there exists a subsequence $\left(v_{2,k_j^2}\right)$ which converges. Then by the previous exercise, $\left(v_{1,k_j^2}\right)$ converges.
Now consider the sequence $\left(v_{3,k_j^2}\right)$. By B-W, it has a convergent subsequence $\left(v_{3,k_j^3}\right)$. etc.

Apply B-W n times, we get $\left(\mathbf{v}_{k_j^n}\right)$ of original $(\mathbf{v}_n)$ s.t. $\left(v_{i,k_j^n}\right)$ converges for $1 \leq i \leq n$. So $\left(\mathbf{v}_{k_j^n}\right)$ converges. $\square$

Example. Let $V = C\,[0,1]$ with $|| \cdot ||_\infty$, and

$$f_n(x) = \begin{cases} 1 - nx & x \in \left[0, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right] \end{cases}$$



If

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

then $(f_n) \to f$ pointwise. Then (f_n) is bounded with respect to $|| \cdot ||_\infty$ but has no convergent subsequence.

Proof. Suppose $\left(f_{n_j}\right) \to g$ uniformly, then $\left(f_{n_j}\right) \to g$ pointwise, so $g = f$. But $f \notin C[0,1]$, so $\left(f_{n_j}\right) \not\to f$ uniformly. $\square$

Definition. We say $A \subset V$ is sequentially compact (s.compact) if any sequence $(\mathbf{v}_n)$ in A has a convergent subsequence $\left(\mathbf{v}_{n_j}\right) \to \mathbf{v} \in A$.

Example. R is not s.compact, since (n) has no convergent subsequence.

Example. $A = (0,2)$ is not s.compact, since $\left(\frac{1}{n}\right) \to 0 \notin A$.

Proposition. Suppose $A \subset V$ is s.compact. Then A is closed in V and bounded.

Proof. We prove the contrapositive:

If A is not closed, then there exists a sequence $(\mathbf{v}_n) \to \mathbf{v}$ with $\mathbf{v}_n \in A$ for all n but $\mathbf{v} \notin A$. By the exercise, any subsequence $\left(\mathbf{v}_{n_j}\right)$ converges to $\mathbf{v} \notin A$. So A is not s.compact.

If A is not bounded, then for all $n \in \mathbb{N}$ we can find $\mathbf{v}_n \in A$ with $||\mathbf{v}_n|| \geq n$. We claim that $\left(\mathbf{v}_{n_j}\right)$ has no convergent subsequence: if $\left(\mathbf{v}_{n_j}\right) \to \mathbf{v}$, then $\exists J$ s.t. $||\mathbf{v}_{n_j} - \mathbf{v}|| < 1$ for all $j > J$. So

$$||v_{n_j}|| \leq ||\mathbf{v}|| + ||\mathbf{v_{n_j}} - \mathbf{v}|| \leq ||\mathbf{v}|| + 1$$

for all $j > J$, but this is impossible since $n_j \geq j$, so $||v_{n_j}|| \geq j \to \infty$ as $j \to \infty$.

It follows that $\mathbf{v}_n$ has no convergent subsequence, so A is not s.compact. $\square$

Theorem. (Heine-Borel) $A \subset \mathbb{R}^n$ is s.compact if and only if A is closed and bounded.

Proof. By the proposition, A is s.compact $\implies$ A is closed and bounded.
Conversely, suppose A is closed and bounded, and $(\mathbf{v}_n)$ is a sequence in A. Then $(\mathbf{v}_n)$ is bounded (since A is). So by B-W, it has a convergent subsequence. Since A is closed, $\mathbf{v} \in A$. So A is s.compact. $\square$

Remark. By previous example, $\bar{B}_1(0)$ in $C[0,1]$ with $||\cdot||_\infty$ is closed and bounded but not s.compact since (f_n) has no convergent subsequence. So Heine-Borel theorem does not hold in general spaces.

Remark. If $A \subset V$ a normed space, then A is s.compact $\iff$ A is compact.

Proposition. Suppose $C \subset V$ is s.compact and $f : C \to W$ is continuous. Then $f(C)$ is s.compact.

Proof. Suppose $(\mathbf{w}_n)$ is a sequence in $f(C)$. Pick $\mathbf{v}_n \in C$ with $f(\mathbf{v}n) = \mathbf{w}_n$. We know C is s.compact, so $(\mathbf{v}_n)$ has a convergent subsequence $\left(\mathbf{v}_{n_j}\right) \to \mathbf{v} \in C$.

Now f is continuous, so $\left(\mathbf{w}_{n_j}\right) = \left(f\left(\mathbf{v}_{n_j}\right)\right) \to (f(\mathbf{v})) \in f(C)$. So $f(C)$ is s.compact. $\square$

We'll use the above to prove maximum value theorem.

Lemma. If $A \subset \mathbb{R}$ is closed and bounded, then $\sup A \in A$.

Proof. A is bounded, so $\sup A$ exists. Pick $x_n \in A$ with $\sup A - \frac{1}{n} \leq x_n \leq \sup A$. Then $(x_n) \to \sup A$. The result follows since A is closed. □

Theorem. (Maximum value theorem) Suppose C is s.compact, $f : C \to \mathbb{R}$ is continuous. Then there exists $\mathbf{v} \in V$ s.t.

$$f(\mathbf{v}) \geq f(\mathbf{v}')$$

for all $\mathbf{v}' \in C$.

Proof. We know $A = f(C)$ is a s.compact subset of $\mathbb{R}$, so it is closed and bounded. So by the lemma, $\sup A$ is in $A = f(C)$. So pick $\mathbf{v} \in C$ with $f(\mathbf{v}) = \sup A$. □

Application: Norms on $\mathbb{R}^n$:

Let $||\cdot||$ be a norm on $\mathbb{R}^n$.

Lemma. The map id:$(\mathbb{R}^n, ||\cdot||_1) \to (\mathbb{R}^n, ||\cdot||)$ is continuous.

Proof. Write $\mathbf{v} = (v_1, ..., v_n) = \sum_{i=1}^n v_i \mathbf{e}_i$. By the triangle inequality,

$$||\mathbf{v}|| \leq \sum_{i=1}^n ||v_i \mathbf{e}_i|| = \sum_{i=1}^n |v_i| ||\mathbf{e}_i|| \leq C \sum_{i=1}^n |v_i| = C||\mathbf{v}||_1$$

Where $C = \max_{1 \leq i} \leq n \{||\mathbf{e}_j||\}$. By criterion of section 1.4, the given map is continuous. □

Corollary. The map $f : (\mathbb{R}^n, ||\cdot||_1) \to \mathbb{R}$ given by $f(\mathbf{v}) = ||\mathbf{v}||$ is continuous.

Theorem. $||\cdot||$ is Lipschitz equivalent to $||\cdot||_1$.

Proof. Let $S = \{\mathbf{v} \in \mathbb{R}^n \mid ||\mathbf{v}_1 = 1\} = g^{-1}(\{1\})$, where $g(\mathbf{v}) = ||\mathbf{v}||_1$.

Now $g : (\mathbb{R}^n, ||\cdot||_1) \to \mathbb{R}$ is continuous, $\{1\}$ is closed in $\mathbb{R}$, so $g^{-1}(\{1\})$ is closed in $(\mathbb{R}^n, ||\cdot||_1)$. S is also obviously bounded in $(\mathbb{R}^n, ||\cdot||_1)$. So S is s.compact by Heine-Borel.

$f : (\mathbb{R}^n, ||\cdot||_1) \to \mathbb{R}$, $f(\mathbf{v}) = ||\mathbf{v}||$ is continuous by corollary. So by maximum value theorem, there exists $\mathbf{v}_\pm \in S$ s.t.

$$C_- = f(\mathbf{v}_-) \leq f(\mathbf{v}) \leq f(\mathbf{v}_+) = C_+$$

for all $\mathbf{v} \in S$, i.e. $C_- \leq \mathbf{v} \leq \mathbb{C}_+$ for all $\mathbf{v} \in S$ where $C_- = ||\mathbf{v}_-|| > 0$ since $\mathbf{v}_- \in S \implies \mathbf{v}_- \neq \mathbf{0} \implies \mathbf{v}_- \neq 0$.

Then for $\mathbf{v} \neq 0$ in $\mathbb{R}^n$, $\mathbf{v}/||\mathbf{v}||_1 \in S$. So

$$0 < C_- \leq ||\frac{\mathbf{v}}{||\mathbf{v}||_1} \leq C_+$$

i.e.

$$C_{-}||\mathbf{v}||_1 \leq ||\mathbf{v}|| \leq C_{+}||\mathbf{v}||_1$$

where $C_{-}, C_{+} > 0$. So the two norms are Lipschitz equivalent. $\square$

Corollary. Any two norms on $\mathbb{R}^n$ are Lipschitz equivalent.

3.2 Completeness

Let V be a normed space, and let $(\mathbf{v}_n)$ be a sequence in V.

Definition. The sequence $(\mathbf{v})_n$ is *Cauchy* if given $\varepsilon > 0$, there exists N s.t. $||\mathbf{v}_n - \mathbf{v}_m|| < \varepsilon$ for all $n, m \geq N$.

Example. If $(\mathbf{v}_n) \to \mathbf{v}$, then $(\mathbf{v}_n)$ is Cauchy.

Proof. Given $\varepsilon > 0$, pick N s.t. $||\mathbf{v}_n - \mathbf{v}|| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then for $n, m \geq N$, by triangle inequality,

$$||\mathbf{v}_n - \mathbf{v}_m|| \leq ||\mathbf{v}_n - \mathbf{v}|| + ||\mathbf{v} - \mathbf{v}_m|| < \varepsilon$$

i.e. $(\mathbf{v}_n)$ is Cauchy. $\square$

Example. Let $s_n = \sum_{i=1}^{n} \frac{1}{i}$. Then s_n diverges. Also it is not Cauchy, even though $|s_n - s_{n+1}| \to 0$ as $n \to \infty$.

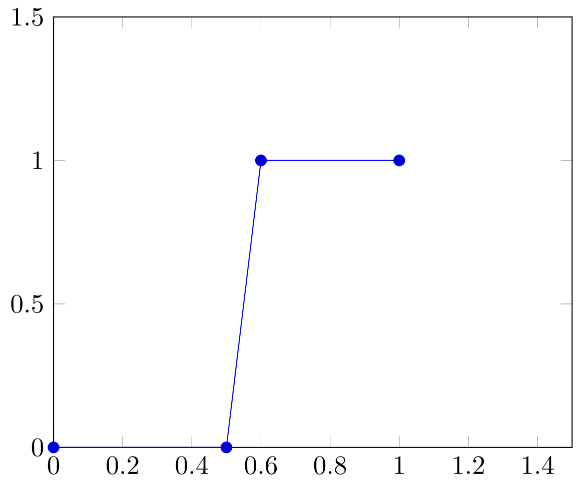
Cauchy sequences *want* to converge.

Example. Given $\varepsilon > 0$, pick N s.t. $||\mathbf{v}_n - \mathbf{v}_m|| < \varepsilon$ for all $n, m \geq N$. Then all but finitely many terms of $(\mathbf{v}_n)$ are contained in $B_\varepsilon(\mathbf{v}_N)$.

However they may not have an element of V to converge to.

Example. Let $V = C[0,1]$ with $||\cdot||_1$. Take

$$f_n = \begin{cases} 0 & x \in [0, 1/2] \\ n(x - 1/2) & x \in [1/2, 1/2 + 1/n] \\ 1 & x \in [1/2 + 1/n, 1] \end{cases}$$



$f(n)$ is Cauchy:
If $m, n \geq > N$, $|f_n(x) - f_m(x)| = 0$ if $x \notin A_n = [1/2, 1/2 + 1/N]$, and < 1 if $x \in A_N$. Then

$$||f_n - f_m||_1 = \int_0^1 |f_n(x) - f_m(x)|dx \leq \int_{1/2}^{1/2+1/N} 1dx = \frac{1}{N}$$

so (f_n) is Cauchy.

Now let

$$f(x) = \begin{cases} 0 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$$

which is not in $C[0,1]$.

If $(f_n) \to g \in C[0,1]$ then $(f_n) \to g$ with respect to $|| \cdot ||_1$ on $[0,1] - A_n$ for any $N > 0$. On the other hand, $(f_n) \to f$ uniformly on $[0,1] - A_N$ for any $N > 0$.

On the other hand, $(f_n) \to f$ uniformly on $[0,1] - A_N$ for any $N > 0$. So $(f_n) \to f$ with respect to $|| \cdot ||_1$ on $[0,1] - A_N$ for all $N > 0$. Therefore $g(x) = f(x)$ for all $x \in [0,1]$. Contradiction.

Definition. A normed space V is *complete* if every Cauchy sequence $(\mathbf{v}_n)$ in V converges to a limit $\mathbf{v} \in V$.

Example. $(C[0,1], || \cdot ||_1)$ is not complete.

Application: Completeness of $\mathbb{R}^n$.

Let V be a normed vector space, and suppose $(\mathbf{v}_n)$ is a Cauchy sequence in V.

Lemma. $(\mathbf{v}_n)$ is bounded. (Exercise)

Lemma. If $(\mathbf{v}_n)$ has a convergent subsequence $(\mathbf{v}_{n_i}) \to \mathbf{v} \in V$, then $(\mathbf{v}_n) \to \mathbf{v}$.

Proof. Given $\varepsilon > 0$, pick M s.t. $||\mathbf{v}_n - \mathbf{v}_m|| < \frac{\varepsilon}{2}$ whenever $n, m > M$. Now $\mathbf{v}_{n_i}$ converges to $\mathbf{v}$, so pick I s.t. $||\mathbf{v}_{n_i} - \mathbf{v}|| < \frac{\varepsilon}{2}$ whenever $i > I$.
So choose $I' > I$ s.t. $n_{I'} \geq M$. Then for $n > n_{I'}$,

$$||\mathbf{v}_n - \mathbf{v}|| \leq ||\mathbf{v}_n - \mathbf{v}_{n_{I'}}|| + ||\mathbf{v}_{n_{I'}} - \mathbf{v}|| < \varepsilon$$

So $(\mathbf{v}_n) \to \mathbf{v}$. □

Theorem. $\mathbb{R}^n$ is complete.

Proof. Suppose $(\mathbf{v}_n)$ is a Cauchy sequence in $\mathbb{R}^n$. By lemma 1, $(\mathbf{v}_n)$ is bounded. By B-W, $(\mathbf{v}_n)$ has a convergent subsequence $(\mathbf{v}_{n_i}) \to \mathbf{v}$. By lemma 2, $(\mathbf{v}_n) \to \mathbf{v}$, i.e. every Cauchy sequence converges. So $\mathbb{R}^n$ is complete. □

Remark. If $||\cdot||$ and $||\cdot||'$ are Lipschitz equivalent, then $(\mathbf{v}_n)$ is Cauchy with respect to the two norms are equivalent. So Completeness with respect to the two norms are equivalent.

Since all norms on $\mathbb{R}^n$ are Lipschitz equivalent, the the theorem holds for any norm.

We saw $(C[0,1], ||\cdot||_1)$ is not complete. What about $(C[0,1], ||\cdot||_\infty)$?

Bounded sequences need not have convergent subsequences.

Theorem. $C[0,1]$ is complete with respect to $||\cdot||_\infty$.

Proof. Given a Cauchy sequence (f_n), we must find $f \in C[0,1]$ s.t. $(f_n) \to f$ uniformly.

Given $\varepsilon > 0$, choose N s.t. $||f_n - f_m|| < \varepsilon/2$ for all $n, m \geq N$. Then if $x \in [0,1]$,

$$\begin{aligned} |f_n(x) - f_m(x) &\leq \max_{x \in [0,1]} |f_n(x) - f_m(x)| \\ &= ||f_n - f_m||_\infty \\ &< \varepsilon/2 < \varepsilon \end{aligned}$$

For $n, m \geq N$.

So $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. But $\mathbb{R}$ is complete. So $\lim_{n\to\infty} f_n(x)$ exists.

Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then $(f_n) \to f$ pointwise.

Now we want to prove $(f_n) \to f$ uniformly. Given $\varepsilon > 0$, and $x \in [0,1]$, pick M (depending on x) s.t. $|f_n(x) - f(x) < \varepsilon/2$ whenever $n \geq M$.

Let $R = \max(N, M)$, then for $n \geq N$,

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_R(x)| + |f_R(x) - f(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for $n, R \geq N$. i.e. $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ i.e. $||f_n - f||_\infty < \varepsilon$. So $(f_n) \to f$ uniformly.

$f_n \in C[0,1] \implies f \in C[0,1]$. So $(f_n) \to f \in C[0,1]$ uniformly. □

3.3 Uniform continuity

Suppose V, W are normed spaces, $A \subset V$.

Definition. $f : A \to W$ is *uniformly continuous* if for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. $||f(\mathbf{v}) - f(\mathbf{v}')|| < \varepsilon$ whenever $||\mathbf{v} - \mathbf{v}'|| < \delta$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f(x+\delta) - f(x) = 2x\delta + \delta^2$. For fixed δ, $2x\delta + \delta^2 \to \infty$ as $x \to \infty$. So $f(x) = x^2$ is not uniformly continuous.

Example. Let $f : (0,1] \to \mathbb{R}$ with $f(x) = \frac{1}{x}$. This is not uniformly continuous as well (consider $x \to 0$).

Theorem. If C is s.compact, and $f : C \to W$ is continuous, then f is uniformly continuous.

Proof. Suppose f is not uniformly continuous. Then there exists $\varepsilon > 0$ s.t. for all $n > 0$ we can find $\mathbf{v}_n, \mathbf{w}_n \in C$ with $||\mathbf{v}_n - \mathbf{w}_n|| < \frac{1}{n}$, and $||f(\mathbf{v}_n) - f(\mathbf{w}_n)|| \geq \varepsilon$ (else f is uniformly continuous).

Since C is s.compact, $(\mathbf{v}_n)$ has a convergent subsequence $(\mathbf{v}_{n_i}) \to \mathbf{v}^* \in C$.

f is continuous and $\mathbf{v}^* \in C$, so $\exists \delta > 0$ s.t. $||f(\mathbf{v}) - f(\mathbf{v}^*)|| < \varepsilon/2$ whenever $\mathbf{v} \in B_\delta(\mathbf{v}^*)$.

If $\mathbf{v}, \mathbf{v}' \in B_\delta(\mathbf{v}^*)$, then

$$\begin{aligned} ||f(\mathbf{v}) - f(\mathbf{v}')|| &\leq ||f(\mathbf{v}) - f(\mathbf{v}^*)|| + ||f(\mathbf{v}^*) - f(\mathbf{v}')|| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

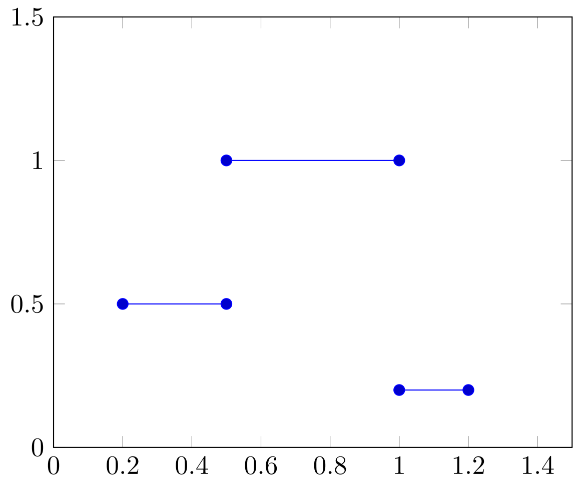
$(\mathbf{v}_{n_i}) \to \mathbf{v}^*$, so pick I_1 s.t. $||\mathbf{v}_{n_i} - \mathbf{v}^*|| < \delta/2$ when $i \geq I_1$.

Pick I_2 s.t. $1/I_2 < \delta/2$. Then for $i \geq \max(I_1, I_2)$, we have $||\mathbf{v}_{n_i} - \mathbf{v}^*|| < \delta/2$ and $||\mathbf{v}_{n_i} - \mathbf{w}_{n_i}|| < \frac{1}{n_i} < \frac{1}{i} < \frac{1}{I_2} < \frac{\delta}{2}$.

So $||\mathbf{w}_{n_i} - \mathbf{v}^*|| < ||\mathbf{w}_{n_i} - \mathbf{v}_{n_i}|| + ||\mathbf{v}_{n_i} - \mathbf{v}^*|| < \delta/2 + \delta/2 = \delta$, i.e. $\mathbf{w}_{n_i}, \mathbf{v}_{n_i} \in B_\delta(\mathbf{v}^*)$, $||f(\mathbf{v}_{n_i}) - f(\mathbf{w}_{n_i})|| \geq \varepsilon$. Contradiction. So f must be uniformly continuous. □

3.4 Application: Integration

Recall from Analysis I: We say $f : [a,b] \to \mathbb{R}$ is piecewise constant if $\exists a = a_0 < a_1 < ... < a_n = b$ and $c_1, ..., c_n \in \mathbb{R}$ s.t. $f(x) = c_i$ if $x \in (a_{i-1}, a_i)$.



Let $P[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is piecewise constant}\}$. If $f \in P[a,b]$ is as above, then

$$I(f) = \sum_{i=1}^{n} c_i (a_i - a_{i-1}) = \text{"} \int \text{"} f$$

Lemma. If $f, g \in P[a,b]$, $\lambda \in \mathbb{R}$, then

$$f - \lambda g \in P[a,b]$$

and $I(f - \lambda g) = I(f) - \lambda I(g)$.

Write $f \geq g$ if $f(x) \geq g(x)$ for all $x \in [a,b]$.

Lemma. If $f \geq 0$, $I(f) \geq 0$.

So if $f, g \in P[a,b], f \geq g$, then $I(f) \geq I(g)$.

Definition. (Riemann Integral) Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Let

$$\mathcal{U}(f) = \{g \in P[a,b] \mid g \geq f\},$$
$$\mathcal{L}(f) = \{g \in P[a,b] \mid g \leq f\}$$

since f is bounded, these are not empty.
Let

$$U(f) = \{I(g) \mid g \in \mathcal{U}(f)\},$$
$$L(f) = \{I(g) \mid g \in \mathcal{L}(f)\}$$

If $g^+ \in \mathcal{U}(f)$ and $g^- \in \mathcal{L}(f)$, then $g^+ \geq f \geq g^-$. So $I(g^+) \geq I(g^-)$. If $u \in U(f)$ and $l \in L(f)$, then $u \geq l$. So $U(f)$ is bounded below, $L(f)$ is bounded above.

Now let

$$u(f) = \inf U(f),$$
$$l(f) = \inf L(f)$$

Note that $u(f) \geq l(f)$.

We say f is Riemann integrable if $u(f) = l(f)$, in which case we define

$$\int_a^b f(x)\, dx = u(f) = l(f)$$

If $f \in P[a,b]$, then $u(f) = I(f) = l(f)$, so f is RI.

Theorem. If $f \in C[a,b]$, then f is RI.

Lemma. Given $\varepsilon > 0$, $\exists g^+ \in \mathcal{U}(f)$ and $g^- \in \mathcal{L}(f)$ s.t. $I(g^+) - I(g^-) < \varepsilon$.

Proof. $[a,b]$ is closed and bounded in $\mathbb{R}$, so it is s.compact. By last lecture's theorem, $f : [a,b] \to \mathbb{R}$ is uniformly continuous.
So pick δ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a}$$

whenever $|x-y| < \delta$. Choose $a = a_0 < a_1 < ... < a_n = b$ such that $a_{i+1} - a_i < \delta$ for all i.

Define

$$c_i^+ = \max_{x \in [a_{i-1}, a_i]} f(x),$$
$$c_i^- = \min_{x \in [a_{i-1}, a_i]} f(x)$$

(These exist by Maximum value theorem) So

$$c_i^+ = f\left(x^+\right) \geq f\left(x^-\right) \forall x \in [a_{i-1}, a_i],$$
$$c_i^- = f\left(x^-\right) \leq f(x) \forall x \in [a_{i-1}, a_i]$$

$x^+, x^- \in [a_{i-1}, a_i] \implies |x^+ - x^-| < \delta$.

Define

$$g^+(x) = c_i^+ \text{ if } x \in [a_{i-1}, a_i),$$
$$g^-(x) = c_i^- \text{ if } x \in [a_{i-1}, a_i)$$

Then $|x^+ - x^-| < \delta \implies c_i^+ - c_i^- < \frac{\varepsilon}{b-a}$ for all i. So to sum up, $g^+ \geq f \geq g^-$ and $g^+ - g^- \leq \frac{\varepsilon}{b-a}$.
Thus $g^+ \in \mathcal{U}(f)$, $g^- \in \mathcal{L}(f)$, and

$$I\left(g^+\right) - I\left(g^-\right) = I\left(g^+ - g^-\right) \leq I\left(\frac{\varepsilon}{b-a}\right) = \varepsilon$$

□

Now prove the theorem:

Proof. $I(g^+) \geq u(f) \geq l(f) \geq I(g^-)$. So $u(f) - l(f) \leq I(g^+) - I(g^-) < \varepsilon$ for all $\varepsilon > 0$, which implies $u(f) = l(f)$. □

Corollary. If $f \in C[a,b]$, $\exists f_k \in P(a,b)$ s.t. $(f_k) \to f$ uniformly on $[a,b]$.

Proof. For each k, choose g_k^+ as in the proof of lemma with $\varepsilon = \frac{1}{k}$. Then $\left(g_k^+\right) \to f$ uniformly. □

Example. (Speed and Distance) Suppose $f\,[a,b] \to \mathbb{R}^n$ is continuous. $f(t) = (f_1(t), ..., f_n(t))$ where all f_i are continuous.
Define $\int_a^b f(t)\,dt = \left(f_1(t)\,dt, ..., \int_a^b f_n(t)\,dt\right)$ (Integrating pointwise).

If $f(t) = \mathbf{v}(t)$ =velocity of a particle in $\mathbb{R}^n$ at time t, then $\mathbf{p}(b) - \mathbf{p}(a) = \int_a^b f(t)\,dt$ is the displacement of particle from its position at $t = a$. $||\mathbf{v}(t)||$ is the speed of particle.

Proposition. If $f : [a,b] \to \mathbb{R}^n$ is continuous, then

$$|| \int_a^b f(t)\,dt|| \le \int_a^b ||f(t)||dt$$

Lemma. If $x_i, y_i \in \mathbb{R}$ satisfy:
(1) $x_i \le y_i$ for all i;
(2) $(x_i) \to x$ and $(y_i) \to y$
Then $x \le y$.

Proof. $y_i - x_i \ge 0, (y_i - x_i) \to y - x \implies y - x \ge 0$. □

Lemma. The proposition holds if f is piecewise constant (maybe not continuous).

Proof. Suppose $f(t) = \mathbf{v}_i$ for $t \in (a_{i-1}, a_i)$. Then

$$\begin{aligned} || \int_a^b f(t)\,dt|| &= ||I(f)|| \\ &= ||\sum_{i=1}^{n} (a_{i+1} - a_i)\,\mathbf{v}_i|| \\ &\le \sum_{i=1}^{n} (a_i - a_{i-1})\,||\mathbf{v}_i|| \\ &= I(||f||) \\ &= \int_a^b ||f||dt. \end{aligned}$$

□

Proof of proposition:

Proof. Choose a sequence of piecewise constant functions $f_k : [a,b] \to \mathbb{R}^n$ s.t. $(f_k) \to f$ uniformly.
Then
$$\int_a^b f_k \to \int_a^b f$$
(uniformly convergence $\implies$ L^1 convergence) and
$$\left(\left|\left| \int_a^b f_k \right|\right| \right) \to \left(\left|\left| \int_a^b f \right|\right| \right)$$
since $||\cdot||$ is continuous.

Also $(||\mathbf{f}_k||) \to ||f||$ uniformly ($||\cdot||$ is continuous). So
$$\left(\int_a^b ||f_k|| \right) \to \int_a^b ||f||$$

So now take $x_k = ||\int_a^b f_k||, x = ||\int_a^b f||, y_k = \int_a^b ||f_k||, y = \int_a^b ||f||$.

Then $x_k \le y_k$, so $x \le y$. $\square$