Practical Nearly-Linear-Time Approximation Algorithms for Hybrid and Overlapping Graph Clustering

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Abstract

In many graph-clustering applications, overwhelming empirical evidence suggests that communities and clusters are naturally overlapping, calling for novel *overlapping graph-partitioning algorithms* (OGP). In this work, we introduce a framework based on two novel clustering objectives, which naturally extend the well-studied notion of conductance to overlapping clusters and to clusters with hybrid vertex- and edge-boundary structure. Our main algorithmic contributions are nearly-linear-time algorithms $O(\log n)$-approximation algorithms for both these objectives. To this end, we show that the cut-matching framework of Khandekar et al. (2014) can be extended to overlapping partitions and give novel cut-improvement primitives that perform a small number of s-t maximum flow computations over the instance graph to detect sparse overlapping partitions near an input partition. Crucially, we implement our approximation algorithm to produce both overlapping and hybrid partitions for large graphs, easily scaling to tens of millions of edges, and test our implementation on real-world datasets against other competitive baselines.

1. Introduction

Detecting communities in real-world networks and clustering similarity graphs are major data mining tasks with a wide range of applications in graph mining, collaborative filtering, and bioinformatics. *Ratio-cut objectives* (also known as quotient-cut objectives) constitute a well-studied and commonly used family of graph partitioning problems (Hagen & Kahng, 1992; Abrahao et al., 2012). A ratio-cut objective measures the quality of a graph cut by the ratio of the weight of the edge cutset to the volume of the smaller side of the partition. Specifically, given a graph $G = (V, E, \mu \in \mathbb{R}^{|V|}_{\geq 0} w \in \mathbb{R}^{|E|}_{\geq 0})$ with non-negative edge weights w and a measure μ over vertices, the ratio-cut objective Ψ_G over partitions $(S, \bar{S})$ of V is defined as:

$$\Psi_G(S, \bar{S}) = \frac{w\big(E(S, \bar{S})\big)}{\min\{\mu(S), \mu(\bar{S})\}}. \tag{1}$$

The ratio-cut minimization problem asks us to minimize this objective over all partitions $(S, \bar{S})$, i.e., determine $\Psi(G) = \min_S \Psi_G(S, \bar{S})$. Ratio-cut objectives play a major role in graph clustering, as they include the widely used expansion $(\forall i \in V, \mu_i = 1)$ and conductance $(\forall i \in V, \mu_i = \sum_{j \sim i} w_{ij})$, which is often taken to be the "gestalt" notion of graph clustering (Leskovec et al., 2009; Zahn, 1971).

In many real-world applications, it is desirable to allow entities to belong to more than one cluster. For instance, in biology a protein may belong to multiple protein complexes (Nepusz et al., 2012), in social networks an agent may be part of multiple communities (Ahn et al., 2010), and a political blog may reflect more than one party affiliation (Latouche et al., 2011). The seminal work of Leskovec et al. (2008) has shown, in numerous large-scale information and social networks, the existence of a core that spans most vertices and lacks community structure (aka "expander-like"), and the existence of numerous small communities with up to few hundreds of nodes that overlap with the core. As a result, standard graph clustering approaches (Abrahao et al., 2012), based on non-overlapping objectives such as ratio cuts, fail to recover the community structure in these ubiquitous datasets. We focus on the following fundamental open question:

> **Question 1.** *Can we design a framework for* overlapping graph partitioning (OGP) *that allows for (i) a principled and intuitive mathematical formulation, together with (ii) solid worst-case approximation algorithms that (iii) scale gracefully to large networks?*

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Proceedings of the 39^{th} *International Conference on Machine Learning*, Baltimore, Maryland, USA, PMLR 162, 2022. Copyright 2022 by the author(s).

Despite a recent a flurry of works on OGP (Ahn et al., 2010; Andersen et al., 2012; Arora et al., 2012; Bonchi et al., 2013; Khandekar et al., 2014; Mishra et al., 2007; Airoldi et al., 2008; Yang & Leskovec, 2013; Gopalan & Blei, 2013; Li et al., 2017; Palla et al., 2012; Tsourakakis, 2015; Whang et al., 2016), all prior works forgo at least one of these desired properties. By contrast, these properties are already satisfied by well-developed theory and software implementations for the non-overlapping ratio-cut objectives (Leighton & Rao, 1999; Arora et al., 2009; Leskovec et al., 2009; Shi & Malik, 2000; Orecchia et al., 2008), of which conductance is a special case. In this work, we provide problem formulations, algorithms and implementations that satisfy all parts of question 1.

Novel overlapping objectives We formulate natural generalizations of ratio-cut objectives for partitioning the graph into *two* overlapping partitions. As with other ratio-cut objectives, the balanced and k-way versions of overlapping problems can be reduced to the 2-way problem in standard ways (Kannan et al., 2004). The key idea behind our generalizations is to redefine the notion of the *boundary* of a cluster to contain *both* edges that leave the cluster and vertices that are shared with other clusters.

Definition 1. *An* overlapping partition[1] $[S,T]$ *of the vertex set* V *consists of two subsets* $S,T \subseteq V$ *such that* $S \cup T = V$. *The corresponding* edge-cutset $\delta_E[S,T]$ *and* vertex-cutset $\delta_V[S,T]$ *are:*

$$\delta_E[S,T] \overset{\text{def}}{=} E(S \setminus T, T \setminus S) \quad \textit{and} \quad \delta_V[S,T] \overset{\text{def}}{=} S \cap T,$$

As long as $S,T \neq S \cap T$, the edge-cutset $\delta_E[S,T]$ and the vertex-cutset $\delta_V[S,T]$ can be thought of as a generalized notion of boundary in V, as their removal disconnects the graph into at least two components associated to $S \setminus T$ and $T \setminus S$. We can now associate two ratio-cut-like measures to an overlapping partition:

$$q_E[S,T] \overset{\text{def}}{=} \frac{w(\delta_E[S,T])}{\min\{\mu(S),\mu(T)\}}, \tag{2}$$

$$q_V[S,T] \overset{\text{def}}{=} \frac{\mu(\delta_V[S,T])}{\min\{\mu(S),\mu(T)\}}. \tag{3}$$

In Section 3, for a parameter $\epsilon \in [0,1)$, we define the ϵ-*overlapping ratio-cut* (ϵ-ORC) problem to be the minimization of the edge ratio-cut $q_E[S,T]$ under the condition that the vertex ratio-cut $q_V[S,T] \leq \epsilon$, i.e., the overlap between S and T contains at most an ϵ-fraction of the measure of the smaller side of $[S,T]$. We also define a version of the problem with softer overlap constraints: for a parameter $\lambda \geq 0$, the λ-*hybrid ratio-cut* problem (λ-HCUT) is the minimization of $q_E[S,T] + \lambda \cdot q_V[S,T]$, i.e., the cost of cutting one unit of vertex boundary is λ times that of a unit of edge boundary.

Algorithm design Both λ-HCUT and ϵ-ORC are easily seen to be NP-hard. Existing metric relaxations and rounding algorithms (Leighton & Rao, 1999; Arora et al., 2004) for graph partitioning problems can be applied to obtain polylogarithmic approximations. However, solving such relaxations requires the computation of dense multicommodity flows on an edge- and vertex-capacitated version of the input graph, which needs quadratic time in the size of the input (Arora et al., 2010). To scale our computations to networks with tens of millions of edges on a single-machine, we rely on the *cut-matching* game of Khandekar, Rao and Vazirani (Khandekar et al., 2009), which computes approximate solutions to the formulation of Arora et al. (2004) by assuming oracle access to a *cut-improvement algorithm* (Andersen & Lang, 2008) for the desired ratio-cut problem. We provide two main **technical contributions**:

- the *first cut-improvement algorithm for* OGP *problems*, generalizing the graph version of Andersen & Lang (2008), while only requiring a polylogarithmic number of s-t maximum flow computations over a vertex-capacitated version of the input graph.
- the *first extension of the cut-matching game framework* to OGP problems, showing that the expander flow of paradigm of Arora et al. (2009) seamlessly ports over to the OGP setting.

Combining these contributions with recent advances in the fast solution of maximum flow problems (Chen et al., 2022), we obtain the first almost-linear-time $O(\log n)$ approximation to λ-HCUT and ϵ-ORC.

Empirical Evaluation We evaluate the performance of our proposed method `cm+improve` on graphs sampled from the Overlapping Stochastic Block Model (Abbe & Sandon, 2015) and on large real-world networks from the SNAP collection (Leskovec & Krevl, 2014). Our results show that `cm+improve` is competitive or outperforms baselines while scaling to graphs with over 10^7 edges.

2. Related work

Overlapping graph clustering. Overlapping community detection has been studied from a statistical viewpoint through the overlapping stochastic block model (Latouche et al., 2011; Abbe & Sandon, 2015). The problem remains largely open for general graphs that do not conform to such simple probabilistic models. Due to the importance of overlapping graph clustering, a variety of rigorous methods have been proposed based on different models of overlapping partitions. Arora et al. (2012) present an average-case analysis approach based on certain random graph models. Balcan

[1] We denote an overlapping partition by $[S,T]$ to clearly differentiate it from non-overlapping partitions $(S,\bar{S})$.

et al. (2012) consider a set-based latent structure, and extend the notion of (α, β)-communities originally proposed by Mishra et al. (2007). Other machine-learning methods also assume that nodes have latent features according to which they decide how to connect. Such methods can be seen as matrix factorization methods, and include notably mixed membership models (Airoldi et al., 2008), BIGCLAM (Yang & Leskovec, 2013), and several others (Andersen et al., 2012; Bonchi et al., 2013; Gopalan & Blei, 2013; Li et al., 2017; Palla et al., 2012; Tsourakakis, 2015; Whang et al., 2016).

Much less is known about algorithms with worst-case guarantees for overlapping clustering. Khandekar et al. (2014) consider the problem of minimizing the sum and the maximum of conductances for a set of communities under the constraint that each node belongs to at least one community. Their approach is based on the tree decomposition of Räcke (Räcke, 2008), with the authors themselves pointing out that their methods do not scale to large graphs.

A long line of work has focused on developing polynomial-time approximation algorithms for ratio-cut minimization, including spectral algorithm based on Cheeger's inequality (Alon & Milman, 1985), the multicommodity-flow-based Leighton-Rao $O(\log n)$ approximation algorithm (Leighton & Rao, 1999), and finally the current state-of-the-art approximation due to Arora, Rao and Vazirani (Arora et al., 2009), which combines spectral and flow techniques. In parallel, practitioners have developed many scalable graph-partitioning heuristics, including the Kernighan-Lin heuristic (Kernighan & Lin, 1970b) that is frequently used as a sub-routine for refining partitions (e.g., (Hendrickson & Leland, 1995)), the widely used METIS software (Karypis & Kumar, 1996; 1998; 1995), Graclus (Dhillon et al., 2007), and KaHIP that imposes balance constraints on the clusters (Sanders & Schulz, 2013).

3. Novel Overlapping Clustering Objectives

We model the input to our OGP formulation as consisting of a weighted undirected graph $G = (V, E, w, \mu)$ with arbitrary non-negative edge weights[2]$\{w_e \in \mathbb{Z}_{\geq 0}\}_{e \in E}$ and arbitrary non-negative vertex weights[2] $\{\mu_v \in \mathbb{Z}_{\geq 0}\}_{v \in V}$. Our main OGP problem is the *the ϵ-overlapping ratio-cut* (ϵ-ORC), which takes a parameter $\epsilon \in [0,1]$ controlling the maximum size of the overlap $\delta_V[S,T]$:

$$\epsilon - \mathrm{ORC}: \min_{S \cup T = V} q_E[S,T] = \min_{S \cup T = V} \frac{w(\delta_E[S,T])}{\min\{\mu(S), \mu(T)\}}$$

$$q_V[S,T] = \frac{\mu(\delta_V[S,T])}{\min\{\mu(S), \mu(T)\}} \leq \epsilon$$

In words, we are attempting to minimize the ratio between weight of the edges between $S \setminus T$ and $T \setminus S$, and the weight of the smaller of S and T, while constraining the weight of the overlap $S \cap T$ to be at most an ϵ-fraction of both S and T. The logic behind the choice of the ϵ-ORC objective is the realization that overlapping partitions fail to be detected by existing algorithms because they do not correspond to either sparse edge cuts or sparse vertex cuts.

Consider the emblematic Zachary's Karate Club social network (Zachary, 1977) in Figure 1 in which two karate clubs S and T overlap on a subset $S \cap T$. This intersection contains a small number of nodes that are well-connected to both communities. At the same time, there also exists a small number of edges directly between $S \setminus T$ and $T \setminus S$, possibly because of second-order interactions between the nodes. In this setting, neither an edge-based graph partitioning algorithm nor a vertex-based one succeeds in detecting the overlapping partition S and T. The former suffers a large penalty if it separates either S or T from $S \cap T$, as a large number of edges is cut. The latter cannot identify $S \cap T$ because it is not a vertex separator, i.e., S and T are not disconnected by the removal of $S \cap T$. Our objective ϵ-ORC enables us to interpolate between the edge- and vertex-based cuts to optimize over *hybrid* cuts, as shown in Subfigure (b).

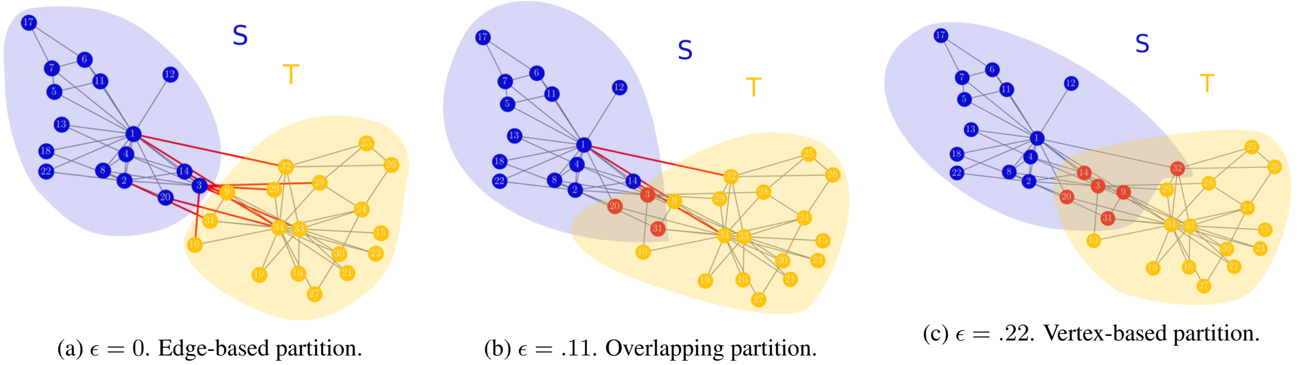
3.1. Hybrid Graph Partitioning

The λ-HCUT problem provides an unconstrained, computationally easier, version of the the ϵ-ORC problem, where the overlap fraction $q_V[S,T]$ is controlled via a penalty term $\lambda \cdot q_V[S,T]$ directly in the objective. For a parameter $\lambda \geq 0$, we define the λ-HCUT objective $q_{G,\lambda}$ as:

$$\begin{aligned} q_{G,\lambda}[S,T] &\overset{\mathrm{def}}{=} q_E[S,T] + \lambda \cdot q_V[S,T] \\ &= \frac{w(\delta_E[S,T]) + \lambda \cdot \mu(\delta_V[S,T])}{\min\{\mu(S), \mu(T)\}} \end{aligned}$$

The $\lambda -$ HCUT problem consists of the minimization of $q_{G,\lambda}[S,T]$ over all overlapping partition $[S,T]$ of V, i.e., determining $q(G, \lambda) = \min_{[S,T]} q_{G,\lambda}[S,T]$. To minimize the objective $q_{G,\lambda}$, we are looking to separate the graph into two disconnected components $S \setminus T$ and $T \setminus S$ by removing a small set of edges $\delta_E[S,T]$ and a small set of vertices $\delta_V[S,T]$. The parameter λ regulates the relative cost of cutting one unit of edge-weight compared to one unit of vertex-weight. By varying λ, the λ-HCUT problem interpolates between edge-based ($\lambda \geq \max_{i \in V}(\Sigma_{i \sim j}\, w_{ij}/\mu_i)$)

[2] We assume integral weights for the rest of the paper. Problems with rational weights can be reduced to the integral case by an appropriate scaling. Our complexity guarantees will depend (logarithmically) on the magnitude of the largest weight.



(a) $\epsilon = 0$. Edge-based partition.

(b) $\epsilon = .11$. Overlapping partition.

(c) $\epsilon = .22$. Vertex-based partition.

Figure 1: Visualizations of overlapping solutions to ϵ-ORC for different values of $\epsilon \in [0, 1]$ on the Karate graph (Zachary, 1977), with μ set to the degree measure. Partition sides are yellow and blue. Overlap is red. Cut edges are also red. Solutions were computed using our algorithm `cm+improve`.

and vertex-based ($\lambda \leq \min_{i \in V}(\Sigma_{i\sim j}\, {w_{ij}}/{\mu_i})$) partitions via hybrid partitions cutting both edges and vertices.

Interpreting the parameter λ as a Lagrangian multiplier yields a simple relation between λ-HCUT and ϵ-ORC: optimal solutions to λ-HCUT yield optimal solutions to the ϵ-ORC problem.

Lemma 1. *For any $\lambda \geq 0$, let $[S, T]$ be an α-approximate optimal solution for the λ-HCUT problem. Define $\epsilon = {\mu(\delta_V[S,T])}/{\min\{\mu(S),\mu(T)\}}$. Then, $[S, T]$ is an optimal solution to the ϵ-ORC problem.*

This lemma can be easily generalized to α-approximate optimal solution, as long as we allow for a bi-criterion approximation for the ϵ-ORC problem, where the output overlapping partition is only required to have $\delta_V[S, T] \leq \alpha\epsilon$. While it may be tempting to use this approach to reduce the optimization of ϵ-ORC to that of λ-HCUT, by performing a search over the Lagrangian multiplier λ, this is not possible in general, as approximately optimal solutions for ϵ-ORC for some values of ϵ may not be approximately optimal for any λ. Fortunately, our algorithmic approach, described in the next section, still allows us to solve both problems, essentially by carrying out an analogue of the the proposed reduction for localized, convex versions of the two problems. For this reason, we first describe an algorithm for the λ-HCUT problem in the next section.

4. Efficient Approximation Algorithms

The λ-HCUT problem is NP-hard, as it generalizes edge-based and vertex-based graph partitioning problems, capturing the minimum-conductance problem as a special case. Spectral methods yield provable non-trivial approximation guarantees only for the minimum-conductance problem, as Cheeger's Inequality does not extend to generic vertex measures other than the degree measure. For this reason, we consider approximation algorithms based on metric relaxations of graph partitioning problems (Leighton & Rao, 1999; Arora et al., 2004). Such relaxations yield polynomial-time poly-logarithmic approximations for both edge- and vertex-based ratio-cut problems (Feige et al., 2005), including Ψ_G. Indeed, these methods can be adapted to yield the same approximation for the λ-HCUT problem. However, the convex programs arising from these relaxations have a cubic number of constraints and generally require the solution of dense multi-commodity flow problems (Arora et al., 2010) over G, drastically limiting the scalability of this approach.

Due to the practical importance of graph partitioning, a number of works have focused on designing scalable algorithms that match the poly-logarithmic approximation ratios afforded by the metric relaxations while only using s-t maximum flow computations, rather than the more time-consuming multi-commodity flows. A particularly simple framework for this reduction is the *cut-matching* game of khandekar2009graph,thesis, which computes approximate solutions to the (Arora et al., 2004) formulation by assuming oracle access to a *cut-improvement algorithm*, which is implemented via the solution of a small number of s-t maximum-flow computations over the instance graph. We follow this approach and design a novel cut-improvement algorithm, `HybridImprove`, for the λ-HCUT problem, and a closely related cut-improvement procedure , `OverlapImprove`, for the ϵ-ORC problem.

For an instance of the λ-HCUT problem G, let the parameter W_G denote the maximum weight $\max\{\max_{e \in E}\{w_e\}, \max_{i \in V}\{\lambda\mu_i\}\}$. Let $T_f(m, n, C)$ be the time complexity of solving a s-t maximum flow problem over an edge- and vertex-capacitated graph with m arcs, n vertices and integral arc capacities bounded by C. The following is our main result, yielding a logarithmic approximation to the λ-HCUT problem.

Theorem 2. *For an input graph $= (V, E, w, \mu)$, the cut-matching framework applied to* `HybridImprove` *yields a $O(\log|V|)$-approximation algorithm for the λ-HCUT problem. The running time is $[O(|E|) + T_f(O(|E|), O(|V|), O(W_G))] \cdot \mathrm{poly}(\log|V| \cdot \log W_G)$.*

By the same method, we also obtain a similar result for the ϵ-ORC problem, with a bi-criterion approximation, which is standard for constrained graph-partitioning problems, e.g., balanced graph partitioning (Leighton & Rao, 1999).

Theorem 3. *For an input graph $= (V, E, w, \mu)$, let $R = \max_{i,j} \mu_i/\mu_j$. The cut-matching framework applied to* `OverlapImprove` *outputs an overlapping partition $[S, T]$ such that $q_V[S, T] \leq (R + 1) \cdot \epsilon$ and for all overlapping partitions $[A, B]$ with $q_V[A, B] \leq \epsilon$*

$$q_E[S, T] \leq O(\log|V|) \cdot q_E[A, B].$$

The running time is $[O(|E|) + T_f\big(O(|E|), O(|V|), O(W_G)\big)] \cdot \mathrm{poly}(\log|V| \cdot \log W_G)$.

If we choose the algorithm of Goldberg & Rao (1998) as our s-t maxflow solver, the total running time for both algorithms becomes $O(|V||E|^{1/2} \cdot \mathrm{poly}(\log|V| \cdot \log W_G)$. To obtain the advertised almost-linear running times, there are two approaches based on approximate maximum-flow computations:

- Following Khandekar et al. (2009) , we can approximately compute the maximum flow by running blocking flows of length up to $O\left({|V|}/{q(G,\lambda)}\right)$ to obtain a running time of $O\left({|E|}/{q(G,\lambda)} \cdot \mathrm{poly}(\log|V| \cdot \log W_G)\right)$.
- By the recent work of Chen et al. (2022), the edge- and vertex-capacitated flow can be computed in $O(|E|^{1+o(1)})$ time.

In our implementation, we use the HIPR implementation (Cherkassky et al., 1994) of the push-relabel method, which has proved to be very efficient in practice.

4.1. Cut Improvement for λ-HCUT

A cut-improvement algorithm (Kernighan & Lin, 1970a; Fiduccia & Mattheyses, 1982; Andersen & Lang, 2008) for a ratio-cut problem takes as input a candidate partition $(S_0, \bar{S}_0)$ and outputs a nearby partition $(S, \bar{S})$ with an improved objective $\Psi_G(S, \bar{S}) \leq \Psi_G(S_0, \bar{S}_0)$. A practical approach to solving HGP is to use a generic partitioning algorithm to find a non-overlapping cut (S, T), where $T = \bar{S}$, together with a *cut improvement* procedure that includes vertices in the overlap $S \cap T$ as to minimize $q_{G,\lambda}$. A natural cut-improvement heuristic, which we refer to as `GreedyImprove`, is to repeatedly loop over all vertices u on the boundary of S and T and greedily include $u \in S \cap T$ if the inclusion decreases the value of the λ-HCUT objective. This heuristic will serve as a competitor to our algorithm in the empirical evaluation of Section 5. Unfortunately, `GreedyImprove` does not yield any global approximation guarantees.

Our first significant algorithmic contribution is the design and analysis of a novel cut-improvement method for the λ-HCUT problem. Our algorithm `HybridImprove` generalizes previous flow-based improvement algorithms (Andersen & Lang, 2008; Lang & Rao, 2004) to the hybrid cut setting. However, our approximation guarantees for `HybridImprove` are much sharper, as previous results cannot be directly deployed in the cut-matching game. Our guarantees are more easily stated if we first extend the definition of the non-overlapping ratio-cut objective Ψ_G to overlapping partitions $[A, B]$ in the following natural way:

$$\Psi_G([A, B]) = \max_{S \subseteq A, \bar{S} \subseteq B} \frac{w\big(E(S, \bar{S})\big)}{\min\{\mu(A), \mu(B)\}} \qquad (4)$$

Here the numerator in the ratio-cut $\Psi_G([A, B])$ for an overlapping partition $[A, B]$ is the worst (maximum) edge-cutset weight over all ways of splitting the overlap $A \cap B$ between $S \subseteq A$ and $\bar{S} \subseteq B$.

Given an input non-overlapping partition $(S_0, \bar{S}_0)$, `HybridImprove` outputs an improved overlapping partition $[S, T]$, together with a certificate that lower-bounds the ratio-cut of partitions near $[S, T]$. This dual certificate takes the form of a bipartite μ-regular graph H between $(S_0, \bar{S}_0)$. Indeed, the dual problem solved by `HybridImprove` is exactly that of routing the largest possible multiple of a bipartite μ-regular graph across the input partition $(S_0, \bar{S}_0)$ into an edge- and vertex-capacitated version of G. The execution of `HybridImprove` only requires a small number of s-t maxflow computations over a directed version on G.

Theorem 4. *Let $G = (V, E, w \in \mathbb{R}^E_{\geq 0}, \mu \in \mathbb{R}^V_{\geq 0})$ be an undirected weighted graph and $(S_0, \bar{S}_0)$ be a non-overlapping partition of G. Assume without loss of generality that $\mu(S_0) \leq \mu(\bar{S}_0)$ and define $\kappa \overset{\mathrm{def}}{=} \mu(S_0)/\mu(\bar{S}_0) \in (0, 1]$. On input $(G, (S_0, \bar{S}_0), \lambda \geq 0)$, the* `HybridImprove` *algorithm outputs:*

- *a weighted graph $H = (V, E_H, u \in \mathbb{R}^{E_H}_{\geq 0}, \mu)$, with the same vertex weights as G, such that*
  - *H is bipartite across $(S_0, \bar{S}_0)$,*
  - *the weighted degree of a vertex i in H is μ_i if $i \in S_0$ and $\kappa \cdot \mu_i$ if $i \in \bar{S}_0$.*
- *an overlapping partition $[S, T]$ such that for all overlapping partitions $[A, B]$,*

$$\frac{q_{G,\lambda}([A, B])}{q_{G,\lambda}([S, T])} \geq \Psi_H([A, B]) \qquad (5)$$

The running time of `HybridImprove` *is $(T_f(O(|E|), O(|V|), O(W_G)) + |E|) \cdot O(\log(W_G \cdot |V|)$.*

A crucial property of `HybridImprove` is the approximation result of Equation 5 for the output overlapping partition $[S, T]$. In particular, it guarantees that *for all* overlapping partitions $[A, B]$, including ones potentially far from $(S_0, \bar{S}_0)$, the objective of the output partition $[S, T]$ is within a factor $\Psi_H([A, B])$ of $q_{G,\lambda}([A, B])$, i.e., the more edges of H cross $[A, B]$, the better approximation to $q_{G,\lambda}([A, B])$ we have. Applying this reasoning to the optimal cut for the λ-HCUT objective yields the following corollary:

Corollary 5. *Let $[A^\star, B^\star]$ be the overlapping partition minimizing $q_{G,\lambda}$. On input $(S_0, \bar{S}_0)$,* `HybridImprove` *outputs an overlapping partition $[S, T]$ which is a $^1/_{\Psi_H([A^\star, B^\star])}$ approximation to the optimum $q_{G,\lambda}([A^\star, B^\star])$.*

Hence, we can obtain a good approximation ratio for the λ-HCUT problem, if we use `HybridImprove` to query a cut $(S_0, \bar{S}_0)$ that forces H to have many edges cross the optimal overlapping partition $[A^\star, B^\star]$. The details of the `HybridImprove` algorithm and a full proof of Theorem 4 can be found in the Appendix.

4.2. Cut Improvement for ϵ-ORC

Recall that it is not generally possible to reduce the ϵ-ORC problem to a sequence of calls to an oracle for the HCUT problem with different λ values. Fortunately, the same reduction strategy works instead when applied at the cut improvement level: we can obtain a cut improvement algorithm `OverlapImprove` for the ϵ-ORC algorithm simply via performing binary search on λ in the `HybridImprove` algorithm. The full details of `OverlapImprove` and the proof of the following analogue of Theorem 4 are described in the Appendix. An exact analogue of Corollary 5 also holds.

Theorem 6. *Under the same assumptions of Theorem 4, let R be the largest ratio between vertex weights, i.e., $R = \max_{i,j} {}^{\mu_i}/_{\mu_j}$. The algorithm* `OverlapImprove` *on input $(G, (S_0, \bar{S}_0), \epsilon \in (0, 1)$ outputs:*

- *a weighted graph $H = (V, E_H, u \in \mathbb{R}^{E_H}_{\geq 0}, \mu)$, with the same properties as in Theorem 4,*
- *an overlapping partition $[S, T]$, $q_V[S, T] \leq (R + 1)\epsilon$, such that for all overlapping partitions $[A, B]$ with $q_V[A, B] \leq \epsilon$:*

$$\frac{q_E([A, B])}{q_E([S, T])} \geq \Psi_H([A, B]). \tag{6}$$

4.3. Reduction to the Cut-Matching Game

The cut-matching game is an interactive game between a cut player $\mathcal{C}$ and a matching player $\mathcal{M}$ over a vertex set V with vertex measure μ. Starting with an empty graph over V, at each iteration t, $\mathcal{C}$ plays a partition $(S_t, \bar{S}_t)$ of V with $\mu(S_t) \leq \mu(\bar{S}_t)$. The matching player $\mathcal{M}$ responds by placing a bipartite μ-regular graph H_t across $(S_t, \bar{S}_t)$, i.e., a bipartite graph such that every vertex $v \in S_t$ has degree μ_v and every vertex $u \in \bar{S}_t$ has degree ${}^{\mu(S_t)}/_{\mu(\bar{S}_t)} \cdot \mu_u$. Let $\bar{H}_T = \frac{1}{T} \cdot \sum_{t=1}^{T} H_t$ be the average of the graphs added by $\mathcal{M}$ up to time T. As T goes to infinity, the goal of the cut player is to maximize the minimum ratio-cut quotient $\Psi(\bar{H}_T, \mu)$, while the matching player aims to minimize the same quantity. In words, the cut player aims to select sparse cuts to force the matching player to make $\bar{H}_t$ more expander-like. Conversely, the matching player will try to add edges to $\bar{H}_t$ while preserving some sparse cuts. The following theorem (Orecchia et al., 2008; Orecchia, 2011) gives an efficient strategy for the cut player, which is based on Matrix Multiplicative Weight Updates (Tsuda et al., 2005).

Theorem 7. *(Orecchia et al., 2008; Orecchia, 2011) There exists a strategy for the cut player $\mathcal{C}$ such that, for any play of the matching player $\mathcal{M}$, we have $\Psi(\bar{H}_T, \mu) \geq \Omega(^1/_{\log |V|})$ for $T = O(\log^2 |V|)$. At time t, the cut $(S_t, \bar{S}_t)$ played by this strategy can be computed as a function of $\bar{H}_t$ in time $O(|E(\bar{H}_t)| \cdot \mathrm{polylog}(\mu(V)))$.*

With this strategy in hand, we are ready to sketch the proof of our main results on the approximation of λ-HCUT (Theorem 2) and ϵ-ORC (Theorem 3). Pseudocode for resulting generic algorithm is given as Algorithm 1. A learning interpretation of these results is that the cut player strategy is using Matrix Multiplicative Weight Updates to boost the local approximation guarantees of `HybridImprove` and `OverlapImprove` to a global guarantee by carefully choosing which cut-improvement problems are solved. The complete proof can be found in the Appendix.

Proof Sketch for Theorem 2 and Theorem 3. The cut-improvement algorithm (`HybridImprove` or `OverlapImprove`) takes the role of the matching player in the cut-matching game, i.e., at every iteration t, the matching player's response to $(S_t, \bar{S}_t)$ is the μ-regular bipartite certificate H_t output by `HybridImprove` on input $(S_t, \bar{S}_t)$. By choosing input cuts $(S_t, \bar{S}_t)$ using the strategy of Theorem 7, we can guarantee that after $T = O(\log^2 |V|)$, we have that for any overlapping partition $[A, B]$

$$\max_{1 \leq t \leq T} \Psi_{H_t}([A, B]) \geq \frac{1}{T} \sum_{t=1}^{T} \Psi_{H_t}([A, B]) \geq \Psi_{\bar{H}_t}([A, B]) = \Omega\left(\frac{1}{\log |V|}\right),$$

where the second inequality is a consequence of the definition in Equation 5. Applying this statement to the optimal overlapping partition $[A^\star, B^\star]$, it must be the case for some $t^\star$ that $\Psi_{H_t}([A^\star, B^\star]) \geq \Omega(^1/_{\log |V|})$. Hence, by Corollary 5, the overlapping partition output by the cut-improvement algorithm at iteration $t^\star$ is a $O(\log|V|)$-approximation to the optimum. □

Algorithm 1 Generic cut-matching game algorithm

```
Input: graph instance $G = (V, E, w, \mu)$
Output: overlapping cut$[S, T]$
$H_0 \leftarrow G$    {Certificate initialization}
for $t \leftarrow 1, \cdots, T = O(\log^2(n))$ do
    $(S_t, \bar{S}_t) \leftarrow \mathcal{C}(H_{t-1})$    {Cut player's move}
    $M_t, [A_t, B_t] \leftarrow \texttt{Improve}(G, S_t, \bar{S}_t)$
    $H_t = H_{t-1} + M_t$    {Certificate update}
end for
return best partition in $\{(A_t, B_t)\}_{t\in[T]}$
```

5. Empirical Evaluation

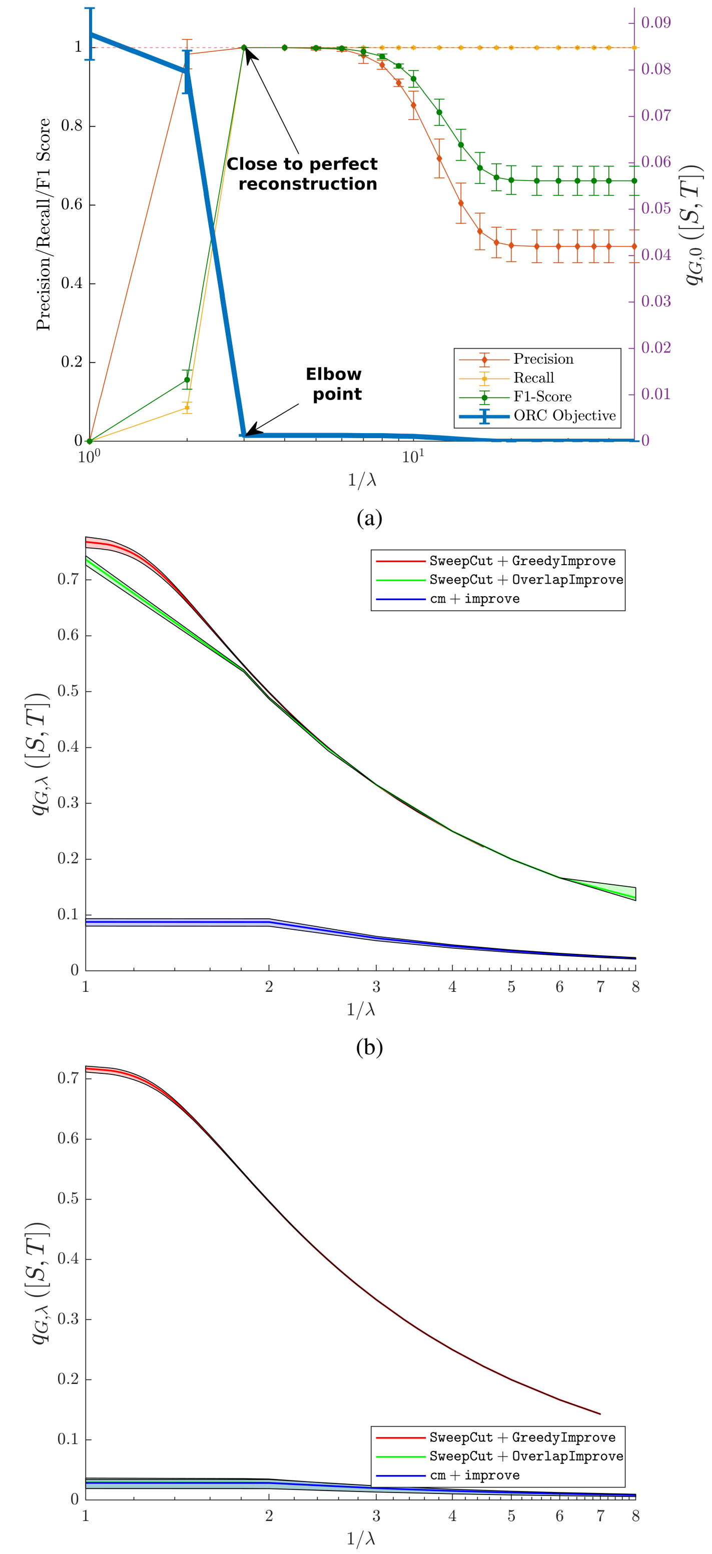
The main challenge in comparing cm+improve with existing algorithms is the lack of closely related methods for OGP problems, as statistical methods are highly tuned to the structure and parameters of the model. BIGCLAM (Yang & Leskovec, 2014), a popular method for detecting overlapping communities performs very poorly in our testbed, likely because it optimizes a very different notion of objective, more akin to detecting smaller obvious clusters, rather than partitioning the whole graph. This challenge is compounded by the lack of other well-defined objective functions for the task of overlapping clustering, which is actually our motivation in defining ϵ-ORC and λ-HCUT. In order to make a meaningful comparison to other algorithms, we post-process the partitions generated by our competitors via one of two HCUT-based cut improvement procedures: the GreedyImprove heuristic described in Section 4.1 or our HybridImprove algorithm. Thanks to this post-processing, we can now expand our set of competitors to include popular algorithms for non-overlapping clustering, such as spectral clustering methods (Von Luxburg, 2007) and METIS, and use our OGP objectives without arbitrarily skewing the playing field.

Our Implementation of cm+improve: The cut-matching strategy of Theorem 7 is implemented in MATLAB, while the cut-improvement algorithm HybridImprove is single-threaded C++, using Goldberg's s-t-maxflow solver HIPR, which implements the push-relabel algorithm (Goldberg; Cherkassky & Goldberg, 1997). Our implementation performs some numerical approximations to minimize the number of calls to HIPR and, as a result, departs slightly from the theoretical description of the HybridImprove algorithm. These optimizations are described in the Appendix. Throughout our evaluation, we set μ to be the degree measure of the graph, as other methods are already tuned to minimize conductance. For experiments requiring the output to be a balanced overlapping partition, we implemented a simple heuristic modification of HybridImprove, by only routing a fraction of the maximum flow in HybridImprove, as suggested by Khandekar et al. (2009). All assets are currently accessible (sup, 2021) and will become publicly available under the BSD license after publication. All experiments were conducted on an institutional cluster on machines with 24 Cores (2x 24 core Intel Xeon Silver 4116 CPU @ 2.10GHz), 48 threads and 128GB RAM.

Competitors: The SweepCut algorithm is the classic spectral approach to graph partitioning: it performs a sweep of the second eigenvector of the normalized Laplacian (Chung, 1997) and outputs the threshold cut with minimum conductance. This is then fed into our overlapping post-processor. METIS (Karypis & Kumar, 1995) is a software suite for solving edge-based graph-partitioning and producing fill reducing orderings for sparse matrices. Its high-quality results, speed and over 25 years of support make it one of the most widely used packages for these tasks. The multi-scale graph-coarsening approach championed by METIS yields an extremely fast algorithm, whose accuracy varies with the choice of randomness used in the coarsening step. For a fair comparison, we run METIS on many random seeds, for a total time comparable to that of our cm+improve on the same instance. The more sophisticated overlap-specific algorithm BIGCLAM (Yang & Leskovec, 2013) solves the ERM problem, where each edge between two nodes comes from a shared community and nodes with no shared communities have a very small chance of connecting. In essentially all cases, we found that BIGCLAM fails to partition the whole graph, often leaving $> 50\%$ nodes in no community. BIGCLAM also tends to output small clusters, even when better, more balanced overlapping partitions clearly exist. This probability reflects a radical difference between BIGCLAM's objective function and standard isoperimetry-based definitions of graph partitioning. We postpone the quantitative results on the comparison of BIGCLAM to cm+improve to the Appendix.

5.1. Overlapping Stochastic Block Model

The first goal of our evaluation is to assess whether our OGP objectives reflect meaningful overlapping clustering structure on datasets where the ground-truth overlapping clusters are known. To address this question, we study the statistical performance of cm+improve in recovering overlapping partitions on graphs generated by the Overlapping Stochastic Block Model (OSBM) of Abbe and Sandon (Abbe & Sandon, 2015), the most generic statistical model of an overlapping clustering. This is an instance of the general stochastic block model in which the ground-truth partition is a tripartition (L, C, R) corresponding to an overlapping partition (S, T) where $L = S \setminus T, R = T \setminus S, C = S \cap T$.



(a)

(b)

(c)

Figure 2: (a) Statistical performance of `cm+improve` in the recovery of the ground-truth overlap C on 5 samples on balanced OSBM with parameters $(|C|, p, \epsilon) = (100, 4, 0.05)$. (b) Comparison on balanced OSBM with parameters $(|C|, p, q, \epsilon) = (10, 4, 4, \epsilon = 0.05)$. (c) Comparison on balanced OSBM with parameters $(|C|, p, q, \epsilon) = (10, 4, 4, \epsilon = 0.05)$.

Each pair of vertices $\{i, j\}$ is added to the edge set independently with probability $p \cdot {}^{\log n}/_{n}$ if both vertices belong to either S or T. If neither S nor T contains both i and j, they are connected with the smaller probability $\epsilon \cdot {}^{\log n}/_{n}$. The ${}^{\log n}/_{n}$ scaling is standard and ensures connectedness of the resulting graph. Besides the values of p and ϵ, our experiments varied the balance of the communities in the generated graphs and the size of the overlap. We also attempted to vary the probability assigned to pairs in the overlap $S \cap T$, but could not detect significant differences in the behavior of the algorithm, A full description of all settings is found in the Appendix.

Results The results were remarkably consistent across the size of the graph. We display here highlights of the results for the smallest graphs with $n = 10^4$. Figure 2(a) shows how the recovery performance of cm+improve changes as ${}^{1}/_{\lambda}$ increases and the size of the overlap grows. On the same y-axis, we also show how the contribution of the edge cutset to $q_{G,\lambda}([S, T])$, i.e., the corresponding ϵ-ORC value. The overlap starts empty for small ${}^{1}/_{\lambda}$ on the left. As ${}^{1}/_{\lambda}$ increases, cm+improve starts including in the overlap $S \cap T$ vertices from the true C, boosting precision. Once the overlap is large enough, the recall follows so that we obtain essentially perfect recovery at $\lambda = 3$. At the same time, the edge cutset has significantly shrunk, as we have switched from cutting edges incident to C to cutting vertices in C.

If we continue increasing the overlap after this point, cm+improve will start adding vertices incident to edges in $E(S \setminus T, T \setminus S)$ to the overlap, until we reach a vertex cut. In this last phase, the edge cutsets decreases slowly, as the vertices added to the overlap do not belong to the C, but are endpoints of the more rare edges in $E(L, R)$. Indeed, the sharp elbow in the ϵ-ORC objective coincides with perfect recovery of the overlap, demonstrating that our method does not require prior knowledge of the overlap size. By contrast, all other algorithms in our test bed generally achieve poor recovery. We focus here on the comparison with SweepCut as the spectral approach comes with strong guarantees in stochastic block models. METIS performs entirely analogously. The example of Figure 2(b) is typical of its behavior when the overlap becomes large enough ($\approxeq \sqrt{n}$)). It shows that cm+improve outperforms both post-processings of SweepCut by an order of magnitude on the λ-HCUT objective. We do not show statistical information here because the precision and recall of SweepCut are both 0 for all values of λ, i.e., even after postprocessing SweepCut fails to find any vertex in the true overlap C. This phenomenon can be explained as follows: because of the sparsity of the ground-truth L and R and the relative density higher density of C, SweepCut finds outputs smaller cuts entirely contained within L or R. As the overlap is large, these cuts cannot be rounded to C by GreedyImprove or HybridImprove.

Figure 2(c) shows the same setup for a smaller ground-truth overlap $|C| = \Theta(\log n)$. In this case, the output of SweepCut is not too far from the optimal overlapping partition. Indeed, the HybridImprove post-processing matches the performance of cm+improve and achieves the same statistical performance. On the other hand, the heuristic GreedyImprove post-processing still fails to recover any vertices of C.

Our results support the overall superiority of global algorithms targeting overlapping measures of graph partitioning, such as cm+improve, over algorithms based on local improvement of edge-based cuts. Such standard approaches appear to fail in detecting overlapping clusters, even in the simple case of OSBM and even when given access to a very powerful overlapping cut-improvement in HybridImprove. We believe that this makes a powerful case for the adoption of OGP objective functions and algorithms in practice.

5.2. Information and Social Networks

In the next set of experiments, we evaluate the performance and efficiency of different methods on the λ-cut objective on a number of social and information networks from the SNAP database (Yang & Leskovec, 2015; Leskovec & Krevl, 2014). We now focus on finding *balanced* overlapping partitions for two reasons: i) our best competitor METIS is biased towards outputting balanced cuts, and ii) such partitions plausibly contain more interesting structural information and could be used to recursively decompose network. We find that cm+improve performs comparably to METIS+HybridImprove, showing that overlapping cuts in real networks tend to be somewhat correlated with sparse edge-based cuts. Unfortunately, the running time of cm+improve quickly becomes infeasible for networks with over 10^7 edges. We are confident that an optimized implementation taking greater advantage of parallelism and randomness will allow cm+improve to scale to even larger graphs. Due to space limitations, full quantitative results appear in the Appendix.

Recursive Bisection: In Section E.4 of the Appendix, we also highlight how recursive application of cm+improve to the DBLP co-authorship graph yields multi-way partitions that recover different areas of Computer Science and detect overlap between different interest communities.

References

Supplementary material, 2021. URL https://drive.google.com/drive/folders/1RK-Q_8_S6LFxmWC9oRhlZAicNWb-gcax?usp=sharing.

Abbe, E. and Sandon, C. Community detection in general stochastic block models: Fundamental limits and efficient algorithms for recovery. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*, pp. 670–688, 2015. doi: 10.1109/FOCS.2015.47.

Abrahao, B., Soundarajan, S., Hopcroft, J., and Kleinberg, R. On the separability of structural classes of communities. In *Proceedings of the 18th ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 624–632, 2012.

Ahn, Y.-Y., Bagrow, J. P., and Lehmann, S. Link communities reveal multiscale complexity in networks. *nature*, 466(7307):761–764, 2010.

Ahuja, R. K., Magnanti, T. L., and Orlin, J. B. *Network Flows: Theory, Algorithms, and Applications*. Prentice hall, 1993.

Airoldi, E. M., Blei, D. M., Fienberg, S. E., and Xing, E. P. Mixed membership stochastic blockmodels. *Journal of machine learning research*, 9(Sep):1981–2014, 2008.

Alon, N. and Milman, V. D. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *Journal of Combinatorial Theory, Series B*, 38(1):73–88, 1985.

Andersen, R. and Lang, K. An algorithm for improving graph partitions. In *SODA '08 Proc. 19th ACM-SIAM Symp. Discret. algorithms*, pp. 651–660, 2008.

Andersen, R., Gleich, D. F., and Mirrokni, V. Overlapping clusters for distributed computation. In *Proceedings of the fifth ACM international conference on Web search and data mining*, pp. 273–282. ACM, 2012.

Arora, S., Rao, S., and Vazirani, U. Expander flows, geometric embeddings and graph partitioning. In *STOC '04 Proc. thirty-sixth Annu. ACM Symp. Theory Comput.*, pp. 222–231, New York, NY, USA, 2004. ACM. ISBN 1-58113-852-0. doi: http://doi.acm.org/10.1145/1007352.1007355.

Arora, S., Rao, S., and Vazirani, U. Expander flows, geometric embeddings and graph partitioning. *Journal of the ACM (JACM)*, 56(2):5, 2009.

Arora, S., Hazan, E., and Kale, S. $O(\sqrt{\log(n)})$ approximation to sparsest cut in $\tilde{O}(n^2)$. *SIAM J. Comput.*, 39: 1748–1771, 01 2010.

Arora, S., Ge, R., Sachdeva, S., and Schoenebeck, G. Finding overlapping communities in social networks: toward a rigorous approach. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pp. 37–54. ACM, 2012.

Balcan, M.-F., Borgs, C., Braverman, M., Chayes, J., and Teng, S.-H. I like her more than you: Self-determined communities. Technical report, 01 2012.

Bonchi, F., Gionis, A., and Ukkonen, A. Overlapping correlation clustering. *Knowledge and information systems*, 35(1):1–32, 2013.

Chen, L., Kyng, R., Liu, Y. P., Peng, R., Gutenberg, M. P., and Sachdeva, S. Maximum Flow and Minimum-Cost Flow in Almost-Linear Time. Technical Report arXiv:2203.00671, arXiv, April 2022. URL http://arxiv.org/abs/2203.00671. arXiv:2203.00671 [cs] type: article.

Cherkassky, B. and Goldberg, A. On implementing the push—relabel method for the maximum flow problem. *Algorithmica*, 19:390–410, 1997.

Cherkassky, B. V., Goldberg, A. V., and Radzik, T. Shortest paths algorithms: theory and experimental evaluation. In *SODA '94*, pp. 516–525, Philadelphia, PA, USA, 1994. Society for Industrial and Applied Mathematics. ISBN 0-89871-329-3. URL http://dl.acm.org/citation.cfm?id=314464.314638.

Chung, F. R. K. *Spectral Graph Theory*. American Mathematical Society, 1997.

Dhillon, I. S., Guan, Y., and Kulis, B. Weighted graph cuts without eigenvectors a multilevel approach. *IEEE transactions on pattern analysis and machine intelligence*, 29(11):1944–1957, 2007.

Feige, U., Hajiaghayi, M., and Lee, J. R. Improved approximation algorithms for minimum-weight vertex separators. In *Proc. thirty-seventh Annu. ACM Symp. Theory Comput. - STOC '05*, 2005. ISBN 1581139608. doi: 10.1145/1060590.1060674.

Fiduccia, C. M. and Mattheyses, R. M. A linear-time heuristic for improving network partitions. In *DAC '82*, pp. 175–181, 1982.

Goldberg, A. Hipr version 3.7. /http://www.avglab.com/andrew/soft.html. Last retrieved December 2019. License: Attribution.

Goldberg, A. V. and Rao, S. Beyond the flow decomposition barrier. *J. ACM*, 45(5):783–797, September 1998. ISSN 0004-5411. doi: 10.1145/290179.290181. URL https://doi.org/10.1145/290179.290181.

Gopalan, P. K. and Blei, D. M. Efficient discovery of overlapping communities in massive networks. *Proceedings of the National Academy of Sciences*, 110(36):14534–14539, 2013.

Hagen, L. and Kahng, A. New spectral methods for ratio cut partitioning and clustering. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 11(9):1074–1085, 1992. doi: 10.1109/43.159993.

Hendrickson, B. and Leland, R. W. A multi-level algorithm for partitioning graphs. *SC*, 95(28):1–14, 1995.

Kannan, R., Vempala, S., and Vetta, A. On clusterings: Good, bad and spectral. *Journal of the ACM (JACM)*, 51 (3):497–515, 2004.

Karypis, G. and Kumar, V. Metis-unstructured graph partitioning and sparse matrix ordering system, version 2.0. 1995.

Karypis, G. and Kumar, V. Parallel multilevel graph partitioning. In *IPPS*, pp. 314–319, 1996.

Karypis, G. and Kumar, V. A fast and high quality multilevel scheme for partitioning irregular graphs. *SIAM J. Sci. Comput.*, 20(1):359–392, December 1998. ISSN 1064-8275. doi: 10.1137/S1064827595287997. URL http://dx.doi.org/10.1137/S1064827595287997.

Kernighan, B. W. and Lin, S. An efficient heuristic procedure for partitioning graphs. *Bell Syst. Tech. J.*, 49(2): 291–307, February 1970a.

Kernighan, B. W. and Lin, S. An efficient heuristic procedure for partitioning graphs. *The Bell system technical journal*, 49(2):291–307, 1970b.

Khandekar, R., Rao, S., and Vazirani, U. Graph partitioning using single commodity flows. *Journal of the ACM (JACM)*, 56(4):19, 2009.

Khandekar, R., Kortsarz, G., and Mirrokni, V. On the advantage of overlapping clusters for minimizing conductance. *Algorithmica*, 69(4):844–863, 2014.

Lang, K. and Rao, S. A flow-based method for improving the expansion or conductance of graph cuts. In Bienstock, D. and Nemhauser, G. (eds.), *Integer Programming and Combinatorial Optimization*, pp. 325–337, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg. ISBN 978-3-540-25960-2.

Latouche, P., Birmelé, E., and Ambroise, C. Overlapping stochastic block models with application to the French political blogosphere. *Annals of Applied Statistics*, 5(1):309–336, 2011. ISSN 19326157. doi: 10.1214/10-AOAS382.

Leighton, T. and Rao, S. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *Journal of the ACM (JACM)*, 46(6):787–832, 1999.

Leskovec, J. and Krevl, A. SNAP Datasets: Stanford large network dataset collection. http://snap.stanford.edu/data, 2014.

Leskovec, J., Lang, K., Dasgupta, A., and Mahoney, M. W. Statistical properties of community structure in large social and information networks. In *Proceeding of the 17th international conference on World Wide Web*, pp. 695–704. ACM, 2008.

Leskovec, J., Lang, K. J., Dasgupta, A., and Mahoney, M. W. Community structure in large networks: Natural cluster sizes and the absence of large well-defined clusters. *Internet Mathematics*, 6(1):29–123, 2009.

Li, P., Dau, H., Puleo, G., and Milenkovic, O. Motif clustering and overlapping clustering for social network analysis. In *IEEE INFOCOM 2017-IEEE Conference on Computer Communications*, pp. 1–9. IEEE, 2017.

Mishra, N., Schreiber, R., Stanton, I., and Tarjan, R. E. Clustering social networks. In *International Workshop on Algorithms and Models for the Web-Graph*, pp. 56–67. Springer, 2007.

Nepusz, T., Yu, H., and Paccanaro, A. Detecting overlapping protein complexes in protein-protein interaction networks. *Nature methods*, 9(5):471, 2012.

Orecchia, L. *Fast Approximation Algorithms for Graph Partitioning using Spectral and Semidefinite-Programming Techniques*. PhD thesis, EECS Department, University of California, Berkeley, May 2011. URL http://www.eecs.berkeley.edu/Pubs/TechRpts/2011/EECS-2011-56.html.

Orecchia, L., Schulman, L. J., Vazirani, U. V., and Vishnoi, N. K. On partitioning graphs via single commodity flows. In *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, STOC '08, pp. 461–470, New York, NY, USA, 2008. Association for Computing Machinery. ISBN 9781605580470. doi: 10.1145/1374376.1374442. URL https://doi.org/10.1145/1374376.1374442.

Palla, K., Knowles, D., and Ghahramani, Z. An infinite latent attribute model for network data. *arXiv preprint arXiv:1206.6416*, 2012.

Räcke, H. Optimal hierarchical decompositions for congestion minimization in networks. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pp. 255–264. ACM, 2008.

Sanders, P. and Schulz, C. Think Locally, Act Globally: Highly Balanced Graph Partitioning. In *Proceedings of the 12th International Symposium on Experimental Algorithms (SEA'13)*, volume 7933 of *LNCS*, pp. 164–175. Springer, 2013.

Shi, J. and Malik, J. Normalized cuts and image segmentation. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 22(8):888–905, 2000.

Tsourakakis, C. Provably fast inference of latent features from networks: with applications to learning social circles and multilabel classification. In *Proceedings of the 24th International Conference on World Wide Web, WWW 2015*, pp. 1111–1121, 2015.

Tsuda, K., Rätsch, G., and Warmuth, M. K. Matrix exponentiated gradient updates for on-line learning and bregman projection. *Journal of Machine Learning Research*, 6(34):995–1018, 2005. URL http://jmlr.org/papers/v6/tsuda05a.html.

Von Luxburg, U. A tutorial on spectral clustering. *Statistics and computing*, 17(4):395–416, 2007.

Whang, J. J., Gleich, D. F., and Dhillon, I. S. Overlapping community detection using neighborhood-inflated seed expansion. *IEEE Transactions on Knowledge and Data Engineering*, 28(5):1272–1284, 2016.

Yang, J. and Leskovec, J. Community-affiliation graph model for overlapping network community detection. In *Proceedings - IEEE International Conference on Data Mining, ICDM*, pp. 1170–1175, 12 2012. ISBN 978-1-4673-4649-8. doi: 10.1109/ICDM.2012.139.

Yang, J. and Leskovec, J. Overlapping community detection at scale: a nonnegative matrix factorization approach. In *Proceedings of the sixth ACM international conference on Web search and data mining*, pp. 587–596, 2013.

Yang, J. and Leskovec, J. Overlapping communities explain core–periphery organization of networks. *Proceedings of the IEEE*, 102(12):1892–1902, 2014.

Yang, J. and Leskovec, J. Defining and evaluating network communities based on ground-truth. *Knowledge and Information Systems*, 42(1):181–213, 2015.

Zachary, W. W. An information flow model for conflict and fission in small groups. *Journal of anthropological research*, 33(4):452–473, 1977.

Zahn, C. T. Graph-theoretical methods for detecting and describing gestalt clusters. *IEEE Transactions on computers*, 100(1):68–86, 1971.

A. Cut Improvement Algorithms

A.1. The `HybridImprove` Algorithm

Specification The `HybridImprove` algorithm takes the following **inputs**:

1. an undirected graph $G = (V, E, w, \mu)$ with non-negative integral edge weights $\{w_e\}_{e \in E}$ and non-negative integral vertex weights $\{\mu_i\}_{i \in V}$.
2. a non-overlapping partition $(S_0, \bar{S}_0)$ of V.
3. a value $\lambda \geq 0$ for which we seek to minimize $q_{G,\lambda}$.

The `HybridImprove` algorithm returns the following **outputs**:

1. an overlapping partition $[S, T]$,
2. a weighted graph $H = (V, E_H, w_H \in \mathbb{R}_{>0}^{E_H})$.

Assume without loss of generality that $\mu(S_0) \leq \mu(\bar{S}_0)$ and define $\kappa \overset{\text{def}}{=} \mu(S_0)/\mu(\bar{S}_0) \in (0, 1)$.

The flow network G_α The algorithm starts by building an auxiliary flow network G_α, parametrized by $\alpha \geq 0$ from G. To support vertex capacities, for each vertex $v \in V$, G_α contains two vertices labeled v_{IN} and v_{OUT}, together with a directed edge $(v_{\text{IN}}, v_{\text{OUT}})$ with capacity $\alpha \cdot \lambda \cdot \mu_i$. Every edge $\{u, v\} \in E$ yields two directed arcs $(u_{\text{OUT}}, v_{\text{IN}})$ and $(v_{\text{OUT}}, u_{\text{IN}})$ of capacity $\alpha \cdot w_{uv}$ in G_α. Finally, G_α contains two auxiliary nodes, a source s and a sink t. They are connected to the rest of graph based on the input partition $(S_0, \bar{S}_0)$ as follows:

- for all $v \in S_0$, there is an arc (s, v_{IN}) with capacity μ_i;
- for all $v \in \bar{S}_0$, there is an arc (v_{OUT}, t) with capacity $\kappa \cdot \mu_i$.

The capacity on the $\bar{S}_0$-side are scaled down by κ to ensure that the total capacity of the trivial source cut equals the total capacity of the trivial sink cut. As we aim to set α to be large enough such that both these cuts are saturated, these capacities can also be interpreted as demands we want to concurrently route from S_0 to $\bar{S}_0$. The construction of G_α is illustrated in Figure 3.

Searching over the α parameter The `HybridImprove` algorithm aims to find the minimum value $\alpha = \alpha^*$ such that a single-commodity flow can be routed from s to t while fully saturating the source cut and the sink cut, i.e., while routing $\alpha \cdot \mu(S_0)$ units of flow. To do so, it performs a binary search over α by testing, for each α whether the required flow can be routed by solving the corresponding s-t maximum-flow. Assuming that the graph G is connected and that the edge- and vertex-weights, together with the parameter λ are integral, α can range between $1/\mu(V)$ and $w(E) + \lambda\mu(V)$, so that $O(\log(|V| \cdot W_G)$ rounds suffice to compute α^*.

Returning the output Once the algorithm has identified the optimal value α^*, it extracts a non-trivial s-t mincut (A, B) in G_{α^*}. Such a mincut is guaranteed to exist by the variational definition of α^*. The output overlapping partition $[S, T]$ of G is formed as following:

1. A vertex $i \in V$ is placed in S if $v_{\text{IN}} \in A$.
2. A vertex $i \in V$ is placed in T if $v_{\text{OUT}} \in B$.

In particularly, vertices for which both conditions hold are placed in the overlap $S \cap T$.

Finally, the algorithm computes a flow-path decomposition of the flow routed into G_α using dynamic trees (Ahuja et al., 1993; Khandekar et al., 2009), to obtain a list of flow paths routed from S_0 to $\bar{S}_0$. The demands routed by these paths are defined to be the graph H returned by `HybridImprove`.

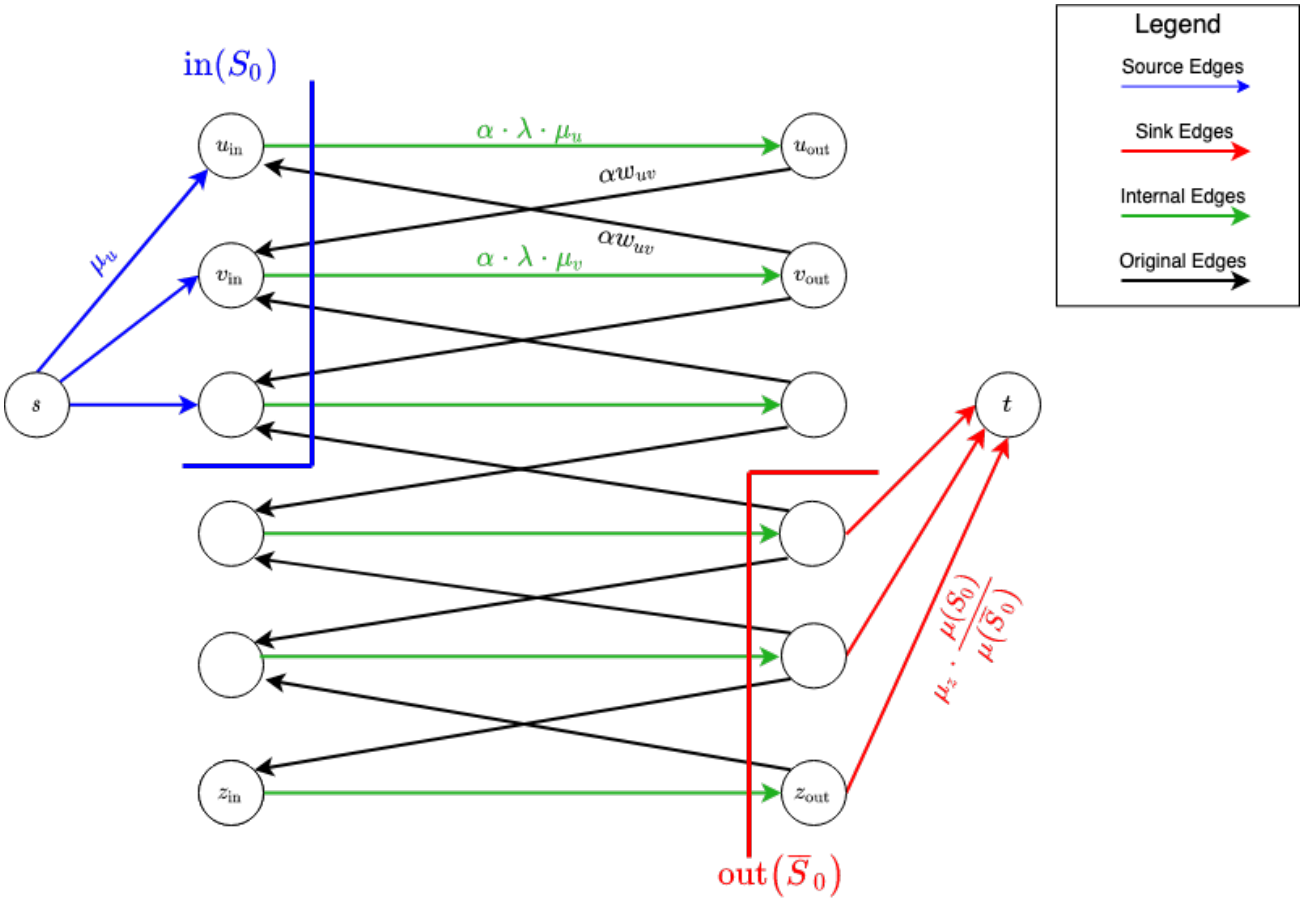


Figure 3: The flow network G_α for a path graph on 6 vertices, for a bisection $(S, \bar{S}_0)$ into connected components.

A.2. The `OverlapImprove` Algorithm

Specification The `OverlapImprove` algorithm takes the following **inputs**:

1. an undirected graph $G = (V, E, w, \mu)$ with non-negative integral edge weights $\{w_e\}_{e \in E}$ and non-negative integral vertex weights $\{\mu_i\}_{i \in V}$.
2. a non-overlapping partition $(S_0, \bar{S}_0)$ of V.
3. a value $\epsilon \in [0, 1]$ for which we seek to minimize $q_{G,\lambda}$.

The `OverlapImprove` algorithm returns the following **outputs**:

1. an overlapping partition $[S, T]$,
2. a weighted graph $H = (V, E_H, w_H \in \mathbb{R}^{E_H}_{>0})$.

Assume without loss of generality that $\mu(S_0) \leq \mu(\bar{S}_0)$ and define $\kappa \overset{\text{def}}{=} {\mu(S_0)}/{\mu(\bar{S}_0)} \in (0, 1)$.

Description The `OverlapImprove` algorithm is obtained by performing a binary search on the input λ of the `HybridImprove` algorithm, starting at $\max_i \sum_{j \sim i} {w_{ij}}/{\mu_i}$. If the output overlapping partition $[S, T]$ has $q_V[S, T] \geq \epsilon$, then λ is reduced. Otherwise, it is increased. The process eventually stops in polylogarithmic iterations for λ^* and α^* such that two s-t mincuts exists in G_{α^*}, corresponding to overlapping partitions $[S_1, T_1]$ with $q_V[S_1, T_1] \leq \epsilon$ and $[S_2, T_2]$ with $q_V[S_2, T_2] \geq \epsilon$. The submodularity of the cut function implies that we must necessarily have $\delta_V[S_1, T_1] \subseteq \delta_V[S_2, T_2]$. Hence, adding a single vertex from the overlap $\delta_V[S_2, T_2]$ to the overlap $\delta_V[S_1, T_1]$ yields a new overlapping partition that also corresponds to a s-t mincut in G_{α^*}. By the bound R on the ratio of weights, we have that the resulting overlapping partition $[S, T]$ has $q_V[S, T] \leq (R + 1)\epsilon$.

B. Cut Improvement Analysis: Proof of Theorem 3

B.1. Valid s-t cuts and corresponding overlapping partitions

We start by proving some simple lemmata about the `HybridImprove` construction, which is essentially a reduction from the overlapping improvement problem to a family of $s - t$ minimum cut problems on bipartite flow networks G_α. To do this end, we define a subset of s-t cuts in G_α that can be put in bijection with overlapping partitions of G.

Definition 2. *An s-t cut (A', B') of G_α is valid if, for all vertices v, $v_{\text{IN}} \in B'$ implies $v_{\text{OUT}} \in B'$. Equivalently, for all v, $v_{\text{OUT}} \in A'$ implies $v_{\text{IN}} \in A'$.*

The relevance of this definition is shown by the following lemma.

Lemma 8. *An s-t mincut (S', T') in G_α is valid.*

Proof. Suppose there exists $v \in V$ such that $v_{\text{IN}} \in B'$ and $v_{\text{OUT}} \in A'$. Including v'_{OUT} in B' decreases the capacity of the s-t cut as the only arc going into v_{OUT} is the arc $(v_{\text{IN}}, v_{\text{OUT}})$, which has strictly positive capacity. □

The bijection between valid s-t cuts in G_α and corresponding overlapping partitions of G is constructed as follows: a valid s-t cut (A', B') maps to the overlapping partition $[A, B]$ such that $v \in A$ if $v_{\text{IN}} \in A'$ and $v \in B$ if $v_{\text{OUT}} \in B'$. Notice that both of these conditions will hold for a vertex in the overlap $A \cap B$. By the validity of (A', B'), we deduce that $A \cup B = V$, so that $[A, B]$ is indeed an overlapping partition of s-t. Similarly, for an overlapping partition $[A, B]$, with $\mu(A) \leq \mu(B)$ we can construct a valid s-t cut (A', B') as follows: if $v \in A \setminus B$, let $v_{\text{IN}}, v_{\text{OUT}} \in A'$; if $v \in B \setminus A$, let $v_{\text{IN}}, v_{\text{OUT}} \in B'$; if $v \in A \cap B$, let $v_{\text{IN}} \in A'$ and $v_{\text{OUT}} \in B'$.

The following lemma describes the relation between the capacity of a valid s-t cut (S', T') in G_α and the edge- and vertex-cutsets of the corresponding overlapping partition $[S, T]$. For a subset C' of vertices in G_α, we denote by C'_{IN} its IN vertices and by C'_{OUT} its OUT vertices.

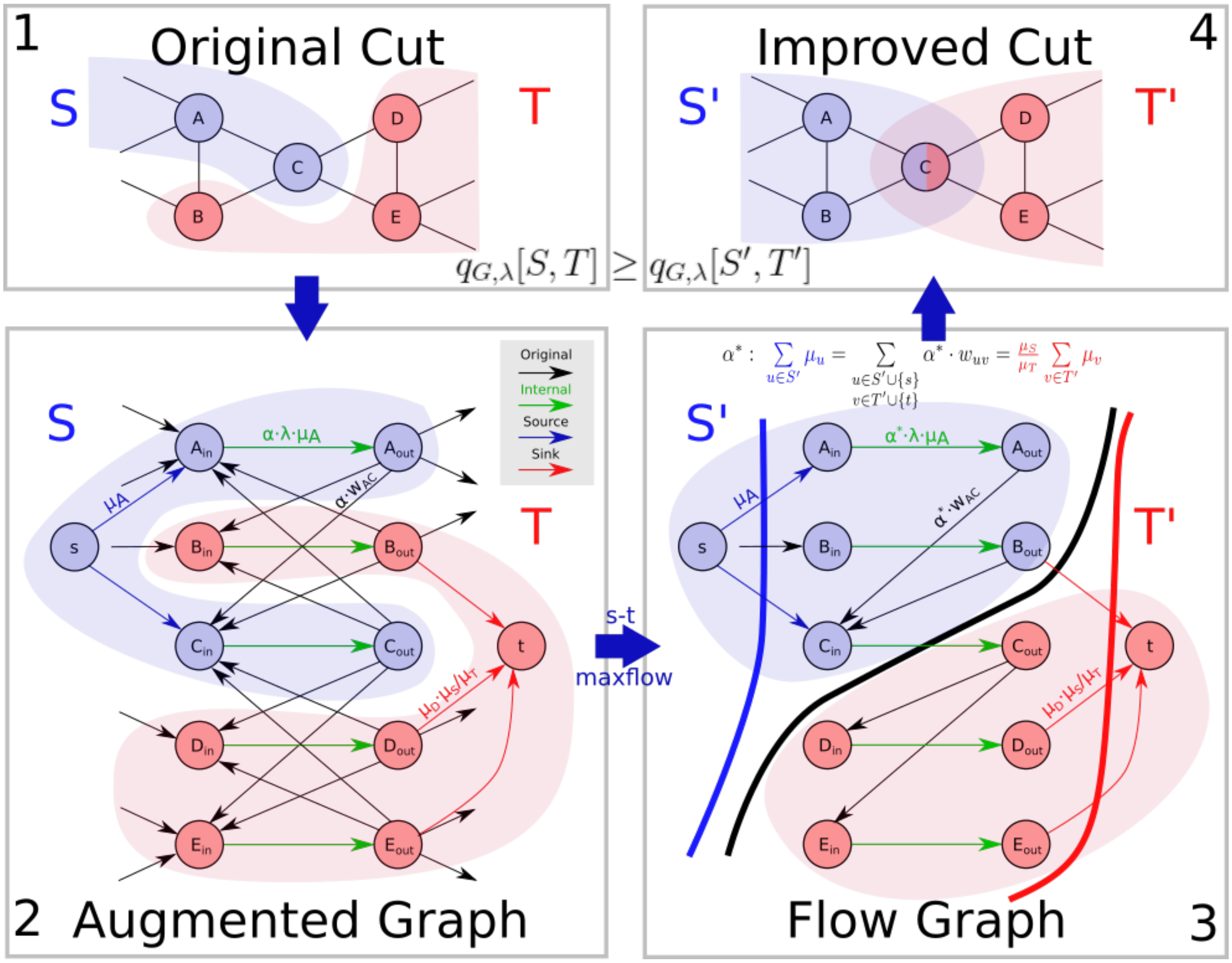


Figure 4: A valid s-t mincut (S', T') of G_α, together with the corresponding overlapping partition.

Lemma 9. *Let (S', T') be a valid s-t cut in G_α and let $[S, T]$ be the corresponding overlapping partition. Then:*

$$\mathrm{cap}_{st}(S', T') = \alpha \cdot w(\delta_E[S, T]) + \alpha \cdot \lambda \cdot \mu(\delta_V[S, T]) + \mu(\bar{S} \cap S_0) + \kappa\mu(\bar{T} \cap \bar{S}_0).$$

Proof. The capacity of (S', T') can be written in terms of the capacities between different subsets of G_α. By the construction of G_α (see also Figure 4), we obtain the following:

$$\mathrm{cap}_{st}(S', T') = \mathrm{cap}_{st}(S'_{\text{IN}}, T'_{\text{OUT}}) + \mathrm{cap}_{st}(S'_{\text{OUT}}, T'_{\text{IN}}) + \mathrm{cap}_{st}(\{s\}, T'_{\text{IN}}) + \mathrm{cap}_{st}(S'_{\text{OUT}}, \{t\}).$$

Notice now that $\mathrm{cap}_{st}(S'_{\text{IN}}, T'_{\text{OUT}}) = \alpha \cdot \lambda \cdot \mu(\delta_V[S, T])$, as the only arcs going from the IN-side to the OUT-side are the internal edges of vertices included in the overlap. Similarly for the second term, $\mathrm{cap}_{st}(S'_{\text{OUT}}, T'_{\text{IN}}) = w(\delta_E[S, T])$, as the only arcs going the opposite way correspond to original edges in $\delta_E[S, T]$. Finally, the last two terms arise from vertices in S_0 that were moved directly to the opposite side $\bar{S}$ and of vertices in $\bar{S}_0$ that were switched over to $\bar{T}$. An example of such a vertex is vertex y in Figure 4. By the choice of capacities from s and to t, we have $\mathrm{cap}_{st}(\{s\}, T'_{\text{IN}}) = \mu(\bar{S} \cap S_0)$ and $\mathrm{cap}_{st}(S'_{\text{OUT}}, \{t\}) = \kappa\mu(\bar{T} \cap \bar{S}_0)$, completing the proof. □

B.2. Splits of overlapping partitions

Next, we discuss the notion of a non-overlapping split of an overlapping partition $[A, B]$, formalizing the notion behind the definition of $\Psi_{G,\mu}$.

Definition 3. *Let $[A, B]$ be an overlapping partition of vertex set V. A non-overlapping split $(C, \bar{C})$ of $[A, B]$ is a non-overlapping partition of V such that $C \subseteq A$ and $\bar{C} \subseteq B$. The non-overlapping split of $[A, B]$ according to a non-overlapping partition $(S, \bar{S})$ is the non-overlapping split that assigns vertices in $A \cap B$ to C or $\bar{C}$ based on their location in $(S, \bar{S})$, i.e.:*

$$C = \bar{B} \cup (A \cap S) \subseteq A$$
$$\bar{C} = \bar{A} \cup (B \cap \bar{S}) \subseteq B$$

We will need the following fact about the relation between flows across $[A, B]$ and splits of $[A, B]$ in G_α.

Lemma 10. *Let $[A, B]$ be an overlapping partition of V and $(C, \bar{C})$ be a split of $[A, B]$. Let (A', B') and $(C', \bar{C}')$ denote the corresponding s-t cuts in G_α. For an s-t maximum flow in G_α, the following holds:*

$$\mathrm{netflow}_{st}(A', B') \geq \mathrm{netflow}_{st}(C', \bar{C}').$$

The same holds with equality if (A', B') is a non-trivial s-t mincut of G_α and $(C, \bar{C})$ is the split according to $(S_0, \bar{S}_0)$.

Proof. Consider any $v \in A \cap B$ in the overlap of $[A, B]$. In the flow network G_α, we have $v_{\text{IN}} \in A'$ and $v_{\text{OUT}} \in B'$. Because all the flow out of v_{IN} and into v_{OUT} runs along the internal arc $(v_{\text{IN}}, v_{\text{OUT}})$, shifting v_{IN} or v_{OUT} to the other side of the s-t can only decrease the netflow. For the second part of the lemma, further assume the same $v \in S_0$. Then equality holds as long as there is no flow into v_{IN} via arcs coming from B'. Similarly, for $v \in \bar{S}_0$, equality holds if there is no flow from v_{OUT} to vertices in A'. This is the case if (A', B') is an s-t mincut of G_α. □

B.3. Flow and cuts in G_{α^*} and their relation with the demand graph H

Next, we use the demand graph H, which has been routed from S_0 to $\bar{S}_0$ in G_{α^*}, to construct lower bounds on the hybrid quotient-cut $q_{G,\lambda}([A, B])$ of an overlapping partition $[A, B]$:

Lemma 11. *For any overlapping partition $[A, B]$ of G:*

$$q_{G,\lambda}([A, B]) \geq \frac{\Psi_{H,\mu}([A, B])}{\alpha^*}$$

Equality is achieved for any overlapping partition $[S, T]$ corresponding to a non-trivial s-t mincut in G_{α^*}, yielding the stronger bound:

$$q_{G,\lambda}([S, T]) = \frac{\Psi_{H,\mu}([A, B])}{\alpha^*} \leq \frac{1}{\alpha^*}$$

Proof. Consider an overlapping partition $[A, B]$ of V and its corresponding s-t mincut (A', B') in G_{α^*}. Take also any non-overlapping split $(C, \bar{C})$ of $[A, B]$ and its corresponding s-t cut $(C', \bar{C})$. By the s-t maxflow-mincut theorem and Lemma 10, for an s-t maxflow on G_{α^*}, we have:

$$\mathrm{cap}_{st}(A', B') \geq \mathrm{netflow}_{st}(A', B') \geq \mathrm{netflow}_{st}(C', \bar{C}').$$

Now, we can easily relate $\mathrm{netflow}_{st}(C', \bar{C}')$ to the cut $(C, \bar{C})$ in H :

$$\mathrm{netflow}_{st}(C', \bar{C}') = w_H(E(C, \bar{C})) + \mu(\bar{C} \cap S_0) + \kappa\mu(C \cap \bar{S}_0). \tag{7}$$

Compare this lower bound on $\mathrm{cap}_{st}(A', B')$ with the result of Lemma 9:

$$\mathrm{cap}_{st}(A', B') = \alpha^* \cdot w(\delta_E[A, B]) + \alpha^* \cdot \lambda \cdot \mu(\delta_V[A, B]) + \mu(\bar{A} \cap S_0) + \kappa\mu(\bar{B} \cap \bar{S}_0). \tag{8}$$

By the definition of $(C, \bar{C})$, we have that $\bar{A} \subseteq \bar{C}$ and $\bar{B} \subseteq C$. Hence, the last two terms in Equation 7 dominate the last two terms of Equation 8. Combining Equation 7 and Equation 8, we then get:

$$\alpha^* \cdot w(\delta_E[A, B]) + \alpha^* \cdot \lambda \cdot \mu(\delta_V[A, B]) \geq w_H(E(C', \bar{C}'))$$

Dividing both sides by $\min\{\mu(A), \mu(B)\}$ completes the proof of the first part of the lemma as:

$$q_{G,\lambda}([A, B]) \geq \frac{\Psi_{H,\mu}([A, B])}{\alpha^*}$$

For the second part, the s-t maximum-flow minimum-cut theorem and Lemma 10 ensure that for a non-trivial s-t minimum cut (S', T') and its split $(C, \bar{C})$ according to S_0:

$$\mathrm{cap}_{st}(S', T') = \mathrm{netflow}_{st}(S', T') = \mathrm{netflow}_{st}(C', \bar{C}').$$

Moreover, we have that $\bar{C} \cap S_0 = \bar{A} \cap S_0$ and $C \cap \bar{S}_0 = \bar{B} \cap \bar{S}_0$, so that the last two terms in Equations 7 and 8 cancel exactly, yielding:

$$\alpha^* \cdot w(\delta_E[S, T]) + \alpha^* \cdot \lambda \cdot \mu(\delta_V[S, T]) = w_H(E(C, \bar{C})).$$

By construction of α^*, we also know that the capacity of any non-trivial s-t minimum cut in $G_{\alpha*}$ equals that of the trivial s-t cut S, which is $\mu(S_0)$. Hence:

$$\begin{aligned} w_H(C, \bar{C}) &= \mu(S_0) - \mu(\bar{A} \cap S_0) - \kappa\mu(\bar{B} \cap \bar{S}_0) \\ &\leq \min\{\mu(A \cap S_0), \kappa\mu(B \cap \bar{S}_0)\} \\ &\leq \min\{\mu(A), \mu(B)\}. \end{aligned}$$

Equivalently, we have that $\Psi_{H,\mu}([S, T]) \leq 1$. Together with the first part of the lemma, this yields:

$$q_{G,\lambda}([S, T]) = \frac{q_H(S, T)}{\alpha^*} \leq \frac{1}{\alpha^*}.$$

□

B.4. Proof of Theorem 3

We are now ready to complete the proof of the main theorem:

Proof. By construction, the demand graph H is bipartite between S_0 and $\bar{S}_0$. Moreover, H is induced by an s-t maximum flow on G_{α^*}, so that each vertex $i \in S_0$ routes μ_i units of flow to $\bar{S}_0$ and each vertex $j \in \bar{S}_0$ routes $\kappa\mu_j$ units of flow to S_0. Hence, the degree in H of $i \in S_0$ is μ_i and the degree in H of $j \in \bar{S}_0$ is $\kappa\mu_j$, as required.

For any overlapping partition (A, B), combining the two part of Lemma 11 ensures that

$$q_{G,\lambda}([A, B]) \geq \frac{\Psi_{H,\mu}([A, B])}{\alpha^*} \geq \Psi_{H,\mu}([A, B]) \cdot q_{G,\lambda}([S, T]),$$

which is the required approximation guarantee. The proof for `OverlapImprove` is entirely analogous.

To bound the running time of `HybridImprove`, we notice that, for a connected [3], α^* must lie in the interval $\left[\frac{1}{|E|W_G}, |V|W_G\right]$, or it is not possible to have the trivial s-t cut have the same capacity as a non-trivial s-t minimum cut. Hence, performing binary search requires $O(\log|V| + \log W_G)$ s-t maxflow computations on graphs G_α in which capacities can be rescaled to be integral and at most $|E|W_G^2$. This yields the first term in the promised running time. The second term $|E|\log(W_G|V|)$ accounts for the computation of graph H, which is achieved by a flow-path decomposition of the s-t maximum flow in G_{α^*} via dynamic trees (Goldberg & Rao, 1998).

□

The following simple corollary show that `HybridImprove` is indeed a cut-improvement algorithm, in that it always improves the initial input partition.

Corollary 12.

$$q_\lambda(S,T) \le q(S_0, \bar{S}_0),$$

Proof. Because H is bipartite across $(S_0, \bar{S}_0)$ and has degrees proportional to μ, we have $q_H(S_0, \bar{S}_0) = 1$. The result then follows from the quotient cut guarantee of the main theorem. □

C. Other Proofs

Lemma 13 (Lemma 1 in main body)**.** *For any $\lambda \ge 0$, let (L, C, R) be an optimal solution for the λ-HCUT problem. Let $S \stackrel{\text{def}}{=} L \cup C$ and $T \stackrel{\text{def}}{=} R \cup C$ and define $\epsilon = \mu(\delta_V(S,T))/\min\{\mu(S),\mu(T)\}$. Then, (S,T) is an optimal solution to the ϵ-ORC problem.*

Proof. Suppose (S,T) is not optimal. Then, there exists a different overlapping clustering (S', T') of smaller objective value with $\mu(S' \cap T') \le \min\{\mu(S), \mu(T)\} \le \epsilon$. Let $L' \stackrel{\text{def}}{=} S' \setminus T'$, $R' \stackrel{\text{def}}{=} T' \setminus S'$ and $C = S' \cap T'$. Hence, we have:

$$q_\lambda(L', C', R') = \frac{w(\delta_E(S',T')) + \lambda \cdot \mu(\delta_V(S',T'))}{\min\{\mu(S'), \mu(T')\}}$$
$$\le \mathrm{ORC}(S',T') + \lambda\epsilon < \mathrm{ORC}(S,T) + \lambda\epsilon = q_\lambda(L,C,R).$$

This contradicts the optimality of (L, C, R). □

Proof of Theorem 2. It remains to prove the running time result. This is a simple consequence of Theorem 5 in the main body. The total number of calls to `HybridImprove` is $T = O(\log^2|V|)$, which yields the polylog bound in the theorem. The total cost of computing the cut strategy is at most $O(\log^2|V|) \cdot \log(|V|W_G) \cdot |E(\bar{H}_T)|$, as $|E(\bar{H}_t)|$ is monotonically increasing. However, by construction $|E(\bar{H}_T)| \le \sum_{t=1}^{T} |E(H_t)| \le O(|E(G)| \cdot \log(W_G|V|))$ by the proof of Theorem 3 in the main body. □

D. Implementation Details

Our implementation is available online (sup, 2021). It departs in a small number of places from the theoretical description of the `HybridImprove` algorithm. We highlight them here:

- Following other implementations of cut-improvement algorithms by a subset of co-authors for standard ratio-cut objectives, rather than performing binary search over α, we initialize α to be the $1/q_{G,\lambda}(S_0,\bar{S}_0)$. After each maxflow operation, we extract the overlapping partition $[A, B]$c from the s-t mincut in G_α and update α to be $1/q_{G,\lambda}(A,B)$.

- We limit the precision of the value α, a rational number, by approximating it up to a 1.001-factor using Farey sequences. When running the maximum flow operation, we scale all capacities in G_α by the denominator in our representation of α to ensure all capacities become integral.

[3] The partitioning problem is trivial if the graph G is disconnected

- For the results in this paper, we use a simpler DFS-based algorithm to compute the flow-path decomposition of the flow routed on G_α. We are currently in the process of implementing dynamic trees to further speed up this operation.

E. Empirical Evaluation

E.1. Comparison with BIGCLAM

Communities	Average community size	Nodes absent
2	3591.5	75931
5	1881.4	78334
20	1030.15	69333
100	594.3	46549

Table 1: BIGCLAM node coverage statistics in cExtractedDblp graph with $n = 83114$ vertices

In order to evaluate our algorithm's performance we considered BIGCLAM (Yang & Leskovec, 2013) and its predecessor AGMFIT (Yang & Leskovec, 2012). The latter was disqualified because it required a quadratic number of iterations, which is computationally infeasible for larger graphs. The successor BIGCLAM replaces the discrete step in the EM algorithm with a continuous one, requiring far fewer iterations. While the running time is no longer a problem, the partitions output by BIGCLAM do not cover the entire graph. A majority of the nodes belongs to no community, even when the algorithm is allowed to output a large number of communities. This could ameliorated if only a few nodes were absent from the communities. However, when a majority of the nodes are unclassified, there is no easy way to convert BIGCLAM's output into a partitioning scheme without radically changing the algorithm. The problem definition of BIGCLAM is very elegant, but unfortunately the current implementation cannot be used as a comparison baseline. It is also conceivable that BIGCLAM may optimize an objective that is inherently different from ratio-cut objectives. Indeed, BIGCLAM has no restriction against including high degree nodes in the overlap, which is sub par in our problem definition, as the cost for a node to be included is proportional to its degree. BIGCLAM may serve as a more useful benchmark when considering the problem of detecting small overlapping communities in the periphery of a large information network . Indicatively, in Table 1, we present the average community size and the total uncovered nodes for a number of settings of the parameter regulating the number of communities output.

E.2. OSBM Experiments

Table 2a describes the coefficients applied to the scaling ${}^{\log n}/_{n}$ to define the probabilities of including edge between different parts of the ground-truth tripartition in the OSBM model. Table 2b displays the different parameters choices for the graph generation in our OSBM experiments.

	L	R	C
L	p	ϵ	p
R	ϵ	p	p
C	p	p	q

(a) Edge probability coefficients between two vertices in the OSBM model.

Size	L-R-C	p-q
10,000	0.45 - 0.45 - 0.10	4-2
30,000	0.45-0.45-0.01	4-4
100,000	0.6-0.3-0.1	4-6
	0.745-0.245-0.01	4-8

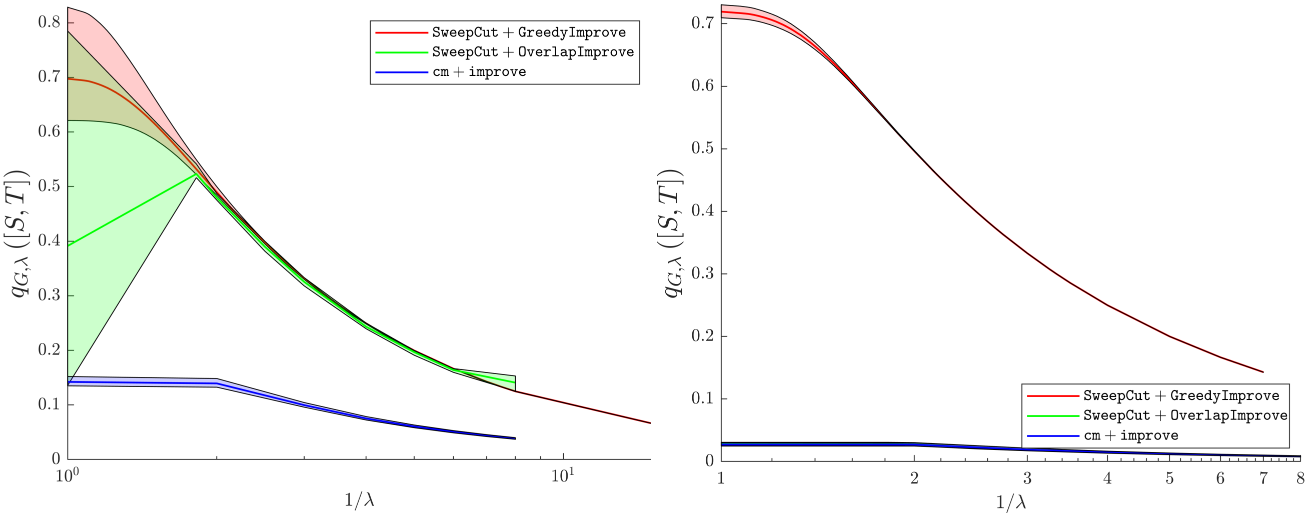
(b) Values used for generating OSBM graphs. $\epsilon = 0.05$.

The results of the experiments on OSBM were remarkably consistent across the size of the graph and the choice of p and q. Here and in the paper, we display results for the smallest graphs with $n = 10^4$. Further study is required to find interesting settings of p and q where threshold phenomena may arise. Figure 5 shows the comparison between $\texttt{cm} + \texttt{improve}$ and SweepCut for the case when the partition is unbalanced, as given by the third and fourth entry of second column in Table 2b. The main body of the paper includes the same results for the balanced choice (first and second entry of second column in

Network	n	m	time
DBLP	83,114	409,541	2-4min
Amazon	334,863	925,872	15-18min
Youtube	1,134,890	2,987,624	55-75min
LiveJournal	3,997,962	17,340,594	5-8h

Table 3: Social networks overlook and range of running times on 5 executions.

Table 2b). As in the balanced case, for large overlap size $|C|$, both versions of the SweepCut algorithm fail to detect the ground-truth overlap and have much larger λ-HCUT values. However, in this case, SweepCut+HybridImprove actually often achieves the same value of cm + improve when only looking for an edge partition, but is unable to maintain that performance when looking for overlaps. Once again, this suggests the need for algorithms that explicitly target overlapping clustering notions, rather than locally improving edge-based notions.



(a) Comparison on unbalanced OSBM with parameters $(|C|, p, q, \epsilon) = (100, 4, 4, \epsilon = 0.05)$.

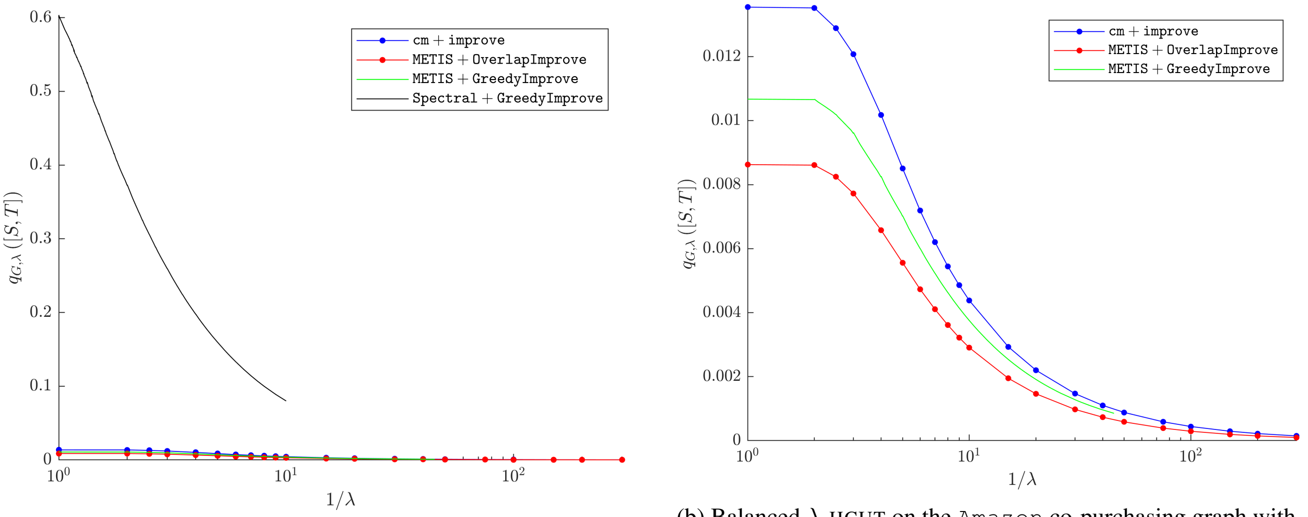
(b) Comparison on unbalanced OSBM with parameters $(|C|, p, q, \epsilon) = (10, 4, 4, \epsilon = 0.05)$

Figure 5: Performance of SweepCut against cm+improve on graphs from the OSBM model with $n = 10^4$. Each graph displays the performance over 5 samples from the model. The shaded area shows the minimum and maximum values over the samples, while the bold curve represents the average.

E.3. Large Social and Information Networks

We now describe the quantitative results of the comparison between cm+improve and METIS for the task of finding balanced overlapping partitions on the large networks in the SNAP database (Leskovec & Krevl, 2014). Table 3 displays our selection of graphs, together with their diverse sizes, and the running time required by cm+improve. Below, we focus on the performance as measured by the λ-HCUT objective $q_{G,\lambda}$.

We start with the Amazon graph in Figure 6. The left subfigure here includes data for SweepCut+GreedyImprove, showing a performance that is two orders of magnitude worse than cm+improve or METIS. The right subfigure excludes SweepCut+GreedyImprove allowing us to have a closer comparison with METIS. It shows that METIS, with either post-processing, slightly outperforms cm+improve, especially for large λ, i.e., edge-based cuts. This is not surprising, as METIS is highly optimized to find sparse balanced edge cuts. The fact that this advantage persists when searching for overlapping clusters may be an indication of the lack of meaningful overlapping structure over balanced cuts for this network.



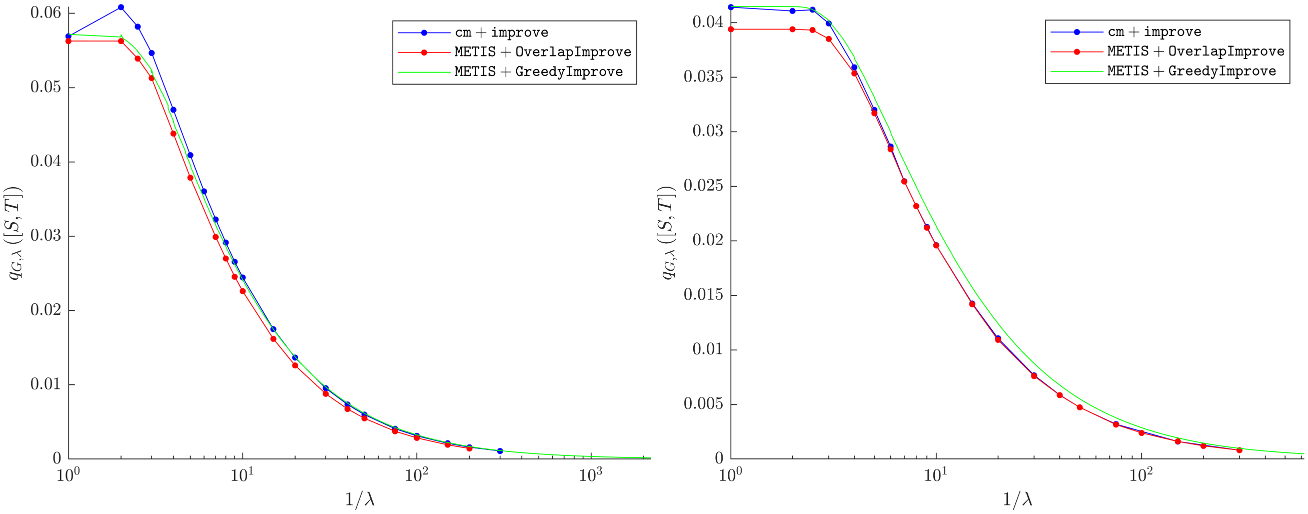
(a) Balanced λ-HCUT on Amazon copurchasing graph.

(b) Balanced λ-HCUT on the Amazon co-purchasing graph without Spectral + GreedyImprove.

Figure 6

It is also possible that cm+improve may require a larger number of iterations to achieve its optimal performance or that our balanced heuristic needs to be refined.

The results for the Youtube and DBLP graphs are shown in Figure 7. For these graphs, cm+improve essentially matches the performance of METIS which is a testament to the power of our algorithm framework, even in the non-overlapping settings. The fact



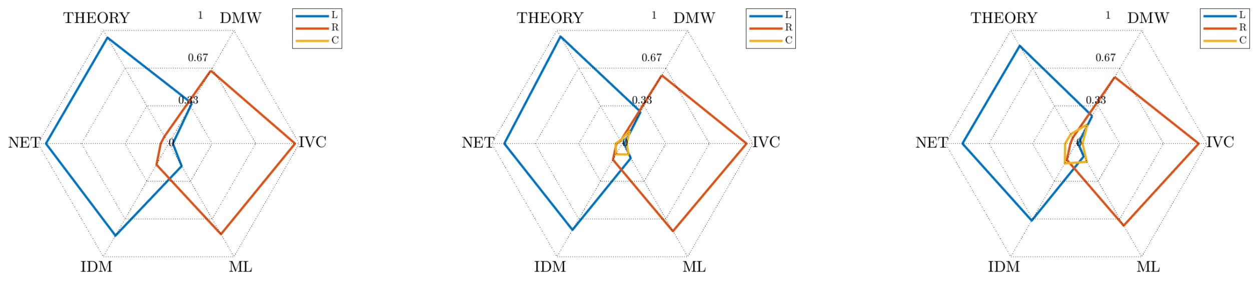
(a) Balanced λ-HCUT on Youtube social graph.

(b) Balanced λ-HCUT on DBLP co-authorship graph.

Figure 7

Future work should further address this comparison by relaxing the balanced constraint or using overlapping balanced clustering recursively to produce overlapping decompositions of the network. Such decompositions may detect meaningful overlapping structure at smaller sizes and in localized areas of the graph.

Visualization of Overlapping Clusters in DBLP co-authorship network The DBLP dataset for co-authorship in the academic field of Computer Science gives us the opportunity to visualize the overlapping communities discovered by cm+improve and compare them with the known clustering based on the venue of the each paper. Specifically, we built the co-authorship network creating for every paper p a weighted clique on the d-coauthors of paper p. The weight of this clique is equal to $\frac{1}{d}$ to ensure that each paper carries the same amount of information, i.e., the resulting random walk on the graphs consists of choosing a paper uniformly at random and sampling a co-author in this paper uniformly. The natural setting for the measure weight μ is then the degree measure of the graph. The results of running the balanced version of `cm+improve` as λ decreases are displayed below. We verified our results against `METIS+HybridImprove`, which outputs essentially the same tripartitions.



(a) $\lambda = 1$. Edge-based partition.

(b) $\lambda = 1/10$. Small overlap.

(c) $\lambda = 1/300$. Large overlap.

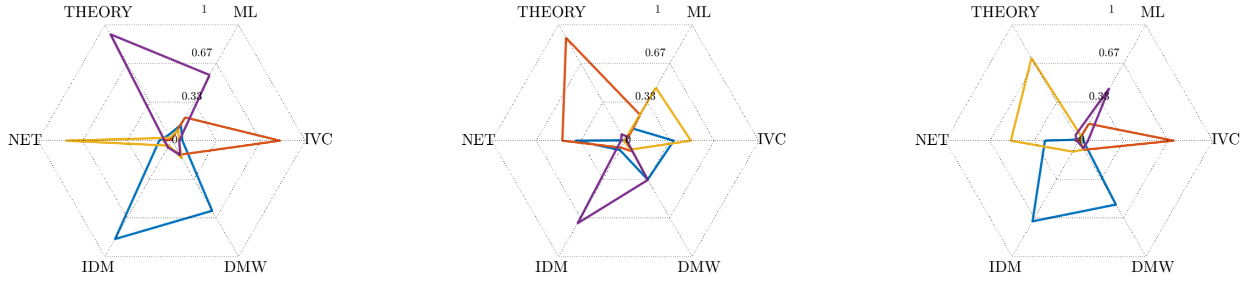
Figure 8: Spider plots showing the composition of the tripartition (L, C, R) output by `cm+improve`. For each set $X \in \{L, C, R\}$ of this partition and each subarea Y of Computer Science represent, we show the total edge volume of X coming from papers in area Y over the total edge volume of papers in Y.

The overlapping partitions found are very-well correlated with the edge-based partition capturing a clustering of subareas corresponding to a left cluster $\{\text{THEORY}, \text{NET}, \text{IDM}\}$ and a right cluster $\{\text{DMW}, \text{IVC}, \text{ML}\}$. As we saw above, this is expected as the real-world networks in our testbed do not appear to exhibit an overlapping balanced clustering that is different from the non-overlapping ones. As the first subfigure shows, a few papers from DMW and ML contribute to the left cluster. Indeed, as the overlap grows nodes with large degree in DMW and ML are the first to be included.

E.4. Recursive Overlapping Bisections

Our algorithm `cm+improve` can be used as a black box to **recursively bisection** the graph and identify multiple overlapping communities. When our algorithm is run without a balance constraint the successive cuts identify the "whiskers" around the core of the graph (Leskovec et al., 2009). For example, running recursive bisectioning on the Amazon dataset requires 52 cuts before a cut of significant volume is returned. All cuts up to that point have hundreds of nodes, meaning less than 1% of the graph.

Balanced multicuts are of greater interest as they partition the graph in more interpretable parts. For example, in the DBLP co-authorship graph we can further refine the found communities and relate them more closely to specific areas.



(a) $\lambda = 1/2$. Edge-based partition.

(b) $\lambda = 1/10$. Small overlap.

(c) $\lambda = 1/300$. Large overlap.

Figure 9: Partitioning the DBLP co-authorship graph in four communities. The edge based partitions sharply correlate to specific computing areas. As the overlap increases communities become more rounded, losing their clear definition.

On the Power of Louvain in the Stochastic Block Model

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Abstract

A classic problem in machine learning and data analysis is to partition the vertices of a network in such a way that vertices in the same set are densely connected and vertices in different sets are loosely connected.

In practice, the most popular approaches rely on local search algorithms; not only for the ease of implementation and the efficiency, but also because of the accuracy of these methods on many real world graphs. For example, the Louvain algorithm – a local search based algorithm – has quickly become the method of choice for clustering in social networks. However, explaining the success of these methods remains an open problem: in the worst-case, the runtime can be up to $\Omega(n^2)$, much worse than what is typically observed in practice, and no guarantee on the quality of its output can be established.

The goal of this paper is to shed light on the inner-workings of Louvain; only if we understand Louvain, can we rely on it and further improve it. To achieve this goal, we study the behavior of Louvain in the famous two-bloc Stochastic Block Model, which has a clear ground-truth and serves as the standard testbed for graph clustering algorithms. We provide valuable tools for the analysis of Louvain, but also for many other combinatorial algorithms. For example, we show that the probability for a node to have more edges towards its own community is $1/2 + \Omega(\min(\Delta(p-q)/\sqrt{np}, 1))$ in the SBM(n, p, q), where Δ is the imbalance. Note that this bound is asymptotically tight and useful for the analysis of a wide range of algorithms (Louvain, Kernighan-Lin, Simulated Annealing etc).

1 Introduction

Local search algorithms are widely-used in machine learning and data analysis, to extract information or optimize models. Among the most classic examples are Gradient Descent for tuning neural networks, Lloyd's method and Expectation-Maximization (EM) for clustering, unsupervised learning and statistical inference. However, understanding the practical success of local search algorithms through a theoretical analysis remains a major open problem. Proving guarantees on the quality of the local optima found by the algorithm and the required running time remain notoriously hard problems. For most of the above mentioned methods it is possible to construct adversarial examples that lead to highly sub-optimal local optima or induce very slow convergence. Nonetheless, many of these worst-case examples are contrived and highly unlikely to arise in real-world scenarios. Therefore, if one seeks to understand the success of local search algorithms, one must go *beyond the worst-case* scenario. This path has been recently explored for various algorithms and numerous papers have recently shown the power of gradient descent, EM, Lloyd's method in various specific contexts.

34th Conference on Neural Information Processing Systems (NeurIPS 2020), Vancouver, Canada.

An illustrative example of the discrepancy between the success of local search techniques versus its theoretical understanding is the case of graph partitioning. Consider the problem where one is given a graph G, and asked to partition it into subgraphs, each of which exhibiting a higher density of edges within the subgraph than towards the rest of the graph. For this problem, the power of local search algorithms was first materialized by simulated annealing heuristics. In the early 70s, Kernighan and Lin [26] presented a simple local search procedure for computing a balanced cut[1] of a graph of small size. The heuristic quickly became a standard tool for VLSI design and is still part of various packages [32]. More recently, the success of the Louvain algorithm [7] for extracting information from social networks, knowledge or similarity graphs has shown that despite a flurry of new techniques, local search algorithms remain the most popular heuristics. However, from a theoretical standpoint, designing approximation algorithms for graph partitioning objectives such as modularity, sparsest cut, bisection, or multicut is a major challenge: under some popular complexity assumptions such as the unique game conjecture or P $\neq$ NP, there is no constant factor polynomial-time approximation algorithms for the above problems.

Modularity and the Louvain algorithm. Introduced in 2008 and designed to detect communities in social networks, the Louvain heuristic has received more than 11400 citations over the last 10 years [19] and is now the method of choice for clustering similarity graphs (see for example the extensive analysis of Lancichinetti and Fortunato [27]). The algorithm is simply a slight refinement of a local search algorithm which aims at optimizing the *modularity* of the current clustering (see Equation 1 and a more detailed presentation of the Louvain algorithm in Section 2). More interestingly, this algorithm is recognized to produce a *good* clustering very fast, outperforming most of the other clustering methods.

Even though this heuristic is now widely used, it is known that it may output arbitrarily bad partitions (in terms of modularity) for some adversarial examples. Even more surprising is the fact that the worst-case running time of the algorithm is $\Omega(n^2)$, a prohibitive running time in practice, but experiments show that it often terminates after $O(n \text{ polylog}(n))$ operations.

Quite surprisingly, our understanding of the success of this heuristic is very poor: no guarantees on the quality of the solution output by Louvain, even for some simple scenarios, have been established. We thus ask: what is the structure of Louvain's solution for real-world graphs?

A natural setting for providing a beyond-worst-case analysis of these local search algorithms is through the classic Stochastic Block Model (see formal definition in SuppMat A) which exhibits a *clear ground-truth clustering* and which has been used to provide a beyond-worst-case-analysis framework in a large number of works.

1.1 Our Results

We focus on the classic Stochastic Block Model with two communities, namely the graph consists of two communities, each consisting of n nodes, and the probability of observing an edge between two nodes of the same (resp. different) community is p (resp. q). We refer the reader to Section 2 for formal definition of the above concepts and the Louvain algorithm. Our results are two-fold. We show, for a large range of parameters, that Louvain recovers the hidden partition and that it converges rapidly.

We first show that if Louvain is initialized properly, namely with an equal-size two-partition with *imbalance* Δ, i.e.: where some part contains $n/2 + \Delta$ vertices of a given community, then it converges in $O(n)$ steps to the correct clustering with high probability assuming ${}^{p-q}/_{\sqrt{p}} \geq c\sqrt{n \log n}/\Delta$, for some constant c. Interestingly, this bound is near-optimal, namely close to the information theoretic threshold ${}^{p-q}/_{\sqrt{p}} \geq \sqrt{{}^{\log n}/_{n}}$ up to constant factors, if $\Delta \geq n/c'$ for some constant c'.

Theorem 1.1 (Warm Start)**.** *Let $\Delta > 0$. Consider a graph $G \sim \mathrm{SBM}(n,p,q)$. Then, there exists a constant c such that, with high probability,* Louvain *initialized with a partition of imbalance Δ recovers the partition $\{V_1, V_2\}$ in $O(n)$ rounds, if $\frac{p-q}{\sqrt{p}} \geq 200\frac{\sqrt{\log n}}{\sqrt{\Delta}} \max\left(1, \frac{\sqrt{(n/2-\Delta)}}{\sqrt{\Delta}}\right)$.*

[1] A balanced cut is a set of edges whose removal splits the graph into two components with equal number of vertices.

We then show that even when Louvain is initialized with a random equal-size two-partition, it converges to the correct clustering with high probability after only $O(n)$ steps, provided that $^{p-q}/_{\sqrt{p}} \geq 100n^{-1/6+\varepsilon}$.

Theorem 1.2 (Cold Start)**.** *Consider a graph $G \sim \mathrm{SBM}(n,p,q)$ and assume $\frac{p-q}{\sqrt{p}} \geq 200n^{-1/6+\varepsilon}$, for some $\varepsilon > 0$. With hight probability,* LOUVAIN *algorithm recovers the partition $\{V_1, V_2\}$ within $O(n)$ rounds.*

To prove these theorems, we provide valuable tools for the analysis of Louvain, but also for a wide range of combinatorial algorithms. For example, we show that the probability for a node to have more edges towards its own community is $1/2 + \Omega(\min(\Delta(p-q)/\sqrt{np}, 1))$ in the SBM(n,p,q), where Δ is the imbalance. Note that this bound is asymptotically tight and useful for the analysis of other local-search based algorithm such as the aforementioned Kernighan-Lin, Simulated Annealing etc.

As a side product of our techniques we also obtain bounds for MAJORITY, which is a simpler version of Louvain, where a node simply moves to the part to which it has the most number of edges. We can show that for $p - q \geq 1/n^{1/4}$, MAJORITY recovers the optimal partition in $O(n^2p)$ steps, which is linear in the graph size. In comparison, the state-of-the-art, [9] showed that MAJORITY if $p - q \geq 1/n^{1/4}$ in dense graphs, namely when the number of edges is $\Omega(n^2)$. In contrast to their techniques, ours does not have any requirement on the density of the graph. Another drawback of their analysis is that it does not imply that the convergence time would be subquadratic. Here we show that it is in fact linear.

1.2 Comparison to Previous Work

Understanding the power of local search for graph cut problems has always been of high interest for the research community. The classic *majority* algorithm has been studied since the work of Kernighan and Lin [26]: the algorithm maintains a two-partition of the graph and swap a node from one side to the other if it has more neighbor in the latter. The research community has first taken an important step towards understanding local search algorithms in the Stochastic Block Model through the work of Jerrum and Sorkin [24, 25] on the metropolis algorithm for graph bisection. They showed that in the Stochastic Block Model with 2 communities, the metropolis algorithm (simulated annealing at some specific fixed temperature) recovers the optimal bisection if $p - q \geq 1/n^{1/6}$ after $O(n^2)$ steps. This was later improved by Carson and Impagliazzo [9] who showed that the standard local search algorithm also recovers the optimal partition if $p - q \geq 1/n^{1/4}$ in dense graphs, namely when the number of edges is $\Omega(n^2)$. However, this result is unsatisfactory in two aspects: first, the proof critically relies on the number of edges being $\Omega(n^2)$, which is for this type of algorithms arguably a strong assumption since the information per node is much higher than in a sparse regime. More importantly, the result did not address the running time of the algorithm (i.e. the convergence time of the process). Thus, in addition to the first analysis of LOUVAIN, our results also improve upon the work of Carson and Impagliazzo on the analysis of the Majority algorithm by addressing sparser regimes, obtaining a strong bound on the running time. More recently, Boumal [8] showed that simulated annealing at the "correct" temperature recovers the correct partition nearly-optimality (namely up a constant factor of the information theoretic threshold). However, the temperature should be set as a function of the model parameters and so this algorithm remains far from practical. More recently, Chin, Rao and Vu [10] and Yun and Proutière [34] have designed local-search-based algorithms that aim at improving a solution obtained via spectral method. Both proofs assume that the initial partition given to the local search algorithm only missclassifies a very tiny fraction of the vertices (only $O(1/p)$ vertices are misclassified in [34], $O(n/10)$ in [10] – Note also that [10] considers an algorithm that is designed to avoid most of the technical issues encountered when analysing local search methods since it at each step it works with 'fresh' edges for which the randomness has not been revealed). Those results are therefore far from addressing the cold start setting, which is the most challenging and interesting for the analysis of local-search heuristics, while our results on the warm start setting are strictly more general.

From a technical standpoint, an important challenge that our work addresses is handling the randomness of the graph through successive local search steps. This is a key step when analysing a local search algorithm since it is particularly hard to deal with the dependencies created by the algorithm, which considers every edge many times. To the best of our knowledge, previous work tackled this issue by carefully designing their algorithms. This is not possible to do when analyzing LOUVAIN, and we therefore must develop new tools. On the one hand, the existing local-search algorithms of [25, 11, 9] are designed such as to avoid this dependency issue, by using at every step ”fresh” edges, for which the randomness has not been revealed until that step. On the other hand, the techniques developed in the series of work dedicated to the Stochastic Block Model mentioned above relies mostly on SDP or spectral graph theory, and do not seem to apply to local-search heuristic. From a performance standpoint, those algorithms recover the partition when ${}^{p-q}/_{\sqrt{p}} = O(\sqrt{{}^{\log n}/_{n}})$.

There is a large body of other work on the Stochastic Block Model and describing it is beyond the scope of this paper. The interested reader may look into the survey of Abbe [1]. The precise understanding of what can be recovered as a function of p and q in the Stochastic Block Model is due to Abbe et al. [2] and Mossel et al. [30]. They prove that recovery is possible if and only if ${}^{p-q}/_{\sqrt{p}} > 2\sqrt{{}^{\log n}/_{n}}$. Classic results encompass the fundamental result of McSherry [29], the augmentation algorithm of Condon and Karp [11]. Iterative methods [16, 28, 35, 17], semi-definite programming [20, 21, 5, 13, 14] and spectral algorithm [1] have been investigated under the Stochastic Block Model. Perhaps, more closely related results are the recent advances on the analysis of the Belief Propagation (BP) algorithm, a much more evolved message-passing than the standard MAJORITY. Some algorithms, based on BP or variants of BP (such as the linearized acyclic BP) have been shown to recover the ground-truth output in the Stochastic Block Model as well [4, 3, 12]. Nonetheless, we believe that these works, while of high importance for the study of BP algorithms do not allow to shed light on simpler heuristics, such as MAJORITY or LOUVAIN which are widely-used in practice and also reasonable models of local-decision dynamics.

1.3 Roadmap

In Section 2 we introduce the algorithms. A formal definition of the Stochastic Block Model can be found in SuppMat A together with some additional notations. In Section 3, we study the behaviour of LOUVAIN when initialized with a large imbalance, and prove Theorem 1.1. In Section 4, we study the algorithm initiated with a random cut, and show Theorem 1.2. All proofs can be found in the supplementary material.

2 Preliminaries and Notations

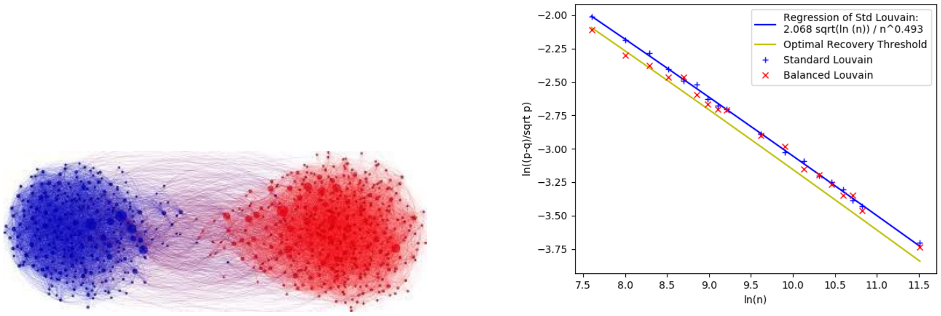
The formal definition of the Stochastic Block Model can be found in SuppMat A. In short, there are two communities each with n nodes. Two nodes from the same community are connected with probability p and nodes from different communities are connected with probability q. The goal is to recover the two communities.

The LOUVAIN Algorithm We now describe the local-search algorithm LOUVAIN ([7]). Although this article focuses on the case with two communities, LOUVAIN is more general and we define it for more communities here. It is a local search technique that aims at finding a partition of the vertices of a given graph that maximizes the *modularity*. For any partition $P = (P_1, \ldots, P_\ell)$, the modularity of P is defined as

$$M(P) = \frac{1}{2m} \sum_{i=1}^{\ell} \sum_{u,v \in P_i} \left(1_{(u,v)} - \frac{\deg(u) \cdot \deg(v)}{2m} \right), \tag{1}$$

where $1_{(u,v)}$ is 1 if and only if there is an edge between vertices u and v, $\deg(u) = \sum_v 1_{(u,v)}$, $2m = \sum_u \deg(u)$. For a vertex u, we let $P(u)$ be its part in the partition P. The Louvain local dynamic is defined as follows. Consider a partition $P = (P_1, \ldots, P_\ell)$. For each vertex u, define $P^{u,i}$ to be the partition where u is removed from $P(u)$ and added to P_i. Define $Q_{u,i}$ as the modularity of $P^{u,i}$ minus the modularity of P and let i^*_u be $\arg\max_i Q_{u,i}$, breaking ties arbitrarily. We say that $Q_{u,i}$ is the swap value for u.

The Louvain algorithm consists of successive steps, where each step is performed as follows. Given a partition P, the algorithm considers all the vertices u such that $P(u) \neq P_{i^*_u}$, picking a pair u, v such that $P(u) = P_{i^*_v}$ and $P(v) = P_{i^*_u}$ at random and then defining a new partition P' obtained by removing u from $P(u)$ and adding it to $P_{i^*_u}$ and removing v from $P(v)$ and adding it to $P_{i^*_v}$ Then,



(a) The figure depicts a graph generated by the Stochastic Block Model. The nodes within the communities are densely connected and nodes of different communities are less densely connect.

(b) The points represent, for a given number of points n, the smallest value of ${}^{p-q}/_{\sqrt{p}}$ for which Louvain (resp. Standard Louvain) succeeds. More details are given in Section 5.

the algorithm performs the next step on partition P'. The algorithm stops when the partition P is such that $P(u) = P_{i_u^*}$, for all u.

For $k = 2$ communities, this is very similar to the classic Hillclimbing procedure considered in [9] and the Metropolis algorithm as temperature 0 considered in [24].

The algorithm we analyze, Balanced Louvain, is a slight modification of the algorithm above (which we call Standard Louvain) in three ways.

1. First, Balanced Louvain starts with a random equi-sized partition $P = (P1, P2)$, whereas Standard Louvain starts with $2n$ parts each containing one node.

2. Second, Standard Louvain moves only one node at a time, whereas Balanced Louvain swaps nodes, to maintain balanced clusters. More precisely, Balanced Louvain select one node with positive swap value in each part, and moves it to the other part.

3. Third, once a local optimum is reached, Standard Louvain merges the nodes of a cluster together and proceeds. Balanced Louvain simply stops when a local optimum is reached.

We now make a case for these adaptations are justified. 1) At the beginning Louvain will quickly reduce clusters until there are only two left. Essentially, Louvain does not encounter any local optima when the number of clusters is strictly larger than two. 2) We adopted this variant of Louvain to avoid having to keep trace of the size imbalance between communities during the process. It can be shown with random walk argument that, assuming 1), the size imbalance stays negligible: This is done in SuppMat F. 3) By considering our simpler procedure, without any contraction, we actually show that for the SBM, the hierarchy has a single level, and the algorithm does not need to escape local optimum. Just like one would hope.

To further justify our adaptations, we show experimentally that our variant, Balanced Louvain, performs just as well as Louvain (see Figure 1b and Section 5).

3 Warm Start

In this section we consider a graph $G \sim \text{SBM}(n, p, q)$ and a partition $V = (P_1, P_2)$ with imbalance Δ. For any $i \in \{1, 2\}$, we refer to the part that contains the larger number of vertices of community i as the *home* of i and we refer to this part as HOME i. Namely, HOME $1 = \text{argmax}_{U \in \{P_1, P_2\}} |U \cap V_1|$ and HOME $2 = \text{argmax}_{U \in \{P_1, P_2\}} |U \cap V_2|$. We say that a vertex is *good* if it is of community i and not in HOME i. A vertex is *bad* if it is of community i and in HOME i. This section is dedicated to the proof of Theorem 1.1. The proof idea is built around showing the following property $\mathcal{P}$. For any given cut with large imbalance Δ, there is no large subset of nodes whose sum of swap values is negative. If the sum of swap values is non-negative, this also means that among the good nodes, there can only be very few with a negative swap value. Therefore, $\mathcal{P}$ implies that most of the good nodes would move to their HOME if chosen. Similarly, we can show that most of the bad nodes prefer to stay in their HOME. Putting both of these facts together, we get that the imbalance is likely to increase after one round. We can show that the imbalance 'performs' a biased random walk and will therefore quickly increase to a size of $n/2$ implying convergence.

One of the main technical challenges of proving $\mathcal{P}$ is the dependencies among the nodes: the swap values of the nodes are correlated due to mutual edges. However, this dependency is weak and we can use Theorem A.1 to obtain strong concentration bounds. So strong, that we can take a Union bound over all cuts with large imbalance taking care of revealed randomness in previous steps. The bulk of the proof is captured by the following lemma.

Lemma 3.1 (proved in SuppMat B)**.** *Consider a graph $G \sim \mathrm{SBM}(n, p, q)$ on n vertices and let $0 \le \Delta < n/2$. Assume $\frac{p-q}{\sqrt{p}} \ge 200\frac{\sqrt{\log n}}{\sqrt{\Delta}} \max\left(1, \frac{\sqrt{(n/2-\Delta)}}{\sqrt{\Delta}}\right)$.*

Fix a partition $(S, V \setminus S)$ with imbalance $\Delta < n/2$, the following holds with probability at least $1 - 3\exp(-5(n/2-\Delta)\log n)$: The number of good vertices with positive swap values is at least $(2/3)(n/2-\Delta)$ and the number of bad vertices with positive swap values is at most $(n/2-\Delta)/3$.

Assuming this lemma, we can prove the main proposition in SuppMat B. Briefly, Lemma 3.1 is used to show that with probability $1 - O(1/n^2)$ all cuts are *improving*, i.e. the probability of increasing the imbalance is at least $2/3$. The imbalance is therefore a random walk on $\mathbb{N}$, with probability $2/3$ of increasing: the time to reach n is thus $O(n)$ with probability $1 - O(1/n^2)$, which concludes the lemma.

The rest of this section is dedicated to the proof of Lemma 3.1. Our strategy is to consider the sum of swap values of a big enough set of vertices S_0. Since this is in expectation way larger than the swap value of a single vertex – it is $|S_0|\Delta(p-q)$ for a set of good vertices – Chernoff bounds allows to show concentration with way higher probability. The first part of Lemma 3.2 shows that the sum of swaps values of at least $\frac{1}{3}(n/2-\Delta)$ bad vertices must be negative. This is used as follows: let S_0 be the set of bad vertices with positive swap value. If S_0 had size bigger than $\frac{1}{3}(n/2-\Delta)$, it would contradict Lemma 3.2. Hence, the number of bad vertices with positive swap value is at most $\frac{1}{3}(n/2-\Delta)$.

Similarly, the second statement of the lemma shows that a big enough group of good vertices must have positive total swap value. This implies as well that, if S_0 is the set of good vertices with negative swap value, S_0 must have size smaller than $\frac{1}{3}(n/2-\Delta)$. Hence, since there are $n/2-\Delta$ good vertices, there must be at least $\frac{2}{3}(n/2-\Delta)$ good vertices with positive swap value.

Lemma 3.2 (proved in SuppMat B)**.** *Consider a graph $G \sim \mathrm{SBM}(n, p, q)$ on n vertices and assume that $\frac{p-q}{\sqrt{p}} \ge 200\frac{\sqrt{\log n}}{\sqrt{\Delta}} \max\left(1, \frac{\sqrt{(n/2-\Delta)}}{\sqrt{\Delta}}\right)$.*

Fix a partition $(S, V \setminus S)$ with imbalance $\Delta < n/2$, and let $S_0 \subseteq S$ be a set of vertices of size $\frac{1}{3}(n/2-\Delta)$.

- *If S_0 consists only of bad vertices, then the sum of swap values of the vertices in S_0 is* at most $-|S_0|\Delta(p-q)/2$ *with probability at least* $1 - 3\exp(-5(n/2-\Delta)\log n)$.
- *If S_0 consists only of good vertices, then the sum of swap values of the vertices in S_0 is* at least $|S_0|\Delta(p-q)/2$ *with probability at least* $1 - 3\exp(-5(n/2-\Delta)\log n)$.

The proof of Lemma 3.1, presented in SuppMat B simply uses the previous lemma as explained previously.

4 Cold Start

We start by giving the intuition. The proof has two parts. In the first part (Section 4.1), we assume that we start with a graph with *fresh* randomness (i.e.: nothing about the random process generating the edges has been revealed so far), and a random partition into 2 parts having imbalance at least Δ. That is, we assume that we have $n/2+\Delta$ nodes of community 1 and $n/2-\Delta$ nodes of community 2 in the first part of the partition, and that we draw edges in the graph according to the Stochastic Block Model. Then, we show that the probability of a node u having more edges to its HOME is at least $p' = 1/2 + 0.018 \cdot \min\left\{{}^{\Delta(p-q)}/\sqrt{np(1-p)}, 1\right\}$. Note that this term is up to constants in the second-order term tight and improves on the result of [11] that did not have the factor $1/\sqrt{p(1-p)}$ in the second-order term. For our results, which allows p to be very small, this term is vital. To obtain it, we use Esseen's inequality together with coupling arguments. From this, we deduce that a large fraction of the node have more edges to their HOMEthan to the other part; which in turn implies that

Louvain has a good probability of moving one node to its HOMEand improve the imbalance. When the imbalance is $\Omega(n/\log^2(n))$ we can appeal to the warm start result. The challenge is thus to show that the following property $\mathcal{P}$ holds: The fraction of nodes that has more edges to its HOME than to the other community is at least $1/2+\Omega\left(\min\left\{\Delta(p-q)/\sqrt{np(1-p)},1\right\}\right)$.

In the second part (Section 4.2), we aim at proving that the probability that property $\mathcal{P}$ holds for all the cuts encountered by the algorithm is indeed high. To do so, we proceed as follows. From Section 4.1 we know that property holds for a random cut of a fresh graph with at least constant probability but this is not good enough because after one iteration of Louvain (i.e.: swapping one vertex from one side to the other) some of the randomness of the graph has been revealed. Hence, the cut reached after one iteration cannot be considered to be on a fresh graph, preventing us from directly applying the above result. We then argue that $\mathcal{P}$ has exponentially small probability of not happening and [2] applying a counting argument, we show in lemma Lemma 4.7 that the property $\mathcal{P}$ holds w.h.p. for all of the partitions encountered by Louvain during the first $n/\log^2(n)$ steps. The counting argument is arguable simple and essentially states that the number of partitions that can possibly be encountered after t steps is at most $2^{t\log n}$ but the probability of a random cut to be problematic is exponentially smaller than that number. Hence, the probability that the initial random cut could lead to a cut for which property $\mathcal{P}$ does not hold is exponentially small. From there on we can use Markov chain theory to argue that the behavior of the imbalance Δ can be modeled by a biased random walk and will quickly increase to a size that is covered by the warm start regime (Section 3). From thereon, the process quickly converges and we obtain the hidden partition.

4.1 The probability of Improving the Cut

The goal of this section is to prove Lemma 4.1, which in substance states that for a graph with imbalance Δ and fresh randomness, the probability of a node u having more edges to its HOME is at least $p' = 1/2 + 0.018\cdot\min\left\{\Delta(p-q)/\sqrt{np(1-p)},1\right\}$ (simplified).

To this end we introduce some notation. Let $X_i(u)$ denote the number of edges from u to part i, for $i\in\{1,2\}$. Each $X_i(u)$ decomposes as the sum of two binomials with different parameters. These variables encapsulate the movement of u: u goes to community i such that $X_i(u)=\max\{X_1(u)+L_1, X_2(u)+L_2\}$, where L_1 and L_2 are the Louvain terms. We seek to calculate the probability that a given $X_1(u)$ is larger than $X_2(u)$, by at least $|L_1|+|L_2|$ (note that L_1 can be negative). To do this we use Esseen's inequality to show that $X_1(u)$ and $X_2(u)$ are distributed very similarly to $Y_1(u)$ and $Y_2(u)$, where $Y_i(u), i\in\{1,2\}$ is the Gaussian equivalent of $X_i(u)$ (see Lemma 4.2). We then introduce ideal Gaussians $\{Z_i(u)\}_{i\in\{1,2\}}$ coupled with $\{Y_i(u)+L_i\}_{i\in\{1,2\}}$. The ideal Gaussians $\{Z_i(u)\}_{i\in\{1,2\}}$ allow us to use some symmetry properties enabling us to bound the probability that node u goes to the other part) tightly up to a constant in the second-order term (Lemma 4.1).

We now give the formal definitions. For a given vertex u with COMMUNITY$(u)=1$, we define the following random variables corresponding to the number of edges u has to part 1

$$X_1(u)\sim B(n/2+\Delta,p)+B(n/2-\Delta,q)\quad\text{and}\quad X_2(u)\quad\sim B(n/2-\Delta,p)+B(n/2+\Delta,q)$$

As mentioned before, the goal of this section is to prove the following lemma, which gives a lower bound on the probability of improving the cut. We will use L^* which will be a bound on the Louvain terms, that is $|L_1|+|L_2|\le\mathrm{L}^*$.

Lemma 4.1. *Assume $|L_1|+|L_2|\le\mathrm{L}^*$. Then, $\mathbb{P}\left[X_1\ge X_2+\mathrm{L}^*\right]$ is at least*

$$1/2+0.018\cdot\min\left\{\frac{\Delta(p-q)}{\sqrt{np(1-p)}},1\right\}-\frac{1}{2\Delta(p-q)}-\frac{\mathrm{L}^*}{2\sqrt{(n/2-\Delta)p(1-p)}}-4\sqrt{\frac{2}{np(1-p)}}$$

In order to prove the lemma, we define the normally distributed random variables corresponding to $X_1(u)$ and $X_2(u)$.

[2] Clearly, there are dependencies between the nodes due to the shared edges and one can't apply a standard Chernoff bound to derive the probability of $\mathcal{P}$ to be satisfied. However, we argue that the dependencies are weak, allowing us to use Theorem A.1 to obtain concentration bounds on the probability of the the above mentioned property $\mathcal{P}$ being true. The obtained probability of $\mathcal{P}$ not holding is exponentially small in $\Delta(p-q)$.

$$Y_1(u) \sim \mathcal{N}((\frac{n}{2}+\Delta)p+(\frac{n}{2}-\Delta)q,\sigma_1) \text{ and } Y_2(u) \sim \mathcal{N}((\frac{n}{2}-\Delta)p+(\frac{n}{2}+\Delta)q,\sigma_2) \quad (2)$$

where $\sigma_1 = (n/2+\Delta)p(1-p)+(n/2-\Delta)q(1-q)$ and $\sigma_2 = (n/2-\Delta)p(1-p)+(n/2+\Delta)q(1-q)$. We will define two normally distributed random variables $Z_1(u)$ and $Z_2(u)$ which we will use to argue that the Louvain term does not influence the outcome. We will assume they have the same law as Y_j. Define $Z_j \overset{d}{=} Y_j$ for $j \in \{1,2\}$.[3]

In the following, we will focus on a particular vertex u, assuming w.l.o.g that $\text{COMMUNITY}(u) = 1$. Hence, we drop the parenthesis from the variables X, Y and Z. We show that the binomials X behave very similarly to their Gaussian counterparts Y, which, together with the Louvain term, we will couple with Z. We then use these similarity between X and Z to prove Lemma 4.1.

In a first step, we relate the Binomials X_j and the Gaussians Y_j.

Lemma 4.2 (proved in SuppMat C)**.** *Assume* $|L_1| + |L_2| \leq \mathrm{L}^*$. *We have for all* i, j $\left|\mathbb{P}\left[\, X_1 \geq X_2 + \mathrm{L}^* \,\right] - \mathbb{P}\left[\, Y_1 \geq Y_2 + \mathrm{L}^* \,\right]\right| \leq 4\sqrt{\frac{2}{np(1-p)}}$.

Note that X_1 and X_2 are Binomial random variables, i.e., the sums of Bernoulli random variables. In the proof, we use Esseen's inequality (Theorem E.1) to convert the Binomials to Gaussians.

In a second step, we relate between the Gaussians Y_j and Z_j.

Lemma 4.3 (proved in SuppMat C)**.** *Assume* $|L_1| + |L_2| \leq \mathrm{L}^*$. *We have* $\left|\mathbb{P}\left[\, Y_1 > Y_2 + \mathrm{L}^* \,\right] - \mathbb{P}\left[\, Z_1 > Z_2 \,\right]\right| \leq \frac{\mathrm{L}^*}{2\sqrt{\mathrm{Var}[\, Y_2 \,]}}$.

In the proof, we show that there exists a coupling such that $\forall j, L \leq \mathrm{L}^* : \mathbb{P}\left[\, Y_j + L = Z_j \,\right] \geq 1 - \frac{L}{2\sigma_2}$. We do this by bounding the total variation distance between the distributions Y_j and Z_j.

Carefully analyzing Gaussians allows us prove the following lemma which gives bounds that are tight up the constant in the second-order term.

Lemma 4.4 (proved in SuppMat C)**.** *We have* $\mathbb{P}\left[\, Z_1 > Z_2 \,\right] \geq 1/2 + 0.018 \cdot \min\left\{\frac{\Delta(p-q)}{\sqrt{np(1-p)}}, 1\right\} - \frac{1}{2\Delta(p-q)}$.

We now have all parts required to prove Lemma 4.1, which we do in the supplementary material. Note that choosing $\mathrm{L}^* = 1$ provides the same guarantees we get for Louvain for MAJORITY. The proof can be found in SuppMat C.

4.2 From Imbalance $\sqrt{n}$ to $\Omega(n/\log^2 n)$

We now use Lemma 4.1 to show that the imbalance rapidly grows. For this, we start with a random cut, which can be assumed to have imbalance $\sqrt{n}$. This, together with Lemma 4.5 that bounds L^*, allows us to compute the probability that a random positive swap improves the cut. We conclude the section in Lemma 4.7, comparing the imbalance with a random walk to show its growth.

The first step stems from the fact that, with constant probability, the imbalance of a random cut is more than $\sqrt{n}$. The probability can be boosted by repetition, since no randomness of the edges is revealed. The next step is to bound the term L^*.[4] This is captured in the following lemma.

Lemma 4.5. *[proved in SuppMat C] Given a random cut with imbalance* Δ*, there exists a constant* c *such that, with probability* $1 - 2\exp(-\frac{\Delta^2(p-q)^2}{cp})$*, it holds that for all vertex* u*,* $|\mathrm{L}(u)| \leq \Delta(p-q)/100$.

We now combine Lemma 4.5 with Lemma 4.1 to get the probability that a swap vertex chosen by the algorithm is good – and note *good* this event. We note *positive* the event that the vertex chosen

[3] In principle, we could avoid introducing the random variables Z_j and only work with Y_j, but to avoid some dependency issues, we chose to introduce the fresh variables Z_js.

[4] For the analysis of MAJORITY we obtain the better bound $(p-q)/\sqrt{p} = \Omega(1/n^{1/4})$ since we can simply use $L^* = 1$.

by the algorithm has positive swap value. Recall that a *good* vertex is a vertex that is not in its home, so that we would like to swap to the other side.

Lemma 4.6. *[proved in SuppMat C] Assume* $\frac{p-q}{\sqrt{p}} \geq 200n^{-1/6+\varepsilon}$*. Fix some imbalance* $\Delta \in [\sqrt{n}, n/\log^2 n]$*. Depending on the size of* Δ *the following holds. There exists constants* c_1, c_2 *such that:*

1. *For* $\Delta(p-q)/\sqrt{n} \leq 1$*, we have w.p.* $1 - \exp\left(-\frac{\Delta^2(p-q)^2}{100}\right)$ *that* $\mathbb{P}\left[\, good \mid positive \,\right] = 1/2 + c_1\frac{\Delta(p-q)}{n}$*, for some constant* c_1*.*
2. *For* $\Delta(p-q)/\sqrt{n} > 1$*, we have w.p.* $1 - \exp\left(-\frac{c_2^2 n}{4}\right)$ *that* $\mathbb{P}\left[\, good \mid positive \,\right] = 1/2 + c_2$*, for some constant* c_2*.*

Now that we know that the algorithm has a good probability to increase the imbalance, we can formalize the convergence with a random walk argument:

Lemma 4.7 (proved in SuppMat C)**.** *Assume* $\frac{p-q}{\sqrt{p}} \geq 100n^{-1/6+\varepsilon}$*. Then, after* $O(n/\log n)$ *steps of the algorithm, we have that* $\Delta = \Omega(n/\log^2 n)$ *with probability* $1 - 1/n$*.*

4.3 From Imbalance $\Omega(n/\log^2 n)$ to convergence

We can finally conclude the proof of Theorem 1.2, combining Lemma 4.7 and the results of Section 3, Theorem 1.1. The proof can be found in the supplementary material.

5 Experiments

We experimentally evaluated the performances of Louvain in the SBM. For the experiment we used the standard Louvain and the vertex-swapping version that we analyze. Our implementations builds on the Louvain implementation of Guillaume [23]. In order to generate the graphs efficiently, we devised a method that draws the the edges from the correct probability distribution in $O(m)$-time instead of $O(n^2)$ time, where m is the number of expected edges $\approx n^2(p+q)$.

In our experiments, we set $q = p/2$. The plotted curve is the smallest value of ${}^{p-q}/_{\sqrt{p}} = \sqrt{p}/2$ for which the algorithm recovers the ground truth at least 8 times out of 15 trials on different random graphs. We use a log-log scale for plots. We added to them the curve $2.068 \log n/n^{0.493}$, found with non-linear least squares to fit Louvain's performances curve.

We make the two following observations. First, the exponent in our analysis $(1/6)$ does not seem tight. It is worth noting that 0.5 is optimal, as it is known that p and q must verify ${}^{p-q}/_{\sqrt{p}} = \Omega(\sqrt{{}^{\log n}/_{n}})$ since otherwise at least one node will have more edges towards the other community (see [2, 30]). The fitted curve has a better asymptotic because of the variance of our experiments. Second, the two experimental curves of Louvain and our slight modification essentially coincide: making the assumption that Louvain uses swap is therefore a fair assumption to make as it simplifies the proof greatly.

6 Broader Impact

We give the first theoretical explanation of Louvain's success. We show that Louvain not only recovers the hidden partition in the stochastic block model successfully, but also does so in linear time and so for a large range of parameters. Interestingly, if Louvain is properly seeded it can recover the parameters nearly up to the information theoretic threshold.

As explained in the introduction, the goal of this paper is to cast a new light on the success of a popular heuristic for clustering, namely LOUVAIN. With more than 10 000 citations, LOUVAIN is the method of choice for graph clustering. Thus, explaining its power and limitation is of primary importance for a large variety of research areas (see for instance Hoffman et al. [22], analyzing the Bible with LOUVAIN, or Wu et al. [33] for drug repositioning). Our work shows that for graphs exhibiting a clear but noisy clustering structure, then Louvain quickly converges to a global optimum (w.r.t. the modularity objective). Therefore, when the clusters maximizing modularity align with the ground-truth clusters, Louvain is indeed a powerful clustering algorithms with a reliable performance.

Finally, our work also improves the theoretical analysis and provides tools for a wide-range of other algorithms including Kernighan-Lin, Majorty and other combinatorial algorithms that rely on moving nodes to communities to which they have the most number of edges. Concretely, we show that the probability for a node to have more edges towards its own community is $1/2 + \Omega(\min(\Delta(p-q)/\sqrt{np}, 1))$ in the SBM$(2n, p, q)$, where Δ is the imbalance. Note that this bound is asymptotically tight. In addition, we also develop strong combinatorial methods that despite dependent variables $(read - 2)$ allow us to analyze a vast amount of cuts. These insights are important for many combinatorial algorithms.

Acknowledgments and Disclosure of Funding

Ce projet a bénéficié d'une aide de l'État gérée par l'Agence Nationale de la Recherche au titre du Programme Appel à projets générique JCJC 2018 portant la référence suivante : ANR-18-CE40-0004-01.

The work of Vincent Cohen-Addad has been partially done while at CNRS.

This work was supported in part by NSF Award Numbers CF-1461559, CCF-0939370, and CCF-1810758 when Frederik Mallmann-Trenn was affiliated with MIT.

References

[1] Emmanuel Abbe. Community detection and stochastic block models: Recent developments. *Journal of Machine Learning Research*, 18(177):1–86, 2018. URL: http://jmlr.org/papers/v18/16-480.html.

[2] Emmanuel Abbe, Afonso S Bandeira, and Georgina Hall. Exact recovery in the stochastic block model. *IEEE Transactions on Information Theory*, 62(1):471–487, 2015.

[3] Emmanuel Abbe and Colin Sandon. Detection in the stochastic block model with multiple clusters: proof of the achievability conjectures, acyclic bp, and the information-computation gap. *arXiv preprint arXiv:1512.09080*, 2015.

[4] Emmanuel Abbe and Colin Sandon. Achieving the ks threshold in the general stochastic block model with linearized acyclic belief propagation. In *Advances in Neural Information Processing Systems*, pages 1334–1342, 2016.

[5] Afonso S Bandeira. Random laplacian matrices and convex relaxations. *Foundations of Computational Mathematics*, 18(2):345–379, 2018.

[6] Andrew C Berry. The accuracy of the gaussian approximation to the sum of independent variates. *Transactions of the american mathematical society*, 49(1):122–136, 1941.

[7] Vincent D Blondel, Jean-Loup Guillaume, Renaud Lambiotte, and Etienne Lefebvre. Fast unfolding of communities in large networks. *Journal of statistical mechanics: theory and experiment*, 2008(10):P10008, 2008.

[8] Nicolas Boumal. Nonconvex phase synchronization. *SIAM Journal on Optimization*, 26(4):2355–2377, 2016.

[9] Ted Carson and Russell Impagliazzo. Hill-climbing finds random planted bisections. In *Proc. 12th Symposium on Discrete Algorithms (SODA 01), ACM press, 2001*, pages 903–909, 2001.

[10] Peter Chin, Anup Rao, and Van Vu. Stochastic block model and community detection in sparse graphs: Aspectral algorithm with optimal rate of recovery. In Peter Grünwald, Elad Hazan, and Satyen Kale, editors, *Proceedings of The 28th Conference on Learning Theory, COLT 2015, Paris, France, July 3-6, 2015*, volume 40 of *JMLR Workshop and Conference Proceedings*, pages 391–423. JMLR.org, 2015. URL: http://proceedings.mlr.press/v40/Chin15.html.

[11] Anne Condon and Richard M. Karp. Algorithms for graph partitioning on the planted partition model. *Random Struct. Algorithms*, 18(2):116–140, March 2001. URL: http://dx.doi.org/10.1002/1098-2418(200103)18:2<116::AID-RSA1001>3.0.CO;2-2, doi:10.1002/1098-2418(200103)18:2<116::AID-RSA1001>3.0.CO;2-2.

[12] Aurelien Decelle, Florent Krzakala, Cristopher Moore, and Lenka Zdeborová. Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications. *Physical Review E*, 84(6):066106, 2011.

[13] Yingjie Fei and Yudong Chen. Exponential error rates of sdp for block models: Beyond grothendieck's inequality. *IEEE Transactions on Information Theory*, 65(1):551–571, 2018.

[14] Yingjie Fei and Yudong Chen. Achieving the bayes error rate in synchronization and block models by sdp, robustly. *IEEE Transactions on Information Theory*, 66(6):3929–3953, 2020.

[15] William Feller. *An Introduction to Probability Theory and Its Applications*. Wiley, 3rd edition, 1968.

[16] Chao Gao, Zongming Ma, Anderson Y Zhang, and Harrison H Zhou. Achieving optimal misclassification proportion in stochastic block models. *The Journal of Machine Learning Research*, 18(1):1980–2024, 2017.

[17] Chao Gao, Zongming Ma, Anderson Y Zhang, Harrison H Zhou, et al. Community detection in degree-corrected block models. *The Annals of Statistics*, 46(5):2153–2185, 2018.

[18] Dmitry Gavinsky, Shachar Lovett, Michael Saks, and Srikanth Srinivasan. A tail bound for read-k families of functions. *Random Struct. Algorithms*, 47(1):99–108, August 2015. URL: http://dx.doi.org/10.1002/rsa.20532, doi:10.1002/rsa.20532.

[19] Google Scholar. https://scholar.google.com/scholar?hl=fr&as_sdt=0%2C5&as_vis=1&q=+"Fast+unfolding+of+communities+in+large+networks", 2020. [Online; accessed 29-may-2020].

[20] Olivier Guédon and Roman Vershynin. Community detection in sparse networks via grothendieck's inequality. *Probability Theory and Related Fields*, 165(3-4):1025–1049, 2016.

[21] Bruce Hajek, Yihong Wu, and Jiaming Xu. Achieving exact cluster recovery threshold via semidefinite programming. *IEEE Transactions on Information Theory*, 62(5):2788–2797, 2016.

[22] Mark Anthony Hoffman, Jean-Philippe Cointet, Philipp Brandt, Newton Key, and Peter Bearman. The (protestant) bible, the (printed) sermon, and the word (s): The semantic structure of the conformist and dissenting bible, 1660–1780. *Poetics*, 68:89–103, 2018.

[23] Jean-Loup Guillaume. https://github.com/jlguillaume/louvain, 2020. [Online; accessed 29-may-2020].

[24] Mark Jerrum and Gregory B. Sorkin. Simulated annealing for graph bisection. In *34th Annual Symposium on Foundations of Computer Science, Palo Alto, California, USA, 3-5 November 1993*, pages 94–103, 1993. doi:10.1109/SFCS.1993.366878.

[25] Mark Jerrum and Gregory B Sorkin. The metropolis algorithm for graph bisection. *Discrete Applied Mathematics*, 82(1-3):155–175, 1998.

[26] B. W. Kernighan and S. Lin. An efficient heuristic procedure for partitioning graphs. *The Bell System Technical Journal*, 49(2):291–307, Feb 1970. doi:10.1002/j.1538-7305.1970.tb01770.x.

[27] Andrea Lancichinetti and Santo Fortunato. Community detection algorithms: a comparative analysis. *Physical review E*, 80(5):056117, 2009.

[28] Yu Lu and Harrison H Zhou. Statistical and computational guarantees of lloyd's algorithm and its variants. *arXiv preprint arXiv:1612.02099*, 2016.

[29] Frank McSherry. Spectral partitioning of random graphs. In *Proceedings 2001 IEEE International Conference on Cluster Computing*, pages 529–537. IEEE, 2001.

[30] Elchanan Mossel, Joe Neeman, and Allan Sly. Belief propagation, robust reconstruction and optimal recovery of block models. In *Conference on Learning Theory*, pages 356–370, 2014.

[31] Mark EJ Newman. Equivalence between modularity optimization and maximum likelihood methods for community detection. *Physical Review E*, 94(5):052315, 2016.

[32] S.P. Ravikumār and C.P. Ravikumar. *Parallel Methods for VLSI Layout Design*. Computer engineering and computer science. Ablex Pub., 1996.

[33] Chao Wu, Ranga C Gudivada, Bruce J Aronow, and Anil G Jegga. Computational drug repositioning through heterogeneous network clustering. *BMC systems biology*, 7(S5):S6, 2013.

[34] Se-Young Yun and Alexandre Proutière. Community detection via random and adaptive sampling. In Maria-Florina Balcan, Vitaly Feldman, and Csaba Szepesvári, editors, *Proceedings of The 27th Conference on Learning Theory, COLT 2014, Barcelona, Spain, June 13-15, 2014*, volume 35 of *JMLR Workshop and Conference Proceedings*, pages 138–175. JMLR.org, 2014. URL: http://proceedings.mlr.press/v35/yun14.html.

[35] Anderson Y Zhang, Harrison H Zhou, et al. Theoretical and computational guarantees of mean field variational inference for community detection. *Annals of Statistics*, 48(5):2575–2598, 2020.