Fast Fourier Transform (FFT)

We saw last time: given $f(x)$, sampled at $x_j = j\,\Delta x = j/N$

$$f(x_j) = \frac{1}{N}\sum_{n=0}^{N-1} F_n\, e^{2\pi i n x_j} \qquad \text{IDFT}$$

periodic:
$$\boxed{F_{n+N} = F_n} \qquad F_n = \sum_{j=0}^{N-1} f(x_j)\, e^{-2\pi i n x_j} \qquad \text{DFT}$$

Number of operations: • each F_n has a sum of N terms
• there are N F_n

So "computational complexity" $\sim N^2$.

Take N even. Observe that:

$$F_n = \sum_{j=0}^{\frac{N}{2}-1} f(x_{2j})\, e^{-2\pi i n x_{2j}} + \sum_{j=0}^{\frac{N}{2}-1} f(x_{2j+1})\, e^{-2\pi i n x_{2j+1}}$$

$$= \sum_{j=0}^{\frac{N}{2}-1} f(x_{2j})\, e^{-2\pi i n \left(j/N/2\right)} + \sum_{j=0}^{\frac{N}{2}-1} f(x_{2j+1})\, e^{-2\pi i n \left(j/N/2\right)} e^{-2\pi i \frac{n}{N}}$$

$$= E_n + e^{-2\pi i n/N} O_n$$

"even" $N/2$ terms — "odd" $N/2$ terms

So far, there is no gain.

But note that E_n, being a DFT with $N/2$ points, has period $N/2$! Similarly for O_n.

$$F_{n+\frac{N}{2}} = E_{n+\frac{N}{2}} + e^{-2\pi i(n+\frac{N}{2})/N} O_{n+\frac{N}{2}}$$

$$\boxed{\begin{aligned} F_{n+\frac{N}{2}} &= E_n - e^{-2\pi i n/N} O_n \\ F_n &= E_n + e^{-2\pi i n/N} O_n \end{aligned}}$$

$n = 0, \cdots \frac{N}{2}-1$

We've gained a little bit. But if $\frac{N}{2}$ is still even, we can do this again! And so on. This is the stadard Cooly-Tukey alogorithm, which works best if $N = 2^P$.

$\Rightarrow N \log N$ complexity

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F = fft(f, N, s)
    if N = 1
        F_0 = f_0
    else
        F_{0,...,N/2-1} = fft(f, N/2, 2s)      % (f_0, f_2s, f_4s, ...)
        F_{N/2,...,N-1} = fft(f+s, N/2, 2s)    % (f_s, f_s+2s, ...)
combine  {  for k = 0 to N/2-1
2 halves {      t = F_k
         {      F_k = t + exp(-2πi k/N) F_{k+N/2}
         {      F_{k+N/2} = t - exp(-2πi k/N) F_{k+N/2}
```