Exercises

3.1 Find the Cholesky factorisation of the matrix

$$A = \begin{pmatrix} 4 & 6 & 2 \\ 6 & 10 & 3 \\ 2 & 3 & 5 \end{pmatrix}.$$

3.2 Use the method of Cholesky factorisation to solve the system of equations

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 4\,, \\ -2x_1 + 5x_2 - 3x_3 &= -7\,, \\ 2x_1 - 3x_2 + 6x_3 &= 10\,. \end{aligned}$$

3.3 Let $n \geq 3$. The $n \times n$ tridiagonal matrix T has the diagonal elements

$$T_{ii} = 2\,, \quad i = 1, 2, \dots, n\,,$$

and the off-diagonal elements

$$T_{i\,i+1} = T_{i+1\,i} = -1\,, \quad i = 1, 2, \dots, n-1\,.$$

In the factorisation $T = LU$, where $L \in \mathbb{R}^{n\times n}$ is unit lower triangular and $U \in \mathbb{R}^{n\times n}$ is upper triangular, show that

$$L_{i+1\,i} = -i/(i+1)\,, \qquad i = 1, 2, \dots, n\,,$$

and find expressions for the elements of U. What is the determinant of T?

3.4 Let $n \geq 3$ and $1 \leq k \leq n$. Define the vector $\boldsymbol{v}^{(k)} \in \mathbb{R}^n$ with elements given by

$$v_i^{(k)} = \begin{cases} i(n+1-k)\,, & i = 1, \dots, k\,, \\ k(n+1-i)\,, & i = k+1, \dots, n\,. \end{cases}$$

Evaluate M_{kj}, the inner product of the vector $\boldsymbol{v}^{(k)}$ with column j of the matrix T defined in Exercise 3. (The inner product $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is defined as the real number $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w}$.) Hence give expressions for the elements of the inverse matrix T^{-1}, and verify that this inverse is symmetric. Find the ∞-norm of the inverse, $\|T^{-1}\|_\infty$, and show that the condition number of T is

$$\kappa_\infty(T) = \frac{1}{2}(n+1)^2\,, \quad n \text{ odd}\,.$$

What is the condition number $\kappa_\infty(T)$ when n is even?

3.5 Given that $n \geq 3$, in the notation of Theorem 3.4 suppose that

$$|b_j| \geq |a_j| + |c_j|, \quad j = 1, 2, \ldots, n,$$

and

$$|c_j| > 0, \quad j = 1, 2, \ldots, n-1,$$

with the convention that $a_1 = 0$ and $c_n = 0$. Show that the factorisation $T = LU$ exists without pivoting, and can be constructed by the Thomas algorithm. Give an example of a matrix T which satisfies these conditions, except that $c_k = 0$ for some $k \in \{1, 2, \ldots, n-1\}$ and such that T is singular and cannot be written in the form $T = LU$ without pivoting.

3.6 Let $n \geq 3$ and suppose that the matrix $T \in \mathbb{R}^{n \times n}$ is tridiagonal. Show that there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that

$$PA = L^{(1)}U^{(1)}$$

where $L^{(1)} \in \mathbb{R}^{n \times n}$ is unit lower triangular with at most two nonzero elements in each row, and $U^{(1)} \in \mathbb{R}^{n \times n}$ is upper triangular with at most three nonzero elements in each row.

3.7 Suppose that the matrix B is Band(p,q), and that there exists a factorisation $B = LU$ without row interchanges. Show that L is Band(p,0) and U is Band(0,q).

3.8 Suppose that $n \geq 4$, that the matrix $A \in \mathbb{R}^{n \times n}$ is Band(3,3), and has the LU factorisation $A = LU$, so that $L \in \mathbb{R}^{n \times n}$ is Band(3,0) and $U \in \mathbb{R}^{n \times n}$ is Band(0,3). Suppose also that $a_{i+2,i} = 0$, $a_{i,i+2} = 0$ for $i = 1, 2, \ldots, n-2$. By considering u_{24} and l_{42}, or otherwise, show that in general the elements $l_{i+2,i}$ and $u_{i,i+2}$ are not zero.