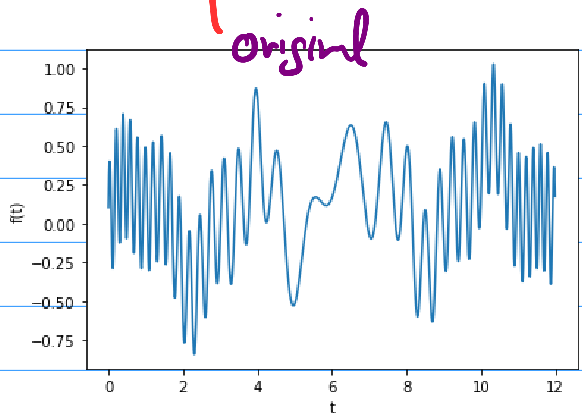
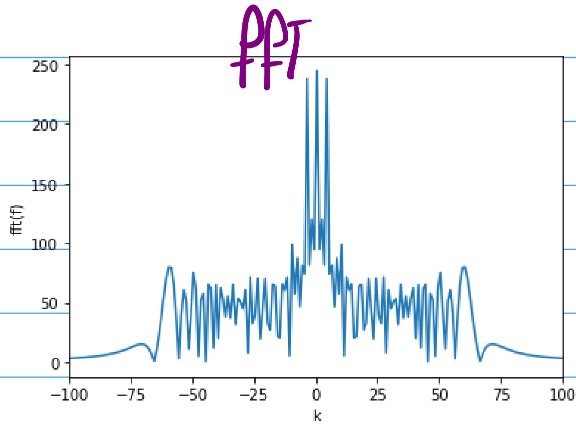
# Lecture 4 - Windowed Fourier Transform

We saw the DFT in last lecture which is an excellent tool for breaking a signal into sum of trigonometric fun's.

The main feature of DFT is that it extracts the frequencies that are present in a signal.

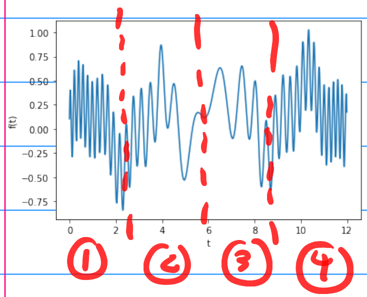
But, FT has a major drawback. It sees the entire signal at once & so does not characterize changes in freq. over time.

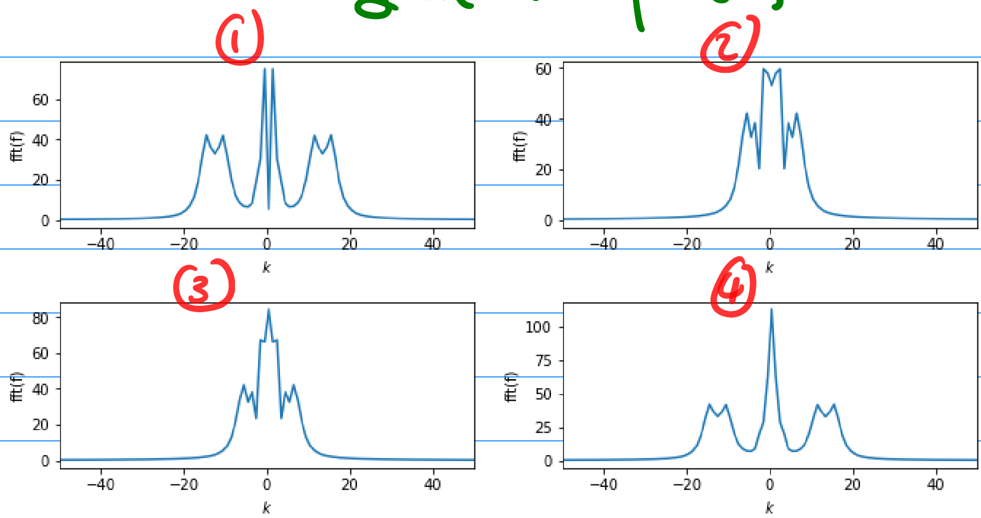



looking at this signal we clearly see that different freq. are active at different times!

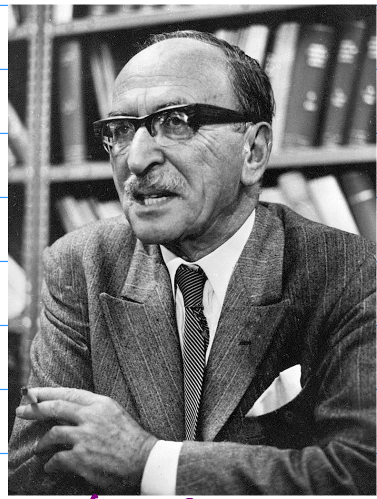
Simple solution: Break signal into pieces.



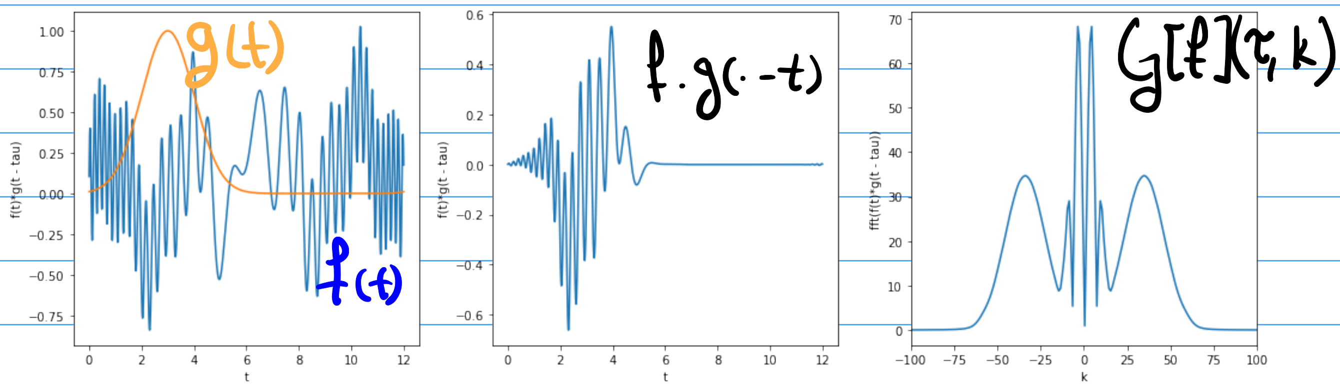



what we just did is an example of time-frequency analysis where the behavior of a signal is analyzed as function of time.

A formal method for localization of FT was proposed by Gabor. The idea is very simple. First, localize the signal using a filter. Then apply FT.



Dénés Gabor




$$\mathcal{G}[f](\tau,k) := \int_0^{2\pi} \underbrace{f(t)\,g(t-\tau)}_{\text{localized signal}}\ \underbrace{\exp(-ikt)}_{\text{fourier kernel}}\,dt$$

on entire real line $\displaystyle = \int_{-\infty}^{\infty} \sim \quad \sim \ dt$

Originally Gabor suggested to take the filter g to be a Gaussian

$$g(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{t^2}{2\sigma^2}\right)$$

But one can use large families of localization filters. In this light such transforms are often refered to as short time Fourier transform (STFT). Some simply call them Gabor transforms.

We pose some requirements on $g$ (stated on real line)

- $g$ is real & symmetric.
- $\|g\|_{L^2} := \left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^{1/2} = 1$, $\|g(\cdot - \tau)\|_{L^2} = 1$, $\forall \tau \in \mathbb{R}$.

then we have that

Not used (a) Gabor transform is linear.

often (b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G[f](\tau,k)\, g(t-\tau) \exp(ikt)\, dk\, d\tau$

For functions defined on closed intervals, ex $[0, 2\pi]$ in prev. lectures, we also ask $g$ to be Periodic on the interval.

Based on the above ideas, we can easily devise a discrete variant of STFT.

Consider discrete frequencies,

$k = m\delta k$ freq step size

& grid in time

$\tau = n\delta t$ time step size

then define the discrete Gabor transform on $[0, 2L]$

$$G[f](m,n) = \frac{1}{2L}\int_0^{2L} f(t)\, g(t-n\delta t) \exp\left(-\pi i \frac{m \delta k\, t}{L}\right) dt$$

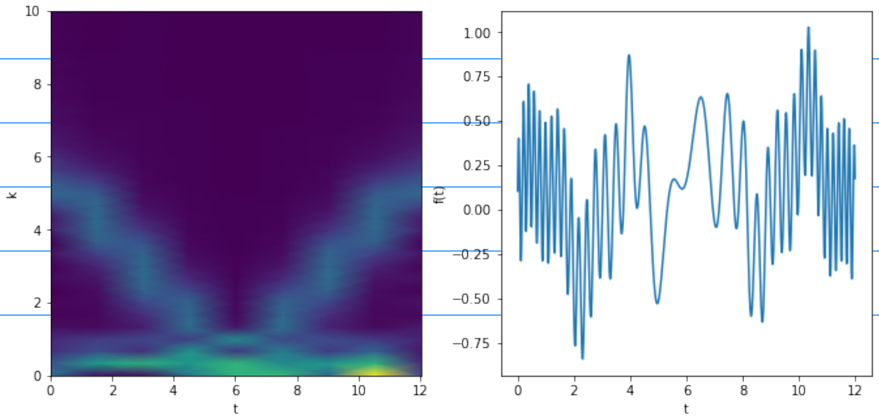
$$= \widehat{f(t)\, g(t-n\delta t)}$$

* Note, once again, there are different normalizations & conventions out there. I am using the one that is most convenient in connecting to FT & DFT.

With this definition the output of Gabor transform is a 2D-array (matrix) where one coordinate is time & another is frequency.

taking the amplitude of the Gabor transform & plotting it as a function of time & freq. results in an spectrogram of the signal.

spectro gram

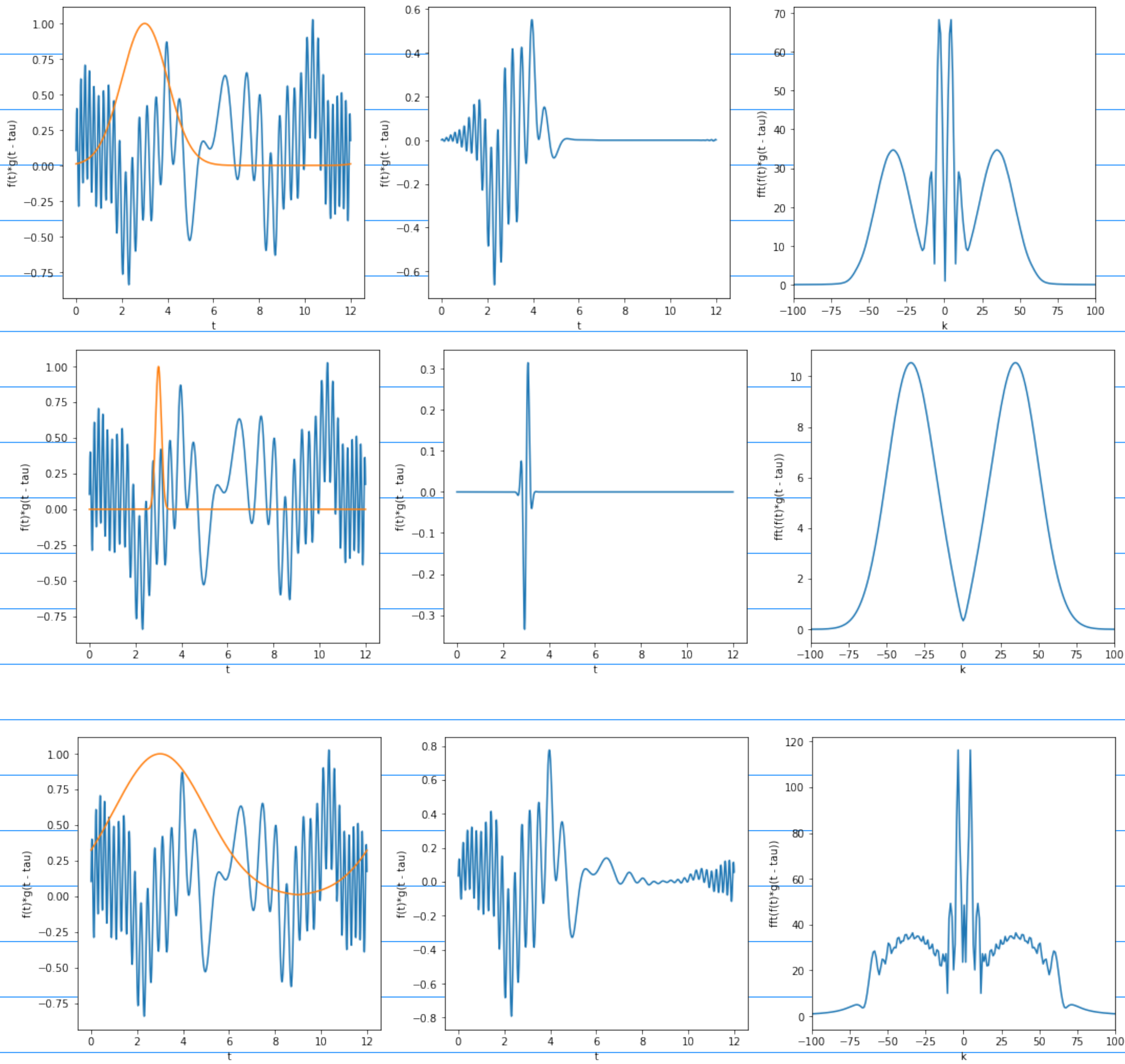



signal

# Drawback of Gabor transform (STFT).

We saw that Gabor transform allows us to study the freq. content of a signal as a function of time.

But there is an issue here regarding the width of the filter g.




If width of g is too small, we don't see low freq part of the signal & in limit of infinitesimal window will get the original signal back. With large width we lose the time resolution & in the limit simply recover the FT which has all freq. at once.

This is a manifestation of an uncertainty priciple; ie, we cannot have both time & freq behavior of a signal at the same time. And so, one needs to compromise.