Lecture 6: Multi-Resolution Analysis with Wavelet Bases.

Last lecture we saw the continuous Wavelet Transform (CWT) & how it helps us perform a systematic analysis of a signal in both time & freq.

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)$$

$$W_\psi[f](a,b) = \int_{-\infty}^{\infty} f(t)\,\psi_{a,b}(t)\,dt.$$

We also mentioned the discrete wavelet transform (DWT) briefly.

Most common convention (including PyWavelets & MATLAB toolbox) is

$$a = 2^{-m}, \quad b = n2^{-m}, \quad \text{integers } -\infty < n, m < +\infty$$

$$\psi_{m,n} = 2^{m/2}\,\psi\left(\frac{t - n2^{-m}}{2^{-m}}\right)$$

$$= 2^{m/2}\,\psi(2^m t - n)$$

Using such functions $\psi_{m,n}$ in the CWT is straightforward & leads to the DWT. Most discrete wavelets used in practice are particular choices of ψ so that $W_\psi[f](m,n)$ can be computed efficiently & accurately.

But, for the DWT to be useful we need ψ to satisfy other properties. Most notably, we want to be able to reconstruct f (or approximate it from its DWT).

Recall DFT:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{i\pi t k}{L}\right)$$

$$c_k = \frac{1}{2L}\int_0^{2L} f(t) \exp\left(\frac{-i\pi t k}{L}\right) dt$$

the key to the reconstruction formula for DFT was the orthogonality of the trigonometric functions.

$$\int_0^{2\pi} \sin(kx)\cos(mx) = 0 \quad \forall k,m$$

$$\int_0^{2\pi} \cos(kx)\cos(mx)\, dx = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases}$$

$$\int_0^{2\pi} \sin(kx)\sin(mx)\, dx = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases}$$

In order for the DWT to be more useful, & the reconstruction formula to hold we ask the $\psi_{m,n}$ to form an orthonormal basis, ie,

$$\int_{-\infty}^{\infty} \psi_{m,n}(t)\,\psi_{k,l}(t)\,dt = \delta_{mk}\,\delta_{nl}$$

where δ_{mn} is the Kronecker delta function

$$\delta_{mn} := \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Let's unpack this a bit.

(i) we have $\int_{-\infty}^{\infty} |\psi_{m,n}(t)|^2\,dt = 1$ (Normalization in $L^2(\mathbb{R})$)

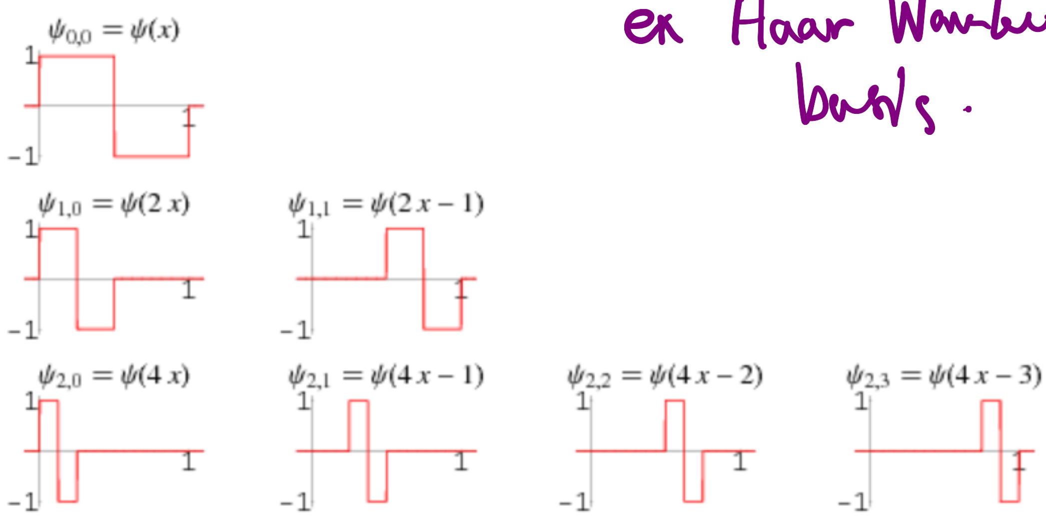
(ii) $\int_{-\infty}^{\infty} \psi_{m,n}(t)\,\psi_{m,(n+n')}(t)\,dt = 0 \quad n' \neq 0$

so translated wavelets in same scale m are orthogonal.

(iii) $\int_{-\infty}^{\infty} \psi_{(m+m'),n}(t)\,\psi_{m,n}(t)\,dt = 0 \quad m' \neq 0.$

so wavelets at different scales are also orthogonal.

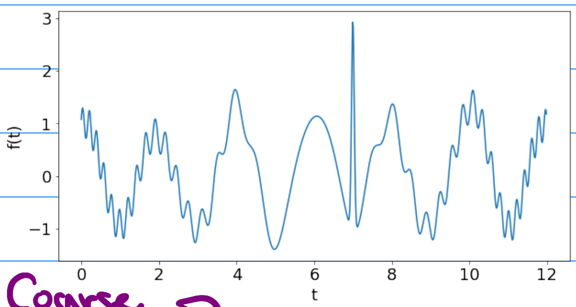
ex Haar Wavelet basis.



Property (ii) lets us extract features of a signal in time.
Property (iii) " " study " at different scales.

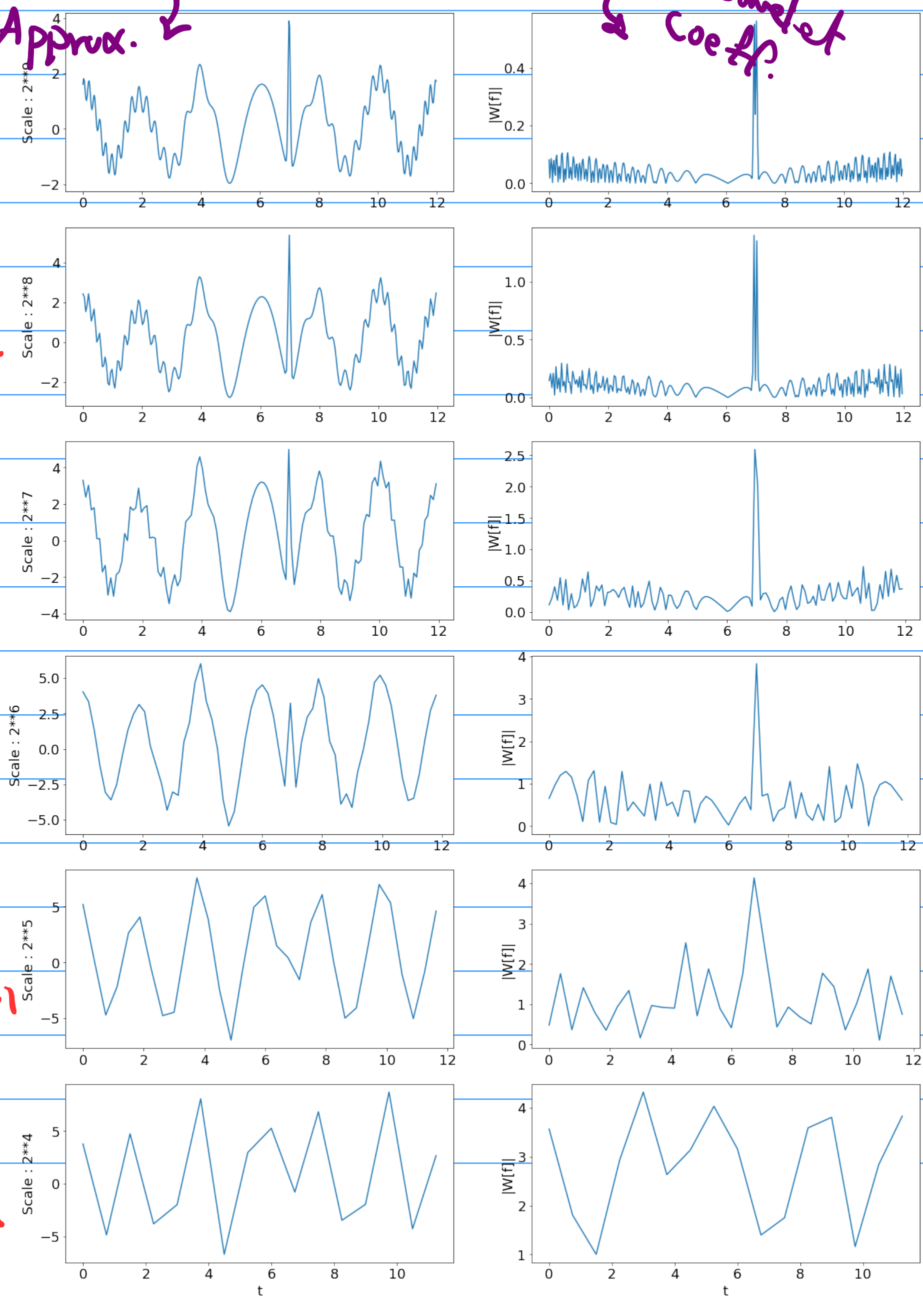
(wider wavelets see low freq. while skinny wavelets see high freq.)

This is called a "Multi-resolution analysis." as the details of a signal are extracted at different scales (resolutions) in time.



Coarse Approx.

Detail / wavelet coeff.



Coarse

V_{m+5} W_{m+5}

V_{m+4} W_{m+4}

V_{m+3} W_{m+3}

V_{m+2} W_{m+2}

V_{m+1} W_{m+1}

V_m W_m

Fine

Thus the DWT Lets us write a signal as the sum of its features at different scales & locations (assume $\psi_{m,n}$ are orthonormal)

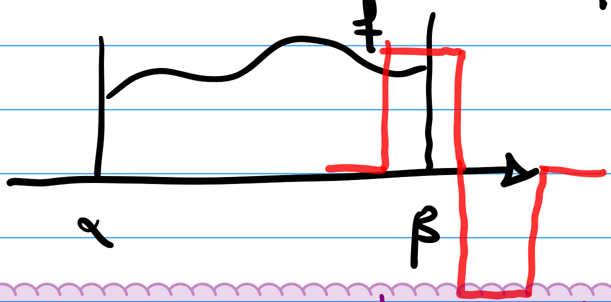
$$f(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} \psi_{m,n}(t)$$

$$c_{m,n} = \int_{-\infty}^{\infty} f(t) \psi_{m,n}(t)\, dt$$

- on intervals $[\alpha,\beta] \subset \mathbb{R}$ we simply change the range of integral.

$$c_{m,n} = \int_{\alpha}^{\beta} f(t) \psi_{m,n}(t)\, dt$$

- Note, one needs to decide what to do about values of f outside $[\alpha,\beta]$.



Number of conventions exist & can be selected in PyWavelets. ex extend f by zero. or take f & ψ to be periodic on $[\alpha,\beta]$.

6.1 Some theory for Multi-resolution analysis

Recall the $L^2(\mathbb{R})$ space of functions

$$L^2(\mathbb{R}) := \{ f:\mathbb{R}\to\mathbb{R} \mid \int_{\mathbb{R}} |f(t)|^2 dt < +\infty \}$$

which is an inner product (Hilbert) space with inner product

$$f,g \in L^2(\mathbb{R}), \quad (f,g) := \int_{-\infty}^{\infty} f(t)g(t)\,dt$$

& norm

$$\|f\|_{L^2(\mathbb{R})} := (f,f)$$

We now consider subspaces of $L^2(\mathbb{R})$ denoted as V_m

$$\{V_m : m\text{-integer}\}$$

that nested

$$\text{Coarse} \ \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \ \text{fine}$$

Think of V_m as span of $\psi_{m,n}$ up to some value m.

We now make requirements on these subspaces.

(i) Union of V_m is $L^2(\mathbb{R})$ (density)

$$\bigcup_{m=-\infty}^{\infty} V_m = L^2(\mathbb{R})$$

(ii) Intersections are empty (Scales are separate)

$$\bigcap_{m=-\infty}^{\infty} V_m = \emptyset$$

(iii) Each subspace picks up a certain resolution

$f(t) \in V_m$ if & only if $f(2x) \in V_{m+1}$

(iv) The space V_0 is generated by translations of a given function.

There exists $\phi \in V_0$ so that

$$\phi_{0,n} := \phi(t-n)$$

form an orthonormal basis (with respect to $L^2(\mathbb{R})$) for V_0 ie,

for any $f \in V_0$ we have $f(t) = \sum_{n=-\infty}^{\infty} c_n \phi_{0,n}(t)$

The function ϕ is called the Scaling function or father Wavelet.

If above conditions hold then we say ϕ generates a multi-resolution analysis (MRA) of $L^2(\mathbb{R})$.

Note: you can extend all of this to $L^2([\alpha,\beta])$ easily.

Note: ϕ is not a wavelet yet! we need to do a bit more.

Since we have the inclusions $V_m \subset V_{m+1}$ it makes sense to consider the orthogonal complement of V_m in V_{m+1}, ie

$$V_{m+1} = V_m \oplus W_m$$

$\{W_m$: fine scale details that are not in $V_m\}$

continuing this expression we get

$$V_{m+1} = (V_{m-1} \oplus W_{m-1}) \oplus W_m$$

$$\vdots$$

$$= V_0 \oplus \left(\bigoplus_{n=0}^{m} W_n \right)$$

Condition (i) implies $V_0 \oplus \left(\bigoplus_{n=0}^{\infty} W_n \right) = L^2(\mathbb{R})$

Continuing to coarser scales V_m with negative m giving

$$\bigoplus_{m=-\infty}^{+\infty} W_m = L^2(\mathbb{R})$$

Then there exists a theorem that tells us, there exists a function $\psi \in W_0$ called the Mother Wavelet so that

$$\psi_{0,n}(t) = \psi(t-n)$$

is an orthonormal basis for W_0 &

$$\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$$

Wavelets we've seen so far.

is an " " " W_m.

In fact we have

$$\psi(t) = \sqrt{2} \sum_{-\infty}^{\infty} (-1)^{n-1} c_{-n-1}\, \phi(2t-n)$$

$$c_n = \sqrt{2} \int_{-\infty}^{\infty} \phi(t)\, \phi(2t-1)\, dt$$

so given ϕ we can always find ψ.