Lecture 7: High-dimensional extensions & image processing

So far in our lectures we have seen two primary signal analysis tools (Fourier & Wavelet transforms) but we exclusively focused on 1D signals (aka time series data). But not all data is 1D, in fact the entire field of image processing is concerned with 2D data.

Recall, the overarching idea of linear signal (data) processing methods is to write a function as a sum of simpler components.

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \gamma_k(x)$$

where γ_k are known functions (ex trig. funs or wavelet basis.

In this light extending linear methods to multiple dimensions is natural.

$\mathcal{X} \subseteq \mathbb{R}^d \quad d \geq 1$

$f: \mathcal{X} \to \mathbb{R}, \quad f \in L^2(\mathcal{X}) := \left\{ \int_{\mathcal{X}} |f(x)|^2 dx < +\infty \right\}$

simply choose a basis $\gamma_k \in L^2(\mathcal{X})$.

So conceptually nothing really changes.

If $\{\delta_k\}$ is a complete orthonormal basis then

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \delta_k(x)$$

$$\delta_k = \int_X f(x) \delta_k(x) \, dx$$

The question is how do we construct high dimensional bases using the 1D ones we have seen before.

We will do this in 2D in the context of image processing. extension to nD is the same & most software such as numpy & PyWavelet already do it for you.

7.1 DFT for Images

Easiest way to extend DFT to mult. dim. is to start from the FT in d-dim.

$f : \mathbb{R}^d \to \mathbb{R}$

$\hat{f} : \mathbb{R}^d \to \mathbb{C}$, $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-2\pi \xi^T x) \, dx$

Observe,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x_1, x_2, \dots, x_d) \exp\left(-i2\pi \sum_{k=1}^{d} \xi_k x_k\right) dx_1 \cdots dx_d$$

$$= \int_{\mathbb{R}^d} f(x_1, \dots, x_d) \prod_{k=1}^{d} \exp(-i2\pi \xi_k x_k)\, dx_1 \cdots dx_d$$

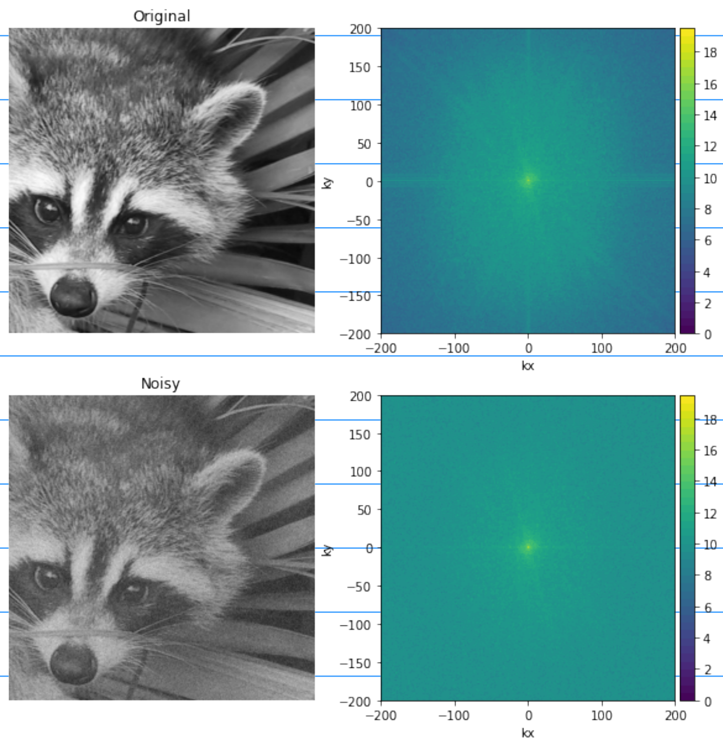
$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, \dots, x_d) \exp(-i2\pi \xi_1 x_1)\, dx_1 \exp(-i2\pi \xi_2 x_2)\, dx_2 \cdots \exp(-i2\pi \xi_d x_d)\, dx_d.$$

That is, we successively compute FT along each coordinate.

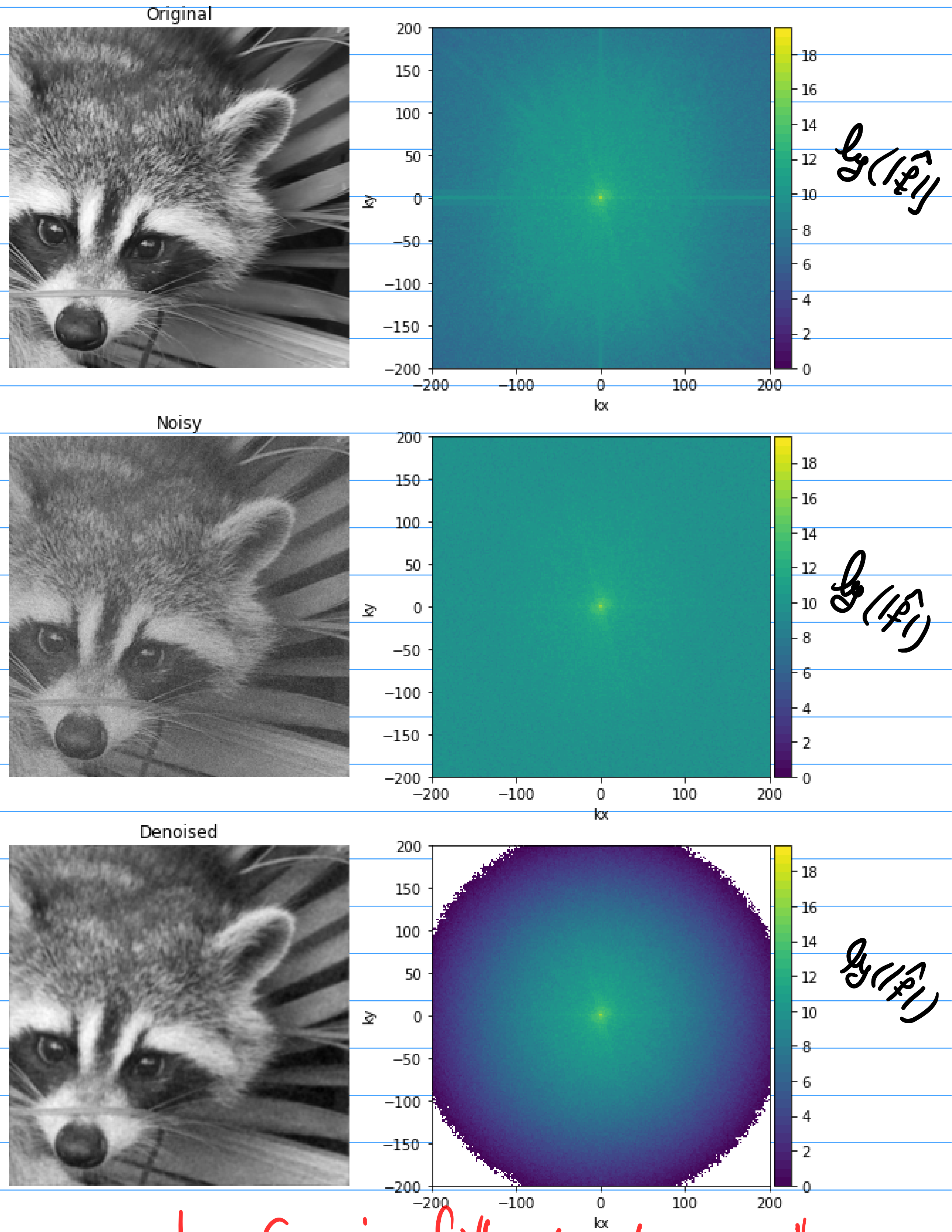
This is precisely how DFT is computed in d-dim.

* fft2 is specialized to 2D. fftn does n-dim.

* We still need fftshift.



Just as we did in 1D, we can manipulate the 2D FFT to filter noise out of an image.



Here we used a Gaussian filter to dampen the high freq. noise in the fft.

In Samy 2D-DFT writes fnc. as

$$f : [0, L_1] \times [0, L_2] \longrightarrow \mathbb{R}$$

$$f(x_1, x_2) = \sum_{k,l=-\infty}^{\infty} c_{kl} \exp\left(i\pi\left(\frac{kx_1}{L_1} + \frac{lx_2}{L_2}\right)\right)$$

7.2 DWT for Images

The same simple idea will allow us to extend the Wavelet transform to multiple dimensions.

ie, apply the transform one coordinate at a time.

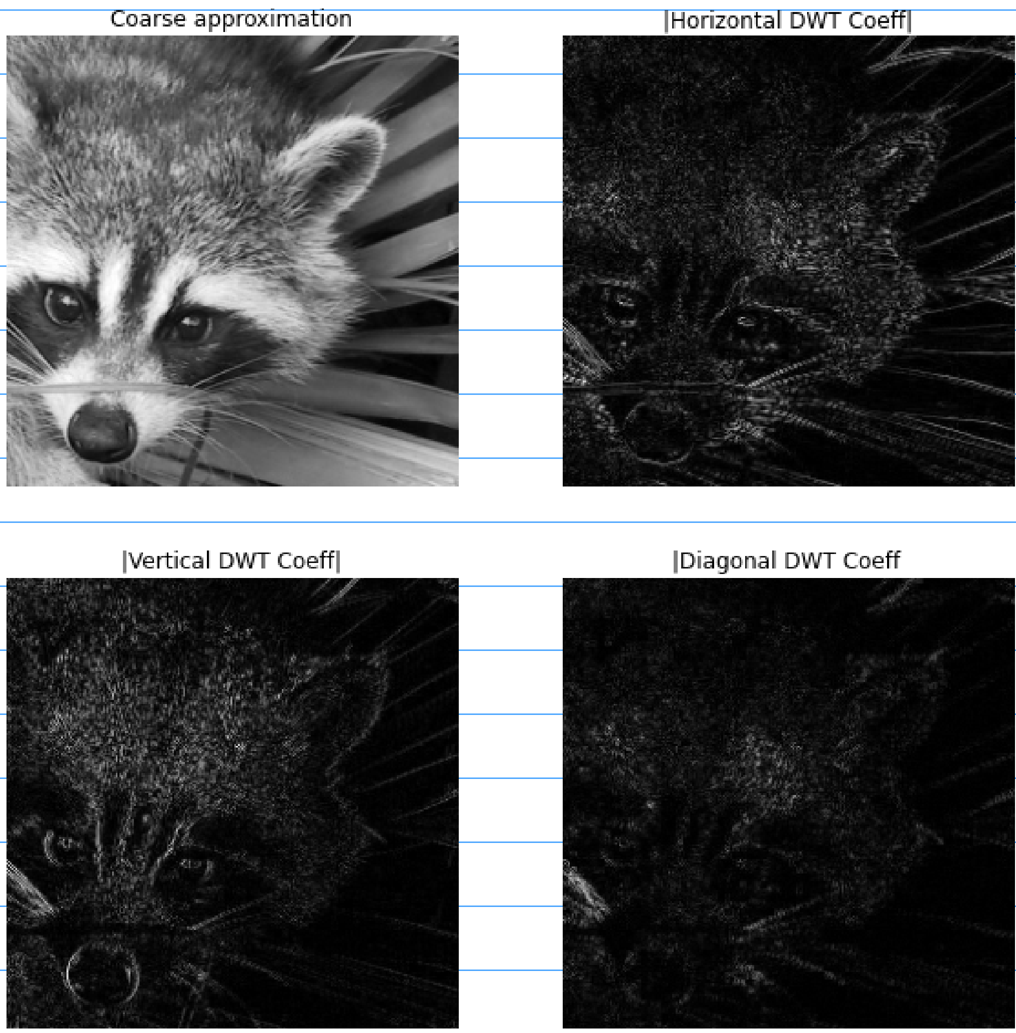
$$W_1[f](a_1, b_1; x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, \psi_{a_1,b_1}(x_1)\, dx_1$$

$$W_2[f](a_1, a_2, b_1, b_2) = \int_{-\infty}^{\infty} W_1[f](a_1, b_1; x_2)\, \psi_{a_2,b_2}(x_2)\, dx_2$$

* Can be extended to d-dim but notation is tedious.

Interestingly, the order in which we do these transforms is informative.

- If we transform horizontally only we find horizontal features in an image.
- If 〃 〃 vertically 〃 〃 〃 vertical 〃 〃 〃.
- Transforming in both directions extracts "diagonal" features.



How does DWT extract horizontal/vertical features?

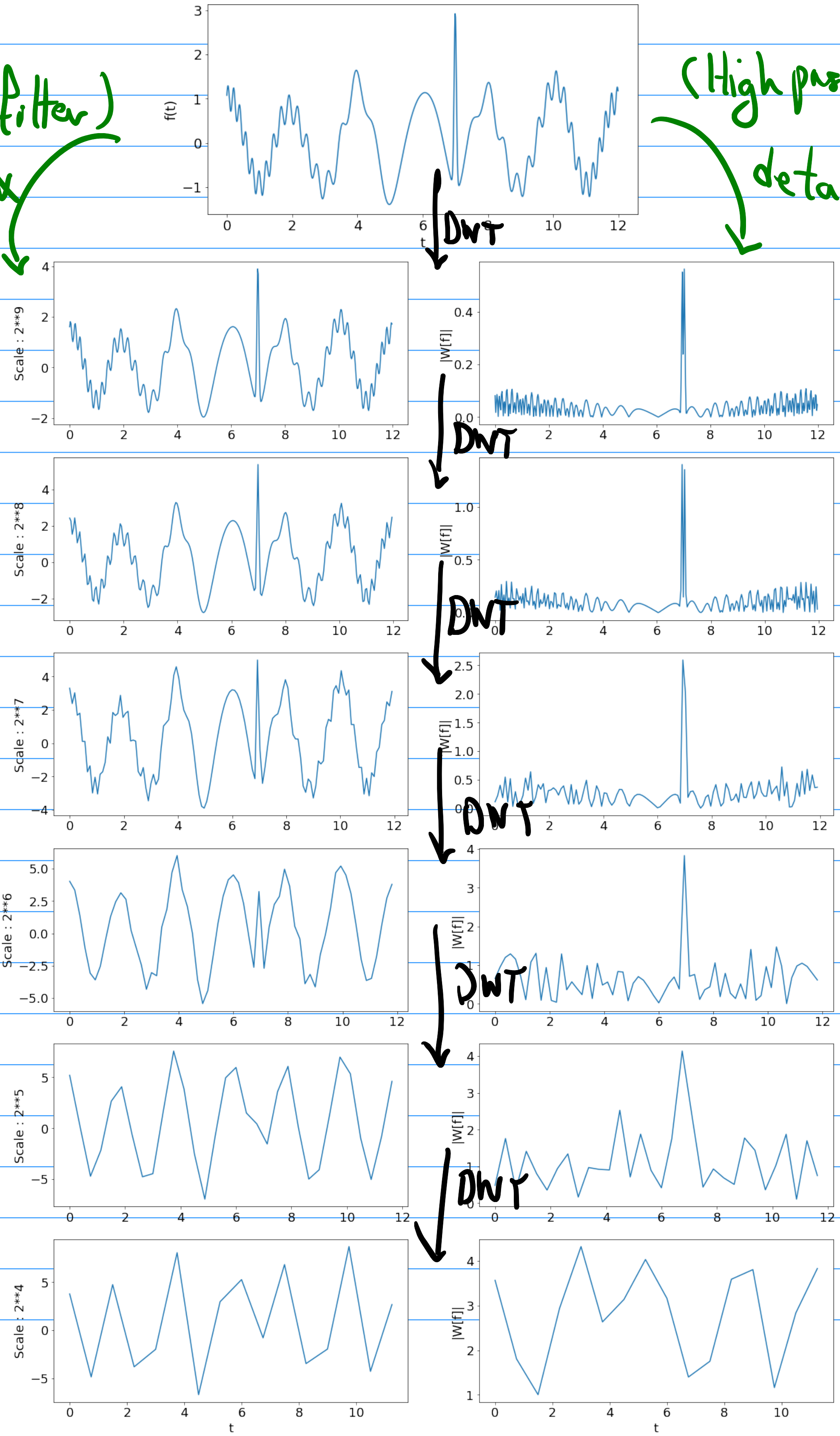
Recall, the way DWT works on 1D functions

(low pass filter)

Coarse approx

(High pass filter)

detail coeff.



LP

LP

LP

⋮

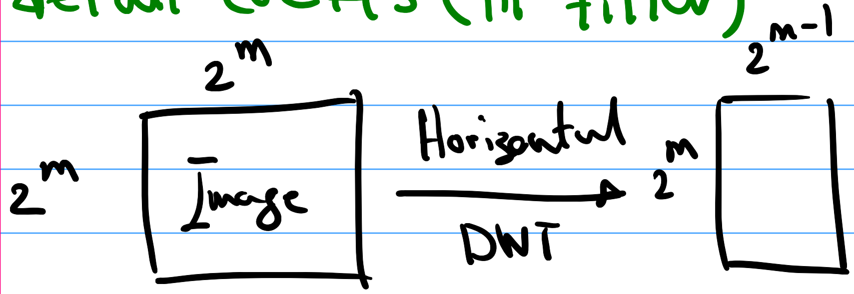
HP

HP

HP

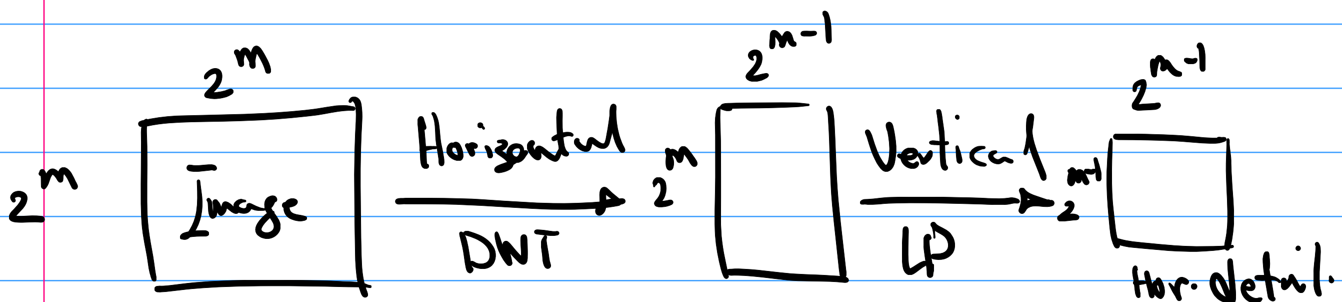
⋮

- So, when we apply DWT to 2D data the Horizontal details are extracted by applying DWT on ROWs of image to extract detail coeffs (HP filter)



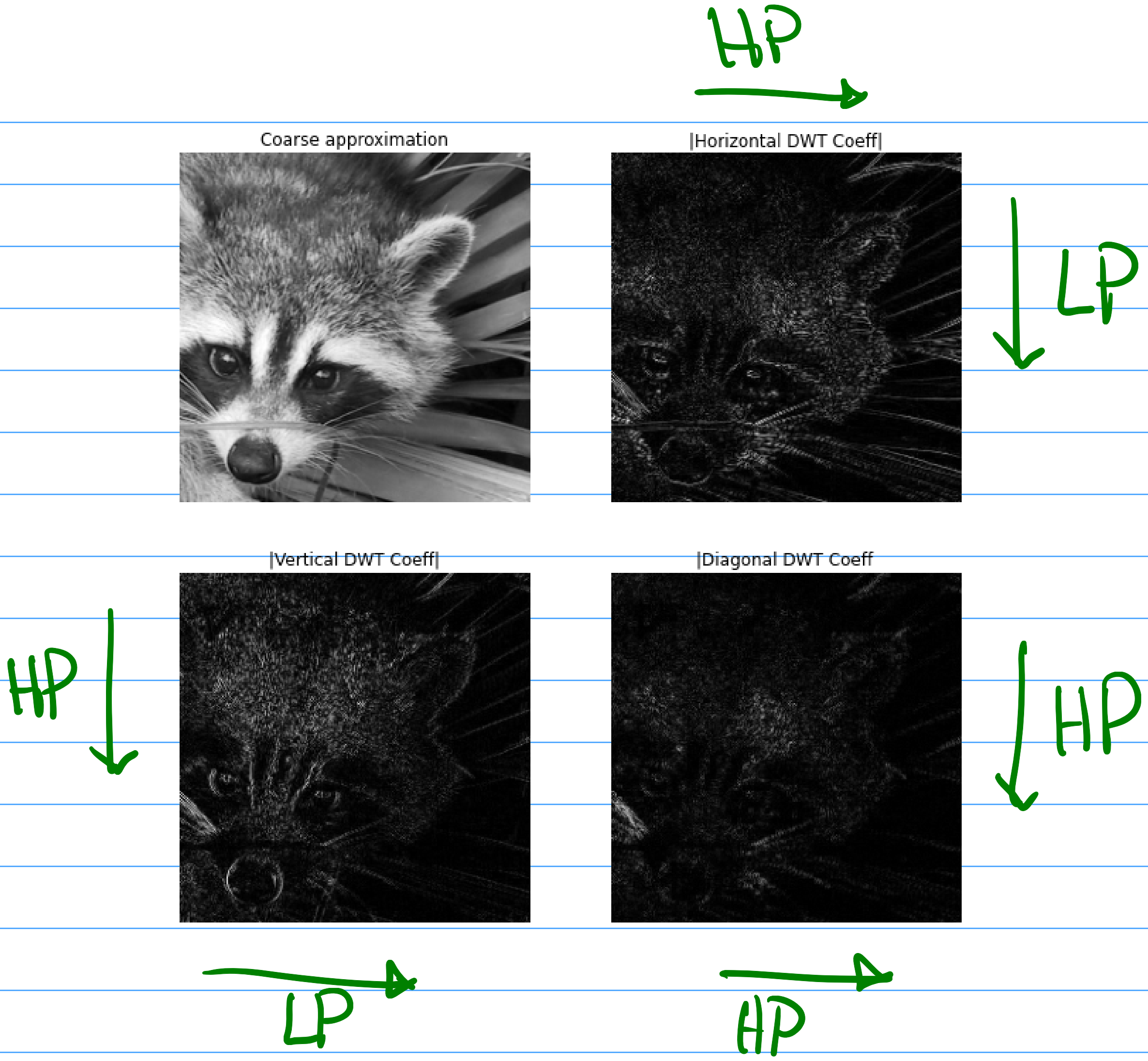
the resulting output is a tall matrix, at half resolution in the horizontal direction.

To fix the aspect ratio we now apply an LP filter, ie Coarse Approx, in the vertical direction



- Vertical detail is computed analogously by applying DWT column-wise first to get detail coeffs. then Coarse approx on columns.

- Diagonal detail is extracted by applying DWT twice to get detail coeff. on both rows & colums.


HP
Coarse approximation
|Horizontal DWT Coeff|
LP
|Vertical DWT Coeff|
|Diagonal DWT Coeff
HP
HP
LP
HP