Lecture 5: The Wavelet Transform

What we've seen so far,

- Fourier transform

$$\hat{f}_k = \frac{1}{2L}\int_0^{2L} f(t)\exp\left(-\frac{i\pi k t}{L}\right)dt$$

(sees all freq. info at once, no time info)

- Gabor transform

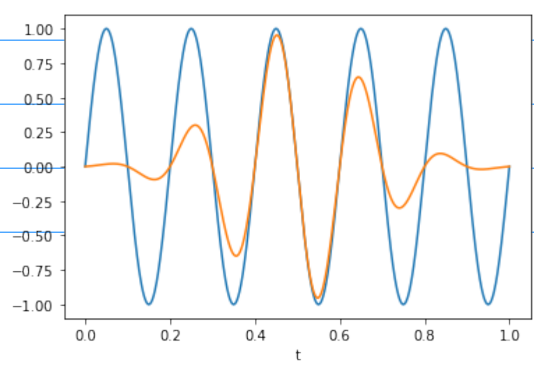
$$G[f](m,k) = \frac{1}{2L}\int_0^{2L} f(t)\,g(t-n\delta t)\exp\left(-\pi i\frac{m\delta k t}{L}\right)dt$$

(sees time & freq. but limited by uncertainty principle & also sensitive to choice of g).

But let us think of Fourier & Gabor transforms from a diff. perspective.

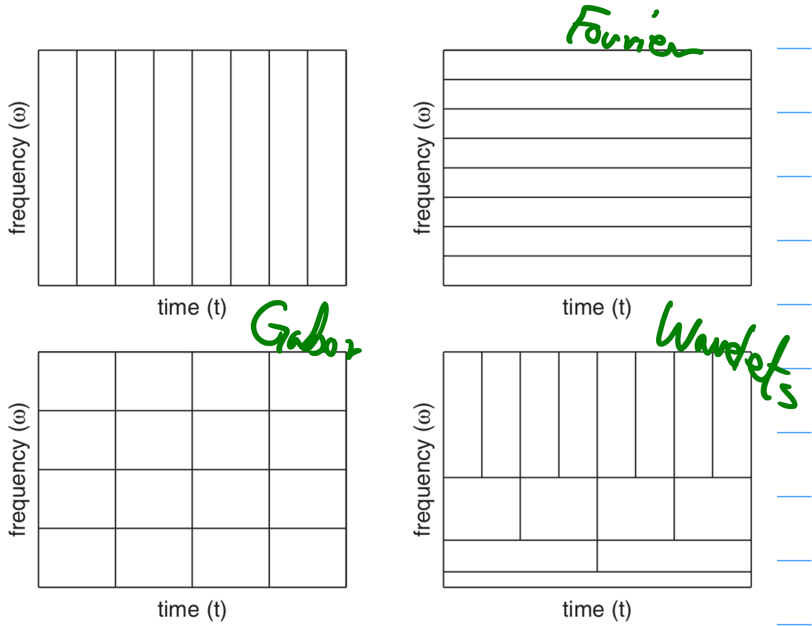
- Fourier computes $\int f(t)\exp\left(-\frac{i\pi k t}{L}\right)dt$
  i.e. "projects" on $\exp\left(-\frac{i\pi k t}{L}\right)$. ← Non-local oscillatory func.
- Gabor computes $\int f(t)\,g(t-\tau)\exp\left(-\frac{i\pi k t}{L}\right)dt$
  i.e. "projects" on $g(t-\tau)\exp\left(-\frac{i\pi k t}{L}\right)$.

localized osc. func.



We saw that the issue with Gabor is that it uses a fixed window size & so we have to compromise between freq. & time resolution of the transform.

There is a simple fix to this, ie change the window size together with frequency in a reasonable manner to get maximum possible information in both time & frequency. This is the main idea of the wavelet transform.



Disclaimer: Wavelets & the wavelet trans. are new tools (mostly developed in the 1990's & 2000's) as such. Many implementations & standards exist. We will only see a small subset of these.

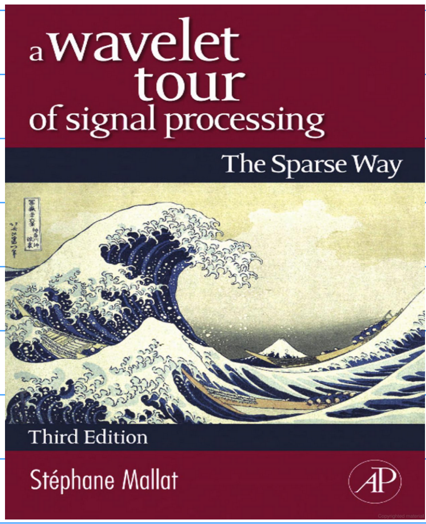
Stephane Mallat
(Ecole Normale Supériere)

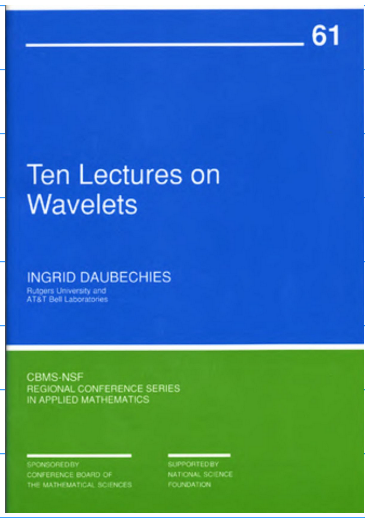
Ingrid Daubechies
(Duke University)

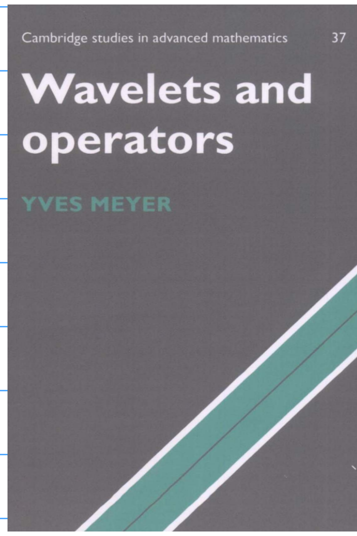
Alfred Haar

(1885 – 1933)

Advanced reading







5.1 The continuous Wavelet transform

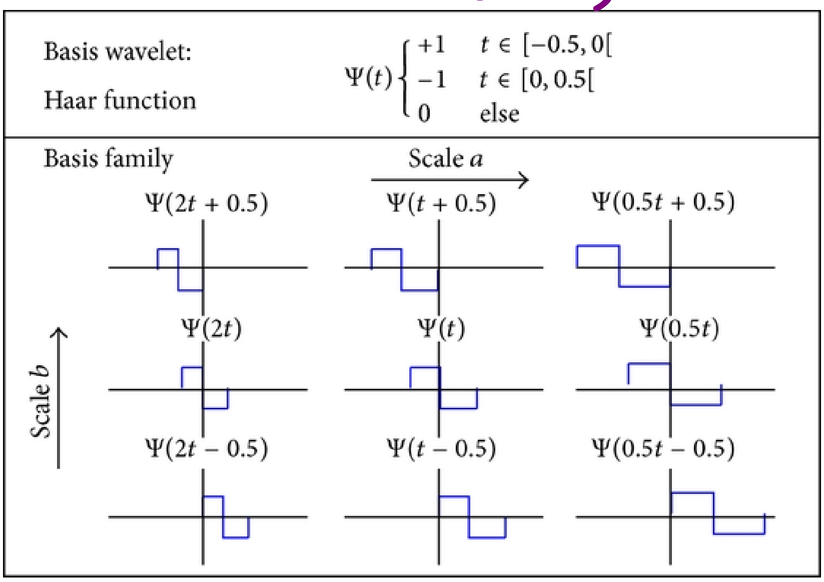
Consider a function $\psi : \mathbb{R} \to \mathbb{R}$ along with its translation/dilation

$$\psi_{a,b}(t) := \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a} \right)$$

translation (shift/time)

dilation/stretching (scale)

- Haar wavelet is the simplest example we will see.



- factor $1/\sqrt{a}$ is a normalization factor ensuring

$$\int_{-\infty}^{\infty} |\psi_{a,b}|^2 \, dt = \int_{-\infty}^{\infty} |\psi|^2 \, dt .$$

Then the Continuous Wavelet transform (CWT) can be defined in a familiar manner

$$W_\psi[f](a,b) := \int_{-\infty}^{\infty} f(t)\, \psi_{a,b}(t)\, dt$$

- Note, like Gabor the CWT represents a 1D signal in 2-variables. (shift/time & scale as opposed to time & frequency).

The CWT has many desirable properties that we mention here. Some should be familiar & hopefully easy to see.

- Linear

$$W[\alpha f + \beta g](a,b) = \alpha W[f](a,b) + \beta W[g](a,b)$$

- Translation (define $T_c f(t) := f(t-c)$)

$$W[T_c f](a,b) = W[f](a,b-c)$$

- Dilation (define $D_c f(t) = 1/_t\, f(t/c)$)

$$W[D_c f](a,b) = \frac{1}{\sqrt{c}} W[f](a/c, b/c)$$

- Inversion (only one not easily shown)

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi[f](a,b)\, da\, db.$$

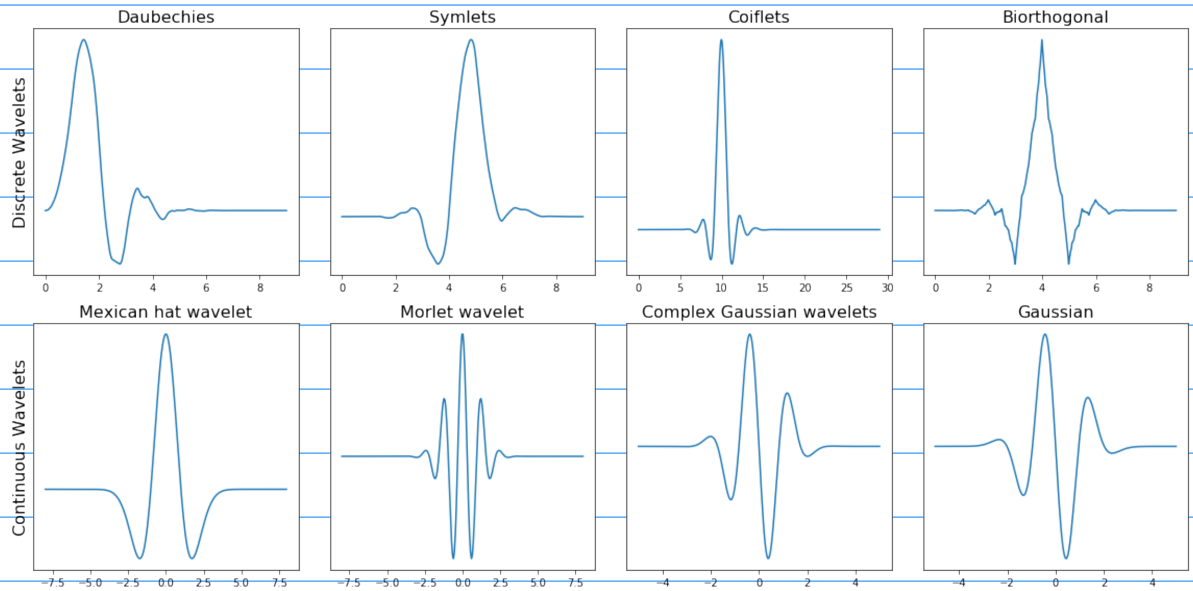
where
$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(k)|^2}{|k|}\, dk.$$

Observe that so far we have taken ψ to be arbitrary. But the inversion formula tells us that restrictions should be imposed.

We say ψ is an admissible wavelet if $C_\psi < +\infty$.

first thing to check if you are designing your own..

Many standard choices of ψ exist & are implemented in software packages (some from PyWavelets package)



We can naturally define a Discrete Wavelet Transform (DWT) from the CWT by choosing discrete values of (a,b) & approximating the integrals.

The convention is as follows.

often 2   often 0, 1, 2, ...

Take constants a_0, b_0 & integers $m, n \geq 0$ then write

$$\psi_{m,n}(t) = a_0^{-m/2}\,\psi(a_0^{-m}t - nb_0)$$

the DWT is then defined as

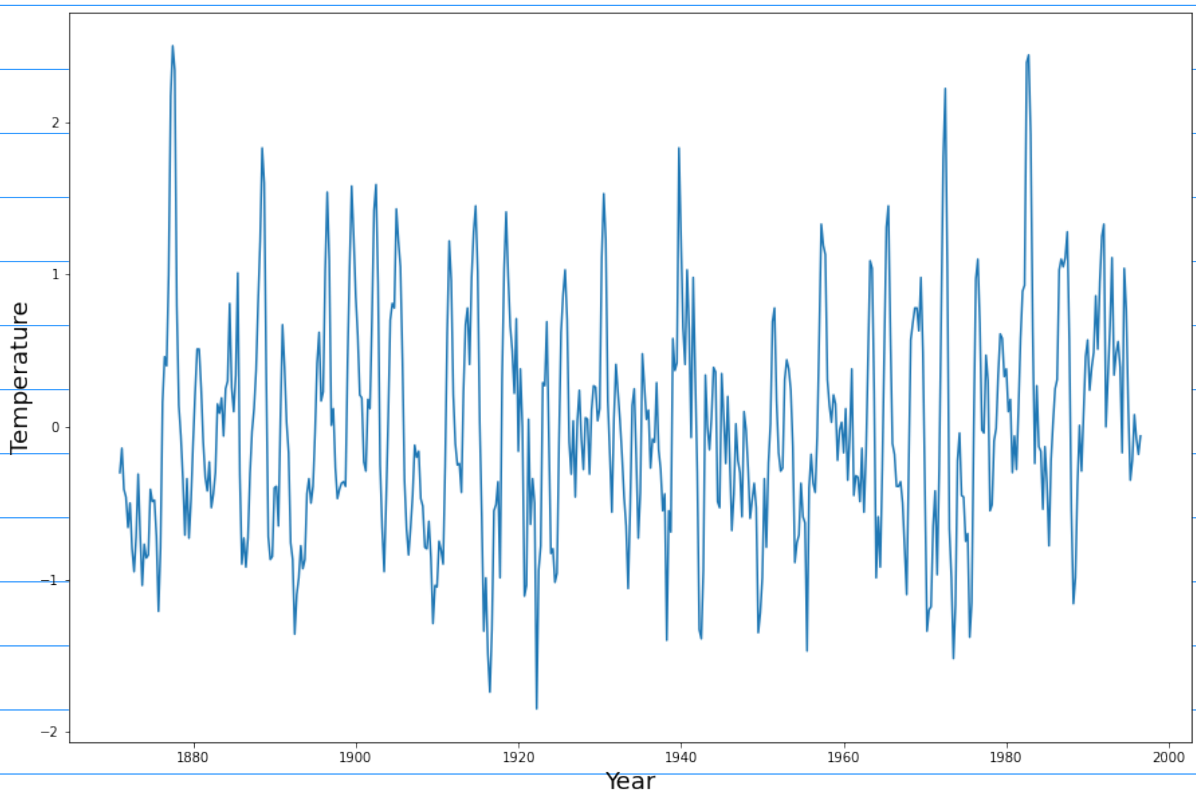
$$W[f](m,n) := \int_{-\infty}^{\infty} f(t)\,\psi_{m,n}(t)\,dt.$$

$$\boxed{\begin{array}{l}\text{If the } \psi_{m,n} \text{ are chosen well, ie, form a complete basis in}\\ L^2(\mathbb{R}) \text{ then we also have the reconstruction formula}\\ \displaystyle f(t) = \sum_{m,n=-\infty}^{\infty} W[f](m,n)\,\psi_{m,n}(t).\end{array}}$$

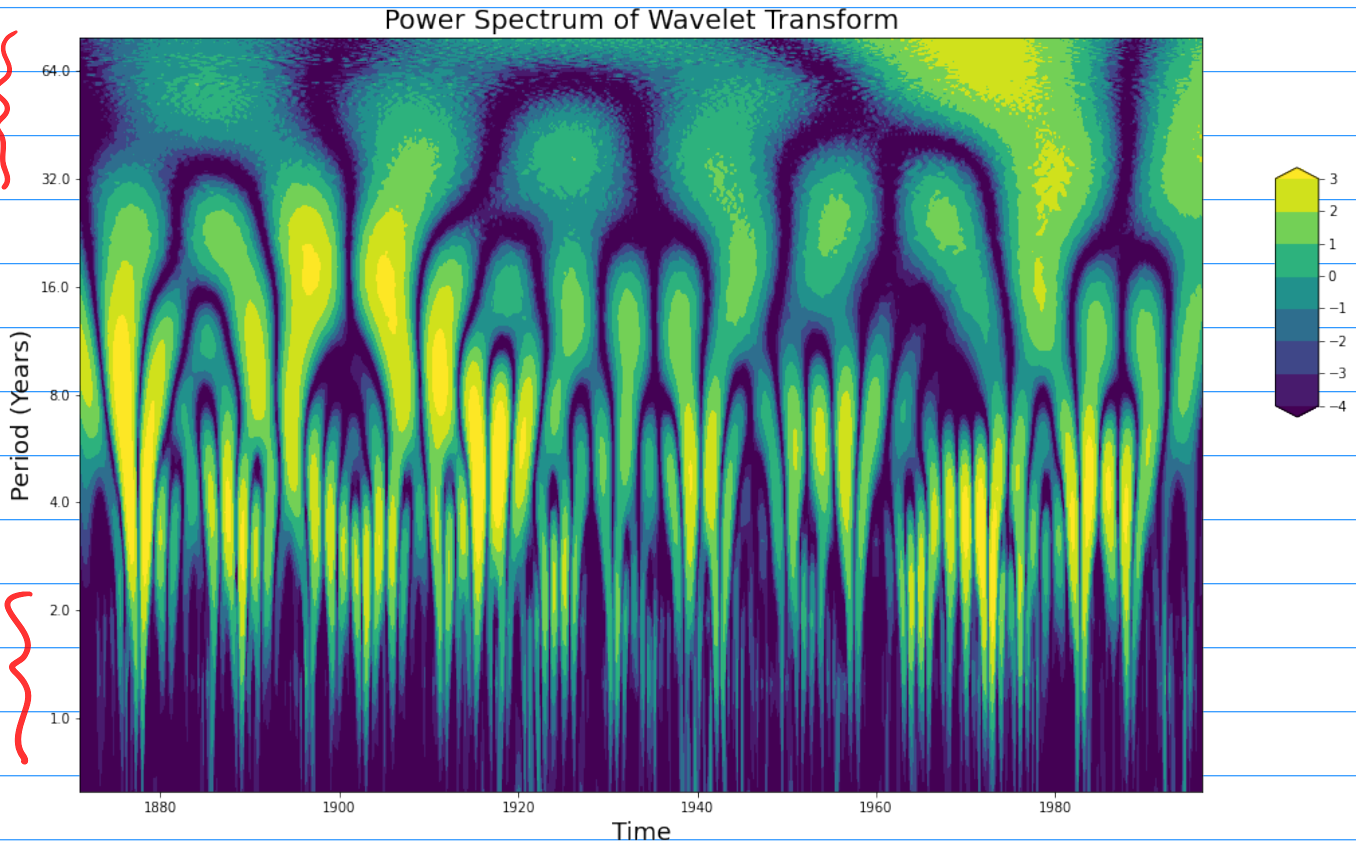
Example application of CWT (Scaleogram) where we plot the square (power) of CWT as a function of time & scales in the signal.

Idea is to visualize where the majority of "energy/content" of a signal lies in both time & scale.

(credit Ahmet Taspinar / ML Fundamentals)

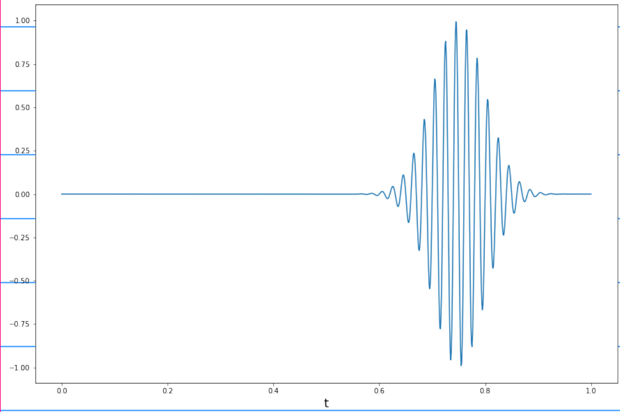


large scale (slow)



small scale fast

$f(t)$ $W[f](a,b)$


1.00
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1.0
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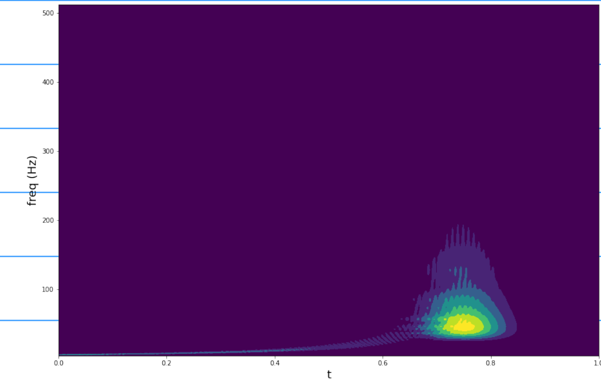
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freq (Hz)
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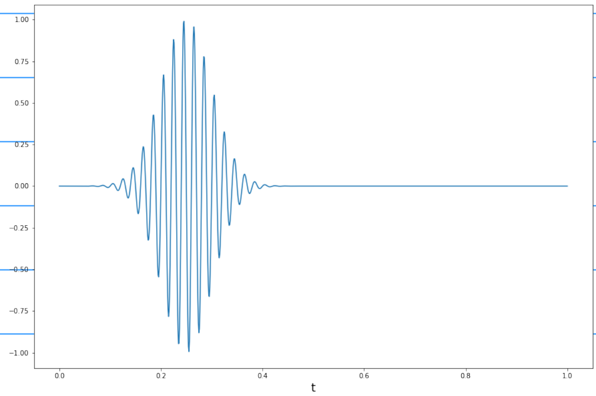
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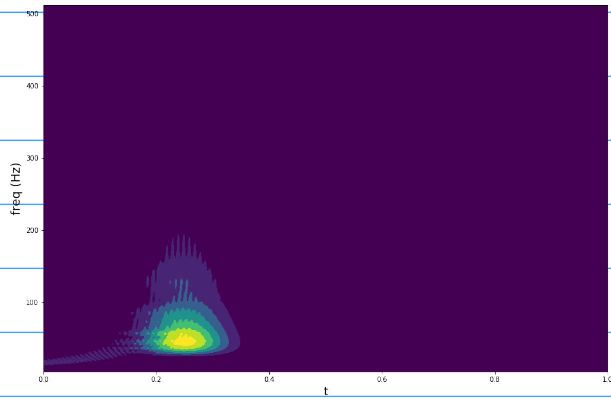
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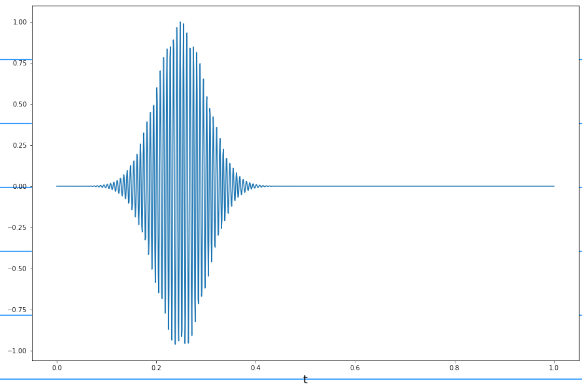
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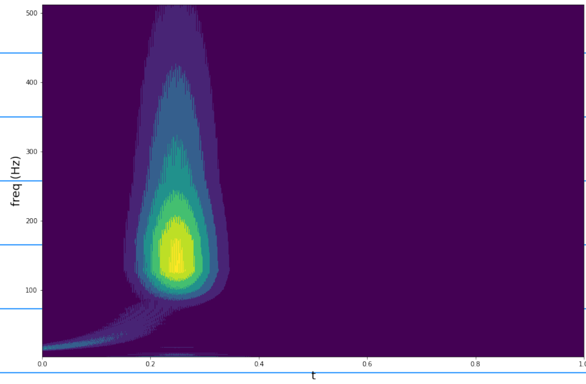
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