Lecture 2 - Introduction to Fourier analysis

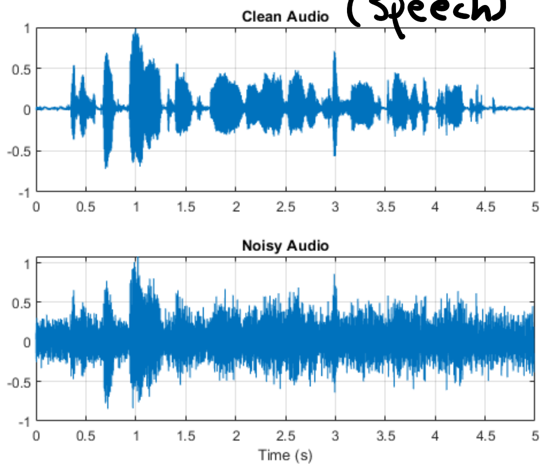
2.1 High level Overview:

Often times we have noisy signals / image / data that needs to be cleaned up or even compressed.

(speech)

$f(x)$

$f(x)$ + noise



One approach to this which is essentially the focus of the field of signal processing is to approximate the signal (data / image / etc) as the sum of simpler components. i.e.

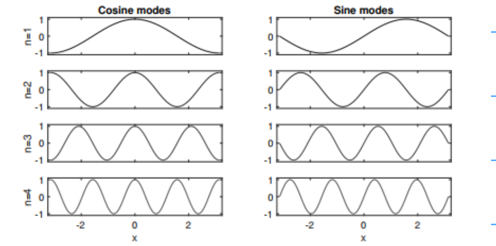
$$f(x) = \sum_{j=0}^{N} a_j \psi_j(x)$$

where $a_j \in \mathbb{R}$ coefficients & $\psi_j(x)$ are appropriate functions chosen by us.

Fourier series / discrete Fourier transform is one approach to this that takes the ψ_j to be trigonometric functions.

$$\psi_k(x) = \sin(kx)$$

or $$\psi_k(x) = \cos(kx)$$

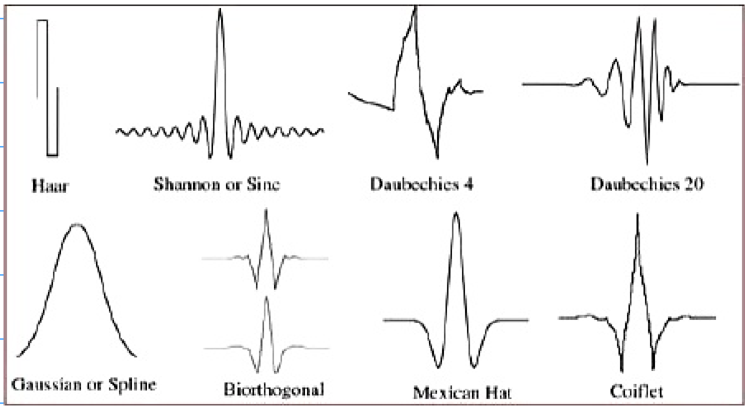


3.1 The first four Fourier sine and cosine modes.

Joseph Fourier

This is not the only decomposition one can use. For example, later in the course we will see Wavelets.

ex. wavelets



Why do we care about Fourier series?

- Widely used in practice & research.
- Fast fourier transform (FFT) alg.
- Deep connections to approximation theory & functional analysis.

2.2 Details of Fourier Series

Suppose we have a function $f:[0,2\pi]\longrightarrow\mathbb{R}$. Then Fourier's idea was to represent f as a sum of sine & cosine functions.

(I) $$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\sin(kx)+b_k\cos(kx)$$

*Note: assumption that f is defined on $[0,2\pi]$ is innocuous & can be replaced with any other compact interval after a change of variables.

This is striking because sin/cos are smooth & periodic functions but the above assertion is still true if $f(x)$ is not periodic or has discontinuities.

The task of finding the coeff.'s a_k, b_k is called the Fourier transformation (FT) of $f(x)$. or its Fourier Series (FS).

To compute the FS we proceed as follows.

Mult (I) by $\cos(lx)$ (or $\sin(lx)$) & integrate

$$\int_0^{2\pi}f(x)\cos(lx)\,dx=\frac{a_0}{2}\int_0^{2\pi}\cos(lx)\,dx$$
$$+\sum_{k=1}^{\infty}a_k\int_0^{2\pi}\sin(kx)\cos(lx)\,dx$$
$$+b_k\int_0^{2\pi}\cos(kx)\cos(lx)\,dx$$

But sin/cos satisfy the following orthogonality conditions (check by yourself)

$$\int_0^{2\pi} \sin(kx)\cos(lx)\,dx = 0$$

$$\int_0^{2\pi} \sin(kx)\sin(lx)\,dx = \begin{cases} 0 & l \neq k \\ \pi & l = k \end{cases}$$

$$\int_0^{2\pi} \cos(kx)\cos(lx)\,dx = \begin{cases} 0 & l \neq k \\ \pi & l = k \end{cases}$$

This implies

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx, \quad k \geq 0$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx, \quad k \geq 1.$$

We can further compress our notation using complex variables & Euler's formula

$$\exp(ix) = \cos(x) + i\sin(x)$$

so we can simply write

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp(ikx), \quad c_k \in \mathbb{C}$$

with

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\exp(-ikx)\,dx$$

- Observe that what we are doing here is breaking $f(x)$ into smaller, simpler components. In particular, these components are periodic functions of successively higher frequency.

- The above integrals are called projections, in the parlance of vector calculus. Here the "basis vectors" that we are projecting on are the $\sin(kx)$ & $\cos(kx)$ & the inner product is defined by the integral

$$f^T g \equiv \langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$$

called L^2 inner product

real analysis notat

Exercise: Extend the above framework to compute the FS of functions $f(x)$ defined on arbitrary intervals $[a, b] \subset \mathbb{R}$.

2.3 Connection to Fourier Transform

FS is a special case of another tool called Fourier transform defined for functions on the whole real line.

Given $f:\mathbb{R} \longrightarrow \mathbb{R}$ we define its Fourier Transform (FT)

$$\mathcal{F}(f) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ix\xi)\, dx$$

& inverse FT for $g:\mathbb{R} \longrightarrow \mathbb{C}$

$$\mathcal{F}^{-1}(g) = \check{g}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) \exp(ix\xi)\, d\xi$$

The fact that $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$ is a consequence of the Fourier integral theorem. (look up)

FT is a continuous version of FS by replacing the summation with an integral. Infact you get the FS by restricting FT to compact intervals $[a,b]$ instead of $(-\infty,\infty)$. This is a somewhat technical result concerning the eigen expansion of the Fourier kernel $K(x,\xi) = \exp(ix\xi)$. We will come back to this later in the course.

2.4 FT Properties

FS & FT have a lot of nice properties.

We state these without proof. Most are easy to show.

$f : [0, 2\pi] \to \mathbb{R}$, $\{\hat{f}_k\}_{k=-\infty}^{\infty}$ - the FS of f

$g : [0, 2\pi] \to \mathbb{R}$ $\{\hat{g}_k\}_{k=-\infty}^{\infty}$ - " " " g

- Linearity

$$\widehat{(af + bg)}_k = a\hat{f}_k + b\hat{g}_k$$

- Shifts/Translations

$$\widehat{(f(x-x_0))}_k = \exp(-ikx_0)\hat{f}_k$$

- Differentiation:

Let $f^{(n)}$ denote the n-th derivative of f then

$$\widehat{(f^{(n)})}_k = (ik)^n \hat{f}_k$$

- Integration:

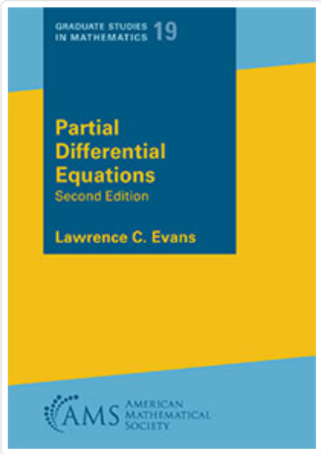
$$\int_0^{2\pi} f(x)\,dx = \hat{f}_0$$

- Convolution: f, g periodic on $[0, 2\pi]$

$$(f*g)(x) := \int_0^{2\pi} f(x-y)g(y)\,dy$$ also 2π periodic

$$\widehat{(f*g)}_k = 2\pi \hat{f}_k \hat{g}_k$$

Further reading (Advanced)



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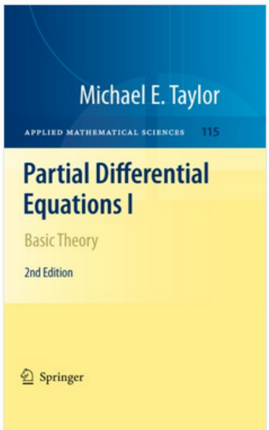

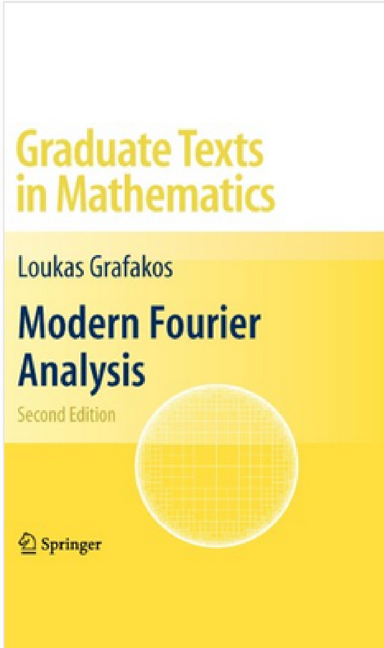

+ any engineering textbook on signal processing.

* Data-driven Modeling & Scientific Computation by N. Kutz (ch 13).