Lecture 23: Introduction to Sparse Recovery

Up to this point in the course we have seen many problems & models, all of which roughly are of the form

$$\text{(*)} \quad \min_{\underline{\beta}} \frac{1}{2}\|A\underline{\beta} - \underline{Y}\|_2^2 + \underbrace{\frac{\lambda}{2}\|\beta\|_2^2}_{\text{Regularization}},$$

ex supervised learning, Ridge regression, graphical SSL, classification, kernel regression broadly.

We also saw, through Cross-Validation, that the choice of $\lambda \geq 0$ has a profound impact on the quality of our solutions. This is super important when A has fewer rows than columns! ie, the system $A\underline{\beta} = \underline{Y}$ is underdetermined.

Observe that if we differentiate (*) wrt $\underline{\beta}$ we get.

$$A^T(A\hat{\beta} - Y) + \lambda\hat{\beta}$$

$$\Rightarrow (A^TA + \lambda I)\hat{\beta} = A^TY.$$

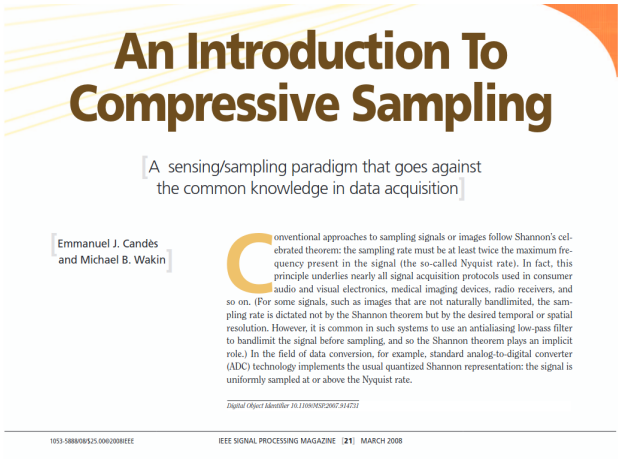
if $A \in \mathbb{R}^{N\times M}$ with $N < M$ then $A^TA \in \mathbb{R}^{M\times M}$ is rank deficient & so not invertible! Then adding λI to A^TA results in an invertible matrix & stabelizes the solution $\hat{\beta}$.

Hence, the term "regularization" since λI makes the problem more regular.

But one might wonder, what is special about $\|\beta\|_2$? why not another norm?

This is an excellent question & as it turns out it has profound implications. Particular choices of the norm, ex $\|\beta\|_p$ with $p \leq 1$ turns out to be very helpful. This is an advanced topic at the intersection of signal processing, convex analysis, applied math & statistics!

This line of thought also gave rise to the field of Compressed Sensing!



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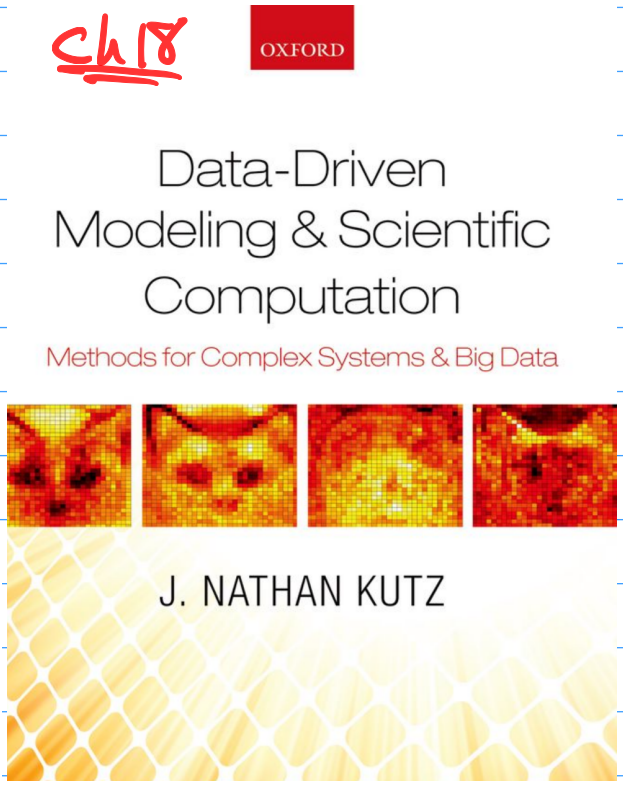
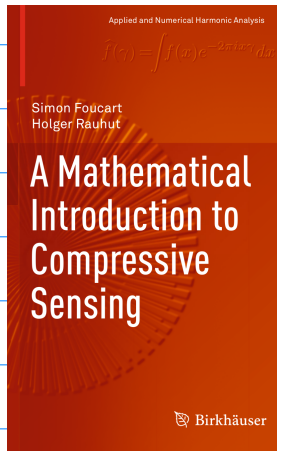
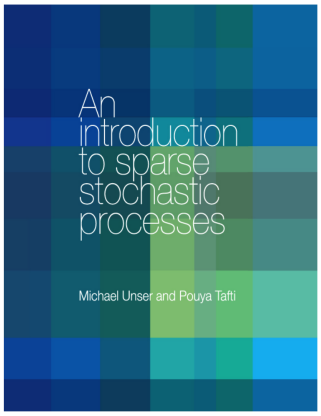






23.1 The Role of Sparsity

The notion of "sparsity" is going to be key for us.

Consider a linear system

$A\beta = Y, \qquad \beta \in \mathbb{R}^N, \; Y \in \mathbb{R}^M, \; A \in \mathbb{R}^{M \times N}$

where $M < N$, so the system is underdetermined so it cannot be solved uniquely.

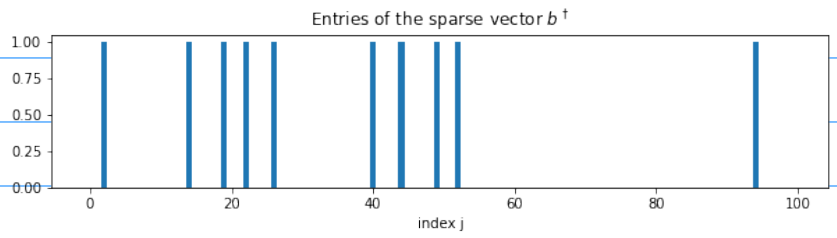
However, suppose that we have additional information that the vector $\underline{\beta}$ is sparse, more precisely it only has s-non-zero entries! (we say $\underline{\beta}$ is s-sparse)

Then, you might wonder whether you could find $\underline{\beta}$ if $s \ll M < N$?

Let us look at a quick example.

A vector $\underline{\beta}^t \in \mathbb{R}^{100}$ with only $s=10$ non-zero entries & a matrix $A \in \mathbb{R}^{40 \times 100}$ with random entries $A_{ij} \overset{iid}{\sim} N(0,1)$. Then set $\underline{y} = A\underline{\beta}^t \in \mathbb{R}^{40}$.

Our goal is then to infer $\underline{\beta}^t$ from $\underline{b}$, i.e., "solve" $A\underline{\beta} = \underline{y}$.



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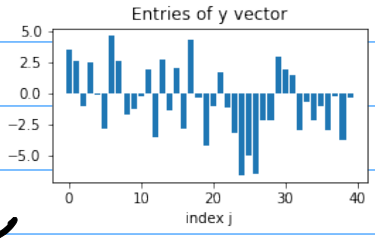
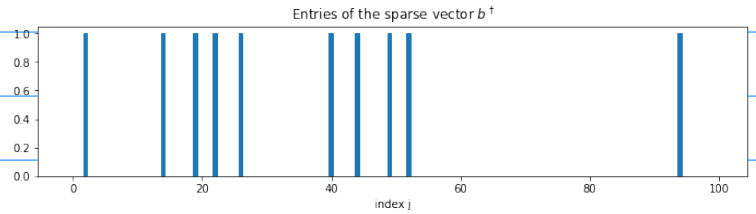
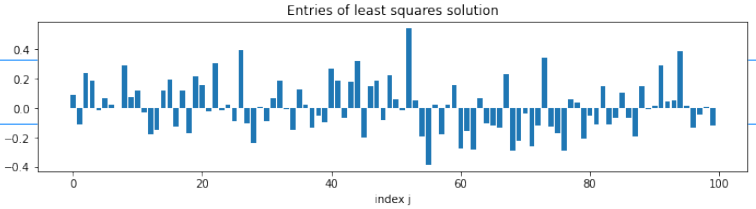


First thing we might try is least squares but this does not work well!

$$\underline{\beta}_{lsq} = \underset{\underline{\beta} \in \mathbb{R}^{100}}{\arg\min} \|A\underline{\beta} - \underline{y}\|_2^2$$

$\underline{\beta}_{lsq}$





We now try a diffent approach call Lasso which solves

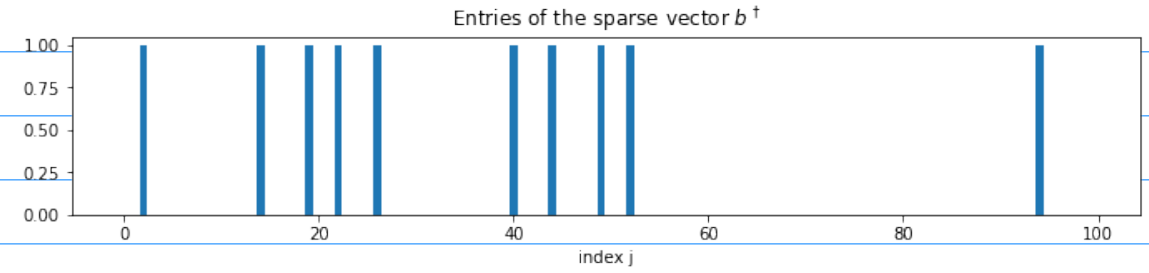
$$\beta_{Lasso} = \underset{\beta \in \mathbb{R}^{100}}{\arg\min} \frac{1}{2} \|A\beta - y\|_2^2 + \lambda \|\beta\|_1,$$

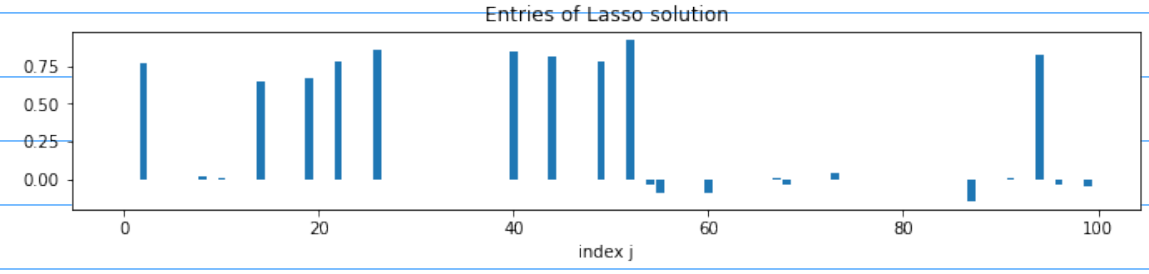
Compare to Ridge

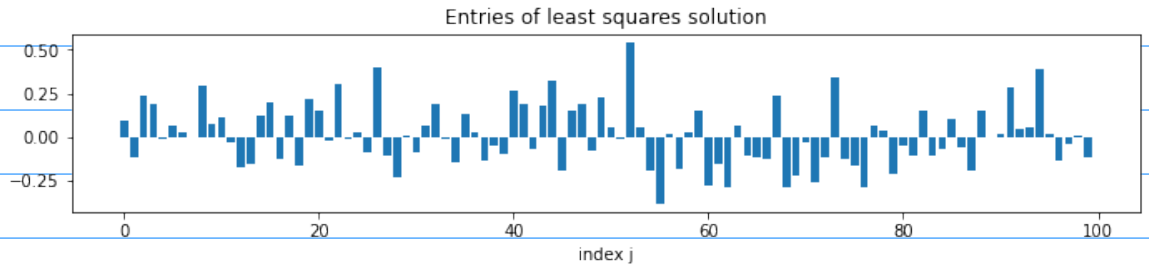
$$\beta_{Ridge} = \arg\min \frac{1}{2} \|A\beta - y\|_2^2 + \lambda \|\beta\|_2^2.$$

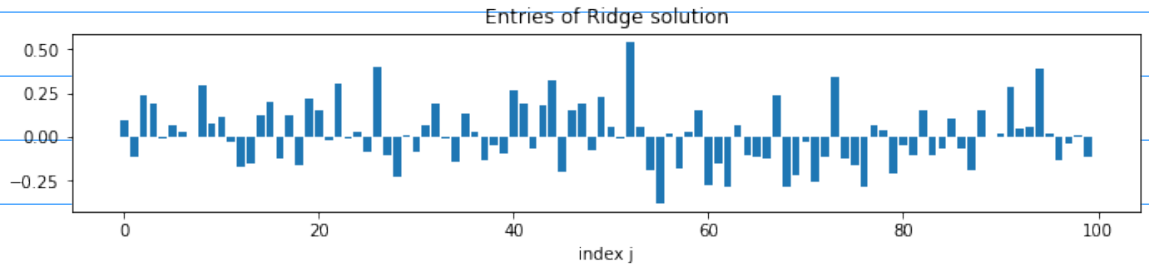
essentially we use a diffent regularization term.

But this ends up having a huge effect!









At an intuitive level the reason for the success of Lasso is that the $\|\cdot\|_1$-norm "promotes/prefers" solutions that are sparse while the $\|\cdot\|_2$-norm does not. Hence, Lasso/1-norm regularization are preferrable when we expect $\underline{x}$ to be sparse!

23.2 Intuition for Lasso

Solving a problem of the form

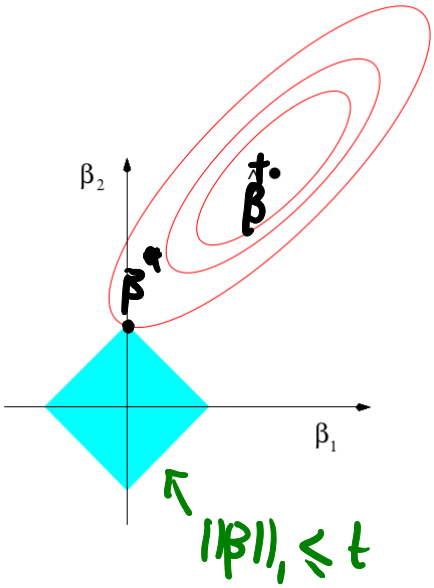
(I) $$\underline{\beta}^* = \underset{\underline{\beta}\in\mathbb{R}^N}{\operatorname{argmin}} \frac{1}{2}\|A\underline{\beta} - \underline{y}\|_2^2 + \lambda\|\underline{\beta}\|_p^p$$

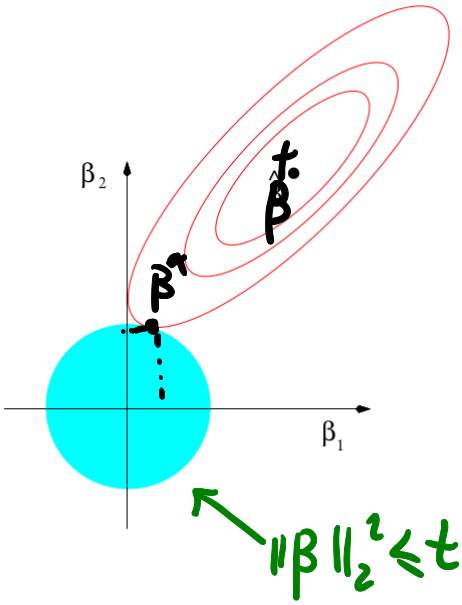
is equivalent to solving

(II) $$\underline{\beta}^* = \operatorname{argmin} \frac{1}{2}\|A\underline{\beta} - \underline{y}\|_2^2$$
$$\text{subject to } \|\underline{\beta}\|_p^p \leq t,$$

The value of t & λ are related to each other, indeed, (I) is the Lagrangian form of (II).

Here we can see the effect of the choice of the norm $\|\cdot\|_p$ in the regularization term.


β_2
β_1
$\hat{\beta}$
$\hat{\beta}^r$
$\|\beta\|_1 \leq t$


β_2
β_1
$\hat{\beta}$
$\hat{\beta}^r$
$\|\beta\|_2^2 \leq t$