Midterm Exams Reviewer

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Variable Separable ODE

Problem 1

Solve $(1+x)dy - ydx = 0$

Solution: Dividing by $(1+x)y$, we get

$$\frac{dy}{y} = \frac{dx}{1+x}$$

from which it follows that

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{dx}{1+x} \\ \ln|y| &= \ln|1+x| + c \\ e^{\ln|y|} &= e^{\ln|1+x|+c} \\ y &= e^c(1+x) \end{aligned}$$

Relabeling the e^c as c then gives us

$$y = c(1+x)$$

Answer: $y = c(1+x)$

Problem 2

Solve $\frac{dy}{dx} = -\frac{x}{y}$

Solution: Multiplying ydx, we get

$$ydy = -xdx$$

and integrating both sides gives us

$$\begin{aligned} \int ydy &= \int -xdx \\ \frac{y^2}{2} &= -\frac{x^2}{2} + c \end{aligned}$$

Solving for y gives us

$$y = \pm\sqrt{-x^2 + 2c}$$

Answer: $y = \pm\sqrt{-x^2 + 2c}$

Problem 3

Solve $\frac{dy}{dx} = e^{3x+2y}$

Solution: Rewriting $e^3x + 2y$ as $e^{3x}e^{2y}$, we get

$$\frac{dy}{dx} = e^{3x}e^{2y}$$

Dividing by e^2ydx we get

$$\frac{dy}{e^{2y}} = e^{3x}dx$$

and integrating both sides gives us

$$\begin{aligned} \int \frac{dy}{e^{2y}} &= \int e^{3x}dx \\ -\frac{1}{2}e^{-2y} &= \frac{1}{3}e^{3x} + c \\ 3e^{-2y} &= -2e^{3x} + c \end{aligned}$$

Answer: $3e^{-2y} = -2e^{3x} + c$

Homogeneous ODE

Problem 4

Solve $2x^3ydx + (x^4 + y^4)dy = 0$

Solution: Each coefficient is a homogeneous function of degree four, so we can let $x = vy$ and substitute to get

$$\begin{aligned}
2(vy)^3yd(vy) + ((vy)^4 + y^4)dy &= 0 \\
2v^3y^4(vdy + ydv) + (v^4y^4 + y^4)dy &= 0 \\
2v^4y^4dy + 2v^3y^5dv + v^4y^4dy + y^4dy &= 0 \\
3v^4y^4dy + 2v^3y^5dv + y^4dy &= 0 \\
y^4(3v^4 + 1)dy + 2v^3y^5dv &= 0
\end{aligned}$$

Now we can separate the variables and integrate to get

$$\begin{aligned}
\int \frac{dy}{y} + \int \frac{2v^3}{3v^4 + 1}dv &= \int 0 \\
\ln|y| + \frac{1}{6}\ln|3v^4 + 1| &= c \\
6\ln|y| + \ln|3v^4 + 1| &= c \\
e^{6\ln|y|} + e^{\ln|3v^4+1|} &= e^c \\
y^6 + (3v^4 + 1) &= c \\
y^6 + \frac{3x^4}{y^4} + 1 &= c \\
y^6 + \frac{3x^4}{y^4} &= c - 1
\end{aligned}$$

Rewriting $c - 1$ as c, we get

$$y^6 + \frac{3x^4}{y^4} = c$$

Answer: $y^6 + \frac{3x^4}{y^4} = c$

Problem 5

Solve $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

Solution: Both $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree two, so we can let $y = ux$ and substitute to get

$$\begin{aligned}
(x^2 + (ux)^2)dx + (x^2 - (ux)^2)d(ux) &= 0 \\
(x^2 + u^2x^2)dx + (x^2 - u^2x^2)[udx + xdu] &= 0 \\
x^2dx + u^2x^2dx + ux^2dx + x^3du - u^3x^2dx - u^2x^3du &= 0 \\
x^2(1+u)dx + x^3(1-u)du &= 0
\end{aligned}$$

Now we can separate the variables and integrate to get

$$\begin{aligned}
\frac{dx}{x} + \frac{1-u}{1+u}du &= 0 \\
\frac{dx}{x} + [-1 + \frac{2}{1+u}]du &= 0 \\
\int \frac{dx}{x} + \int [-1 + \frac{2}{1+u}]du &= \int 0 \\
\ln|x| - u + 2\ln|1+u| &= c \\
\ln|x| + \frac{y}{x} - 2\ln|1 + \frac{y}{x}| &= c
\end{aligned}$$

We can rewrite c as $\ln c$, and then using the properties of logarithms, we can get

$$ln\frac{(x+y)^2}{cx} = \frac{y}{x}$$

The definition of logarithms then yeilds

$$(x+y)^2 = cxe^{\frac{y}{x}}$$

Answer: $(x+y)^2 = cxe^{\frac{y}{x}}$

Problem 6

Solve $(x-y)dx + xdy = 0$

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