Probabilistic Graphical Models : Homework 2

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1 Exercise 1

1.1 Question 1

The implied factorization for any joint distribution $p \in \mathcal{L}(G)$ is :

$$p(x, y, z, t) = p(x)p(y)p(z|x, y)p(t|z)$$

Lets take $X \sim \mathcal{B}(p), Y \sim \mathcal{B}(p), Z = X \oplus Y, T = Z$. It is clear that $X \perp\!\!\!\perp Y$ and that $X \perp\!\!\!\perp Y \not| Z$ because with X and Z we can determine Y. Therefore since $Z = T$, $X \perp\!\!\!\perp Y \not| T$

1.2 Question 2

1.2.a

We consider $Z \sim \mathcal{B}(\pi)$ with $X \perp\!\!\!\perp Y \mid Z$ and $X \perp\!\!\!\perp Y$. We can write $p(x, y)$ in two ways:

$$\begin{aligned} p(x, y) &= p(x, y \mid z = 0)p(z = 0) + p(x, y \mid z =)p(z = 1) \\ &= p(x \mid z = 0)p(y \mid z = 0)p(z = 0) + p(x \mid z = 1)p(y \mid z = 1)p(z = 1) \end{aligned}$$

And

$$\begin{aligned} p(x, y) &= p(x)p(y) \\ &= [p(x \mid z = 0)p(z = 0) + p(x \mid z = 1)p(z = 1)][p(y \mid z = 0)p(z = 0) + p(y \mid z = 1)p(z = 1)] \end{aligned}$$

Then we take the difference between those two expressions of $p(x, y)$ and factorize by $p(z = 0)p(z = 1) \neq 0$

$$0 = p(x \mid z = 0)p(y \mid z = 0) - p(x \mid z = 1)p(y \mid z = 1) + p(x \mid z = 0)p(y \mid z = 1) + p(x \mid z = 1)p(y \mid z = 0)$$

1.2.b

2 Exercise 2

2.1 Question 1

Let $G = (V, E)$ be a DAF, and $i \to j$ be a covered edge of G. We consider $G = (V, E')$ where $E' = (E \setminus \{i \to j\}) \cup \{j \to i\}$.

$$\begin{aligned}
p(x_j \mid x_{\pi_j^G}) p(x_i \mid x_{\pi_i^G}) &= p(x_j \mid x_{\pi_i^G}, x_i) p(x_i \mid x_{\pi_i^G}) \\
&= p(x_i \mid x_{\pi_i^G}, x_j) p(x_j \mid x_{\pi_i^G}) && \textit{(Bayes)} \\
&= p(x_i \mid x_{\pi_i^{G'}}) p(x_j \mid x_{\pi_j^{G'}})
\end{aligned}$$

Since we haven't modified any other edges, we have proven that $\mathcal{L}(G) = \mathcal{L}(G')$

2.2 Question 2

Let $G = (V, E)$ a directed tree and $\tilde{G}$ the symmetrized graph (which is equal to moralized graph). The cliques of $\tilde{G}$ are by the definition of a tree the set $\mathcal{C} = \{\pi_x \mid x \in V\} \cup V$. Now let $p \in \mathcal{L}(\tilde{G})$ and consider the ψ such that $\sum_x \prod_{c \in \mathcal{C}} \psi_c(x_c) = 1$

$$\begin{aligned}
p(x) &= \prod_{c \in \mathcal{C}} \psi_c(x_c) \\
&= \prod_{x \in V} \psi_{x_i}(x_i) \psi_{x_i, \pi_{x_i}}(x_i, \pi_{x_i})
\end{aligned}$$

We can define $f(x_i, x_{\pi_{x_i}}) = \prod_{x \in V} \psi_{x_i}(x_i) \psi_{x_i, \pi_{x_i}}(x_i, \pi_{x_i})$ and therefor $p \in \mathcal{L}(G)$. So $\mathcal{L}(\tilde{G}) \subset \mathcal{L}(G)$
Now let $p \in \mathcal{L}(G)$

$$\begin{aligned}
p(x) &= \prod_{x \in V} p(x \mid x_{\pi_x}) \\
&= \prod_{x \in V} \frac{p(\pi_x \mid x)}{p(\pi_x)} p(x)
\end{aligned}$$

We can define $\psi_x(x) = p(x)$ and $\psi_{\pi_x}(\pi_x) = \frac{p(\pi_x | x)}{p(\pi_x)}$ and therefor $p \in \mathcal{L}(\tilde{G})$. So $\mathcal{L}(G) \subset \mathcal{L}(\tilde{G})$
Finally $\mathcal{L}(G) = \mathcal{L}(\tilde{G})$