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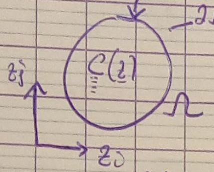
$\boxed{\text{Elasticité et Plasticité des Matériaux hétérogènes}}$

Rappel :

$\underline{\xi}(\underline{z}) = \underline{\underline{E}} \cdot \underline{z}$ CL Uniforme en déformation (Hashin)

$\underline{z}$: vecteur position à l'échelle microscopique



$$\begin{cases} \forall \underline{z} \in \Omega \quad \underline{\underline{\sigma}}(\underline{z}) = \underline{\underline{\underline{\underline{c}}}}(\underline{z}) : \underline{\underline{\varepsilon}}(\underline{z}) \\ \underline{\underline{\varepsilon}}(\underline{z}) = \frac{1}{2}\left(\underline{\nabla}\underline{\xi} + {}^t\underline{\nabla}\underline{\xi}\right) \\ \operatorname{div} \underline{\underline{\sigma}} = 0 \\ \forall \underline{z} \in \partial\Omega \quad \underline{\xi}(\underline{z}) = \underline{\underline{E}} \cdot \underline{z} \end{cases}$$

$\forall \underline{z} \in \Omega \quad \exists\, \underline{\underline{\underline{\underline{A}}}}(\underline{z}) \qquad \underline{\underline{\varepsilon}}(\underline{z}) = \underline{\underline{\underline{\underline{A}}}}(\underline{z}) : \underline{\underline{E}}$

avec $A_{ijkl} = A_{jikl}$, $A_{ijkl} = A_{ijlk}$ } symétrie mineure

$A_{ijkl} \neq A_{klij}$ } symétrie majeure

$$\begin{array}{lccc} \text{Macro} & \underline{\underline{E}} & \xleftrightarrow{\ ?\ } & \underline{\underline{\Sigma}} \\ & \underline{\underline{\underline{\underline{A}}}} \updownarrow \text{Loc} & & \updownarrow \text{Hom} \\ \text{Micro} & \underline{\underline{\varepsilon}}(\underline{z}) & \xleftrightarrow[\underline{\underline{\underline{\underline{c}}}}]{} & \underline{\underline{\sigma}}(\underline{z}) \end{array}$$

Important

$$\boxed{\int_\Omega f \frac{\partial g}{\partial z_j}\, dV = -\int_\Omega \frac{\partial f}{\partial z_j} g\, dV + \int_{\partial\Omega} f g\, n_j\, dS}$$

$\xi_i = E_{ij} z_j \qquad \int_\Omega \frac{\partial \xi_i}{\partial z_j} dV = \int_{\partial\Omega} \xi_i n_j\, dS = \int_{\partial\Omega} E_{ik} z_k n_j\, dS = E_{ik} \int_{\partial\Omega} z_k n_j\, dS$

$= E_{ik} \int_\Omega \underbrace{\frac{\partial z_k}{\partial z_j}}_{\delta_{kj}} dV$

$= |\Omega|\, E_{ij}$

Donc $\overline{\left(\frac{\partial \xi_i}{\partial z_j}\right)} = E_{ij}$

Travail de déformation

MACRO $\quad \underline{\underline{\Sigma}} : \underline{\underline{E}}\, (\Omega)$

MICRO $\quad \int_\Omega \underline{\underline{\sigma}}(\underline{z}) : \underline{\underline{\varepsilon}}(\underline{z})\, dV$

Condition de cohérence énergétique

$$\underline{\underline{\Sigma}} : \underline{\underline{E}} = \overline{\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}}$$

$$\int_\Omega \sigma_{ij} \frac{\partial \xi_i}{\partial z_j} dV = -\int_\Omega \underbrace{\frac{\partial \sigma_{ij}}{\partial z_j}}_{=0} \xi_i\, dV + \int_{\partial\Omega} \sigma_{ij} \xi_i n_j\, dS$$

$$= \int_{\partial\Omega} \sigma_{ij} E_{ik} z_k n_j\, dS = E_{ik} \int_{\partial\Omega} \sigma_{ij} z_k n_j\, dS$$

$$= E_{ik} \times \int_\Omega (\sigma_{ij} z_k)_{,j}\, dV = E_{ik} \int_\Omega \sigma_{ij} \frac{\partial z_k}{\partial z_j} dV = E_{ik} \int_\Omega \sigma_{ij}\, \delta_{kj}\, dV$$

$$= E_{ij} \int_\Omega \sigma_{ij}\, dV$$

$$\cancel{|\Omega|}\ \overline{\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}} = \cancel{|\Omega|}\ \overline{\sigma_{ij}} \times E_{ij}$$

$\underline{\underline{E}} = \overline{\underline{\underline{\varepsilon}}}$ $\qquad \Sigma_{ik} = \overline{\sigma_{ik}}\ ; \ \underline{\underline{\Sigma}} = \overline{\underline{\underline{\sigma}}}$

$$\overline{\underline{\underline{\sigma}}(\underline{z})} = \overline{\underline{\underline{\underline{\underline{C}}}}(\underline{z}) : \underline{\underline{\underline{\underline{A}}}}(\underline{z}) : \underline{\underline{E}}}$$

$$\underline{\underline{\Sigma}} = \overline{\underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\underline{\underline{A}}}}} : \underline{\underline{E}}$$

$$\underline{\underline{\underline{\underline{C}}}}^{hom} = \overline{\underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\underline{\underline{A}}}}} \qquad \text{avec } \overline{\underline{\underline{\underline{\underline{A}}}}} = \underline{\underline{\underline{\underline{1}}}}$$

biphasé (f_1, f_2 ; $f_1 + f_2 = 1$)

$$\underset{\approx}{C}^{hom} = \frac{1}{|\Omega|} \left(\int_{\Omega_1} C_1 : A\, dV + \int_{\Omega_2} C_2 : A\, dv \right)$$

$$= f_1 \underset{\approx}{C_1} : \overline{\underset{\approx}{A}}^{(1)} + f_2 \underset{\approx}{C_2} : \overline{\underset{\approx}{A}}^{(2)}$$

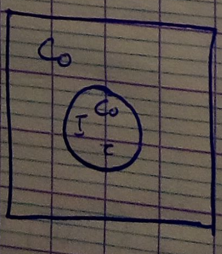
$$f_1 \overline{\underset{\approx}{A}}^{(1)} = I - f_2 \overline{\underset{\approx}{A}}^{(2)}$$

d'où $\underset{\approx}{C}^{hom} = \underset{\approx}{C_1} + f_2 \left(\underset{\approx}{C_2} - \underset{\approx}{C_1} \right) : \overline{\underset{\approx}{A}}^{(2)}$

Notions de Polarisation

Milieu infini ($\mathbb{R}^3$) élastique linéaire $\qquad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}(\underline{z}) : \underline{\underline{\varepsilon}} = \underset{\approx}{C_0} : \underline{\underline{\varepsilon}} + \underline{\underline{\tau}}$

$\underline{\underline{\tau}}$ précontrainte dans un [illegible]



$\underline{\underline{\varepsilon}} = 0$ à l'∞

$\forall \underline{z} \in I : \underline{\underline{\sigma}} = C_0 : \underline{\underline{\varepsilon}} + \underline{\underline{\tau}}$

$\underline{z} \in \mathbb{R}^3 \setminus I : \underline{\underline{\sigma}} = C_0 : \underline{\underline{\varepsilon}}$

$$\begin{cases} \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C_0}}}} : \underline{\underline{\varepsilon}} + \underline{\underline{\tau}}\, \chi^I(\underline{z}) \quad \begin{cases} 1 \text{ dans } I \\ 0 \text{ sinon} \end{cases} \\ \text{div } \underline{\underline{\sigma}} = 0 \\ \underline{\xi} \to 0 \text{ à l'infini} \end{cases}$$

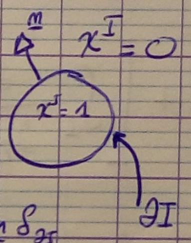
$$\operatorname{div}\left(\underset{\approx}{C_0} : \nabla\underline{\xi}\right) + \underbrace{\operatorname{div}\left(\underline{\underline{\tau}}\, \chi^I\right)}_{\underline{\underline{\tau}} \cdot \nabla(\chi^I)} = 0$$

$$\int \frac{\partial \chi^I}{\partial z_j} \varphi \, dV = -\int \chi^I \frac{\partial \varphi}{\partial z_j} dV \qquad \left(\text{définition de } \frac{\partial \chi^I}{\partial z_j}\right)$$

$$= -\int_I \frac{\partial \varphi}{\partial z_j} dV = -\int_{\partial I} \varphi n_j \, dS .$$

Dirac sur une surface : $\int \delta_{\partial I}(z) \, \varphi(z) \, dV = \int_{\partial I} \varphi(z) \, dS$

D'où $\frac{\partial \chi^I}{\partial z_j} = -n_j \, \delta_{\partial I}$



$$\nabla(\chi^I) = -\underline{n}\, \delta_{\partial I}$$

On obtient donc

forces à distance

$$\operatorname{div}\left(\underset{\approx}{C_0} : \nabla\underline{\xi}\right) - \underline{\underline{\tau}} \cdot \underline{n}\, \delta_{\partial I} = 0$$

$$\int_I -\left(\underline{\underline{\tau}} \cdot \underline{n}\, \delta_{\partial I}(\underline{z})\right) \cdot \underline{\xi}(\underline{z}) \, dV = \int_{\partial I} \underline{\xi}(\underline{z}) \cdot \left(-\underline{\underline{\tau}} \cdot \underline{n} \, dS\right)$$

Fonction de Green

force ponctuelle $\underline{f}$ à l'origine de l'espace

$$\operatorname{div}\left(\underset{\approx}{C_0} : \nabla\underline{\xi}\right) + \underline{f}\, \delta_0(\underline{z}) = 0$$

$$\int \underline{f}\, \delta_0(z) \cdot \underline{\xi} \, dV = \underline{f} \cdot \underline{\xi}(0)$$

Solution : $(\forall z \in \mathbb{R}^3) \quad \underline{\xi}(\underline{z}) = \underline{\underline{G}}(\underline{z}) \cdot \underline{f}$

$$\underline{f} = \underline{e}_m \longrightarrow \xi_l = G_{lm}$$

$$\frac{\partial}{\partial z_j}\left(C^0_{ijkl} \frac{\partial (G_{lm})}{\partial z_k}\right) + \delta_{im}\, \delta_0(\underline{z}) = 0$$

$$\boxed{C^0_{ijkl} \frac{\partial^2}{\partial z_j \partial z_k}(G_{lm}) + \delta_{im}\, \delta_0(\underline{z}) = 0}$$

Si on se place en $\underline{z}'$ $\underline{\xi}(\underline{z}) = \underline{\underline{G}}(\underline{z}-\underline{z}')\cdot \underline{f}$

$$\underline{\xi}(\underline{z}) = \int_{\partial I} \underline{\underline{G}}(\underline{z}-\underline{z}')\cdot(-\underline{\underline{\tau}}\cdot \underline{n}'\, dS_{z'})$$

on cherche à exprimer $\underline{\underline{\varepsilon}}(\underline{z}) = -\underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \underline{\underline{\tau}}$ ← Tenseur de Hill

$$\xi_i(\underline{z}) = -\int_{\partial I} G_{ik}(\underline{z}-\underline{z}')\, \tau_{kl}\, n_l'\, dS_{z'}$$

$$= -\left[\int_I \frac{\partial}{\partial z'_l}\left(G_{ik}(\underline{z}-\underline{z}')\right) dV_{z'}\right] \tau_{kl}$$

$$= +\left[\int_I \frac{\partial}{\partial z_l}\left(G_{ik}(\underline{z}-\underline{z}')\right) dV_{z'}\right] \tau_{kl}$$

$$= \frac{\partial}{\partial z_l}\left(\int_I G_{ik}(z-z')\, dV_{z'}\right) \tau_{kl}$$

$$\frac{\partial \xi_i(\underline{z})}{\partial z_j} = \frac{\partial^2}{\partial z_l \partial z_j}\left(\int_I G_{ik}(z-z')\, dV_{z'}\right) \tau_{kl}$$

$$\varepsilon_{ij}(\underline{z}) = \frac{\partial^2}{\partial z_j \partial z_l}\left(\int_I G_{ik}(z-z')\, dV_{z'}\right)\Big|_{(ij)(kl)} \tau_{kl}$$

Notation

$$\left(a_{|(ij)} = \frac{1}{2}(a_{ij}+a_{ji})\right)$$

$$P_{ijkl}(\underline{z}) = -\frac{\partial^2}{\partial z_j \partial z_l}\left(\int_I G_{ik}(z-z')\, dV_{z'}\right)\Big|_{(ij)(kl)}$$

si I est ellipsoïde alors $I : z \in I \rightarrow P_{ijkl}(\underline{z})$ est uniforme

$$\underbrace{\underline{\underline{\underline{\underline{P}}}}\big|_I}_{\text{restriction de } P \text{ à } I} = \underline{\underline{\underline{\underline{P}}}}^I$$ tenseur de Hill de I

Fourier polarisé ($T \to 0$ à l'∞)

$$\begin{cases} \underline{z} \in I : & \underline{j} = -\underline{\underline{k}} \nabla T + \underline{\tau}^I \\ \underline{z} \notin I : & \underline{j} = -\underline{\underline{k}} \cdot \nabla T \end{cases} \quad \Big\| \quad \underline{j} = -\underline{\underline{k}} \nabla T + \underline{\tau}^I \chi^I \qquad \underline{\underline{k}} = k \underline{\underline{\delta}}$$

$$\operatorname{div} \underline{j} = 0 \Rightarrow \underbrace{-k \Delta T}_{\text{Laplacien}} + \underline{\tau}^I \cdot \nabla(\chi^I) = 0$$

$$-k \Delta T - \underline{\tau}^I \cdot \underline{n}\, \delta_{\partial I}(\underline{z}) = 0$$

problème élémentaire :

$$-k \Delta T + \delta_0(\underline{z}) = 0 \quad \text{(fonction de Green)}$$

$$\Delta\left(\frac{1}{r}\right) = ? \qquad r = \sqrt{z_1^2 + z_2^2 + z_3^2} \qquad \Delta = \operatorname{div} \operatorname{grad}$$

$$\frac{\partial r}{\partial z_i} = \frac{z_i}{r}$$

$$\frac{\partial}{\partial z_i}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \frac{\partial r}{\partial z_i} = -\frac{z_i}{r^3}$$

$$\frac{\partial^2}{\partial z_i^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} - z_i \times (-3) r^{-4} \frac{\partial r}{\partial z_i} = \frac{3 z_i^2}{r^5} - \frac{1}{r^3}$$

$r \neq 0 \quad \Delta\left(\frac{1}{r}\right) = 0$ mais Δ en $r = 0$

$$\int_{B(0,R)} \Delta\left(\frac{1}{r}\right) dV = \int_{r=R} \nabla\left(\frac{1}{r}\right) \cdot \underline{e_r}\, dS \qquad \nabla\left(\frac{1}{r}\right) = -\frac{\underline{z}}{r^3}$$

$$= \int_{r=R} -\frac{\underline{e_r}}{R^2} \cdot \underline{e_r}\, dS$$

$$= -4\pi$$

Donc $\Delta\left(\frac{1}{r}\right) = -4\pi\, \delta_0(\underline{z})$

$$G(\underline{z}) = -\frac{1}{4\pi k} \times \frac{1}{r}$$

$$T(\underline{z}) = \int_{\partial I} G(\underline{z} - \underline{z}')\left(-\tau_j^I \cdot n_j'\, dS_{z'}\right) = -\int_I \frac{\partial}{\partial z_j'}\left(G(\underline{z} - \underline{z}')\right) dV_{z'}\, \tau_j^I$$

$$= \frac{\partial}{\partial z_j}\left(\int_I G(\underline{z} - \underline{z}')\, dV_{z'}\right) \tau_j^I$$

$$\frac{\partial T}{\partial z_i}(\underline{z}) = \underbrace{\frac{\partial^2}{\partial z_i \partial z_j}\left(\int_I G(\underline{z}-\underline{z}')\,dV_{z'}\right)}_{P_{ij}(\underline{z})} \tau_j^I$$

$$\left|\begin{array}{l} \underline{\nabla} T(\underline{z}) = \underline{\underline{P}}(\underline{z}) \cdot \underline{\tau} \\ \underline{\underline{\varepsilon}}(\underline{z}) = -\underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \underline{\underline{\tau}} \end{array}\right.$$

si I est une sphère de centre O et de rayon a

$$\phi(\underline{z}) = \int_I \frac{dV_{z'}}{|\underline{z}-\underline{z}'|} = \frac{2\pi a^2}{3}\left(3 - \frac{r^2}{a^2}\right)$$

$$\int_{B(O,a)} \frac{r^2\,dr\,d\theta\,d\varphi \sin\theta}{r} = 4\pi \int_0^a r\,dr = 2\pi a^2$$

Quel est le tenseur de Hill de la sphère $\underline{\underline{P}}^{sphère}$

$$\underline{\underline{P}}^{sphère} = \frac{1}{3k}\underline{\underline{\delta}} \qquad \underline{\underline{\delta}} = \underline{\underline{1}} \qquad P_{ij}$$

$$P_{ij} = -\frac{1}{4\pi k} \times \frac{\partial^2}{\partial z_i \partial z_j}(\phi(\underline{z})) = -\frac{1}{4\pi k} \times -\frac{2\pi}{3} \times \underbrace{\frac{\partial^2}{\partial z_i \partial z_j}(r^2)}_{2\delta_{ij}}$$

Polarisation d'une région I

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} + \underline{\underline{\tau}}\, \chi^I$$

$$\forall z \quad \underline{\underline{\varepsilon}}(\underline{z}) = -\underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \underline{\underline{\tau}}$$

si I est un ellipsoïde $\underline{\underline{\underline{\underline{P}}}}_{|I} = \underline{\underline{\underline{\underline{P}}}}^I$

Exercice

$\mathbb{R}^3$ polarisation uniforme d'une sphère

variation de T uniforme dans $S(O,R)$

$$\Delta T = \theta$$

$$\underline{\underline{\sigma}} = \underset{\approx}{C} : \underline{\underline{\varepsilon}} - k\theta\, \underline{\underline{\delta}} \quad \text{dans } S(O,R)$$

$$\underline{\underline{\sigma}} = \underset{\approx}{C} : \underline{\underline{\varepsilon}} \quad \text{dans } \mathbb{R}^3 \setminus S(O,R)$$

$$\underset{\approx}{C} = 3k\, \underset{\approx}{J} + 2\mu\, \underset{\approx}{K} \quad , \quad \underset{\approx}{P}^{sphère} = \frac{\alpha}{3k}\underset{\approx}{J} + \frac{\beta}{2\mu}\underset{\approx}{K}$$

$\alpha \in]0,1[$ $\beta \in]\frac{2}{5}, \frac{3}{5}[$

I.c) $\underline{\underline{\sigma}} = -k\theta\, \underline{\underline{\delta}}$ $\alpha = \dfrac{3k}{3k+4\mu}$ $\beta = \dfrac{6}{5}\dfrac{k+2\mu}{3k+4\mu}$

$$\underline{\underline{\varepsilon}}^{sph} = -\left(\frac{\alpha}{3k}\underline{\underline{J}} + \frac{\beta}{2\mu}\underline{\underline{K}}\right) : \left(-k\theta\,\underline{\underline{\delta}}\right)$$

$$= \frac{\alpha}{3k}k\theta\,\underline{\underline{\delta}} = \frac{k\theta}{3k+4\mu}\underline{\underline{\delta}}$$

* $\underline{\xi} = f(r)\,\underline{e_r}$

$$\underbrace{\mathrm{div}\left(\lambda\, \mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{\delta}} + 2\mu\,\underline{\underline{\varepsilon}}\right)}_{\lambda \underline{\nabla}(\mathrm{tr}\,\underline{\underline{\varepsilon}})} = (\lambda + 2\mu)\,\underline{\nabla}(\mathrm{tr}\,\underline{\underline{\varepsilon}})$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{\nabla}}\,\underline{\xi} = \begin{pmatrix} f'(r) & & \\ & \frac{f(r)}{r} & \\ & & \frac{f(r)}{r} \end{pmatrix}_{e_r, e_\theta, e_\varphi}$$

$$\mathrm{div}(\underline{\underline{\varepsilon}}) = \mathrm{div}({}^t\underline{\underline{\nabla}}\,\underline{\xi}) = \left(\frac{\partial}{\partial x_j}\left(\frac{\partial \xi_j}{\partial x_i}\right)\right) e_i = \underline{\nabla}(\mathrm{tr}\,\underline{\underline{\varepsilon}})$$

$\left.\begin{array}{l} r<R \\ r>R \end{array}\right\}$ $\mathrm{div}\,\underline{\underline{\sigma}} = (\lambda+2\mu)\,\underline{\nabla}\,\mathrm{tr}(\underline{\underline{\varepsilon}})$

$$\Rightarrow \underline{\nabla}\,\mathrm{tr}(\underline{\underline{\varepsilon}}) = 0$$

$r<R$ $f' + 2\dfrac{f}{r} = A$ $f(r) = \dfrac{A}{3}r + \cancel{\dfrac{B}{r^2}}$ Pas de divergence en

Donc $A' = 0$ et

$r>R$ $f' + \dfrac{2f}{r} = A'$ $f(r) = \cancel{\dfrac{A'r}{3}} + \dfrac{B'}{r^2}$

+ continuité en R $\Rightarrow$ $\dfrac{A}{3}R = \dfrac{B'}{R^2}$

+ continuité du vecteur contrainte $\underline{\underline{\sigma}}\cdot\underline{n}$

$r<R$ $\sigma_{rr} = \lambda A + 2\mu \times \dfrac{A}{3} - k\theta = (\lambda + \frac{2}{3}\mu)A - k\theta$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \frac{A}{3} & & \\ & \frac{A}{3} & \\ & & \frac{A}{3} \end{pmatrix}$$

$r>R$ $\sigma_{rr} = -4\mu\dfrac{B'}{r^3}$

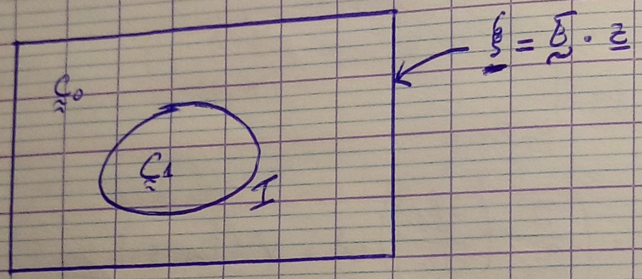
$$\underline{\underline{\varepsilon}} = \begin{pmatrix} -2\frac{B'}{r^3} & & \\ & \frac{B}{r^3} & \\ & & \frac{B}{\ } \end{pmatrix}$$

Contraintes de $\underline{\underline{\sigma}}_{R}$ et
Sur contraintes de σ_{rr}

$$\underbrace{(\lambda + \tfrac{2}{3}\mu)}_{K} A - k\theta = -4\mu \frac{A}{3} \qquad \text{car} \quad \frac{B'}{R^3} = \frac{A}{3}$$

Donc $\quad A(K + \tfrac{4}{3}\mu) = k\theta$

$$\frac{A}{3} = \frac{k\theta}{3K + 4\mu}$$



En se servant des informations de la dernière fois avec un $\underline{\xi} = 0$
on a $\underline{\underline{\varepsilon}}(\underline{z}) = -\underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \underline{\underline{\tau}} + \underline{\underline{E}}$ (Par superposition)

$$\underset{\approx}{C}_1 : \underset{\sim}{\varepsilon} = \underset{\approx}{C}_0 : \underset{\sim}{\varepsilon} + \overbrace{\delta \underset{\approx}{C} : \underset{\sim}{\varepsilon}}^{\underline{\underline{\tau}}\,?} \quad \text{avec} \quad \delta \underset{\approx}{C} = \underset{\approx}{C}_1 - \underset{\approx}{C}_0$$

hypothèse $\underline{\underline{\varepsilon}}$ uniforme dans la région I

$$\underline{\underline{\varepsilon}}^I = -\underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \delta\underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}^I + \underline{\underline{E}}$$

$$\underline{\underline{\varepsilon}}^I = \left(\underline{\underline{\underline{\underline{I}}}} + \underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \delta\underline{\underline{\underline{\underline{C}}}}\right)^{-1} : \underline{\underline{E}}$$

Si I est un ellipsoïde l'hypothèse est vérifiée alors P est uniforme
Si I n'est pas un ellipsoïde alors l'hypothèse n'est pas vérifiée.

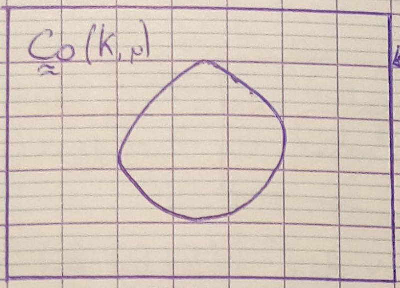
Si I est un ellipsoïde

$$\underline{\underline{\varepsilon}}^I = \left(\underline{\underline{\underline{\underline{I}}}} + \underline{\underline{\underline{\underline{P}}}}^I : \delta\underline{\underline{\underline{\underline{C}}}}\right)^{-1} : \underline{\underline{E}}$$

$$P^I_{ijkl} = -\frac{\partial^2}{\partial z_j \partial z_l}\left(\int_I G^0_{ik}(\underline{z} - \underline{z}')\, dV\right)_{(ij)|(kl)}$$

de la forme mais pas de la taille — Dépend de l'orientation — Dépend de C^0 ... C^1

__Exemple__ : Un pore sphérique dans un milieu infini soumis à une déformation uniaxiale



$\underline{\underline{E}}\cdot\underline{z}$

avec $\underline{\underline{E}} = E\,\underline{e_3}\otimes\underline{e_3}$

$E<0$

On cherche la déformation dans le pore

$$\underline{\underline{\varepsilon}}^I = \left(\mathbb{I} + \mathbb{P}^I : \delta\mathbb{C}\right)^{-1} : \underline{\underline{E}}$$

et $\delta\mathbb{C} = 0 - \mathbb{C}_0$

$$\mathbb{P}^I = \mathbb{P}^{sph} = \frac{\alpha}{3k}\mathbb{J} + \frac{\beta}{2\mu}\mathbb{K} \quad \text{et} \quad \mathbb{C}_0 = 3k\,\mathbb{J} + 2\mu\,\mathbb{K}$$

Donc $\mathbb{I} + \mathbb{P}^I : \delta\mathbb{C} = (1-\alpha)\mathbb{J} + (1-\beta)\mathbb{K}$

D'où $\left(\mathbb{I} - \mathbb{P}^I : \delta\mathbb{C}\right)^{-1} = \frac{1}{1-\alpha}\mathbb{J} + \frac{1}{1-\beta}\mathbb{K}$

$$\mathbb{J} : \underline{\underline{E}} = \frac{E}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{K} : \underline{\underline{E}} = \frac{E}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\varepsilon_{33}^I = \frac{E}{3}\left(\frac{1}{1-\alpha} + \frac{2}{1-\beta}\right)$$

$$\varepsilon_{11}^I = \varepsilon_{22}^I = \frac{E}{3}\left(\frac{1}{1-\alpha} - \frac{1}{1-\beta}\right) = \frac{E}{3}\,\frac{\alpha-\beta}{(1-\alpha)(1-\beta)}$$

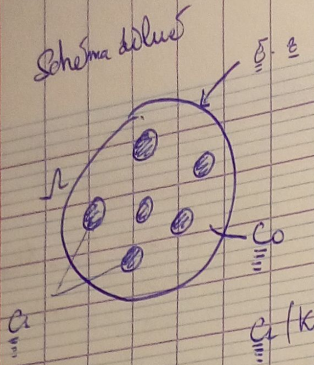
$$\alpha - \beta = \frac{1}{3k+4\mu}\left(3k - \frac{6}{5}(k+2\mu)\right) = \frac{1}{5(3k+4\mu)}(9k - 12\mu)$$

$$= \frac{3}{5}\cdot\frac{1}{3k+4\mu}(3k - 4\mu)$$

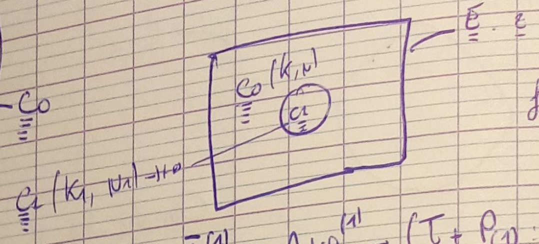
$$3k = \frac{E}{1-2\nu}, \quad 4\mu = \frac{2E}{1+\nu}$$

$$\alpha > \beta \Leftrightarrow \nu > \frac{1}{5}$$

Schéma dilué



$$C^{hom} = \overline{C : A} = C_0 + f_1 (C_1 - C_0) : \bar{A}^{(1)}$$



$f_1 << 1 \quad \bar{\varepsilon}^{(1)}_{réel} = \varepsilon^{I}_{aux}$

$(I + P : \delta C)^{-1} : E$

$$\bar{A}^{(1)} \simeq A_{dil}^{(1)} = (I + P_{(1)} : \delta C)^{-1}$$

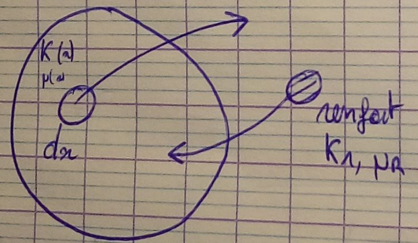
$$C^{hom}_{dil} = C_0 + f_1 (C_1 - C_0) : (I + P_1 : (C_1 - C_0))^{-1}$$

$$3 k^{hom}_{dil} = 3k + f_1 (3(k_1 - k)) \times \frac{1}{1 + \frac{\alpha}{3k}(3(k_1 - k))}$$

$$k^{hom}_{dil} \xrightarrow[k_1 \to +\infty]{} k + f_1 \frac{k}{\alpha} = k\left(\frac{f_1}{\alpha} + 1\right)$$

De même $\mu^{hom}_{dil} = \mu \left(1 + \frac{f_1}{\beta}\right)$

Schéma différentiel (plusieurs itérations du schéma dilué)



$$C^{hom}(\alpha + d\alpha) = C^{hom}(\alpha) + d\alpha \left(C_R - C^{hom}(\alpha)\right) : \left(1 + P_1(\alpha) : (C_R - C^{hom}(\alpha))\right)^{-1}$$

$$k^{hom}(\alpha + d\alpha) = k^{hom}(\alpha) + d\alpha \left(k_R - k^{hom}(\alpha)\right) \times \frac{1}{1 + \frac{\alpha(\alpha)}{k^{hom}(\alpha)}\left(k_R - k^{hom}(\alpha)\right)}$$

$$\begin{cases} \dfrac{d k^{hom}}{d\alpha} = \dfrac{k^{hom}(\alpha)}{\alpha(\alpha)} \\ \dfrac{d \mu^{hom}}{d\alpha} = \dfrac{\mu^{hom}(\alpha)}{\beta(\alpha)} \end{cases}$$

et $k^{hom}(0) = k_0$; $\mu^{hom}(0) = \mu_0$

$$\frac{dk}{d\mu} = \frac{k}{\mu} \times \frac{\beta}{\alpha} = \frac{k}{\mu} \times \frac{6}{5} \times \frac{k + 2\mu}{3k} = \frac{2}{5}\frac{k}{\mu} + \frac{4}{5}$$

Pour une solution particulière

on pose $k = \alpha\mu \rightarrow \frac{4}{3}\mu$

$$\alpha = \frac{5}{2}\alpha + \frac{4}{5}$$

$$\alpha = \frac{4}{3}$$

Solution homogène

$$\frac{dk}{d\mu} = \frac{2}{5}\frac{k}{\mu} \qquad \ln\left(\frac{k}{k_0}\right) = \frac{2}{5}\ln\left(\frac{\mu}{\mu_0}\right) \qquad \frac{k}{k_0} = \left(\frac{\mu}{\mu_0}\right)^{\frac{2}{5}}$$

$$k = \frac{4}{3}\mu + \left(k_0 - \frac{4}{3}\mu_0\right)\left(\frac{\mu}{\mu_0}\right)^{\frac{2}{5}}$$

dans la limite $f_1 \rightarrow 1$ on aura $k, \mu >> k_0, \mu_0$

et donc $k \sim \frac{4}{3}\mu$

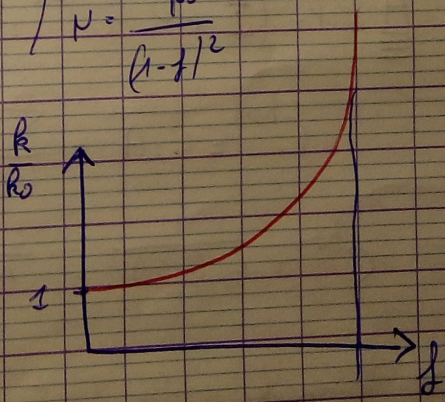
$\rightarrow \alpha, \beta \rightarrow \frac{1}{2}$

$$df_1 = -f_1\,dx + dx = (1-f_1)\,dx \Rightarrow \boxed{dx = \frac{df_1}{1-f_1}}$$

Donc $\dfrac{dk}{k} = 2\dfrac{df}{1-f}$

$$\ln\left(\frac{k}{k_0}\right) = -2\ln(1-f) \Leftrightarrow \begin{cases} k = \dfrac{k_0}{(1-f)^2} \\ \mu = \dfrac{\mu_0}{(1-f)^2} \end{cases}$$

$$K_{dil} = k_0\left(1 + \frac{f_1}{2}\right)$$



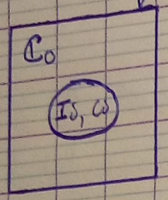
$$\underline{\dot{\overline{\varepsilon}}}^{0}_{hom} \neq \underline{\overline{\varepsilon}}^{0}_{réel} \qquad \underline{\overline{\varepsilon}}^{(1)}_{réel} \approx \underline{\overline{\varepsilon}}^{(1)}_{hom}$$

Schéma de Mori Tanaka

$\begin{cases} \text{une matrice} \rightarrow \text{phase n°0} \\ n \text{ phases inclusionnaires} \end{cases}$

problème(s) auxiliaire(s)

$\underline{\xi} = \underline{\underline{E_0}} \cdot \underline{x}$



$1 \leq i \leq n$

$$\overline{\underline{\underline{\varepsilon}}}^{m}_{réel} = \overline{\underline{\underline{\varepsilon}}}^{m}_{aux} = \underline{\underline{E_0}} \quad \text{et} \quad \overline{\underline{\underline{\varepsilon}}}^{i}_{réel} = \underline{\underline{\varepsilon}}^{I_i}_{aux} \quad (n \text{ relations})$$

$$\text{or} \quad \underline{\underline{\varepsilon}}^{I_i}_{aux} = \Big(\mathbb{I} + \underline{\underline{\underline{\underline{P}}}}_0^{I_i} : \underbrace{\delta\mathbb{C}}_{c_i - c_0}\Big)^{-1} : \underline{\underline{E_0}}$$

$$\overline{\underline{\underline{\varepsilon}}}_{réel} = \underline{\underline{E}}$$

$$f_0 \underline{\underline{E_0}} + \sum_{i=1}^{n} f_i (\mathbb{I} + \mathbb{P}_i : \delta \mathbb{C}_i)^{-1} : \underline{\underline{E_0}} = \underline{\underline{E}}$$

$$\underbrace{\sum_{i=0}^{n} f_i (\mathbb{I} + \mathbb{P}_i : \delta \mathbb{C}_i)^{-1}}_{\overline{(\mathbb{I} + \mathbb{P}_i : \delta \mathbb{C}_i)^{-1}}} : \underline{\underline{E_0}} = \underline{\underline{E}}$$

Donc $\underline{\underline{E_0}} = \overline{(\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}}^{-1} : \underline{\underline{E}}$

$$\overline{\underline{\underline{\underline{\underline{A}}}}}^{(i)}_{réel} = (\mathbb{I} + \mathbb{P}_i \delta \mathbb{C}_i)^{-1} : \overline{(\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}}^{-1} \qquad \forall i \in [\![0, n]\!]$$

$$\mathbb{C}^{hom} = \overline{\mathbb{C} : \mathbb{A}} = \sum_{i=0}^{n} f_i \mathbb{C}_i : \overline{\mathbb{A}}^{(i)} \underset{MT}{=} \Big[\sum_{i=0}^{n} f_i \mathbb{C}_i : (\mathbb{I} + \mathbb{P}_i \delta \mathbb{C}_i)^{-1}\Big] : \overline{(\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}}^{-1}$$

$$\mathbb{C}^{hom} = \overline{\mathbb{C} : (\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}} : \overline{(\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}}^{-1}$$

Dans le cas du schéma dilué on aurait eu :

$$\mathbb{C}^{hom}_{dil} = \mathbb{C}_0 : \Big(\mathbb{I} - \sum_{i=1}^{n} f_i \overline{\mathbb{A}}^{(i)}\Big) + \sum_{i=1}^{n} f_i \, \mathbb{C}_i : \overline{\mathbb{A}}^{(i)}$$

$$= \mathbb{C}_0 + \sum_{i=1}^{n} f_i (\mathbb{C}_i - \mathbb{C}_0) : (\mathbb{I} + \mathbb{P}_i \delta \mathbb{C}_i)^{-1}$$

$$= \mathbb{C}_0 + \overline{\delta\mathbb{C} : (\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}}$$

$$\mathbb{C}^{hom}_{MT} = \mathbb{C}_0 + \overline{\delta\mathbb{C} : (\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}} : \overline{(\mathbb{I} + \mathbb{P}\delta\mathbb{C})^{-1}}^{-1}$$

Application du schéma de MT avec des renforts infiniment rigides plongés dans une matrice

$$\underline{\underline{P_0}}^{\wedge} = \frac{\alpha_0}{3k_0}\underline{\underline{J}} + \frac{\beta_0}{2\mu_0}\underline{\underline{K}}$$

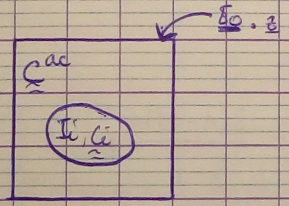
$$3k^{hom} = \left[(1-f)\cdot 3k_0 + f \times 3k_1 \frac{1}{1+\frac{\alpha_0}{3k_0}3(k_1-k_0)}\right] \times \frac{1}{1-f+f\times\frac{1}{1+\frac{\alpha_0}{3k_0}(k_1-k_0)}}$$

quand $K_1 \to +\infty$

$$k^{hom} = \frac{1}{1-f}\left((1-f)k_0 + f\frac{k_0}{\alpha_0}\right) = k_0 + \frac{f}{1-f}\frac{k_0}{\alpha_0}$$

$$\mu^{hom} = \mu_0 + \frac{f}{1-f}\frac{\mu_0}{\beta_0}$$

Schéma Auto cohérent $\quad$ m phases $\{1, \dots, m\}$ $\quad$ (schéma pour les polycristaux)



$$\overline{\varepsilon}^{i}_{réel\,(Ac)} = \underline{\underline{\varepsilon}}^{I_i} = (I + P_i^{Ac}\delta C_i)^{-1} : \underline{\underline{E_0}} = (I + P_i^{Ac}\delta C_i)^{-1} : \overline{(I+P\delta C)^{-1}}^{-1} : E$$

$$\sum_{i=1}^{m} f_i \overline{\varepsilon}^{(i)} = \underline{\underline{E}} \Longrightarrow \overline{(I+P\delta C)^{-1}} : \underline{\underline{E_0}} = \underline{\underline{E}}$$

$$C^{hom} = \overline{C:(I+P\delta C)^{-1}} : \overline{(I+P\delta C)^{-1}}^{-1}$$

si $\forall i \in \{1, \dots, m\}$ $\underline{\underline{P}}^{i}_{Ac} = \underline{\underline{P}}_{Ac}$ alors $\overline{(I+P\delta C)^{-1}} = I$ et donc $C^{Ac} = \overline{C:(I+P\delta C)^{-1}}$

$$C^{Ac} = \overline{C:(I+P\delta C)^{-1}} : \overline{(I+P\delta C)^{-1}}^{-1}$$

$$\overline{C^{Ac}:(I+P\delta C)^{-1}} = \overline{C:(I+P\delta C)^{-1}}$$

$$0 = \overline{\delta C:(I+P\delta C)^{-1}} \Longrightarrow \overline{(P\delta C):(I+P\delta C)^{-1}} = 0$$

$$\overline{[(\underline{\underline{I}}+P\delta C) - \underline{\underline{I}}] : \overline{(\underline{\underline{I}}+P\delta C)^{-1}}} = 0$$

$$C^{AC} = \overline{C : (\underline{\underline{I}}+P\delta C)^{-1}} : \overline{(\underline{\underline{I}}+P\delta C)^{-1}}^{\,-1}$$

$$\underset{\equiv}{P}_p^{AC} = \underset{\equiv}{P}_1^{AC} = \underset{\equiv}{P}^{AC} = \frac{\alpha_{AC}}{3k_{AC}} \underset{\equiv}{J} + \frac{\beta_{AC}}{2\mu_{AC}} \underset{\equiv}{K}$$

$$k^{AC} = (1-f)\, k_S \times \frac{1}{1+\frac{\alpha^{AC}}{k^{AC}}(k_S - k_{AC})} \qquad \mu^{AC} = (1-f)\,\mu_S \times \frac{1}{1+\frac{\beta^{AC}}{\mu^{AC}}(\mu_S - \mu_{AC})}$$

$k_S \gg \mu_S$ $\qquad k^{AC} = (1-f)\dfrac{k^{AC}}{\alpha^{AC}} \implies \alpha^{AC} = (1-f) = \dfrac{3k^{AC}}{3k^{AC}+4\mu^{AC}}$

$$k^{AC} = \frac{4}{3} \times \frac{1-f}{f}\, \mu^{AC}$$

$$\beta^{AC} = \frac{6}{5}\,\frac{k^{AC}+2\mu^{AC}}{3k^{AC}+4\mu^{AC}}$$

$\beta^{AC} = \dfrac{2+f}{5}$ $\quad$ et donc $\mu_{AC} = \dfrac{3(1-2f)}{3-f}\,\mu_S$

⚠ $\mu_{AC} \leq 0$ si $f > \frac{1}{2}$

homogénéiser un composite $\qquad$ (Levin)

Maintenant on a : $\underset{=}{\sigma}(\underline{z}) = \underset{\equiv}{C}(\underline{z}) : \underset{=}{\varepsilon}(\underline{z}) + \boxed{\underset{=}{\sigma_0}(\underline{z})}$

si $\underset{=}{\sigma_0}(\underline{z}) = 0 \longrightarrow \underset{=}{\Sigma} = \overline{\underset{=}{\sigma}} = \overline{\underset{\equiv}{C} : \underset{\equiv}{A}} : \underset{=}{E} \qquad (\underset{=}{E} = \overline{\underset{=}{\varepsilon}})$

Problèmes sans précontraintes (ie $\underset{=}{\sigma_0}(\underline{z}) = 0$)

$$\begin{cases} \underset{=}{\varepsilon}'(\underline{z}) = \underset{\equiv}{A}(\underline{z}) : \underset{=}{E}' \\ \underset{=}{\sigma}'(\underline{z}) = \underset{\equiv}{C} : \underset{\equiv}{A} : \underset{=}{E}' \end{cases}$$

$$\overline{\underset{=}{\sigma}(\underline{z}) : \underset{=}{\varepsilon}'(\underline{z})} = \overline{\underset{=}{\varepsilon}'(\underline{z}) : \underset{\equiv}{C}(\underline{z}) : \underset{=}{\varepsilon}(\underline{z})} + \overline{\underset{=}{\sigma_0}(\underline{z}) : \underset{=}{\varepsilon}'(\underline{z})}$$

Lemme de Hill $\Downarrow$ (car il faut SA et CA) pour appliquer le lemme de Hill

$\underset{=}{\Sigma} = \overline{\underset{=}{\sigma}} \qquad \overline{\underset{=}{\varepsilon}'} = \underset{=}{E}'$

⚠ ε' champ à divergence nul mais $\mathbb{C}\,\varepsilon$ ne l'est pas forcément.

Lemme de Hill

Donc $\overline{\underline{\underline{\varepsilon}}'(z) : \underline{\underline{\mathbb{C}}}(z) : \underline{\underline{\varepsilon}}(z)} = \underbrace{\overline{\underline{\underline{\mathbb{C}}} : \underline{\underline{\varepsilon}}'}}_{\underline{\underline{E}}' : \underline{\underline{\underline{\underline{\mathbb{C}}}}}^{hom}} : \underbrace{\overline{\underline{\underline{\varepsilon}}}}_{\underline{\underline{E}}}$

$\overline{\underline{\underline{\sigma}}_0(z) : \underline{\underline{\varepsilon}}'(z)}$ $\quad \underline{\underline{\varepsilon}}'(z) = \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{E}}'$ et $\underline{\underline{E}}'$ est constant donc on peut le sortir de la moyenne.

$\quad \|$

$\overline{\underline{\underline{\sigma}}_0(z) : \underline{\underline{\underline{\underline{A}}}}} : \underline{\underline{E}}'$

$(\forall E')$ D'où $\underline{\underline{\Sigma}} : \underline{\underline{E}}' = \left(\underline{\underline{\underline{\underline{\mathbb{C}}}}}^{hom} : \underline{\underline{E}} + \overline{\underline{\underline{\sigma}}_0 : \underline{\underline{\underline{\underline{A}}}}}\right) : \underline{\underline{E}}'$

Donc $\underline{\underline{\Sigma}} = \underbrace{\underline{\underline{\underline{\underline{\mathbb{C}}}}}^{hom}}_{\overline{\mathbb{C} : A}} : \underline{\underline{E}} + \overline{\underline{\underline{\sigma}}_0 : \underline{\underline{\underline{\underline{A}}}}}$

Matériau biphasé

$$\left.\begin{array}{l} (1)\ f_1 ;\ \underline{\underline{\sigma}} = \underline{\underline{\mathbb{C}}}_1 : \underline{\underline{\varepsilon}} - \underline{\underline{k}}_1 \theta \\ (2)\ f_2 ;\ \underline{\underline{\sigma}} = \underline{\underline{\mathbb{C}}}_2 : \underline{\underline{\varepsilon}} - \underline{\underline{k}}_2 \theta \end{array}\right\} \quad \underline{\underline{\Sigma}} = \overline{\mathbb{C} : A} : \underline{\underline{E}} - \underline{\underline{k}}^{hom} \theta$$

Montrer que $\underline{\underline{k}}^{hom}(\underline{\underline{k}}_1, \underline{\underline{k}}_2, \mathbb{C}^{hom}, \mathbb{C}_1, \mathbb{C}_2)$

$$\underline{\underline{k}}^{hom} = f_1 \underline{\underline{k}}_1 : \overline{A}^1 + f_2 \underline{\underline{k}}_2 : \overline{A}^2 = \underline{\underline{k}}_1 + f_2 (\underline{\underline{k}}_2 - \underline{\underline{k}}_1) : \overline{A}^2$$

$$\underline{\underline{\underline{\underline{\mathbb{C}}}}}^{hom} = f_1 \mathbb{C}_1 : \overline{A}^1 + f_2 \underline{\underline{\underline{\underline{\mathbb{C}}}}}_2 : \overline{A}^2 = \mathbb{C}_1 + f_2 (\mathbb{C}_2 - \mathbb{C}_1) : \overline{A}^2$$

D'où $\underline{\underline{k}}^{hom} = \underline{\underline{k}}_1 + \cancel{f_2} (\underline{\underline{k}}_2 - \underline{\underline{k}}_1) : \frac{1}{\cancel{f_2}} (\underline{\underline{\mathbb{C}}}_2 - \underline{\underline{\mathbb{C}}}_1)^{-1} : (\underline{\underline{\underline{\underline{\mathbb{C}}}}}^{hom} - \underline{\underline{\mathbb{C}}}_1)$

Dans le cas isotrope $\underline{\underline{k}}_i = k_i \underline{\underline{\delta}}$

$$k^{hom} = k_1 + (k_2 - k_1) \times \frac{K^{hom} - K_1}{K_2 - K_1}$$

$$\frac{k^{hom} - k_1}{k_2 - k_1} = \frac{K^{hom} - K_1}{K_2 - K_1}$$

Rappel de la polarisation

Ω élasticité uniforme $\underline{\underline{c}}_0$

$$\underline{\underline{\sigma}} = \underline{\underline{c}}_0 : \underline{\underline{\varepsilon}} + \underline{\underline{\tau}}(\underline{z})$$

(déplacement) $\underline{\xi} = 0 \; (\partial \Omega$

$\mathrm{div}\, \underline{\underline{\sigma}} = 0$

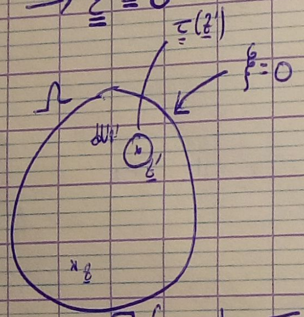
$\underline{\underline{\varepsilon}} = (\underline{\nabla}\, \underline{\xi})_{sym}$

$I \subset \Omega = \mathbb{R}^3$ $\quad \tau(\underline{z}) = \underline{\underline{\tau}}^I \chi^I(\underline{z})$

$\underline{\underline{\varepsilon}}(\underline{z}) = -\underline{\underline{\underline{\underline{P}}}}(\underline{z}) : \underline{\underline{\tau}}^I$ avec $P_{ijkl}(\underline{z}) = -\dfrac{\partial^2}{\partial z_j \partial z_l} \displaystyle\int_I G^0_{ik}(\underline{z} - \underline{z}')\, dV_{\underline{z}'} \Big|_{(i,j)(k,l)}$ f° du matériau $\underline{\underline{c}}^0$

Cas borné (Ω borné)

$\underline{\underline{\tau}}$ uniforme $\rightarrow \underline{\underline{\varepsilon}} = 0$



$$d\underline{\underline{\varepsilon}}(\underline{z}) = -\underline{\underline{\underline{\underline{\Gamma}}}}(\underline{z}, \underline{z}') : \underline{\underline{\tau}}(\underline{z}')$$

$$\underline{\underline{\varepsilon}}(\underline{z}) = -(\Gamma * \underline{\underline{\tau}})(\underline{z}) \qquad \Gamma * \underline{\underline{\tau}}(\underline{z}) = \int \underline{\underline{\underline{\underline{\Gamma}}}}(\underline{z}, \underline{z}') : \underline{\underline{\tau}}(\underline{z}')\, dV_{\underline{z}'}$$

si $\Omega = \mathbb{R}^3$ $\underline{\underline{\underline{\underline{\Gamma}}}}(\underline{z} - \underline{z}') : \underline{\underline{\tau}}(\underline{z}')$

- $\Gamma * \overline{\overline{\tau}} = 0$
- $\overline{\Gamma * \tau} = 0$

Cas infini $\Omega = R^3$ $\quad \varepsilon \to 0$ à l'∞

$$dP^{\underline{z}'}_{ijkl}(\underline{z}) = -\frac{\partial^2 G_{ik}}{\partial z_j \partial z_l}(\underline{z} - \underline{z}')\, dV_{\underline{z}'} \Big|_{(ij)(kl)}$$

Donc $\Gamma^{\infty,0}_{ijkl}(\underline{z}) = -\dfrac{\partial^2 G^0_{ik}}{\partial z_j \partial z_l}(\underline{z}) \Big|_{(ij)(kl)}$ car $-d\underline{\underline{\underline{\underline{P}}}}^{\underline{z}'}(\underline{z}) : \underline{\underline{\tau}}(\underline{z}') = d\underline{\underline{\varepsilon}}(\underline{z})$

$G(r) \sim \frac{1}{r}$

$$\underline{\underline{\varepsilon}}(\underline{z}) = -\int_{\mathbb{R}^3 \setminus B(\underline{z}, \varepsilon)} \underline{\underline{\underline{\underline{\Gamma}}}}^{\infty,0}(\underline{z} - \underline{z}') : \underline{\underline{\tau}}(\underline{z}')\, dV_{\underline{z}'} - \underline{\underline{\underline{\underline{P}}}}^{\text{sphère}} : \underline{\underline{\tau}}(\underline{z})$$

$\sim$ sphérique

Valeur principale en prenant la limite quand $\varepsilon \to 0$

$$-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus \cancel{B(z,\varepsilon)}} \underline{\underline{\underline{\underline{\Gamma}}}}^{\infty,0}(\underline{z}-\underline{z}') : \underline{\underline{\tau}}(\underline{z}')\,dV_{z'}$$

SS à l'∞ C° type Hashin $\quad \underline{\xi} = \underline{\underline{E}} \cdot \underline{z}$ $\qquad \underline{\underline{\varepsilon}} = \underline{\underline{E}} - \Gamma^{\infty} * \underline{\underline{\tau}}$

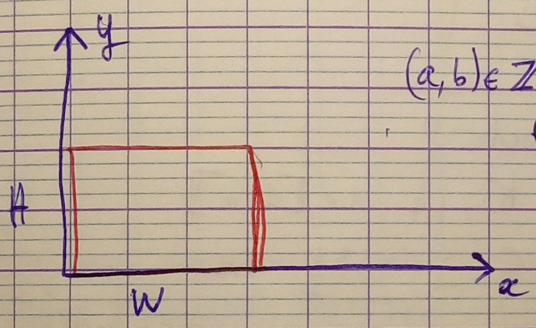
1) Choix d'une élasticité de référence $\mathbb{C}_0$

2) $\underline{\underline{\underline{\underline{C}}}}(\underline{z}) : \underline{\underline{\varepsilon}}(\underline{z}) = \underline{\underline{\underline{\underline{C}}}}_0 : \underline{\underline{\varepsilon}}(\underline{z}) + \underbrace{(\underline{\underline{\underline{\underline{C}}}}(\underline{z}) - \underline{\underline{\underline{\underline{C}}}}_0) : \underline{\underline{\varepsilon}}(\underline{z})}_{\underline{\underline{\tau}}(\underline{z})}$

Solution :

$$\underline{\underline{\varepsilon}}(\underline{z}) = \underline{\underline{E}} - \left(\Gamma^{\infty,0} * (\delta \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}})\right)(\underline{z})$$

$$\underline{\underline{\varepsilon}} + \Gamma^{\infty,0} * (\delta C : \underline{\underline{\varepsilon}}) = \underline{\underline{E}} \qquad (L.S) \text{ ou } (L.S.D)$$



$(a,b) \in \mathbb{Z}^2$

$$\underline{k}_{a,b} = 2\pi\left(\frac{a}{W}\underline{e_x} + \frac{b}{H}\underline{e_y}\right)$$

Décomposition en série de Fourier

$$u(\underline{z}) = \sum_{a=-\infty}^{+\infty}\sum_{b=-\infty}^{+\infty} \hat{u}(\underline{k}_{ab}) \exp(i\underline{k}_{a,b}\cdot\underline{z})$$

avec $\quad \hat{u}(\underline{k}_{a,b}) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(\underline{z}) \exp(-i\underline{k}_{a,b}\cdot\underline{z})\, dV_z$

Algorithme de Moulinec - Suquet (2001)

$$\underline{\underline{\varepsilon}}^{n+1} = \underline{\underline{E}} - \Gamma^0 * (\delta C : \varepsilon^n) \quad \text{avec } \underline{\underline{\varepsilon}}^0 = \underline{\underline{E}}$$

1 - calculer $\tau^n = (\mathbb{C}(\underline{z}) - \mathbb{C}_0) : \underline{\underline{\varepsilon}}^n(\underline{z})$ (Espace Réel)

2 - calculer $\hat{\tau}^n(\underline{k}_{a,b}) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \underline{\underline{\tau}}^n(\underline{z}) \exp(-i\underline{k}_{a,b}\cdot\underline{z})\, dV_z$

3 - calculer $\hat{\varepsilon}^{n+1}(\underline{k}_{a,b}) = \underline{\underline{E}} \qquad (a,b) = (0,0)$

$\hat{\varepsilon}^{n+1}(\underline{k}_{a,b}) = -\hat{\underline{\underline{\underline{\underline{\Gamma}}}}}^0(\underline{k}_{a,b}) : \hat{\underline{\underline{\tau}}}^n(\underline{k}_{a,b}) \qquad (a,b) \neq (0,0)$

4 - Calculer $\varepsilon^{n+1}(\underline{z}) = \displaystyle\sum_a \sum_b \hat{\varepsilon}^{n+1}(\underline{k}_{a,b}) \exp(-i\underline{k}_{a,b})$

Ω v.er $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{c}}}}(\underline{x}) : \underline{\underline{\varepsilon}}$

énergie élastique

$\underline{\underline{W}} = \frac{1}{2} \underline{\underline{E}} : \underline{\underline{\underline{\underline{C}}}}^{hom} : \underline{\underline{E}} \, |\Omega| = \int_\Omega \frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} \, dV_x$

Soit t un paramètre d'élasticité

MACRO $\frac{\partial W}{\partial t} = \frac{1}{2} \underline{\underline{E}} : \frac{\partial C^{hom}}{\partial t} : \underline{\underline{E}} \, |\Omega|$

micro $\frac{\partial W}{\partial t} = \int_\Omega \underline{\underline{\varepsilon}}_{,t} : \underbrace{\underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}}_{\underline{\underline{\sigma}}} \, dV + \int_\Omega \frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{c}}}}_{,t} : \underline{\underline{\varepsilon}} \, dV$

$\underline{\underline{\varepsilon}}_{,t} = \left(\underline{\nabla}(\underline{\xi}_{,t})\right)_{sym}$ $\quad \partial\Omega : \begin{cases} \underline{\xi}(\underline{z}) = \underline{\underline{E}} \cdot \underline{z} \\ \underline{\xi}_{,t}(\underline{z}) = 0 = \underline{\underline{0}} \cdot \underline{z} \end{cases}$

On peut appliquer le lemme de Hill sur la 1ère intégrale

$= |\Omega| \overline{\underline{\underline{\sigma}}} : \overline{\underline{\underline{\varepsilon}}_{,t}} = 0$, $\overline{\underline{\underline{\varepsilon}}_{,t}} = 0$

$\int_{\Omega_r} \varepsilon_d : \varepsilon_d \, dV$ $\quad$ Moment d'ordre 2

Ω m-phases isotropes $\quad k_r, \mu_r \quad r \in [\![1, m]\!]$

On $\underline{\underline{\underline{\underline{c}}}}_r = 3k_r \underline{\underline{\underline{\underline{J}}}} + 2\mu_r \underline{\underline{\underline{\underline{K}}}}$; $\underline{\underline{\underline{\underline{c}}}}_{j,\mu_r} = 2\underline{\underline{\underline{\underline{K}}}}$ si $j = r$

$= \underline{\underline{\underline{\underline{0}}}}$ sinon

$\int \frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{c}}}}_{,t} : \underline{\underline{\varepsilon}} \, dV = \sum_{j=1}^{m} \int_{\Omega_j} \frac{1}{2} \underline{\underline{\varepsilon}} : \frac{\partial}{\partial t}(\underline{\underline{\underline{\underline{c}}}}_j) : \underline{\underline{\varepsilon}} \, dV$

déviatorique

$= \int_{\Omega_r} \frac{1}{2} \underline{\underline{\varepsilon}} : 2\underline{\underline{\underline{\underline{K}}}} : \underline{\underline{\varepsilon}} \, dV = \int_{\Omega_r} \varepsilon_d : \varepsilon_d \, dV = 2 f_r \overline{\frac{1}{2} \varepsilon_d : \varepsilon_d}^{(r)} = \frac{1}{2} \underline{\underline{E}} : \frac{\partial C^{hom}}{\partial \mu_r} : \underline{\underline{E}}$

$\frac{1}{2} \underline{\underline{\Sigma}} : \underline{\underline{\underline{\underline{S}}}}^{hom} : \frac{\partial C^{hom}}{\partial \mu_r} : \underline{\underline{\underline{\underline{S}}}}^{hom} : \underline{\underline{\Sigma}}$ $\quad \frac{\partial C^{hom}}{\partial \mu_r} : S^{hom} + C^{hom} : \frac{\partial S^{hom}}{\partial \mu_r} = 0$

$C^{hom} : S^{hom} = \mathbb{I}$

d'où $-\frac{1}{2} \underline{\underline{\Sigma}} : \frac{\partial S^{hom}}{\partial \mu_r} : \underline{\underline{\Sigma}} = \frac{1}{2} \underline{\underline{E}} : \frac{\partial C^{hom}}{\partial \mu_r} : \underline{\underline{E}}$

Ω m-phases élastiques isotropes $(k_r, \mu_r)_{r=1,\dots,m}$

phase r : critère de rupture $\frac{1}{2} \underline{\underline{\sigma}}_d : \underline{\underline{\sigma}}_d \leq k_r^2$ (local)

$\Downarrow$

$\overline{\frac{1}{2} \sigma_d : \sigma_d}^r \leq k_r^2$ (global)

$\sigma_d = \text{dev}(\underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}) = (3k_r \underline{\underline{\underline{\underline{J}}}} : \underline{\underline{\varepsilon}} + 2\mu_r \underline{\underline{\underline{\underline{K}}}} : \underline{\underline{\varepsilon}}) = 2\mu_r \underline{\underline{\varepsilon}}_d$

$\overline{\frac{1}{2} \sigma_d : \sigma_d}^{(r)} = 4\mu_r^2 \overline{\frac{1}{2} \varepsilon_d : \varepsilon_d}^{(r)} \leq k_r^2$

$\forall r \quad -\frac{\mu_r^2}{f_r}\left(\underline{\underline{\Sigma}} : \frac{\partial S^{hom}}{\partial \mu_r} : \underline{\underline{\Sigma}}\right) \leq k_r^2$

Cas du biphasé solide / pore

$\Sigma_m = \frac{1}{3} tr\, \underline{\underline{\Sigma}} \qquad \Sigma_d = \sqrt{\frac{1}{2} \underline{\underline{\Sigma}}_d : \underline{\underline{\Sigma}}_d}$

porosité f et la phase solide est incompressible $(k_s \to \infty)$

$k^{hom} = \mu\, K(f)$

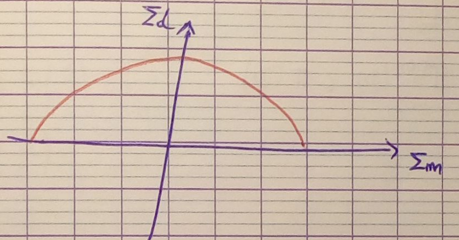
$\mu^{hom} = \mu\, \pi(f)$

$S^{hom} = \frac{1}{\mu}\left(\frac{1}{3K(f)}\underline{\underline{J}} + \frac{1}{2\pi(f)}\underline{\underline{K}}\right) \qquad \frac{\partial S^{hom}}{\partial \mu} =$

$\underline{\underline{\Sigma}} : \left(\frac{1}{3K}\underline{\underline{J}} + \frac{1}{2\pi}\underline{\underline{K}}\right) : \underline{\underline{\Sigma}} \leq (1-f) k_s^2$

$\underline{\underline{J}} : \underline{\underline{\Sigma}} = \left(\frac{1}{3} tr\, \underline{\underline{\Sigma}}\right) \underline{\underline{1}}$

$$\boxed{\frac{\Sigma_m^2}{K(f)} + \frac{\Sigma_d^2}{\pi(f)} \leq (1-f) k_s^2}$$



Résistance des matériaux hétérogènes

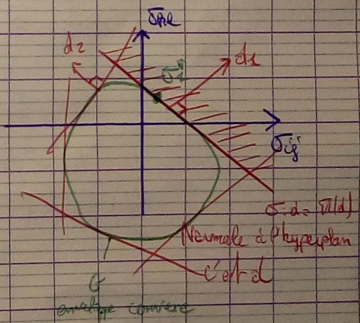
Critère d'admissibilité $f(\underline{\underline{\sigma}}) \leq 0 \qquad f$ convexe

$\Updownarrow$

$\underline{\underline{\sigma}} \in G = f^{-1}(\mathbb{R}^-) \iff \forall \underline{\underline{d}} \quad \underline{\underline{\sigma}} : \underline{\underline{d}} \leq \pi(\underline{\underline{d}})$

fonction d'appui de G : $\pi(\underline{\underline{d}}) = \sup(\underline{\underline{\sigma}} : \underline{\underline{d}},\ \underline{\underline{\sigma}} \in G)$

↳ convexe et positivement homogène de degré 1



$t > 0 \quad \pi(t\underline{\underline{d}}) = t\pi(\underline{\underline{d}})$

formule d'Euler $\quad \frac{\partial \pi}{\partial \underline{\underline{d}}}(t\underline{\underline{d}}) : \underline{\underline{d}} = \pi(\underline{\underline{d}})$

$\frac{\partial \pi}{\partial \underline{\underline{d}}}(\underline{\underline{d}}) : \underline{\underline{d}} = \pi(\underline{\underline{d}})$

$\underline{\underline{\sigma}}^* \in G \quad \underline{\underline{\sigma}}^* : \underline{\underline{d}} = \pi(\underline{\underline{d}})$

f^{o} convexe est au dessus de sa tangente

$\pi(\underline{\underline{d}}') \geq \frac{\partial \pi}{\partial \underline{\underline{d}}}(\underline{\underline{d}}) : (\underline{\underline{d}}' - \underline{\underline{d}}) + \pi(\underline{\underline{d}})$

$\pi(\underline{\underline{d}}') \geq \frac{\partial \pi}{\partial \underline{\underline{d}}}(\underline{\underline{d}}) : \underline{\underline{d}}'$

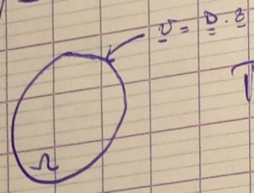
Donc $\in G$ car $\underline{\underline{\sigma}} : \underline{\underline{d}} \leq \pi(\underline{\underline{d}})$

Donc $\boxed{\underline{\underline{\sigma}}^* = \frac{\partial \pi}{\partial \underline{\underline{d}}}(\underline{\underline{d}})}$

regarder au sens des distributions (livre biblio)

v.e.r Ω $(\forall \underline{z} \in \Omega)$ $G(\underline{z}) \rightarrow \Pi(\underline{\underline{d}}, \underline{z})$ /

$\underline{\underline{\Sigma}} \in G^{hom} = \{ \underline{\underline{\Sigma}}, \exists \underline{\underline{\sigma}}(\underline{z}), \underline{\underline{\Sigma}} = \overline{\underline{\underline{\sigma}}}, \operatorname{div} \underline{\underline{\sigma}} = \underline{0}, \forall \underline{z} \in \Omega \ \underline{\underline{\sigma}}(\underline{z}) \in G(\underline{z}) \}$



Π^{hom} ?

$$\Pi(\underline{\underline{D}}) = \inf_{\underline{v} \text{ C.A.D}} \frac{1}{|\Omega|} \int_\Omega \Pi(\underline{\underline{d}}) \, dV$$

Soit $\underline{\underline{\Sigma}} \in G^{hom}$

$$\underline{\underline{\Sigma}} : \underline{\underline{D}} \overset{\text{Th Hill}}{=} \frac{1}{|\Omega|} \int_\Omega \underbrace{\underline{\underline{\sigma}} : \underline{\underline{d}}}_{\leq \Pi(\underline{\underline{d}})} \, dV \leq \frac{1}{|\Omega|} \int_\Omega \Pi_{\underline{z}}(\underline{\underline{d}}) \, dV$$

$$\boxed{\underline{\underline{\Sigma}} : \underline{\underline{D}} \leq \Pi(\underline{\underline{D}})}$$

Donc $\Pi^{hom}(\underline{\underline{D}}) \leq \Pi(\underline{\underline{D}})$

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} = \underline{0} \\ \underline{\underline{\sigma}} = \frac{\partial \pi}{\partial \underline{\underline{d}}}(\underline{\underline{d}}, \underline{z}) \\ \underline{\underline{d}} = (\nabla \underline{v})_{sym} \\ \underline{v} = \underline{\underline{D}} \cdot \underline{z} \quad (\partial \Omega) \end{cases}$$

$\underline{\underline{\Sigma}} = \overline{\underline{\underline{\sigma}}} \Rightarrow \underline{\underline{\Sigma}} \in G^{hom}$

Donc $\underline{\underline{\Sigma}} : \underline{\underline{D}} \leq \Pi^{hom}(\underline{\underline{D}})$

lemme de Hill

$\overline{\pi(\underline{\underline{d}})} = \overline{\underline{\underline{\sigma}} : \underline{\underline{d}}} = \underline{\underline{\Sigma}} : \underline{\underline{D}} \leq \Pi^{hom}(\underline{\underline{D}})$

Donc $\underline{\Pi}(\underline{\underline{D}}) \leq \Pi^{hom}(\underline{\underline{D}})$ d'où $\overline{\Pi}(\underline{\underline{D}}) = \Pi^{hom}(\underline{\underline{D}})$

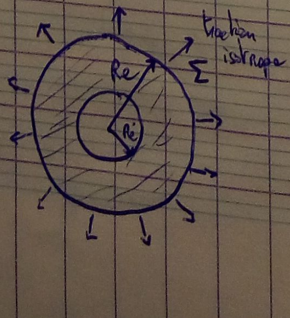
Modèle de Gurson

porosité f, phase solide : von Mises $f(\underline{\underline{\sigma}}) = \frac{3}{2} \underline{\underline{\sigma_d}} : \underline{\underline{\sigma_d}} - \sigma_0^2 \leq 0$ ou $\sigma_{eq} - \sigma_0 \leq 0$

$d_{eq} = \sqrt{\frac{2}{3} \underline{\underline{d_d}} : \underline{\underline{d_d}}}$ $\sigma_{eq} = \sqrt{\frac{3}{2} \underline{\underline{\sigma_d}} : \underline{\underline{\sigma_d}}}$

$$\Pi(\underline{\underline{d}}) = \begin{cases} +\infty & \text{si } \operatorname{tr}(\underline{\underline{d}}) \neq 0 \\ \sigma_0 d_{eq} & \text{si } \operatorname{tr}(\underline{\underline{d}}) = 0 \end{cases}$$

$f = \left(\frac{R_i}{R_e}\right)^3$



$\underline{\underline{\Sigma}} = \Sigma \underline{\underline{\delta}}$

$$\underline{\underline{\Sigma}} : \underline{\underline{D}} \leq \Pi^{hom}(\underline{\underline{D}}) \leq \frac{1}{|\Omega|} \int_\Omega \Pi(\underline{\underline{d}}) \, dV$$