State Space Methods

Lecture 5: introducing reference signals, anti-windup, optimal control

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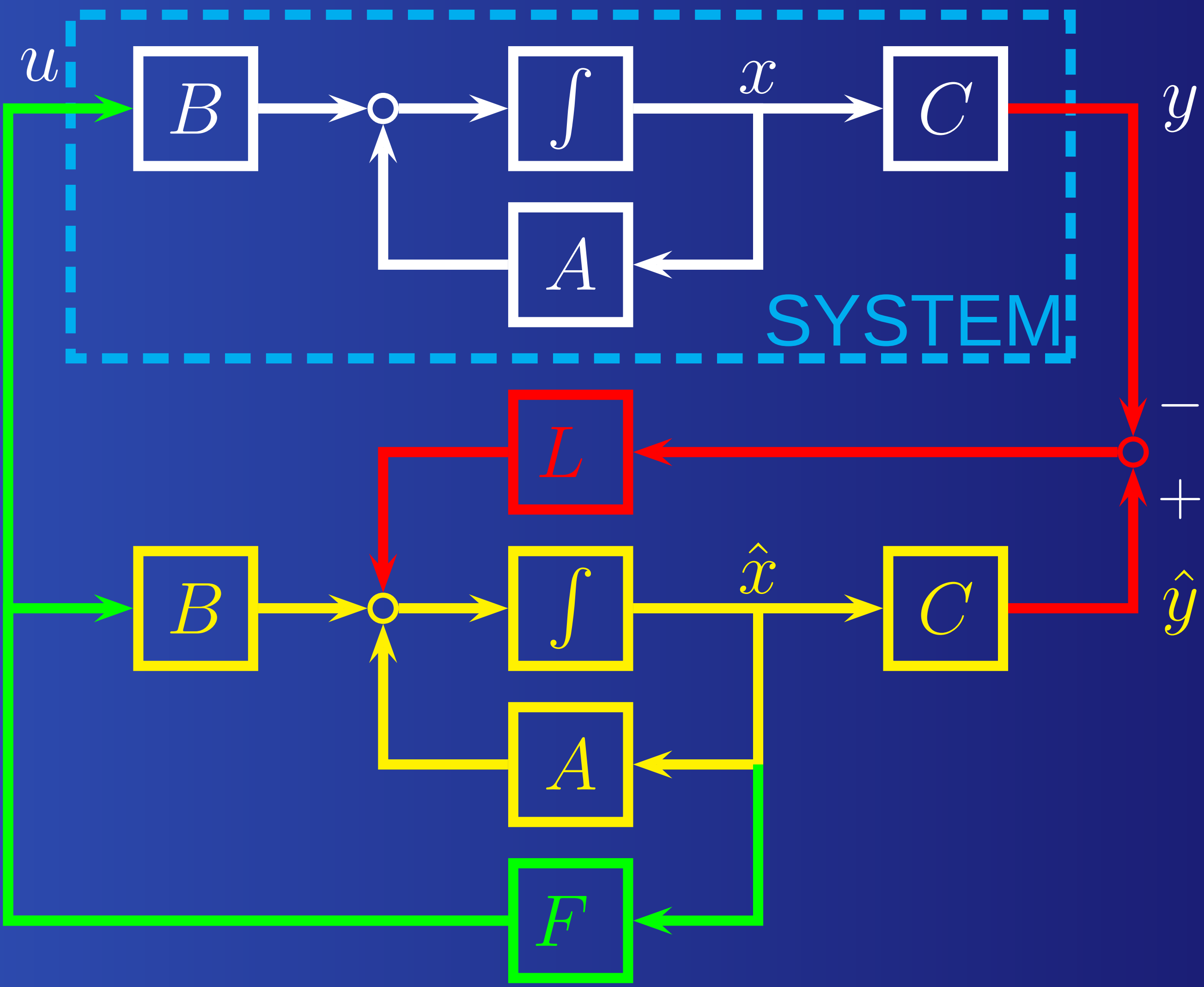
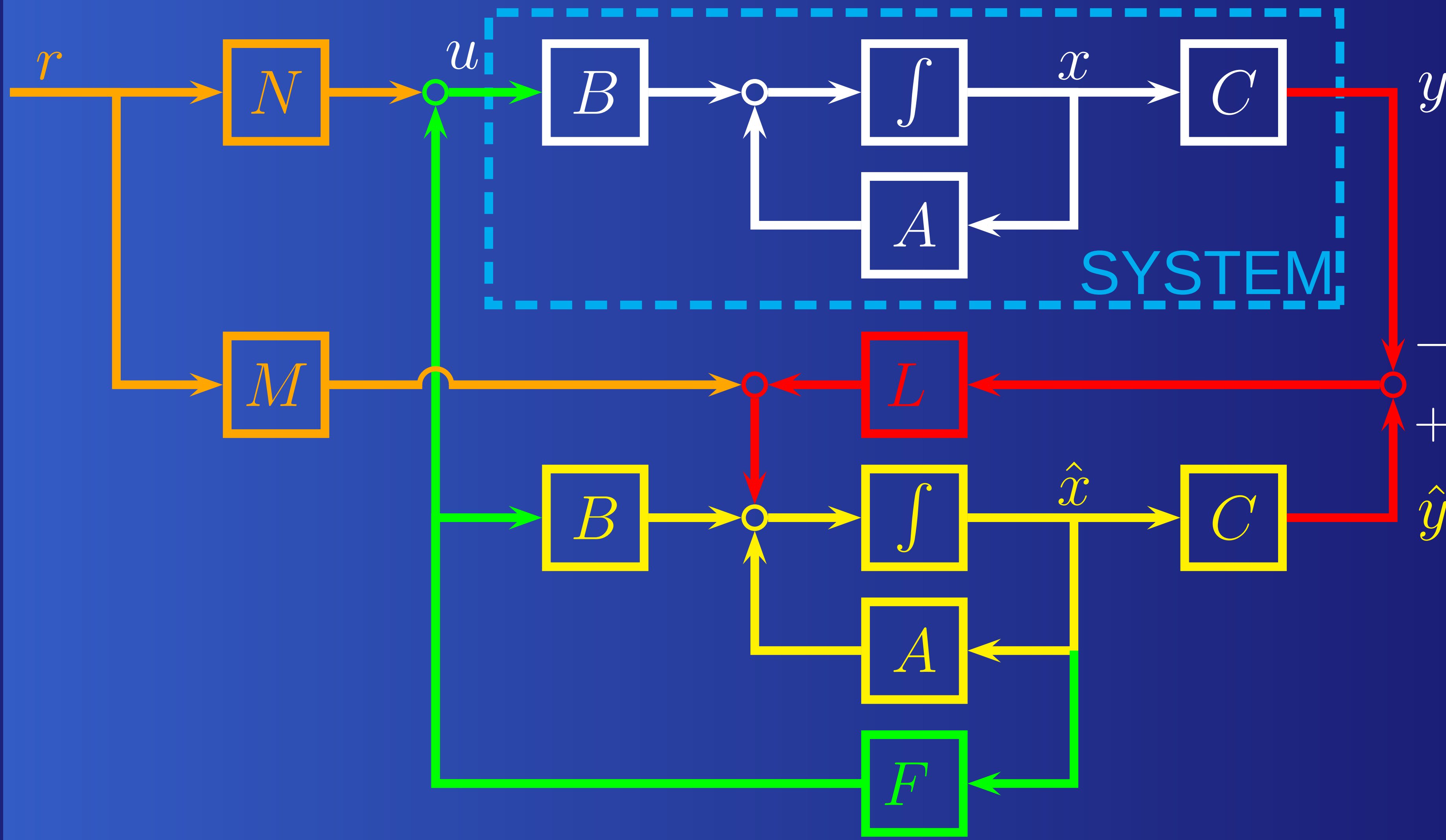
Introducing reference signals (1)



Introducing reference signals (1)



Introducing reference signals (2)

System:

$$\begin{aligned} \dot{x} &= Ax + B(F\hat{x} + Nr) \\ y &= Cx \end{aligned}$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr$$

Introducing reference signals (2)

System:

$$\begin{aligned}\dot{x} &= Ax + B(F\hat{x} + Nr)\\ y &= Cx\end{aligned}$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr$$

$$\dot{\begin{pmatrix} x \\ \hat{x} \end{pmatrix}} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} BN \\ M \end{pmatrix} r$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

Zeros of systems

We have previously introduced this result:

LEMMA. A square (#inputs=#outputs) system with a state space model of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

has a zero with value $z \in \mathbb{C}$ only if

$$\det \begin{pmatrix} A - zI & B \\ C & D \end{pmatrix} = 0$$

Zero assignment

$$\det \begin{pmatrix} A_{\mathsf{cl}} - zI & B_{\mathsf{cl}} \\ C_{\mathsf{cl}} & D_{\mathsf{cl}} \end{pmatrix} = 0$$

Zero assignment

$$\det \begin{pmatrix} A_{\mathsf{cl}} - zI & B_{\mathsf{cl}} \\ C_{\mathsf{cl}} & D_{\mathsf{cl}} \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & BF & BN \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

$$\det \begin{pmatrix} A_{\mathsf{cl}} - zI & B_{\mathsf{cl}} \\ C_{\mathsf{cl}} & D_{\mathsf{cl}} \end{pmatrix} = 0$$

$$\det \begin{pmatrix} A - zI & BF & BN \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

$$\det\begin{pmatrix} A_{\mathsf{cl}} - zI & B_{\mathsf{cl}} \\ C_{\mathsf{cl}} & D_{\mathsf{cl}} \end{pmatrix} = 0$$

$$\det\begin{pmatrix} A - zI & BF & BN \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} A - zI & BF - BNN^{-1}F & BN \\ -LC & A + BF + LC - zI - MN^{-1}F & M \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

$$\det\begin{pmatrix} A-zI & BF & BN \\ -LC & A+BF+LC-zI & M \\ C & 0 & 0 \end{pmatrix}=0$$

$$\det\begin{pmatrix} A-zI & BF-BNN^{-1}F & BN \\ -LC & A+BF+LC-zI-MN^{-1}F & M \\ C & 0 & 0 \end{pmatrix}=0$$

$$\det\begin{pmatrix} A-zI & 0 & BN \\ -LC & A+BF+LC-MN^{-1}F-zI & M \\ C & 0 & 0 \end{pmatrix}=0$$

Zero assignment

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$$\det\begin{pmatrix} A - zI & 0 & BN \\ -LC & A + BF + LC - MN^{-1}F - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

$$\det\begin{pmatrix} A-zI & BF-BNN^{-1}F & BN \\ -LC & A+BF+LC-zI-MN^{-1}F & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} A-zI & 0 & BN \\ -LC & A+BF+LC-MN^{-1}F-zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} A-zI & 0 & B \\ -LC & A+BF+LC-\tilde{M}F-zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

Zero assignment

$$\det\begin{pmatrix} A - zI & 0 & BN \\ -LC & A + BF + LC - MN^{-1}F - zI & M \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{pmatrix} = 0$$

$$\begin{cases} \det\begin{pmatrix} A - zI & B \\ C & 0 \end{pmatrix} = 0 \quad \text{or} \\ \det\left(A + BF + LC - \tilde{M}F - zI\right) = 0 \end{cases}$$

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Zero assignment

LEMMA. If $\tilde{M}$ is an 'observer gain' such that the characteristic polynomial of the matrix $A_{\text{za}} + \tilde{M}C_{\text{za}}$ has the characteristic polynomial

$$\det\left(sI - \left(A_{\text{za}} + \tilde{M}C_{\text{za}}\right)\right) = (s - z_1)\cdots(s - z_n)$$

with $A_{\text{za}} = A+BF+LC$ and $C_{\text{za}} = -F$, then the numbers $z_1, \ldots, z_n$ are all zeros of the closed loop transfer function from r to y.

Algorithm for zero assignment

1. Design $\tilde{M}$ assigning zeros close to the cut-off frequency of the Bode plot, such that the ’horisontal’ part is extended.

Algorithm for zero assignment

2. Compute N such that the DC-value of the transfer function from r to y is unity:

$$N = -\left(C_{\mathsf{cl}} A_{\mathsf{cl}}^{-1} \tilde{B}_{\mathsf{cl}}\right)^{-1}$$

where

$$A_{\mathsf{cl}} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad \tilde{B}_{\mathsf{cl}} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}$$

$$C_{\mathsf{cl}} = \begin{pmatrix} C & 0 \end{pmatrix}$$

Algorithm for zero assignment

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where

$$A_{\text{cl}} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad \tilde{B}_{\text{cl}} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}$$

$$C_{\text{cl}} = \begin{pmatrix} C & 0 \end{pmatrix}$$

3. Compute $M = MN^{-1}N = \tilde{M}N$.

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Example: zero assignment (1)

We consider again the system

$$\dot{x} = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u$$
$$y = \begin{pmatrix} -3 & 2 \end{pmatrix} x$$

A state feedback F that assign poles in $\{-3, -4\}$ and an observer gain L that assigns poles in $\{-9, -12\}$ are given by:

$$F = \begin{pmatrix} 22 & -16 \end{pmatrix}, \quad L = \begin{pmatrix} -122 \\ -192 \end{pmatrix}$$

We would like to assign zeros from r to y in $\{-3, -4\}$ to cancel the poles from F.

Example: zero assignment (2)

With these values of F and L we obtain:

$$A_{\text{za}} = A + BF + LC = \begin{pmatrix} 412 & -279 \\ 646 & -437 \end{pmatrix}$$

$$C_{\text{za}} = -F = \begin{pmatrix} -22 & 16 \end{pmatrix}$$

An 'observer gain' that assigns poles in $\{-3, -4\}$ for $A_{\text{za}} + \tilde{M} C_{\text{za}}$ is

$$\tilde{M} = \begin{pmatrix} 7.0460 \\ 10.8133 \end{pmatrix}$$

Example: zero assignment (3)

N can be computed as:

$$N =$$

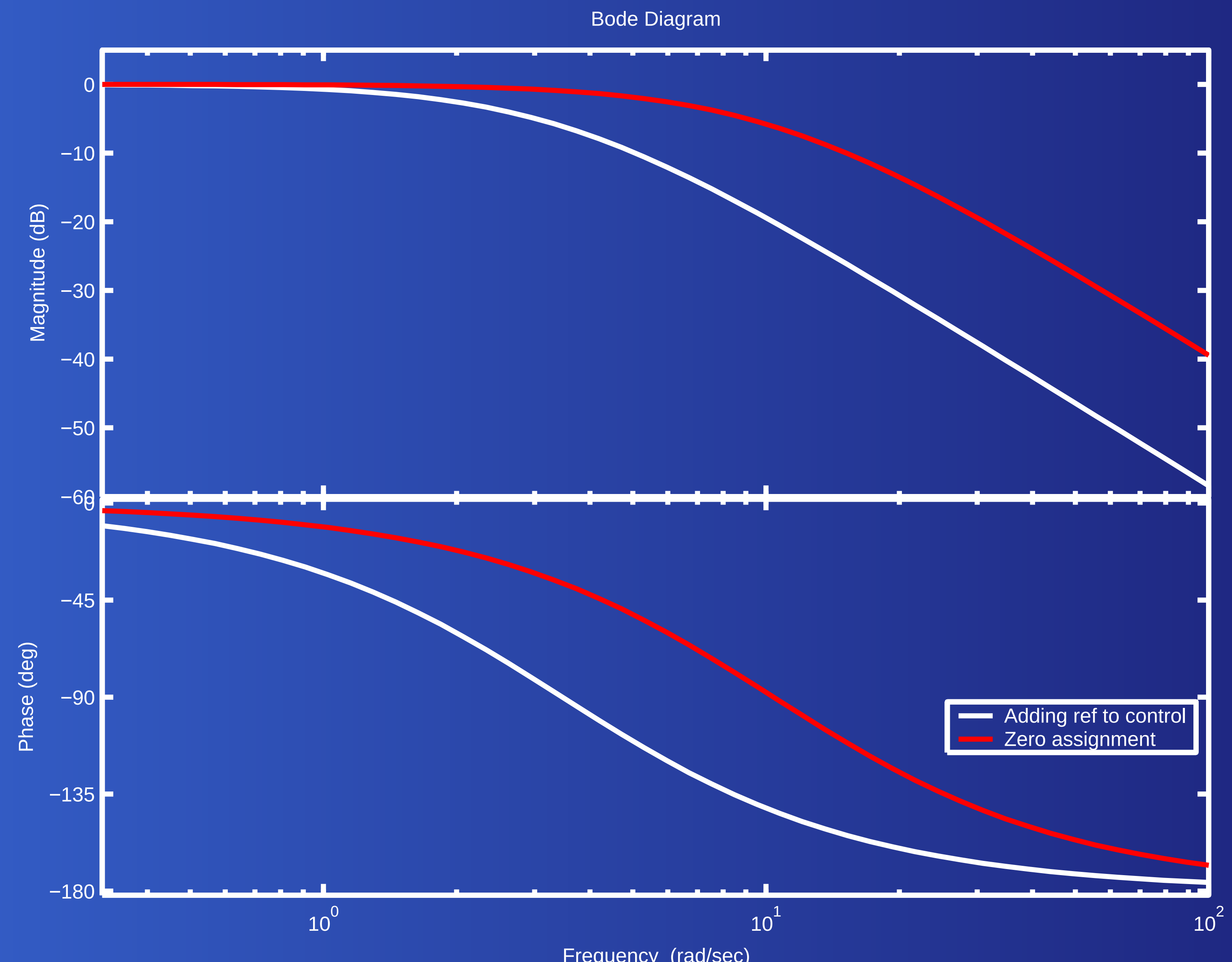
$$-\left(\begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}^{-1} \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}\right)^{-1}$$

$$= 108$$

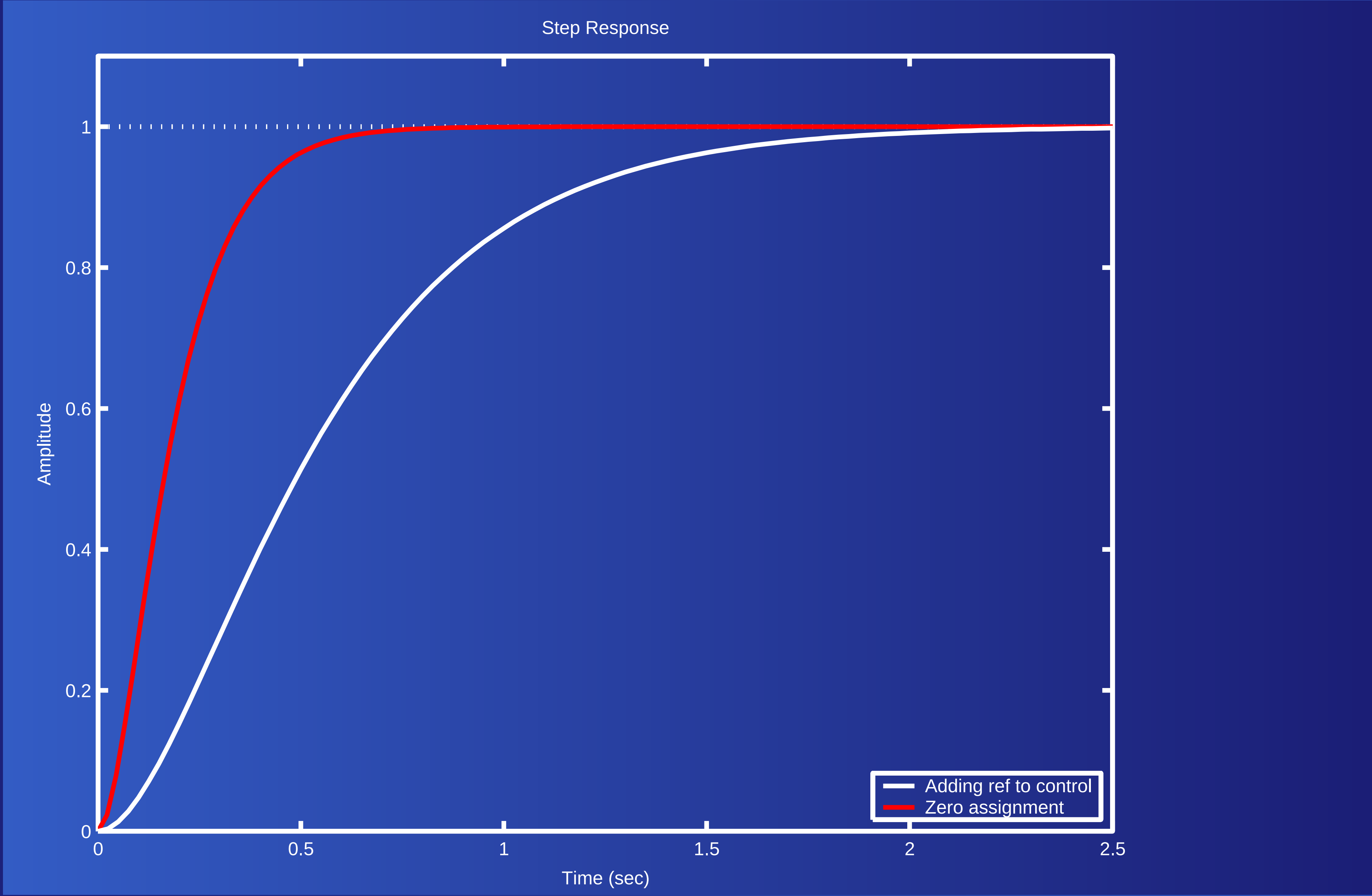
M is obtained from:

$$M = \tilde{M}N = \begin{pmatrix} 7.0460 \\ 10.8133 \end{pmatrix} \cdot 108 = \begin{pmatrix} 760.97 \\ 1167.84 \end{pmatrix}$$

Example: Bode plot



Example: step response



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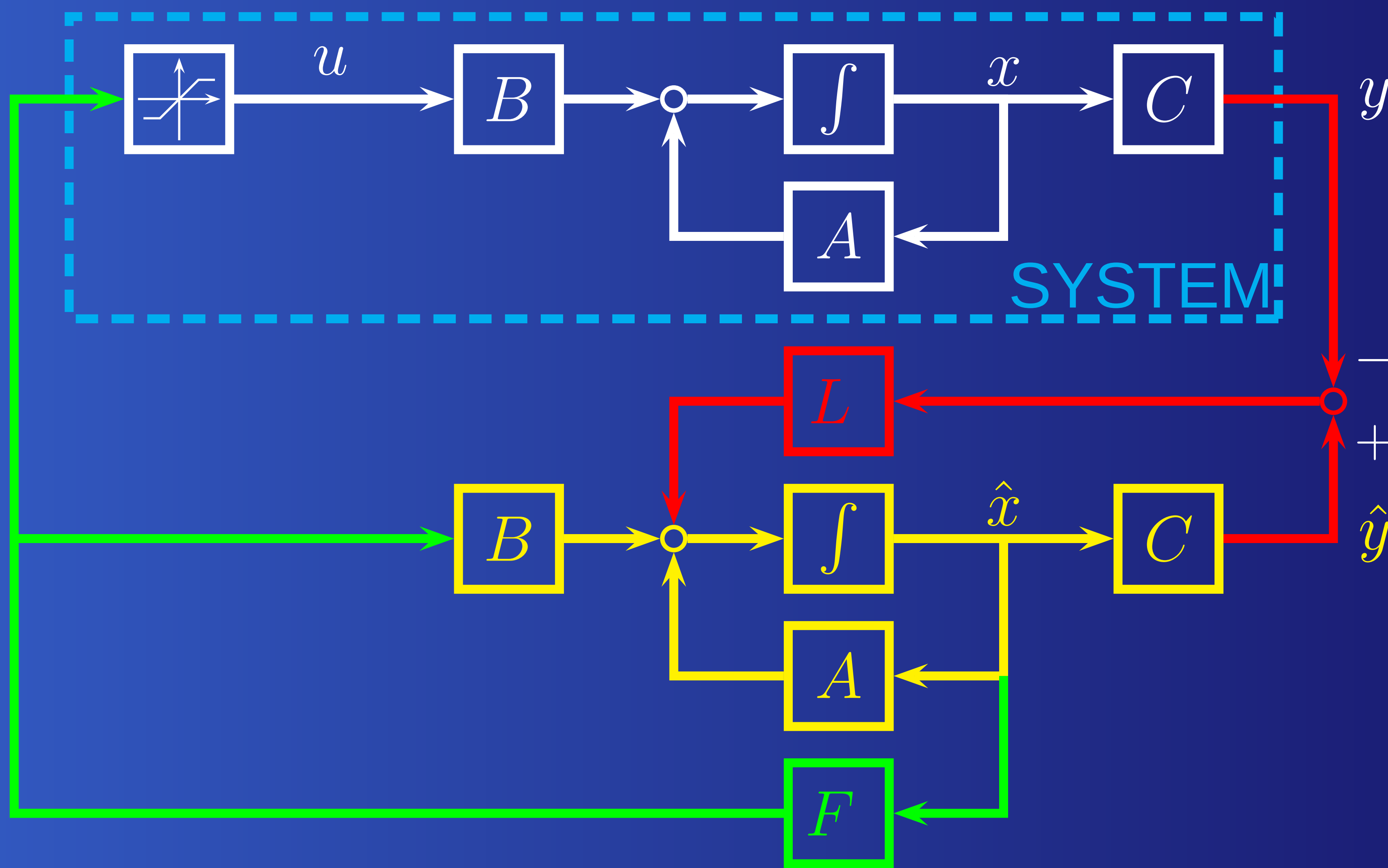
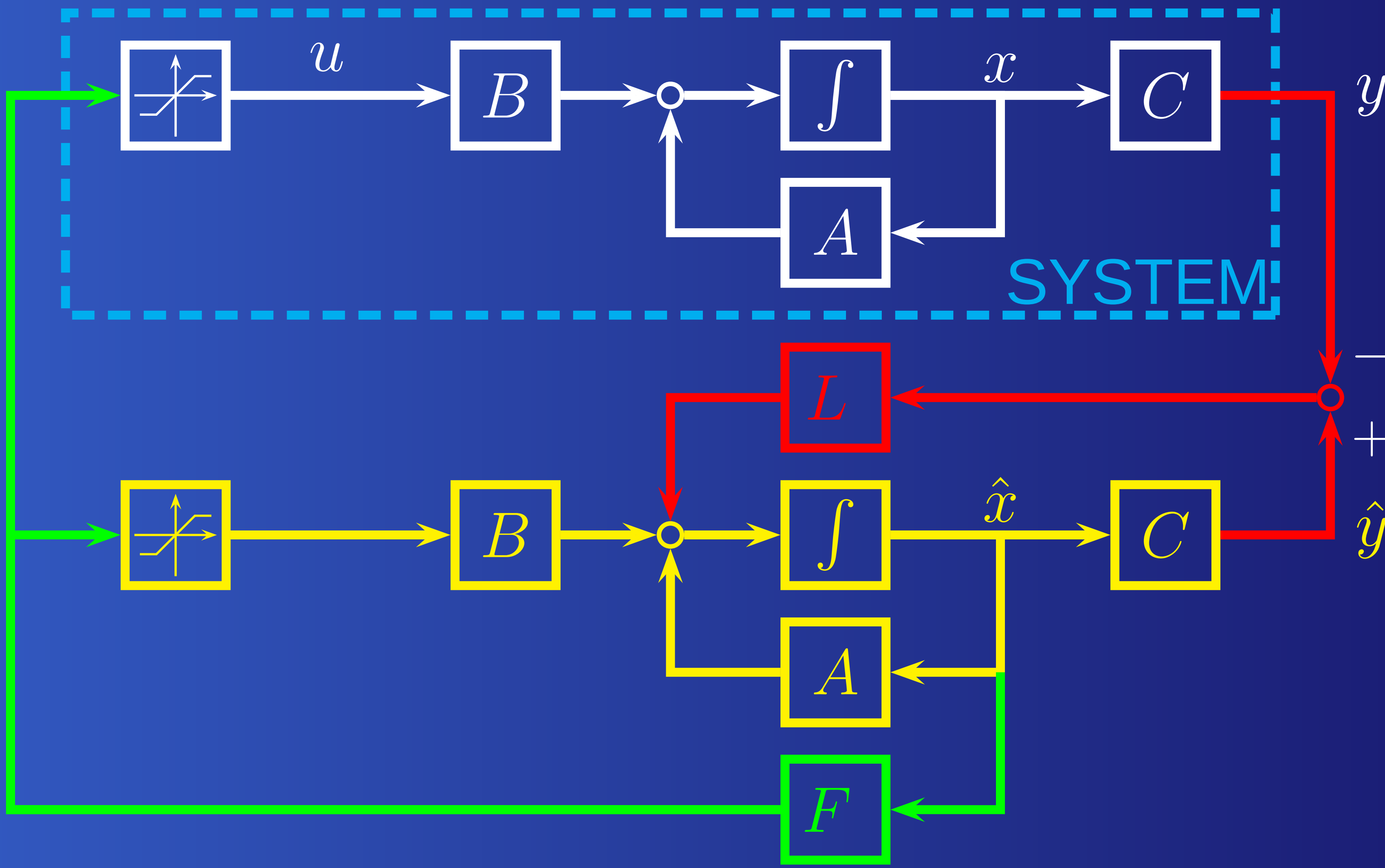
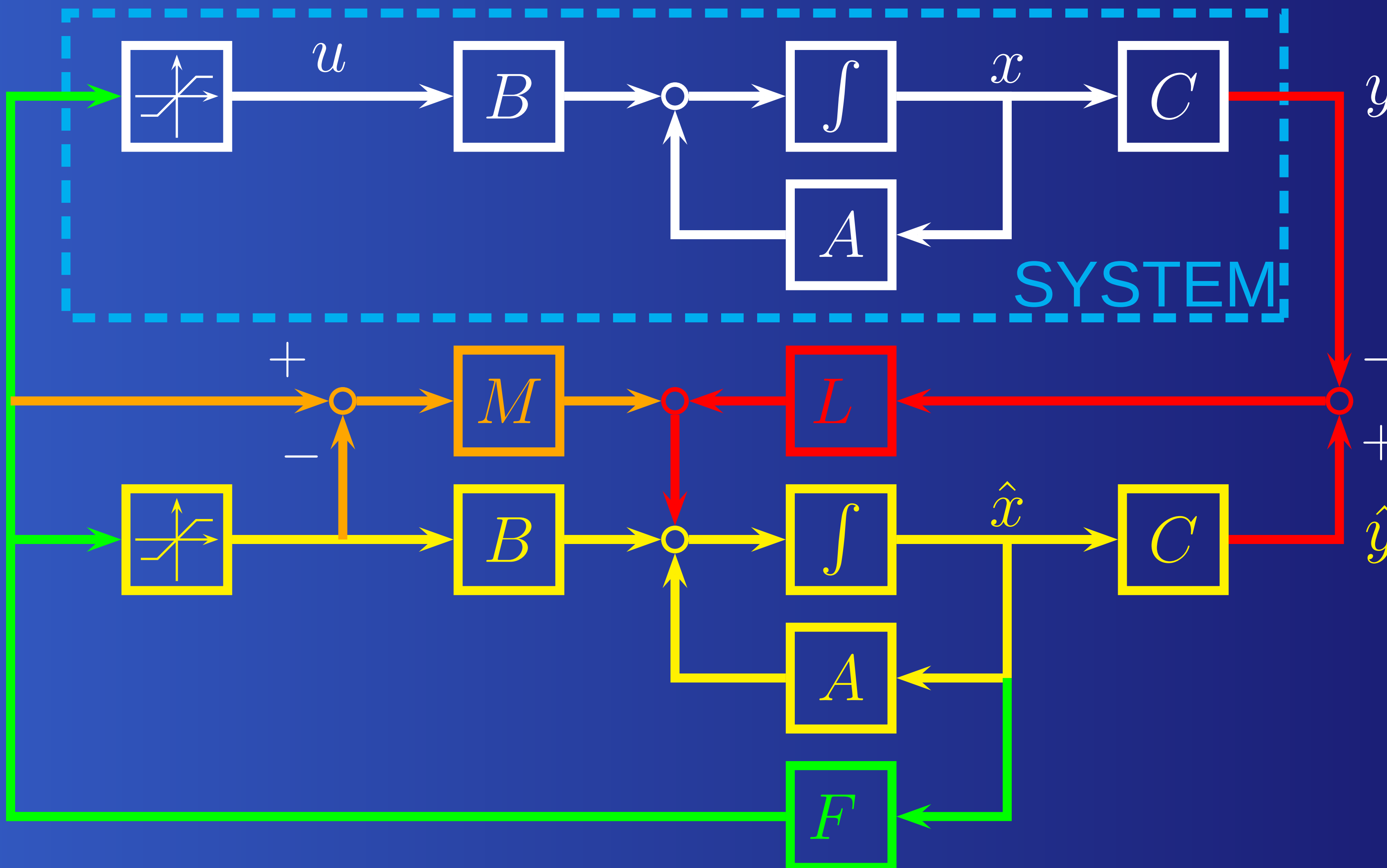
Anti-windup architecture



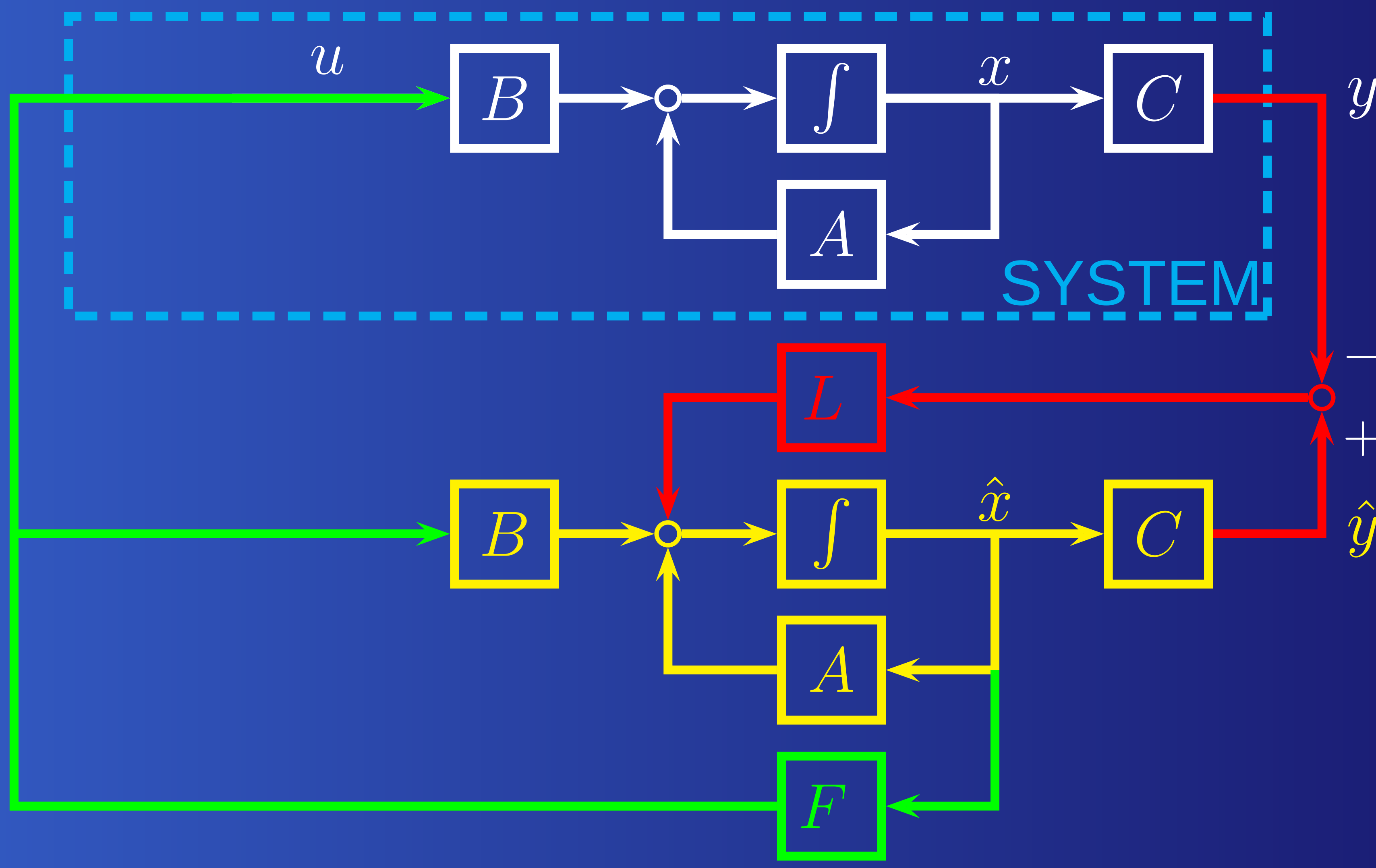
Anti-windup architecture



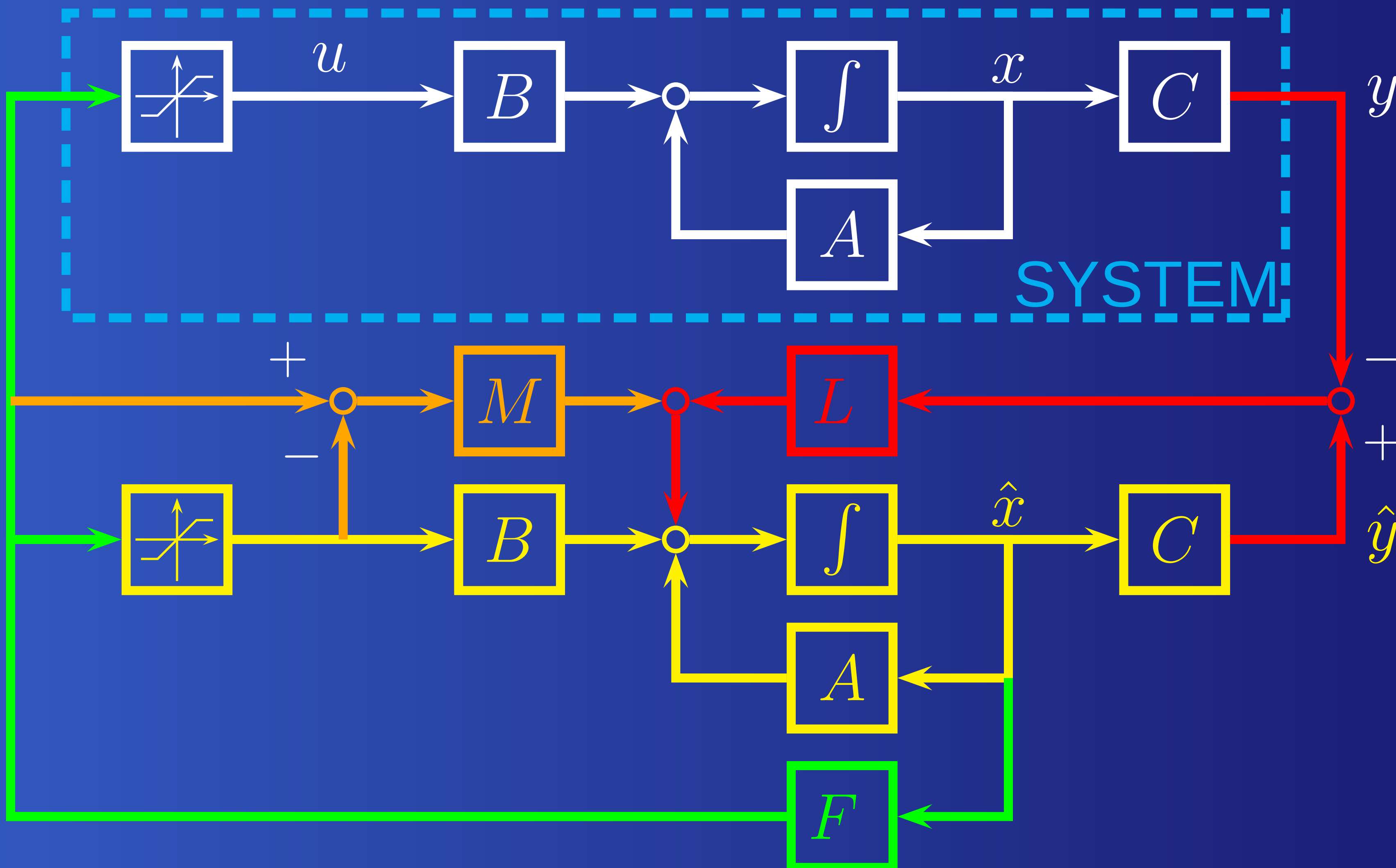
Anti-windup architecture



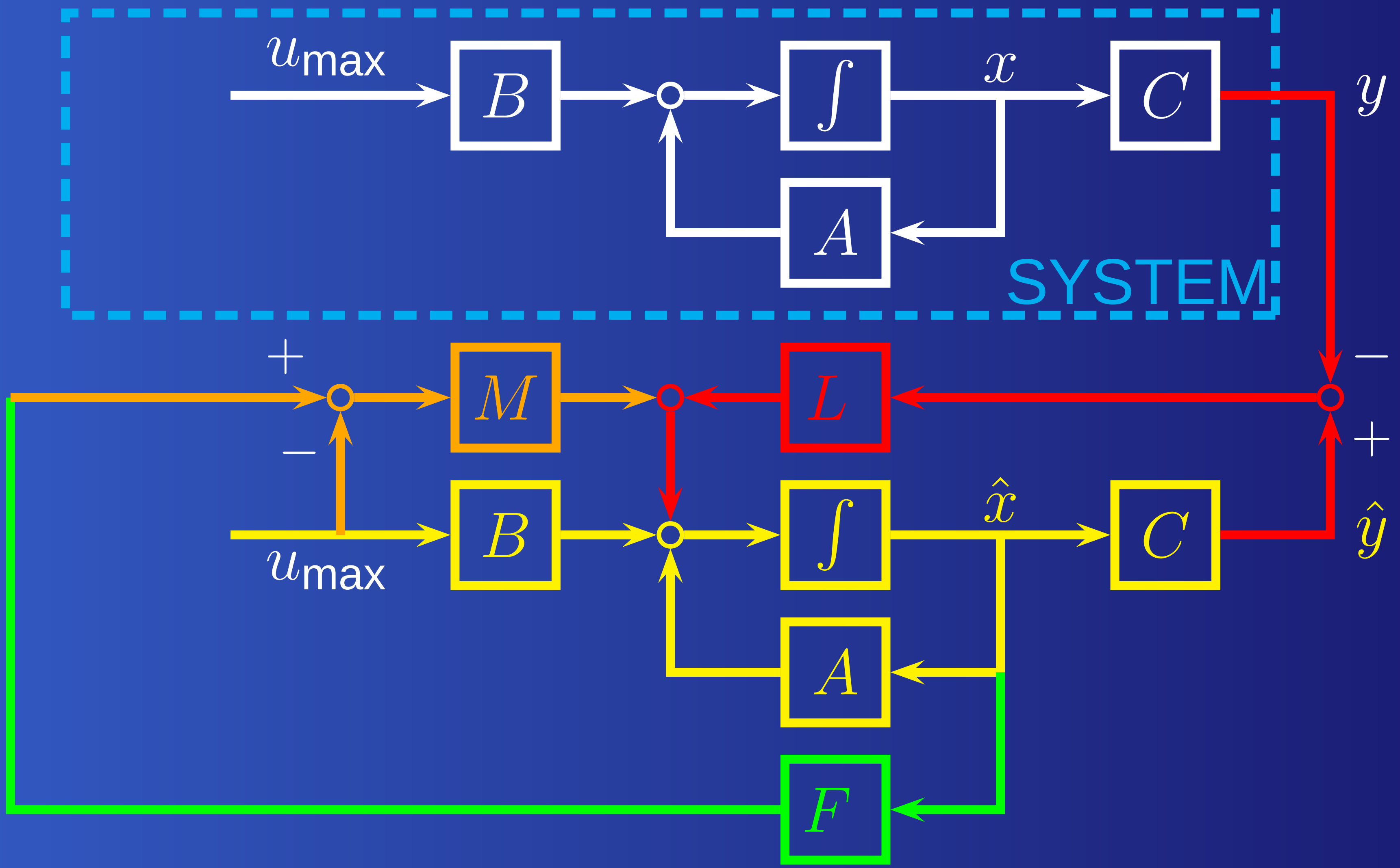
Anti-windup architecture, nominal



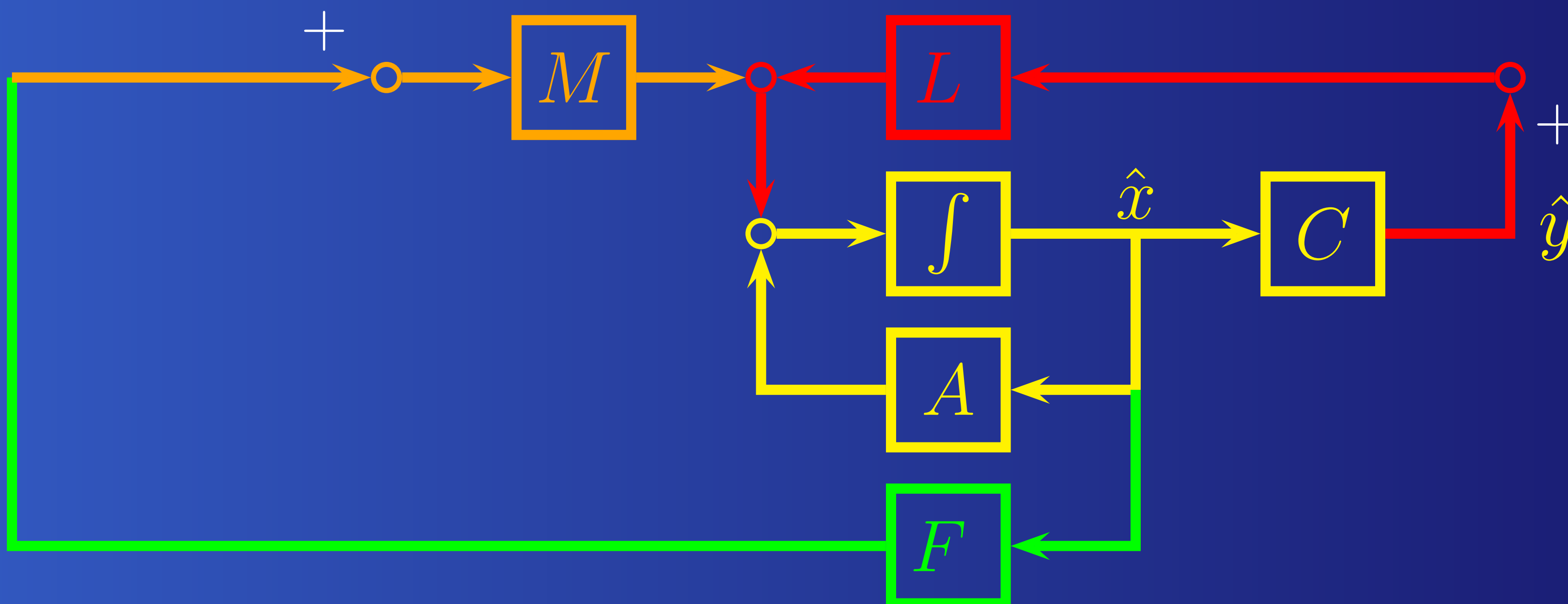
Anti-windup architecture, saturated



Anti-windup architecture, saturated



Anti-windup architecture, saturated



Designing saturation gain

Dynamics of controller during saturation:

$$\dot{\hat{x}} = A\hat{x} + LC\hat{x} + MF\hat{x}$$

or

$$\dot{\hat{x}} = \left(A + LC + MF\right)\hat{x}$$

Designing saturation gain

Dynamics of controller during saturation:

$$\dot{\hat{x}} = A\hat{x} + LC\hat{x} + MF\hat{x}$$

or

$$\dot{\hat{x}} = (A + LC + MF)\,\hat{x}$$

Determining M can be recognized as an observer gain design problem:

$$\dot{\hat{x}} = \left(\tilde{A} + \tilde{L}\tilde{C}\right)\hat{x}$$

with $\tilde{A} = A + LC$, $\tilde{L} = M$, and $\tilde{C} = F$, from which the unknown $\tilde{L} = M$ can be chosen to assign any desired poles to the saturated controller.

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Optimal control

We consider a linear control system of the form:

$$
\begin{aligned}
\dot{x} &= Ax + Bu\,, \quad x(0) = x_0 \\
y &= Cx
\end{aligned}
$$

A control law for such a system is said to be *optimal*, if it minimizes the cost functional:

$$\mathcal{J} = \int_0^{\infty} x^T Q x + u^T R u \, dt$$

where $Q = Q^T$ is a positive semi-definite matrix and $R = R^T$ is a positive definite matrix.

The algebraic Riccati equation

An *Algebraic Riccati Equation* is a second order matrix equation in an indeterminate $P = P^T \in \mathbb{R}^{n \times n}$ of the form:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are matrices, $R = R^T \in \mathbb{R}^{m \times m}$ is a positive definite matrix, and $Q = Q^T \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix.

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where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are matrices, $R = R^T \in \mathbb{R}^{m\times m}$ is a positive definite matrix, and $Q = Q^T \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix. P is called a *stabilizing solution* to the ARE, if it satisfies the equation, and further satisfies that the eigenvalues of $A - BR^{-1}B^T P$ are in the open left half plane.

Optimal state feedback control

THEOREM. Consider a linear system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

Let P be a stabilizing solution to the ARE:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Then the optimal state feedback law is given by:

$$u = Fx \quad \text{where} \quad F = -R^{-1}B^T P$$

Output variance minimization

Introducing $y = Cx$ into a cost functional of the type

$$\mathcal{J} = \int_0^{\infty} \rho y^T y + u^T u \, dt \,, \quad \rho \in \mathbb{R}$$

this can be written as an optimal control problem

$$\begin{aligned} \mathcal{J} &= \int_0^{\infty} \rho y^T y + u^T u \, dt \\ &= \int_0^{\infty} \rho x^T C^T C x + u^T u \, dt \\ &= \int_0^{\infty} x^T Q x + u^T R u \, dt \,, \quad Q = \rho C^T C \,, R = I \end{aligned}$$

Optimal state estimation

Given the system

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Gw \\
y &= Cx + Du + v
\end{aligned}
$$

with unbiased process noise w and measurement noise v with covariances

$$
\mathcal{E}\{ww^T\} = Q\,, \quad \mathcal{E}\{vv^T\} = R
$$

Then an optimal state estimator is given by:

$$
\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)
$$

Optimal state estimation

with unbiased process noise w and measurement noise v with covariances

$$\mathcal{E}\{ww^T\} = Q\,, \quad \mathcal{E}\{vv^T\} = R$$

Then an optimal state estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$

where

$$L = -PC^T R^{-1}$$

P is a stabilizing solution to the ARE:

$$AP + PA^T - PC^T R^{-1} CP + Q = 0$$

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Example: optimal control (1)

We consider once again the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

Computing an optimal state feedback for the cost functional:

$$\mathcal{J} = \int_0^\infty 800\, y^T y + u^T u \, dt$$

can be done with the MATLAB™ command

```
Fopt = -lqr(A,B,800*C'*C,1)
```

Example: optimal control (2)

This yields the result:

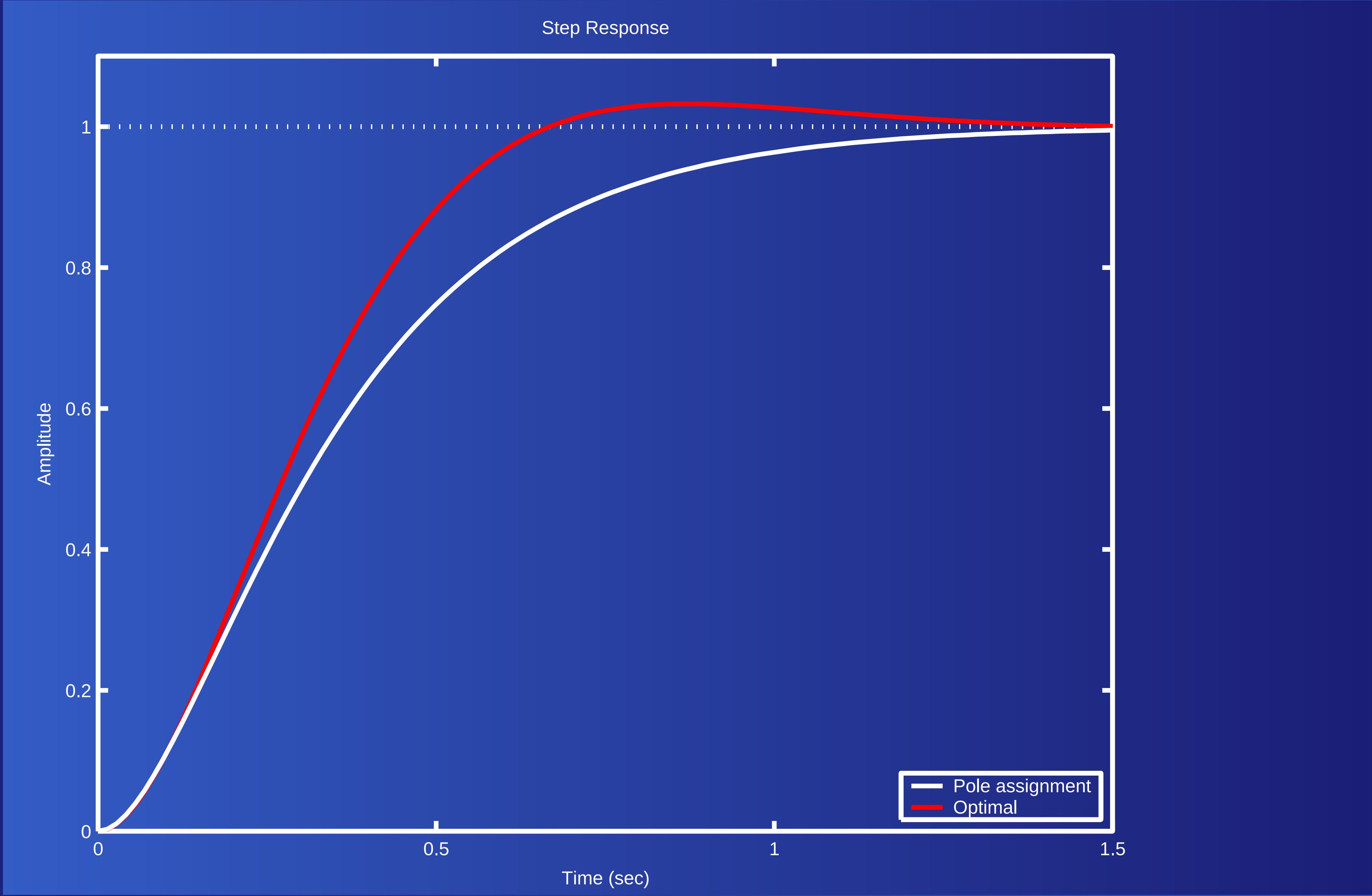
$$F_{\text{opt}} = \begin{pmatrix} 69.3536 & -47.8542 \end{pmatrix}$$

In comparison, a pole assignment with the poles $\{-4, -8\}$ leads to the gain:

$$F = \begin{pmatrix} 72 & -51 \end{pmatrix}$$

A first glance would suggest that the pole assignment with its larger gains would have faster dynamics. However, the optimal feedback assigns complex poles, giving a better rise-time.

Example: optimal control (3)



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- Controllability

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- State feedback design (pole assignment)

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