State Space Methods

Lecture 4: reduced order observers, integral control

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Contents

- **The reduced order observer**
- Reduced order observer based control
  - Example: reduced order control
- Integral control
  - Example: integral control

The reduced order observer (1)

Possibly following a state space transformation, a state space model can be partioned as:

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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\end{aligned}
$$

Writing out the equations, we obtain:

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\begin{aligned}
\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\
\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \\
y &= x_1
\end{aligned}
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y &= \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
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Writing out the equations, we obtain:

$$
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\dot{y} &= A_{11}y + A_{12}x_2 + B_1u \\
\dot{x}_2 &= A_{21}y + A_{22}x_2 + B_2u
\end{aligned}
$$

The reduced order observer (2)

By rearranging the equation for $\dot{x}_1 = \dot{y}$:

$$\dot{y} \;=\; A_{11}y + A_{12}x_2 + B_1u$$

The reduced order observer (2)

By rearranging the equation for $\dot{x}_1 = \dot{y}$:

$$\underbrace{A_{12}x_2}_{\text{unknown}} = \underbrace{\dot{y} - A_{11}y - B_1 u}_{\text{known}}$$

it can be seen as a 'measurement equation'. The corresponding state estimation equation is formed from the equation for $\dot{x}_2$:

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$$\begin{aligned}\dot{\hat{x}}_2 &= A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2) \\ &= A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - (\dot{y} - A_{11}y - B_1u))\end{aligned}$$

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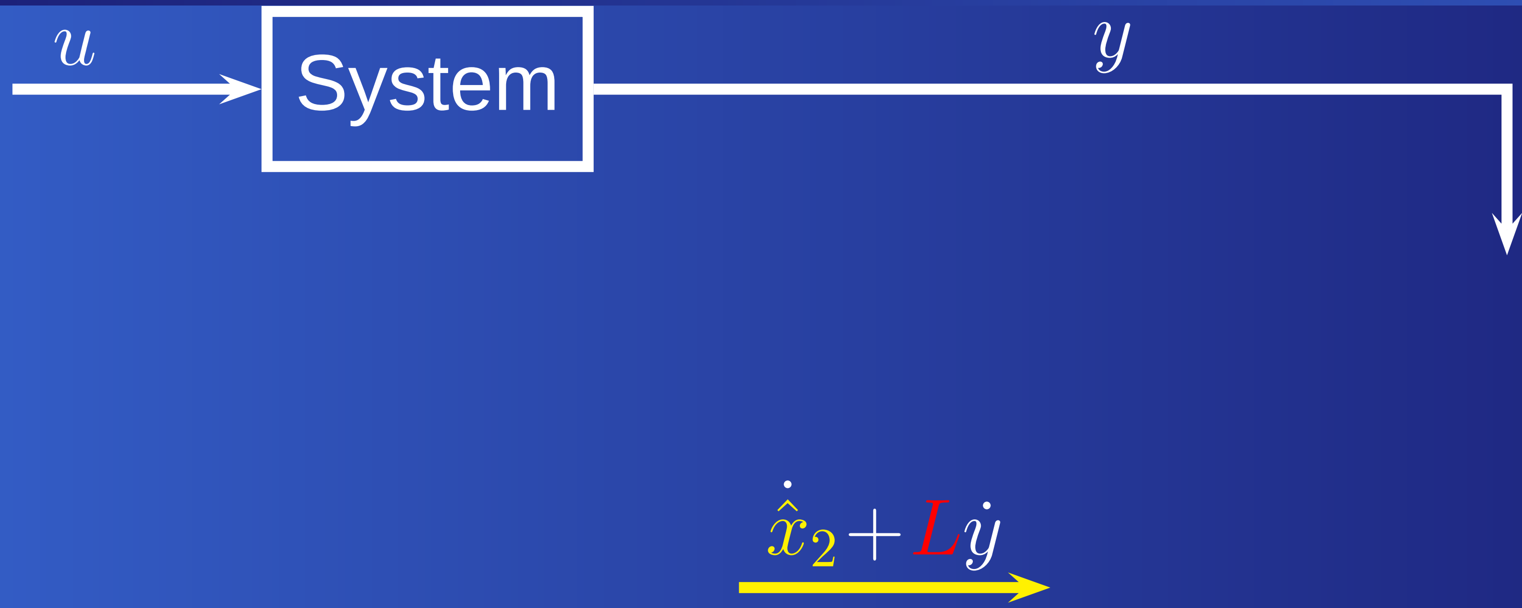
Rearranging the terms, we obtain:

$$\dot{\hat{x}}_2 + L\dot{y} = (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y + (B_2 + LB_1)u$$

The reduced order observer (3)



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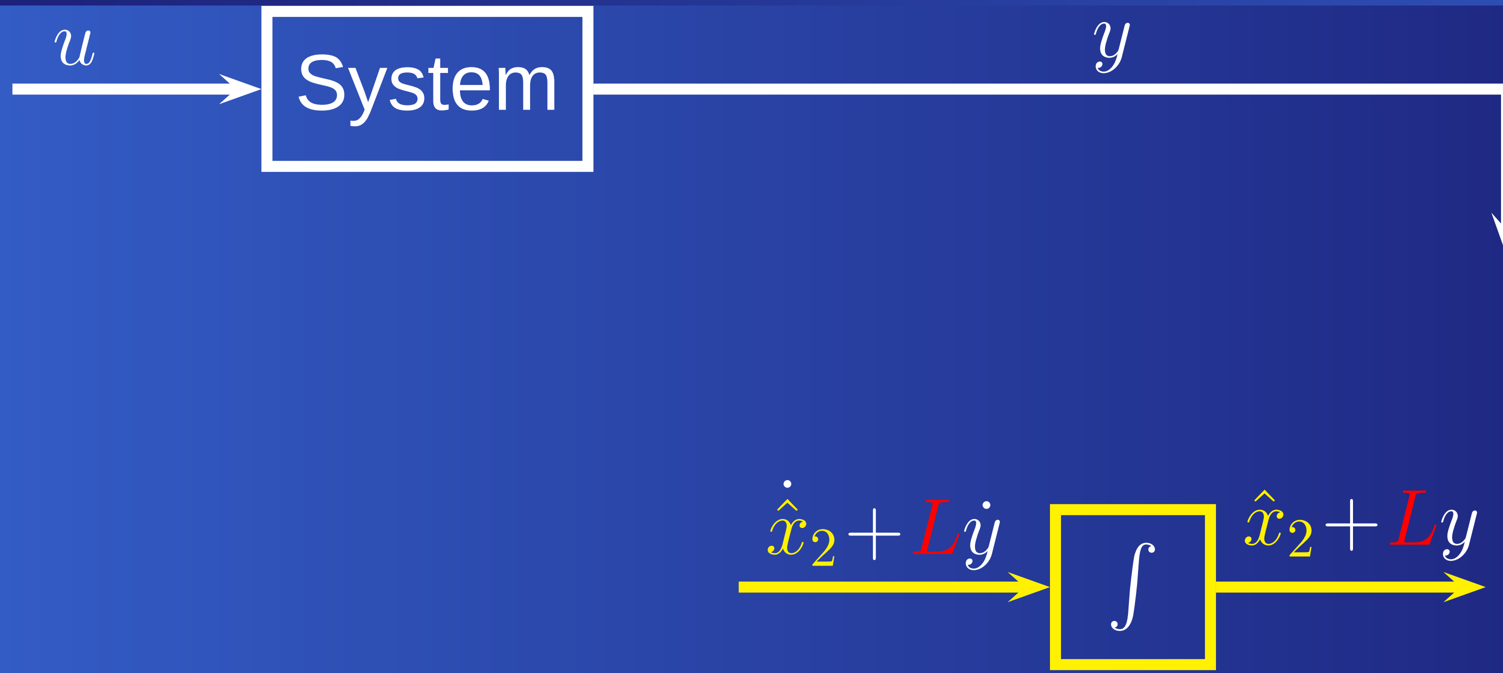

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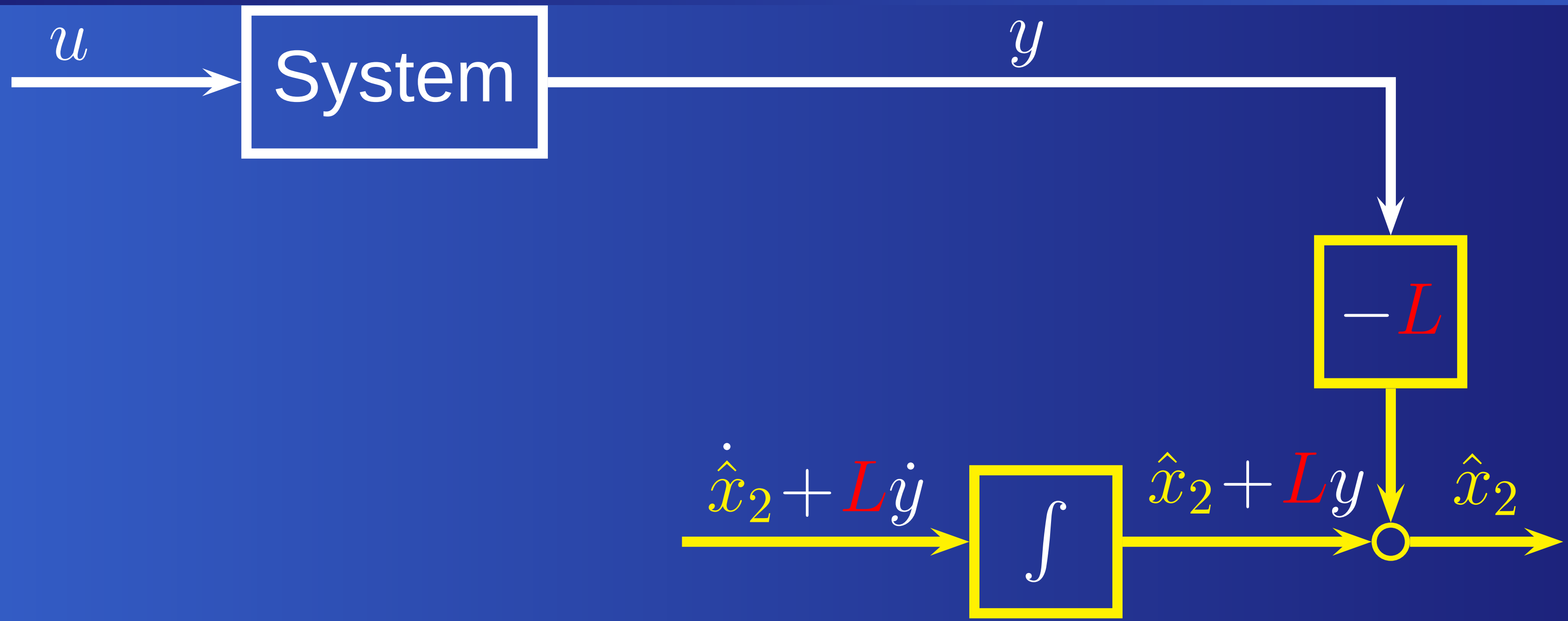

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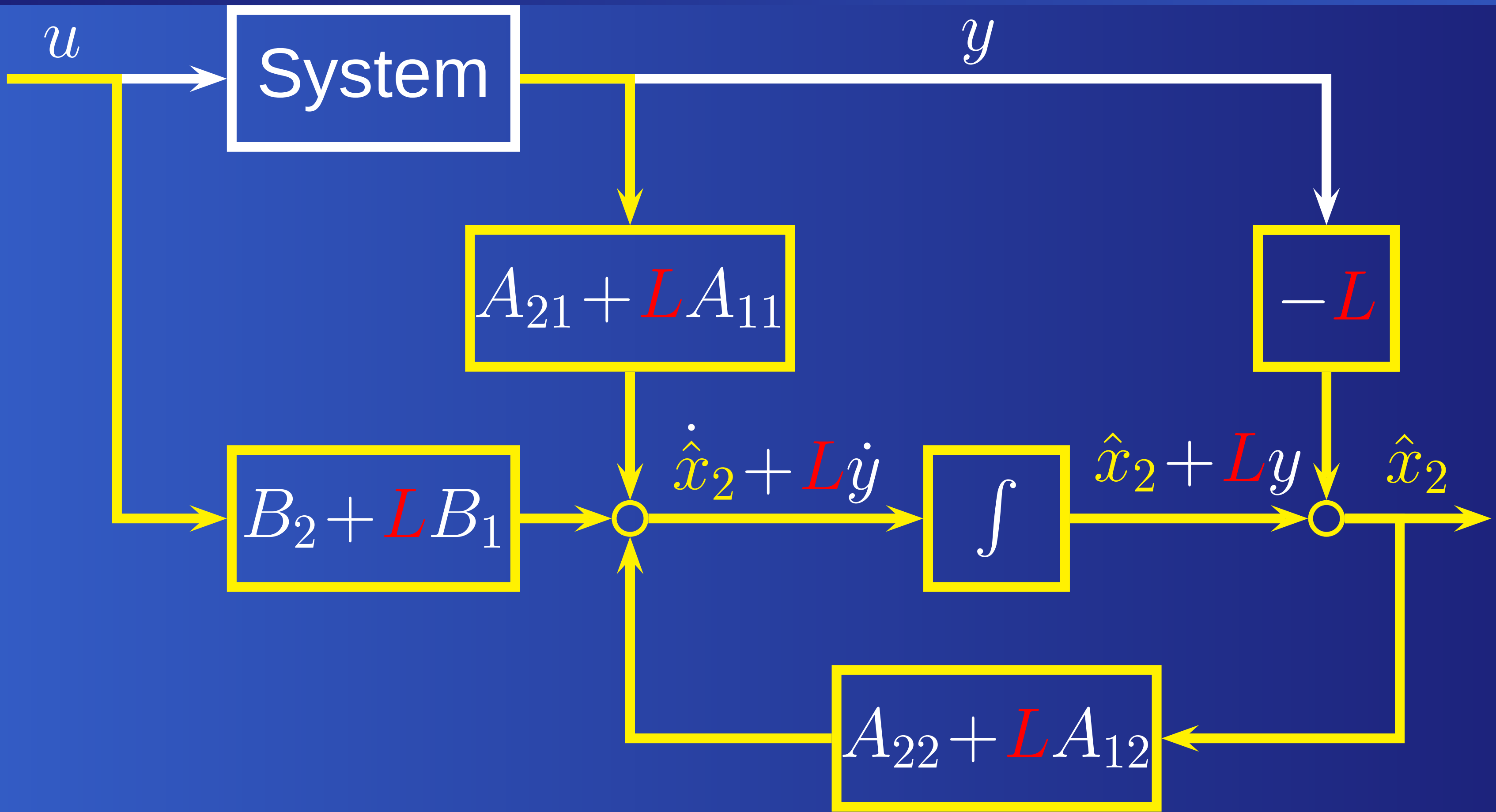

$$\dot{\hat{x}}_2+L\dot{y}=(A_{22}+LA_{12})\hat{x}_2+(A_{21}+LA_{11})y+(B_2+LB_1)u$$

The reduced order observer (4)

System equation for x_2:

$$\dot{x}_2 = A_{21}y + A_{22}x_2 + B_2u$$

Observer equation:

$$\dot{\hat{x}}_2 = A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2)$$

The reduced order observer (4)

System equation for x_2:

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Observer equation:

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Estimation error: $e = \hat{x}_2 - x_2$.

$$\begin{aligned}
\dot{e} &= \dot{\hat{x}}_2 - \dot{x}_2 \\
&= A_{21}y + A_{22}\hat{x}_2 + B_2u + L(A_{12}\hat{x}_2 - A_{12}x_2) \\
&\qquad\quad - (A_{21}y + A_{22}x_2 + B_2u) \\
&= (A_{22} + LA_{12})(\hat{x}_2 - x_2) = (A_{22} + LA_{12})\,e
\end{aligned}$$

The reduced order observer (5)

THEOREM. Assume that the auxiliary system

$$\dot{x}_2 = A_{22}x_2\,, \quad y = A_{12}x_2$$

is observable. Then there exists an observer gain L such that $A_{22} + LA_{12}$ is stable.

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is observable. Then there exists an observer gain L such that $A_{22} + LA_{12}$ is stable.
With this observer gain, the observer

$$\dot{\hat{x}}_2 = A_{21} y + A_{22} \hat{x}_2 + B_2 u + L(A_{12} \hat{x}_2 - A_{12} x_2)$$

is guaranteed to give an estimate $\hat{x}_2$ which converges to x_2 at a rate given by the eigenvalues of $A_{22} + LA_{12}$.

Contents

- The reduced order observer
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Reduced order obs. based control (1)

Based on the estimates of a reduced order observer, the feedback law becomes:

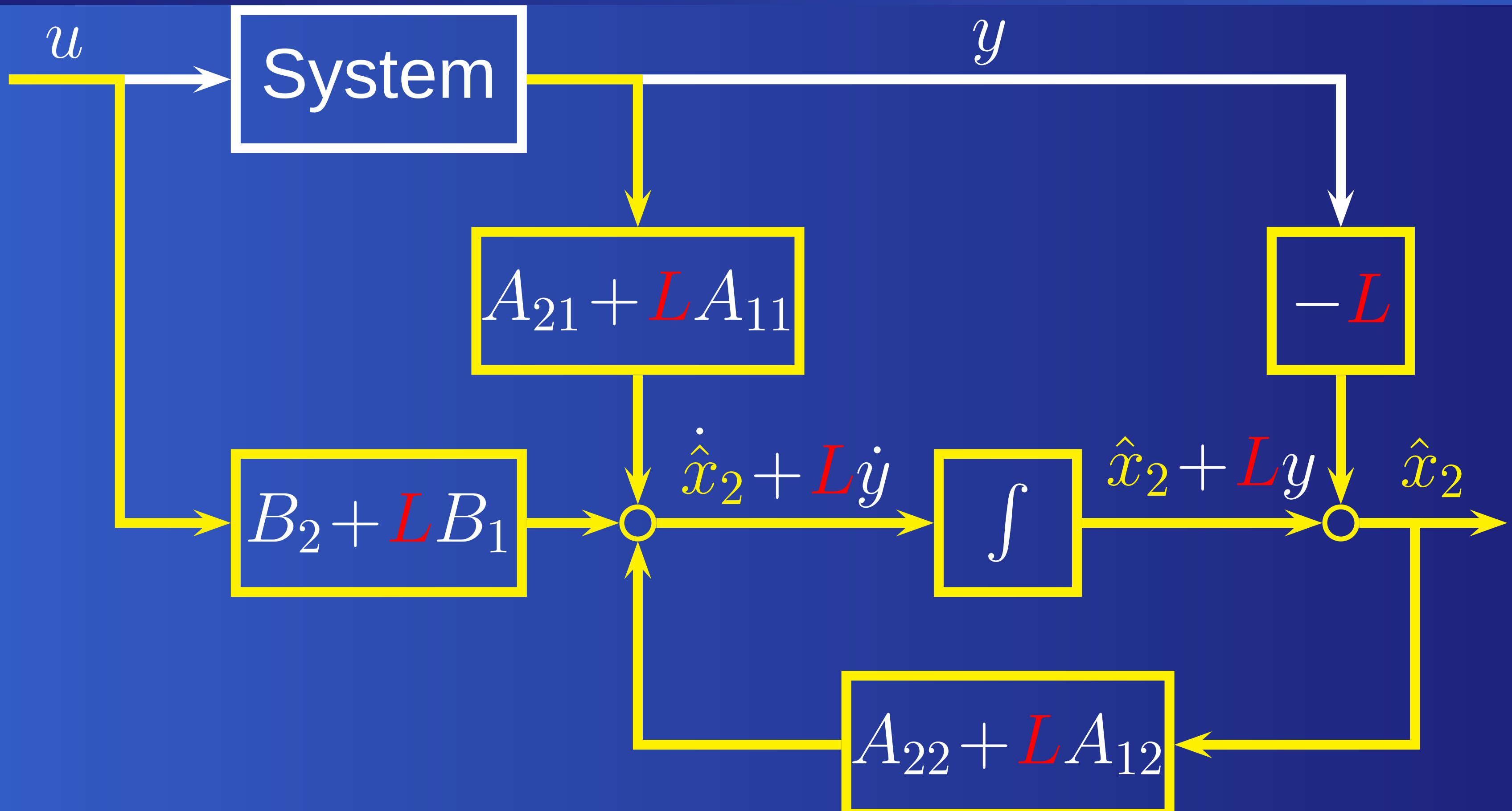
$$u = F \begin{pmatrix} y \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \end{pmatrix} \begin{pmatrix} y \\ \hat{x}_2 \end{pmatrix} = F_1 y + F_2 \hat{x}_2$$

The resulting closed loop system has poles equal to the eigenvalues of the two matrices:

$$A + BF \quad \text{and} \quad A_{22} + LA_{12}$$

This is the reduced order version of the *separation theorem*!

Reduced order obs. based control (2)



$$\dot{\hat{x}}_2 + L\dot{y} = (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y + (B_2 + LB_1)u$$

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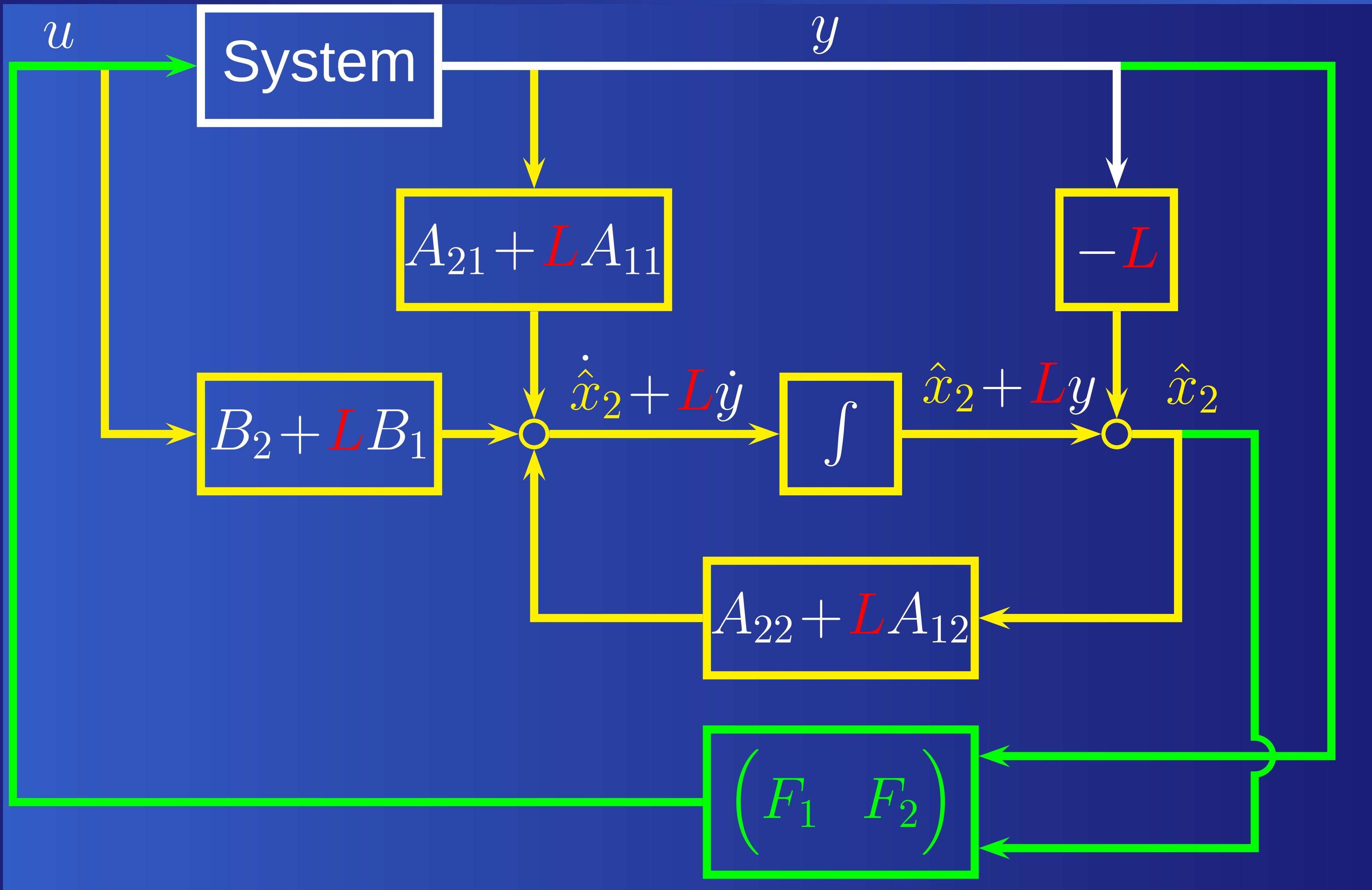

Algorithm for red. order control

1. Design a state feedback matrix F, such that the eigenvalues of $A + BF$ corresponds to desired poles.

Algorithm for red. order control

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2. Transform, if necessary, the system to a form where the output equation has the form

   $$y = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

   For a single output system, transformation to observable canonical form is one possible choice.

Algorithm for red. order control

3. Partition the transformed system matrices:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = T^{-1}AT\,, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = T^{-1}B$$

$$\begin{pmatrix} F_1 & F_2 \end{pmatrix} = FT$$

where T is the transform matrix.

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4. Design L such that $A_{22} + LA_{12}$ is a stable matrix with desired observer poles as eigenvalues.

Algorithm for red. order control

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5. Construct the reduced order observer:

$$\begin{aligned}\dot{\hat{x}}_2 + L\dot{y} = (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y \\ + (B_2 + LB_1)u\end{aligned}$$

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6. Close the loop by the feedback law:

$$u = F_1 y + F_2 \hat{x}_2$$

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Example: red. order control (1)

We consider again the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x \end{aligned}$$

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1. We have already computed a state feedback which assigns poles in $\{-4, -5\}$:

$$F = \begin{pmatrix} 42 & -30 \end{pmatrix}$$

Example: red. order control (1)

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$$F = \begin{pmatrix} 42 & -30 \end{pmatrix}$$

2. $CT = \begin{pmatrix} I & 0 \end{pmatrix}$ can be achieved by transforming to observable canonical form, which is obtained by:

$$T = \begin{pmatrix} -5 & 2 \\ -7 & 3 \end{pmatrix}$$

Example: red. order control (1)

3. Partitioning gives:

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) = T^{-1}AT = \left(\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right) = T^{-1}B = \left(\begin{array}{c} 0 \\ \hline 1 \end{array}\right)$$

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4. The observer pole is chosen as -5 :

$$A_{22} + LA_{12} = 0 + L1 = -5 \Longrightarrow L = -5$$

Example: red. order control (1)

5. The reduced order observer equation:

$$\dot{\hat{x}}_2 + L\dot{y} = (A_{22} + LA_{12})\hat{x}_2 + (A_{21} + LA_{11})y + (B_2 + LB_1)u$$

becomes:

$$\dot{\hat{x}}_2 + (-5)\dot{y} = (0-5 \cdot 1)\hat{x}_2+(-2+(-5) \cdot (-3))y +(1+(-5) \cdot 0)u$$

or

$$\dot{\hat{x}}_2 - 5\dot{y} = -5\hat{x}_2 + 13y + u$$

Example: red. order control (1)

6. The feedback law becomes:

$$u = F_1 y + F_2 \hat{x}_2 = 0y + (-6)\hat{x}_2 = -6\hat{x}_2$$

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Taking Laplace transform of the observer eq.:

$$\dot{\hat{x}}_2 - 5\dot{y} = -5\hat{x}_2 + 13y + u$$

and substituting the feedback law gives:

$$s\hat{x}_2 - 5sy = -5\hat{x}_2 + 13y - 6\hat{x}_2$$

which implies:

$$(s + 11)\hat{x}_2 = (5s + 13)y$$

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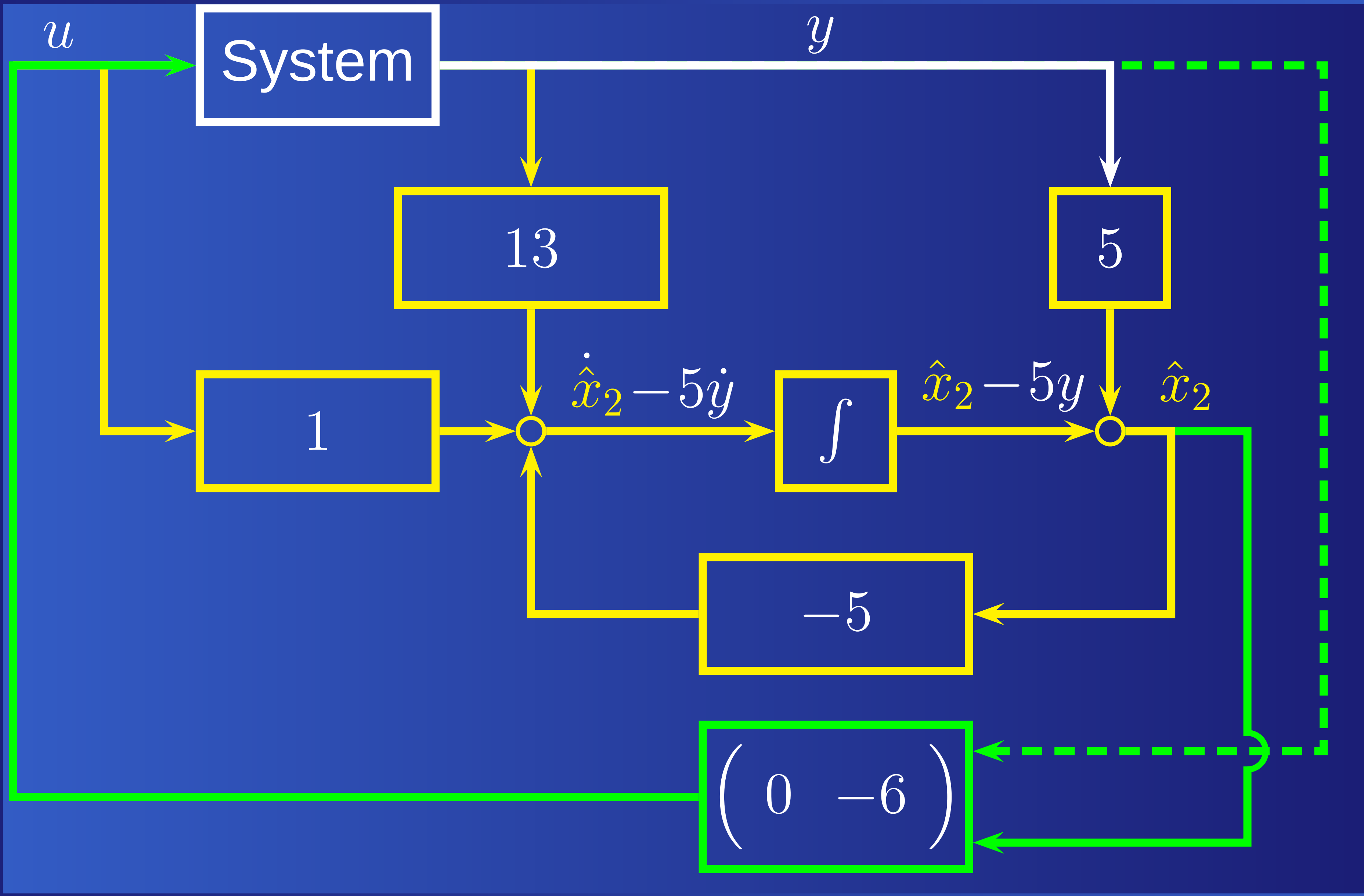
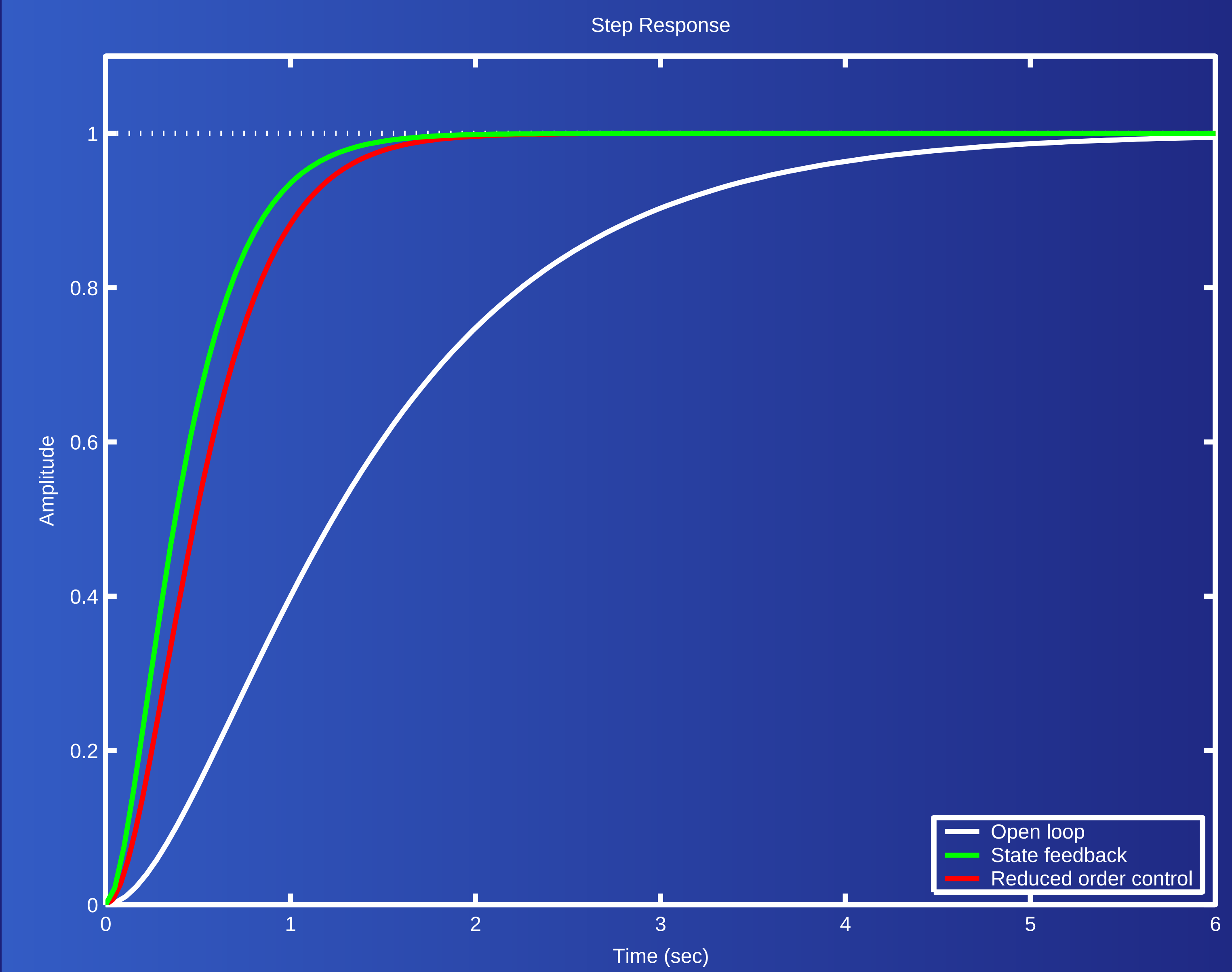
which implies:

$$(s+11)\hat{x}_2 = (5s+13)y \Rightarrow \boxed{u = -6\hat{x}_2 = -6\frac{5s+13}{s+11}y}$$

Example: red. order control (2)



Example: red. order control (3)



Contents

- The reduced order observer
- Reduced order observer based control
  - Example: reduced order control
- **Integral control**
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Integral control (1)

We consider a state space system of the form:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

for which we wish to design a feedback law:

$$u(t) = F x(t) + F_I x_I(t)$$

where

$$x_I(t) = \int_0^t y(\tau) - r(\tau)\, d\tau$$

or

$$\dot{x}_I(t) = y(t) - r(t)$$

Integral control (2)

The equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{x}_I &= y - r \\ y &= Cx \end{aligned}$$

can be combined into an extended state model:

$$\begin{aligned} \dot{\begin{pmatrix} x \\ x_I \end{pmatrix}} &= \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -I \end{pmatrix} r \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} \end{aligned}$$

Integral control (2)

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$$\begin{aligned} \begin{pmatrix} \dot{x} \\ x_I \end{pmatrix} &= \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -I \end{pmatrix} r \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} \end{aligned}$$

for which the feedback law becomes:

$$u = F x + F_I x_I = \begin{pmatrix} F & F_I \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix}$$

Integral control (3)

Thus, the integral control problem has been reduced to a conventional state feedback problem:

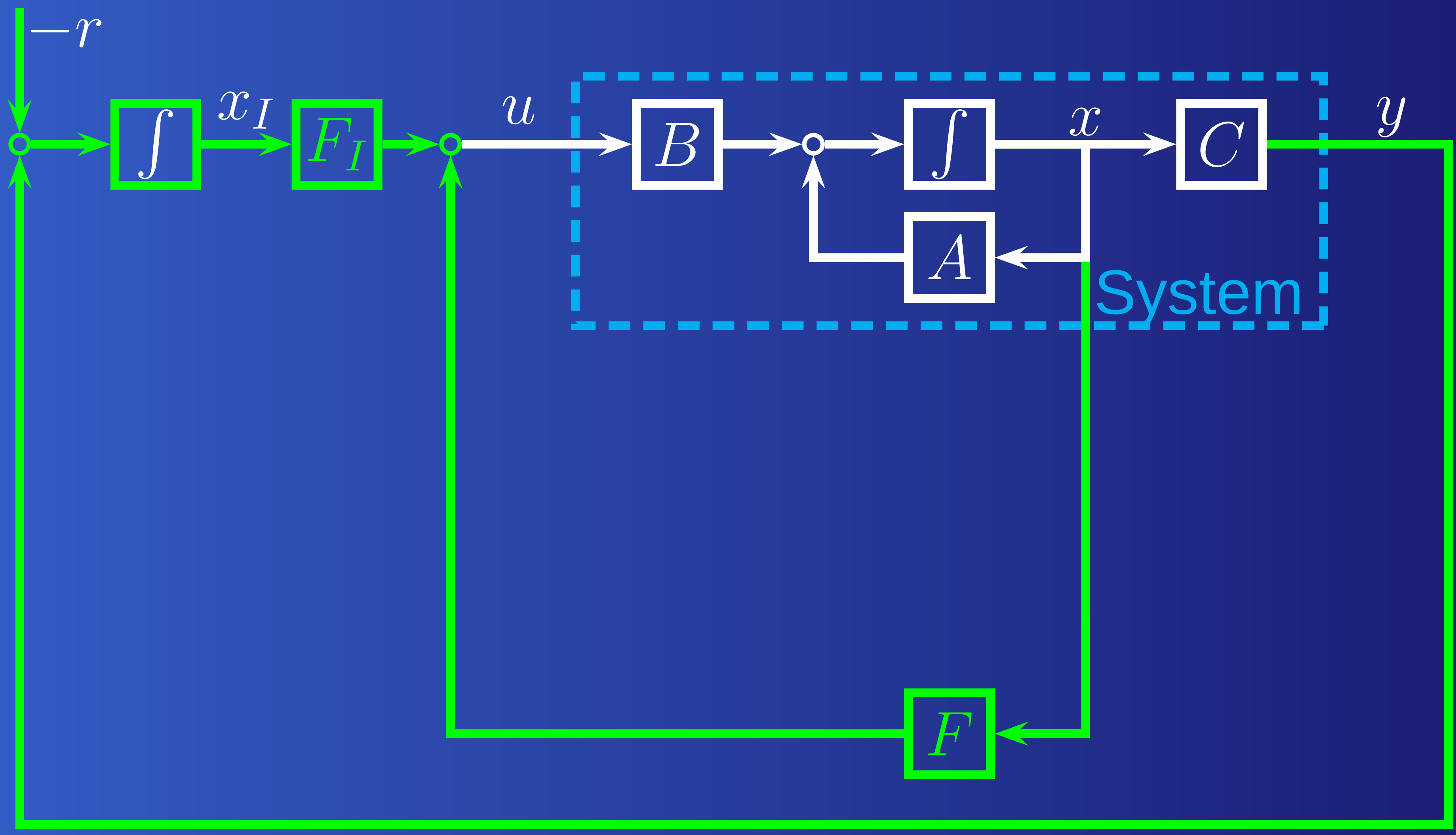
$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u \\ y &= C_e x_e \end{aligned}$$

for which we have to design a state feedback $u = F_e x_e$, where:

$$F_e = \begin{pmatrix} F & F_I \end{pmatrix}, \quad x_e = \begin{pmatrix} x \\ x_I \end{pmatrix}$$

$$A_e = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, B_e = \begin{pmatrix} B \\ 0 \end{pmatrix}, C_e = \begin{pmatrix} C & 0 \end{pmatrix}$$

Integral control (4)



Integral control (5)

If the states are unavailable for feedback, they can be estimated by e.g. a full order observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

where L is chosen such that $A + LC$ is stable with desirable eigenvalues.

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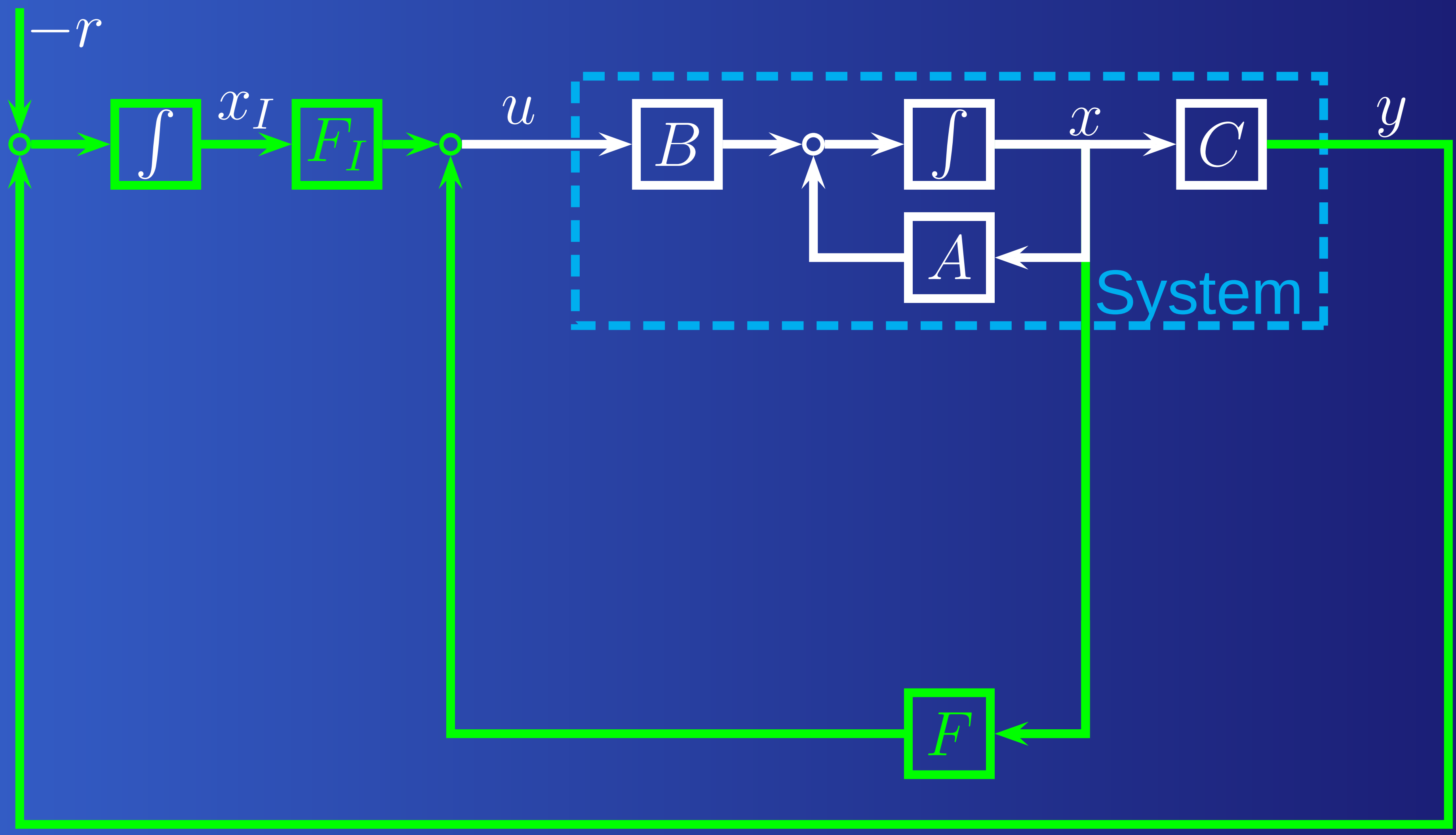
where L is chosen such that $A + LC$ is stable with desirable eigenvalues.

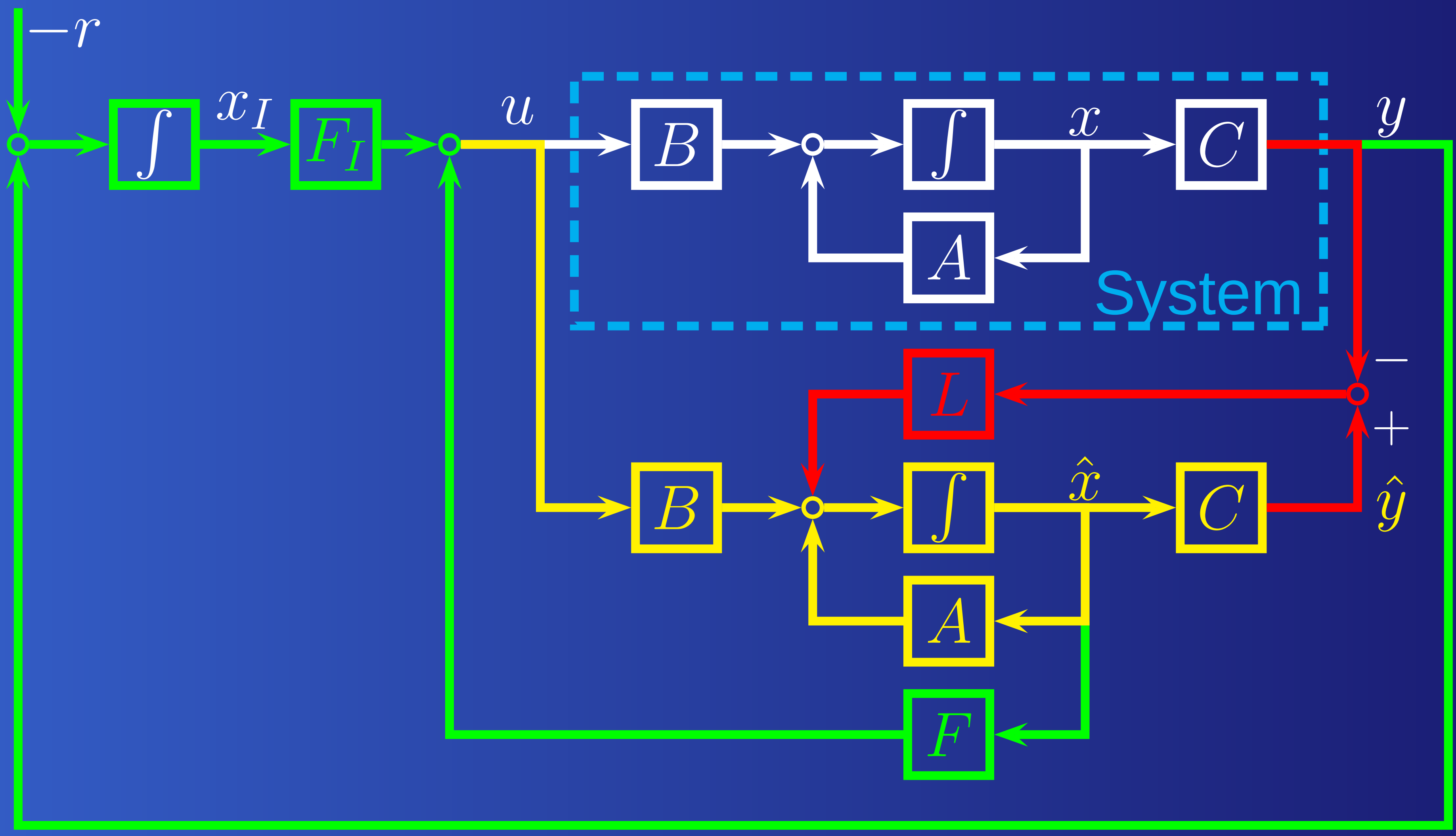
Separation result: The closed loop poles of such an observer based integral control scheme consist of the eigenvalues of

$$A_e + B_eF_e \quad \text{and of} \quad A + LC$$

Integral control (6)



Integral control (6)



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Example: integral control (1)

We consider again the system

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for which we have already computed an observer gain assigning poles in $\{-4, -5\}$:

$$L = \begin{pmatrix} -6 \\ -12 \end{pmatrix}$$

Example: integral control (2)

The extended system becomes:

$$A_e = \left(\begin{array}{c|c} A & 0 \\ \hline C & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & -5 & 0 \\ \hline -3 & 2 & 0 \end{array}\right)$$

$$B_e = \left(\begin{array}{c} B \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} 2 \\ 3 \\ \hline 0 \end{array}\right)$$

$$C_e = \left(\begin{array}{c|c} C & 0 \end{array}\right) = \left(\begin{array}{cc|c} -3 & 2 & 0 \end{array}\right)$$

Example: integral control (3)

Using e.g. controllable canonical form, an extended state feedback can be found, which assigns poles in $\{-3, -4, -5\}$:

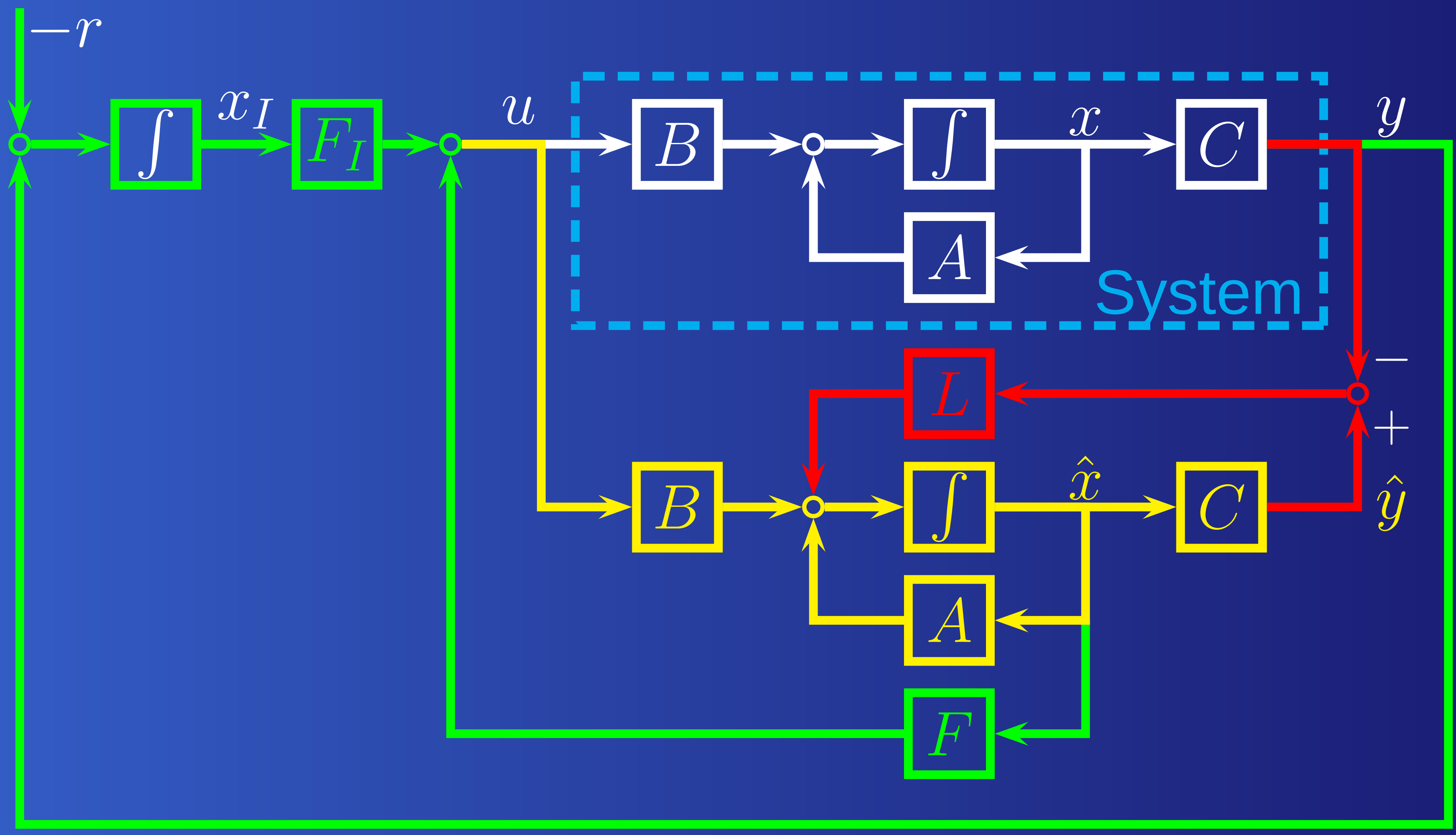
$$F_e = \begin{pmatrix} 117 & -81 & -60 \end{pmatrix}$$

$$\Rightarrow \; F = \begin{pmatrix} 117 & -81 \end{pmatrix}, \qquad F_I = -60$$

The resulting controller can be shown to have the transfer function:

$$-\frac{1}{6s} \cdot \frac{55s^2 + 207s + 200}{s^2 + 18s + 119}$$

Integral control (4)



Example: integral control (5)