State Space Methods

Lecture 2: controllability and state feedback

Jakob Stoustrup

jakob@control.aau.dk
www.control.aau.dk/∼jakob/

Section for Automation & Control

Department of Electronic Systems

Aalborg University

Denmark

Contents

- **Controllability**
- Controllable canonical form
- State feedback and pole assignment

Controllability (1)

A continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $u(t)$ such that for some $T > 0$, $x(T) = \xi$.

A discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $(u(0), u(1), \ldots)$ such that for some $N > 0$, $x(N) = \xi$.

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$x(1) = Ax(0) + Bu(0)$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$x(1) = Ax(0) + Bu(0)$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$
$$x(2) = Ax(1) + Bu(1)$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$\begin{aligned} x(1) &= Bu(0) \\ x(2) &= Ax(1) + Bu(1) \end{aligned}$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$x(1) = Bu(0)$$
$$x(2) = ABu(0) + Bu(1)$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$\begin{aligned} x(1) &= Bu(0) \\ x(2) &= ABu(0) + Bu(1) \\ x(3) &= Ax(2) + Bu(2) \end{aligned}$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$\begin{aligned} x(1) &= Bu(0) \\ x(2) &= ABu(0) + Bu(1) \\ x(3) &= Ax(2) + Bu(2) \end{aligned}$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \qquad x(0) = 0$$

and iterate:

$$\begin{aligned} x(1) &= Bu(0) \\ x(2) &= ABu(0) + Bu(1) \\ x(3) &= A^2Bu(0) + ABu(1) + Bu(2) \end{aligned}$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$\begin{aligned}
x(1) &= Bu(0) \\
x(2) &= ABu(0) + Bu(1) \\
x(3) &= A^2Bu(0) + ABu(1) + Bu(2) \\
&\vdots \\
x(n) &= Ax(n-1) + Bu(n-1)
\end{aligned}$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$\begin{aligned}
x(1) &= Bu(0) \\
x(2) &= ABu(0) + Bu(1) \\
x(3) &= A^2Bu(0) + ABu(1) + Bu(2) \\
&\vdots \\
x(n) &= A\color{green}{x(n-1)} + Bu(n-1)
\end{aligned}$$

Controllability (2)

We consider the discrete time system

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0$$

and iterate:

$$\begin{aligned}
x(1) &= Bu(0) \\
x(2) &= ABu(0) + Bu(1) \\
x(3) &= A^2Bu(0) + ABu(1) + Bu(2) \\
&\vdots \\
x(n) &= A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)
\end{aligned}$$

Controllability (3)

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

Controllability (3)

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

$$x(n) = \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

Controllability (3)

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

$$x(n) = \underbrace{\begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix}}_{\text{Controllability matrix}} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

Controllability (3)

Writing the equation

$$x(n) = A^{n-1}Bu(0) + \ldots + ABu(n-2) + Bu(n-1)$$

in matrix form we obtain:

$$x(n) = \underbrace{\begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix}}_{\text{Controllability matrix}} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

When is $x(n) = \xi$ solvable for any $\xi \in \mathbb{R}^n$?

Controllability (4)

THEOREM. A system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{(continuous time)} \\ x(k+1) = Ax(k) + Bu(k) & \text{(discrete time)} \end{cases}$$

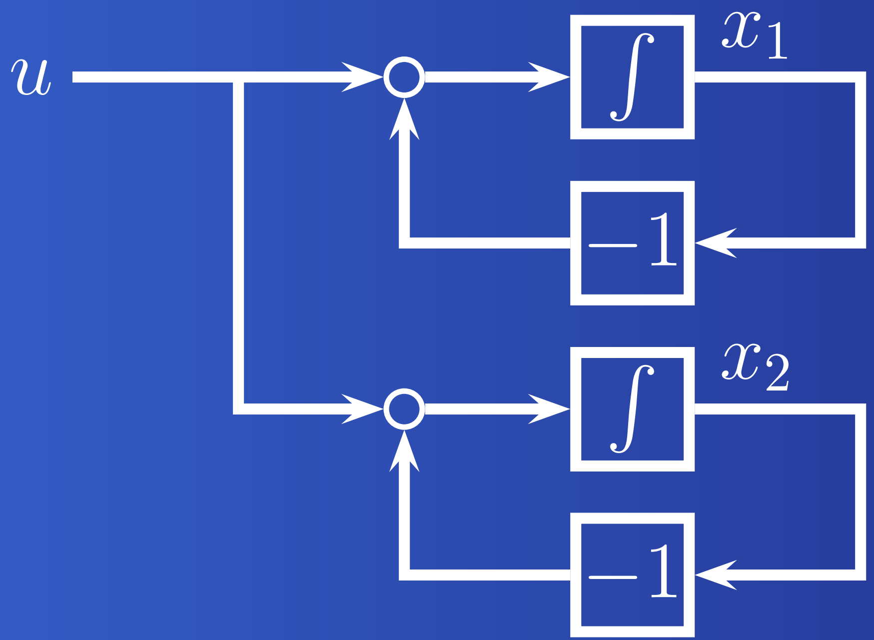
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is controllable if and only if

$$\operatorname{rank} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = n$$

For $m = 1$ this reduces to

$$\det \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \neq 0$$

Example: parallel connection (1)



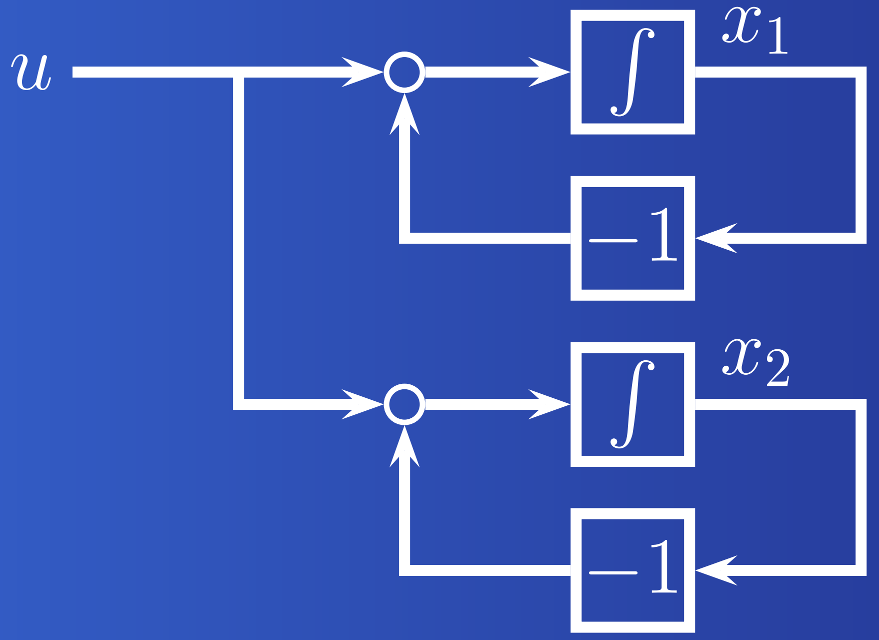
State space equations:

$$\left\{ \begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -x_2 + u \end{aligned} \right\}$$

State space equations in matrix form:

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

Example: parallel connection (2)



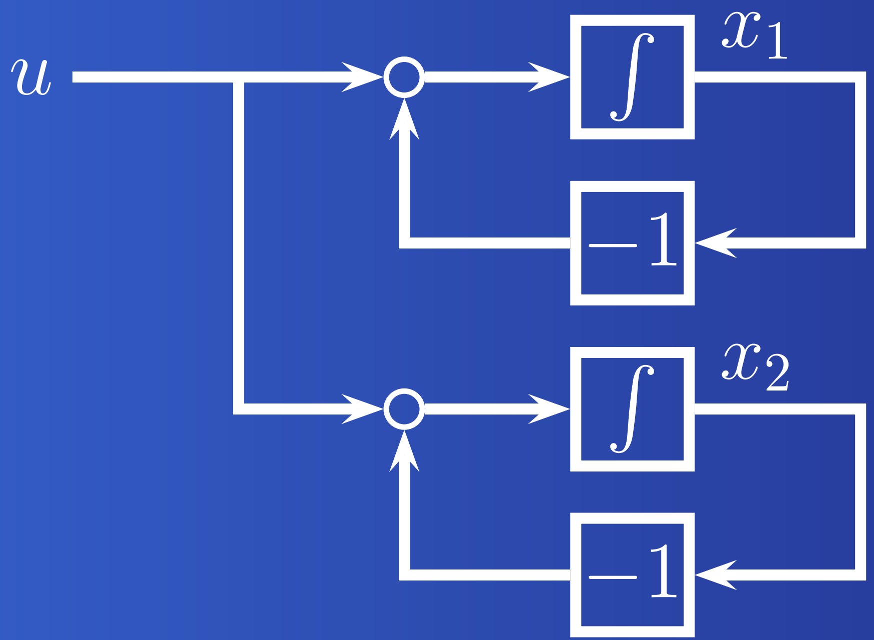
$$\dot{x} = Ax + Bu$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Example: parallel connection (2)



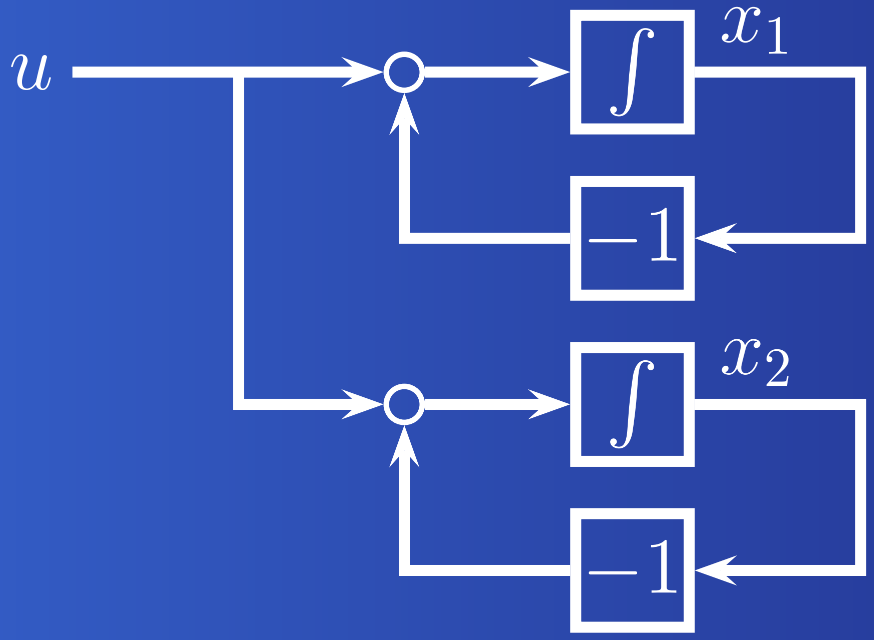
$$\dot{x} = Ax + Bu$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \det(\mathcal{C}) = 0$$

Example: parallel connection (2)



$$\dot{x} = Ax + Bu$$

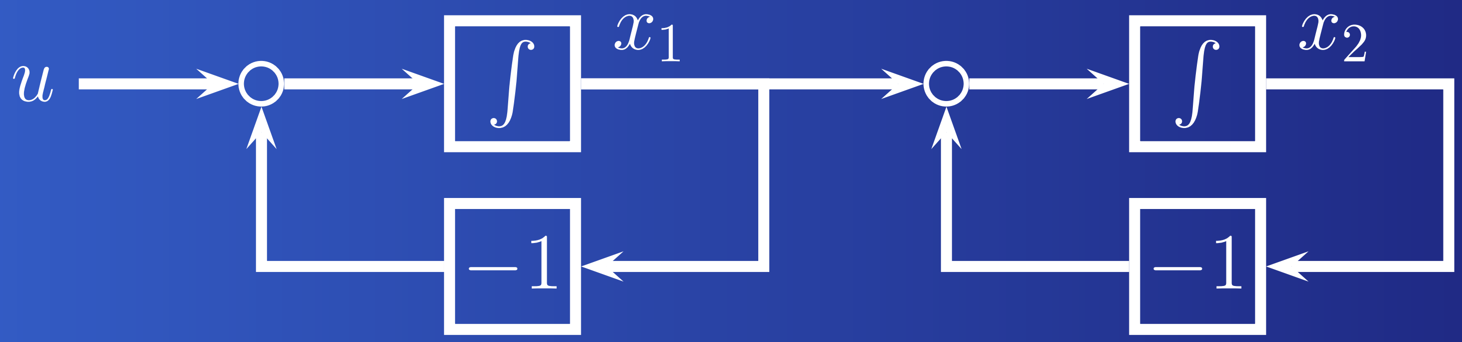
$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \det(\mathcal{C}) = 0$$

$\mathrm{rank}(\mathcal{C}) = 1 < 2 \Longrightarrow$ uncontrollable

Example: series connection



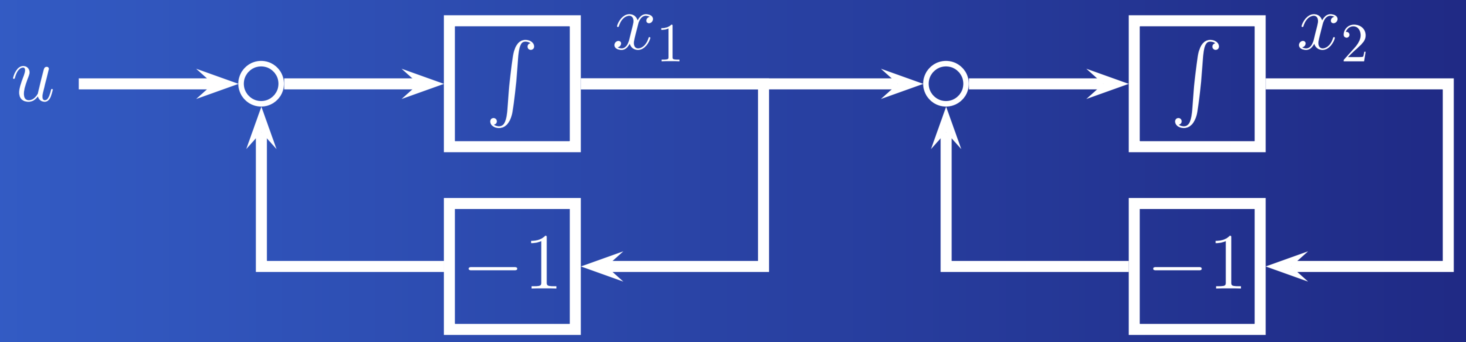
State equations:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + x_1 \end{cases}$$

State space model in matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Example: series connection

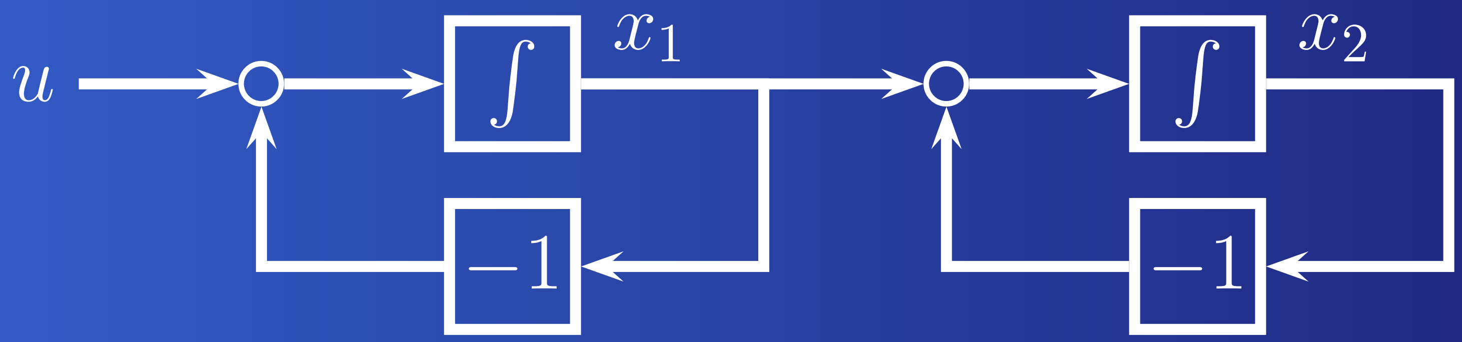


$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Controllability analysis

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Example: series connection

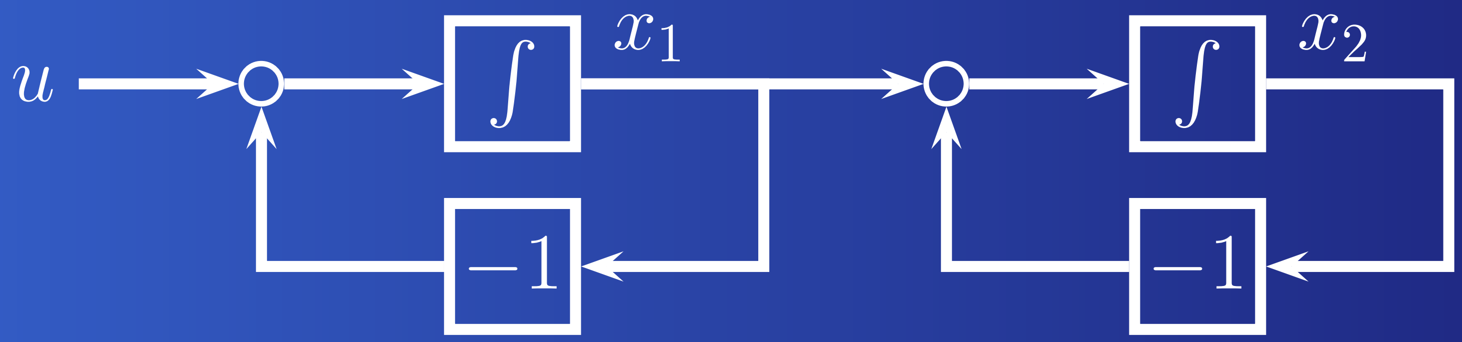


$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Controllability analysis

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

Example: series connection



$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Controllability analysis

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

$\operatorname{rank}(\mathcal{C}) = 2 \implies$ controllable

Contents

- Controllability
- **Controllable canonical form**
- State feedback and pole assignment

Controllable canonical form (1)

Any controllable *single input* system can be written in the form:

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

where

$$A_c = \left(\begin{array}{c|c} \multicolumn{2}{c}{a^T} \\ \hline I_{n-1} & 0_{(n-1)\times 1} \end{array}\right)\,, \quad B_c = \left(\begin{array}{c} 1 \\ \hline 0_{(n-1)\times 1} \end{array}\right)$$

and where $a \in \mathbb{R}^{n\times 1}$, $a^T = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$. It can be shown that

$$\det(\lambda I - A_c) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

Controllable canonical form (2)

For $n = 3$ the controllable canonical form becomes:

$$A_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), \quad B_c = \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right)$$

which is indeed controllable:

$$\mathcal{C}_c = \begin{pmatrix} B_c & A_c B_c & A_c^2 B_c \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_1^2 + a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det(\mathcal{C}) = 1 \neq 0 \Longrightarrow$ system is controllable.

Controllable canonical form (3)

Given a state space model of a controllable system:

$$\dot{x} = Ax + Bu\,, \quad x \in \mathbb{R}^n\,, u \in \mathbb{R}$$

we wish to find a basis transformation $x = Tx_c$, such that:

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,, u \in \mathbb{R}$$

where $A_c = T^{-1}AT$ and $B_c = T^{-1}B$, is in controllable canonical form. We can solve for T^{-1} by rewriting these equations as

$$A_c T^{-1} = T^{-1}A \quad \text{and} \quad B_c = T^{-1}B$$

Controllable canonical form (4)

We consider $n = 3$, and introduce the following notation for the rows of T^{-1}

$$T^{-1} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} , \qquad s_1, s_2, s_3 \in \mathbb{R}^{1\times n}$$

Then we can rewrite the transformation equations $A_c T^{-1} = T^{-1} A$ and $T^{-1} B = B_c$ as:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A \, , \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Controllable canonical form (5)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A\,, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

Controllable canonical form (5)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ \boxed{s_2} \\ s_3 \end{pmatrix} A \, , \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{matrix} \boxed{s_1 = s_2 A} \\ \end{matrix} \right\} , \left\{ \begin{matrix} \\ \\ \end{matrix} \right\}$$

Controllable canonical form (5)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ \boxed{s_3} \end{pmatrix} A\,, \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ \boxed{s_2 = s_3 A} \end{array} \right\}, \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\}$$

Controllable canonical form (5)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A \, , \quad \begin{pmatrix} \boxed{s_1} \\ s_2 \\ s_3 \end{pmatrix} B = \begin{pmatrix} \boxed{1} \\ 0 \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\} \, , \quad \left\{ \begin{array}{c} \boxed{s_1 B = 1} \\ \\ \\ \end{array} \right\}$$

Controllable canonical form (5)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A \, , \quad \begin{pmatrix} s_1 \\ \boxed{s_2} \\ s_3 \end{pmatrix} B = \begin{pmatrix} 1 \\ \boxed{0} \\ 0 \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{c} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\} , \left\{ \begin{array}{c} s_1 B = 1 \\ \boxed{s_2 B = 0} \\ \end{array} \right\}$$

Controllable canonical form (5)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} A, \quad \begin{pmatrix} s_1 \\ s_2 \\ \boxed{s_3} \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ \boxed{0} \end{pmatrix}$$

yields:

$$\left\{ \begin{array}{c} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \left\{ \begin{array}{c} s_1 B = 1 \\ s_2 B = 0 \\ \boxed{s_3 B = 0} \end{array} \right\}$$

Controllable canonical form (6)

Combining the equations

$$\left\{ \begin{array}{c} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \left\{ \begin{array}{c} s_1 B = 1 \\ s_2 B = 0 \\ s_3 B = 0 \end{array} \right\}$$

we obtain

$$s_3 \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

yielding the solution

$$s_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathcal{C}^{-1}, \; s_2 = s_3 A, \; s_1 = s_2 A$$

for nonsingular $\mathcal{C} = \begin{pmatrix} B & AB & A^2B \end{pmatrix}$.

Example: companion form (1)

We consider the system

$$\dot{x} = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u$$
$$y = \begin{pmatrix} -3 & 2 \end{pmatrix} x$$

having the controllability matrix

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

Example: companion form (2)

We compute the rows of T^{-1} by

$$s_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathcal{C}^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -7 & 5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \end{pmatrix}$$

$$s_1 = s_2 A = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}$$

Thus,

$$T^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad \Longrightarrow \quad T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

Example: companion form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$$

and

Example: companion form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right)$$

and

Example: companion form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right)$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example: companion form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right)$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$$

Example: companion form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right) \Rightarrow \det(\lambda I - A) = \lambda^2 + 3\lambda + 2$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ \hline 0 \end{pmatrix}$$

Example: companion form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \left(\begin{array}{cc} -3 & -2 \\ \hline 1 & \multicolumn{1}{|c}{0} \end{array} \right) \Rightarrow \det(\lambda I - A) = (\lambda + 1)(\lambda + 2)$$

and

$$B_c = T^{-1}B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \left(\begin{array}{c} 1 \\ \hline 0 \end{array} \right)$$

Contents

- Controllability
- Controllable canonical form
- **State feedback and pole assignment**

State feedback (1)

For a state space model

$$\dot{x} = Ax + Bu$$

a *state feedback* is a feedback of the form

$$u = Fx$$

Combining these two equations, we obtain:

$$\dot{x} = Ax + BFx = (A + BF)x$$

Thus, the result of a state feedback is a system with a modified system matrix, and thus with modified poles.

State feedback (2)

For a single input system in companion form, a state feedback takes a particular simple form:

$$A_c = \begin{pmatrix} a_1 & a_2 & a_3 \\ \hline 1 & 0 & \multicolumn{1}{|c}{0} \\ 0 & 1 & \multicolumn{1}{|c}{0} \end{pmatrix}, \quad B_c = \begin{pmatrix} 1 \\ \hline 0 \\ 0 \end{pmatrix}$$

Applying the feedback $u = Fx$ with

$$F_c = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$$

State feedback (3)

We obtain:

$$A_c + B_c F_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) + \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right) \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} a_1 + f_1 & a_2 + f_2 & a_3 + f_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

State feedback (4)

Thus, the characteristic polynomium has been changed from

$$\det(\lambda I - A_c) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

to

$$\det(\lambda I - (A_c + B_c F_c)) = \lambda^n - (a_1 + f_1)\lambda^{n-1} - \ldots - (a_n + f_n)$$

By choosing $f_1, \ldots, f_n$ appropriately, *any* closed loop pole configuration can be obtained. This is known as *pole assignment*.

Algorithm for pole assignment

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ be given.

1. Choose desired closed loop polynomial $\det(\lambda I - (A + BF)) = \lambda^n + a_{\text{cl},1}\lambda^{n-1} + \ldots + a_{\text{cl},n}$.

Algorithm for pole assignment

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ be given.

1. Choose desired closed loop polynomial $\det(\lambda I - (A + BF)) = \lambda^n + a_{\text{cl},1}\lambda^{n-1} + \ldots + a_{\text{cl},n}$.
2. Determine T, such that $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ are in controllable canonical form.

Algorithm for pole assignment

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ be given.

1. Choose desired closed loop polynomial
   $\det(\lambda I - (A + BF)) = \lambda^n + a_{\text{cl},1}\lambda^{n-1} + \ldots + a_{\text{cl},n}$.
2. Determine T, such that $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ are in controllable canonical form.
3. Determine open loop polynomial
   $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$

Algorithm for pole assignment

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ be given.

1. Choose desired closed loop polynomial $\det(\lambda I - (A+BF)) = \lambda^n + a_{\text{cl},1}\lambda^{n-1} + \ldots + a_{\text{cl},n}$.
2. Determine T, such that $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ are in controllable canonical form.
3. Determine open loop polynomial $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$
4. Define $F_c = \begin{pmatrix} a_1 - a_{\text{cl},1} & \ldots & a_n - a_{\text{cl},n} \end{pmatrix}$.

Algorithm for pole assignment

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ be given.

1. Choose desired closed loop polynomial $\det(\lambda I - (A + BF)) = \lambda^n + a_{\text{cl},1}\lambda^{n-1} + \ldots + a_{\text{cl},n}$.
2. Determine T, such that $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ are in controllable canonical form.
3. Determine open loop polynomial $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$
4. Define $F_c = \begin{pmatrix} a_1 - a_{\text{cl},1} & \ldots & a_n - a_{\text{cl},n} \end{pmatrix}$.
5. Compute resulting feedback gain $F = F_c T^{-1}$.

Example: pole assignment (1)

We consider again the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} -3 & 2 \end{pmatrix} x\end{aligned}$$

for which we would like to move the poles to $\{-4, -5\}$.

Example: pole assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$

Example: pole assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow A_c = \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right), B_c = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$

Example: pole assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow A_c = \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right), B_c = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$

Example: pole assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow A_c = \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right), B_c = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $F_c = \begin{pmatrix} 3-9 & 2-20 \end{pmatrix} = \begin{pmatrix} -6 & -18 \end{pmatrix}$

Example: pole assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow A_c = \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right), B_c = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $F_c = \begin{pmatrix} 3-9 & 2-20 \end{pmatrix} = \begin{pmatrix} -6 & -18 \end{pmatrix}$
5. $$F = F_c T^{-1} = \begin{pmatrix} -6 & -18 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$
   $$= \begin{pmatrix} 42 & -30 \end{pmatrix}$$

Example: pole assignment (3)