Model of the firm with capital, debt, fixed adjustment cost and irreversible investment

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A firm chooses its investment in physical capital and short-term debt to maximizes its intertemporal discounted profits. The Bellman equation writes

$$V(z_t, b_t, k_t) = d_t + \beta \mathbb{E}\left[V(z_{t+1}, b_{t+1}, k_{t+1})\right]$$

where we define:

1. Profits: $d_t = y_t + (1-\delta)k_t - b_t(1+r) - \phi(k_{t+1}, k_t) + b_{t+1} - k_{t+1} - c_f$
2. Output: $y_t = z_t k_t^\alpha$, where z_t is the total factor productivity, following an AR-1 stochastic process
3. Investment cost: $\phi(k_{t+1}, k_t) = \frac{\psi}{2}\left(\frac{k_{t+1}-(1-\delta)k_t}{k_t}\right)^2 k_t + \mathbf{1}_{k_{t+1}>(1-\delta)k_t} k_t \xi$. Note that the fixed investment cost creates a non-convexity in the investment problem.

The firm is also subject to the following constraints:

1. Irreversibility constraint on capital: $k_{t+1} \geq (1-\delta)k_t$.
2. Non-negativity of profits: $d_t \geq 0$ (equivalent to an impossibility to issue equity).
3. Collateral constraint on debt: $b_{t+1} \geq \nu k_t$.

1 A nested-VFI solution

Because of the fixed-investment cost, there is an inaction region in the policy function for physical capital. This problem turns out quite hard to solve, and for the sake of simplicity, let us assume that the collateral constraint on debt actually writes $b_{t+1} \geq \bar{b}$.

It is thus useful to rewrite the Bellman equation as the maximum between investing, and thus paying the cost ξk_t, and not investing:

$$V(z_t, b_t, k_t) = \max\left[V^{ina}(z_t, m_t, k_t), V^{inv}(z_t, b_t, k_t)\right],$$

where

$$V^{ina}(z_t, m_t, k_t) = \max_{b_{t+1}} d_t + \beta \mathbb{E}\left[V(z_{t+1}, b_{t+1}, (1-\delta)k_t)\right]$$
$$\text{s.t.}$$
$$m_t = y_t - b_t(1+r)$$

m_t denote the level of cash on hands, and

$$V^{inv}(z_t, b_t, k_t) = \max_{k_{t+1}} V^{ina}\left(z_t, \tilde{m}, \frac{k_{t+1}}{(1-\delta)}\right)$$
$$\text{s.t.}$$
$$\tilde{m} = y_t + (1-\delta)k_t - b_t(1+r) - \phi(k_{t+1}, k_t) - k_{t+1}$$

We could solve this problem using standard VFI method, or a nested-EGM as in Druedahl (2021), by first solving the inaction problem, which is a simple one-choice maximization problem, and conditional on this value function V^{ina}, we could solve the problem V^{inv}. The key intuition to note here is that we can decompose the maximization problem into a sequential choice: you first choose k_{t+1}, and then b_{t+1}, knowing you would then decide not to change your stock of physical capital again.

We can do so because in the investment-stage, we can compute the new cash-on-hands level that will be received in the inaction stage, and scale the stock of physical capital $k_t = \frac{k_{t+1}}{1-\delta}$ so that in the RHS of the inaction problem, we will now have $V(z_{t+1}, b_{t+1}, k_{t+1})$.

This trick, however, does not work when we include the collateral constraint $b_{t+1} \geq \nu k_t$. With this additional constraint, when we "plug" the value $\frac{k_{t+1}}{1-\delta}$ in the $V^{ina}(z_t, \tilde{m}_t, \frac{k_{t+1}}{1-\delta})$, it will erroneously relax the borrowing constraint in the inaction stage. How to solve this issue?