Imperfect Market Selection

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1 The Model

The present model is a continuous-time version of the workhorse model of firm dynamics by Hopenhayn (1992) with convex adjustment costs to labor. Let us comment on these assumptions. First, the continuous time assumption brings a lot of analytical tractability and speeds up considerably the numerical resolution of the model, as discussed in Achdou et al. (2020). Second, firms face labor adjustments frictions. As a consequence, workforce size is adjusting slowly and becomes a state variable of the firm.

Taken together, these assumptions entail that firms can experience large and persistent mismatches between their revenues and costs. The idea is the following. After a series of negative productivity shock, firms' revenues decrease by a lot while costs only adjust partially due to the adjustment frictions. In fact, the mismatch between costs and revenues may be so large that firms anticipate they will earn negative profits for a protracted amount of time and hence prefer to exit immediately.

Therefore, the model is able to match the fact that exit risk is not null for high productivity firms, as observed in the data. Even high productive firms - which are far away from the exit boundary - might experience a long series of bad negative shocks. However, note that the more productive, the longer the series of bad productivity shocks needed and hence the smaller the probability that this happens. Hence, exit risk is smoothly decreasing in the level of firm productivity. By contrast, the baseline Hopenhayn model predicts a degenerate, staircase distribution of exit risk.

1.1 Setup

Time is continuous and discounted at the risk-free interest rate r. The economy is populated by a continuum of firms indexed by their productivity z and their workforce size l. Productivity follows a Markov process, which I detail later. Let define $g_t(z,l)$ the distribution of firms over productivity and workforce size at t.

Technology and Market Structure. Firms are risk-neutral. To operate in the market, they must pay a fixed per-period operating cost c_f. Active firms use l units of labor to produce

$$y = zl^{\alpha} \quad \text{with} \quad \alpha < 1$$

units of output. Note that the production technology features decreasing returns to scale, which entails here that firms' optimal size is non-degenerate.

The output market and the labor markets are competitive and thus the output price p and the wage rate w are taken as given by the firm. I consider a partial equilibrium version of the model where the output price p and the wage rate w are exogenously determined. A more general model would allow for both prices to be determined endogenously. This is left for future research.

I define the firm current gross profits as follows

$$\pi(z,l) = pzl^{\alpha} - wl - c_f$$

Labor adjustment costs. Firms face state-dependent labor adjustment frictions. They must pay a convex adjustment costs $c(x)$ to adjust their workforce by an amount x. In the following, I assume standard quadratic adjustment costs

$$c(x) = \frac{\phi}{2} \cdot x^2$$

where the parameter ϕ control the size of adjustment costs. This entails that the law of motion of the firm workforce size l_t simply writes

$$\dot{l}_t = x_t$$

where I adopt the convention that $\dot{x} = dx/dt$ denotes the time derivative of variable x.

Productivity process. Firm productivity follows a diffusion process. In particular, I assume that the log productivity of firms $\log z_t$ follows an Ornstein-Uhlenbeck process. This is the natural continuous time analog of an AR(1) processes.

$$d\log z_t = -\theta \log z_t dt + \sigma dW_t$$

where where θ controls the persistence of productivity and σ is the standard deviation of the shocks. The stationary distribution of $\log z_t$ is then a Normal distribution with mean zero and variance $\nu = \sigma^2/(2\theta)$.

Finally, the process followed by firm productivity z_t (not $\log z_t$) can be deduced from Ito's lemma. It writes

$$dz_t = \left(-\theta \log z_t + \frac{\sigma^2}{2}\right) z_t dt + \sigma z_t dW_t$$

For future reference, I define the productivity process trend $\mu(z) \equiv \left(-\theta \log z_t + \sigma^2/2\right)$ and the productivity process standard deviation $\sigma(z) \equiv \sigma z_t$

Exit and entry decision. Active firms decide optimally when to exit. I define $\tilde{v}(z,l)$ as the value of an active firm with productivity z and workforce size l. Firms exit the market whenever the value of being active $\tilde{v}(z,l)$ falls below a scrap value v^*. Finally, let $v(z,l)$ be the unconditional value of a firm with productivity z and workforce l. It follows from the endogenous exit decision that

$$v(z,l) = \max\{\tilde{v}(z,l), v^*\}$$

Entrants decide optimally when to enter the market. Potential entrants must pay a fixed cost c_e to enter. Moreover, potential entrants do not observe their future productivity level, which is drawn from a the distribution $\Psi(z)$ with support $[\underline{z}, \bar{z}]$ and associated density $\psi(z)$. I choose lower and upper bounds $\underline{z}$ and $\bar{z}$ to be such that the firm never exits immediately after entering.

The net expected gain from entry writes

$$\int_{\underline{z}}^{\bar{z}} \psi(z) v(z,l) dz - c_e$$

The standard assumption of free entry would entail that the mass of entrants is such that the net expected gain from entry is null. However, this is numerically very unstable. I adopt an alternative specification presented in Moll (2017).

In this alternative specification, the mass of entrants writes

$$m_t = \exp \eta \left(\int_{\underline{z}}^{\bar{z}} \psi(z) v(z,l) dz - c_e \right)$$

The parameter η controls the elasticity of entry to net expected gain from entry. The standard free entry case is recovered when $\eta \to +\infty$.

1.2 The HJB equations

Value to stay. The value of being active with current productivity z and workforce size l satisfies the following Hamilton-Jacobi-Bellman equation (HJB hereafter)

$$r\tilde{v}(z,l) = \max_x \left\{ \pi(z,l) - c(x) + \partial_l v(z,l)x + \mu(z)\partial_z v(z,l) + \frac{\sigma(z)^2}{2}\partial_{zz} v(z,l) \right\}$$

HJB equations are simply the continuous time equivalent of discrete time Bellman equations. The equation states that the flow value $r\tilde{v}(z,l)$ must equalize the sum of flow payoff $\pi(z,l) - c(x)$ and expected capital gains coming from changes in state.

Capital gains are the sum of three terms. The first term accounts for the deterministic change in workforce size l, which happens with intensity x and entails a change in value $\partial_l v(z,l)$. The last two terms account for stochastic changes in productivity z. Stochastic changes in productivity happen with intensity $\mu(z)$ and entail a change $\partial_z v(z,l)$. The third term features standard deviation σ and follows the application of the Ito lemma.

Note that the unconditional firm value $v(z,l)$ rather than the value of being active $\tilde{v}(z,l)$ appears on the right-hand side of the equation above. This accounts for the fact that the firm might decide to exit after changes in state variables have taken place.

Optimal adjustment decision. The firm must decide an optimal amount of labor adjustment x^*. In continuous time, increments in states are independent from each other. Hence, the FOC is simply

$$-c'(x^*) + \partial_l v(z,l) = 0$$

Using the definition of $c(x)$ and rearranging terms yields the policy function of labor adjustment

$$x^*(z,l) = \frac{1}{\phi}\partial_l v(z,l)$$

Unconditional value. Let us define $\mathcal{S}$ the region of states where the firm stays. By definition, for state variables in the stay region, the unconditional and conditional value functions are equal $v(z,l) = \tilde{v}(z,l)$. Hence, the problem faced by the firm writes

$$(z,l) \in \mathcal{S} \ : \quad v(z,l) \geq v^*, \quad rv(z,l) = \max_x \left\{ \pi(z,l) - c(x) + \partial_l v(z,l)x + \mu(z)\partial_z v(z,l) + \frac{\sigma(z)^2}{2}\partial_{zz}v(z,l) \right\}$$
$$(z,l) \notin \mathcal{S} \ : \quad v(z,l) = v, \quad rv(z,l) \geq \max_x \left\{ \pi(z,l) - c(x) + \partial_l v(z,l)x + \mu(z)\partial_z v(z,l) + \frac{\sigma(z)^2}{2}\partial_{zz}v(z,l) \right\}$$

This formulation of the problem allows to get rid of the value to stay $\tilde{v}(z,l)$ and provides a description in terms of the unconditional value $v(z,l)$ only. It brings numerical tractability.

These inequalities can be rewritten more compactly as follows.

$$\min \left\{ rv(z,l) - \max_x \left\{ \pi(z,l) - c(x) + \partial_l v(z,l)x + \mu(z)\partial_z v(z,l) + \frac{\sigma(z)^2}{2}\partial_{zz}v(z,l) \right\}, v(z,l) - v^* \right\} = 0$$

This equation is called an "HJB variational Inequality" or HJBVI in short. Solving the HJBVI yields the exit decision of the firm, that is the stay region $\mathcal{S}$.

Evolution of the firm distribution. The evolution of the mass of firms across the state space $p_t(z,l)$ is governed by a Fokker-Planck equation

$$\partial_t g(z,l) = -\partial_l \left[x(z,l) g_t(z,l) \right] - \partial_z \left[\mu(z) g_t(z,l) \right] + \frac{1}{2}\partial_{zz}\left[\sigma^2(z) g_t(z,l) \right]$$

2 Numerical Solution

In this section, I explain the numerical method chosen to solve the model described above. It contains two different parts. The first part explains the method for solving the HJBVI. The second part explains the method for solving the stationary firm distribution.

2.1 Solving the HJBVI

To solve the HJBVI, I use a finite difference method. The finite difference is the continuous-time equivalent of the value function iteration (VFI) method. Indeed, in both cases the general idea of the method is to let the time-dependent value function relax to its steady-state value. The method presented here follow closely different lecture notes from Moll and the technical appendix of Achdou et al. (2020).

The value function is approximated on a grid of $I \times J$ points, where I is the number of points along the productivity dimension and J is the number of points along the labor dimension. I use equispaced grids, with dz the distance between points along the productivity dimension and dl the distance between points along the labor size dimension. Moreover, I use the notation $v_{i,j} \equiv v(z_i, l_j)$, etc.

The approximated problem writes

$$\min\left\{rv_{i,j} - \pi_{i,j} + \frac{1}{2\phi}\left(\partial_l v_{i,j}\right)^2 + \mu_i \partial_z v_{i,j} + \frac{\sigma_i)^2}{2}\partial_{zz} v_{i,j}, v_{i,j} - v^*\right\} = 0$$

The numerical solution uses three key ingredients

- the upwind scheme gives a robust approximation of the value function derivatives
- the implicit method gives a robust and efficient updating scheme of the value function
- the linear complementarity problem (LCP) solver solves the discretized HJBVI

Upwind scheme. Solving the HJBVI equation above requires computing numerical derivatives. The partial derivative along the labor dimension $\partial_l v$ is approximated using an upwind method, ie. using either a forward or backward difference depending on the sign of the drift

$$\partial_{l,B} v(z_i, l_j) \approx \frac{v_{i,j} - v_{i,j-1}}{dl}$$
$$\partial_{l,F} v(z_i, l_j) \approx \frac{v_{i,j+1} - v_{i,j}}{dl}$$

The drift is computed using either forward or backward difference.

$$x_{i,j,F} = \phi^{-1}\partial_{l,F} v_{i,j} \quad , x_{i,j,B} = \phi^{-1}\partial_{l,B} v_{i,j}$$

For concave value functions, the upwind scheme uses the forward difference when the forward drift is positive and the backward drift when backward drift is negative.

$$\partial_l v_{i,j} = \partial_{l,F} v_{i,j} \mathbb{1}\{x_{i,j,F} > 0\} + \partial_{l,B} v_{i,j} \mathbb{1}\{x_{i,j,B} < 0\} + 0 \cdot \mathbb{1}\{x_{i,j,F} \leq 0 \leq x_{i,j,B}\}$$

The upwind scheme sets the partial derivative at zero when trends are of opposite signs : intuitively, this means that we are ”on top of a hill”. Note that cases where $x_{i,j,B} < x_{i,j,F}$ are not possible. It follows from the assumption of concavity, which entails $\partial_{lB} v_{ij} < \partial_{lB} v_{ij}$ and therefore $x_{i,j,F} < x_{i,j,B}$.

For value functions which are not concave, however, the upwind scheme is a bit more complex. Indeed, situations where $x_{i,j,F} < 0$ and $x_{i,j,B} > 0$ become possible and we need a rule to decide which difference to use. The general idea is to use the difference which gives the largest gain in terms of Hamiltonian $H_{i,j,s} = -c(x_{i,j,s}) - x_{i,j,s} \partial_{l,s} v_{i,j}$ for $s = B, F$. It is likely that the value function displays some non-concavity around the exit region. Thus, I use this more general form of the upwind scheme.

Similarly, I use an upwind method for approximating the partial derivatives along the productivity dimension. The second-order derivative is computed using a central difference

$$
\begin{aligned}
\partial_{z,B} v(z_i, l_j) &\approx \frac{v_{i,j} - v_{i-1,j}}{dz} \\
\partial_{z,F} v(z_i, l_j) &\approx \frac{v_{i+1,j} - v_{i,j}}{dz} \\
\partial_{zz} v(z_i, l_j) &\approx \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(dz)^2}
\end{aligned}
$$

Implicit method. I use an implicit method to update the value function in the stay region. I start with an initial guess $v^0 = (v^0_{i,j})_{i,j}$ and then update $v^n_{i,j}$ for $n \geq 1$ in the stay region according to

$$
\frac{v^{n+1}_{i,j} - v^{n+}_{i,j}}{\Delta} + r v^{n+1}_{i,j} = \pi_{i,j} - c(x^n_{i,j}) + x^n_{i,j} v^{n+1}_{i,j} + \mu_i \partial_z v^{n+1}_{i,j} + \frac{\sigma_i}{2} \partial_{zz} v^{n+1}_{i,j}
$$

Note the $n+1$ superscript on the right-hand side of the equation, which is specific to the implicit method. By contrast, the explicit method uses the value function at iteration n on the right-hand side of the equation. The parameter Δ is the step size of the adjustment along the time-dimension. It controls the magnitude of the update. The advantage of the implicit method is that it is robust to large step sizes. By contrast, the explicit method needs sufficiently small step sizes and thus is both less robust and efficient.

Substituting the approximated partial derivatives yields

$$
\begin{aligned}
\frac{v^{n+1}_{i,j} - v^{n+}_{i,j}}{\Delta} + r v^{n+1}_{i,j} =& \pi_{i,j} - c(x^n_{i,j}) + (x^n_{i,j})^+ \frac{v^{n+1}_{i,j+1} - v^{n+1}_{i,j}}{dl} + (x^n_{i,j})^- \frac{v^{n+1}_{i,j} - v^{n+1}_{i,j-1}}{dl} \\
&+ \mu_i^+ \frac{v^{n+1}_{i+1,j} - v^{n+1}_{i,j}}{dz} + \mu_i^- \frac{v^{n+1}_{i,j} - v^{n+1}_{i-1,j}}{dz} + \frac{\sigma_i}{2} \frac{v^{n+1}_{i+1,j} - 2v^{n+1}_{i,j} + v^{n+1}_{i-1,j}}{(dz)^2}
\end{aligned}
$$

Collecting terms with the same superscripts, one finds

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n+}}{\Delta} = rv_{i,j}^{n+1} + \pi_{i,j} - c(x_{i,j}^{n}) + \alpha_{i,j}v_{i,j-1}^{n+1} + (\beta_{i,j} + \nu_i)v_{i,j}^{n+1} + \gamma_{i,j}v_{i,j+1}^{n+1} + \chi_i v_{i-1,j}^{n+1} + \zeta_i v_{i+1,j}^{n+1}$$

with the loadings defined as follows

$$\alpha_{i,j} = -\frac{(x_{i,j}^n)^-}{dl}, \quad \beta_{i,j} = -\frac{(x_{i,j}^n)^+}{dl} + \frac{(x_{i,j}^n)^-}{dl}, \quad \gamma_{i,j} = -\frac{(x_{i,j}^n)^+}{dl}$$
$$\chi_i = -\frac{\mu_i^-}{dz} + \frac{\sigma_i}{2(dz)^2}, \quad \nu_i = \frac{\mu_i^-}{dz} - \frac{\mu_i^+}{dz} - \frac{\sigma_i}{(dz)^2}, \quad \zeta_i = \frac{\mu_i^+}{dz} + \frac{\sigma_i}{2(dz)^2}$$

In matrix form, this can be rewritten as follows

$$\frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta} + r\boldsymbol{v}^{n+1} = \pi^n - c(x^n) + \boldsymbol{A}(\boldsymbol{v}^{\boldsymbol{n}})\boldsymbol{v}^{n+1} \tag{1}$$

where $\boldsymbol{v}$ is a $I \times J$ vector and $\boldsymbol{A}$ is a $(I \times J)$ squared matrix. I do not write explicitly the matrix A and let the interest reader check the numerical appendix from Achdou et al. (2020). The matrix $\boldsymbol{A}$ is a key variable for the finite difference method. It summarizes all information about possible transitions across the state space. In fact, it can be checked that it is (right) transition rate matrix, that is its columns sum to zero.

The linear complementary problem. The discretized HBJVI can be solved as a linear complementarity problem, for which numerical solvers exist. Indeed, let substitute equation (2) in equation (1)

$$\min\left\{\frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta} + r\boldsymbol{v}^{n+1} - \pi^n + c(\boldsymbol{x}^n) - \boldsymbol{A}\boldsymbol{v}^{n+1}, \boldsymbol{v}^{n+1} - v^*\right\} = 0$$

This can be rewritten as follows

$$\min\{\boldsymbol{B}\boldsymbol{z} + \boldsymbol{q}, \boldsymbol{z}\} = 0$$

where the matrices and vectors $\boldsymbol{B}$, $\boldsymbol{q}$, $\boldsymbol{z}$ are defined as follows

$$\boldsymbol{B} \equiv \left((\frac{1}{\Delta} + r\right)I - \boldsymbol{A}, \quad \boldsymbol{q} \equiv -\pi^n - \frac{\boldsymbol{v}^n}{\Delta} + \boldsymbol{B}\boldsymbol{v}^*, \quad \text{and} \quad \boldsymbol{z} \equiv \boldsymbol{v}^{n+1} - \boldsymbol{v}^*$$

Formulated this way, the discretized HJBVI problem can be solved as a LCP problem. The solution of the LCP solver is $\boldsymbol{z}^*$, from which $\boldsymbol{v}^{n+1} = \boldsymbol{z} + \boldsymbol{v}^*$ is easily recovered. In other words, the LCP solver yields the updated value function.

As with the VFI method, the value function is updated until the point where the value function has converged. That is, the value function is updated until the distance between $\boldsymbol{v}^{n+1}$

and v^n - as measured by the L^∞ norm - is small enough. In the following, I denote by $\boldsymbol{v}$ the stationary value function.

Exit region The exit region can be computed by finding the region of the state space where $\boldsymbol{v} = \boldsymbol{v}^*$.

2.2 Solving the Stationary Firm Distribution.

Without entry and exit. In the absence of entry and exit, the stationary distribution of firms over the state space can be easily computed once the HJB has been solved. Indeed, the evolution of the firm distribution is governed by A', the transpose of the transition matrix from the discretized HJB. Then, the discretized Fokker-Planck writes

$$\partial_t \boldsymbol{p}_t = \boldsymbol{A}_t' \boldsymbol{p}_t$$

where $\boldsymbol{p}_t$ is a $I \times J$ column vector. The reason is that the Fokker-Planck equation solves the same problem as the HJB but forward in time rather than backward in time. Intuitively, taking the the transpose of the transition matrix reverses the flows' direction : for example, $i \to j$ (forward) vs. $j \to i$ (backward). More technically, this follows from the fact that the backward HJB operator is the self-adjoint operator of the forward Fokker-Planck operator.

From there, it is easy to compute the steady-state distribution p. At the steady-state, the time derivative term is equal to zero and hence the stationary distribution solves

$$0 = \boldsymbol{A}' \boldsymbol{p}$$

The stationary distribution is simply the eigenvector v_0 associated with the eigenvalue zero $\lambda_0 = 0$, which always exists because $\boldsymbol{A}$ is a transition matrix. The eigenvector is normalized such that its elements sum to unity, ie. such that it is a probability distribution.

With entry and exit. However, the problem becomes a bit more involved when firms enter and exit. First, an additional entry term shows up in the Fokker-Planck equation. In the stay region, the discretized steady-state Fokker-Planck equation becomes

$$0 = \sum_j A_{j,i} p_j + m\psi_i \quad i \in \mathcal{S}$$

where ψ_i is the probability to enter at the point i of the state space and m is the mass of entry, which at steady-state is also equal to the mass of exit. Note that firms do not enter directly in the exit region, that is we have $\psi_i = 0$ for $i \notin \mathcal{S}$.

Second, the transition matrix must be adjusted to ensure there is no mass of firms in the exit region. Hence, let me define the transformed transition matrix $\boldsymbol{B}$, which takes entry and exit into account

$$
\begin{aligned}
i \in \mathcal{S}, \forall j \quad & B_{i,j} = A_{i,j} \\
i \notin \mathcal{S}, \forall j \quad & B_{i,j} = 0
\end{aligned}
$$

The second line states that the transitions rates from the exit region are set equal to zero. It does not mean that firms do not exit, but rather that they do not appear in the distribution of *active* firms. Moreover, recall that exiters are are *immediately* replaced by new entrants, such that the mass of active firms is constant at steady-state.

Finally, the discretized steady-state distribution of active firms $\boldsymbol{p}$ is the solution to the modified Fokker-Planck equation

$$
0 = \boldsymbol{B}'\boldsymbol{p} + m\boldsymbol{\psi} \quad \implies \quad \boldsymbol{p} = -m(\boldsymbol{B}')^{-1}\boldsymbol{\psi}
$$

where the matrix $\boldsymbol{B}$ is singular, because its determinant is equal to zero.

Exit rates and exit distribution. The mass of entry/exit m can also be recovered from the steady-state distribution as follows

$$
m = \sum_{i \notin \mathcal{S}} \sum_{j} A_{j,i} p_j \cdot dz \cdot dl
$$

Intuitively, this corresponds to the sum of the flows $A_{j,i}p_j$ from any region of the state space j to the exit region $i \notin \mathcal{S}$. The product $dz \cdot dl$ accounts for the size of the state space regions over which the mass of firms is distributed. Here, state space regions are rectangles of width dz and length dl.

Similarly, the distribution of exiting firms is given by the vector $(m_i)_i$ with

$$
m_i = \mathbb{1}(i \notin \mathcal{S}) \left(\sum_{j} A_{j,i} p_j \cdot dz \cdot dl \right)
$$

This corresponds to the sum of all transitions from any state space region j to state space region i, where $i \notin \mathcal{S}$.

3 Main Results

3.1 Parametrization

The table below lists the values of parameters. The model is not calibrated and hence all parameters are fixed.

r	θ	σ^2	α	c_f	w	p	ϕ	V^*	c_e	η
0.05	0.05	0.1	0.5	0.5	1.0	1.0	10	0	2.0	10

As in discrete time settings, the value of flow parameters are implicitly defined relative to a given time scale. Here, I set the period to be one year, so that the interest rate is $r = 0.05$.

The parameters of the productivity process are set as follows. The persistence of the productivity process θ is set such that the equivalent yearly discrete-time persistence is $\rho = 0.95$. The discrete-time persistence is related to its continuous-time counterpart by the following formula $\rho = e^{-\theta}$.[1], which implies $\theta = 0.05$. The variance of shocks to log-productivity σ^2 is set such that the variance of the unconditional distribution of productivity $\log \mathcal{N}(0, \text{Var})$ is equal to Var = 0.3. The relation between Var and σ^2 is the following: $\sigma^2 = 2\theta\text{Var}$, and hence the variance of shocks is equal to $\sigma^2 = 0.03$. These parametrization choices are in line with the rest of the literature. For example, Ottonello and Winberry (2020) find relatively similar parameter values for the productivity process when they fit their model to firm micro-data.

The parameters controlling technology and market structure are set as follows. The wage rate is set at $w = 1$, the price at $p = 1$, the fixed-cost at $c_f = 0.5$ and the labor elasticity at $\alpha = 0.5$. Together, these values of the parameters imply a relatively low optimal size of firms. This choice is mostly motivated by numerical reasons, so that the firm distribution remains within the bounds of the grid. The parameter controlling the size of adjustment cost is equal to $\phi = 10$. This likely corresponds to a relatively high level of adjustment cost, but the goal of the present work is only to present qualitative evidence that adjustment costs can match exit rate across the whole firm distribution. Typically, this parameter should be calibrated with micro-data.

The parameters controlling entry and exit are set as follows. I set the value of exit to $V^* = 0$. Note that this isomorphic to changing the level of fixed-costs c_f. I set the fixed cost of entry $c_e = 2$ and the elasticity of entry with respect to expected entry gain $\eta = 10$, so that the mass of entrants is a bit below one thousand (≈ 700).

[1] The left-hand side of corresponds to the autocorrelation of the productivity process after one-period.

3.2 Results

I present the numerical solution of the model described above. In particular, I comment on the properties of the solutions and their economic interpretation.

The value function. The figure 1 below plots the the value function.

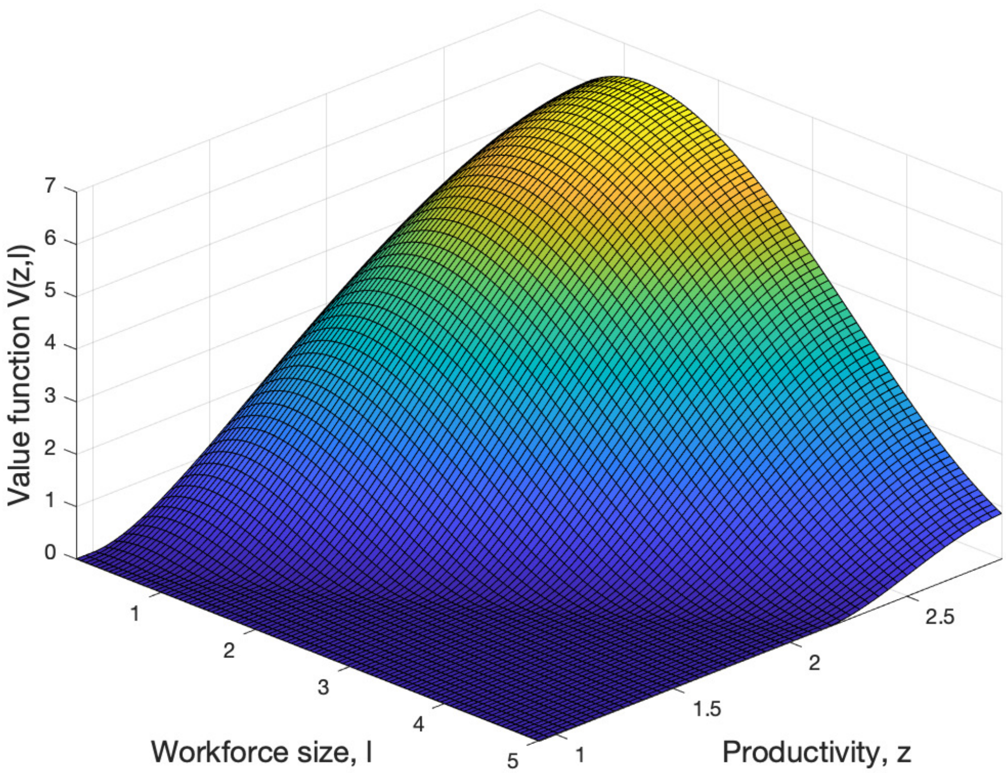

Figure 1: The value function

The value function $V(z,l)$ is strictly increasing in productivity z, as is standard. However, the value function is non-monotonic in workforce size l. It is increasing in workforce size for low levels of l and then starts decreasing in workforce size for high levels of l. That is, the value function displays an inverted-U pattern along the workforce size dimension. Economically, the inverted-U pattern follows from the assumption of decreasing returns. Indeed, decreasing returns imply that for high levels of workforce size, the marginal product of labor is lower than the wage. Therefore, marginal profits are negative and profits start decreasing.

The inverted-U pattern of the value function in the l-direction implies that, for a given productivity level z, there exists an optimal firm size l^* which maximizes the value function $l^*(z) = \arg\max_l V(z,l)$. Although not easily observable here, it is the case that optimal firm $l^*(z)$ size grows with productivity z. Without adjustment costs, firms would simply set their level of labor force such that they are at their optimal size ; that is, they set $l = l^*(z)$ at all dates. In the presence of adjustment costs, however, firms might deviate for long time from their optimal size.

Exit region. Note that the value function is equal to zero for a large region of the state space. This corresponds in fact to the exit region, as the scrap value of firms is set to zero. The figure 2 below plot of the exit region along the two dimensions of the state space.

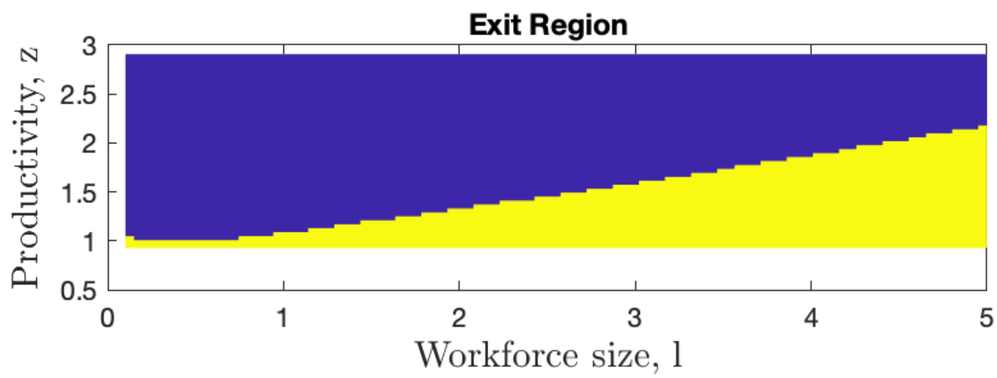

Figure 2: The exit region

The region in blue corresponds to the stay region whereas the region in yellow corresponds to the exit region. The exit productivity threshold $\bar{z}(l)$ is increasing in labor size. This means that productive firms with labor size large enough are susceptible to exit. By contrast, the exit productivity threshold $\bar{z}(l) = \bar{z}$ is constant in the model without adjust frictions. An implication is that firms which are productive enough (above the threshold $\bar{z}$) never choose to exit.

Labor adjustment decision. The figure 3 below plots the numerical approximation of the labor adjustment decision.

The optimal labor adjustment x follows a U-shaped pattern in the l-direction. That is, for a given productivity level, labor adjustment $x(l)$ is first decreasing and then increasing in l. Let me comment more precisely on the different regions of the labor adjustment policy function.

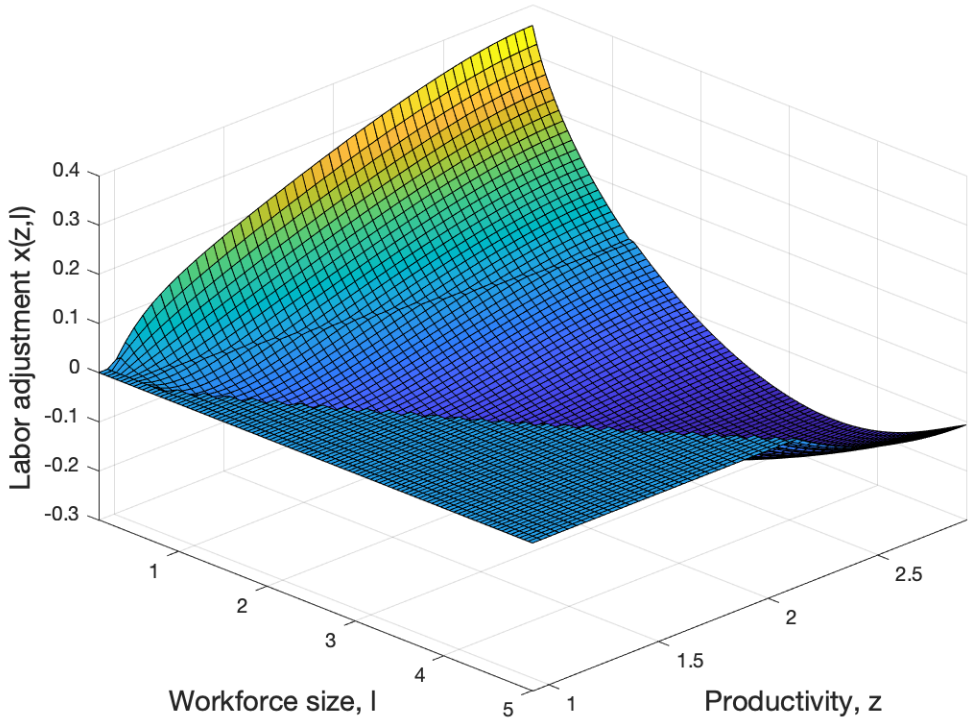

Figure 3: The optimal labor adjustment

- For workforce size below optimal size $l < l^*$, the labor adjustment is positive and decreasing in l. Firms want to increase their workforce in order to grow to their optimal size $l^*(z)$. The closer they get to their optimal size, the lower is the size of labor adjustment needed.

- When firms reach their optimal size $l = l^*$, they set labor adjustment at zero ; that is, we have $x(l^*) = 0$. They do not want their current size to change, since it is optimal.

- For workforce size slightly above the optimal size $l > l^*$, the labor adjustment is negative and decreasing in l. Firms want to reduce their workforce size in order to get back to the optimal workforce size l^*. In addition, the higher the current workforce size l, the larger the size of labor adjustment that is needed.

- For workforce a lot larger than optimal size $l \gg l^*$, the labor adjustment is still negative but labor adjustment is increasing in workforce size l. That is, firms still want to decrease their workforce size but they are not willing to pay for large adjustments. Indeed, firms are closer to the exit frontier and hence the expected benefits of adjusting labor are diminished.

- For $l \ggg l^*$, the firm exits and hence labor adjustment is set to zero.

Moreover, let us note that optimal labor adjustment is slightly increasing in productivity z. This comes from the fact that the cross derivative $\partial_{zl} V(z,l)$ is positive but small.

Stationary distribution of active firms. The figure 4 below plots the stationary distribution of active firms across the state space.

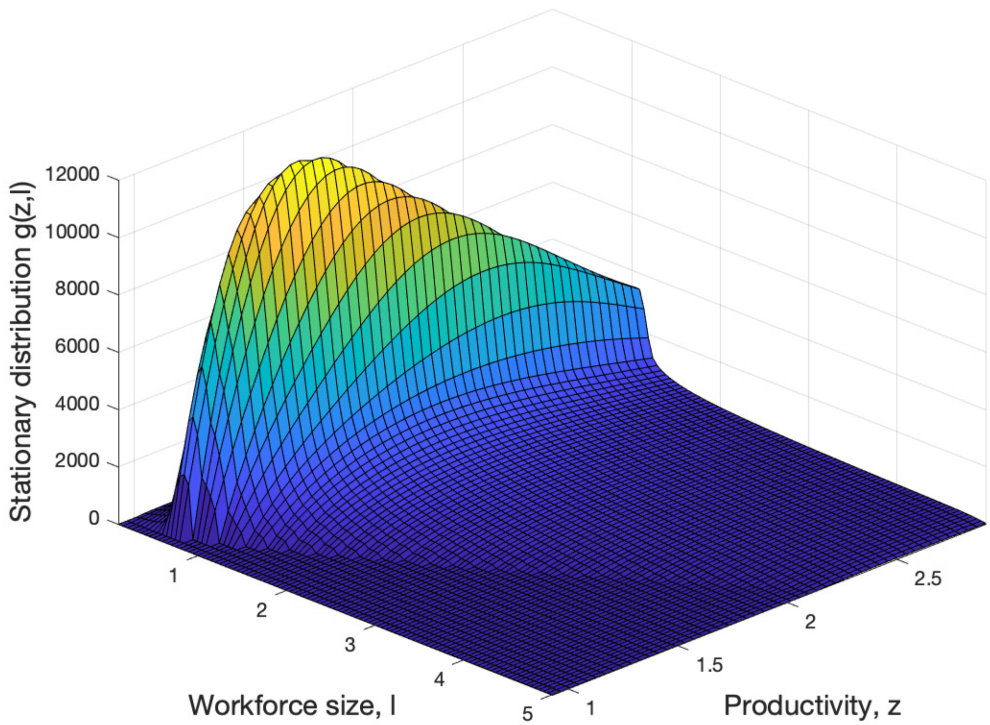

Figure 4: The stationary distribution of active firms

The distribution of active firms in the z-direction is a truncated log-normal distribution. The marginal density is first increasing fast and reaches its peak around $z \approx 1.5$ Then, the density starts decreasing with a relatively low rate of decay, consistent with the fat tails of the log-normal distribution.

The distribution of active firms in the l-direction looks like a truncated Laplace distribution. Laplace distribution are very concentrated at their median value and feature exponential decays in their tail. They are also known as "tent-like" distributions, given their two-sided linear decay in log-log plots. Consistent with this hypothesis, the marginal density seems to decay exponentially.

Note that there is a (strictly) positive mass of firms near the exit boundary, ie. in state space regions with large workforce size relative to productivity. As a consequence, some firms end up exiting in equilibrium. This result is not a trivial: firms could have chosen instead to be so careful in their choice of labor that they never arrive anywhere around the exit boundary. However, the marginal distribution of labor is much more thin-tailed than the marginal distribution of productivity. As a result, only (relatively) few firms end up close to the exit frontier.

Stationary distribution of exiting firms. The figure 5 below plots the stationary distribution of exiting firms across the state space.

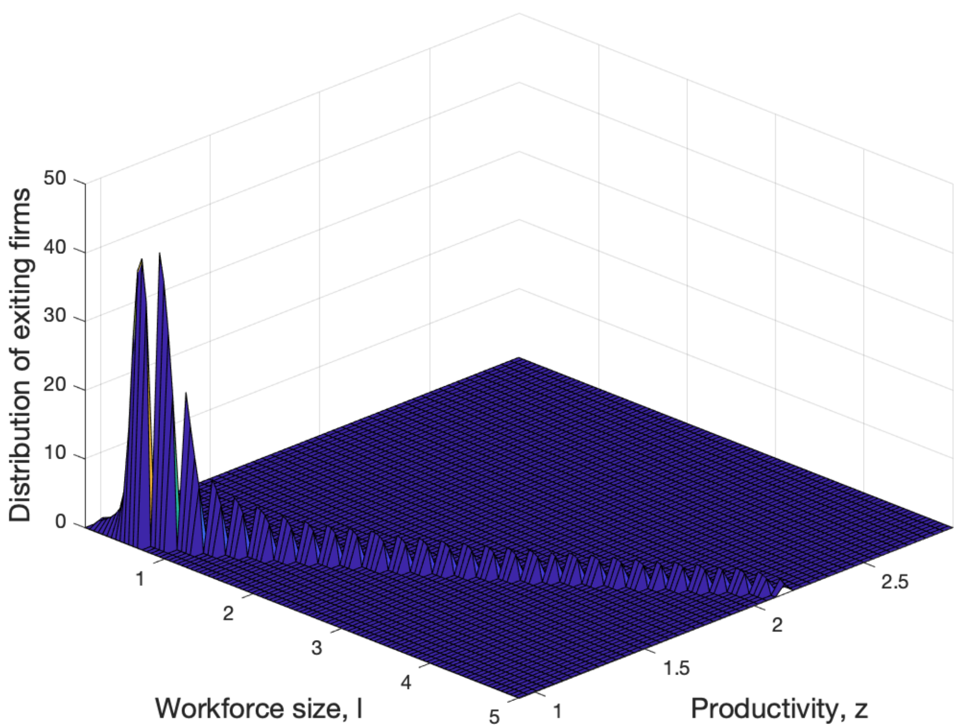

Figure 5: The stationary distribution of exiting firms

Exiting firms are distributed along a line, which corresponds to the exit frontier. This follows from the assumption of diffusion process for productivity. Indeed, the discretization of continuous time diffusion processes approximates them as Poisson process with very small increments. Productivity makes either one step up or one step down the productivity grid, but never moves two steps or more. In other words, the productivity never ”jumps” as it does in discrete time settings. This feature of continuous time diffusion processes ”sparsifies” the transition matrix and hence is at the heart of the speed gains of continuous time methods. However, it constitutes limitation as, plausibly, the real-world distribution of exiting firms is not concentrated on such thin line. Future research would allow for more general ”jump-diffusion” processes for productivity.

Interestingly, the distribution of firms is first increasing then decreasing along the frontier. First, it increases quite fast, with a peak around $l \approx 1$ and $z \approx 1$. Then, it decreases slowly and seems to reach a constant value that is independent from productivity or workforce size. Unlike the Hopenhayn model, the distribution is smoothly decreasing and features positive mass at the top of the distribution. That is, productive firms exit too. Although it happens with a lower probability, their size implies they have a large effect on aggregate productivity.