1 Recap: Limits of sequences

Definition 1 (Limit of a sequence)**.** The sequence u_n tends to or approaches a limit l if and only if for any positive real number ϵ, there is some rank N such that any term u_n of the sequence with $n > N$ must be less than ϵ away from l, i.e.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \in \mathbb{N},\ n > N \implies |u_n - l| < \epsilon.$$

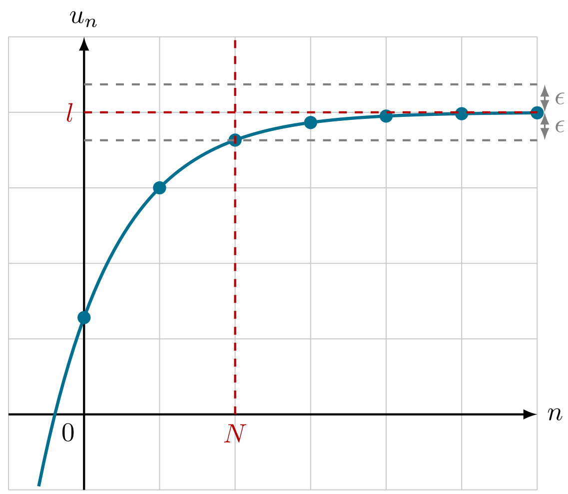

Figure 1: The sequence (u_n) approaches a limit l.

Theorem 1 (Cauchy's criterion)**.** A sequence of numbers (u_n) is said to be Cauchy if for any positive real number ϵ, there is some rank N such that any pair of terms u_m and u_n with $m, n \geq N$ must differ by less than ϵ, i.e.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall m, n \geq N,\ |u_m - u_n| < \epsilon.$$

A sequence of real numbers is Cauchy if and only if it is convergent.

2 Sequences of functions

2.1 Pointwise convergence

For each natural number $n \in \mathbb{N}$, define a function $f_n : A \to \mathbb{R}$, where A is their domain. We denote this sequence of functions as (f_n).

Example 1. For example, we can let $A = [0, +\infty)$ and consider the sequence $f_n(x) = x/(n+1)$. As n increases, the sequence appears to approach a zero function $f(x) = 0$.

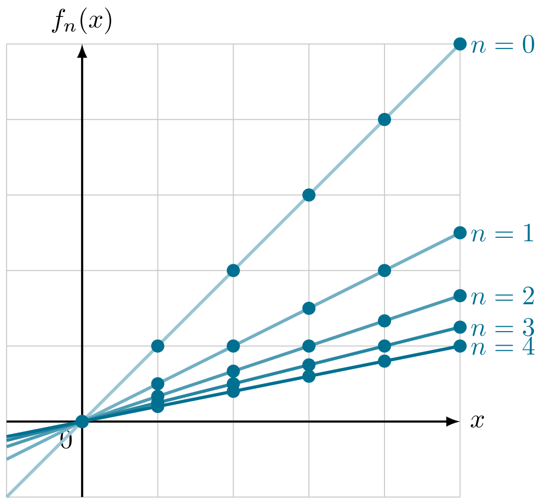

Figure 2: The sequence of functions (f_n), where $f_n(x) = x/(n+1)$, approaches a zero function $f(x) = 0$.

Definition 2. For a sequence of functions (f_n), if the limit

$$\lim_{n\to\infty} f_n(a)$$

exists for all $a \in A$, then the sequence (f_n) is said to converge pointwise to the limit

$$f(x) = \lim_{n\to\infty} f_n(a)$$

which is called the pointwise limit of the sequence.

Definition 3. A sequence of functions (f_n) is said to be pointwise Cauchy if $(f_n(a))$ is Cauchy for all inputs a.

Definition 4. A sequence of functions is pointwise Cauchy if and only if it is pointwise convergent.

Example 2. Let $A = [0, +\infty)$ and $f_n(x) = x/(n+1)$. The sequence converges pointwise to $f(x) = 0$.

Proof. For all $a \in A$ we have

$$\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \frac{a}{n+1} = 0. \qquad \square$$

Example 3. Let $A = [0, +\infty)$ and $f_n(x) = 2x + \sin(nx)/n$. The sequence converges pointwise to $f(x) = 2x$.

Proof. For all $a \in A$ we have

$$\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \left(2a + \frac{\sin(na)}{n}\right) = 2a. \qquad \square$$

Remark 1. Pointwise limits don't always exist. For instance, for the sequence of functions $f_n(x) = x^n$ defined in $x \in [0, +\infty)$, the function $f_n(x)$

- tends to 0 for $0 \leq x < 1$;
- tends to 1 for $x = 1$; and
- tends to positive infinity with no finite limit for $x > 1$,

so this sequence has no pointwise limit.

Remark 2. Continuity is not necessarily preserved under pointwise convergence. Consider again the sequence of functions $f_n(x) = x^n$, this time defined in $x \in [0, 1]$. Although each term of the sequence is continuous, its limit

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

is not continuous.

2.2 Uniform convergence

Uniform convergence provides an alternative definition of convergence for sequences of functions, where continuity is preserved.

Definition 5. A sequence of functions (f_n) is said to converge uniformly to a function f on the interval A if for all positive real number $\epsilon > 0$, there is some rank N such that any term f_n of the sequence with $n > N$ satisfies $|f(x) - f_n(x)| < \epsilon$ for all x in A, i.e.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ \forall x \in A,\ |f(x) - f_n(x)| < \epsilon.$$

Remark 3. Note the different order of quantifiers used in pointwise and uniform convergence.

$$\color{red}{\forall x \in A,}\ \forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ |f_n(x) - f(x)| < \epsilon \qquad \text{(pointwise convergence)}$$
$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ \color{red}{\forall x \in A,}\ |f_n(x) - f(x)| < \epsilon \qquad \text{(uniform convergence)}$$

Theorem 2. Uniform convergence implies pointwise convergence, to the same limit. However, the converse is not true.

Theorem 3. If the sequence $(f_n(x))$ of continuous functions on A converges uniformly on A towards a function f, then f is also continuous.

2.3 Preservation of integrals under convergence

For a sequence of functions, does the integral of the limit equal the limit of the integrals?

Theorem 4. Given that a sequence of functions (f_n) converges pointwise to a limit f, the integral of the limit does not necessarily equal the limit of the integrals of terms in (f_n). In other words, the equality

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx$$

does not necessarily hold.

Proof. Consider the sequence of functions defined by

$$f_n(x) = \begin{cases} n & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{otherwise} \end{cases}$$

which converges pointwise to the zero function $f(x) = 0$. Therefore, the integral of the limit, evaluated from 0 to 1 is zero.

$$\int_0^1 f(x)\,dx = 0$$

However, each function in the sequence has an integral of 1.

$$\begin{aligned}
\int_0^1 f_n(x)\,dx &= \int_0^{1/n} f_n(x)\,dx + \int_{1/n}^1 f_n(x)\,dx \\
&= \int_0^{1/n} n\,dx + \int_{1/n}^1 0\,dx \\
&= [nx]_0^{1/n} \\
&= 1
\end{aligned}$$

□

Theorem 5. Given that a sequence of functions (f_n) converges uniformly to a limit f, the limit of the integrals of terms in (f_n) **must exist** and be equal to the integral of the limit, i.e.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

2.4 Preservation of derivatives under convergence

Theorem 6. Given that a sequence of differentiable functions (f_n) converges pointwise or uniformly to a differentiable limit f, the limit of the derivatives of terms in (f_n) might not exist.

Example 4. Let $f_n(x) = \sin(nx)/\sqrt{n}$. The sequence (f_n) converges pointwise and uniformly to the zero function. However, the limit

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \sqrt{n}\cos(nx)$$

does not exist.

Theorem 7. Suppose a sequence of differentiable functions (f_n) converges to a function f. If (f_n') converges uniformly on $[a, b]$ (i.e. the limit of derivatives exists), then

- the convergence of (f_n) is also uniform; and
- the limit of derivatives equals the derivative of the limit, with $\lim_{n\to\infty} f_n'(x) = f'(x)$.