Introductory Mathematics for Computer Science (COMP0011)

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Contents

1 Complex numbers

The foundation of the *complex numbers* is given by the imaginary unit i, defined either as $i = \sqrt{-1}$ or as $i^2 = -1$.

A complex number z can be written as $a + bi$, where $a, b \in \mathbb{R}$. The real numbers a and b are known as the *real part* and the *complex part* of z respectively.

The set of all complex numbers is denoted as $\mathbb{C}$. Note that the set of real numbers $\mathbb{R}$ is a subset of $\mathbb{C}$. See figure 1.

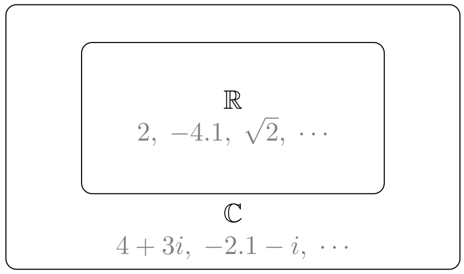

Figure 1: The set of real numbers $\mathbb{R}$ is a subset of the set of complex numbers $\mathbb{C}$. All real numbers are complex numbers.

1.1 Basic arithmetic with complex numbers, and complex conjugates

To add or subtract two complex numbers, we deal with the real and imaginary parts separately.

$$(2+3i) + (5-8i) = (2+5) + (3+(-8))i = 7 - 5i \qquad \text{(Addition)}$$
$$(2+3i) - (5-8i) = (2-5) + (3-(-8))i = -3 + 11i \qquad \text{(Subtraction)}$$

The multiplication of complex numbers is also straightforward as long as we bear in mind that $i^2 = -1$.

$$\begin{aligned}(3+4i)(-2+3i) &= -6 + 9i - 8i + 12i^2 \\ &= (-6-12) + (9-8)i \\ &= -18 + i\end{aligned}$$

To divide a complex number by another, e.g.

$$\frac{a+bi}{c+di}$$

we multiply both the numerator and denominator by $c - di$, which is obtained by flipping the sign of the imaginary part of the denominator. For example, if we want to compute

$$\frac{2+3i}{5-4i},$$

we flip the sign of the imaginary part of $5 - 4i$ to get $5 + 4i$. We then multiply both the numerator and denominator of the fraction by this $5 + 4i$ to get

$$\begin{aligned}\frac{2+3i}{5-4i} &= \frac{(2+3i)(5+4i)}{(5-4i)(5+4i)} \\ &= \frac{10 + 8i + 15i - 12}{25 + 20i - 20i + 16} \\ &= \frac{-2 + 23i}{41} \\ &= \frac{-2}{41} + \frac{23}{41}i.\end{aligned}$$

Notice how multiplying $5 - 4i$ with $5 + 4i$ produces the real number 41. By flipping the sign of the imaginary part of a complex number, we obtain what's called its *complex conjugate*. The complex conjugate of z is denoted as $\overline{z}$. By writing z as $a + bi$, we can easily prove that the product of any complex number with its conjugate must equal a real number:

$$z \times \overline{z} = (a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R}.$$

1.2 Visualising complex numbers

Given some complex number $z = x + yi$, we can treat its real and imaginary parts as Cartesian coordinates, thus mapping it to a point on the 2D plane.

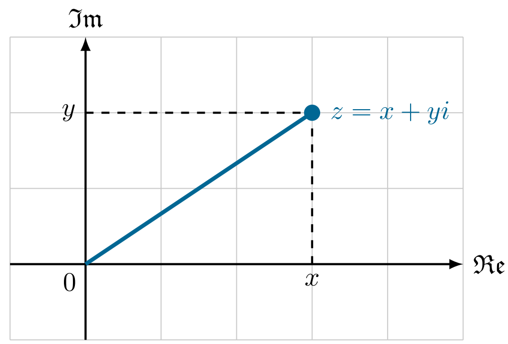

Figure 2: The complex number $z = x + yi$ as a point on the 2D plane

1.3 Exponential form

Recall that it is possible to express a point on a 2D plane using polar coordinates (R, θ) as well. Indeed, given any complex number $z = x + yi$, we can find its corresponding pair of values R and θ.

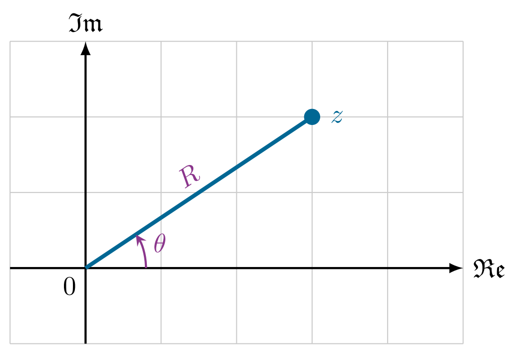

Figure 3: The position of a complex number on the 2D plane can be represented using polar coordinates.

Based on this idea, we introduce a new notation as follows.

> If the position of a complex number z on the 2D plane can be represented by the polar coordinates (R, θ), then we have
>
> $$z = R \times e^{i\theta}$$
>
> where $R, \theta \in \mathbb{R}$ and $R \geq 0$.
>
> R is called the *absolute value* or *modulus* of z and is denoted as $|z|$. This represents the point's position from the origin.

θ is called the *argument* of z and is denoted as $\arg(z)$. This represents the angle from horizontal.

This way of representing complex numbers is known as the *exponential form*. (This is a natural result of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

Now consider two complex numbers expressed in exponential form.

$$z_1 = R_1 \times e^{i\theta_1}$$
$$z_2 = R_2 \times e^{i\theta_2}$$

These two numbers are considered equal if both of the following conditions hold.

$$R_1 = R_2$$
$$\theta_1 = \theta_2 + 2k\pi \qquad \text{(for some } k \in \mathbb{Z}\text{)}$$

Note that the red part is necessary because a rotation of 2π radians has no effect on a point's position.

The exponential form makes the multiplication and division of complex numbers a lot easier.

Multiplication

$$(1 \times e^{\frac{\pi}{6}i}) \times (2 \times e^{-\frac{\pi}{4}i}) = 2 \times e^{\frac{\pi}{6}i - \frac{\pi}{4}i}$$
$$= 2 \times e^{-\frac{\pi}{12}i}$$

Division

$$\frac{1 \times e^{\frac{\pi}{6}i}}{2 \times e^{-\frac{\pi}{4}i}} = \frac{1}{2} \times \frac{e^{\frac{\pi}{6}i}}{e^{-\frac{\pi}{4}i}}$$
$$= 2 \times e^{\frac{5\pi}{12}i}$$

1.4 Converting between Cartesian and exponential forms

The methods used to convert between the Cartesian form $x + yi$ and the exponential form $R \times e^{i\theta}$ are outlined below.

- **Given the Cartesian form of a complex number, find its exponential form.**

  Given the Cartesian form $z = x + yi$, we can find the modulus using Pythagoras' theorem.

  $$|z| = \sqrt{x^2 + y^2}$$

  The argument can be found using the arctangent.

  $$\arg(z) = \arctan\left(\frac{y}{x}\right)$$

- **Given the exponential form of a complex number, find its Cartesian form.**

  Given the exponential form $z = R \times e^{i\theta}$, we can find the Cartesian coordinates using simple trigonometry.

  $$x = R\cos\theta$$
  $$y = R\sin\theta$$

To speed up conversion processes, it is often useful to memorize the Cartesian coordinates of some special points on the unit circle. See figure 4 and table 1.

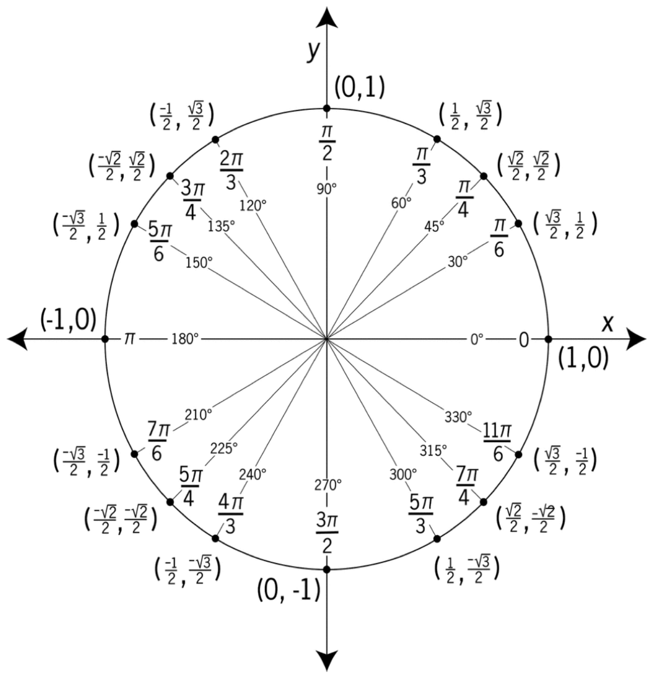

Figure 4: It is important to know the coordinates of points on the circle corresponding to classic angles.

θ (radians)	$\pi/6$	$\pi/4$	$\pi/3$
θ (degrees)	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$

Table 1: The values of $\sin\theta$ and $\cos\theta$ for some classic angles θ.

1.5 Visualising arithmetic on complex numbers

When visualised on the 2D plane, the addition of complex numbers is similar to that of vectors. We join the arrows in a tip-to-tail manner in order to determine the sum, as shown in figure 5.

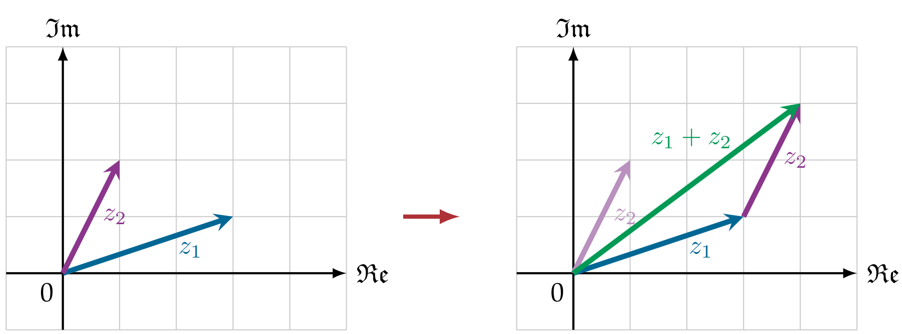

Figure 5: Addition of complex numbers.

The above figure also illustrates another key idea. Notice how in the figure on the right, the vectors of

z_1, z_2 and $z_1 + z_2$ form a triangle. This means their absolute values must fulfil the triangle inequality.

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

To visualise multiplication we consider the exponential form. As shown in figure 6, when two complex numbers are multiplied, their arguments are added together to produce a rotation, while their moduli are multiplied.

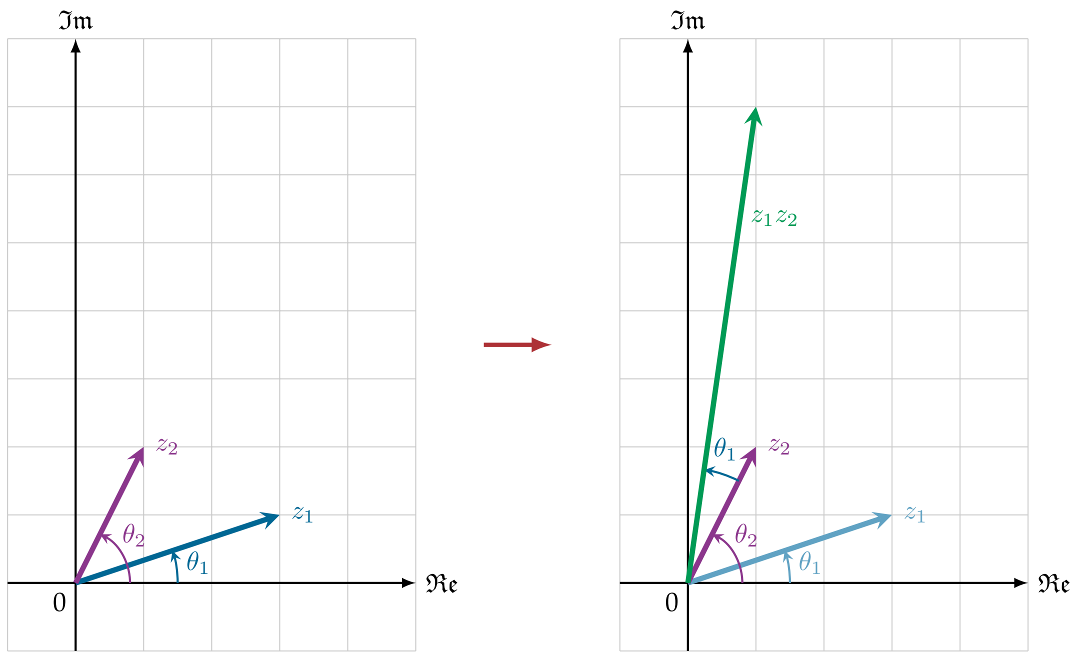

Figure 6: Multiplication of complex numbers.

1.6 Roots of unity

The *roots of unity* are the solutions to the equation

$$z^n = 1, \tag{1}$$

where n is a positive integer.

Solving this equation for values of n such as 2 and 4 is straightforward:

$$\begin{aligned} n = 2 &\implies z^2 = 1 \implies z = 1 \text{ or } -1 \\ n = 4 &\implies z^4 = 1 \implies z = 1,\ i,\ -1 \text{ or } -i \end{aligned}$$

but solving it for other values of n requires us to express z in its exponential form, i.e.

$$z = R \times e^{i\theta}. \qquad (R, \theta \in \mathbb{R} \text{ and } R \geq 0)$$

This allows us to rewrite the equation as

$$\begin{aligned} (R \times e^{i\theta})^n &= 1 \times e^{0i} \\ R^n \times e^{in\theta} &= 1 \times e^{0i} \end{aligned}$$

which yields the following.

$$\begin{cases} R^n = 1 \\ n\theta = 0 + 2k\pi = 2k\pi \end{cases} \qquad (\text{for some } k \in \mathbb{Z})$$

Since $R \geq 0$ and $R \in \mathbb{R}$, we must have $R = 1$. Furthermore, the second equation gives us

$$\theta = \frac{2k\pi}{n}$$

i.e.

$$\theta \in \left\{ \cdots,\ -3 \cdot \frac{2\pi}{n},\ -2 \cdot \frac{2\pi}{n},\ -\frac{2\pi}{n},\ 0,\ \frac{2\pi}{n},\ 2 \cdot \frac{2\pi}{n},\ 3 \cdot \frac{2\pi}{n},\ \cdots \right\}$$

which seemingly means that there are infinitely many roots of unity. However, this is impossible because by the fundamental theorem of algebra, equation (1) (which is a polynomial equation of degree n) can only have n solutions.

To resolve this apparent paradox, let us visualise the problem on a 2D plane. For the sake of simplicity let us assume $n = 3$. We know that all solutions to (1) must have a modulus of $R = 1$, so they must lie on the unit circle.

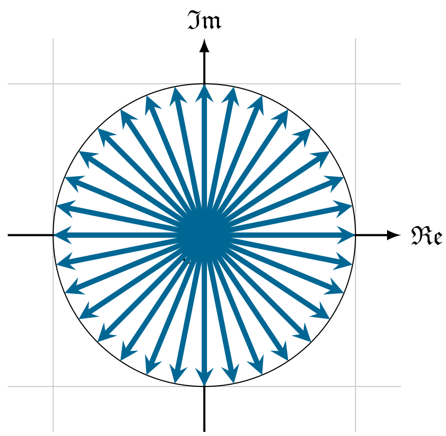

Figure 7: The roots of unity must lie somewhere on the unit circle.

We want to find the angles θ such that if we start at the point 1 and then rotate anticlockwise by θ radians $n = 3$ times, we end up back at 1.

- Obviously we can have $\theta = 0$.
- Another obvious solution is $\theta = 2\pi/3$. If we rotate by this angle 3 times, we will have completed a full 2π radians, bringing us back to the initial point.
- Moreover, we can also have $\theta = 4\pi/3$. Rotating by this angle 3 times creates a total rotation of 4π radians (i.e. 2 full cycles), bringing us once again back to the starting point.
- Continuing this pattern, it appears that $\theta = 6\pi/3$ is also a solution. However, this is in fact the same as $\theta = 0$, since angles differing by 2π are considered equivalent. The same applies for $\theta = 8\pi/3$ (equivalent to $2\pi/3$), $\theta = 10\pi/3$ (equivalent to $4\pi/3$), and so on.

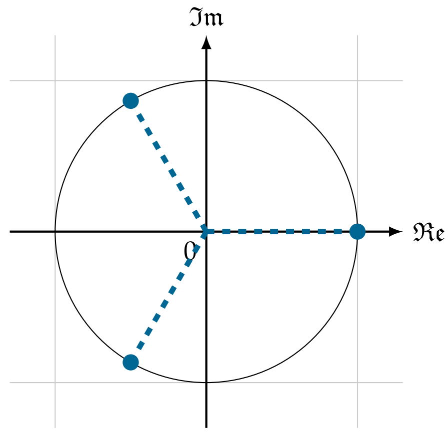

Figure 8: The third roots of unity.

This resolves the above paradox — we were right in thinking that the possible values of θ are given by

$$\theta = \frac{2k\pi}{n}$$

or

$$\theta \in \left\{\cdots,\ -3\cdot\frac{2\pi}{n},\ -2\cdot\frac{2\pi}{n},\ -\frac{2\pi}{n},\ 0,\ \frac{2\pi}{n},\ 2\cdot\frac{2\pi}{n},\ 3\cdot\frac{2\pi}{n},\ \cdots\right\}$$

but these solutions are not all distinct. To make sure we only count distinct solutions, we impose the range $0 \leq k < n$, giving us

$$\theta = \frac{2k\pi}{n} \qquad (k \in \mathbb{N} \text{ and } k < n)$$

or

$$\theta \in \left\{0,\ \frac{2\pi}{n},\ 2\cdot\frac{2\pi}{n},\ 3\cdot\frac{2\pi}{n},\ \cdots,\ (n-1)\cdot\frac{2\pi}{n}\right\}.$$

This yields the solutions

$$z = 1 \times e^{\frac{2k\pi}{n}i} = e^{\frac{2k\pi}{n}i} \qquad (k \in \mathbb{N} \text{ and } k < n)$$

or

$$z \in \left\{0,\ e^{\frac{2\pi}{n}i},\ e^{\frac{4\pi}{n}i},\ \cdots,\ e^{\frac{2(n-1)\pi}{n}i}\right\}.$$

2 Continuous functions

A function f maps elements of a set A to elements of another set B. We denote this as $f : A \to B$. In practice, most functions we consider will have type $\mathbb{R} \to \mathbb{R}$.

If a function maps a number of x to its square x^2, we can denote this by $x \mapsto x^2$. (Note the difference in the arrow symbol used — the symbol $\mapsto$ is read as "maps to".)

A function $y = f(x)$ can be represented graphically as the set of points (x, y). See figure 9.

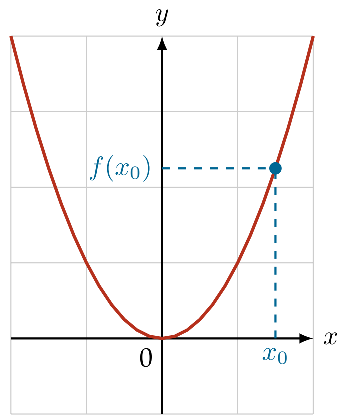

Figure 9: The graph of the function $y = x^2$.

In the next few subsections, we will be looking at some classic mathematical functions.

2.1 Trigonometric functions

Consider a point P on the unit circle. If we let θ be the angle between OP and the horizontal axis, then the coordinates of P can be expressed as $(\cos\theta,\ \sin\theta)$. This is illustrated in figure 10.

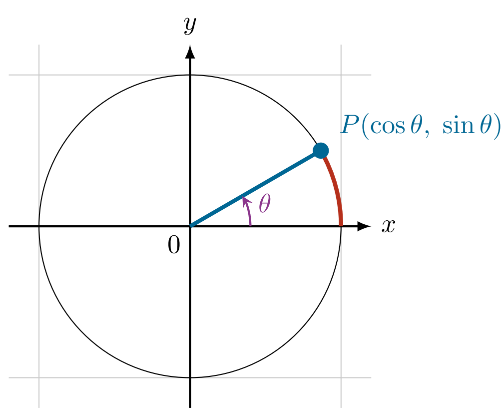

Figure 10: The trigonometric functions $\cos\theta$ and $\sin\theta$ can be defined using the unit circle. Note that if we are measuring θ in radians, then the length of the arc highlighted in red must be equal to θ.

We've previously seen the values of $\sin\theta$ and $\cos\theta$ for some classic angles θ in table 1. Plotting these functions on a graph results in figure 11.

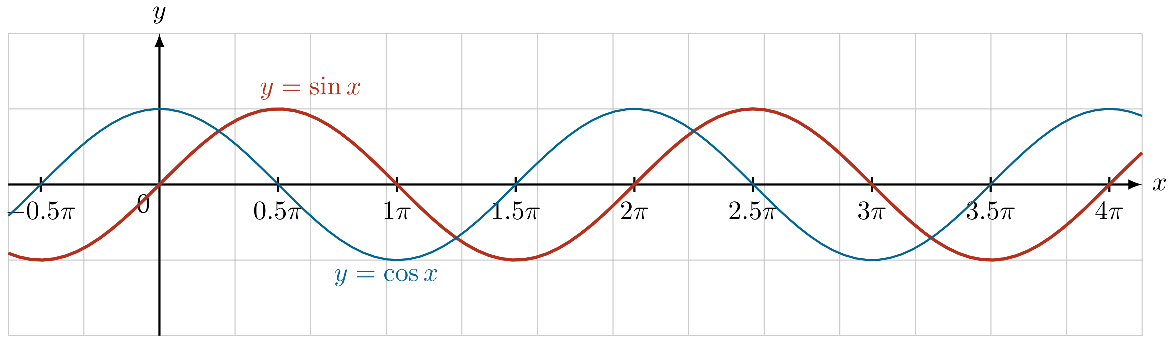

Figure 11: The graph of the functions $\sin x$ and $\cos x$.

2.2 Exponential and logarithm

One way to define the exponential function $\exp$ is as follows.

$$\begin{aligned} \exp(x+y) &= \exp(x) \cdot \exp(y) \\ \exp(0) &= 1 \\ \frac{d}{dx}\exp(x) &= \exp(x) \end{aligned}$$

Note that the first two relationships can be satisfied by any function of the form $f(x) = a^x$ where $a \in \mathbb{R}$. However, if we take all three conditions into account, the only function satisfying them is $\exp(x) = e^x$, where $e = 2.71828\cdots$ is Euler's number.

The exponential function $\exp(x) = e^x$ is plotted in figure 12. Note that:

- For all values of x, we have $\exp(x) > 0$.
- When x is negative, $\exp(x)$ is very small.
- The value of $\exp(x)$ grows very fast as x increases.

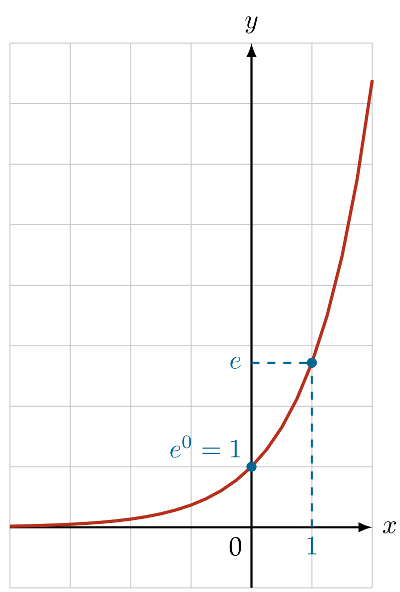

Figure 12: The graph of the function $y = \exp(x) = e^x$.

The natural logarithm $\ln x$ is the inverse of the exponential, meaning that $\ln(e^x) = x$. This results in the following properties.

$$\begin{aligned}
\ln(ab) &= \ln a + \ln b \\
\ln\left(\frac{a}{b}\right) &= \ln a - \ln b \\
\ln 1 &= 0 \\
\ln e &= 1 \\
a^x &= e^{x \ln a} \\
\ln(a^x) &= x \ln a
\end{aligned}$$

Since $e^x > 0$ for all x, the natural logarithm $\ln x$ is only defined for positive values of x.

The plot of $y = \ln x$ is given in figure 13. Note that:

- For $x < 1$, we have $\ln x < 0$.
- The curve intersects the x-axis at $(1, 0)$.
- For $x > 1$, the value of $\ln x$ grows very slowly as x increases.

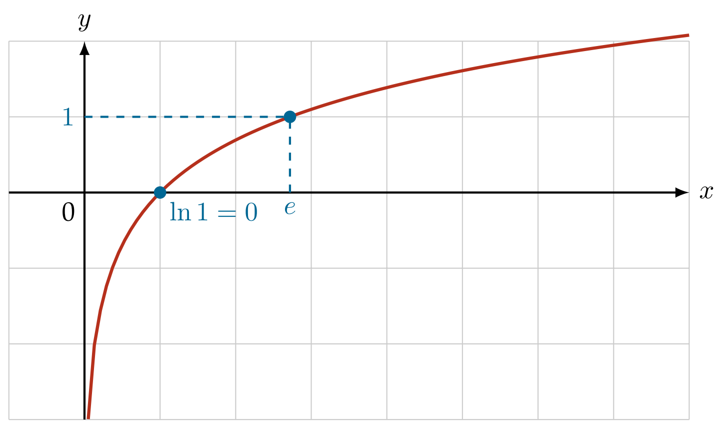

Figure 13: The graph of the function $y = \exp(x) = e^x$.

2.3 Introduction to limits

The idea of limits is simple.

> As x approaches a value, $f(x)$ also approaches a value — both possibly infinite. To denote this we write $f(x) \to b$ as $x \to a$, or $\lim_{x\to a} f(x) = b$.

This gives us four different cases.

- As x approaches infinity, $f(x)$ also approaches infinity. This means that $f(x)$ can become arbitrarily large for large enough x, i.e.

  $$\forall d > 0,\ \exists c > 0,\ x > c \Rightarrow f(x) > d.$$

  See figure 14. (Restricting c and d to positive values is not strictly necessary, but it does make our lives easier in some cases.)

- As x approaches a value a, $f(x)$ approaches infinity. This means that $f(x)$ can become arbitrarily large for x close enough x, i.e.

  $$\forall d > 0,\ \exists \eta > 0,\ 0 < |x - a| < \eta \Rightarrow f(x) > d.$$

In other words, for any value d, $f(x)$ can be greater than d as long as the distance between x and a is less than some value η. See figure 15.

- As x approaches infinity, $f(x)$ approaches a value b. This means that $f(x)$ can get arbitrarily close to b for large enough x, i.e.

$$\forall\epsilon > 0,\ \exists c > 0,\ x > c \Rightarrow |f(x) - b| < \epsilon.$$

  See figure 16.

- As x approaches a value a, $f(x)$ approaches a value b. This means that $f(x)$ can become arbitrarily close to b for x close enough to a, i.e.

$$\forall\epsilon > 0,\ \exists\eta > 0,\ 0 < |x - a| < \eta \Rightarrow |f(x) - b| < \epsilon.$$

  See figure 17.

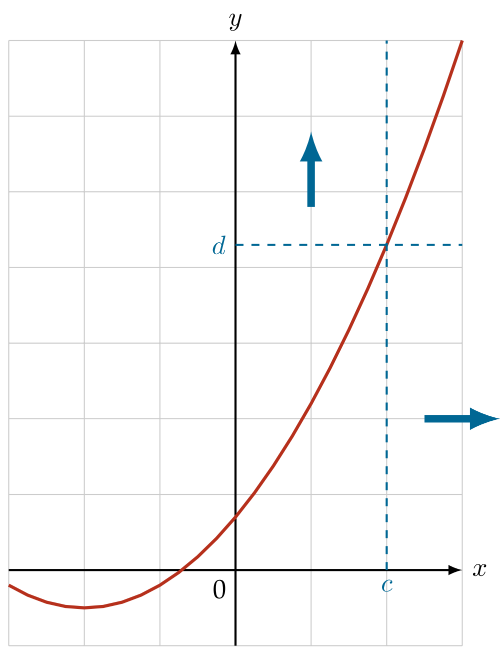

Figure 14: As x approaches infinity, so does $f(x)$.

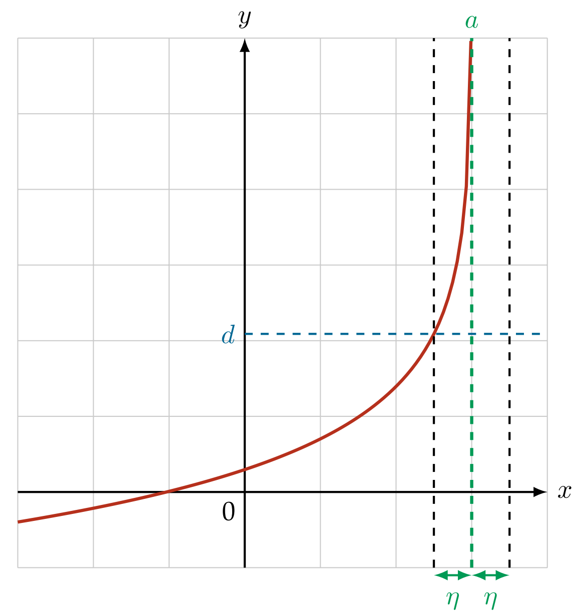

Figure 15: As x approaches a, $f(x)$ approaches infinity.

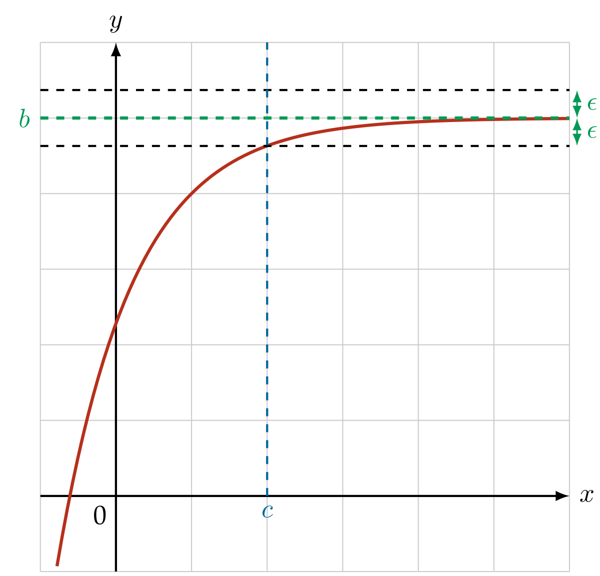

Figure 16: As x approaches infinity, $f(x)$ approaches b.

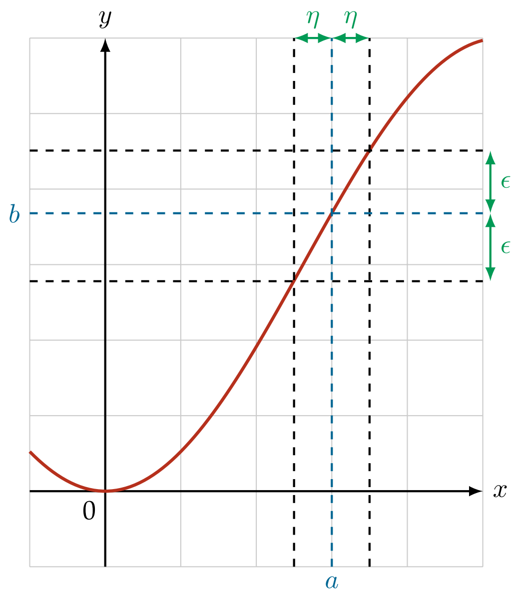

Figure 17: As x approaches a, $f(x)$ approaches b.

A limit that exists is known as a *finite* limit. Finite limits can be combined in a natural way.

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

We use the following rules to handle infinities.

$$a \times \infty = \infty$$
$$\frac{a}{\infty} = 0$$

If a limit involves x approaching zero, we may sometimes have to specify the direction in which x is approaching it, i.e. whether it is approaching zero as a positive number (from the right) or as a negative number (from the left).

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

There are certain cases where we *cannot* combine limits. These are called *indeterminate forms*, and there is no general rule for figuring out what these indeterminate forms evaluate to. Examples of indeterminate forms are given below.

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0 \times \infty,\ \infty - \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

2.4 Little o and big O notation

Here we introduce two types of notation: little o and big O.

Little o notation.

We write $f = o(g)$ near a value b if $\lim_{x \to b} \frac{f(x)}{g(x)} = 0$. This means that f is negligible when compared with g (or that g is more significant than f) near b.

For example, we say that $x = o(x^2)$ near ∞ because $\lim_{x \to \infty} x/x^2 = 0$.

Big O notation.

We write $f = O(g)$ near a value b if $\lim_{x \to b} \left| \frac{f(x)}{g(x)} \right|$ is bounded. We can express the idea of being "bounded" more precisely as

$$\exists M \in \mathbb{R}, \lim_{x \to b} \left| \frac{f(x)}{g(x)} \right| < M.$$

This means that f and g have a similar growth rate near b.

For example, we say that $3x = O(x + 1)$ near ∞ because $\lim_{x \to \infty} \left| \frac{3x}{x+1} \right| < \frac{3x}{x} = 3$.

Notice that by definition, we have $f = o(g) \;\Rightarrow\; f = O(g)$. This is because if $f = o(g)$ is true, then the limit $\lim_{x \to b} \frac{f(x)}{g(x)}$ must equal zero and is therefore bounded, which gives us $f = O(g)$.

2.5 Continuity

A function f is continuous if for all a where $f(a)$ is defined, we have $\lim_{x \to a} f(x) = f(a)$.

In practice, this means that the graph of $y = f(x)$ is a single unbroken curve. The exponential and logarithm functions, for example, are both continuous.

An important result of this is the *intermediate value theorem*.

Intermediate value theorem.

Assume for a continuous function f that $a < b$ and $f(a) < f(b)$. For any value y such that $f(a) < y < f(b)$, there exists a (not necessarily unique) value x such that $a < x < b$ and $f(x) = y$.

See figure 18 and 19.

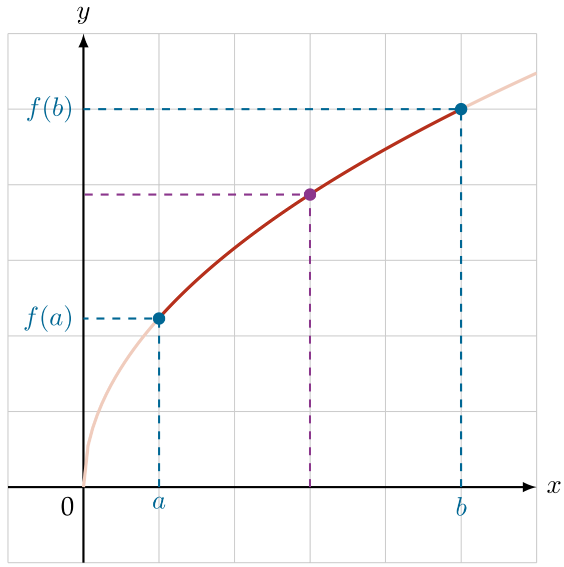

Figure 18: The intermediate value theorem.

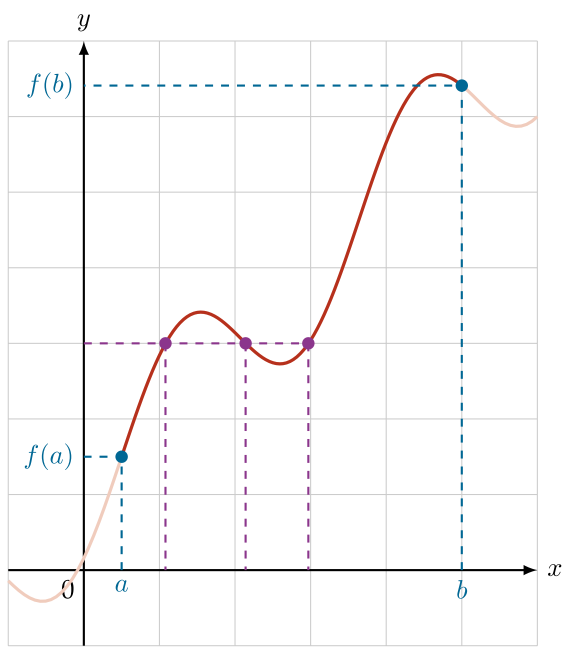

Figure 19: In the intermediate value theorem, for a given value y, the value of x does not necessarily have to be unique.