# Introductory Mathematics for Computer Science (COMP0011)

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## Contents

# 1 Differential calculus

## 1.1 What is a derivative?

Differential calculus is the study of infinitesimal variations. Consider for instance a function $y = f(x)$. Given some interval from $a$ and $a + h$, we can study the variation of $f$ in this interval by considering the ratio

$$\frac{f(a+h) - f(a)}{h}.$$

This is visualised in figure 1.

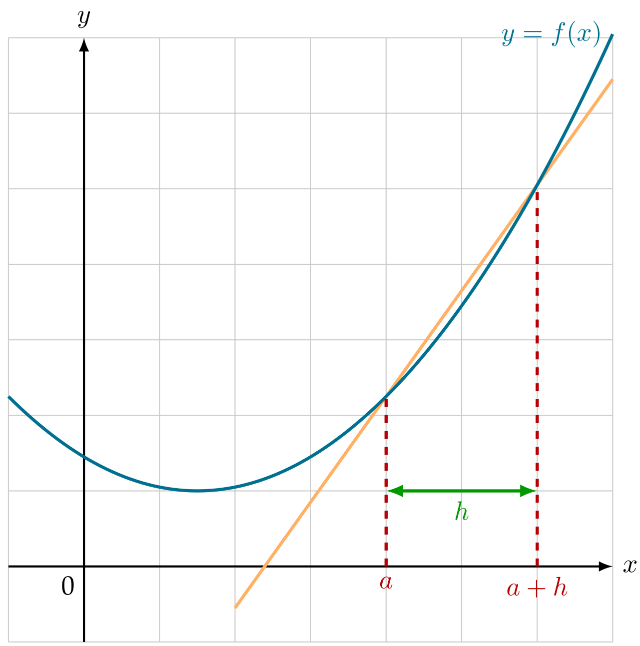


Figure 1: Studying the variation of a function $f$ between $x = a$ and $x = a + h$. The ratio $\frac{f(a+h)-f(a)}{h}$ can be visualised as the slope of the orange line which passes through the points $(a, f(a))$ and $(a + h, f(a + h))$.

To study the variations close to $a$, we consider the limit of this ratio as $h$ tends to zero. See figure 2.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit is referred to as the *derivative* of $f$ in $a$. If this limit exists, then the function $f$ is said to be *differentiable* in $a$.

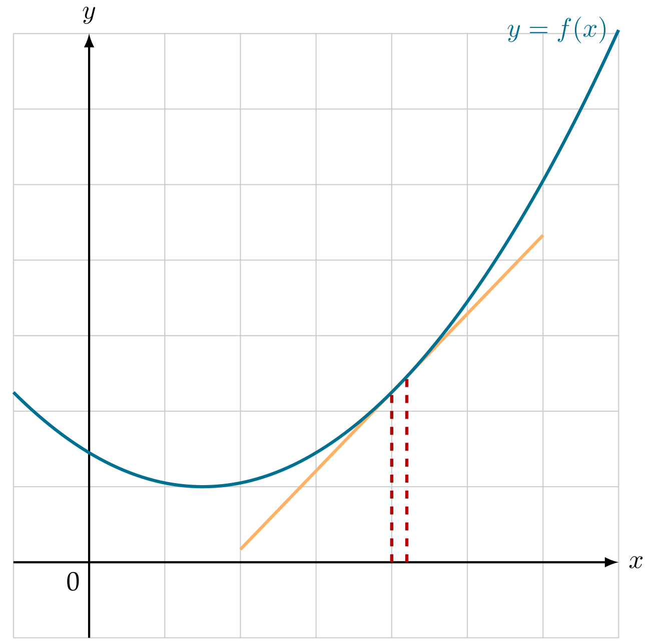


Figure 2: To find the derivative of a function $f$ in $a$, we consider the slope of the orange line as the length of the interval $h$ tends to zero.

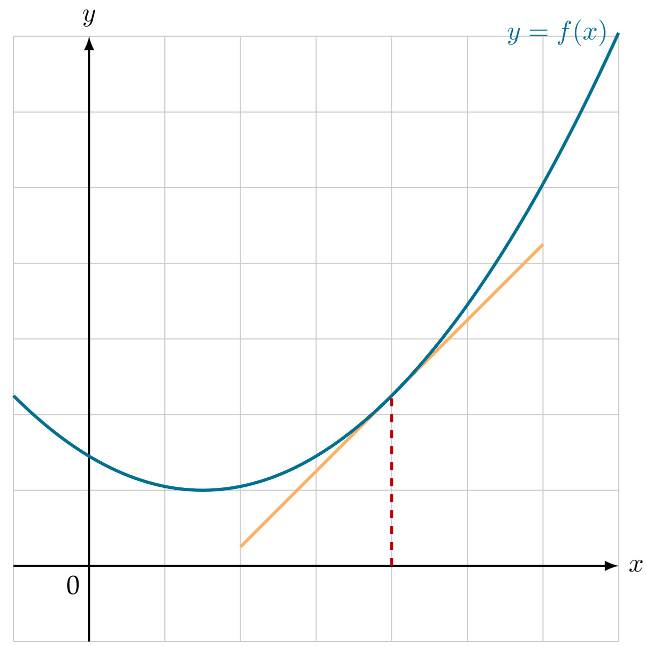


Figure 3: The derivative of a function $f$ in a point $a$ can be thought of as the slope of the tangent to the graph of $f$ at the point where $x = a$.

The derivative of a function $f$ in some value $a$ is denoted as $f'(a)$.

Note that the derivative of $f$ is itself a function, which is denoted here as $f'$. We can also write this as

$$\frac{df}{dx} \quad \text{or} \quad f^{(1)}.$$

Note that $\frac{df}{dx}$ is not a quotient or fraction. It can be read as applying a differentiation operator $\frac{d}{dx}$ to a function $f$. The differentiation operator specifies the variable of differentiation, which in this case is $x$.

Note that

- If the derivative of a function $f$ is $g$, i.e.
$$f' = g$$
then $f$ is called the *primitive* or *antiderivative* of $g$.

- Applying the differentiation operator to a function $f$ twice, i.e.
$$\frac{d}{dx}\left(\frac{df}{dx}\right)$$
can be written more compactly as
$$\frac{d^2 f}{dx^2}.$$
This produces a *second-order derivative*, which is denoted as $f''(x)$ or $f^{(2)}(x)$.

## 1.2 Finding the derivative of a function

To find the derivative of a function, we can use the definition of the derivative as a limit. Consider for instance the function $f(x) = x^2$. We can find the derivative of $f$ by computing the limit

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= \lim_{h \to 0} 2x \\
&= 2x
\end{aligned}$$

which means that the derivative of $f(x) = x^2$ at any point $a$ is given by $f'(a) = 2a$. In other words, the tangent to the graph of $f(x)$ when $x = a$ must have a slope of $2a$.

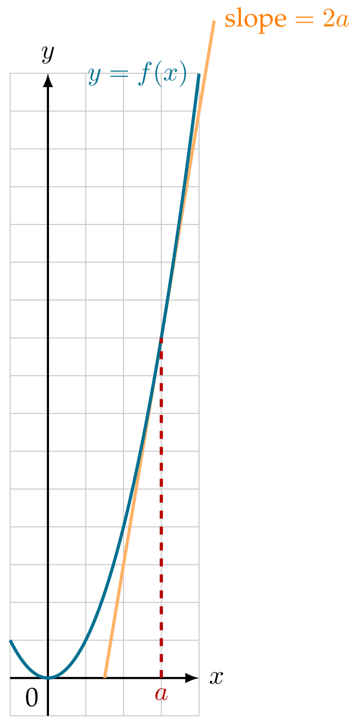


Figure 4: The derivative of $x^2$ is $2x$. This means the derivative of $x^2$ (or the slope of the tangent to the graph of $y = x^2$) at any point $a$ is $2a$.

Finding derivatives as a limit can be cumbersome and tedious. To speed this up, it might be helpful to memorise a few differentiation rules, as listed below. Here, $f$ and $g$ are differentiable functions, $c$ is a constant, and $n$ is any real number.

$$c \xrightarrow{\frac{d}{dx}} 0 \tag{1}$$

$$x \xrightarrow{\frac{d}{dx}} 1 \tag{2}$$

$$\sin x \xrightarrow{\frac{d}{dx}} \cos x \tag{3}$$

$$\cos x \xrightarrow{\frac{d}{dx}} -\sin x \tag{4}$$

$$e^x \xrightarrow{\frac{d}{dx}} e^x \tag{5}$$

$$\ln x \xrightarrow{\frac{d}{dx}} \frac{1}{x} \tag{6}$$

$$f(x) \pm g(x) \xrightarrow{\frac{d}{dx}} f'(x) \pm g'(x) \tag{7}$$

$$c \cdot f(x) \xrightarrow{\frac{d}{dx}} c \cdot f'(x) \tag{8}$$

$$x^n \xrightarrow{\frac{d}{dx}} nx^{n-1} \tag{9}$$

$$f(x) \cdot g(x) \xrightarrow{\frac{d}{dx}} f(x) \cdot g'(x) + g(x) \cdot f'(x) \tag{10}$$

$$\frac{f(x)}{g(x)} \xrightarrow{\frac{d}{dx}} \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2} \tag{11}$$

$$f(g(x)) \xrightarrow{\frac{d}{dx}} f'(g(x)) \cdot g'(x) \tag{12}$$

Several notes:

- From equation (7) we have

$$f(x) + g(x) \xrightarrow{\frac{d}{dx}} f'(x) + g'(x).$$

  Combining this with equation (8) we see that differentiation is a linear operation.

- The power rule (equation (9)) can be applied with any real number $n$. A few examples are shown below.

$$\frac{1}{x} = x^{-1} \xrightarrow{\frac{d}{dx}} -x^{-2} = -\frac{1}{x^2}$$
$$\sqrt{x} = x^{\frac{1}{2}} \xrightarrow{\frac{d}{dx}} \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

- The quotient rule (equation (11)) can be derived from the power rule (equation (9)), the product rule (equation (10)), and the chain rule (equation (12)).

An example problem is shown below.

**Problem.** Calculate $\dfrac{d}{dx}\, 3e^{x^2+2}$.

**Solution 1.** Let $f(x) = 3e^x$ and $g(x) = x^2 + 2$. We want to find $\frac{d}{dx} f(g(x))$.

Note that $f'(x) = 3e^x$ and $g'(x) = 2x$. Hence, we have

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= f'(g(x)) \cdot g'(x) && \text{(chain rule)} \\ &= f'(x^2+2) \cdot 2x \\ &= 3e^{x^2+2} \cdot 2x \\ &= 6xe^{x^2+2}. \end{aligned}$$

**Solution 2.** Those who are more fluent with the rules of differentiation can simply apply the rules to find:

$$\begin{aligned} \frac{d}{dx} 3e^{x^2+2} &= 3 \cdot \left(e^{x^2+2}\right) \cdot (2x) \\ &= 6xe^{x^2+2}. \end{aligned}$$

Generalising from this example problem, we can show that a function of the form

$$f(x) = c \cdot e^{g(x)} \qquad \text{(where } c \text{ is a constant)}$$

must have the derivative

$$\begin{aligned} f'(x) &= c \cdot e^{g(x)} \cdot g'(x) \\ &= f(x) \cdot g'(x). \end{aligned}$$

This will be useful for solving differential equations later.

## 1.3 More on differentiability

Recall that a function $f(x)$ is said to be differentiable if the derivative of $f'(x)$ exists when $x = a$.

We note the following theorem.

**Differentiability is stronger than continuity.**

If a function $f$ is differentiable in $a$, then $f$ is continuous in $a$.

We can prove this as follows.

**Proof that differentiability implies continuity.**

We assume differentiability, meaning that the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. We want to prove that

$$\lim_{x \to a} f(x) = f(a).$$

Consider the following limit.

$$\begin{aligned}
\lim_{x \to a} (f(x) - f(a)) &= \lim_{x \to a} \left( (f(x) - f(a)) \cdot \frac{x-a}{x-a} \right) & (\because x - a \neq 0) \\
&= \lim_{x \to a} \left( \frac{f(x) - f(a)}{x-a} \cdot (x-a) \right) \\
&= \left( \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \right) \cdot \left( \lim_{x \to a} (x-a) \right) \\
&= f'(a) \cdot 0 \\
&= 0
\end{aligned}$$

Rearranging, we get

$$\lim_{x \to a} f(x) = f(a)$$

which concludes the proof.

Intuitively, we can think of differentiability as whether the tangent to the graph of a function $f$ at a point $a$ exists. For instance, the absolute value function $|x|$ is not differentiable at $x = 0$ because at $x = 0$, the graph of $y = |x|$ consists of a sharp angular point where no tangent can be drawn. See figure 5.

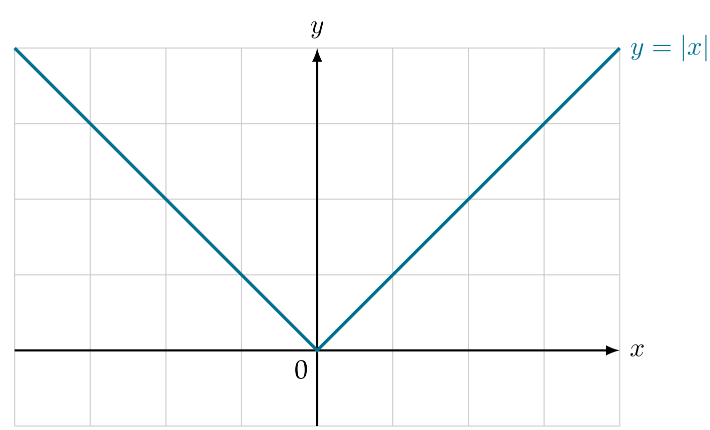


Figure 5: The absolute value function is non-differentiable at $x = 0$.

## 1.4 Stationary points

If we have $\frac{df}{dx} > 0$ at some point $a$, we say that $f$ is *increasing* at that point. Conversely, if we have $\frac{df}{dx} < 0$ at some point $a$, we say that $f$ is *decreasing* at that point. But what if $\frac{df}{dx} = 0$?

When this happens, we say that $f$ has a *stationary point* at $a$. This could mean a number of things:

- A local minimum;
- A local maximum; or
- An inflection point[1].

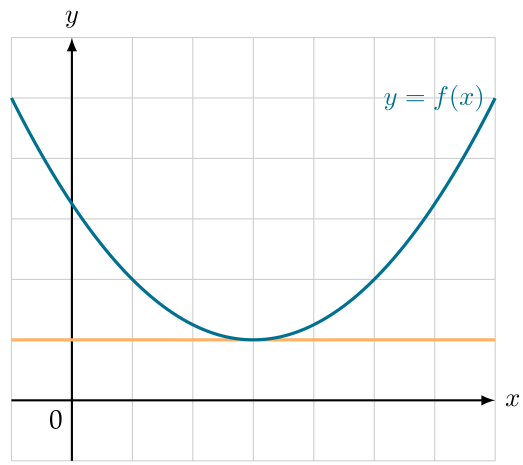


Figure 6: At a local minimum, the derivative of a function is zero.

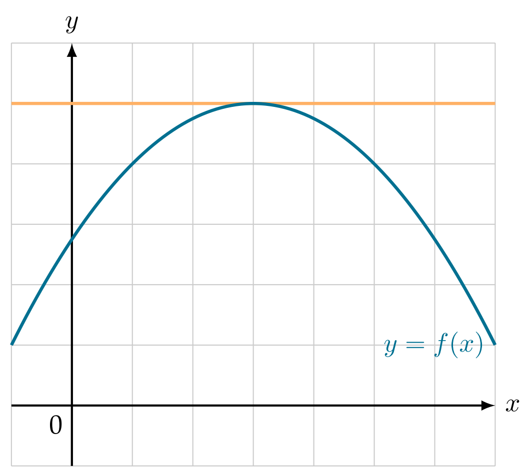


Figure 7: At a local maximum, the derivative of a function is zero.

[1] Also called a "point of inflection".

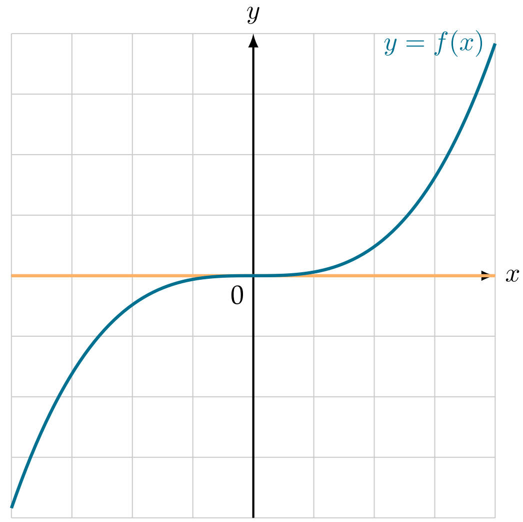


Figure 8: At an inflection point, the derivative of a function is zero.

There are several ways to distinguishing between these cases, as will be demonstrated using the following function.

$$f(x) = 2x^3 - 9x^2 + 12x - 4$$

To find the stationary points of $f$, we first compute its derivative and set it equal to zero.

$$\begin{aligned} f'(x) = 6x^2 - 18x + 12 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x &= 1 \text{ or } 2 \end{aligned}$$

This means that this function has two stationary points: one at $x = 1$ and one at $x = 2$. But are these local maxima, local minima, or inflection points?

One straightforward way for working this out is the *first derivative test*, where we consider the sign of the derivative around the stationary point.

| $x$ | 0 | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | 0 | $-$ | 0 | $+$ |

We can infer from the table above that $f$ has a local maximum at $x = 1$ and a local minimum at $x = 2$, which can be verified using a graph.

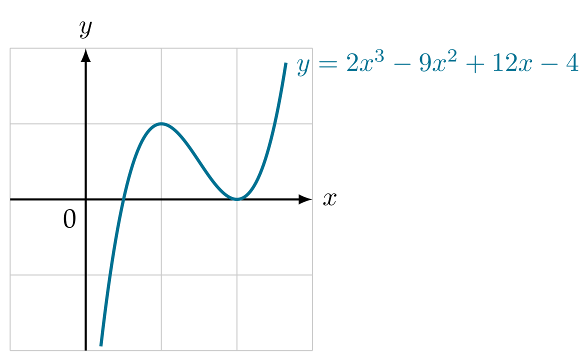


Figure 9: The function $f(x) = 2x^3 - 9x^2 + 12x - 4$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

Another way to identify the nature of a stationary point is the *second derivative test*. This involves computing the second derivative of the function at each stationary point.

- If the second derivative is positive, then the function has a local minimum at that point.
- If the second derivative is negative, then the function has a local maximum at that point.
- If the second derivative is zero, then the test is inconclusive. The point can be a local maximum, a local minimum, or an inflection point.

Applying this test to the function above, we have

$$f''(x) = 12x - 18$$

which gives us

$$f''(1) = -6 < 0$$
$$f''(2) = 6 > 0$$

This leads to the same result as before: $f$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

## 1.5 Differential equations

Differential equations are ones that involve both functions and their derivatives. Recall from earlier that a function of the form

$$f(x) = c \cdot e^{g(x)}$$

must have the derivative

$$f'(x) = c \cdot e^{g(x)} \cdot g'(x) = f(x) \cdot g'(x)$$

where $c \in \mathbb{R}$. We will make use of this relationship a lot when solving differential equations, as we will see in the following example problems.

**Problem.** Solve $f(x) = f'(x)$.

**Solution.** $f(x) = C \cdot e^x$ where $C \in \mathbb{R}$.

**Problem.** Solve $f'(x) = -3f(x)$.

**Solution.** $f(x) = C \cdot e^{-3x}$ where $C \in \mathbb{R}$.

Now consider the slightly more complicated case of having to solve

$$f'(x) = a(x) \cdot f(x).$$

To do this, we find the primitive (antiderivative) of $a(x)$, which we will denote $A(x)$. In other words, we have $A'(x) = a(x)$. The solutions to this differential equation are then given by $f(x) = C \cdot e^{A(x)}$, where $C \in \mathbb{R}$. An example is given below.

**Problem.** Solve $f'(x) = (2x + 1) \cdot f(x)$.

**Solution.** The antiderivative of $2x + 1$ is $x^2 + x$, so the solution to the equation is given by $f(x) = C \cdot e^{x^2+x}$ where $C \in \mathbb{R}$.

These solutions, which contain unspecified multiplicative constants, are called *general solutions* of the differential equation. Sometimes we might be asked to find specific values for these constants by using initial conditions.

**Problem.** What is the solution of $f'(x) = (2x+1) \cdot f(x)$ that verifies $f(0) = 3$?

**Solution.** Continuing from the previous example problem, we know that the solution to the differential equation is given by $f(x) = C \cdot e^{x^2+x}$, where $C \in \mathbb{R}$. We can find the value of $C$ by using the initial condition $f(0) = 3$.

$$\begin{aligned} C \cdot e^{0^2+0} &= 3 \\ C &= 3 \end{aligned}$$

This gives us the specific solution $f(x) = 3 \cdot e^{x^2+x}$.

## 2 Integral calculus

Calculating discrete sums is rather easy arithmetic, but how about infinite sums? Is it possible to, say, compute the area under a given curve?

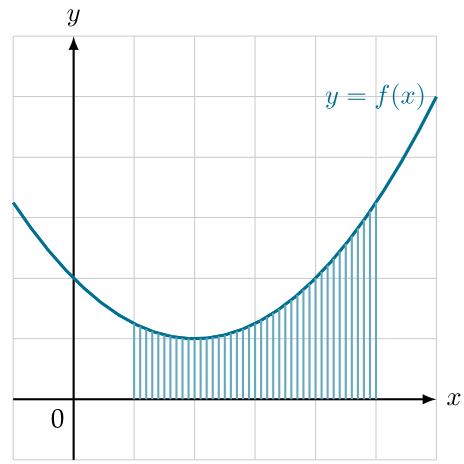


Figure 10: The area under a quadratic curve.

Let us consider a simpler example, where the curve in question is merely a straight line. Suppose we have a linear function $f(x) = 2x$. How can we calculate the area under its graph, evaluated between $x = 0$ and some $x = a$?

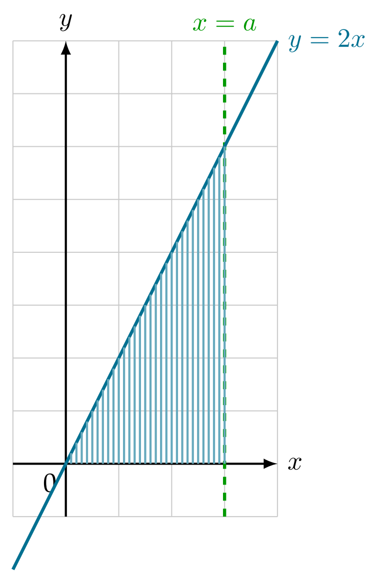


Figure 11: The area under the straight line given by the function $y = 2x$.

Geometry tells us that the area of this triangle can be calculated as

$$\frac{1}{2}(a)(2a) = a^2.$$

If we rename our variable $a$ as $x$, we see that the area under the straight line $y = 2x$ evaluated between $0$ and some real number $x$ can be expressed as $x^2$. Notice that differentiating $x^2$ gives us $2x$, which is our original linear function!

This is by no means a coincidence — in fact, the process of finding the area under a curve is the inverse operation of differentiation. This process of finding a primitive or antiderivative of a function is known as *integration*, which will be the focus of this section.

## 2.1 What is an indefinite integral?

Recall from the previous section that if $f$ is the derivative of $F$, then $F$ is called a *primitive* or *antiderivative* of $f$. For example, $x^2$ is a primitive of $2x$.

$$x^2 \rightleftarrows 2x$$

However, that this is not the only primitive of $2x$. For instance, $x^2 + 5$ and $x^2 - 3$ are both valid antiderivatives. In fact, as we shall prove below, all primitives of $f$ differ by a constant.

> **Theorem.** for any function $f$, all primitives of $f$ differ by a constant.
>
> **Proof.** Let $F_1$ and $F_2$ be two primitives of $f$. We want to prove that $F_1 - F_2$ is a constant.
>
> To do this, we consider the derivative of the expression $F_1 - F_2$.
>
> $$\begin{aligned}(F_1 - F_2)' &= F_1' - F_2' \\ &= f - f \\ &= 0.\end{aligned}$$
>
> Since $(F_1 - F_2)' = 0$, the difference $F_1 - F_2$ must be a constant.

From this theorem, we conclude that $x^2 + C$ is a primitive of $2x$ for any constant $C$. We can express this fact using an *indefinite integral*, as shown below.

$$\int 2x\,dx = x^2 + C$$

Here, the constant $C$ is called the *constant of integration*[2].

## 2.2 Rules of integration

We can modify our rules of differentiation from the last section into rules of integration, some of which are shown below.

$$c \xrightarrow{\int} cx + C \tag{1}$$

$$x \xrightarrow{\int} \frac{1}{2}x^2 + C \tag{2}$$

$$x^p \xrightarrow{\int} \frac{1}{p+1}x^{p+1} + C \tag{3}$$

$$e^x \xrightarrow{\int} \frac{1}{p+1}e^x + C \tag{4}$$

$$\frac{1}{x} \xrightarrow{\int} \ln|x| + C \tag{5}$$

$$\sin x \xrightarrow{\int} -\cos x + C \tag{6}$$

$$\cos x \xrightarrow{\int} \sin x + C \tag{7}$$

Here, $c$ is a constant, $p$ is a real number, and $C$ is the constant of integration[3]. We've left out the product rule and chain rule for now, but we'll see how they can be applied to integration later on.

[2]Or: the *integration constant*.

[3]The rules for integration are essentially the reverse of the rules for differentiation, with a few exceptions. For example, the integral of $1/x$ is $\ln|x|$ rather than $\ln x$, as the natural logarithm is only defined for positive numbers.

## 2.3 Integration by substitution

*Integration by substitution* is a powerful technique for integrating functions that are not immediately obvious. Consider, for example, the integral below.

$$\int 2x\sqrt{x^2+1}\,dx$$

This integral is seems a little tricky at first glance, but we can make it easier by substituting $u = x^2+1$. This substitution gives us

$$\frac{du}{dx} = 2x.$$

Not so rigorously, we can rearrange this equation to obtain

$$du = 2x\,dx$$

the right-hand-side of which is part of our original integral. We can now rewrite our integral in terms of $u$ like so:

$$\begin{aligned}\int 2x\sqrt{x^2+1}\,dx &= \int \sqrt{u}\,du \\ &= \int u^{\frac{1}{2}}\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C\end{aligned}$$

which gives us the answer.

Upon closer examination, one can find that integration by substitution is but the chain rule in disguise, as shown below.

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= f'(g(x))\cdot g'(x) && \text{(chain rule)} \\ \int \frac{d}{dx}f(g(x))\,dx &= \int f'(g(x))\cdot g'(x)\,dx && \text{(integrate both sides)} \\ f(g(x)) + C &= \int f'(g(x))\cdot g'(x)\,dx && \text{(integral \& derivative cancel out)} \\ \int f'(g(x))\cdot g'(x)\,dx &= f(g(x)) + C && \text{(integration by substitution)}\end{aligned}$$

## 2.4 Integration by parts

Another useful technique for integration is *integration by parts*. As an example, consider the following integral.

$$\int x\cos x\,dx$$

We know that

$$\frac{d}{dx}\sin x = \cos x$$

which again after some not-so-rigorous rearrangement gives us

$$d(\sin x) = \cos x\,dx.$$

This allows us to transform our original integral into

$$\int x\cos x\,dx = \int x\,d(\sin x)$$

but now we're stuck.

Fortunately, integration by parts tells us that whenever we have an integral of the form

$$\int f(x)\, d(g(x))$$

we can rewrite it as

$$\begin{aligned} & f(x) \cdot g(x) - \int g(x)\, d(f(x)) \\ = & f(x) \cdot g(x) - \int g(x) f'(x)\, dx. \end{aligned}$$

Hence, we can continue our calculation as follows.

$$\begin{aligned} \int x\, d(\sin x) &= x \cdot \sin x - \int \sin x\, d(x) \\ &= x \cdot \sin x - \int \sin x\, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

Comparing this technique to our rules of differentiation, we see that integration by parts is essentially the product rule in reverse. We demonstrate this below.

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x) \qquad \text{(product rule)}$$

$$\int \frac{d}{dx}(f(x) \cdot g(x))\, dx = \int (f(x) \cdot g'(x) + g(x) \cdot f'(x))\, dx \qquad \text{(integrate both sides)}$$

$$f(x) \cdot g(x) + C = \int (f(x) \cdot g'(x) + g(x) \cdot f'(x))\, dx \qquad \text{(integral \& derivative cancel out)}$$

$$f(x) \cdot g(x) + C = \int f(x) \cdot g'(x)\, dx + \int g(x) \cdot f'(x)\, dx \qquad \text{(integral of sum)}$$

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x)\, dx \qquad \text{(rearranging to eliminate } C\text{)}$$

$$\int f(x) \cdot\ d(g(x)) = f(x) \cdot g(x) - \int g(x) \cdot\ d(f(x)) \qquad \text{(integration by parts)}$$

## 2.5 What is a definite integral?

Consider a function $f$ that is continuous and defined on the interval $[a, b]$. The area under the graph of $f$ between $a$ and $b$ can be calculated using a *definite integral*, denoted by

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where $F$ is a primitive[4] of $f$. For convenience, we also write

$$F(b) - F(a) = [F(x)]_a^b.$$

Note that where the graph is under the horizontal axis, the area is measured in negative values. Therefore, we have the following.

$$\begin{aligned} \int_0^6 (4 - x)\, dx &= 6 \\ \int_0^{2\pi} \sin x\, dx &= 0 \end{aligned}$$

[4] Any primitive will work here as the constant of integration will ultimately cancel out.

See figures 12 and 13.

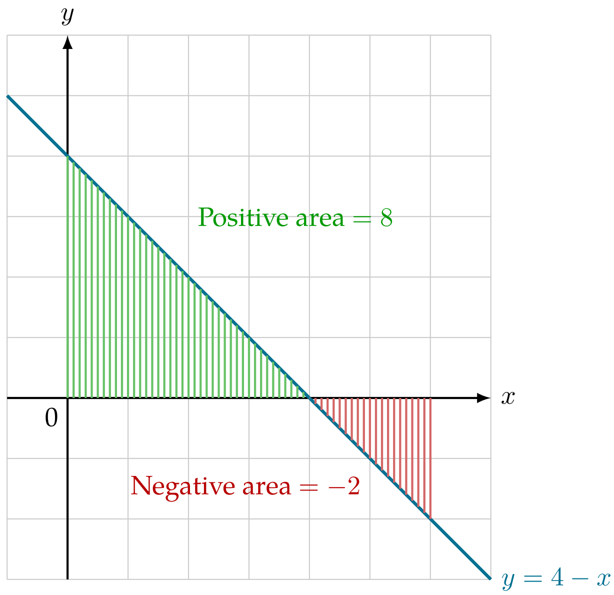


Figure 12: In the context of definite integrals, the area under a curve is always signed.

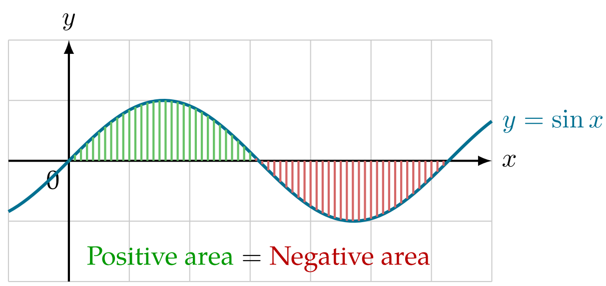


Figure 13: Positive and negative areas may cancel out.

Mathematically, we can compute the integrals above as follows.

$$\begin{aligned}\int_0^6 (4-x)\,dx &= \left[4x - \frac{1}{2}x^2\right]_0^6 \\ &= (24-18) - 0 \\ &= 6\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \sin x\,dx &= [-\cos x]_0^{2\pi} \\ &= -1 - (-1) \\ &= 0\end{aligned}$$

Both integration by substitution and integration by parts can be applied to definite integration. For example, consider the integral below.

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx$$

We proceed with integration by substitution. Let $u = x^2 + 1$, so $\frac{du}{dx} = 2x$, or $du = 2xdx$. When $x = 0$, we have $u = 1$; and when $x = 2$, we have $u = 5$. This allows us to rewrite our integral as

$$
\begin{aligned}
\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx &= \int_1^5 \frac{1}{2\sqrt{u}}\,du \\
&= \frac{1}{2}\int_1^5 u^{-\frac{1}{2}}\,du \\
&= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^5 \\
&= \frac{1}{2}(2\sqrt{5} - 2\sqrt{1}) \\
&= \sqrt{5} - 1
\end{aligned}
$$

which gives us the answer.

Now suppose we want to evaluate the integral below.

$$\int_0^e x \ln x \, dx$$

We proceed with integration by parts.

$$
\begin{aligned}
\int_0^e x \ln x \, dx &= \frac{1}{2}\int_0^e 2x \ln x \, dx \\
&= \frac{1}{2}\int_0^e \ln x \, d(x^2) \\
&= \frac{1}{2}\left( [x^2 \ln x]_0^e - \int_0^e x^2 \, d(\ln x) \right) \\
&= \frac{1}{2}\left( e^2 - \int_0^e x^2 \cdot \frac{1}{x}\, d(x) \right) \\
&= \frac{1}{2}\left( e^2 - \int_0^e x \, dx \right) \\
&= \frac{1}{2}\left( e^2 - \left[\frac{1}{2}x^2\right]_0^e \right) \\
&= \frac{1}{2}\left( e^2 - \frac{1}{2}e^2 \right) \\
&= \frac{1}{4}e^2
\end{aligned}
$$

## 2.6 Numerical methods for computing definite integrals

In practice, how can we approximate the area under a given curve when the corresponding function is unknown (or is horrendous)? One way to do this is to estimate that area using rectangular bars. See figure 14.

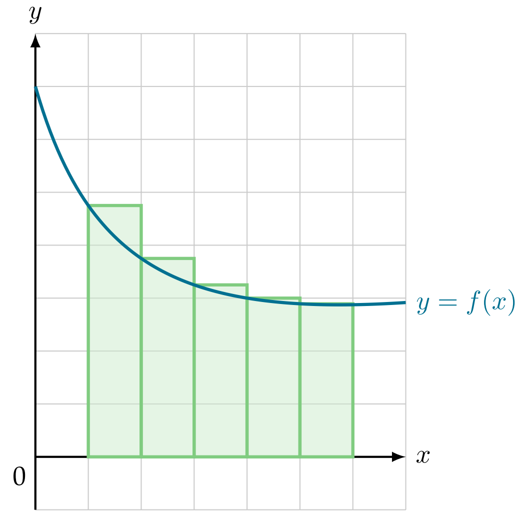


Figure 14: Using rectangles to approximate the area under the curve $y = f(x)$ between $x = 1$ and $x = 5$.

For a total of $n$ rectangles, the area under the curve $y = f(x)$ between $x = a$ and $x = b$ can be approximated as

$$\sum_{i=0}^{n-1} \underbrace{\frac{b-a}{n}}_{\text{Width}} \cdot \underbrace{f\left(a + \frac{b-a}{n} \cdot i\right)}_{\text{Height of } i\text{-th rectangle}}.$$

As $n$ approaches infinity, the subdivisions become more and more precise, and this sum gradually approaches the true area under the curve.

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{b-a}{n} \cdot f\left(a + \frac{b-a}{n} \cdot i\right) = \int_a^b f(x)\,dx$$

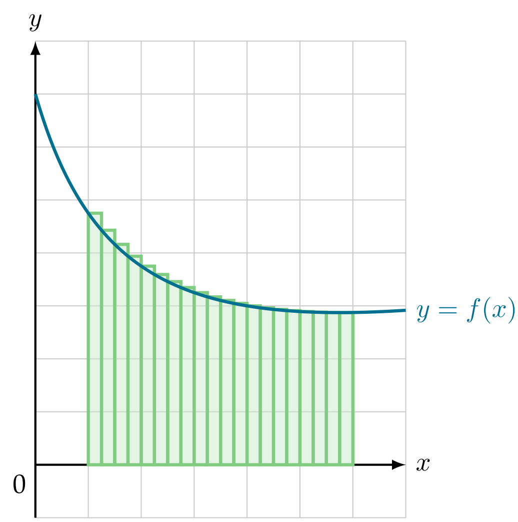


Figure 15: As $n$ approaches infinity, the subdivisions become more and more precise, and the sum of the rectangular areas gradually approaches the true area under the curve.

If we want to obtain an even more accurate result, we can use trapeziums instead of rectangles. See figure 16.

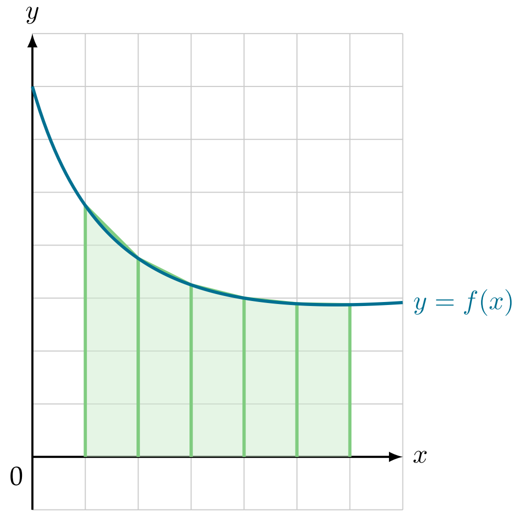


Figure 16: Using trapeziums to approximate the area under the curve.

### 2.7 What is an improper integral?

An *improper integral* is a definite integral that violates the usual assumptions we make when performing integration. An improper integral can take any one of the following forms.

$$\int_a^\infty f(x)\,dx \qquad \text{(see figure 17)}$$

$$\int_{-\infty}^b f(x)\,dx \qquad \text{(see figure 18)}$$

$$\int_{-\infty}^\infty f(x)\,dx \qquad \text{(see figure 19)}$$

$$\int_a^b f(x)\,dx \text{ where } f(x) \text{ is undefined somewhere on } [a,b] \qquad \text{(see figure 20)}$$

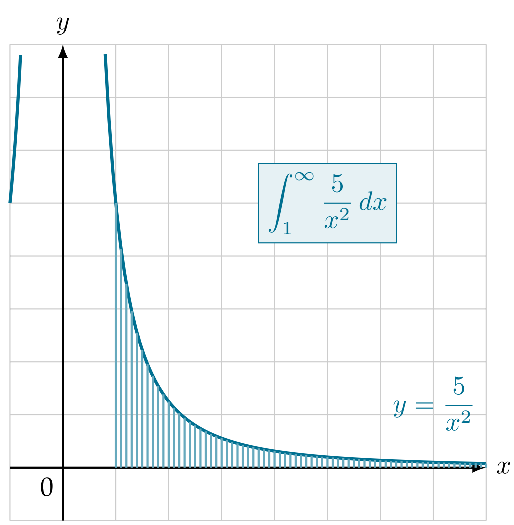


Figure 17: An improper integral evaluated on the interval $[1, \infty)$.

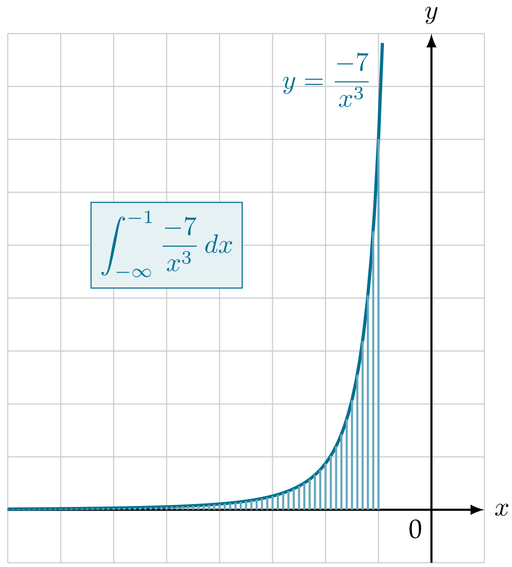


Figure 18: An improper integral evaluated on the interval $(-\infty, -1]$.

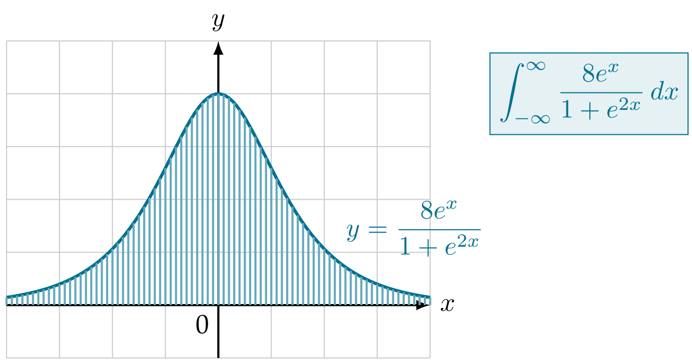


Figure 19: An improper integral evaluated on the interval $(-\infty, \infty)$.

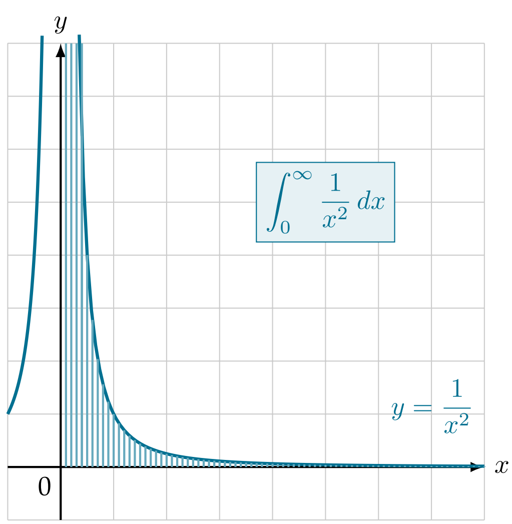


Figure 20: An improper integral involving an asymptote at $x = 0$, where the function $y = 1/x^2$ is undefined.

All of the following can be formally defined as follows.

- If $\int_a^t f(x)\,dx$ exists for all $t > a$, then we define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

  assuming that this limit exists and is finite.

- Similarly, if $\int_t^b f(x)\,dx$ exists for all $t < b$, then we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

  assuming that this limit exists and is finite.

- If for some $c \in \mathbb{R}$, the improper integrals $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ both exist, then we define the following.

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

- Improper integrals involving asymptotes are defined in a similar manner using limits.

Note that these integrals are well-defined only when the corresponding limits exist. For instance, the integral

$$\int_0^\infty 2\,dx$$

is undefined. See figure 21.

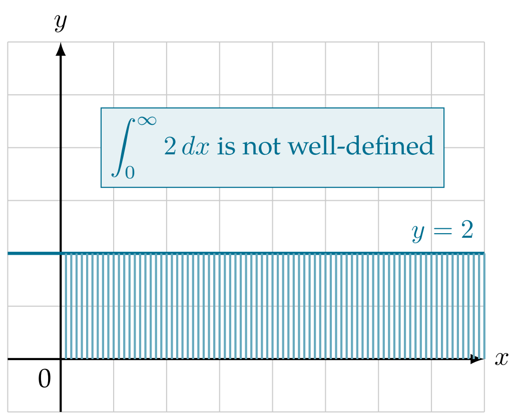


Figure 21: An improper integral involving an asymptote at $x = 0$, where the function $y = 1/x^2$ is undefined.

Sometimes, it might be difficult to directly tell whether an improper integral is well-defined. Take for example the following integral.

$$\int_1^\infty \frac{1}{x}\,dx$$

To determine whether this is a valid integral, we calculate

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx \\ &= \lim_{t\to\infty}[\ln x]_1^t \\ &= \lim_{t\to\infty}(\ln t - \ln 1) \\ &= \lim_{t\to\infty}\ln t\end{aligned}$$

which does not exist. Therefore, this integral is undefined.

On the contrary, consider the following integral.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

This time, we calculate

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx \\ &= \lim_{t\to\infty}[-x^{-1}]_1^t \\ &= \lim_{t\to\infty}\left(-\frac{1}{t}-(-1)\right) \\ &= \lim_{t\to\infty}\left(1-\frac{1}{t}\right) \\ &= 1\end{aligned}$$

which means the integral is defined and equals to $1$.

In general we have the following theorem.

> **Theorem.** Let $a > 0$. The integral
>
> $$\int_a^\infty \frac{1}{x^p}\,dx$$
>
> is defined if $p > 1$ and undefined if $p \leq 1$. On the other hand, the integral
>
> $$\int_0^a \frac{1}{x^p}\,dx$$
>
> is defined if $p < 1$ and undefined if $p \geq 1$.

This is summarised in table 1 and visualised in figure 22. As we approach infinity, the functions with higher exponents become "flat" enough to be integrable. The opposite occurs near zero, where lower exponents give flatter and integrable curves.

| $p$ | $< 1$ | $= 1$ | $> 1$ |
|---|---|---|---|
| $\int_a^\infty \frac{1}{x^p}\,dx$ defined? | ✗ | ✗ | ✓ |
| $\int_0^a \frac{1}{x^p}\,dx$ defined? | ✓ | ✗ | ✗ |

Table 1: A table showing when improper integrals of the function $1/x^p$ are defined. Assume $a > 0$.

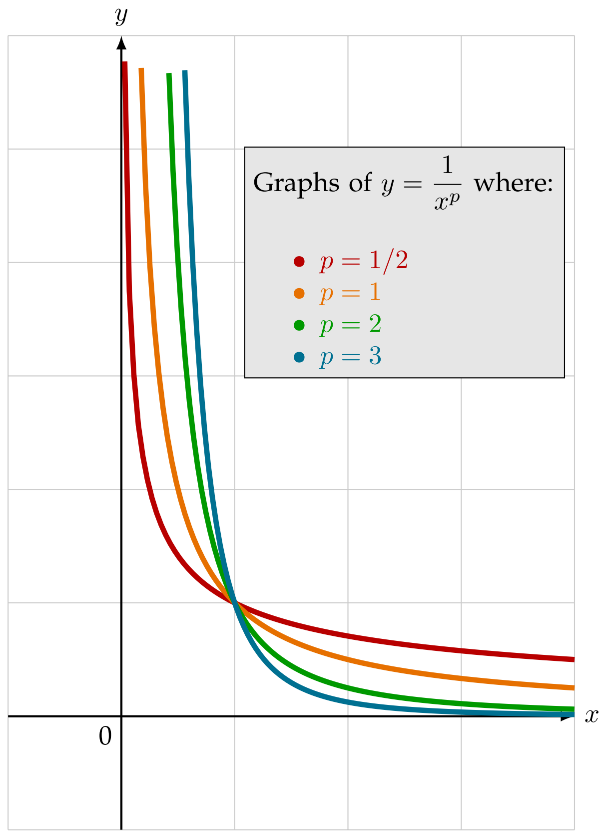


Figure 22: Graphs of $y = 1/x^p$ with varying values of $p$.