# Introductory Mathematics for Computer Science (COMP0011)

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Year 1 Term 2, 2024–25

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## Contents

# 1 Complex numbers

The foundation of the *complex numbers* is given by the imaginary unit $i$, defined either as $i = \sqrt{-1}$ or as $i^2 = -1$.

A complex number $z$ can be written as $a + bi$, where $a, b \in \mathbb{R}$. The real numbers $a$ and $b$ are known as the *real part* and the *complex part* of $z$ respectively.

The set of all complex numbers is denoted as $\mathbb{C}$. Note that the set of real numbers $\mathbb{R}$ is a subset of $\mathbb{C}$. See figure 1.

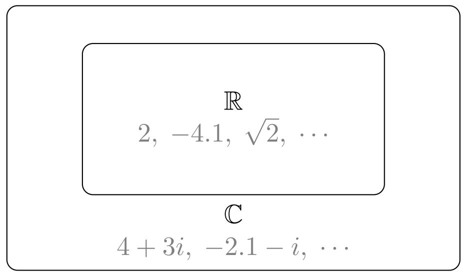


Figure 1: The set of real numbers $\mathbb{R}$ is a subset of the set of complex numbers $\mathbb{C}$. All real numbers are complex numbers.

## 1.1 Basic arithmetic with complex numbers, and complex conjugates

To add or subtract two complex numbers, we deal with the real and imaginary parts separately.

$$(2+3i) + (5-8i) = (2+5) + (3+(-8))i = 7 - 5i \qquad \text{(Addition)}$$
$$(2+3i) - (5-8i) = (2-5) + (3-(-8))i = -3 + 11i \qquad \text{(Subtraction)}$$

The multiplication of complex numbers is also straightforward as long as we bear in mind that $i^2 = -1$.

$$\begin{aligned} (3+4i)(-2+3i) &= -6 + 9i - 8i + 12i^2 \\ &= (-6-12) + (9-8)i \\ &= -18 + i \end{aligned}$$

To divide a complex number by another, e.g.

$$\frac{a+bi}{c+di}$$

we multiply both the numerator and denominator by $c - di$, which is obtained by flipping the sign of the imaginary part of the denominator. For example, if we want to compute

$$\frac{2+3i}{5-4i},$$

we flip the sign of the imaginary part of $5 - 4i$ to get $5 + 4i$. We then multiply both the numerator and denominator of the fraction by this $5 + 4i$ to get

$$\begin{aligned} \frac{2+3i}{5-4i} &= \frac{(2+3i)(5+4i)}{(5-4i)(5+4i)} \\ &= \frac{10 + 8i + 15i - 12}{25 + 20i - 20i + 16} \\ &= \frac{-2+23i}{41} \\ &= \frac{-2}{41} + \frac{23}{41}i. \end{aligned}$$

Notice how multiplying $5 - 4i$ with $5 + 4i$ produces the real number $41$. By flipping the sign of the imaginary part of a complex number, we obtain what's called its *complex conjugate*. The complex conjugate of $z$ is denoted as $\overline{z}$. By writing $z$ as $a + bi$, we can easily prove that the product of any complex number with its conjugate must equal a real number:

$$z \times \overline{z} = (a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R}.$$

## 1.2 Visualising complex numbers

Given some complex number $z = x + yi$, we can treat its real and imaginary parts as Cartesian coordinates, thus mapping it to a point on the 2D plane.

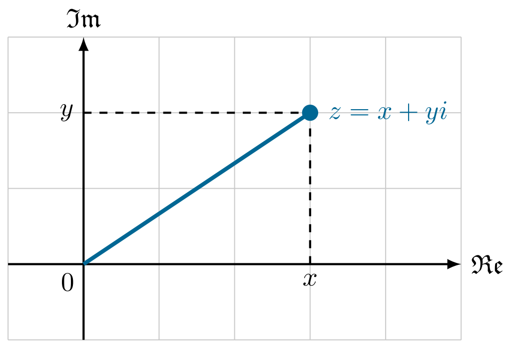


Figure 2: The complex number $z = x + yi$ as a point on the 2D plane

## 1.3 Exponential form

Recall that it is possible to express a point on a 2D plane using polar coordinates $(R, \theta)$ as well. Indeed, given any complex number $z = x + yi$, we can find its corresponding pair of values $R$ and $\theta$.

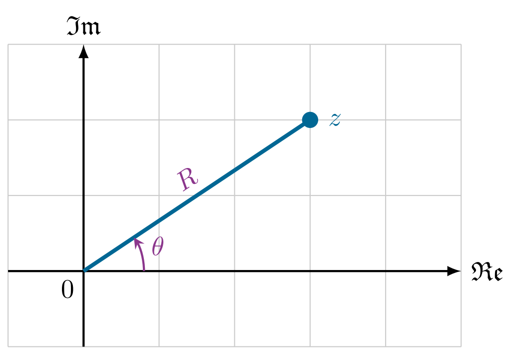


Figure 3: The position of a complex number on the 2D plane can be represented using polar coordinates.

Based on this idea, we introduce a new notation as follows.

> If the position of a complex number $z$ on the 2D plane can be represented by the polar coordinates $(R, \theta)$, then we have
> $$z = R \times e^{i\theta}$$
> where $R, \theta \in \mathbb{R}$ and $R \geq 0$.
>
> $R$ is called the *absolute value* or *modulus* of $z$ and is denoted as $|z|$. This represents the point's position from the origin.

$\theta$ is called the *argument* of $z$ and is denoted as $\arg(z)$. This represents the angle from horizontal.

This way of representing complex numbers is known as the *exponential form*. (This is a natural result of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

Now consider two complex numbers expressed in exponential form.

$$z_1 = R_1 \times e^{i\theta_1}$$
$$z_2 = R_2 \times e^{i\theta_2}$$

These two numbers are considered equal if both of the following conditions hold.

$$R_1 = R_2$$
$$\theta_1 = \theta_2 \color{red}{+2k\pi} \qquad \text{(for some } k \in \mathbb{Z}\text{)}$$

Note that the red part is necessary because a rotation of $2\pi$ radians has no effect on a point's position.

The exponential form makes the multiplication and division of complex numbers a lot easier.

**Multiplication**

$$\begin{aligned}(1 \times e^{\frac{\pi}{6}i}) \times (2 \times e^{-\frac{\pi}{4}i}) &= 2 \times e^{\frac{\pi}{6}i - \frac{\pi}{4}i} \\ &= 2 \times e^{-\frac{\pi}{12}i}\end{aligned}$$

**Division**

$$\begin{aligned}\frac{1 \times e^{\frac{\pi}{6}i}}{2 \times e^{-\frac{\pi}{4}i}} &= \frac{1}{2} \times \frac{e^{\frac{\pi}{6}i}}{e^{-\frac{\pi}{4}i}} \\ &= 2 \times e^{\frac{5\pi}{12}i}\end{aligned}$$

## 1.4 Converting between Cartesian and exponential forms

The methods used to convert between the Cartesian form $x + yi$ and the exponential form $R \times e^{i\theta}$ are outlined below.

- **Given the Cartesian form of a complex number, find its exponential form.**

  Given the Cartesian form $z = x + yi$, we can find the modulus using Pythagoras' theorem.

  $$|z| = \sqrt{x^2 + y^2}$$

  The argument can be found using the arctangent.

  $$\arg(z) = \arctan\left(\frac{y}{x}\right)$$

- **Given the exponential form of a complex number, find its Cartesian form.**

  Given the exponential form $z = R \times e^{i\theta}$, we can find the Cartesian coordinates using simple trigonometry.

  $$x = R\cos\theta$$
  $$y = R\sin\theta$$

To speed up conversion processes, it is often useful to memorize the Cartesian coordinates of some special points on the unit circle. See figure 4 and table 1.

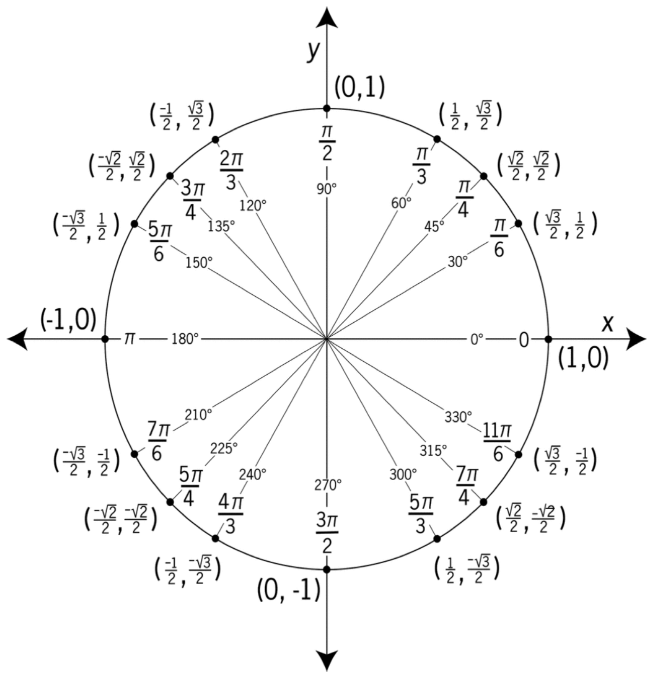


Figure 4: It is important to know the coordinates of points on the circle corresponding to classic angles.

| $\theta$ (radians) | $\pi/6$ | $\pi/4$ | $\pi/3$ |
|---|---|---|---|
| $\theta$ (degrees) | $30°$ | $45°$ | $60°$ |
| $\sin\theta$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos\theta$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |

Table 1: The values of $\sin\theta$ and $\cos\theta$ for some classic angles $\theta$.

## 1.5 Visualising arithmetic on complex numbers

When visualised on the 2D plane, the addition of complex numbers is similar to that of vectors. We join the arrows in a tip-to-tail manner in order to determine the sum, as shown in figure 5.

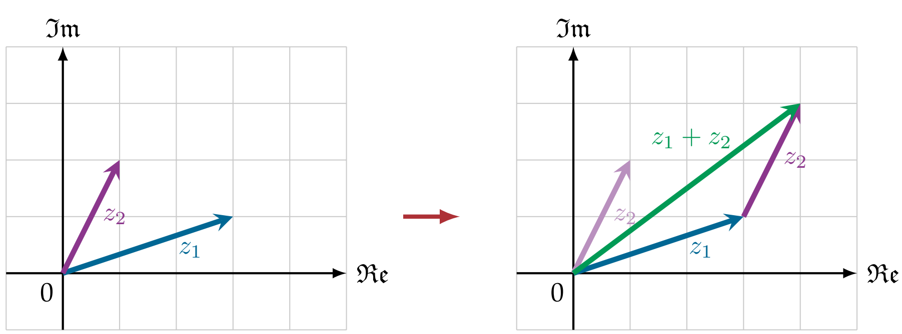


Figure 5: Addition of complex numbers.

The above figure also illustrates another key idea. Notice how in the figure on the right, the vectors of

$z_1$, $z_2$ and $z_1 + z_2$ form a triangle. This means their absolute values must fulfil the triangle inequality.

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

To visualise multiplication we consider the exponential form. As shown in figure 6, when two complex numbers are multiplied, their arguments are added together to produce a rotation, while their moduli are multiplied.

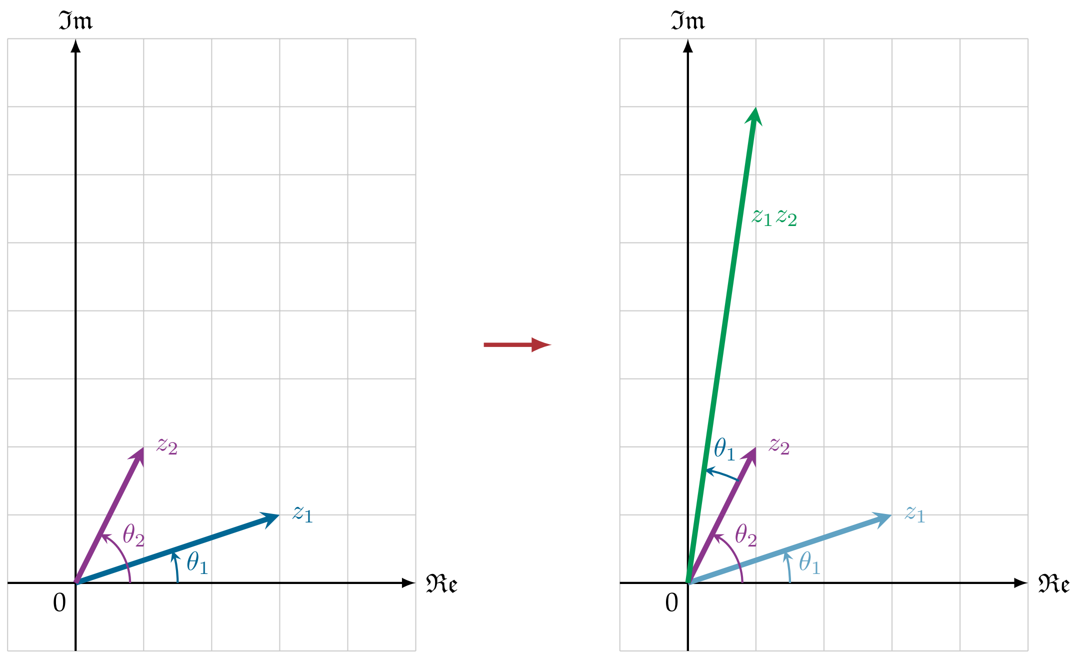


Figure 6: Multiplication of complex numbers.

## 1.6 Roots of unity

The *roots of unity* are the solutions to the equation

$$z^n = 1, \tag{1}$$

where $n$ is a positive integer.

Solving this equation for values of $n$ such as 2 and 4 is straightforward:

$$\begin{aligned} n = 2 &\implies z^2 = 1 \implies z = 1 \text{ or } -1 \\ n = 4 &\implies z^4 = 1 \implies z = 1,\ i,\ -1 \text{ or } -i \end{aligned}$$

but solving it for other values of $n$ requires us to express $z$ in its exponential form, i.e.

$$z = R \times e^{i\theta}. \qquad (R, \theta \in \mathbb{R} \text{ and } R \geq 0)$$

This allows us to rewrite the equation as

$$\begin{aligned} (R \times e^{i\theta})^n &= 1 \times e^{0i} \\ R^n \times e^{in\theta} &= 1 \times e^{0i} \end{aligned}$$

which yields the following.

$$\begin{cases} R^n = 1 \\ n\theta = 0 + 2k\pi = 2k\pi \end{cases} \qquad (\text{for some } k \in \mathbb{Z})$$

Since $R \geq 0$ and $R \in \mathbb{R}$, we must have $R = 1$. Furthermore, the second equation gives us

$$\theta = \frac{2k\pi}{n}$$

i.e.

$$\theta \in \left\{ \cdots, \; -3 \cdot \frac{2\pi}{n}, \; -2 \cdot \frac{2\pi}{n}, \; -\frac{2\pi}{n}, \; 0, \; \frac{2\pi}{n}, \; 2 \cdot \frac{2\pi}{n}, \; 3 \cdot \frac{2\pi}{n}, \; \cdots \right\}$$

which seemingly means that there are infinitely many roots of unity. However, this is impossible because by the fundamental theorem of algebra, equation (1) (which is a polynomial equation of degree $n$) can only have $n$ solutions.

To resolve this apparent paradox, let us visualise the problem on a 2D plane. For the sake of simplicity let us assume $n = 3$. We know that all solutions to (1) must have a modulus of $R = 1$, so they must lie on the unit circle.

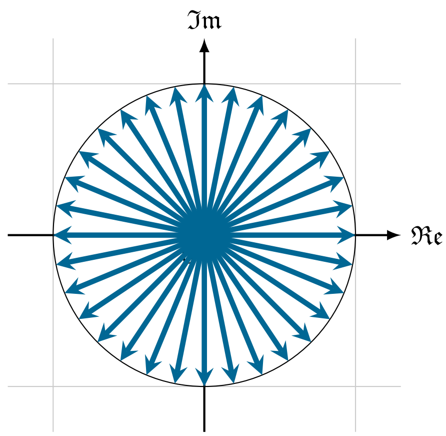


Figure 7: The roots of unity must lie somewhere on the unit circle.

We want to find the angles $\theta$ such that if we start at the point $1$ and then rotate anticlockwise by $\theta$ radians $n = 3$ times, we end up back at $1$.

- Obviously we can have $\theta = 0$.
- Another obvious solution is $\theta = 2\pi/3$. If we rotate by this angle $3$ times, we will have completed a full $2\pi$ radians, bringing us back to the initial point.
- Moreover, we can also have $\theta = 4\pi/3$. Rotating by this angle $3$ times creates a total rotation of $4\pi$ radians (i.e. 2 full cycles), bringing us once again back to the starting point.
- Continuing this pattern, it appears that $\theta = 6\pi/3$ is also a solution. However, this is in fact the same as $\theta = 0$, since angles differing by $2\pi$ are considered equivalent. The same applies for $\theta = 8\pi/3$ (equivalent to $2\pi/3$), $\theta = 10\pi/3$ (equivalent to $4\pi/3$), and so on.

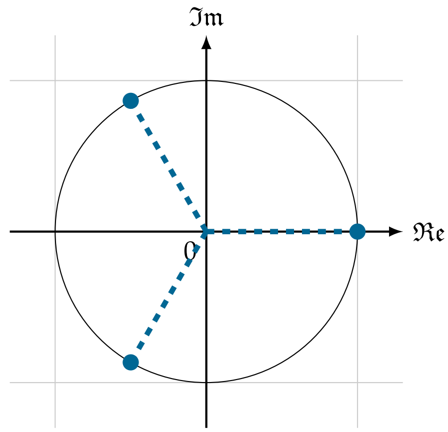


Figure 8: The third roots of unity.

This resolves the above paradox — we were right in thinking that the possible values of $\theta$ are given by

$$\theta = \frac{2k\pi}{n}$$

or

$$\theta \in \left\{\cdots,\ -3 \cdot \frac{2\pi}{n},\ -2 \cdot \frac{2\pi}{n},\ -\frac{2\pi}{n},\ 0,\ \frac{2\pi}{n},\ 2 \cdot \frac{2\pi}{n},\ 3 \cdot \frac{2\pi}{n},\ \cdots\right\}$$

but these solutions are not all distinct. To make sure we only count distinct solutions, we impose the range $0 \leq k < n$, giving us

$$\theta = \frac{2k\pi}{n} \qquad (k \in \mathbb{N} \text{ and } k < n)$$

or

$$\theta \in \left\{0,\ \frac{2\pi}{n},\ 2 \cdot \frac{2\pi}{n},\ 3 \cdot \frac{2\pi}{n},\ \cdots,\ (n-1) \cdot \frac{2\pi}{n}\right\}.$$

This yields the solutions

$$z = 1 \times e^{\frac{2k\pi}{n} i} = e^{\frac{2k\pi}{n} i} \qquad (k \in \mathbb{N} \text{ and } k < n)$$

or

$$z \in \left\{0,\ e^{\frac{2\pi}{n} i},\ e^{\frac{4\pi}{n} i},\ \cdots,\ e^{\frac{2(n-1)\pi}{n} i}\right\}.$$

## 2 Continuous functions

A function $f$ maps elements of a set $A$ to elements of another set $B$. We denote this as $f : A \to B$. In practice, most functions we consider will have type $\mathbb{R} \to \mathbb{R}$.

If a function maps a number $x$ to its square $x^2$, we can denote this by $x \mapsto x^2$. (Note the difference in the arrow symbol used — the symbol $\mapsto$ is read as "maps to".)

A function $y = f(x)$ can be represented graphically as the set of points $(x, y)$. See figure 9.

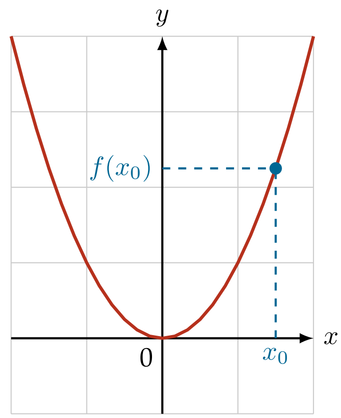


Figure 9: The graph of the function $y = x^2$.

In the next few subsections, we will be looking at some classic mathematical functions.

### 2.1 Trigonometric functions

Consider a point $P$ on the unit circle. If we let $\theta$ be the angle between $OP$ and the horizontal axis, then the coordinates of $P$ can be expressed as $(\cos\theta,\ \sin\theta)$. This is illustrated in figure 10.

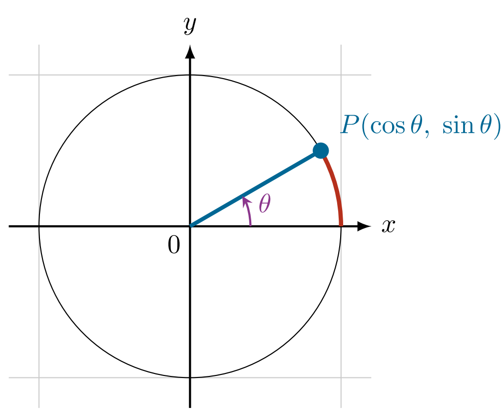


Figure 10: The trigonometric functions $\cos\theta$ and $\sin\theta$ can be defined using the unit circle. Note that if we are measuring $\theta$ in radians, then the length of the arc highlighted in red must be equal to $\theta$.

We've previously seen the values of $\sin\theta$ and $\cos\theta$ for some classic angles $\theta$ in table 1. Plotting these functions on a graph results in figure 11.

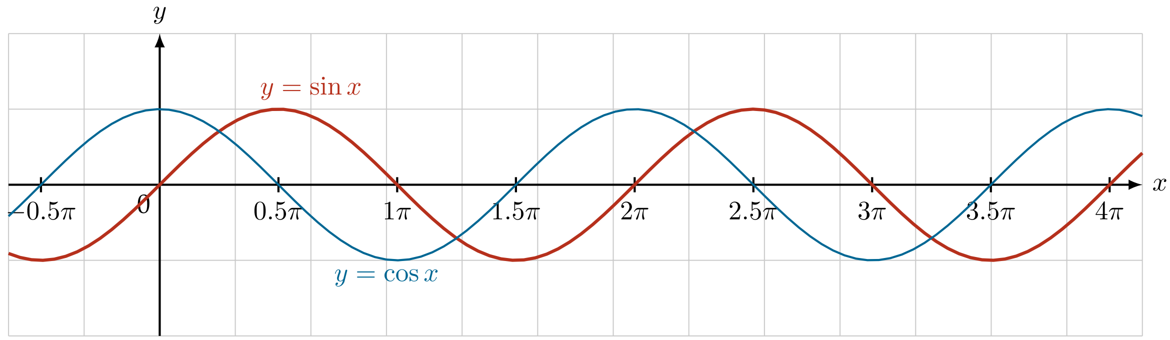


Figure 11: The graph of the functions $\sin x$ and $\cos x$.

## 2.2 Exponential and logarithm

One way to define the exponential function $\exp$ is as follows.

$$\begin{aligned} \exp(x+y) &= \exp(x) \cdot \exp(y) \\ \exp(0) &= 1 \\ \frac{d}{dx}\exp(x) &= \exp(x) \end{aligned}$$

Note that the first two relationships can be satisfied by any function of the form $f(x) = a^x$ where $a \in \mathbb{R}$. However, if we take all three conditions into account, the only function satisfying them is $\exp(x) = e^x$, where $e = 2.71828\cdots$ is Euler's number.

The exponential function $\exp(x) = e^x$ is plotted in figure 12. Note that:

- For all values of $x$, we have $\exp(x) > 0$.
- When $x$ is negative, $\exp(x)$ is very small.
- The value of $\exp(x)$ grows very fast as $x$ increases.

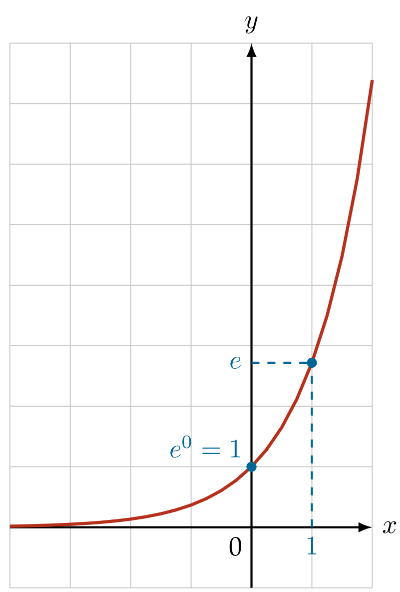


Figure 12: The graph of the function $y = \exp(x) = e^x$.

The natural logarithm $\ln x$ is the inverse of the exponential, meaning that $\ln(e^x) = x$. This results in the following properties.

$$\begin{aligned}
\ln(ab) &= \ln a + \ln b \\
\ln\left(\frac{a}{b}\right) &= \ln a - \ln b \\
\ln 1 &= 0 \\
\ln e &= 1 \\
a^x &= e^{x \ln a} \\
\ln(a^x) &= x \ln a
\end{aligned}$$

Since $e^x > 0$ for all $x$, the natural logarithm $\ln x$ is only defined for positive values of $x$.

The plot of $y = \ln x$ is given in figure 13. Note that:

- For $x < 1$, we have $\ln x < 0$.
- The curve intersects the $x$-axis at $(1, 0)$.
- For $x > 1$, the value of $\ln x$ grows very slowly as $x$ increases.

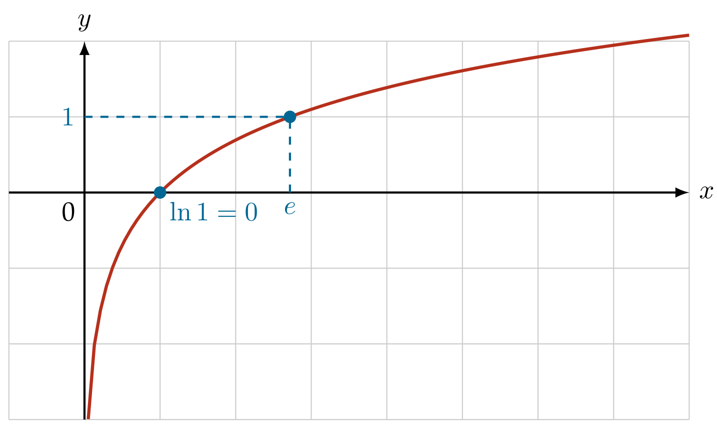


Figure 13: The graph of the function $y = \exp(x) = e^x$.

## 2.3 Introduction to limits

The idea of limits is simple.

> As $x$ approaches a value, $f(x)$ also approaches a value — both possibly infinite. To denote this we write $f(x) \to b$ as $x \to a$, or $\lim_{x \to a} f(x) = b$.

This gives us four different cases.

- As $x$ approaches infinity, $f(x)$ also approaches infinity. This means that $f(x)$ can become arbitrarily large for large enough $x$, i.e.

  $$\forall d > 0,\ \exists c > 0,\ x > c \Rightarrow f(x) > d.$$

  See figure 14. (Restricting $c$ and $d$ to positive values is not strictly necessary, but it does make our lives easier in some cases.)

- As $x$ approaches a value $a$, $f(x)$ approaches infinity. This means that $f(x)$ can become arbitrarily large for $x$ close enough $x$, i.e.

  $$\forall d > 0,\ \exists \eta > 0,\ 0 < |x - a| < \eta \Rightarrow f(x) > d.$$

In other words, for any value $d$, $f(x)$ can be greater than $d$ as long as the distance between $x$ and $a$ is less than some value $\eta$. See figure 15.

- As $x$ approaches infinity, $f(x)$ approaches a value $b$. This means that $f(x)$ can get arbitrarily close to $b$ for large enough $x$, i.e.

$$\forall \epsilon > 0,\ \exists c > 0,\ x > c \Rightarrow |f(x) - b| < \epsilon.$$

  See figure 16.

- As $x$ approaches a value $a$, $f(x)$ approaches a value $b$. This means that $f(x)$ can become arbitrarily close to $b$ for $x$ close enough to $a$, i.e.

$$\forall \epsilon > 0,\ \exists \eta > 0,\ 0 < |x - a| < \eta \Rightarrow |f(x) - b| < \epsilon.$$

  See figure 17.

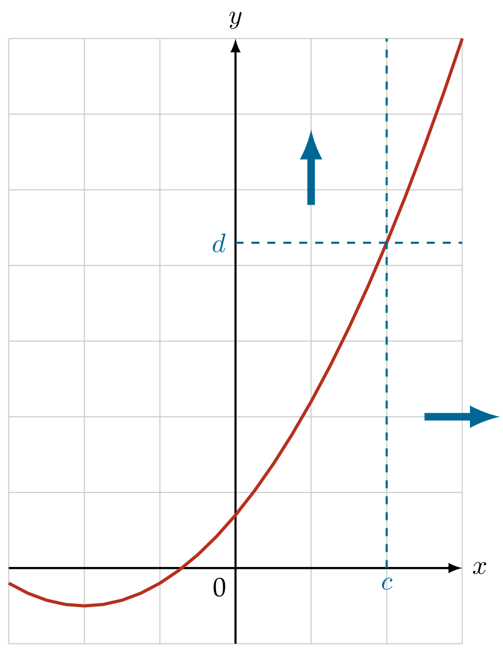


Figure 14: As $x$ approaches infinity, so does $f(x)$.

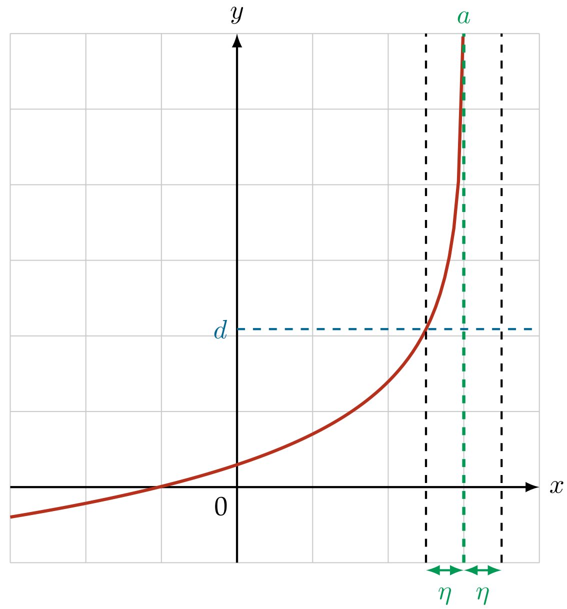


Figure 15: As $x$ approaches $a$, $f(x)$ approaches infinity.

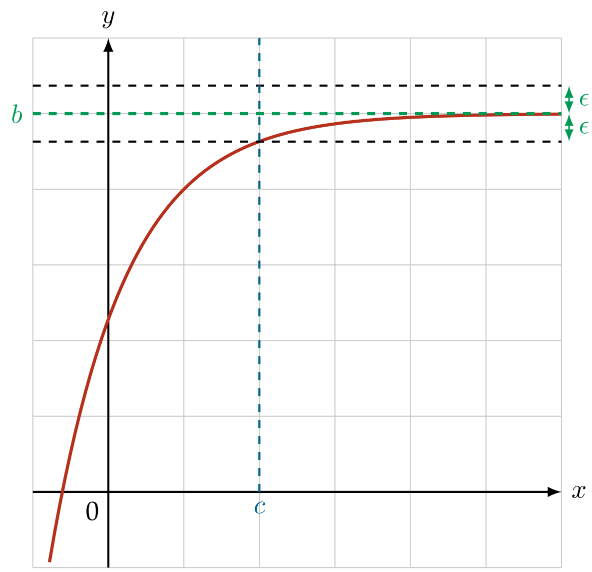


Figure 16: As $x$ approaches infinity, $f(x)$ approaches $b$.

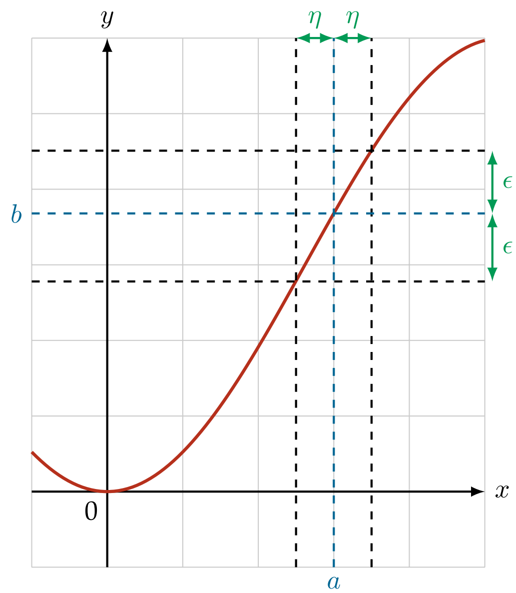


Figure 17: As $x$ approaches $a$, $f(x)$ approaches $b$.

A limit that exists is known as a *finite* limit. Finite limits can be combined in a natural way.

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

We use the following rules to handle infinities.

$$a \times \infty = \infty$$
$$\frac{a}{\infty} = 0$$

If a limit involves $x$ approaching zero, we may sometimes have to specify the direction in which $x$ is approaching it, i.e. whether it is approaching zero as a positive number (from the right) or as a negative number (from the left).

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

There are certain cases where we *cannot* combine limits. These are called *indeterminate forms*, and there is no general rule for figuring out what these indeterminate forms evaluate to. Examples of indeterminate forms are given below.

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0 \times \infty,\ \infty - \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

## 2.4 Little o and big O notation

Here we introduce two types of notation: little $o$ and big $O$.

**Little o notation.**

We write $f = o(g)$ near a value $b$ if $\lim_{x \to b} \frac{f(x)}{g(x)} = 0$. This means that $f$ is negligible when compared with $g$ (or that $g$ is more significant than $f$) near $b$.

For example, we say that $x = o(x^2)$ near $\infty$ because $\lim_{x \to \infty} x/x^2 = 0$.

**Big O notation.**

We write $f = O(g)$ near a value $b$ if $\lim_{x \to b} \left| \frac{f(x)}{g(x)} \right|$ is bounded. We can express the idea of being "bounded" more precisely as

$$\exists M \in \mathbb{R}, \lim_{x \to b} \left| \frac{f(x)}{g(x)} \right| < M.$$

This means that $f$ and $g$ have a similar growth rate near $b$.

For example, we say that $3x = O(x+1)$ near $\infty$ because $\lim_{x \to \infty} \left| \frac{3x}{x+1} \right| < \frac{3x}{x} = 3$.

Notice that by definition, we have $f = o(g) \Rightarrow f = O(g)$. This is because if $f = o(g)$ is true, then the limit $\lim_{x \to b} \frac{f(x)}{g(x)}$ must equal zero and is therefore bounded, which gives us $f = O(g)$.

## 2.5 Continuity

A function $f$ is continuous if for all $a$ where $f(a)$ is defined, we have $\lim_{x \to a} f(x) = f(a)$.

In practice, this means that the graph of $y = f(x)$ is a single unbroken curve. The exponential and logarithm functions, for example, are both continuous.

An important result of this is the *intermediate value theorem*.

**Intermediate value theorem.**

Assume for a continuous function $f$ that $a < b$ and $f(a) < f(b)$. For any value $y$ such that $f(a) < y < f(b)$, there exists a (not necessarily unique) value $x$ such that $a < x < b$ and $f(x) = y$.

See figure 18 and 19.

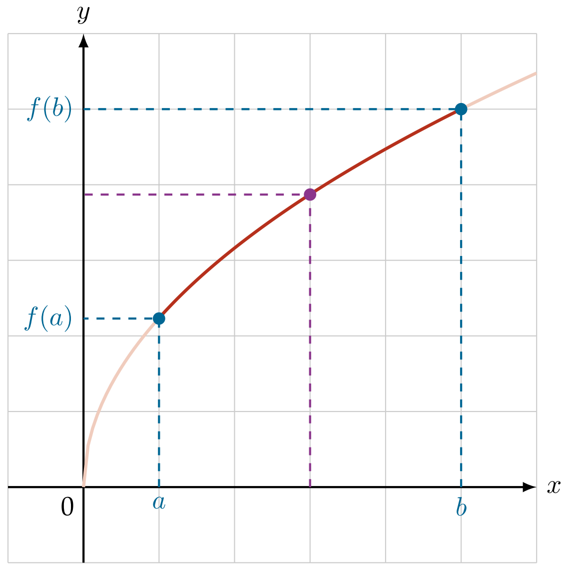


Figure 18: The intermediate value theorem.

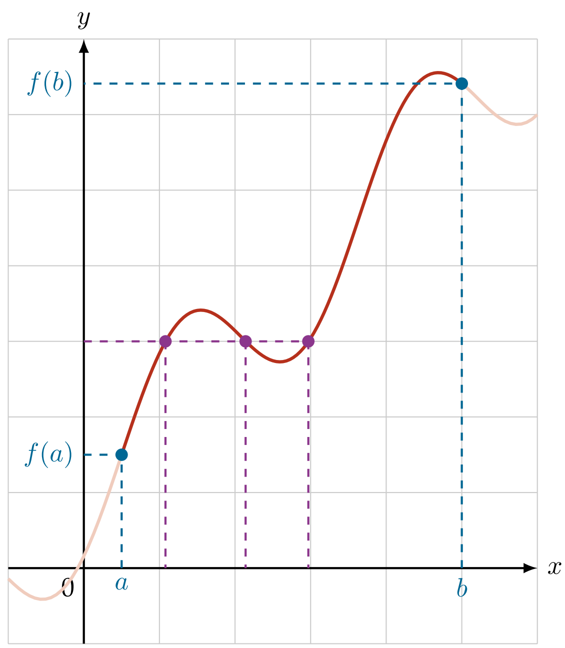


Figure 19: In the intermediate value theorem, for a given value $y$, the value of $x$ does not necessarily have to be unique.