Introductory Mathematics for Computer Science (COMP0011)

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Contents

1 Sequences and series

1.1 What is a sequence?

A *sequence* is essentially an ordered list of numbers. Examples include the harmonic sequence and the Fibonacci sequence, both of which are shown below.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots \qquad \text{(Harmonic sequence)}$$

$$0, 1, 1, 2, 3, 5, 8, 13, \cdots \qquad \text{(Fibonacci sequence)}$$

We denote the sequence $u_0, u_1, u_2, u_3, \cdots$ as (u_n), with $n \in \mathbb{N}$. Note that zero-based indexing is used here.

For instance, we can define the harmonic sequence using

$$u_n = \frac{1}{n+1}$$

while the Fibonacci sequence can be described by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n.$$

1.2 Arithmetic, geometric, monotone, bounded and recursive sequences

An *arithmetic sequence* is one where each term is obtained by adding a constant increment d to the previous term. The general term of an arithmetic sequence (u_n) is given by

$$u_n = a + nd$$

where $a = u_0$ is the first term of the sequence.

A *geometric sequence* is one where each term is obtained by multiplying the previous term by a constant ratio r. The general term of a geometric sequence (u_n) is given by

$$u_n = ar^n$$

where $a = u_0$ is the first term of the sequence.

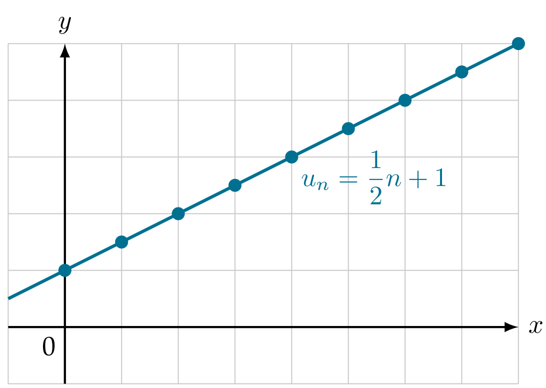

Figure 1: The graph of an arithmetic sequence.

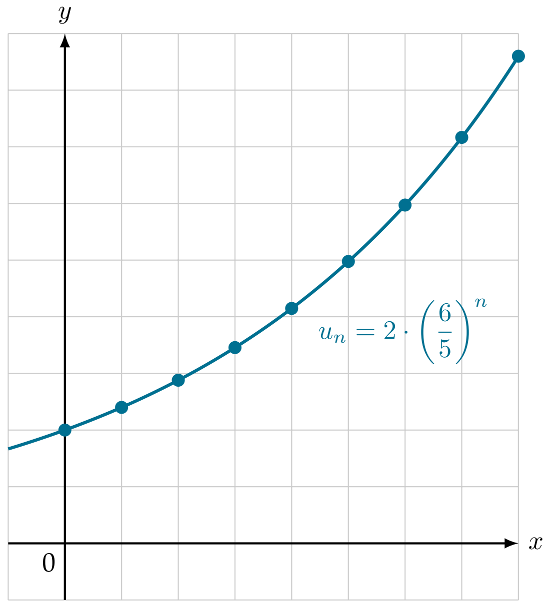

Figure 2: The graph of an geometric sequence.

A sequence (u_n) is said to be...

- ...*increasing* if for all $n \in \mathbb{N}$, we have $u_{n+1} \geq u_n$.
- ...*decreasing* if for all $n \in \mathbb{N}$, we have $u_{n+1} \leq u_n$.
- ...*monotone* if it is either increasing or decreasing.
- ...*bounded* if there exists some real number M such that for all $n \in \mathbb{N}$, we have $|u_n| \leq M$.

Notice that in non-strict inequalities (as opposed to strict inequalities) are used in the first two definitions. This means that a constant sequence is both increasing and decreasing.

In the last definition, both strict or non-strict inequalities can be used — they are equivalent in this context.

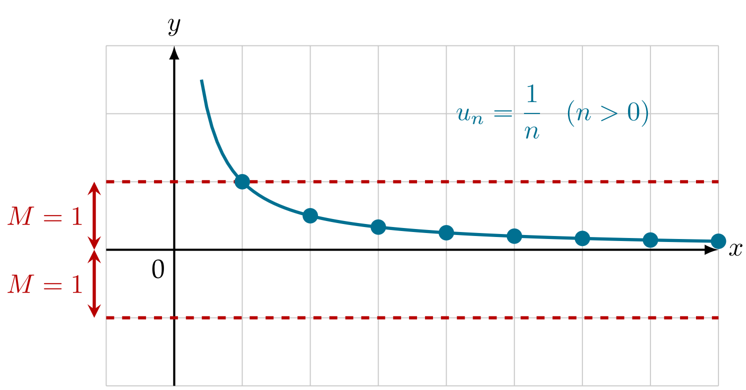

Figure 3: The graph of a decreasing and bounded sequence. Note that none of the terms have an absolute value that exceeds $M = 1$, as shown by the red dashed line.

A *recursive sequence* is defined by a recurrence relation along with some initial values. In other words, each term of the sequence is expressed as a function of the previous terms. For instance, the Fibonacci sequence can be defined as follows.

$$\begin{cases} u_0 = 0 \\ u_1 = 1 \\ u_{n+2} = u_{n+1} + u_n \end{cases}$$

1.3 Convergence of a sequence

Some sequences (u_n) have a limit as n approaches infinity[1]. For instance, as shown in figure 3, the harmonic sequence has a limit of zero. In other words, it *converges* to zero.

We say that a sequence (u_n) converges to a limit L if the terms u_n can get arbitrarily close to L for sufficiently large values of n, i.e.

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ n > n_0 \implies |u_n - L| < \epsilon$$

where n_0 is called a *rank*.

For example, we can prove that the harmonic sequence converges to zero as follows.

Problem. Prove that the harmonic sequence converges to zero, i.e.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Intuition. Recall the definition of a limit of a sequeence.

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ n > n_0 \implies |u_n - L| < \epsilon$$

Substituting $u_n = 1/n$ and $L = 0$, we can simplify the statement above to

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ n > n_0 \implies \frac{1}{n} < \epsilon.$$

Notice that we can rearrange the inequality at the end as follows.

$$\begin{aligned} \frac{1}{n} &< \epsilon \\ n &> \frac{1}{\epsilon} \end{aligned}$$

This means that if we choose $n_0 = 1/\epsilon$, then the inequality holds for all $n > n_0$.

Solution. Let $\epsilon > 0$. If we take $n_0 = 1/\epsilon$, then for all $n > n_0$, we have

$$\begin{aligned} n &> n_0 \\ n &> \frac{1}{\epsilon} \\ \epsilon &> \frac{1}{n} \\ \frac{1}{n} &< \epsilon \\ u_n &< \epsilon \\ |u_n - 0| &< \epsilon \end{aligned}$$

[1]Note that as sequences as discrete, it doesn't make sense to consider limits as n approaches any value other than infinity. Consequently, whenever we mention a limit of a sequence, it is implicitly assumed that we mean the limit as n goes to infinity.

which concludes the proof.

Note that

- Arithmetic sequences do not converge unless the increment is zero.
- Geometric sequences converge if the ratio r satisfies either of the following conditions.

$$r = 1$$
$$-1 < r < 1$$

  When $r = 1$, the sequence is a constant sequence, so it converges to that constant term. When $-1 < r < 1$, the sequence converges to zero.

- The convergence of a recursive sequence depends on several factors, including the properties of the function used to define the recurrence relation.

1.4 Asymptotic behaviour of sequences

Like with functions, we can compare the behaviour of sequences near infinity using little o and big O notation. For example, we have:

$$\begin{aligned} 2n + 5 &= o(n^2) \\ 2n + 5 &= O(n) \\ \frac{1}{n} &= o(1). \end{aligned}$$

1.5 Important thereoms about sequences and their limits

Here we introduce a couple of theorems regarding the limit of sequences.

- **Theorem of monotone convergence.** A monotone sequence converges if and only if it is bounded.

  For example, consider $u_n = 1/n!$. Since

$$u_{n+1} = \frac{1}{(n+1)!} = \frac{1}{n+1} \cdot u_n \leq u_n$$

  the sequence is decreasing. Furthermore, we have

$$\left| \frac{1}{n!} \right| < 1$$

  so the sequence is bounded. Therefore, (u_n) must converge.

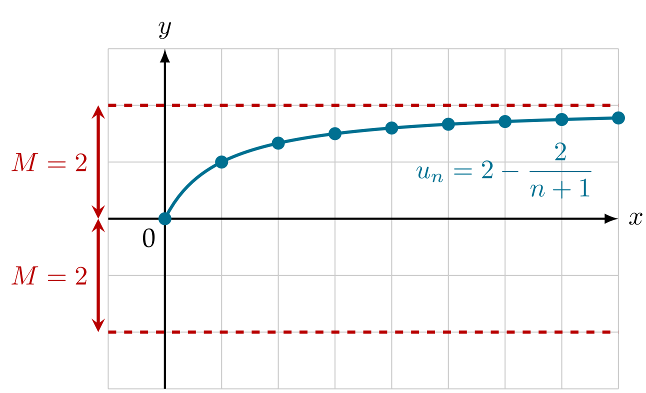

Figure 4: The graph of a increasing and bounded sequence that converges to 2.

- **Squeeze theorem.** Let (u_n), (v_n), and (w_n) be sequences such that for all $n \in \mathbb{N}$, we have

  $$u_n \leq v_n \leq w_n.$$

  If both (u_n) and (w_n) converge to the same limit L, then (v_n) also converges to L.

  For example, suppose we want to compute the limit of the following sequence. (Assume $n > 0$.)

  $$v_n = \frac{\sin n}{n}$$

  Doing this using the classic definition of a limit can be quite difficult. Now consider the following sequences.

  $$u_n = -\frac{1}{n}$$
  $$w_n = \frac{1}{n}$$

  Notice that that both of these sequences converge to zero. Also, for all $n \in \mathbb{N}$, we have

  $$-1 \leq \sin n \leq 1$$
  $$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$
  $$u_n \leq v_n \leq w_n$$

  By the squeeze theorem, the sequence (v_n) also converges to zero. See figure 5.

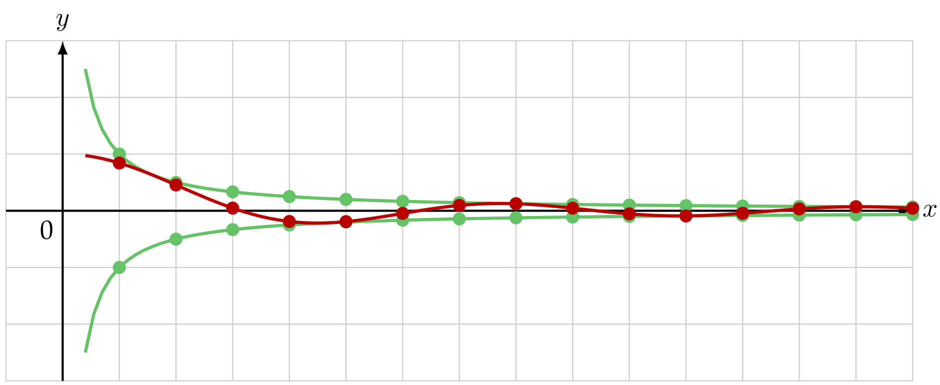

Figure 5: A demonstration of the squeeze theorem. The red curve represents the sequence $v_n = (\sin x)/x$. The green curves represent the sequences $u_n = -1/n$ and $w_n = 1/n$.

1.6 What is a series?

Consider the following sequence.

$$1, 2, 3, 4, 5, 6, \cdots$$

What happens when we add up the first k terms of this sequence?

$$k = 1 \qquad \underbrace{1}_{1} + 2 + 3 + 4 + 5 + 6 + \cdots$$

$$k = 2 \qquad \underbrace{1 + 2}_{3} + 3 + 4 + 5 + 6 + \cdots$$

$$k = 3 \qquad \underbrace{1 + 2 + 3}_{6} + 4 + 5 + 6 + \cdots$$

$$k = 4 \qquad \underbrace{1 + 2 + 3 + 4}_{10} + 5 + 6 + \cdots$$

$$k = 5 \qquad \underbrace{1 + 2 + 3 + 4 + 5}_{15} + 6 + \cdots$$

Each of these sums is called a *partial sum*, and together they form a new sequence, which in this case is the triangular numbers.

$$1, 3, 6, 10, 15, 21, \cdots$$

Whenever we take an existing sequence (u_n) and compute its partial sums $\left(\sum_{n=0}^{k=n} u_k\right)$ to form a new sequence, this new sequence is called a *series*, denoted as $\sum u_n$. The series of triangular numbers in particular can be described by the general term $S_n = n(n+1)/2$. (This is a special case where we opt for one-based indexing!)[2]

Now consider an arithmetic sequence

$$u_n = a + nd$$

where $a = u_0$ is the first term. The series of this sequence is given by

$$\begin{aligned} u_0 + u_1 + u_2 + \cdots + u_n &= a + (a+d) + (a+2d) + \cdots + (a+nd) \\ &= (n+1)a + (1+2+\cdots+n)d \\ &= (n+1)a + \frac{n(n+1)}{2}d \qquad \text{(using the formula for triangular numbers)} \end{aligned}$$

What about the partial sums of a geometric sequence instead? Consider the sequence

$$u_n = ar^n$$

where again $a = u_0$ is the first term. Denote the sum $u_0 + u_1 + u_2 + \cdots + u_n$ as S_n. We can write

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^n \\ rS_n &= ar + ar^2 + ar^3 + \cdots + ar^{n+1} \end{aligned}$$

Subtracting the bottom equation from the top one gives

$$(1-r)S_n = a - ar^{n+1}$$

which gives us the following formula for the series of a geometric sequence.

$$S_n = a + ar + ar^2 + \cdots + ar^n = \frac{a}{1-r}(1 - r^{n+1})$$

1.7 Convergence of a series

Some series do not converge.

$$1 + 2 + 3 + 4 + \cdots \to \infty$$

However, some series do converge — for instance, consider the following series. As the number of terms approach infinity, the sum approaches 2.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

To see why, we first note that this is a geometric series with partial sums

$$\begin{aligned} S_n &= \frac{1}{1 - \frac{1}{2}}\left(1 - \left(\frac{1}{2}\right)^{n+1}\right) \\ &= 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right) \end{aligned}$$

As n approaches infinity, the term in red approaches zero, so the sum converges to 2. We can generalise this result as follows.

[2] The formula can be proved as follows. Suppose we want to find $S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$. This can be rewritten as $S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$. We can add these two equations together to get $2S_n = (n+1) + (n+1) + \cdots + (n+1) = n(n+1)$. Dividing both sides by 2 yields $S_n = n(n+1)/2$.

Convergence of a geometric series.

A geometric series $\sum ar^n$ converges to $a/(1-r)$ if $-1 < r < 1$.

In other words, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

when $-1 < r < 1$.

This can be further generalised into what is called the *ratio test* (also known as d'Alembert's criterion). But first, let us define the term *absolute convergence*. A series $\sum u_n$ is said to be *absolutely convergent* if the series $\sum |u_n|$ converges.

And now, the ratio test.

Ratio test for convergence of a series.

To determine whether a series $\sum a_n$ converges (assuming $a_n \neq 0$ for all n), we consider the sequence of ratios $\dfrac{|a_{n+1}|}{|a_n|}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 &\implies \text{the series diverges} \\
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 &\implies \text{the series converges absolutely} \\
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1 &\implies \text{no conclusion reached}
\end{aligned}$$

For example, consider the series $\sum(n+1)/2^n$. Applying the ratio test, we consider the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+2)/2^{n+1}}{(n+1)/2^n} &= \lim_{n\to\infty} \frac{n+2}{2(n+1)} \\
&= \lim_{n\to\infty} \frac{(n+1)+1}{2(n+1)} \\
&= \lim_{n\to\infty} \frac{(n+1)+1}{2(n+1)} \\
&= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2(n+1)}\right) \\
&= \frac{1}{2} \\
&< 1
\end{aligned}$$

which means the series is absolutely convergent. Since all terms are positive, the series is also convergent.

We also have a comparison test for series convergence. Consider two series $\sum a_n$ and $\sum b_n$, where $a_n, b_n \geq 0$. Then,

- If we have $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent (smaller than a convergent series).

- If we have $a_n \geq b_n$ and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent (larger than a divergent series).

As an example, consider the series $\sum 1/(2^n+1)$. We can easily show that $1/(2^n+1) \leq \frac{1}{2^n}$. Since the series $\sum 1/2^n$ is convergent, the series $\sum 1/(2^n+1)$ is also convergent.

Similarly, consider the series $\sum_{n\geq 3}(\ln n)/n$. When $n \geq 3$, we have $\ln n \geq 1$, so $(\ln n)/n \geq 1/n$. Since the series $\sum 1/n$ is divergent, the series $\sum(\ln n)/n$ is also divergent.

2 Differential calculus

2.1 What is a derivative?

Differential calculus is the study of infinitesimal variations. Consider for instance a function $y = f(x)$. Given some interval from a and $a + h$, we can study the variation of f in this interval by considering the ratio

$$\frac{f(a+h) - f(a)}{h}.$$

This is visualised in figure 6.

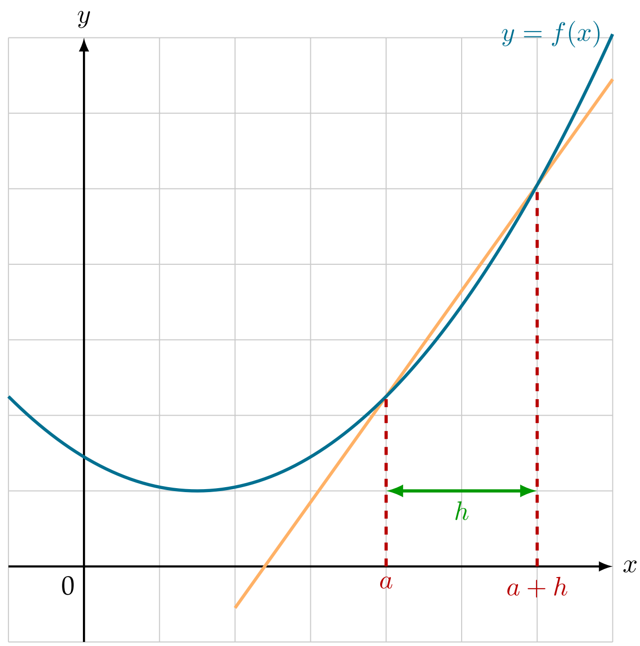

Figure 6: Studying the variation of a function f between $x = a$ and $x = a + h$. The ratio $\frac{f(a+h)-f(a)}{h}$ can be visualised as the slope of the orange line which passes through the points $(a, f(a))$ and $(a + h, f(a + h))$.

To study the variations close to a, we consider the limit of this ratio as h tends to zero. See figure 7.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit is referred to as the *derivative* of f in a. If this limit exists, then the function f is said to be *differentiable* in a.

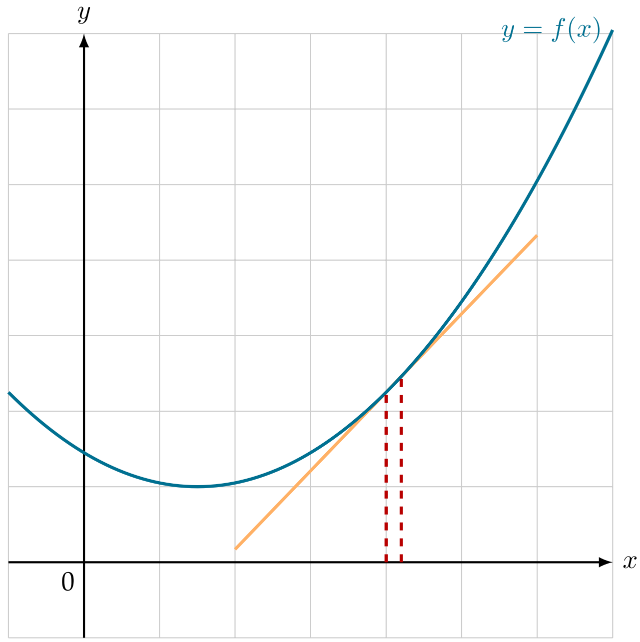

Figure 7: To find the derivative of a function f in a, we consider the slope of the orange line as the length of the interval h tends to zero.

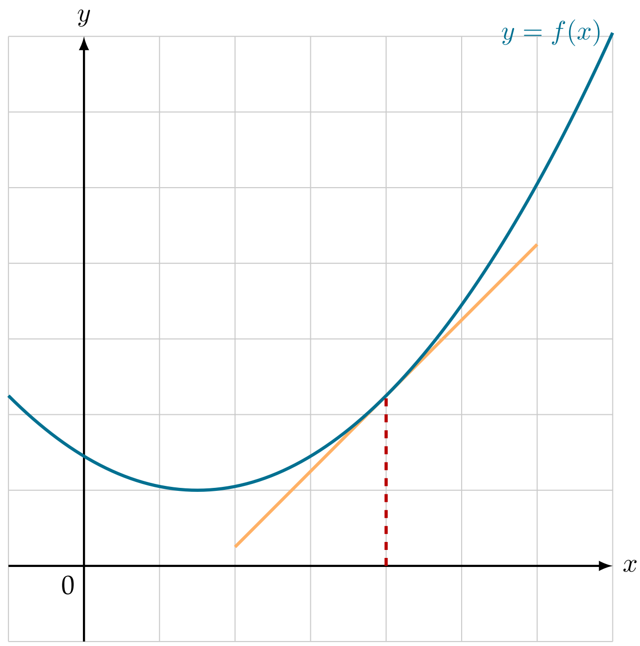

Figure 8: The derivative of a function f in a point a can be thought of as the slope of the tangent to the graph of f at the point where $x = a$.

The derivative of a function f in some value a is denoted as $f'(a)$.

Note that the derivative of f is itself a function, which is denoted here as f'. We can also write this as

$$\frac{df}{dx} \quad \text{or} \quad f^{(1)}.$$

Note that $\frac{df}{dx}$ is not a quotient or fraction. It can be read as applying a differentiation operator $\frac{d}{dx}$ to a function f. The differentiation operator specifies the variable of differentiation, which in this case is x.

Note that

- If the derivative of a function f is g, i.e.
$$f' = g$$
then f is called the *primitive* or *antiderivative* of g.

- Applying the differentiation operator to a function f twice, i.e.
$$\frac{d}{dx}\left(\frac{df}{dx}\right)$$
can be written more compactly as
$$\frac{d^2 f}{dx^2}.$$
This produces a *second-order derivative*, which is denoted as $f''(x)$ or $f^{(2)}(x)$.

2.2 Finding the derivative of a function

To find the derivative of a function, we can use the definition of the derivative as a limit. Consider for instance the function $f(x) = x^2$. We can find the derivative of f by computing the limit

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= \lim_{h\to 0} 2x \\
&= 2x
\end{aligned}$$

which means that the derivative of $f(x) = x^2$ at any point a is given by $f'(a) = 2a$. In other words, the tangent to the graph of $f(x)$ when $x = a$ must have a slope of $2a$.

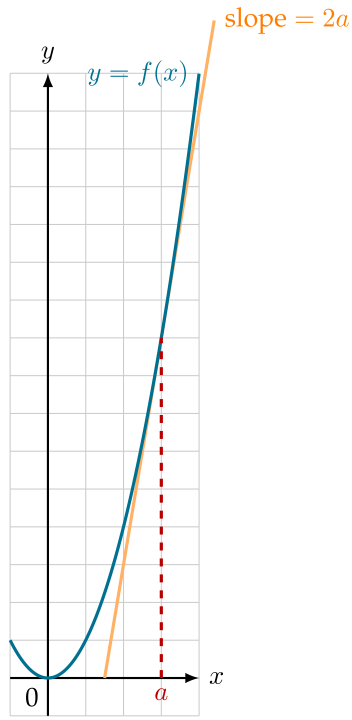

Figure 9: The derivative of x^2 is $2x$. This means the derivative of x^2 (or the slope of the tangent to the graph of $y = x^2$) at any point a is $2a$.

Finding derivatives as a limit can be cumbersome and tedious. To speed this up, it might be helpful to memorise a few differentiation rules, as listed below. Here, f and g are differentiable functions, c is a constant, and n is any real number.

$$c \xrightarrow{\frac{d}{dx}} 0 \tag{1}$$

$$x \xrightarrow{\frac{d}{dx}} 1 \tag{2}$$

$$\sin x \xrightarrow{\frac{d}{dx}} \cos x \tag{3}$$

$$\cos x \xrightarrow{\frac{d}{dx}} -\sin x \tag{4}$$

$$e^x \xrightarrow{\frac{d}{dx}} e^x \tag{5}$$

$$\ln x \xrightarrow{\frac{d}{dx}} \frac{1}{x} \tag{6}$$

$$f(x) \pm g(x) \xrightarrow{\frac{d}{dx}} f'(x) \pm g'(x) \tag{7}$$

$$c \cdot f(x) \xrightarrow{\frac{d}{dx}} c \cdot f'(x) \tag{8}$$

$$x^n \xrightarrow{\frac{d}{dx}} nx^{n-1} \tag{9}$$

$$f(x) \cdot g(x) \xrightarrow{\frac{d}{dx}} f(x) \cdot g'(x) + g(x) \cdot f'(x) \tag{10}$$

$$\frac{f(x)}{g(x)} \xrightarrow{\frac{d}{dx}} \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2} \tag{11}$$

$$f(g(x)) \xrightarrow{\frac{d}{dx}} f'(g(x)) \cdot g'(x) \tag{12}$$

Several notes:

- From equation (7) we have
$$f(x) + g(x) \xrightarrow{\frac{d}{dx}} f'(x) + g'(x).$$
Combining this with equation (8) we see that differentiation is a linear operation.

- The power rule (equation (9)) can be applied with any real number n. A few examples are shown below.
$$\frac{1}{x} = x^{-1} \xrightarrow{\frac{d}{dx}} -x^{-2} = -\frac{1}{x^2}$$
$$\sqrt{x} = x^{\frac{1}{2}} \xrightarrow{\frac{d}{dx}} \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

- The quotient rule (equation (11)) can be derived from the power rule (equation (9)), the product rule (equation (10)), and the chain rule (equation (12)).

An example problem is shown below.

Problem. Calculate $\frac{d}{dx}\, 3e^{x^2+2}$.

Solution 1. Let $f(x) = 3e^x$ and $g(x) = x^2 + 2$. We want to find $\frac{d}{dx} f(g(x))$.

Note that $f'(x) = 3e^x$ and $g'(x) = 2x$. Hence, we have
$$\begin{aligned} \frac{d}{dx} f(g(x)) &= f'(g(x)) \cdot g'(x) && \text{(chain rule)} \\ &= f'(x^2+2) \cdot 2x \\ &= 3e^{x^2+2} \cdot 2x \\ &= 6xe^{x^2+2}. \end{aligned}$$

Solution 2. Those who are more fluent with the rules of differentiation can simply apply the rules to find:
$$\begin{aligned} \frac{d}{dx} 3e^{x^2+2} &= 3 \cdot \left(e^{x^2+2}\right) \cdot (2x) \\ &= 6xe^{x^2+2}. \end{aligned}$$

Generalising from this example problem, we can show that a function of the form
$$f(x) = c \cdot e^{g(x)} \qquad \text{(where } c \text{ is a constant)}$$
must have the derivative
$$\begin{aligned} f'(x) &= c \cdot e^{g(x)} \cdot g'(x) \\ &= f(x) \cdot g'(x). \end{aligned}$$

This will be useful for solving differential equations later.

2.3 More on differentiability

Recall that a function $f(x)$ is said to be differentiable if the derivative of $f'(x)$ exists when $x = a$.

We note the following theorem.

Differentiability is stronger than continuity.

If a function f is differentiable in a, then f is continuous in a.

We can prove this as follows.

Proof that differentiability implies continuity.

We assume differentiability, meaning that the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. We want to prove that

$$\lim_{x \to a} f(x) = f(a).$$

Consider the following limit.

$$\begin{aligned}
\lim_{x \to a} (f(x) - f(a)) &= \lim_{x \to a} \left((f(x) - f(a)) \cdot \frac{x-a}{x-a} \right) && (\because x - a \neq 0) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x-a} \cdot (x-a) \right) \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x-a} \right) \cdot \left(\lim_{x \to a} (x-a) \right) \\
&= f'(a) \cdot 0 \\
&= 0
\end{aligned}$$

Rearranging, we get

$$\lim_{x \to a} f(x) = f(a)$$

which concludes the proof.

Intuitively, we can think of differentiability as whether the tangent to the graph of a function f at a point a exists. For instance, the absolute value function $|x|$ is not differentiable at $x = 0$ because at $x = 0$, the graph of $y = |x|$ consists of a sharp angular point where no tangent can be drawn. See figure 10.

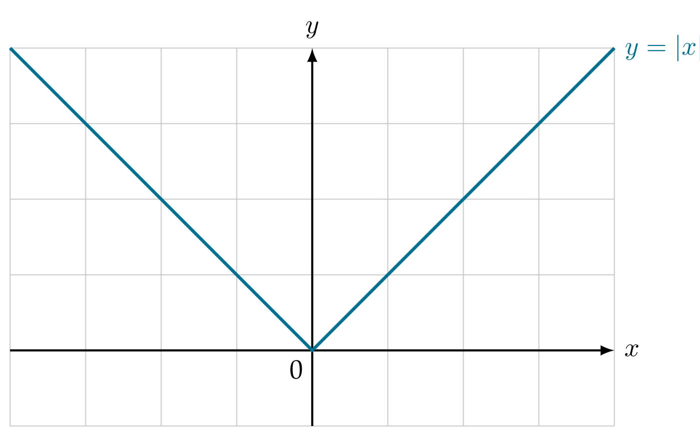

Figure 10: The absolute value function is non-differentiable at $x = 0$.

2.4 Stationary points

If we have $\frac{df}{dx} > 0$ at some point a, we say that f is *increasing* at that point. Conversely, if we have $\frac{df}{dx} < 0$ at some point a, we say that f is *decreasing* at that point. But what if $\frac{df}{dx} = 0$?

When this happens, we say that f has a *stationary point* at a. This could mean a number of things:

- A local minimum;
- A local maximum; or
- An inflection point[3].

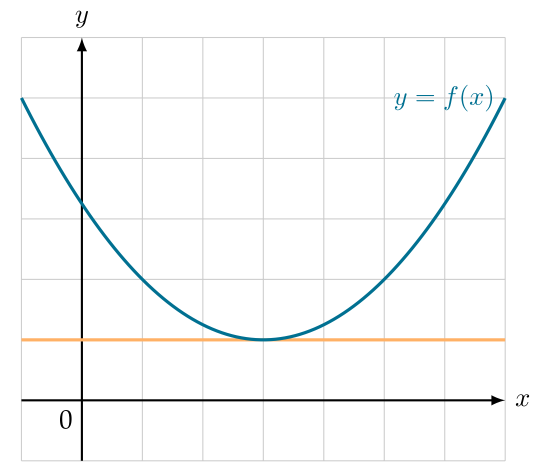

Figure 11: At a local minimum, the derivative of a function is zero.

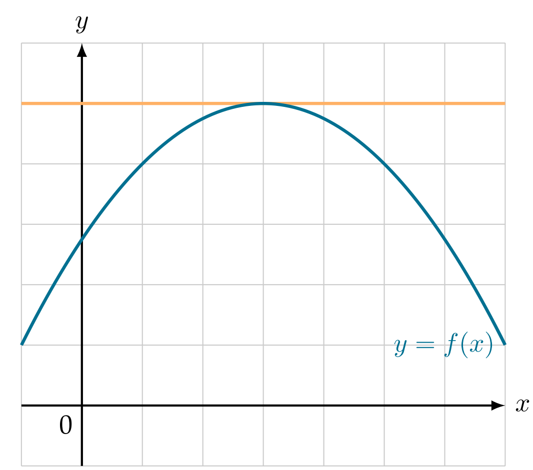

Figure 12: At a local maximum, the derivative of a function is zero.

[3] Also called a "point of inflection".

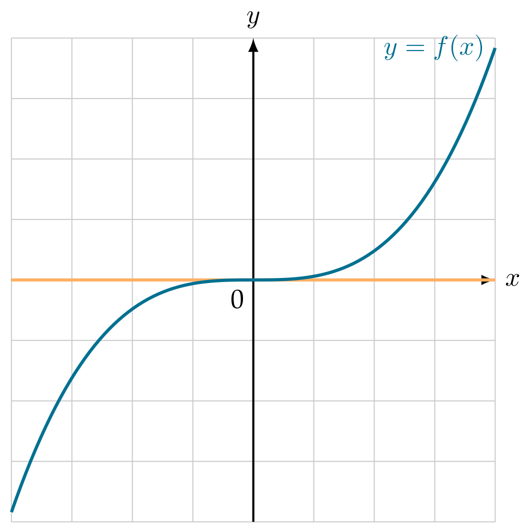

Figure 13: At an inflection point, the derivative of a function is zero.

There are several ways to distinguishing between these cases, as will be demonstrated using the following function.

$$f(x) = 2x^3 - 9x^2 + 12x - 4$$

To find the stationary points of f, we first compute its derivative and set it equal to zero.

$$\begin{aligned} f'(x) = 6x^2 - 18x + 12 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x &= 1 \text{ or } 2 \end{aligned}$$

This means that this function has two stationary points: one at $x = 1$ and one at $x = 2$. But are these local maxima, local minima, or inflection points?

One straightforward way for working this out is the *first derivative test*, where we consider the sign of the derivative around the stationary point.

x	0	1	1.5	2	3
$f'(x)$	$+$	0	$-$	0	$+$

We can infer from the table above that f has a local maximum at $x = 1$ and a local minimum at $x = 2$, which can be verified using a graph.

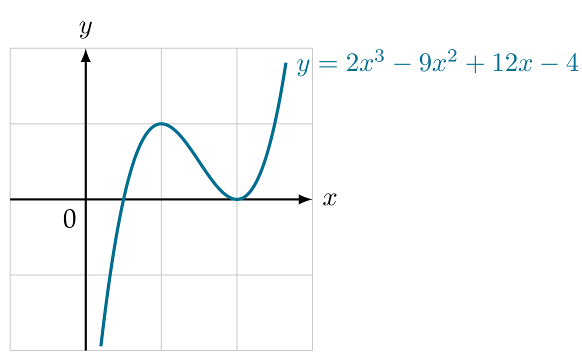

Figure 14: The function $f(x) = 2x^3 - 9x^2 + 12x - 4$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

Another way to identify the nature of a stationary point is the *second derivative test*. This involves computing the second derivative of the function at each stationary point.

- If the second derivative is positive, then the function has a local minimum at that point.
- If the second derivative is negative, then the function has a local maximum at that point.
- If the second derivative is zero, then the test is inconclusive. The point can be a local maximum, a local minimum, or an inflection point.

Applying this test to the function above, we have

$$f''(x) = 12x - 18$$

which gives us

$$\begin{aligned} f''(1) &= -6 < 0 \\ f''(2) &= 6 > 0 \end{aligned}$$

This leads to the same result as before: f has a local maximum at $x = 1$ and a local minimum at $x = 2$.

2.5 Differential equations

Differential equations are ones that involve both functions and their derivatives. Recall from earlier that a function of the form

$$f(x) = c \cdot e^{g(x)}$$

must have the derivative

$$f'(x) = c \cdot e^{g(x)} \cdot g'(x) = f(x) \cdot g'(x)$$

where $c \in \mathbb{R}$. We will make use of this relationship a lot when solving differential equations, as we will see in the following example problems.

Problem. Solve $f(x) = f'(x)$.

Solution. $f(x) = C \cdot e^x$ where $C \in \mathbb{R}$.

Problem. Solve $f'(x) = -3f(x)$.

Solution. $f(x) = C \cdot e^{-3x}$ where $C \in \mathbb{R}$.

Now consider the slightly more complicated case of having to solve

$$f'(x) = a(x) \cdot f(x).$$

To do this, we find the primitive (antiderivative) of $a(x)$, which we will denote $A(x)$. In other words, we have $A'(x) = a(x)$. The solutions to this differential equation are then given by $f(x) = C \cdot e^{A(x)}$, where $C \in \mathbb{R}$. An example is given below.

Problem. Solve $f'(x) = (2x + 1) \cdot f(x)$.

Solution. The antiderivative of $2x + 1$ is $x^2 + x$, so the solution to the equation is given by $f(x) = C \cdot e^{x^2+x}$ where $C \in \mathbb{R}$.

These solutions, which contain unspecified multiplicative constants, are called *general solutions* of the differential equation. Sometimes we might be asked to find specific values for these constants by using initial conditions.

Problem. What is the solution of $f'(x) = (2x + 1) \cdot f(x)$ that verifies $f(0) = 3$?

Solution. Continuing from the previous example problem, we know that the solution to the differential equation is given by $f(x) = C \cdot e^{x^2+x}$, where $C \in \mathbb{R}$. We can find the value of C by using the initial condition $f(0) = 3$.

$$\begin{aligned} C \cdot e^{0^2+0} &= 3 \\ C &= 3 \end{aligned}$$

This gives us the specific solution $f(x) = 3 \cdot e^{x^2+x}$.