Convex Optimisation
Homework 1

2.12. Let $H_n(a,b) = \{x \in \mathbb{R}^n \mid a^T x \le b\}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.
($H_n(a,b)$ define a halfspace and is convex)

a) $A(\alpha,\beta) = \{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\}$ convex ?

As $A = H(a,\beta) \cap H(-a,-\alpha)$, A is convex. (intersection of convex sets)

b) $B(\alpha,\beta) = \{x \in \mathbb{R}^n \mid \forall i \in [\![1,n]\!],\ \alpha_i \le x_i \le \beta_i\}$ convex ?

with $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ($e_{ii} = 1$, $e_{ij} = 0$ pour $i \ne j$), then B can be expressed as

$$B = \bigcap_{i=1}^{n} H(-e_i, -\alpha_i) \cap H(e_i, \beta_i)$$

B is therefore convex.

c) $C(a_1,b_1,a_2,b_2) = \{x \in \mathbb{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$ is clearly convex as

$$C = H(a_1,b_1) \cap H(a_2,b_2)$$

d) $D(x_0,S) = \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \le \|x-y\|_2\ \forall y \in S\}$ convex ?

~~Let $x_1, x_2 \in D(x_0,S)$, let's prove that $\forall t \in [0,1]$, $tx_1+(1-t)x_2$ is in $D(x_0,S)$~~

$\cancel{\|tx_1+(1-t)x_2-x_0\|_2 = \|t(x_1-x_0)+(1-t)(x_2-x_0)\|_2}$

$\cancel{\le t\|x_1-x_0\|_2 + (1-t)\|x_2-x_0\|_2}$ ~~(Triangle inequality)~~

$$D(x_0,S) = \bigcap_{y \in S} \{x \in \mathbb{R}^n \mid \|x-x_0\|_2^2 \le \|x-y\|_2^2\}$$

$$= \bigcap_{y \in S} \{x \in \mathbb{R}^n \mid (x-x_0)^T(x-x_0) \le (x-y)^T(x-y)\} \quad \text{by def of } \|\cdot\|_2^2$$

$$= \bigcap_{y \in S} \{x \in \mathbb{R}^n \mid (-2x_0 + 2y)^T x \le y^T y - x_0^T x_0\}$$

$$= \bigcap_{y \in S} H(2(y-x_0),\ \|y\|_2^2 - \|x_0\|_2^2)$$

D is therefore convex.

e) $E(S,T) = \{x \in \mathbb{R}^n \mid dist(x,S) \leq dist(x,T)\}$ convex?

E is not convex: let's see a conter exemple in $\mathbb{R}$:

$S = \mathbb{R} \setminus [0;1]$

$T = [0;1]$

Then we have clearly $E(S,T) = S$ which is not convex.

f) $F(S_1, S_2) = \{x \in \mathbb{R}^n \mid x + S_2 \subset S_1\}$ with S_1 convex.

Let $x, y \in F(S_1, S_2)$, $t \in [0;1]$

Let $s \in tx + (1-t)y + S_2$: $s = tx + (1-t)y + s'$, $s' \in S_2$

then $s = t(x+s') + (1-t)(y+s')$.

as $x \in F(S_1, S_2)$, $x + s' \in S_1$ (because $x + S_2 \subset S_1$)

and $y + s' \in S_1$ (same reasom)

Thanks to S_1 convexity, $s \in S_1$

Therefore $tx + (1-t)y + S_2 \subset S_1$

$tx + (1-t)y \in F(S_1, S_2)$

$\Rightarrow F(S_1, S_2)$ is convex.

g) $G(\theta, a, b) = \{x \in \mathbb{R}^n \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$, $a \neq b$, $0 \leq \theta \leq 1$. ?

$G(\theta, a, b) = \{x \in \mathbb{R}^n \mid \underbrace{(1-\theta^2)x^Tx + 2(\theta - a)^T x + \|a\|^2 - \theta\|b\|^2}_{f(x),\ f \text{ convex.}} \leq 0\}$ (same as d))

Let $x, y \in G$, then $f(tx + (1-t)y) \leq t\underbrace{f(x)}_{\leq 0} + (1-t)\underbrace{f(y)}_{\leq 0}$

≤ 0

therefore $tx + (1-t)y \in G \Rightarrow$ G is convex.

3.21.

a) Let $g_i : \mathbb{R}^n \to \mathbb{R}$
$x \to \| A^i x - b^i \|$

g_i is convex ~~composition~~ (directly demonstrate with triangle inequality)

Then $f(x) = \max(g_i(x))$ is convex because pointwise maximum preserve convexity.

b) $f(x) = \sum_{i=1}^{r} |x|_{[i]}$ (sum of the r largest components)

$$= \max_{1 \le i_1 < \dots < i_r \le n} \sum_{k=1}^{r} |x|_{i_k}$$

$$= \max_{i_1 < \dots < i_r} g(x)$$

where $g(x) = \sum_{k=1}^{r} |x_{i_k}|$ which is also clearly convex thanks to the triangle inequality.

Therefore f is convex.

3.32

a) Let f, g convex functions, non decreasing and positives.
show that $f \cdot g$ is convex: $(f \cdot g(x) = f(x) g(x))$

Let $x, y \in \mathbb{R}^n$, $t \in [0,1]$

fg convex $\iff f \cdot g(tx + (1-t)y) \le t f \cdot g(x) + (1-t) f \cdot g(y)$

$$\iff \underbrace{-f \cdot g(tx + (1-t)y) + t f \cdot g(x) + (1-t) f \cdot g(y)}_{D_t(x,y)} \ge 0.$$

as f and g convex and positives,

$$f \cdot g(tx + (1-t)y) = \underbrace{f(tx + (1-t)y)}_{\ge 0} \underbrace{g(tx + (1-t)y)}_{\ge 0}$$
$$\le (t f(x) + (1-t) f(y))(t g(x) + (1-t) g(y))$$

$$D_t(x,y) \ge t fg(x) + (1-t) f \cdot g(y) - (t f(x) + (1-t) f(y))(t g(x) + (1-t) g(y))$$

$$D_t(x,y) \geq t\, fg(x) + (1-t) fg(y) - t^2 fg(x) - (1-t)^2 fg(y) - t[1-t]\big(f(x)g(y) + f(y)g(x)\big)$$

$$\geq t(1-t)\big[f.g(x) + f.g(y) - f(x)g(y) + f(y)g(x)\big]$$

$$\geq t(1-t)\big(g(y) - g(x)\big)\big(f(y) - f(x)\big)$$

As f, g are both non increasing $g(y) - g(x)$ as the same sign as $f(y) - f(x)$
therefore $D_t(x,y) \geq 0$

And $\underline{f.g}$ is convex.

b) Show that fg is concave when f, g are concave, positive functions with $f \nearrow$ and $g \searrow$.

$f.g$ concave $\Leftrightarrow$ $D_t(x,y) \leq 0$

or as f, g concave and positives,

$$f(tx + (1-t)y)\, g(tx + (1-t)y) \geq \big(t f(x) + (1-t) f(y)\big)\big(t g(x) + (1-t) g(y)\big)$$

$$\Rightarrow D_t(x,y) \leq t\, f.g(x) + (1-t) fg(y) - \underline{\qquad\qquad}$$

$\vdots$ same calcul as in a)

$$\leq \underbrace{t(1-t)\big(g(y) - g(x)\big)\big(f(y) - f(x)\big)}_{\leq 0 \text{ as } f \nearrow \text{ and } g \searrow}$$

≤ 0

Therefore $\underline{f.g}$ is concave.

c) Let f convex, non decreasing and positive function
g concave, non increasing and positive function.

Then $\frac{1}{g}$ ~~...~~ non decreasing and positive function
~~...~~
~~...~~

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And moreover $\frac{1}{g}$ is convex !

$$\frac{1}{g(tx+(1-t)y)} \leq \frac{1}{tg(x)+(1-t)g(y)}$$ (because g concave and $x \mapsto \frac{1}{x}$ decreasing)

$$\leq t\frac{1}{g(x)} + (1-t)\frac{1}{g(y)}$$ (because $x \mapsto \frac{1}{x}$ is convex)

Therefore $\frac{f}{g} = f \cdot \frac{1}{g}$ is convex (from a))

3.36

a) ~~…~~ is equivalent to b) with $r=1$.

b) $f^*(y) = \sup_{x \in \mathbb{R}^n} x^T y - f(x)$

$$= \sup_{x\in\mathbb{R}^n} \underbrace{\sum_{i=1}^n x_i y_i - \sum_{i=1}^r x_{[i]}}_{g_y(x)} \qquad , r \in [\![1,n]\!].$$

* if $\exists y_i < 0$, then with $x_t = -te_i$, $g_y(x) = -ty_i$ $\boxed{\text{if } r<n}$ $t \geq 0$

$g_y(x) \underset{t\to+\infty}{\longrightarrow} +\infty$ so $f^*(y) = +\infty$.

* if $\exists y_i > 1$ then $x_t = te_i \Rightarrow g_y(x_t) = (y_i - 1)x_t \underset{t\to+\infty}{\longrightarrow} +\infty$ $(t \geq 0)$

Let assume that $y \in [0;1]$ and with $x_t = t\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

$$g_y(x_t) = t\left(\sum_1^n y_i - r\right)$$

if $\sum_1^n y_i > r$ $\quad g_y(x_t) \underset{t\to+\infty}{\longrightarrow} +\infty$

$\sum_1^n y_i < r$ $\quad g_y(x_t) \underset{t\to-\infty}{\longrightarrow} +\infty$

So $f^*(y) = +\infty$ with $\sum y_i \neq r$. and with $y \notin [0,1]$.

Now assume that $\sum y_i = r$. and $y \in [0,1]$.

Let $x \in \mathbb{R}_n$ and $i_1 \dots i_n$ a permutation of $[\![1;n]\!]$

such that $x_{i_1} \geq x_{i_2} \dots \geq x_{i_n}$

$$g_y(x) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]}$$

$$= \sum_{i=1}^{n} x_i y_i - \sum_{k=1}^{r} x_{i_k}$$

$$= \sum_{k=1}^{n} x_{i_k} y_{i_k} - \sum_{k=1}^{r} x_{i_k}$$

$$= \sum_{k=1}^{r} \underbrace{(y_{i_k} - 1)}_{\leq 0} x_{i_k} + \sum_{k=r+1}^{n} x_{i_k} y_{i_k}$$

We have $\forall k \leq r \ x_{i_k} \geq x_{i_r}$ and $\forall k \geq r+1 \ x_{i_k} \leq x_{i_r}$

So $g_y(x) \leq - \sum_{k=1}^{r} (1-y_{i_k}) x_{i_r} + \sum_{k=r+1}^{n} x_{i_r} y_{i_k}$

$$\leq x_{i_r} \left(\sum_{k=1}^{n} y_{i_k} - \sum_{k=1}^{r} 1 \right)$$

$$\leq x_{i_r} \left(\underbrace{\sum_{i=1}^{n} y_i - r}_{=0} \right)$$

$$\leq 0$$

And $g_y(0) = 0$

therefore $f^*(y) = \begin{cases} 0 & \text{if } y \in [0,1]^n \text{ and } \sum y_i = r \\ +\infty & \text{otherwise.} \end{cases}$

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case $r = 1$ (answer to a))

$$f^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \text{ and } \sum y_i = 1 \\ +\infty & \text{otherwise.} \end{cases}$$ ($\sum y_i = 1$ and $y \geq 0 \Rightarrow y \leq 1$)

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case $r = n$

$$g_y(x) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i$$

$$= \sum (y_i - 1) x_i$$

if $\exists y_i \neq 1$ then with $x_t = te_i$, $g_y(x_t) = (y_i - 1)t \xrightarrow[t \to \pm\infty]{} +\infty$

And with $y = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, $g_y(x) = 0 \ \forall x$.

$$f^*(y) = \begin{cases} 0 & \text{if } y = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \\ +\infty & \text{otherwise.}\end{cases}$$

$$= \begin{cases} 0 & \text{if } y \in [0,1]^n \text{ and } \sum y_i = n \quad (\Leftrightarrow y = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}) \\ +\infty & \text{otherwise.}\end{cases}$$

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c) $f^*(y) = \sup_{x \in \mathbb{R}} \underbrace{(xy - \max_i (a_i x + b_i))}_{g_y(x)}$

we can assume that $\forall i<j$, $a_i < a_j$. indeed if $a_i = a_j$ then b_i has to be different than b_j (otherwise it's redundant) but then ~~if ...~~ only one is relevant (~~...~~ the one with the bigger b_i is always above the other).

It's then easy to see that there are m linear pieces in f:

$[-\infty, x_1]$, $[x_i, x_{i+1}]$ $i \in [1, m-2]$ $[x_{m-1}, +\infty]$

$\hookrightarrow a_1 x + b_1$ $\quad \hookrightarrow a_{i+1} x + b_{i+1}$ $\quad \hookrightarrow a_m x + b_m$

where $(x_i, f(x_i))$ is the intersection point of line defined by $\begin{cases} y = a_i x + b_i \\ y = a_{i+1} x + b_{i+1}\end{cases}$

$$\Rightarrow x_i = \frac{b_i - b_{i+1}}{a_{i+1} - a_i}$$

* Now if $y < a_1$ with ~~...~~
with $x < x_1$ then $g_y(x) = xy - a_1 x - b_1$

$$= \underbrace{(y - a_1)}_{<0} x$$

$$g_y(x) \underset{x \to -\infty}{\longrightarrow} +\infty$$

Done $f^*(y) = +\infty$

* if $y > a_m$, with $x > x_{m-1}$ $g_y(x) = x \underbrace{(y - a_m)}_{>0} - b_m \underset{x \to +\infty}{\longrightarrow} +\infty$

$$\Rightarrow f^*(y) = +\infty$$

* $a_1 \leq y \leq a_m$

~~...~~ g_y is increasing until x reaches the linear piece where f grows too much.

Let assume $a_i \leq y \leq a_{i+1}$

for $x \leq x_i$ $\quad g_y$ is non decreasing (as the slope of f is smaller than y)

$x \geq x_i$ $\quad g_y$ is non increasing

Therefore $f^*(y) = g_y(x_i)$

$$= y\frac{b_i - b_{i+1}}{a_{i+1}-a_i} - a_i\frac{b_i - b_{i+1}}{a_{i+1}-a_i} - b_i$$

$$= (y - a_i)\, x_i - b_i$$

f^* is also a linear piece wise function

$$f^*(y) = \begin{cases} +\infty & y < a_1 \text{ or } y > a_m \\ (y-a_i)x_i - b_i & \text{if } a_i \leq y \leq a_{i+1} \end{cases}$$

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d) $f^*(y) = \sup\limits_{x \in \mathbb{R}_{++}} \underbrace{(yx - x^p)}_{g_y(x)}$

* $p > 1$

if $y \leq 0$, $\quad g_y$ is non increasing on $\mathbb{R}_{++}$

therefore $f^*(y) = g_y(0) = 0$.

if $y > 0$ $\quad$ let's find g_y maximum:

$g_y'(x) = y - p x^{p-1}$, $\quad g_y'(x) = 0 \Leftrightarrow x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

	0		$\left(\frac{y}{p}\right)^{\frac{1}{p-1}}$		$+\infty$
↳ g_y'	y	$+$	0	$-$	$-\infty$
g_y			$g(x)$		

The maximum is reached at $\left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

$$f^*(y) = g_y\left(\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\right) = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

$$= y^{\frac{p}{p-1}}\left[\left(\frac{1}{p}\right)^{\frac{1}{p-1}} - \left(\frac{1}{p}\right)^{\frac{p}{p-1}}\right]$$

$$= y^{\frac{p}{p-1}}\left(p\left(\frac{1}{p}\right)^{\frac{p}{p-1}} - \left(\frac{1}{p}\right)^{\frac{p}{p-1}}\right)$$

$$= (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

$$f^*(y) = \begin{cases} (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

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* $p<0$

if $y > 0$, $g_y(x) = yx - x^p \xrightarrow[x\to+\infty]{} +\infty$ $\quad$ ($x^p \xrightarrow[x\to+\infty]{} 0$: $p<0$)

Done $f^*(y) = +\infty$

if $y \leq 0$ let's compute the maximum of g_y.

$$g_y'(x) = 0 \iff x = \left(\frac{y}{\underset{\geq 0}{p}}\right)^{\frac{1}{p-1}}$$

	0^+	$\left(\frac{y}{p}\right)^{\frac{1}{p-1}}$	$+\infty$
g_y'	$+\infty$ $+$	0	$-$ y
g_y	$\nearrow$		$\searrow$

$$f^*(y) = g_y\left(\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\right) = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

$$\text{So } f^*(y) = \begin{cases} (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

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e) $$f^*(y) = \sup_{x\in\mathbb{R}^n_{++}} \Big(\underbrace{y^T x + \left(\prod x_i\right)^{\frac{1}{n}}}_{g_y(x)} \Big)$$

* if $\exists\, y_i \geq 0$ with $x_t = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} + t e_i$

$$g_y(x_t) = \sum y_i + t y_i + (1+t)^{\frac{1}{n}} \xrightarrow[t\to+\infty]{} +\infty$$

$\Rightarrow f^*(y) = +\infty$

$\ast$ $y < 0$ Consider $x_t = \left(-\frac{t}{y_i}\right)_i$, $t > 0$

$$g_y(x_t) = -nt + \left(\prod -\frac{t}{y_i}\right)^{\frac{1}{n}}$$

$$= t\left(\left(\prod -\frac{1}{y_i}\right)^{\frac{1}{n}} - n\right)$$

$\ast$ if $\left(\prod -\frac{1}{y_i}\right)^{\frac{1}{n}} > n$

$$g_y(x_t) \xrightarrow[t \to +\infty]{} +\infty$$

$$\Rightarrow f^*(y) = +\infty$$

$\ast$ $\left(\prod -\frac{1}{y_i}\right)^{\frac{1}{n}} \leq n$

Then $g_y(x) = \sum x_i y_i + \left(\prod x_i\right)^{\frac{1}{n}}$

but the geometric mean inequality states that

$$\frac{1}{n}\sum_{i=1}^{n} x_i(-y_i) \geq \left(\prod_{i=1}^{n} x_i(-y_i)\right)^{\frac{1}{n}}$$

$$\geq \left(\prod -y_i\right)^{\frac{1}{n}} \left(\prod x_i\right)^{\frac{1}{n}}$$

$$\geq \frac{1}{n}\left(\prod x_i\right)^{\frac{1}{n}} \qquad \left(\prod -y_i \geq \frac{1}{n} \Leftrightarrow \prod -\frac{1}{y_i} \leq n\right)$$

$\Leftrightarrow \quad 0 \geq g_y(x)$

and $g_y(0^+) = 0$

Therefore $f^*(y) = 0$

$$f^*(y) = \begin{cases} 0 & \text{if } y < 0 \ \left(\prod -y_i\right)^{\frac{1}{n}} \geq \frac{1}{n} \\ +\infty & \text{otherwise} \end{cases}$$

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f) $f^*(y,u) = \sup\limits_{\substack{x \in \mathbb{R}^n, t \in \mathbb{R} \\ t > \|x\|_2}} \underbrace{y^t x + \log(t^2 - x^t x) + ut}_{g_{y,u}(x,t)}$

$\ast$ if $u \geq 0$, then with $x = 0$, $g_{y,u}(0,t) = ut + \log(t^2) \xrightarrow[t \to +\infty]{} +\infty$

so $f^*(y,u) = +\infty$

$(\alpha > 0)$

Now if $u \geq -\|y\|_2$ then with $x_\alpha = \alpha y$

$t_\alpha = \|x_\alpha\|_2 + 1 = \alpha \|y\|_2 + 1$

$$g_{y,u}(x_\alpha, t_\alpha) = \alpha \sum y_i^2 + \alpha u \|y\|_2 + u + \log\left((\alpha\|y\|_2+1)^2 - \alpha^2 y^t y\right)$$

$$= \alpha\left(\|y\|_2^2 + u\|y\|_2\right) + u + \log\left(1 + 2\alpha\|y\|_2\right)$$

$$= \alpha\|y\|_2 \left(\underbrace{\|y\|_2 + u}_{\geq 0}\right) + u + \log\left(1 + 2\alpha\|y\|_2\right)$$

so $g_{y,u}(x_\alpha, t_\alpha) \xrightarrow[\alpha \to \infty]{} +\infty$

$$f^*(y, u) = +\infty$$

Therefore $u < -\|y\|_2 \leq 0$.

As $-g_{y,u}$ is coercive* and continue, $g_{y,u}$ has a unique global maximizer.

Let's compute the derivatives of $g_{y,u}$:

$$\nabla_x g_{y,u}(x,t) = y - \frac{2x}{t^2 - x^t x} \quad , \quad \nabla_t g_{y,u}(x,t) = u + \frac{2t}{t^2 - x^t x}$$

Let x^*, t^* an extremum of $g_{y,u}$. then:

$$\begin{cases} \nabla_x g_{y,u}(x^*, t^*) = 0 \\ \nabla_t g_{y,u}(x^*, t^*) = 0 \end{cases} \iff \begin{cases} y = \dfrac{2x^*}{t^2 - \|x^*\|^2} \\ u = \dfrac{-2t^*}{t^{*2} - \|x^*\|^2} \end{cases}$$

We can notice than $u^2 - y^t y = \dfrac{4}{t^{*2} - \|x^*\|^2}$

$$\iff t^{*2} - \|x^*\|^2 = \frac{4}{u^2 - \|y\|^2} \quad (1)$$

$$(1) \overset{(y \neq 0)}{\iff} (t^{*2} - \|x^*\|^2)\, y = \frac{4y}{u^2 - \|y\|^2}$$

$$\iff x^* = \frac{2y}{u^2 - \|y\|^2}$$

$$(1) \overset{(u \neq 0)}{\iff} (t^* - \|x^*\|)\, u = \frac{4u}{u^2 - \|y\|^2}$$

$$\iff t^* = \frac{-2u}{u^2 - \|y\|^2}$$

$g_{y,u}$ has only one extremum (maximum) at x^*, t^*

with $x^* = \dfrac{2y}{u^2 - y^t y}$, $t^* = \dfrac{-2u}{u^2 - y^t y}$

$$g_{y,u}(x^*, t^*) = \frac{2y^t y}{u^2 - y^t y} - \frac{2u^2}{u^2 - y^t y} + \log\left(\frac{4u^2 - 4y^t y}{(u^2 - y^t y)^2}\right)$$

$$= -2 + \log(4) - \log(\underbrace{u^2 - y^t y}_{>0})$$

>0 because $u < -\|y\|_2 \leq 0$
$\Rightarrow u^2 > \|y\|_2^2$.

~~or as $g_{y,u}(0,t) = ut + \log(t^2)$ $\underset{t\to+\infty}{\longrightarrow} -\infty$ $(u<0)$~~

~~$\Rightarrow \exists x, t$ ($x=0$, t big enough) $g_{y,u}(x,t) < g_{y,u}(x^*, t^*)$~~

~~Therefore x^*, t^* is the maximum of $g_{y,u}$~~

$$\Rightarrow f^*(y,u) = 2(\log(2) - 2) - \log(u^2 - y^t y)$$

$$f^*(y,u) = \begin{cases} 2(\log(2) - 2) - \log(u^2 - y^t y) & \text{if } u < -\|y\|_2 \\ +\infty & \text{otherwise.} \end{cases}$$

$*$ $-g_{y,u}$ coercive :

if $y^t x < 0$ then $g_{y,u}(x,t) < 0 + ut + \log(\underbrace{t^2 - x^t x}_{\leq t^2})$

$\leq ut + \log(t^2)$

if $\|(x,t)\|_2 \to +\infty$, as $t \geq \|x\|_2$ then t also grows to infinity.

$$\Rightarrow \lim_{\|x,t\|_2 \to +\infty} (g_{y,u}(x,t)) \leq \lim_{\|x,t\|_2 \to +\infty} (ut + \log(t^2))$$

$$\leq \lim_{t \to +\infty} (ut + \log(t^2))$$

$$\leq -\infty \qquad (u<0)$$

Therefore $-g_{y,u}$ is coercive

if $y^t x \geq 0$, then $y^t x = |y^t x| \leq \|y\|_2 \|x\|_2$ (cauchy schwartz)

$$g_{y,u}(x,t) \leq \|y\|_2 \|x\|_2 + ut + \log(t^2)$$

$$\leq \underbrace{(\|y\|_2 + u)}_{<0} t + \log(t^2) \qquad (\|x\|_2 \leq t)$$

$\Rightarrow \lim\limits_{\|x,t\| \to +\infty} g_{y,u}(x,t) = -\infty$. $\Rightarrow -g_{y,u}$ is coercive .