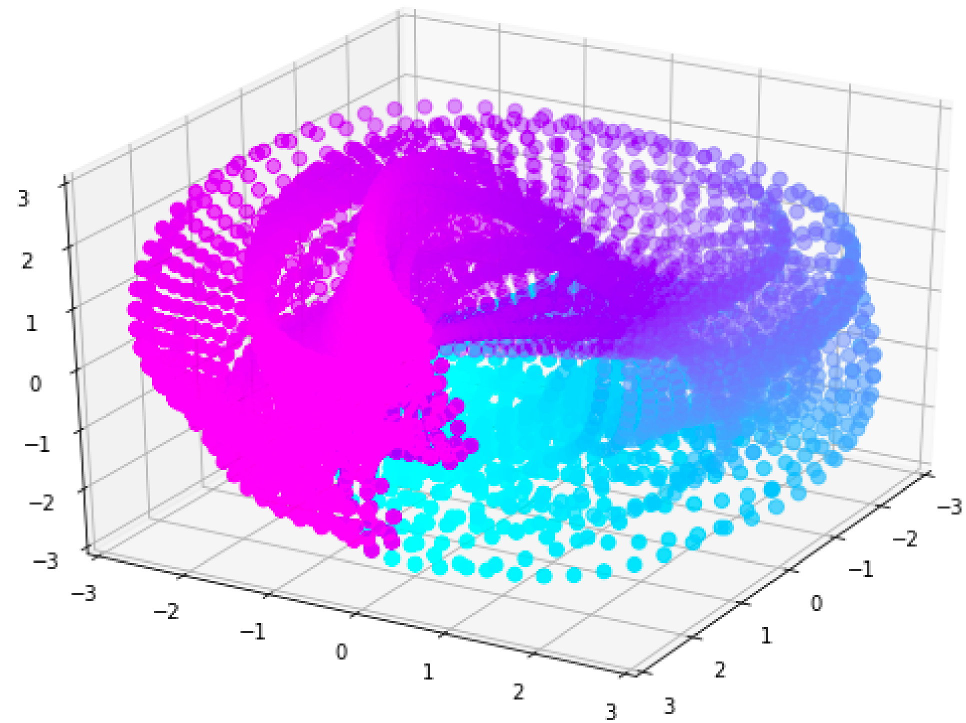
Tangent bundle estimation of immerged manifolds

Raphaël Tinarrage



Introduction (1/3): Immerged manifolds

Abstract manifold $\mathcal{M}_0 \xrightarrow{\quad u \quad} \mathcal{M} \subset \mathbb{R}^n$

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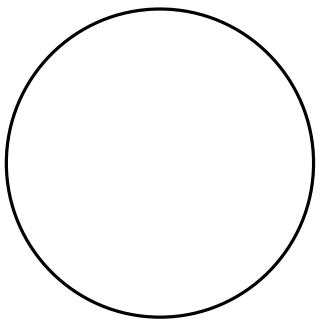
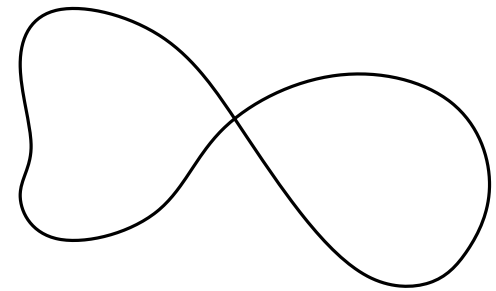
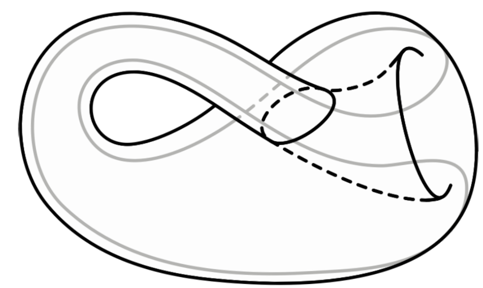
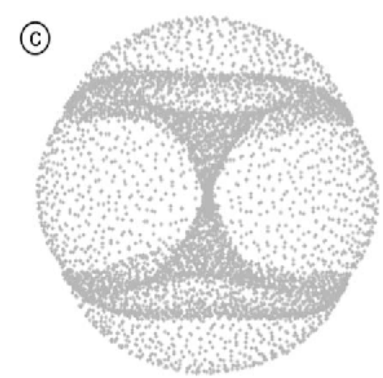
 $\longrightarrow$ 

Klein bottle $\longrightarrow$ 

Klein bottle $\cup\, \mathbb{S}_2$ $\longrightarrow$ 

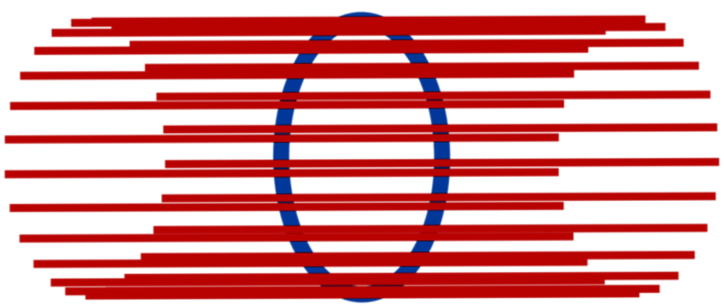
Introduction (2/3): Vector bundles

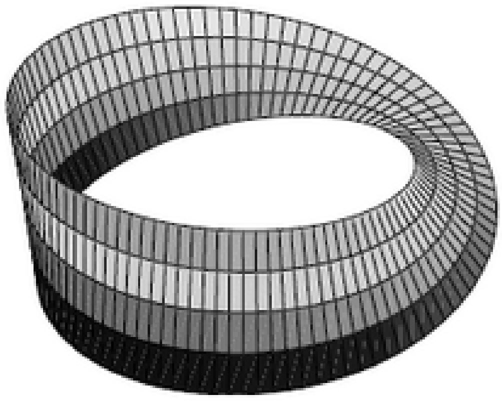
Vector bundle over $\mathcal{M}_0$:
it is a topological space E, with $p : E \to \mathcal{M}_0$ continuous surjective such that

- for all $x_0 \in \mathcal{M}_0$, $p^{-1}(\{x_0\})$ is given a structure of vector space,

and $\forall x_0 \in \mathcal{M}_0, \exists U \subset \mathcal{M}_0$ neighborhood of x_0, $\exists k \geq 0$,
$\exists \phi : U \times \mathbb{R}^k \to p^{-1}(U)$ homeomorphism such that $\forall y_0 \in \mathcal{M}_0$,

- $v \mapsto \phi(y_0, v)$ is an isomorphism (of vector spaces)
- $\forall v \in \mathbb{R}^k, p \circ \phi(y_0, v) = y_0$





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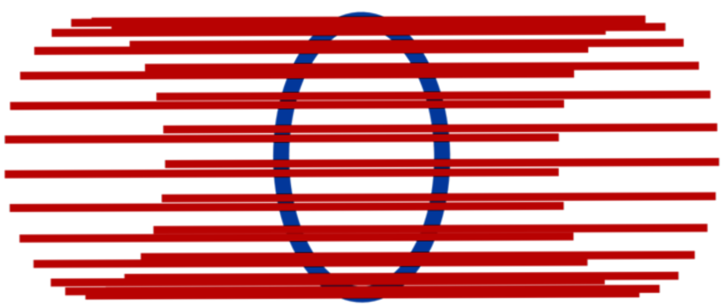
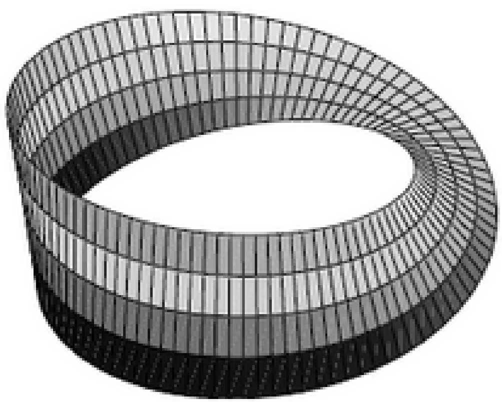




Some vector bundles can be described by a map $\mathcal{M}_0 \to G_{d,n}$

Introduction (3/3): Tangent bundle

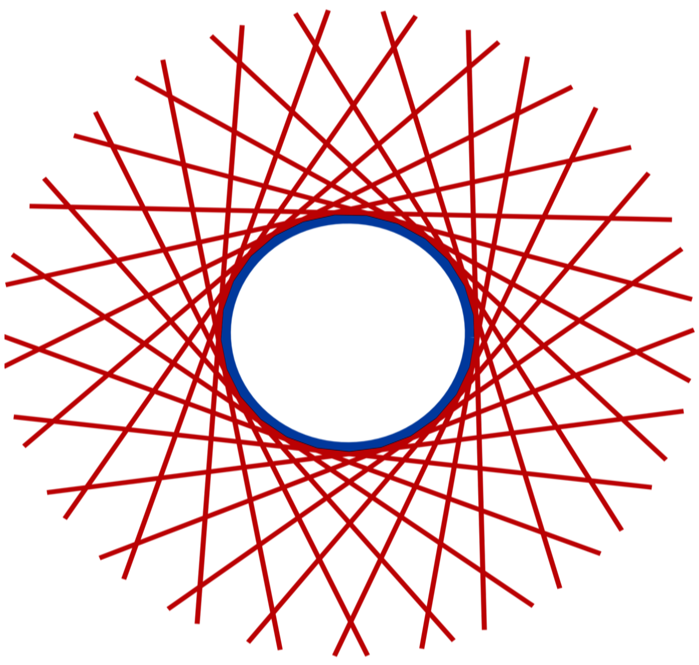
Tangent bundle:

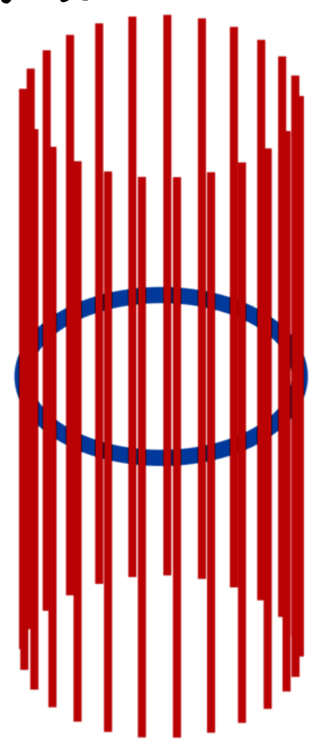
(Definition for an abstract manifold): it is a vector bundle with total space the set $\bigcup_{x_0 \in \mathcal{M}_0} T_{x_0}\mathcal{M}_0$, endowed with a vector bundle structure...

(Definition for embedded manifold $\mathcal{M}_0 \to \mathcal{M}$: the tangent bundle is given by

$$T\mathcal{M} = \{(x,v), x \in \mathcal{M}, v \in T_x\mathcal{M}\}$$

and $p : (x,v) \in T\mathcal{M} \mapsto x \in \mathcal{M}$.





$T\mathcal{M}$ is a submanifold of dimension $2d$ of $\mathbb{R}^n \times \mathbb{R}^n$

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Another point of view:

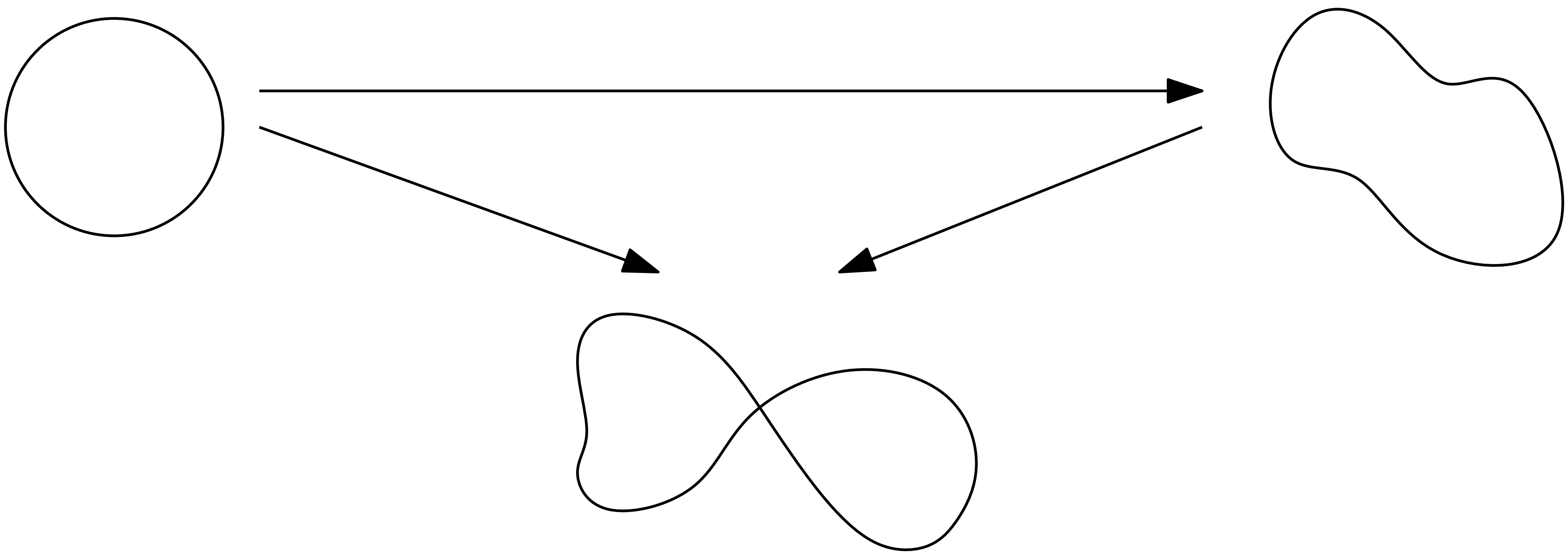
Define $\check{\mathcal{M}} \subset \mathbb{R}^n \times G_{d,n}$ as

$$\check{\mathcal{M}} = \{(x, T_x\mathcal{M}), x_0 \in \mathcal{M}_0\}.$$

It is a submanifold of dimension d of $\mathbb{R}^n \times G_{d,n}$.
$\check{\mathcal{M}}$ can also be seen as a submanifold of $\mathbb{R}^n \times \mathbb{R}^{n^2}$.

Motivations (1/2): recover the diffeomorphism class of $\mathcal{M}_0$

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[rr, "\check{u} : x_0 \mapsto (u(x_0), p_{T_x\mathcal{M}})"] \arrow[dr, "u"'] & & \check{\mathcal{M}} \subset \mathbb{R}^n \times M_n(\mathbb{R}) \arrow[dl, "\pi : (x, T) \mapsto x"] \\
& \mathcal{M} \subset \mathbb{R}^n &
\end{tikzcd}$$



Assumption: $\forall x_0, y_0 \in \mathcal{M}_0$, $T_x\mathcal{M} = T_y\mathcal{M} \implies x_0 = y_0$

$\mathcal{M}_0$ and $\check{\mathcal{M}}$ are diffeomorphic

Motivations (2/2): characteristic classes

The tangent bundle of $\mathcal{M}_0$ contains more precise information about the diffeomorphism class of $\mathcal{M}_0$ than just the cohomology ring $H^*(\mathcal{M}_0)$.

Example:
The Stiefel-Whitney class is a functor

$$\mathrm{VectorBundles}_{\mathcal{M}_0} \longrightarrow H^*(\mathcal{M}_0, \mathbb{Z}_2)$$

If we consider the tangent bundle,

- the first Stiefel-Whitney class $w_1 \in H^1(\mathcal{M}_0, \mathbb{Z}_2)$ is zero iff $\mathcal{M}_0$ is orientable.
- the Stiefel-Whitney numbers are all zero iff $\mathcal{M}_0$ is the boundary of a compact manifold.

Model

Assumptions:

Differential geometric assumptions:

- $\mathcal{M}_0$ is a C^2-manifold of dimension d
- $u : \mathcal{M}_0 \to \mathbb{R}^n$ is an immersion, $\mathcal{M} = u(\mathcal{M}_0)$
- $\forall x_0, y_0 \in \mathcal{M}_0$, $T_x\mathcal{M} = T_y\mathcal{M} \implies x_0 = y_0$

Riemannian geometric assumption:

- $\mathcal{M}_0$ is endowed (by pull-back) with a riemmanian structure, and $\forall x_0 \in \mathcal{M}_0, ||II_{x_0}|| \leq \rho$

Measure assumption:

- μ_0 is a Radon measure on $\mathcal{M}_0$ with density f_0 which is L_0-Lipschitz and bounded by $0 < f_{\min} \leq f_{\max}$

Model

Varifold:

Let $\check{u} : \mathcal{M}_0 \to \mathbb{R}^n \times M_n(\mathbb{R})$ be defined by $\check{u}(x_0) = (u(x_0), \frac{1}{d+2} p_{T_x \mathcal{M}})$.
Consider

$$\check{\mathcal{M}} = \check{u}(\mathcal{M}_0),$$

$$\check{\mu}_0 = \check{u}_* \mu_0.$$

Goal:

We observe a measure ν on $\mathbb{R}^n$ such that $W_1(\mu, \nu) \leq \epsilon$.
Infer from ν the measure $\check{\mu}_0$.

Overview of the method

- For all $x \in \mathrm{supp}(\nu)$, compute $\overline{\Sigma}_\nu(x)$ the normalized local covariance matrix.
- Consider the measure $\check{\nu} = \nu \otimes \delta_{\overline{\Sigma}_\nu(x)}$.
- Show that $W_1(\check{\nu}, \check{\mu}_0)$ small.
- Do persistent homology on $\check{\nu}$.

Reach (1/6)

Reach: Let $A \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n \mapsto d(x, A) = \inf_{a \in A} \|x - a\|$ the distance function to A.
The medial axis of S is defined as

$$\mathrm{med}(A) = \{x \in \mathbb{R}^n, \exists a, b \in A, \|x - a\| = \|x - b\| = d(x, A)\}$$

and the reach of A as

$$\mathrm{reach}(A) = \inf_{a \in A} d(a, \mathrm{med}(A)).$$

It is a scale at which A has good behaviour: with $r < \mathrm{reach}(A)$ and $x \in A$,

- $A \cap \overline{\mathcal{B}}(x, r)$ is contractible
- the projection onto A is well defined on the neighborhood A^r
- tangent space estimation works [Arias-Castro et al, Spectral Clustering based on local PCA]

Reach (2/6)

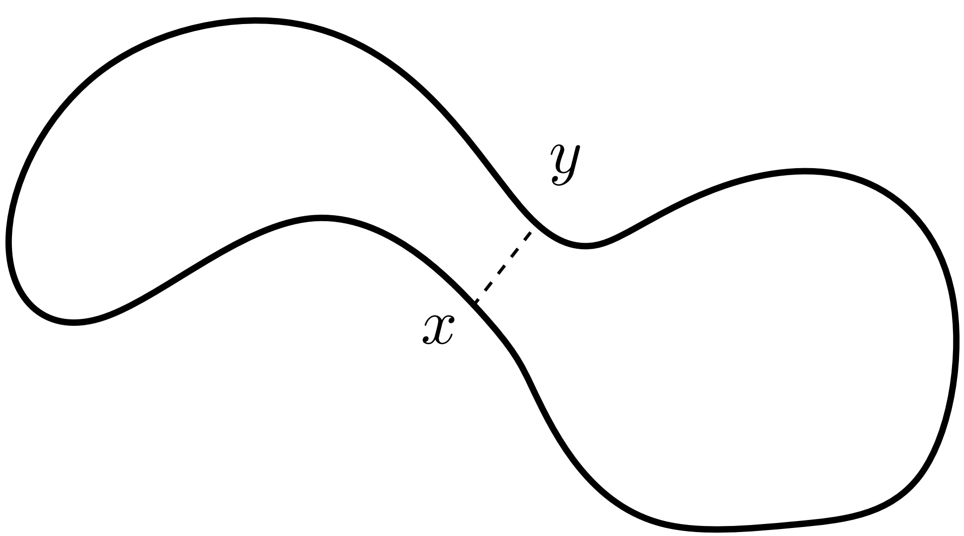
Theorem (Aamari):

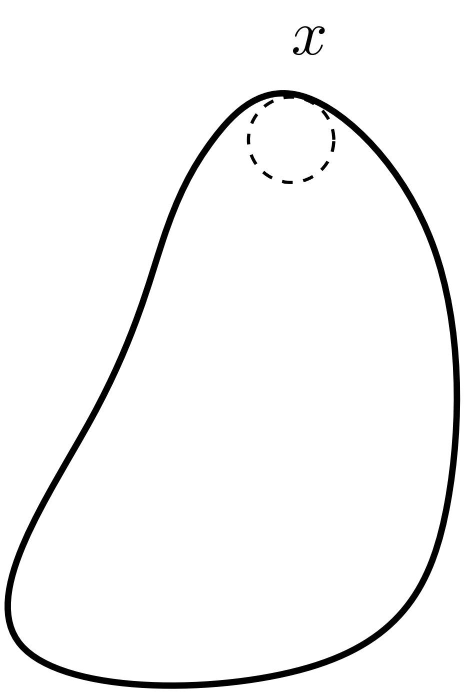
Let M be a compact submanifold. Then

$$\operatorname{reach}(M) = \frac{\lambda}{2} \wedge \frac{1}{\rho},$$

where

- $\lambda = \inf\{\|x - y\|, (x, y)$ bottleneck$\}$ (i.e. $\mathcal{B}(\frac{x+y}{2}, \frac{\|x-y\|}{2}) \cap \mathcal{M} = 0$)
- $\rho = \sup_{x \in \mathcal{M}} ||II_x||$



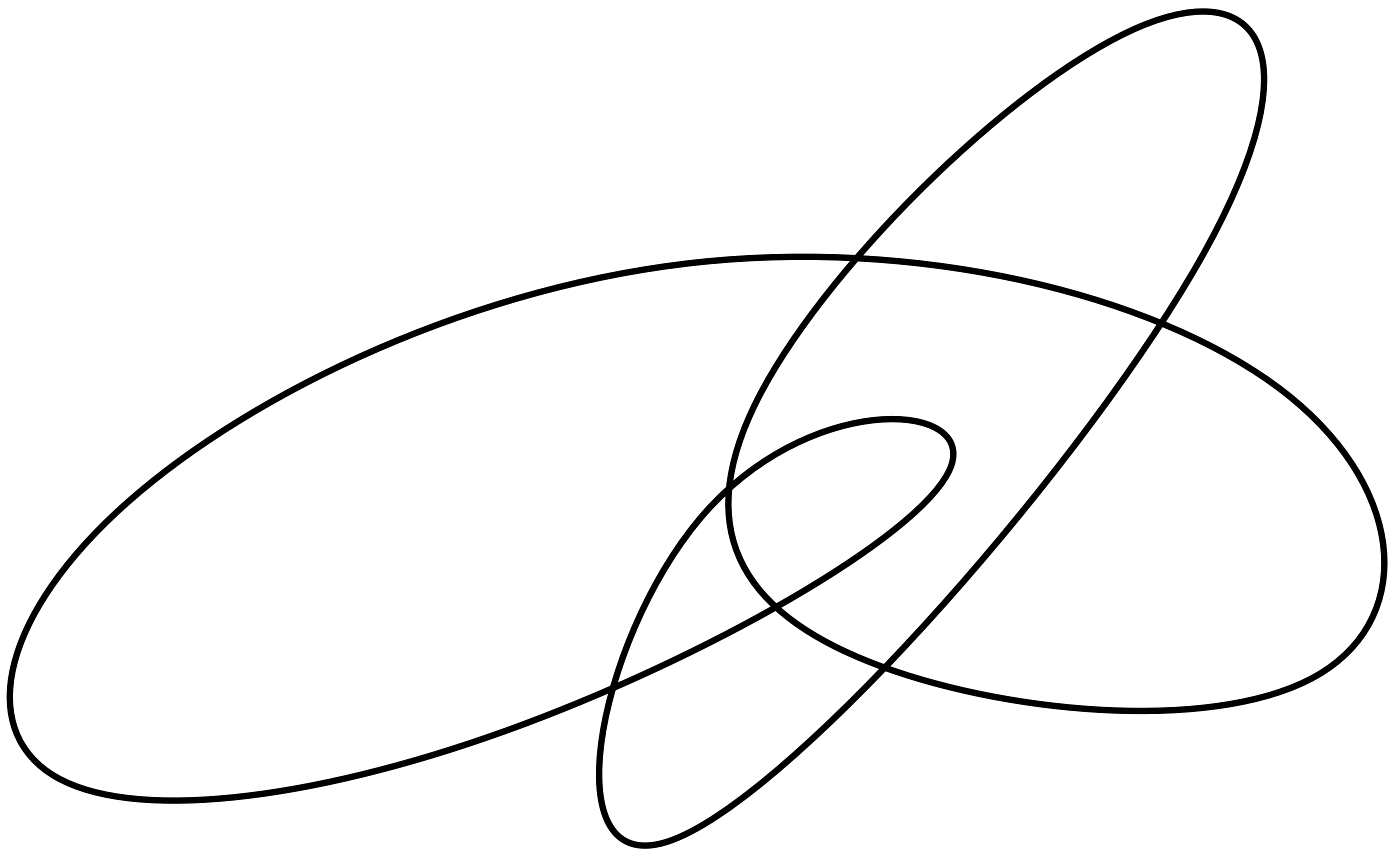


Reach (3/6)

Normal reach:

For every $x_0 \in \mathcal{M}_0$, we set

$$\lambda(x_0) = \inf\{\|x - y\|, y_0 \in \mathcal{M}_0, y_0 \neq x_0, x - y \perp T_y\mathcal{M}\}.$$



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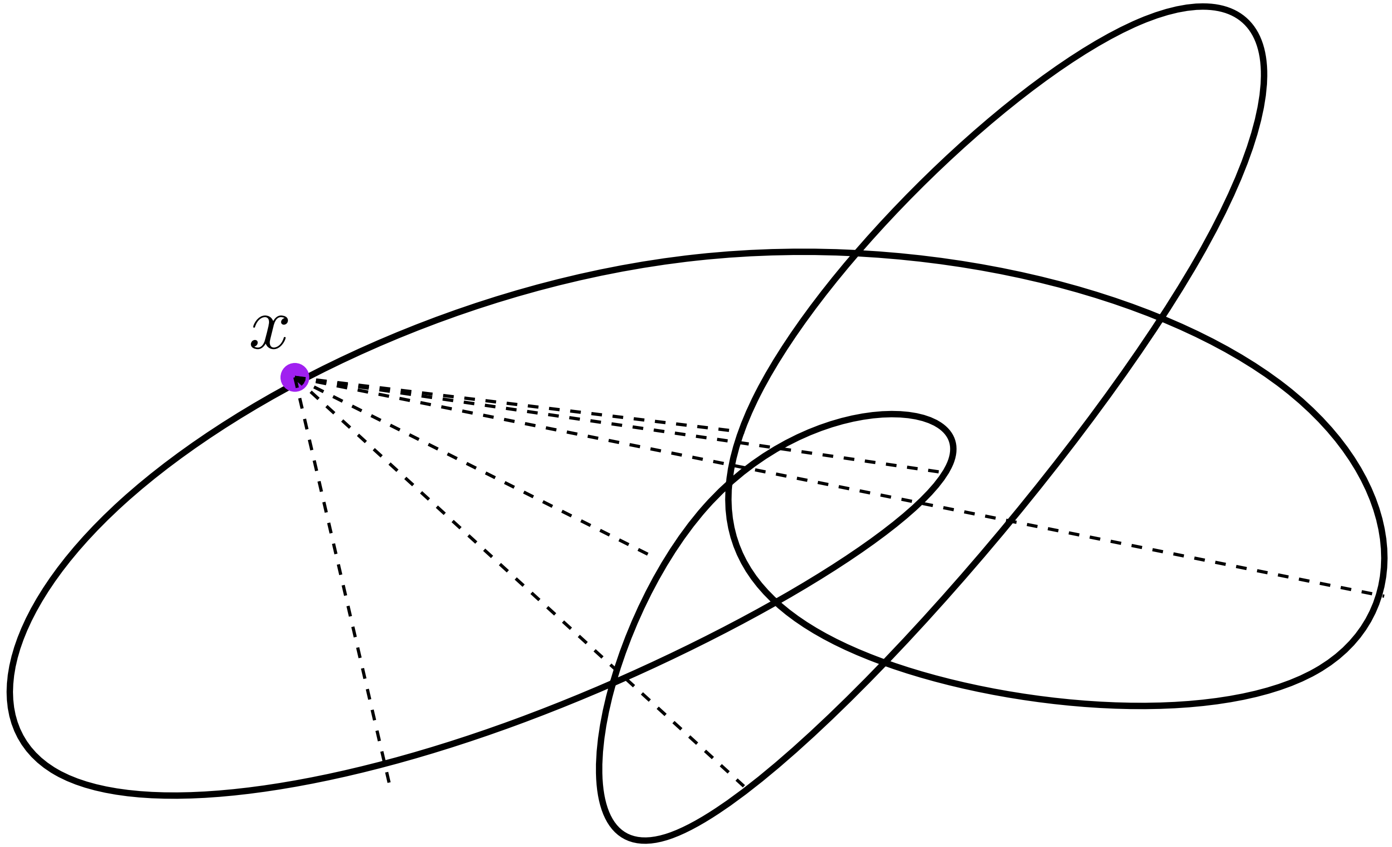

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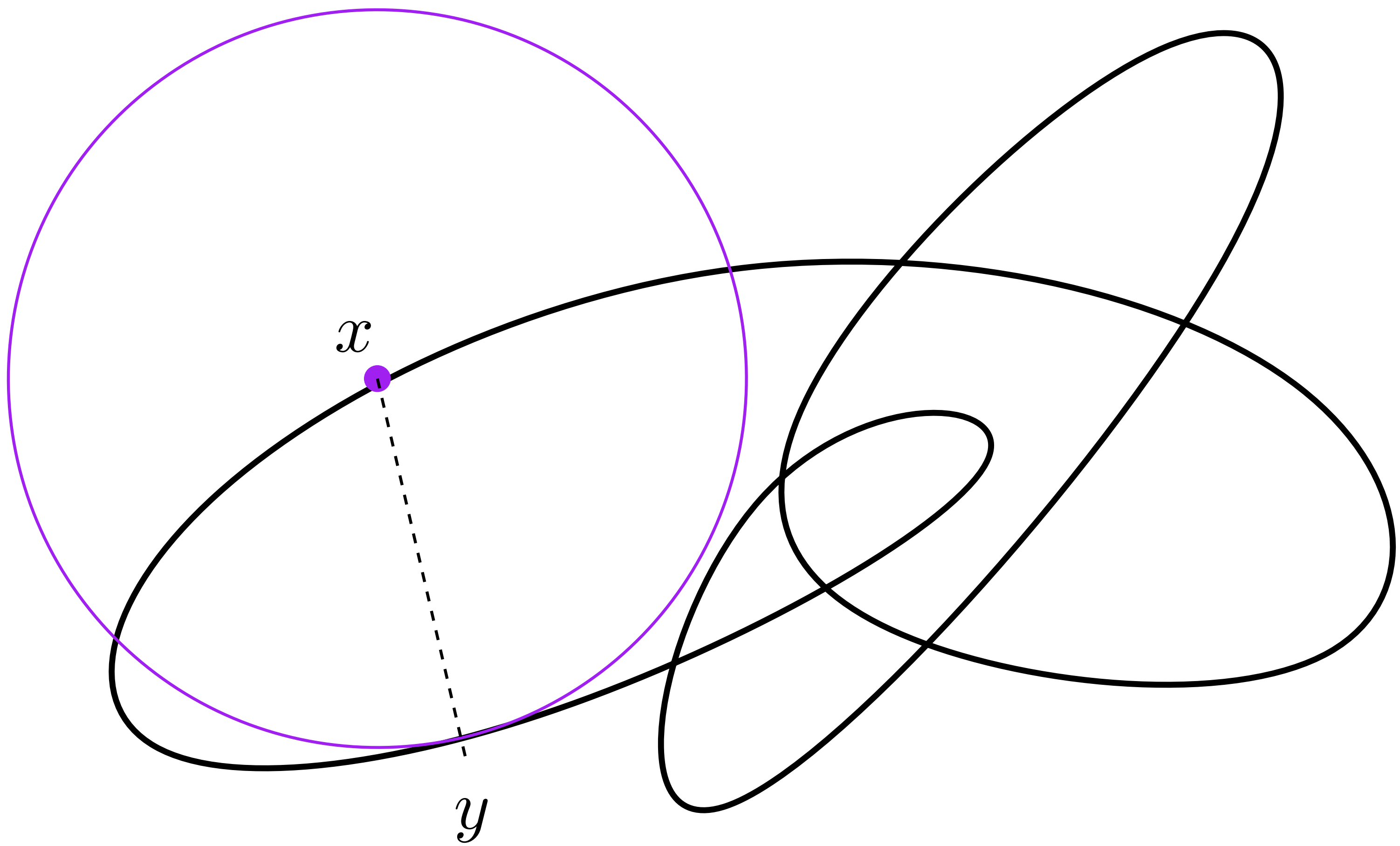

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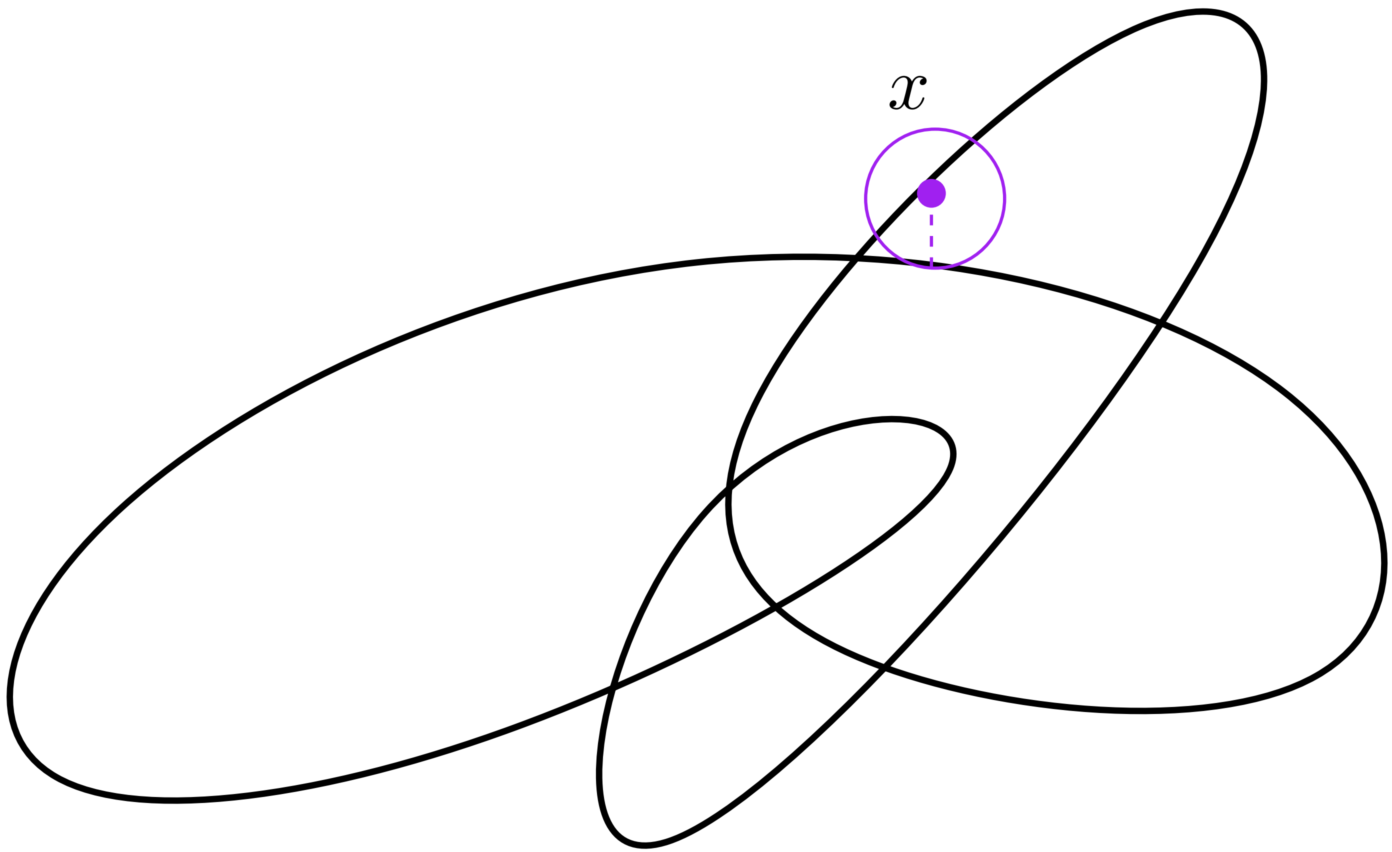
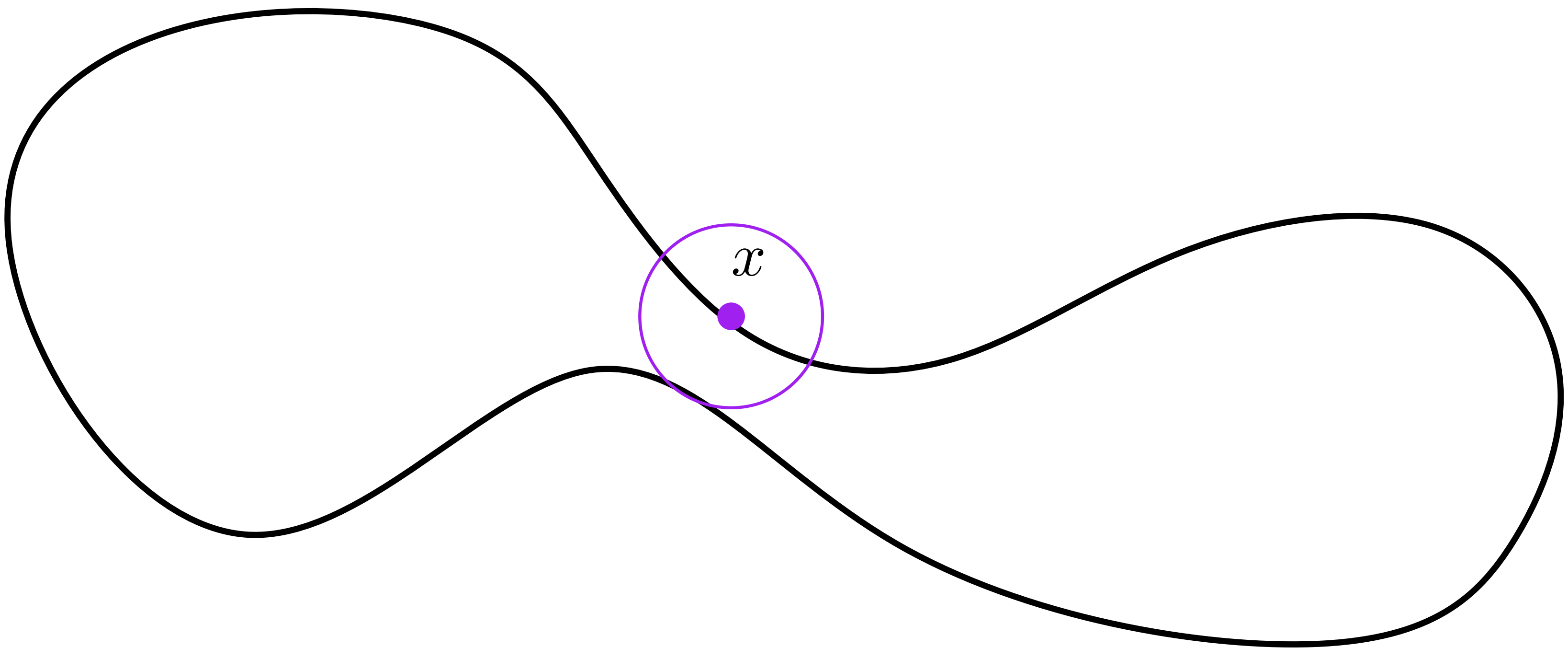


Reach (4/6)

Proposition:

Let $x_0 \in \mathcal{M}_0$ and $r < \frac{1}{4\rho} \wedge \lambda(x_0)$. Then $\overline{\mathcal{B}}(x,r) \cap \mathcal{M}$ is a set of reach at least $\frac{1}{2\rho}$.



Reach (5/6)

Lemma:

Let $x_0, y_0 \in \mathcal{M}_0$. Denote $r = \|x - y\|$ and $\delta = d_{\mathcal{M}_0}(x_0, y_0)$. Suppose that $\|x - y\| < \frac{1}{2\rho} \wedge \lambda(x_0)$.
Then

$$\delta \leq c(\rho r) r \qquad \text{where} \qquad c(t) = \frac{1}{t}(1 - \sqrt{1 - 2t}).$$

Proof:

Show that $u^{-1}(\mathcal{M} \cap \mathcal{B}(x, r))$ is connected.
Show that $u^{-1}(\mathcal{M} \cap \mathcal{B}(x, r)) \cap \partial\mathcal{B}_{\mathcal{M}_0}(x_0, c(\rho r) r + \epsilon) = \emptyset$ for ϵ small enough, following [Niyogi et al., Finding the Homology of Submanifolds with High Confidence from Random Samples].

Reach (6/6)

Proposition:

Let $r \leq \frac{1}{4\rho} \wedge \lambda(x)$ and $x_0 \in \mathcal{M}_0$. We have

$$\mu(\mathcal{B}(x,r)) \geq ar^d$$

$$|\frac{\mu(\mathcal{B}(x,r))}{V_d r^d} - f(x)| \leq cr$$

Moreover, $s \geq 0$ is such that $s \leq r$, we also have

$$\mu(\mathcal{B}(x,r) \setminus \mathcal{B}(x,s)) \leq br^{d-1}(r-s)$$

(with $a = V_d(\frac{47}{48})^d f_{\min}$, $b = \frac{4d}{3}(\frac{17}{16})^d f_{\max}$, and
$c = V_d(L + \frac{d}{4} f_{\max}\rho(\frac{17}{16})^{d-1} + 2d f_{\max}\rho\frac{17}{16}(\frac{3}{2})^{d-1})$.)

Tangent space estimation (1/6)

Definition:

Let $r > 0$ and $x \in \operatorname{supp}(\mu)$. The *local covariance matrix of* μ *around* x *at scale* r is the following matrix:

$$\Sigma_\mu(x) = \int_{\overline{\mathcal{B}}(x,r)} (x-y)^{\otimes 2} \frac{\mathrm{d}\mu(y)}{\mu(\overline{\mathcal{B}}(x,r))}$$

and the *normalized local covariance matrix* is

$$\overline{\Sigma}_\mu(x) = \frac{1}{r^2}\Sigma_\mu(x).$$

Notation: for any $x \in \mathbb{R}^n$, $x^{\otimes 2} = x^t x \in M_n(\mathbb{R})$.

Tangent space estimation (2/6): Consistency

Proposition:
Let $x_0 \in \mathcal{M}_0$ and $r \leq \lambda(x_0) \wedge \frac{1}{4\rho}$. Denote $T = T_x\mathcal{M}$ and by p_T the linear projection on T. Then

$$\|\frac{\Sigma_\mu(x)}{r^2} - \frac{1}{d+2} p_T\|_{\mathrm{F}} \leq cr$$

with $c = (\frac{8}{7})^2\rho + \frac{\mathrm{lip}(f) J_{\max} + \frac{d}{4}\rho f_{\max}}{f_{\min} J_{\min}} + \frac{f_{\max} J_{\max}}{f_{\min} J_{\min}} d + \frac{C}{f_{\min} J_{\min}}$.

$\longrightarrow$ The tangent space at x is well estimated provided that $r \leq \lambda(x_0)$

Tangent space estimation (3/6): Stability

Consider μ, ν close in Wasserstein distance.

The distance $\|\overline{\Sigma}_\mu(x) - \overline{\Sigma}_\nu(x)\|_{\mathrm{F}}$ is defined only when $x \in \mathrm{supp}(\mu) \cap \mathrm{supp}(\nu)$.

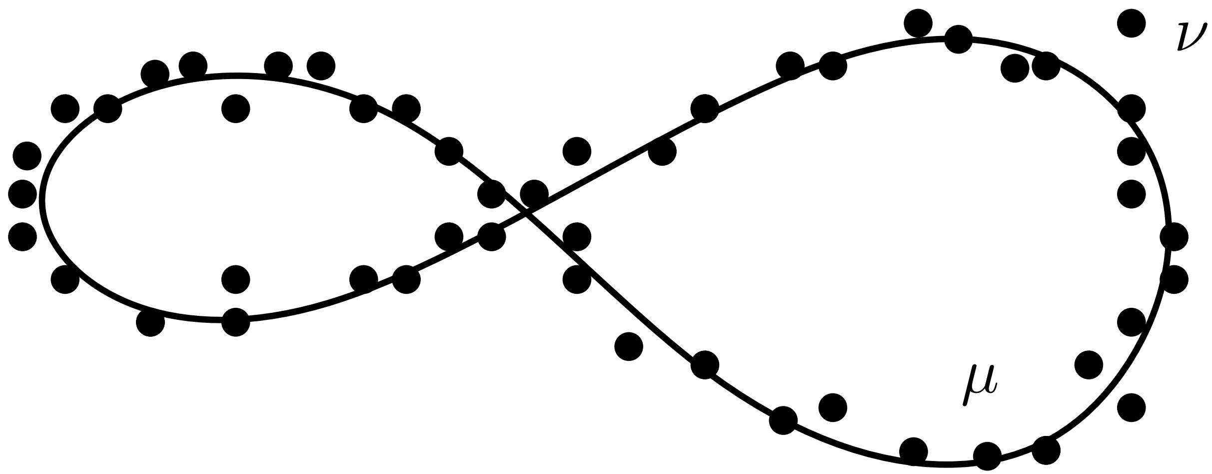
We will study the stability via the measures $\check{\mu}$ and $\check{\nu}$.

We define

$$\check{\mu} = \mu \otimes \delta_{\overline{\Sigma}_\mu(x)}$$

$$\check{\nu} = \nu \otimes \delta_{\overline{\Sigma}_\nu(x)}$$

We wish to bound $W_1(\check{\mu}, \check{\nu})$.



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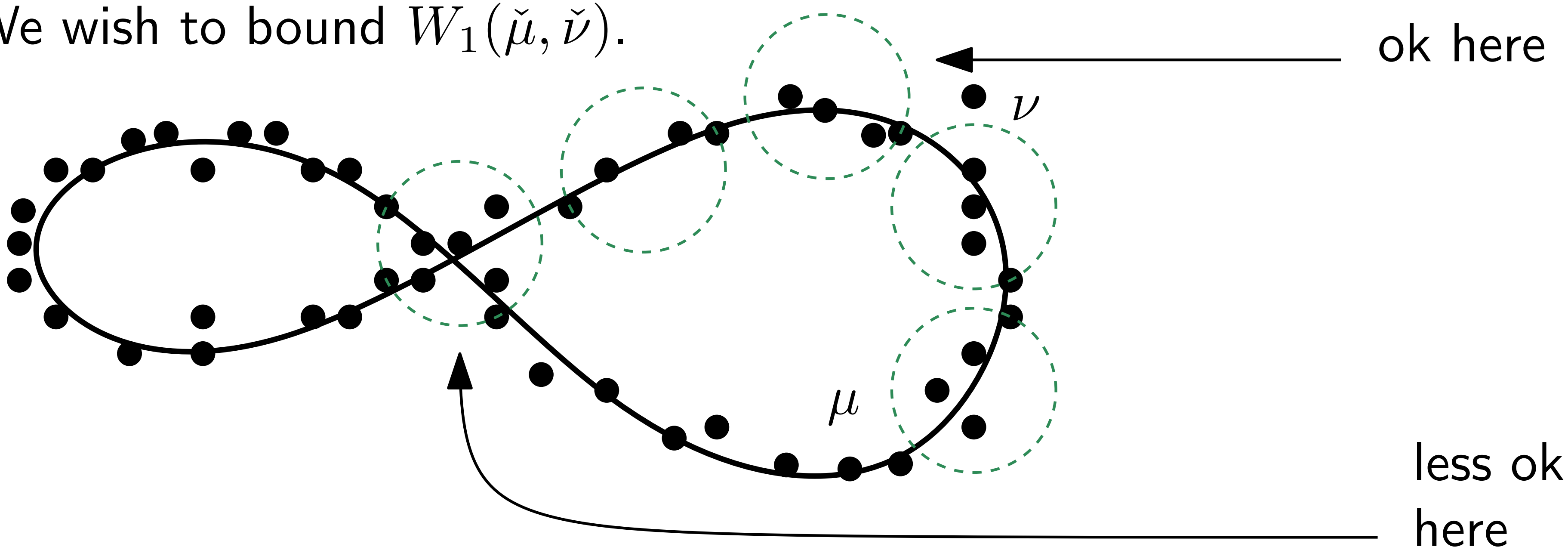
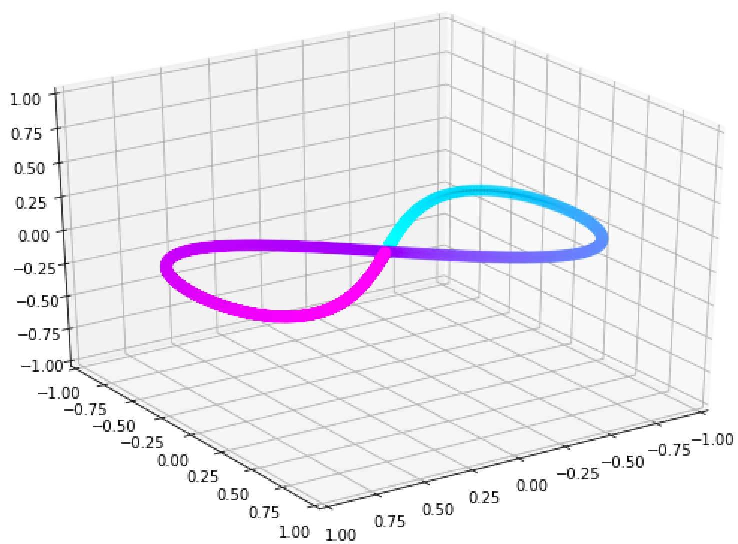


Tangent space estimation (4/6): Norm on $\mathbb{R}^n \times M_n(\mathbb{R})$

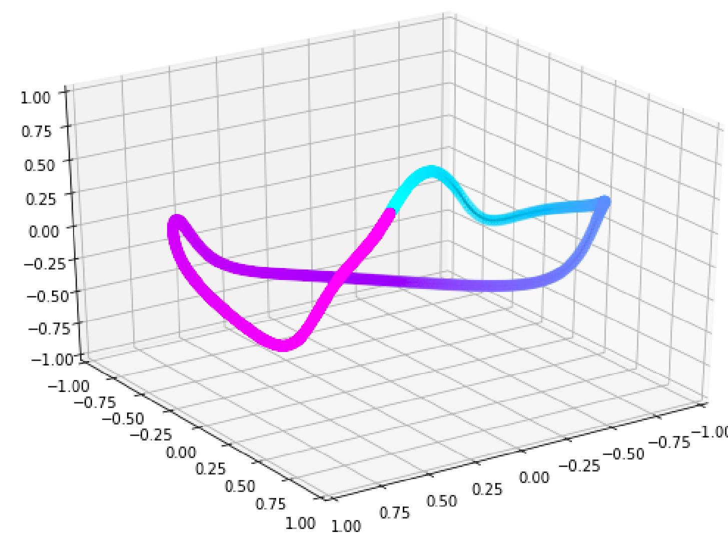
The Wasserstein norm $W_1(\check{\mu}, \check{\nu})$ is defined for the norm $\| \cdot \|_\gamma$ over $\mathbb{R}^n \times M_n(\mathbb{R})$, where

$$\|(x, A)\|_\gamma^2 = \|x\|^2 + \gamma^2 \|A\|_{\mathrm{F}}^2.$$

The parameter γ controls how much is the tangent information important, compared to the spatial one.



$\gamma = 0$



$\gamma = 1$

Tangent space estimation (5/6): Stability

Proposition:

Consider ν any Radon measure on $\mathbb{R}^n$. μ is as before.
Let $w = W_1(\mu, \nu)$. Suppose that $r \leq \frac{1}{4\rho}$ and $w \leq \min(a^{\frac{d+1}{d}}, 1)(\frac{r}{4})^{d+1}$.
If we suppose that $w \leq 1$, then

$$W_1(\check{\mu}, \check{\nu}) \leq \frac{2\gamma}{r} c' \mu(\lambda^r) w^{\frac{1}{2(d+1)}} + \frac{2\gamma}{r} c w^{\frac{1}{d+1}} + w.$$

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Proof: Build an obvious transport plan $\check{\pi}$ between $\check{\mu}$ and $\check{\nu}$ from a transport plan π between μ and ν. Write

$$W_1(\check{\mu}, \check{\nu}) \leq \int \|x - y\| + \gamma \|\overline{\Sigma}_\mu(x) - \overline{\Sigma}_\nu(y)\|_{\mathrm{F}} \mathrm{d}\pi(x, y)$$

Show that, for every $x \in \mathrm{supp}(\mu)$ and $y \in \mathrm{supp}(\nu)$,

$$\|\overline{\Sigma}_\mu(x) - \overline{\Sigma}_\nu(y)\|_{\mathrm{F}} \leq \frac{2}{r}(\|x - y\| + W_1(\overline{\mu_x}, \overline{\nu_y})),$$

where $\overline{\mu_x}$ is μ restricted to $\mathcal{B}(x, r)$ and normalized.

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Conclude

$$\begin{aligned} W_1(\check{\mu}, \check{\nu}) &\leq \int (1 + \frac{2\gamma}{r})\|x - y\| + \frac{2\gamma}{r} W_1(\overline{\mu_x}, \overline{\nu_y}))\mathrm{d}\pi(x, y) \\ &= (1 + \frac{2\gamma}{r})W_1(\mu, \nu) + \frac{2\gamma}{r}\int W_1(\overline{\mu_x}, \overline{\nu_y})\mathrm{d}\pi(x, y). \end{aligned}$$

Tangent space estimation (6/6): Approximation

Theorem:

Let ν be any Radon measure on $\mathbb{R}^n$, and μ as before. Choose $r < \frac{1}{4\rho}$.
Let $w = W_1(\mu, \nu)$. Suppose that $w \leq \min(a^{\frac{d+1}{d}}, 1)(\frac{r}{4})^{d+1}$ and $w \leq 1$.
Then

$$W_1(\check{\nu}, \check{\mu}_0) \leq 2\gamma(1 + \frac{c''}{r} w^{\frac{1}{2(d+1)}})\mu(\lambda^r) + \gamma c' r + \frac{2\gamma}{r} c w^{\frac{1}{d+1}} + w$$

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Proof: Write

$$W_1(\check{\nu}, \check{\mu}_0) \leq W_1(\check{\nu}, \check{\mu}) + W_1(\check{\mu}, \check{\mu}_0)$$

Use stability

Use consistency

Tangent space estimation (6/6): Approximation

Theorem:

Let ν be any Radon measure on $\mathbb{R}^n$, and μ as before. Choose $r < \frac{1}{4\rho}$.

Let $w = W_1(\mu, \nu)$. Suppose that $w \leq \min(a^{\frac{d+1}{d}}, 1)(\frac{r}{4})^{d+1}$ and $w \leq 1$.

Then

$$W_1(\check{\nu}, \check{\mu}_0) \leq 2\gamma(1 + \frac{c''}{r} w^{\frac{1}{2(d+1)}})\mu(\lambda^r) + \gamma c' r + \frac{2\gamma}{r} c w^{\frac{1}{d+1}} + w$$

Does this go to 0 as r does?

Quantification of λ

Proposition (not proven):
There exists a constant c such that for all $r <$ ...,

$$\mu_0(\lambda^r) \leq cr$$

As a consequence, the bound

$$W_1(\check{\nu}, \check{\mu}_0) \leq 2\gamma(1 + \frac{c''}{r} w^{\frac{1}{2(d+1)}})\mu(\lambda^r) + \gamma c' r + \frac{2\gamma}{r} c w^{\frac{1}{d+1}} + w$$

goes to 0 as r and w do (with $\frac{w}{r^{d+1}} \to 0$).

Persistent homology of $\check{\mathcal{M}}$ (1/2)

$\check{\mathcal{M}}$ is the support of $\check{\mu}_0$.

We have seen that $\check{\nu}$ is Wassertein-close to $\check{\mu}_0$ (but $\mathrm{supp}(\check{\nu})$ is not Hausdorff-close to $\mathrm{supp}(\check{\mu}_0)$).

Let us apply measured-based filtrations to $\check{\nu}$ in order to recover $\check{\mathcal{M}}$.

DTM-based filtrations

Given $E = \mathbb{R}^N$ and μ a Radon measure on E, the 1-DTM-filtration is a filtration of the space E that is stable in Wasserstein distance.
It depends on a parameter $m \in [0, 1)$.

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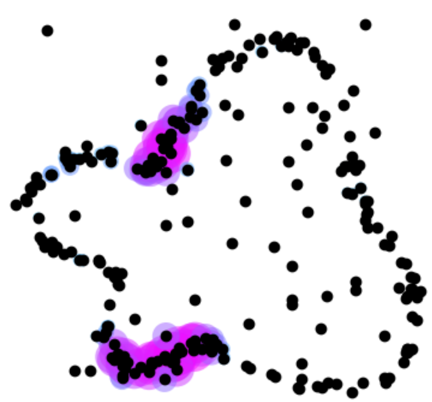

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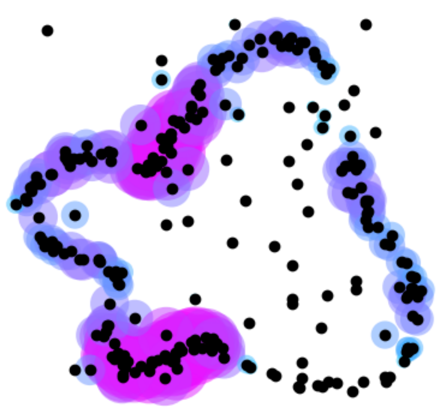

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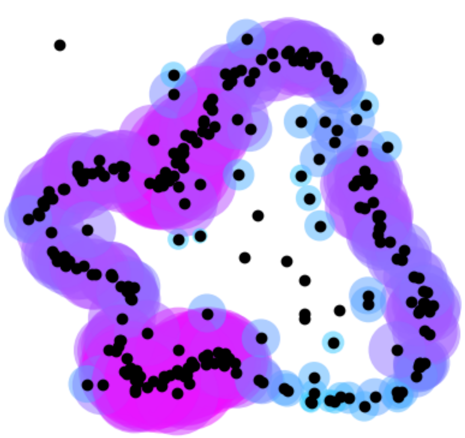

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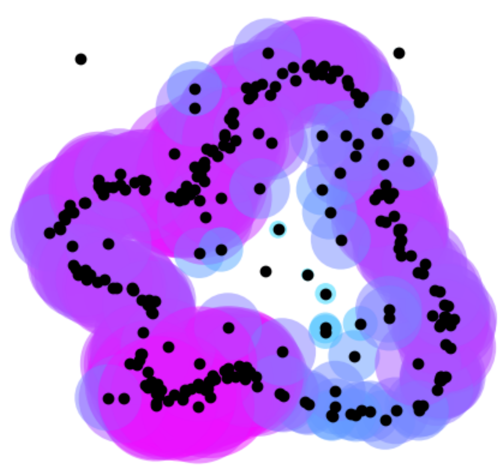

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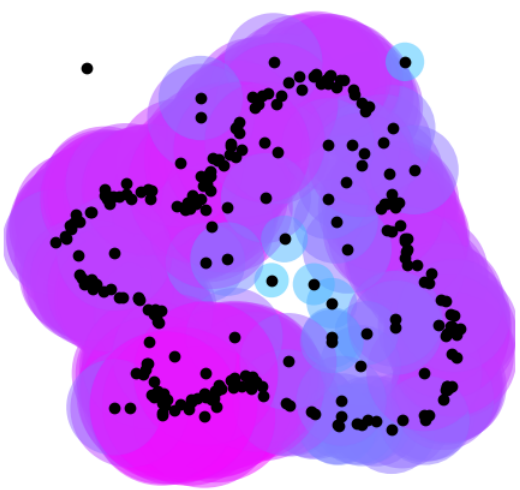

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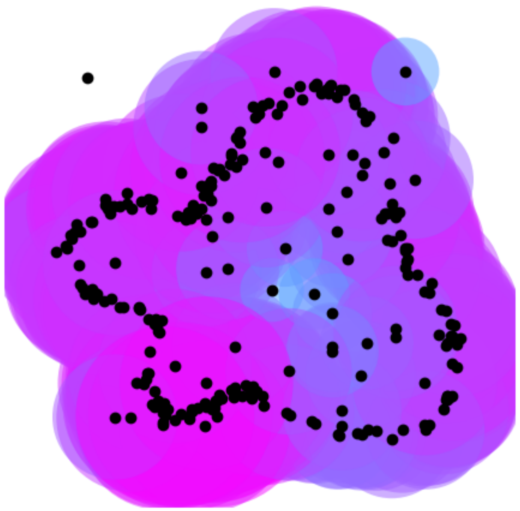

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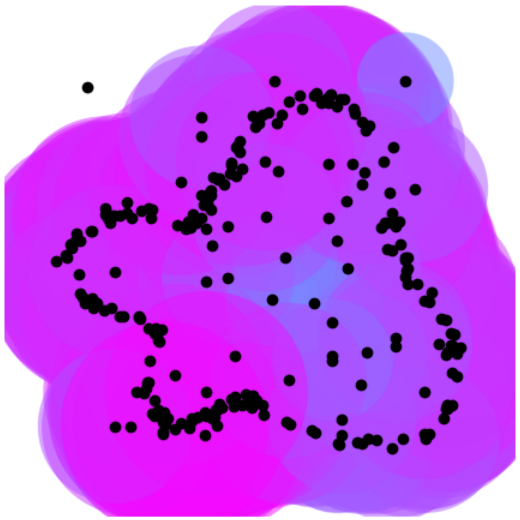

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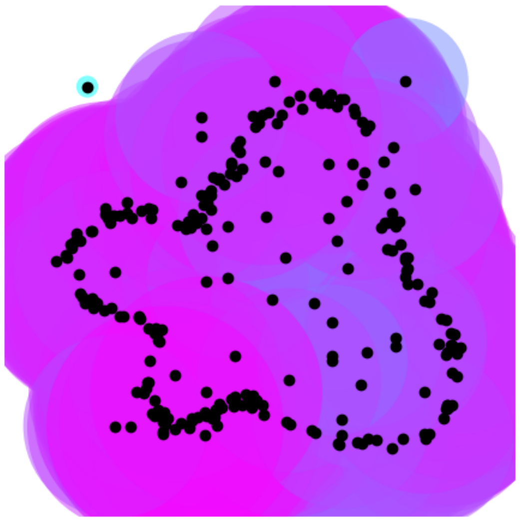
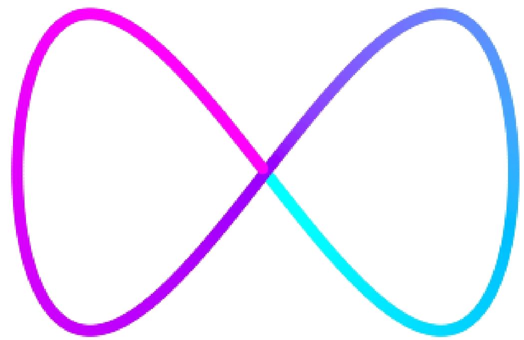


Persistent homology of $\check{\mathcal{M}}$ (2/2)

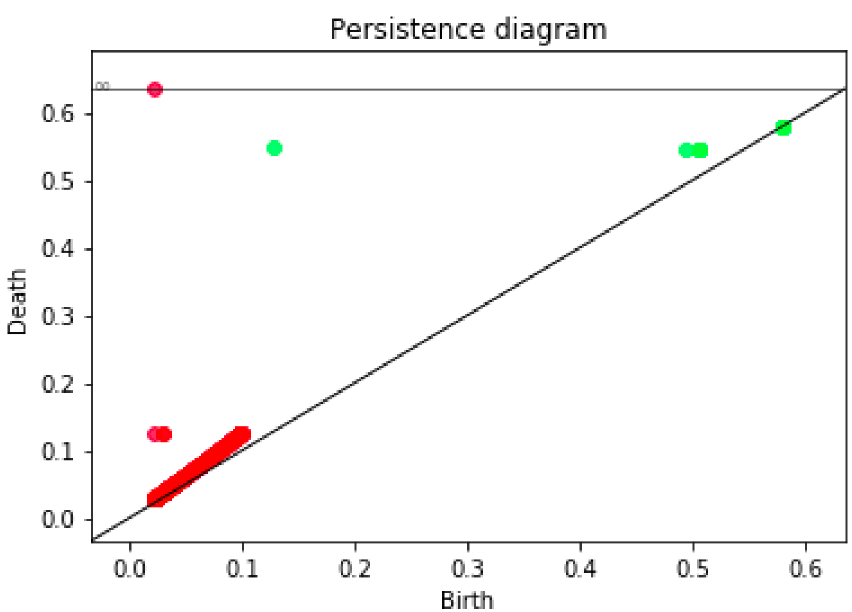
measures on
$\mathbb{R}^n \times M_n(\mathbb{R})$

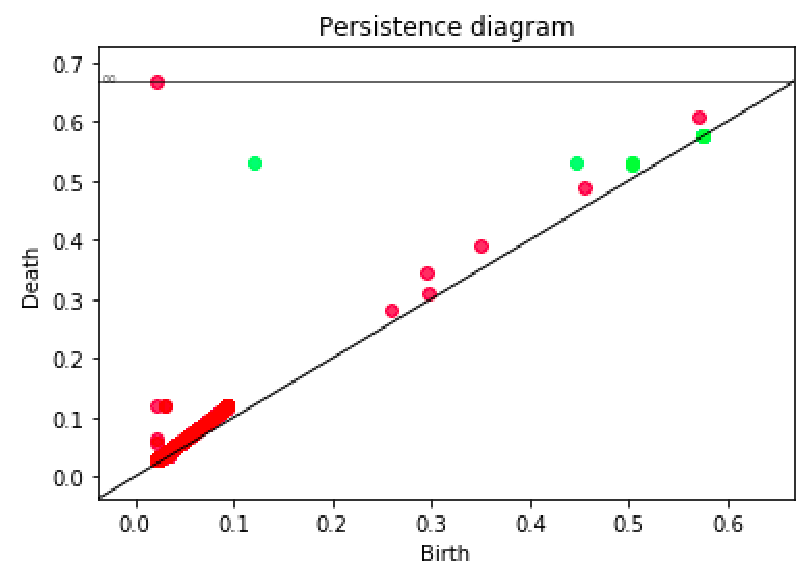


$\check{\mu}_0$

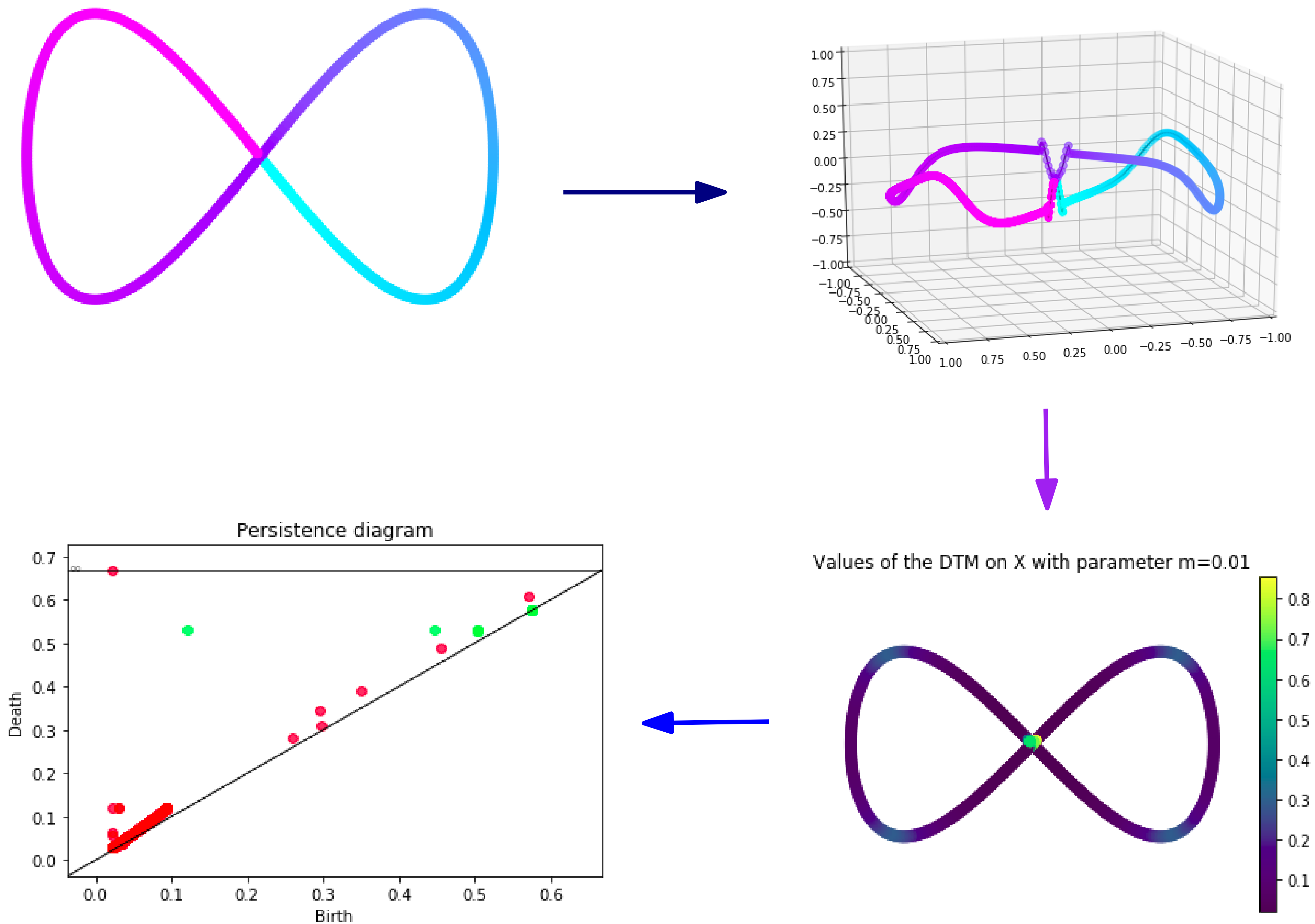
$\check{\nu}$

DTM-based
filtration





Thank you


Persistence diagram
Death
Birth
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
∞
Values of the DTM on X with parameter m=0.01
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
1.00 0.75 0.50 0.25 0.00 −0.25 −0.50 −0.75 −1.00