Persistent Stiefel-Whitney classes

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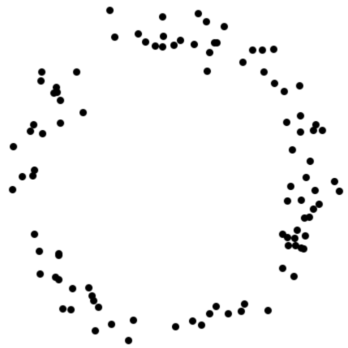
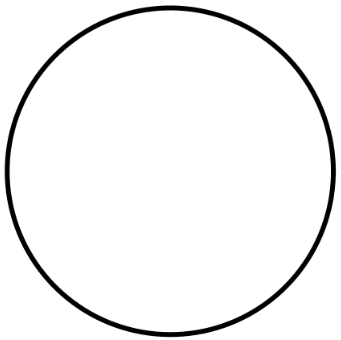
Applied Algebraic Topology Research Network seminar, 09/12/2020

We observe a point cloud X, that we suppose close to a submanifold $\mathcal{M}$.

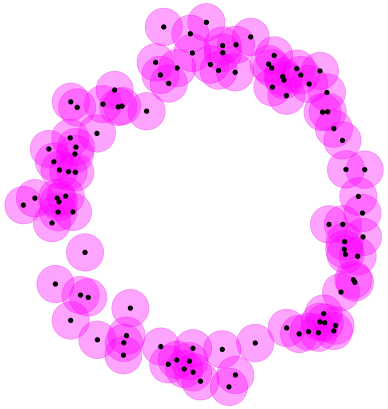
Point cloud X



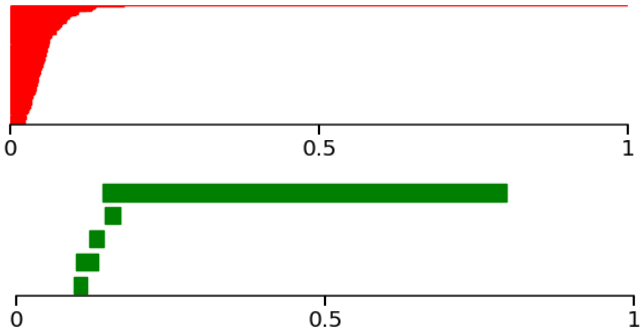
Submanifold $\mathcal{M}$



Let X^t denote the t-thickening of X.
We consider the Čech filtration of X, denoted $V[X] = (X^t)_{t\geq 0}$.

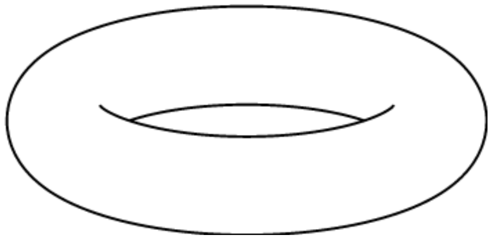


$\longrightarrow \quad \mathbb{V} \quad \longrightarrow$

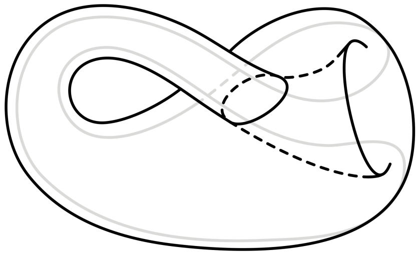


Filtration $V[X]$ | Persistence module $\mathbb{V}[X]$ | $\mathrm{Barcode}[X]$

Persistent homology allows to estimate the *homology* of a space. However, over $\mathbb{Z}/2\mathbb{Z}$, homology may not be fine enough to distinguish between
- non-homeomorphic spaces.
- non-homotopy equivalent spaces.



Torus



Klein bottle

$\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/2\mathbb{Z}$, 0, ...

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Other topological invariants

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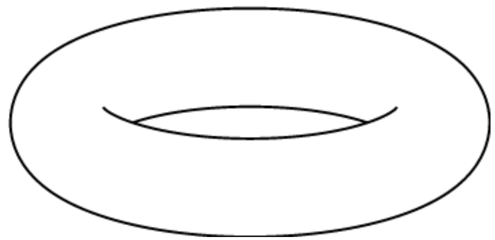
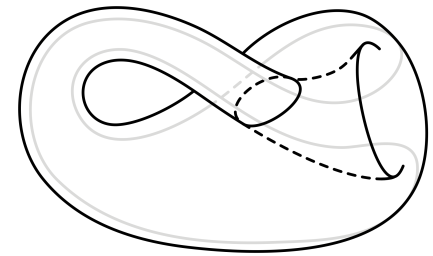
	 Torus	 Klein bottle
$H_i(\mathcal{M}, \mathbb{Z}/2\mathbb{Z}), i \geq 0$	$\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/2\mathbb{Z}$, 0, ...	$\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/2\mathbb{Z}$, 0, ...
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$H^*(\mathcal{M}, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}[x,y]/\langle x^2, y^2 \rangle$	$\mathbb{Z}/2\mathbb{Z}[x,y]/\langle x^3, y^2, x^2y \rangle$
$w_1(\tau)$	0	x

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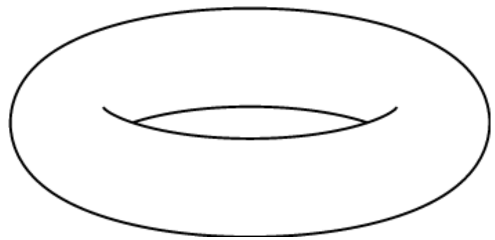
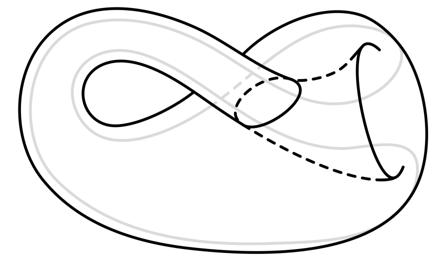
homology groups over $\mathbb{Z}/p\mathbb{Z}$	 Torus	 Klein bottle
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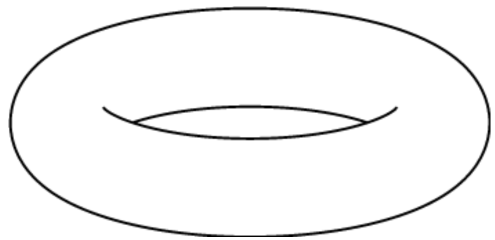
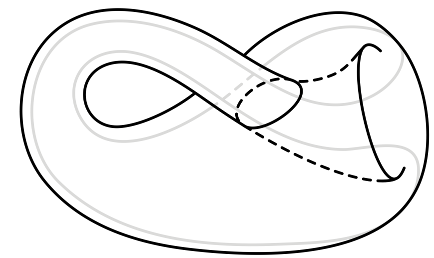
homology groups over $\mathbb{Z}$

	 Torus	 Klein bottle
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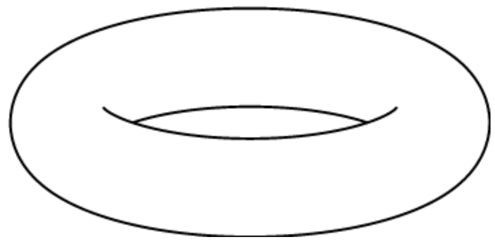
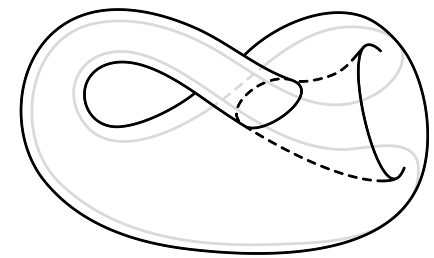
cohomology algebra over $\mathbb{Z}/2\mathbb{Z}$	 Torus	 Klein bottle
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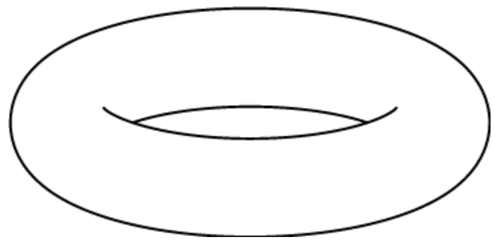
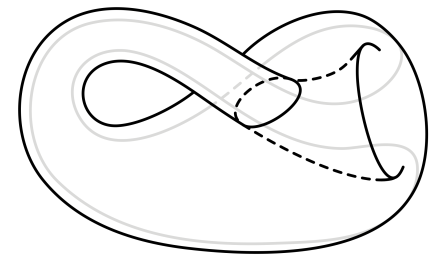
	 Torus	 Klein bottle
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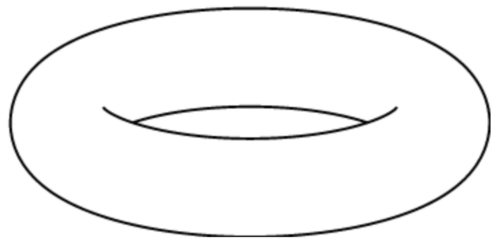
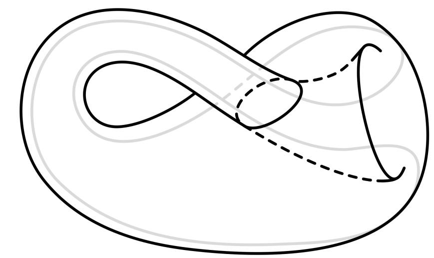
first Stiefel-Whitney class of tangent bundle	 Torus	 Klein bottle
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Aim of this talk:
To build a persistent framework for Stiefel-Whitney classes.

I - Stiefel-Whitney classes

II - Persistent Stiefel-Whitney classes

III - Algorithmic considerations

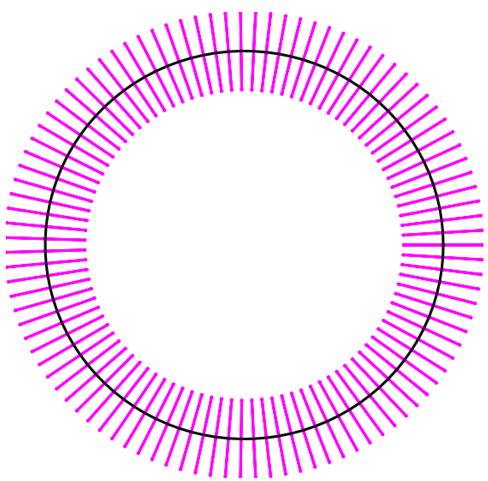
Vector bundles

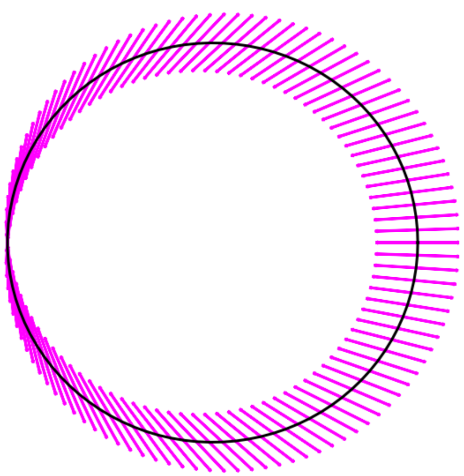
Definition:
A vector bundle (of dimension d) over X is a surjection $\pi : E \to X$, with E a topological space, such that:

- the fibers $\pi^{-1}(\{x\}), x \in X$, are vector spaces of dimension d,
- π satisfies a local triviality condition.

Local triviality condition: for all $x \in X$, there exists a neigborhood $U \subset X$ and a homeomorphism $h \colon U \times \mathbb{R}^d \to \pi^{-1}(U)$ such that for all $y \in U$, $h(y, \cdot)$ is an isomorphism of vector spaces.

Normal bundle of the circle





Möbius strip (universal bundle)

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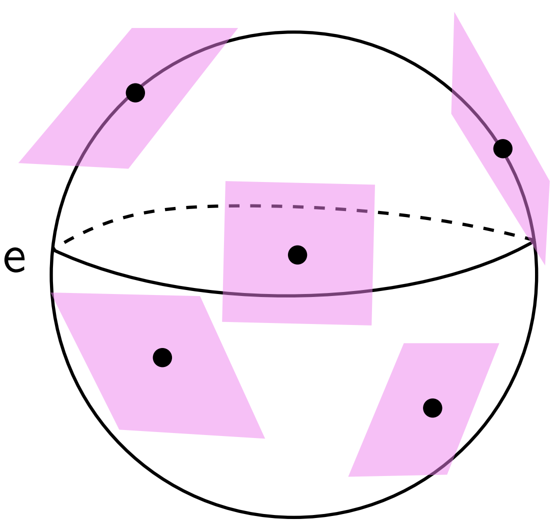
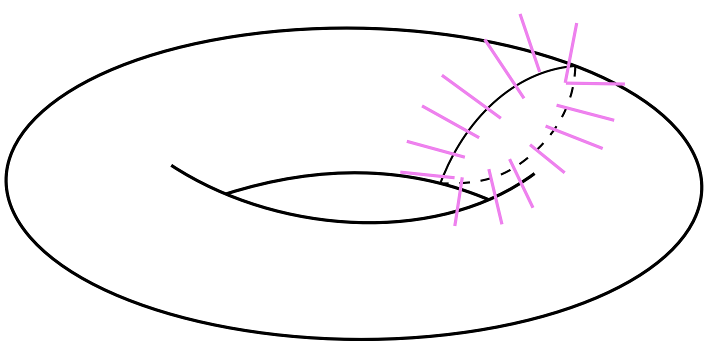
Tangent bundle of sphere





Normal bundle of the torus

Stiefel-Whitney classes (axioms)

For every vector bundle $\pi\colon E \to X$, there exists a sequence of cohomology classes

$$w_0(\pi) \in H^0(X, \mathbb{Z}/2\mathbb{Z}),$$
$$w_1(\pi) \in H^1(X, \mathbb{Z}/2\mathbb{Z}),$$
$$w_2(\pi) \in H^2(X, \mathbb{Z}/2\mathbb{Z}),$$
$$w_3(\pi) \in H^3(X, \mathbb{Z}/2\mathbb{Z}),$$
$$\ldots$$

that satisfy the following axioms:

- **Axiom 1:** $w_0(\pi)$ is equal to $1 \in H^0(X, \mathbb{Z}/2\mathbb{Z})$, and if π is of dimension d then $w_i(\pi) = 0$ for $i > d$.
- **Axiom 2:** if $f\colon \pi \to \rho$ is a bundle map, then $w_i(\pi) = f^*(w_i(\rho))$, where $f^*\colon H^*(X) \leftarrow H^*(Y)$ is the map induced in cohomology by f.
- **Axiom 3:** if π, ρ are vector bundles over the same base space X, then for all $k \in \mathbb{N}$, $w_k(\pi \oplus \rho) = \sum_{i=0}^{k} w_i(\pi) \smile w_{k-i}(\rho)$ (cup product).
- **Axiom 4:** $w_1(\gamma_1^1) \neq 0$, where γ_1^1 denotes the Möbius strip bundle over the circle.

Stiefel-Whitney classes (axioms)

For every vector bundle $\pi\colon E \to X$, there exists a sequence of cohomology classes

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$$\ldots$$

Basic properties:

- If two bundles are isomorphic, then their Stiefel-Whitney classes are equal.
- If π admits a (nowhere vanishing) section, then $w_d(\pi) = 0$.
- If π admits k independent (nowhere vanishing) sections, then $w_d(\pi) = \ldots w_{d-k+1}(\pi) = 0$.

Topological information:

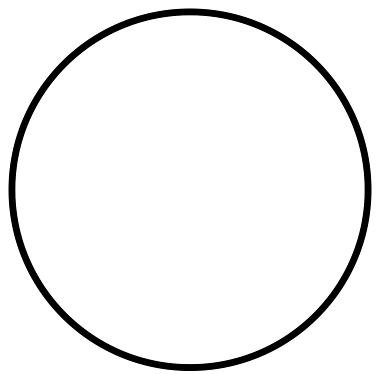
- If τ is the tangent bundle of a manifold $\mathcal{M}$, then $\mathcal{M}$ is orientable if and only if $w_1(\tau) = 0$.
- $\mathcal{M}$ admits a spin structure if and only if $w_1(\tau) = 0$ and $w_2(\tau) = 0$.

The Grassmann manifolds

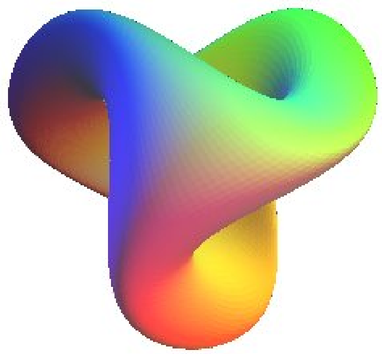
Let $d, n \geq 1$.

The *Grassmannian* $\mathcal{G}_d(\mathbb{R}^n)$ is the set of d-dimensional linear subspaces of $\mathbb{R}^n$.

It can be endowed with a manifold structure, of dimension $d(n-d)$.

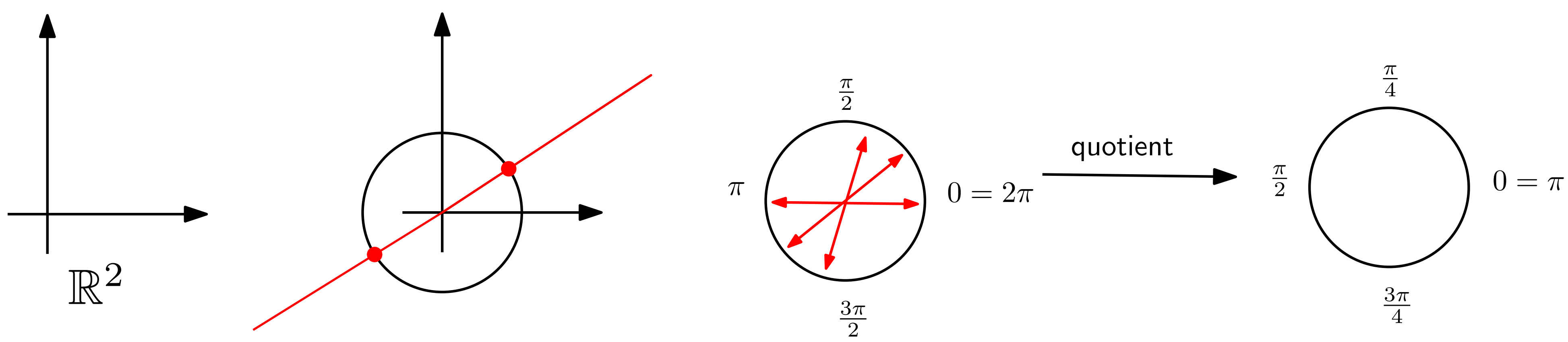


$\mathcal{G}_1(\mathbb{R}^2)$



$\mathcal{G}_1(\mathbb{R}^3) \simeq \mathcal{G}_2(\mathbb{R}^3)$

A construction of $\mathcal{G}_1(\mathbb{R}^2)$:



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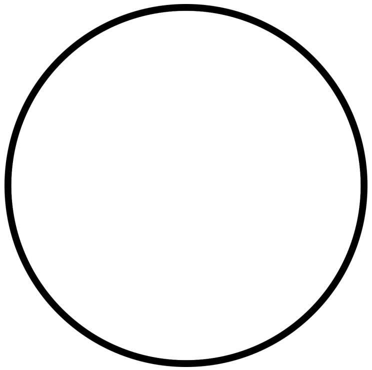

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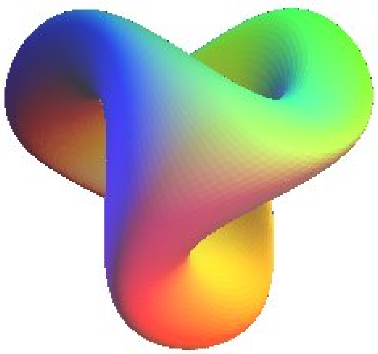

$\mathcal{G}_1(\mathbb{R}^3) \simeq \mathcal{G}_2(\mathbb{R}^3)$

Let $\mathbb{R}^\infty$ denotes the space of sequences of real numbers that are zero from some point.
We can also define the *infinite Grassmannian* $\mathcal{G}_d(\mathbb{R}^\infty)$.

The infinite Grassmannian has $\mathbb{Z}/2\mathbb{Z}$-cohomology

$$H^*(\mathcal{G}_d(\mathbb{R}^\infty)) = \mathbb{Z}/2\mathbb{Z}[w_1, ..., w_d]$$

where w_i has degree i.

In particular, $H^*(\mathcal{G}_1(\mathbb{R}^\infty)) = \mathbb{Z}/2\mathbb{Z}[w_1]$.

The Grassmann manifolds

Let $d, n \geq 1$.
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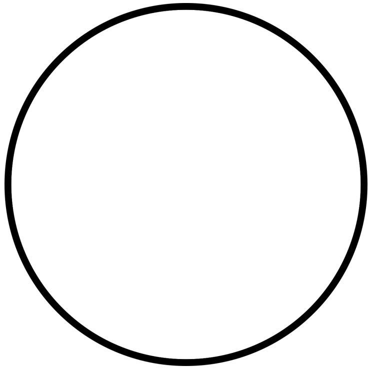
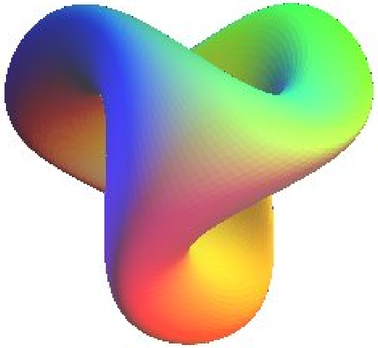


$\mathcal{G}_1(\mathbb{R}^2)$



$\mathcal{G}_1(\mathbb{R}^3) \simeq \mathcal{G}_2(\mathbb{R}^3)$

Let $\mathrm{M}(\mathbb{R}^n)$ be the space of $n \times n$ matrices.
For every linear subspace $T \subset \mathbb{R}^n$, let p_T denotes the orthogonal projection matrix on T.

The application $T \in \mathcal{G}_d(\mathbb{R}^n) \longmapsto p_T \in \mathrm{M}(\mathbb{R}^n)$ is an embedding.

Hence $\mathcal{G}_d(\mathbb{R}^n)$ can be seen as a submanifold of $\mathrm{M}(\mathbb{R}^n)$.

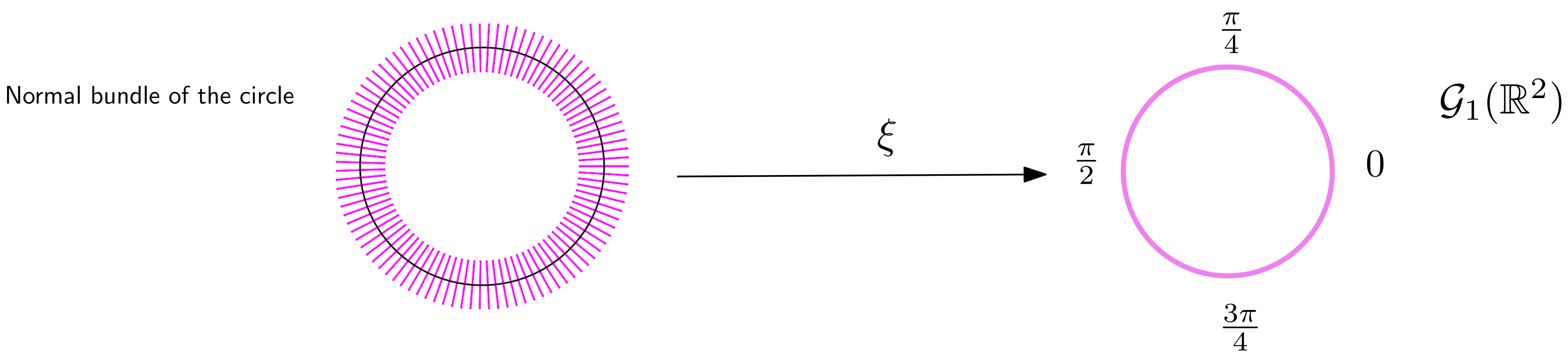
Vector bundles (2^{nd} definition)

Correspondence vector bundles / classifying maps:

Let X is a topological space. From any continuous map $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^n)$, we can build a d-dimensional vector bundle structure on X.

Conversely, for any vector bundle $\pi\colon E \to X$, there exists a corresponding map $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^\infty)$, called a *classifying map*.

Moreover, if X is compact, we can choose $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^m)$ for m large enough.



(Second) Definition:
A vector bundle over X is a continuous map $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^\infty)$ or $\xi : X \to \mathcal{G}_d(\mathbb{R}^m)$.

Stiefel-Whitney classes (construction)

Let $\left| \begin{array}{l} \xi\colon X \to \mathcal{G}_d(\mathbb{R}^\infty) \text{ be a vector bundle, and} \\ \xi^*\colon H^*(X, \mathbb{Z}/2\mathbb{Z}) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^\infty), \mathbb{Z}/2\mathbb{Z}) \text{ the map induced in cohomology.} \end{array} \right.$

Recall that $H^*(\mathcal{G}_d(\mathbb{R}^\infty)) = \mathbb{Z}/2\mathbb{Z}[w_1, ..., w_d]$.

The Stiefel-Whitney classes of the vector bundle $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^\infty)$ can be defined as

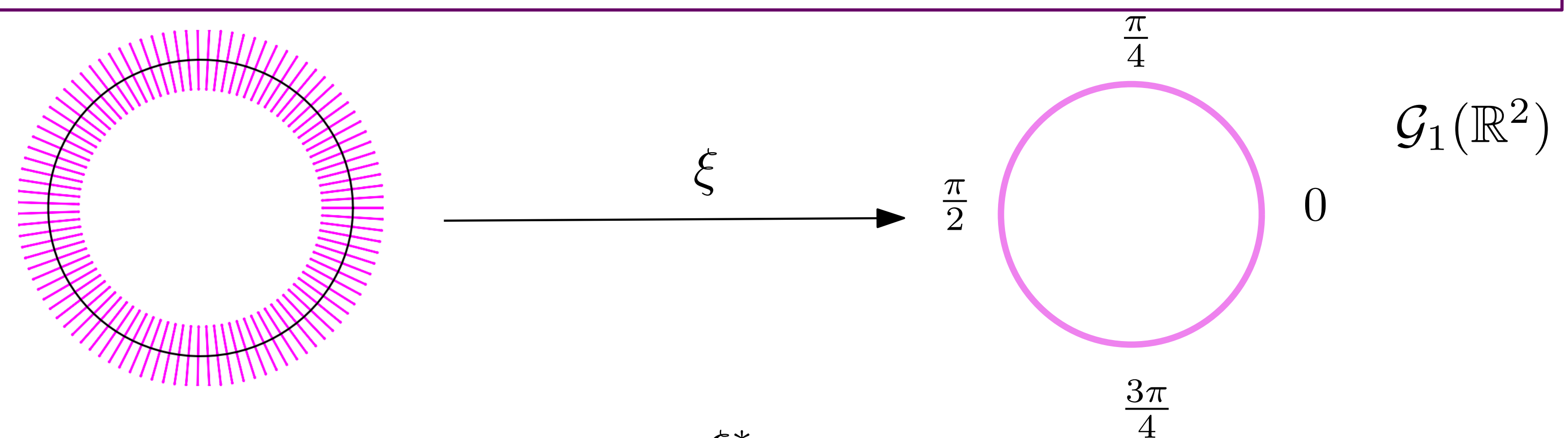
$$w_0(\xi) = \xi^*(\omega_0)$$

$$w_1(\xi) = \xi^*(\omega_1)$$

$$w_2(\xi) = \xi^*(\omega_2)$$

$$\dots$$

Normal bundle of the circle



$$H^1(\mathbb{S}_1, \mathbb{Z}/2\mathbb{Z}) \xleftarrow{\ \xi^*\ } H^1(\mathcal{G}_d(\mathbb{R}^\infty), \mathbb{Z}/2\mathbb{Z})$$

$$w_1(\xi) = 0 \longleftarrow\!\shortmid\ w_1$$

Let $\left| \begin{array}{l} \xi\colon X \to \mathcal{G}_d(\mathbb{R}^\infty) \text{ be a vector bundle, and} \\ \xi^*\colon H^*(X, \mathbb{Z}/2\mathbb{Z}) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^\infty), \mathbb{Z}/2\mathbb{Z}) \text{ the map induced in cohomology.} \end{array} \right.$

Recall that $H^*(\mathcal{G}_d(\mathbb{R}^\infty)) = \mathbb{Z}/2\mathbb{Z}[w_1, ..., w_d]$.

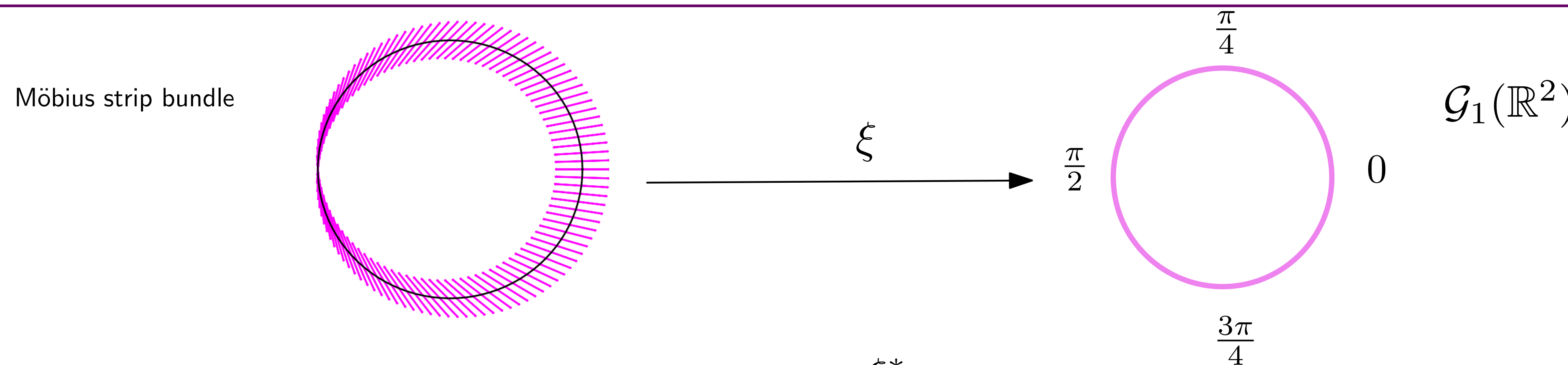
The Stiefel-Whitney classes of the vector bundle $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^\infty)$ can be defined as

$$w_0(\xi) = \xi^*(\omega_0)$$

$$w_1(\xi) = \xi^*(\omega_1)$$

$$w_2(\xi) = \xi^*(\omega_2)$$

$$\dots$$



$$H^1(\mathbb{S}_1, \mathbb{Z}/2\mathbb{Z}) \xleftarrow{\xi^*} H^1(\mathcal{G}_d(\mathbb{R}^\infty), \mathbb{Z}/2\mathbb{Z})$$

$$w_1(\xi) = 1 \longleftarrow\!\shortmid\; w_1$$

I - Stiefel-Whitney classes

II - Persistent Stiefel-Whitney classes

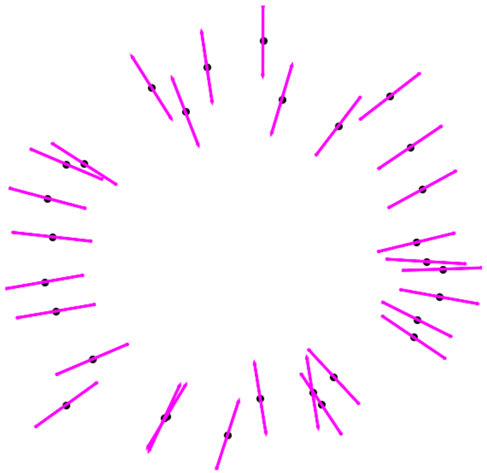
III - Algorithmic considerations

Adopting a persistent viewpoint

Sampling model for vector bundles:

Let $n, m, d > 0$.

We observe | a point cloud $X \subset \mathbb{R}^n$
and a map $\xi \colon X \to \mathcal{G}_d(\mathbb{R}^m)$.



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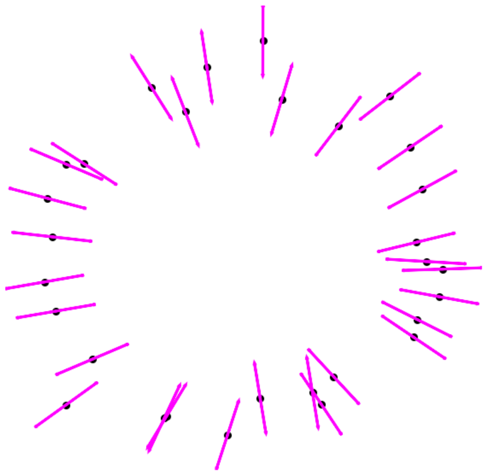
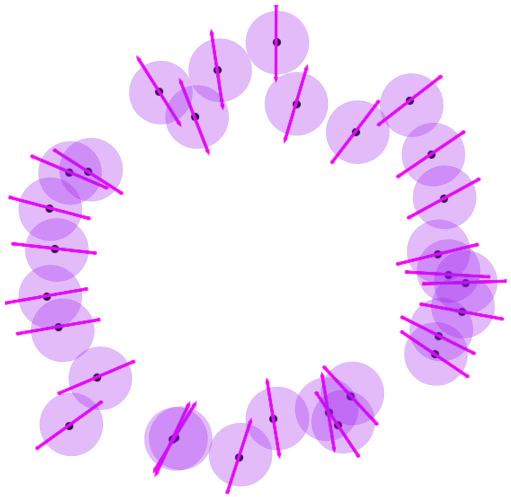


Defining a vector bundle filtration:

Let $(X^t)_{t\geq 0}$ be the Čech filtration of X.

We want to define maps $\xi^t \colon X^t \to \mathcal{G}_d(\mathbb{R}^m)$.



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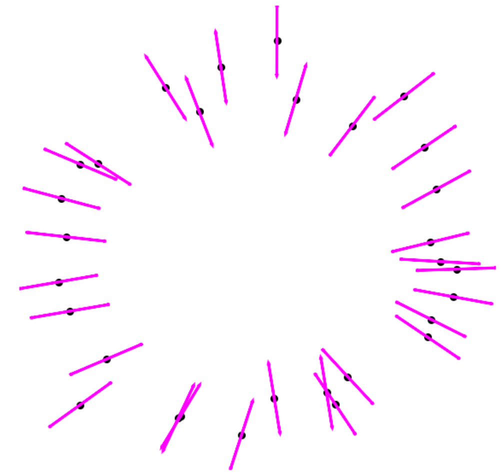

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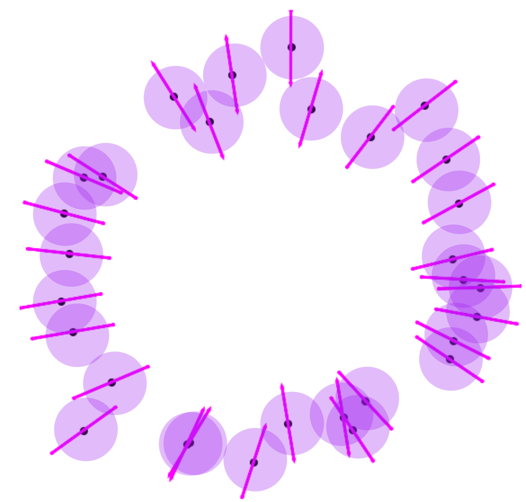

Nothing interesting to do here...

Sampling model for vector bundles:

Let $n, m, d > 0$.

We observe | a point cloud $X \subset \mathbb{R}^n$ **and** a map $\xi\colon X \to \mathcal{G}_d(\mathbb{R}^m)$. $\Longleftrightarrow$ a point cloud $\check{X} \subset \mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m)$.

$$\check{X} = \{(x, \xi(x)), x \in X\}$$

A persistent viewpoint (2^{nd} attempt)

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- By embedding $\mathcal{G}_d(\mathbb{R}^m) \hookrightarrow \mathrm{M}(\mathbb{R}^m)$, we can see $\check{X}$ as a subset of $\mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m)$.
- Let $(\check{X}^t)_{t \geq 0}$ be the Čech filtration of $\check{X}$ in the ambient space $\mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m)$, endowed with the metric $\|(x, A)\| = \sqrt{\|x\|_2^2 + \|A\|_{\mathrm{F}}^2}$.
- We can define extended maps ξ^t as follows:

$$\begin{aligned} \xi^t \colon \quad \check{X}^t &\longrightarrow \mathcal{G}_d(\mathbb{R}^m) \\ (x, A) &\longmapsto \mathrm{proj}\left(A, \mathcal{G}_d(\mathbb{R}^m)\right) \end{aligned}$$

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Definition:
The data of $(\check{X}^t)_{t\geq 0}$ and $(\xi^t \colon \check{X}^t \to \mathcal{G}_d(\mathbb{R}^m))_{t\geq 0}$ is called the *Čech bundle filtration* of $\check{X}$.

Let $\left| \begin{array}{l} \check{X} \subset \mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m), \\ (\check{X}^t)_t, (\xi^t)_t \text{ its Čech bundle filtration,} \\ i \geq 0. \end{array} \right.$

For every $t \geq 0$, $(\check{X}^t, \xi^t)$ is a vector bundle.
Its i^{th} Stiefel-Whitney class can be defined as

$$w_i(\xi^t) = (\xi^t)^*(w_i),$$

where $(\xi^t)^* \colon H^*(\check{X}^t) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^m))$.

Definition:
The i^{th} *persistent Stiefel-Whitney class* of $\check{X}$ is the collection $w_i(\check{X}) = (w_i(\xi^t))_{t \geq 0}$.

Persistent Stiefel-Whitney classes

Let $\left| \begin{array}{l} \check{X} \subset \mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m), \\ (\check{X}^t)_t, (\xi^t)_t \text{ its Čech bundle filtration,} \\ i \geq 0. \end{array} \right.$

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Definition:
The i^{th} *persistent Stiefel-Whitney class* of $\check{X}$ is the collection $w_i(\check{X}) = (w_i(\xi^t))_{t\geq 0}$.

Issue: ξ^t is not well-defined for every $t \geq 0$...

Maximal filtration value

The extended maps ξ^t are defined as

$$\begin{aligned} \xi^t\colon \quad \check{X}^t &\longrightarrow \mathcal{G}_d(\mathbb{R}^m) \\ (x, A) &\longmapsto \operatorname{proj}\left(A, \mathcal{G}_d(\mathbb{R}^m)\right) \end{aligned}$$

But $\operatorname{proj}\left(A, \mathcal{G}_d(\mathbb{R}^m)\right)$ does not make sense if A lies in the medial axis of $\mathcal{G}_d(\mathbb{R}^m)$.

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Lemma

For any $A \in \mathrm{M}(\mathbb{R}^m)$, let A^s denote the matrix $A^s = \frac{1}{2}(A + {}^t A)$, and let $\lambda_1(A^s), \dots, \lambda_n(A^s)$ be the eigenvalues of A^s in decreasing order.
The distance from A to $\operatorname{med}\left(\mathcal{G}_d(\mathbb{R}^m)\right)$ is $\frac{\sqrt{2}}{2}\left|\lambda_d(A^s) - \lambda_{d+1}(A^s)\right|$.

The persistent Stiefel-Whitney class $(w_i(\xi^t))_t$ is defined for every $t \in [0, t^{\mathsf{max}})$.

Lifebar

Let $\check{X} \subset \mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m)$, and $w_i(\check{X})$ its i^{th} persistent Stiefel-Whitney class.

Definition

The *lifebar* of the persistent Stiefel-Whitney class $w_i(\check{X}) = (w_i(\xi^t))_{t<t^{\max}}$ is the set

$$\left\{t \in [0, t^{\max}), w_i(\xi^t) \neq 0\right\}.$$

the lifebar is an interval!

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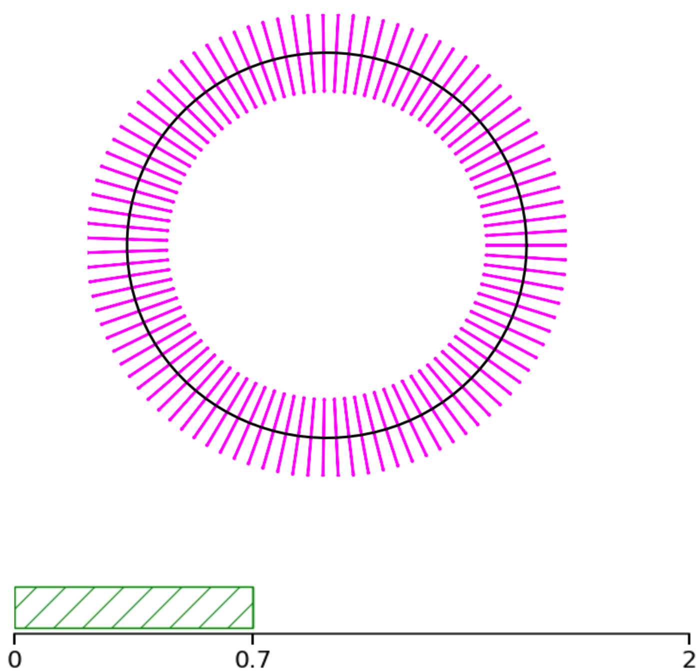
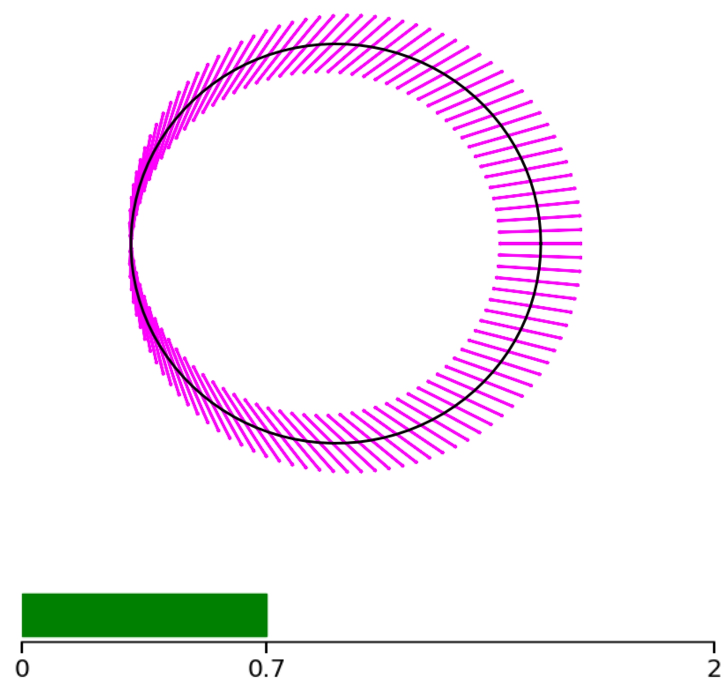
Example: lifebars of first persistent Stiefel-Whitney classes

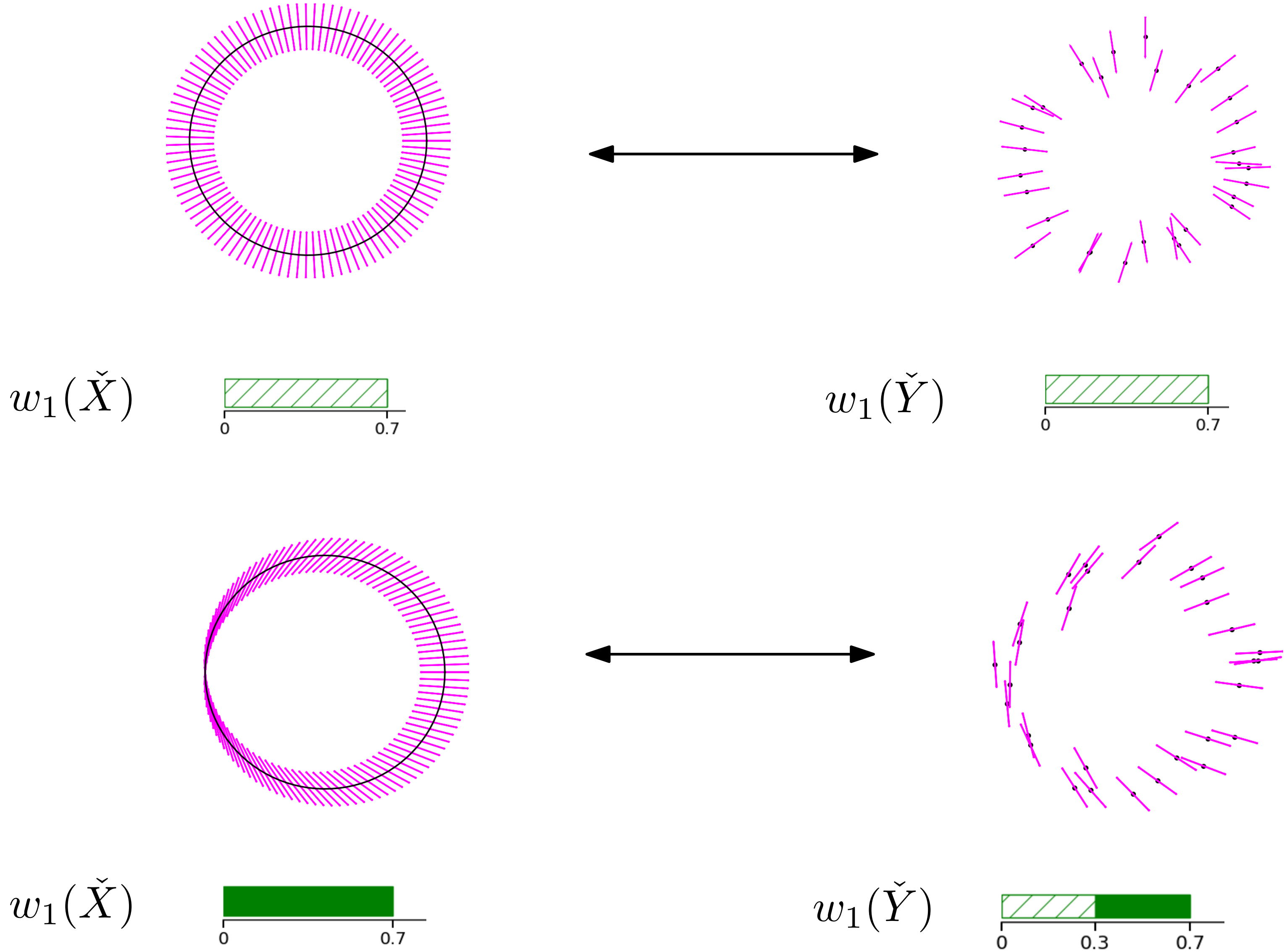




Stability

Theorem

If two subsets $\check{X}, \check{Y} \subset \mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m)$ satisfies $\mathrm{d_H}\left(\check{X}, \check{Y}\right) \leq \epsilon$, then for all $i \geq 0$, the lifebars of their i^{th} Stiefel-Whitney classes are ϵ-close.



Consistency

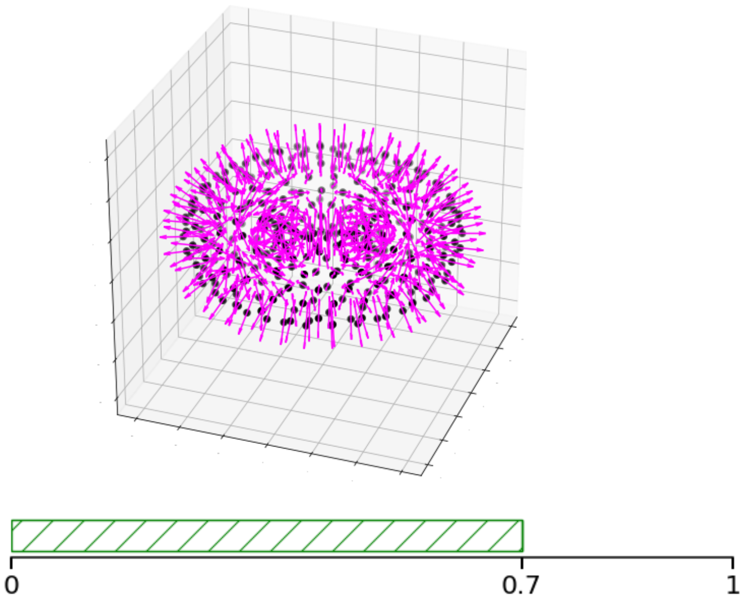
If $u\colon \mathcal{M}_0 \to \mathcal{M} \subset \mathbb{R}^n$ is an immersion and $\xi\colon \mathcal{M}_0 \to \mathcal{G}_d(\mathbb{R}^m)$ a vector bundle, consider the set

$$\check{\mathcal{M}} = \{(u(x_0), \xi(x_0)), x_0 \in \mathcal{M}_0\} \subset \mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m).$$

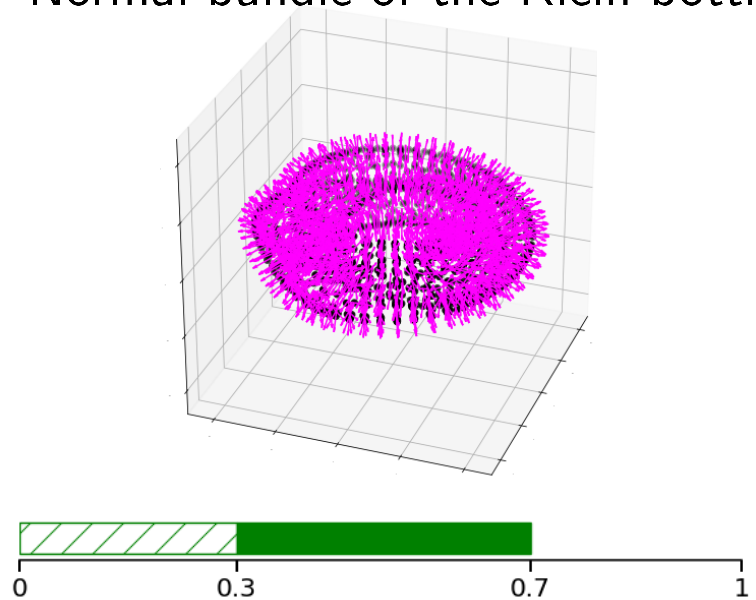
Theorem

Let $X \subset \mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m)$ be any subset such that $\mathrm{d_H}\left(X, \check{\mathcal{M}}\right) \leq \epsilon$. Then for every $t \in [4\epsilon, \mathrm{reach}\left(\check{\mathcal{M}}\right) - 3\epsilon)$, the composition of inclusions $\mathcal{M}_0 \hookrightarrow \check{\mathcal{M}} \hookrightarrow X^t$ induces an isomorphism $H^*(\mathcal{M}_0) \leftarrow H^*(X^t)$ which sends the i^{th} persistent Stiefel-Whitney class $w_i^t(X)$ of the Čech bundle filtration of X to the i^{th} Stiefel-Whitney class of $(\mathcal{M}_0, p)$.

Normal bundle of the torus



Normal bundle of the Klein bottle



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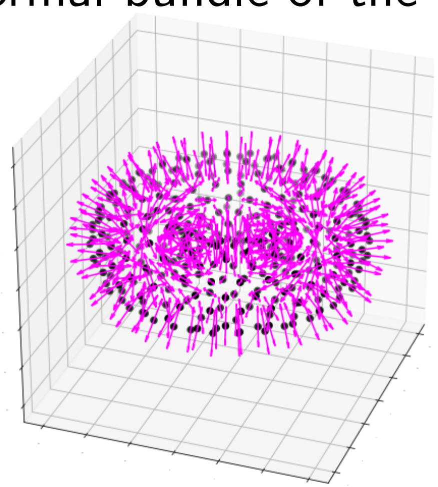
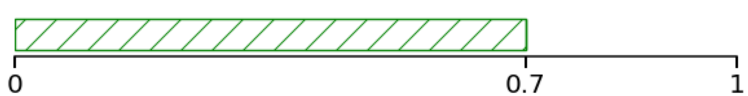
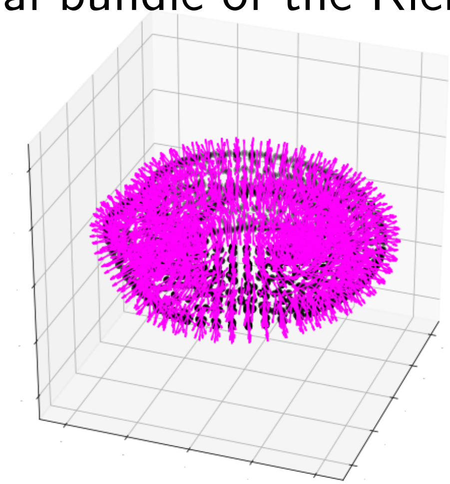




orientable

non-orientable

Normal bundle of the Klein bottle



I - Stiefel-Whitney classes

II - Persistent Stiefel-Whitney classes

III - Algorithmic considerations

Simplicial approximation

Let $\left|\begin{array}{l} \check{X} \subset \mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m) \text{ or } \check{X} \subset \mathbb{R}^n \times \mathrm{M}(\mathbb{R}^m), \\ (\check{X}^t)_t, (\xi^t)_t \text{ its Čech bundle filtration,} \\ (w_i(\xi^t))_t \text{ its } i^{\text{th}} \text{ persistent Stiefel-Whitney class.} \end{array}\right.$

$$\begin{array}{rccc} \xi^t\colon & \check{X}^t & \longrightarrow & \mathcal{G}_d(\mathbb{R}^m) \\ (\xi^t)^*\colon & H^*(\check{X}^t) & \longleftarrow & H^*(\mathcal{G}_d(\mathbb{R}^m)) \\ & w_i(\xi^t) & \longleftarrow\!\shortmid & w_i \end{array}$$

Problem:
Compute $w_i(\xi^t)$ on a computer.

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Compute $w_i(\xi^t)$ on a computer.

Suppose that we have triangulations S^t of $\check{X}^t$ and G of $\mathcal{G}_d(\mathbb{R}^m)$.

nerve of the union of balls

see later

Simplicial approximation

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$$
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$$

Problem:
Compute $w_i(\xi^t)$ on a computer.

Suppose that we have triangulations S^t of $\check{X}^t$ and G of $\mathcal{G}_d(\mathbb{R}^m)$.
Denote their topological realizations $|S^t|$ and $|G|$.

$$
\begin{tikzcd}
\check{X}^t \arrow[r, "\xi^t"] & \mathcal{G}_d(\mathbb{R}^m) \\
{|S^t|} \arrow[u, "\wr"] \arrow[r, dashed] & {|G|} \arrow[u, "\wr"]
\end{tikzcd}
\qquad
S^t \overset{?}{\dashrightarrow} G
$$

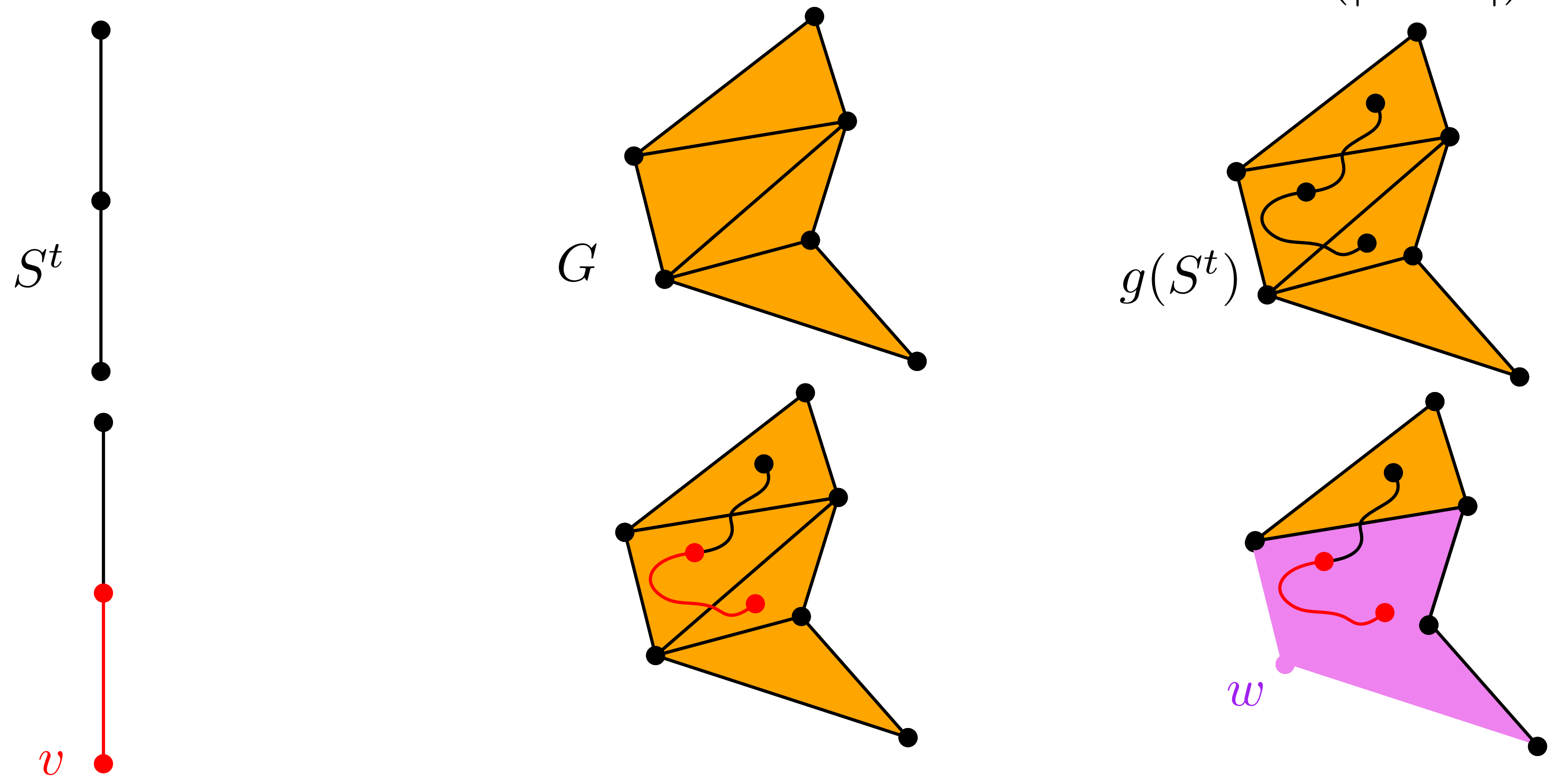
We look for a simplicial map $p^t\colon S^t \to G$ that 'corresponds to' ξ^t.

Star condition

$$
\begin{tikzcd}
\check{X}^t \arrow[r, "\xi^t"] & \mathcal{G}_d(\mathbb{R}^m) \\
{|S^t|} \arrow[u] \arrow[r, dashed, "g"] & {|G|} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
S^t \arrow[r, dashed, "?"] & G
\end{tikzcd}
$$

The map g satisfies the *star condition* if:

for every vertex $v \in S^t$, there exists a vertex $w \in G$ such that $g\left(\left|\overline{\mathrm{St}}(v)\right|\right) \subseteq |\mathrm{St}(w)|$.



Star condition

$$\begin{tikzcd}
\check{X}^t \arrow[rr, "\xi^t"] & & \mathcal{G}_d(\mathbb{R}^m) \\
{|S^t|} \arrow[u] \arrow[rr, dashed, "g"] \arrow[rr, bend right, "{|f|}"'] & & {|G|} \arrow[u]
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S^t \xrightarrow{\quad f \quad} G$$

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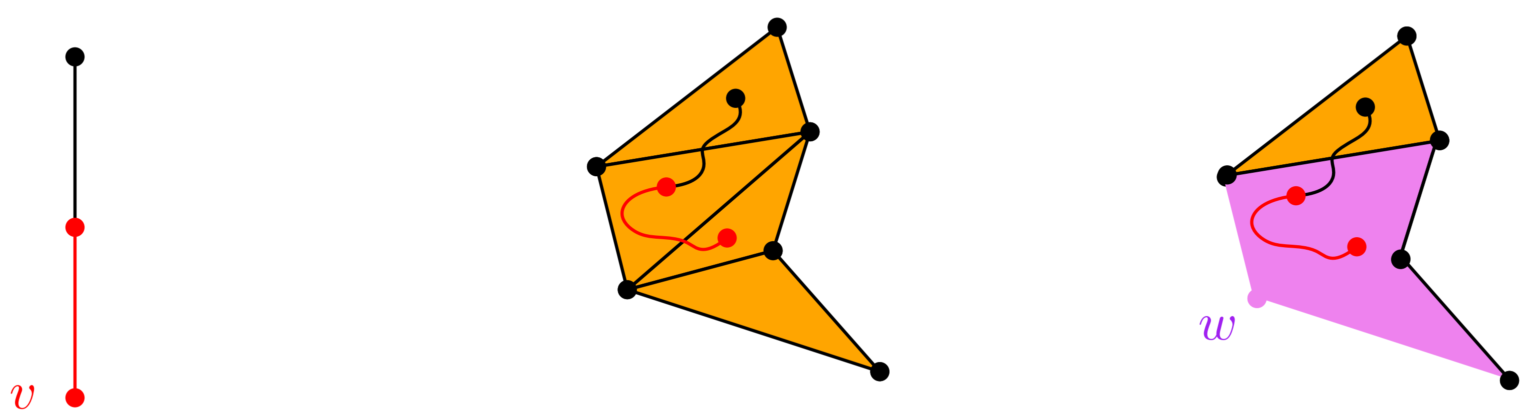
If this is the case, let $f\colon S^t \to G$ be any map between vertex sets such that:

for every vertex $v \in S^t$, we have $g\left(\left|\overline{\mathrm{St}}(v)\right|\right) \subseteq |\mathrm{St}(f(v))|$.

Such a map f is called a *simplicial approximation* to g. It is a simplicial map.
Its topological realization $|f|$ is homotopic to g.



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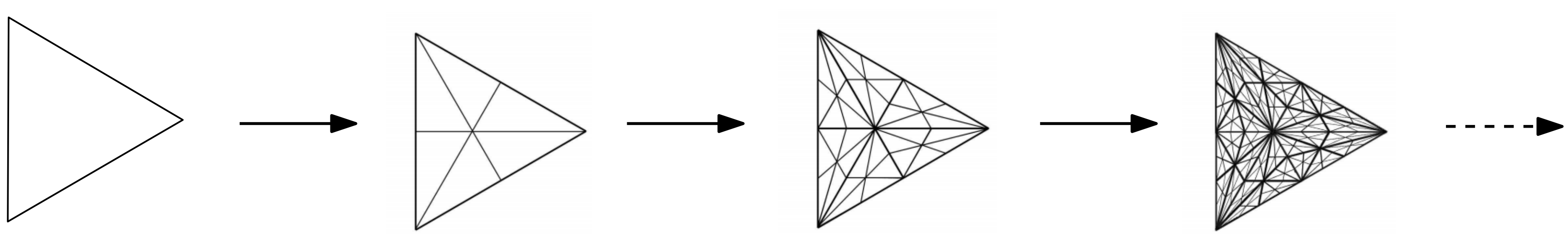
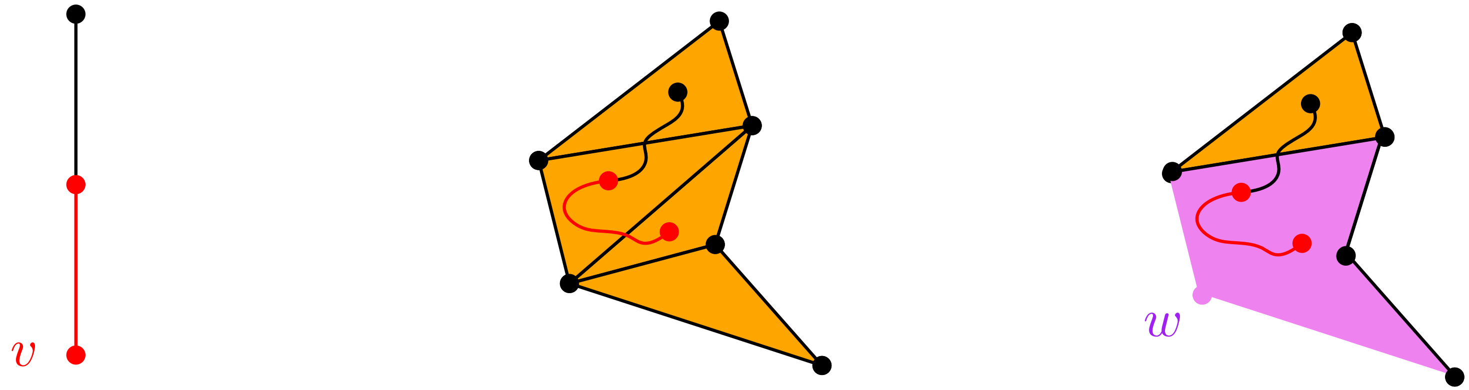
Remark:
If g does not satisfy the star condition, we can apply barycentric subdivisions to S^t.



Weak star condition

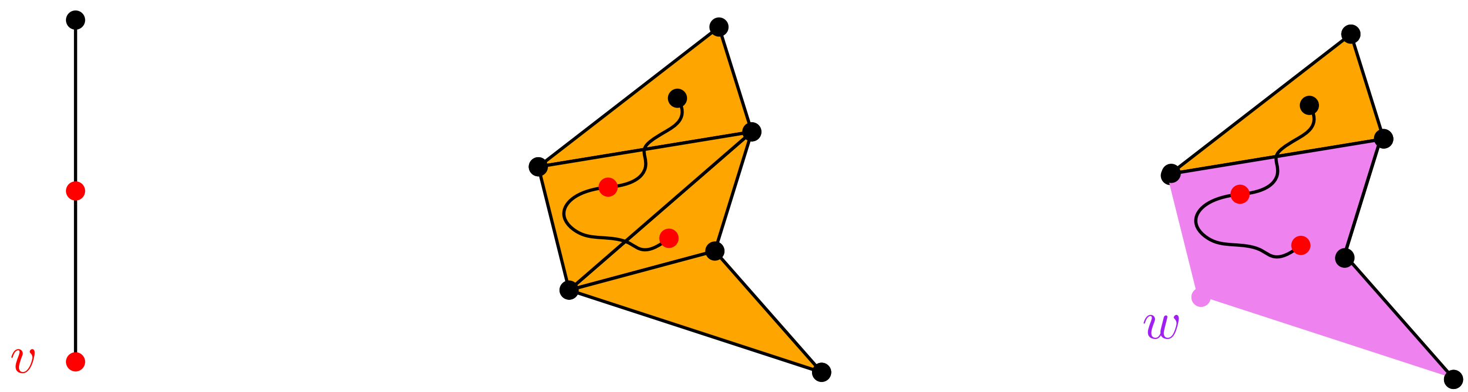
In practice, we cannot check whether g satisfies the star condition...



$$g\left(\left|\overline{\mathrm{St}}(v)\right|\right) \subseteq |\mathrm{St}(w)|?$$

The map g satisfies the *weak star condition* if:

for every vertex $v \in S^t$, there exists a vertex $w \in G$ such that $g\left(\left|\mathrm{vertices}(\overline{\mathrm{St}}(v))\right|\right) \subseteq |\mathrm{St}(w)|$.



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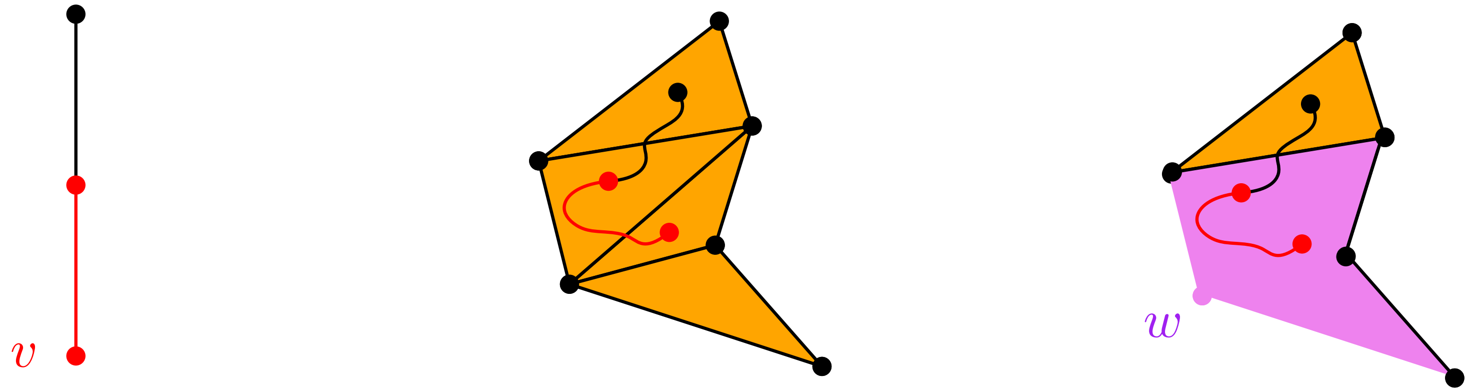

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> for every vertex $v \in S^t$, there exists a vertex $w \in G$ such that $g\left(\left|\mathrm{vertices}(\overline{\mathrm{St}}(v))\right|\right) \subseteq |\mathrm{St}(w)|$.

If this is the case, let $f\colon S^t \to G$ be any map between vertex sets such that:

> for every vertex $v \in S^t$, we have $g\left(\left|\mathrm{vertices}(\overline{\mathrm{St}}(v))\right|\right) \subseteq |\mathrm{St}(f(v))|$.

Such a map f is called a *weak simplicial approximation* to g. It is a simplicial map.

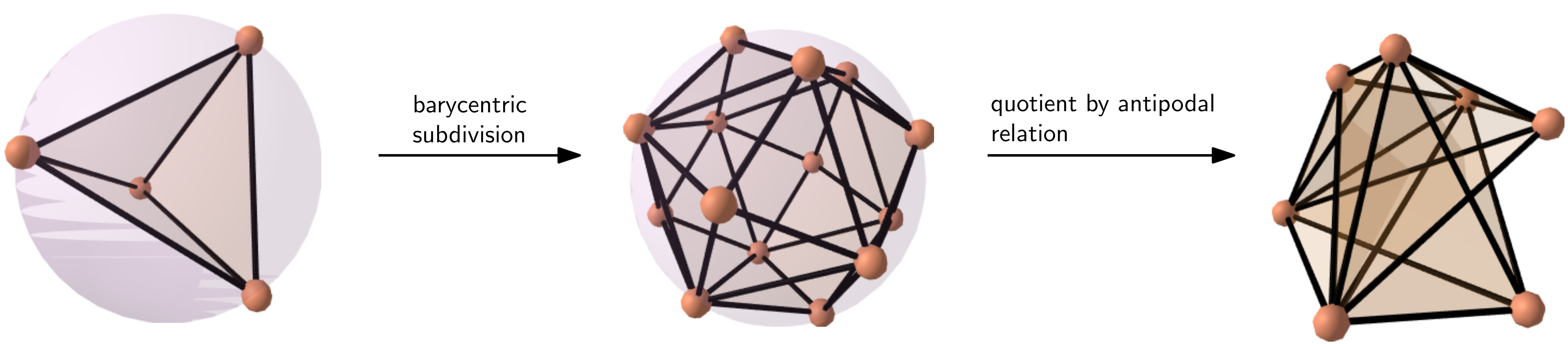
Proposition:
If S^t is subdivised enough, then any weak simplicial approximation is a simplicial approximation.

Triangulations of the Grassmannian

The Grassmaniann $\mathcal{G}_d(\mathbb{R}^n)$ has a well-known CW-complex structure.

However, I had some troubles finding explicit triangulations of $\mathcal{G}_d(\mathbb{R}^n)$.

What is known: triangulations of $\mathcal{G}_1(\mathbb{R}^n)$, the *projective spaces*.



boundary of n-simplex

$\mathcal{G}_1(\mathbb{R}^n)$

Triangulations of the Grassmannian

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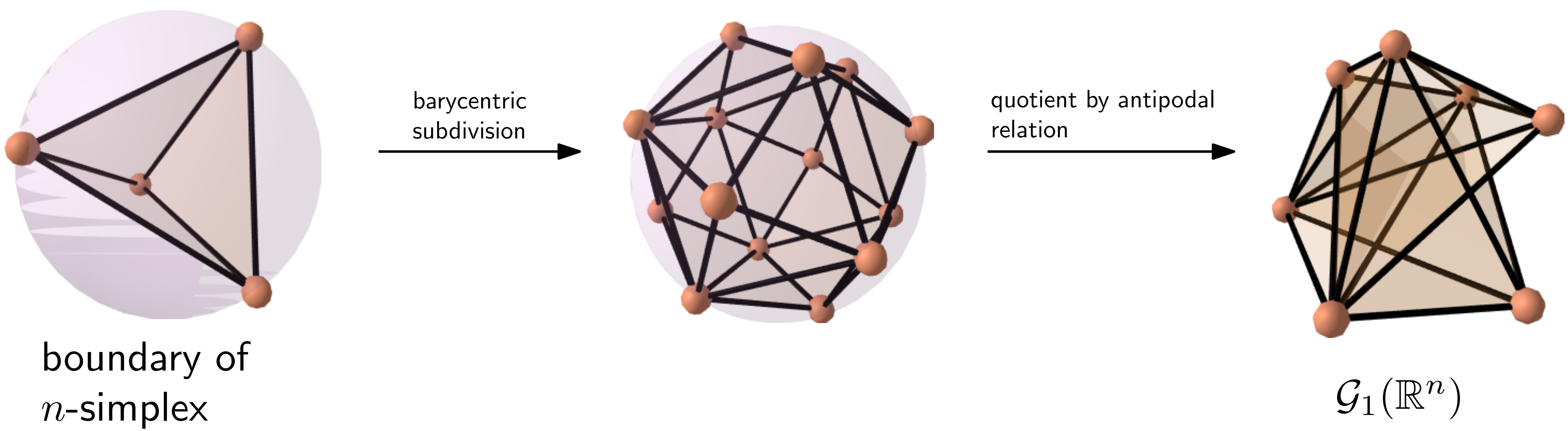


In practice, we will only consider the case $d = 1$.

An algorithm for $w_1(\xi^t)$, t fixed

Consider the map $\xi^t\colon \check{X}^t \to \mathcal{G}_1(\mathbb{R}^m)$. We want to compute $w_1(\xi^t) = (\xi^t)^*(w_1)$.

Reminder: $H^1(\mathcal{G}_1(\mathbb{R}^m)) = \langle w_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

We have to find the image of $(\xi^t)^*\colon H^1(X^t) \leftarrow H^1(\mathcal{G}_1(\mathbb{R}^m))$

- Compute a triangulation S^t of $\check{X}^t$
- Compute a triangulation G of $\mathcal{G}_1(\mathbb{R}^m)$
- Check whether ξ^t satisfies the weak star condition
- If not, subdivise barycentric
- Compute a weak simplicial approximation f to ξ^t
- Compute the induced map in simplicial cohomology $f^*\colon H^1(S^t) \leftarrow H^1(G)$

$\longrightarrow$ The image of f^* is $w_1(\xi^t)$
(seen in simplicial cohomology)

Computing the lifebar

Let $\check{X} \subset \mathbb{R}^n \times \mathcal{G}_d(\mathbb{R}^m)$,
$(\check{X}^t)_t, (\xi^t)_t$ its Čech bundle filtration,
$(w_i(\xi^t))_t$ its i^{th} persistent Stiefel-Whitney class.

We have seen how to compute $w_1(\xi^t)$, t fixed.

Recall that the lifebar of $w_1(X)$ is the set

$$\left\{t < t_{\mathsf{max}}, w_1(\xi^t) \neq 0\right\}.$$

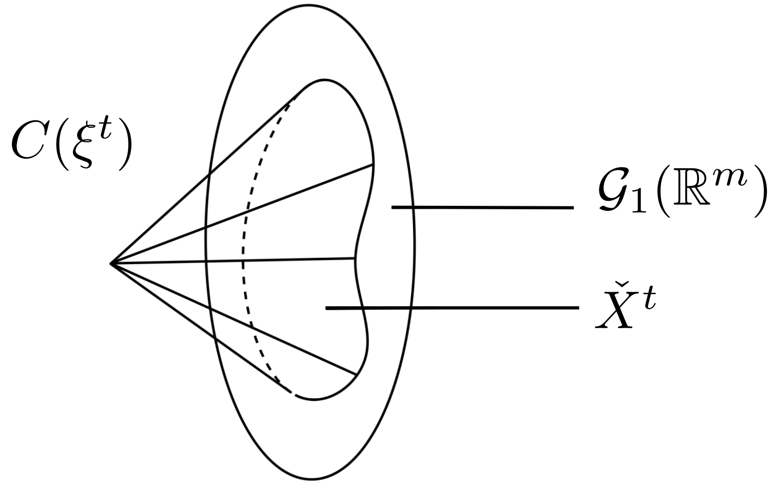
Three possibilities for computing the lifebar:

- Compute $w_1(\xi^t)$ for several values of t, and check whether $w_1(\xi^t) = 0$ (dichotomic search)
- Use the persistent image algorithm of [Cohen-Steiner, Edelsbrunner, Harer, Morozov]
- Use the formula on the next page

Computing the lifebar

Reminder: $H^1(\mathcal{G}_1(\mathbb{R}^m)) = \mathbb{Z}/2\mathbb{Z}$.
We have to find the image of $(\xi^t)^*\colon H^1(\check{X}^t) \leftarrow H^1(\mathcal{G}_1(\mathbb{R}^m))$

Let $C(\xi^t)$ be the mapping cone of $\xi^t\colon \check{X}^t \to \mathcal{G}_1(\mathbb{R}^m)$.



We have a long exact sequence

$$\dots \longrightarrow H^k(\check{X}^t) \longrightarrow H^{k+1}(C(\xi^t)) \longrightarrow H^{k+1}(\mathcal{G}_1(\mathbb{R}^m)) \longrightarrow H^{k+1}(\check{X}^t) \longrightarrow \dots$$

We deduce that

$$\mathrm{rank}((\xi^t)^*) = \sum_{k=1}^{+\infty} (-1)^k \Big(\dim H^k(\check{X}^t) - \dim H^{k+1}(C(\xi^t)) + \dim H^{k+1}(\mathcal{G}_1(\mathbb{R}^m)) \Big)$$

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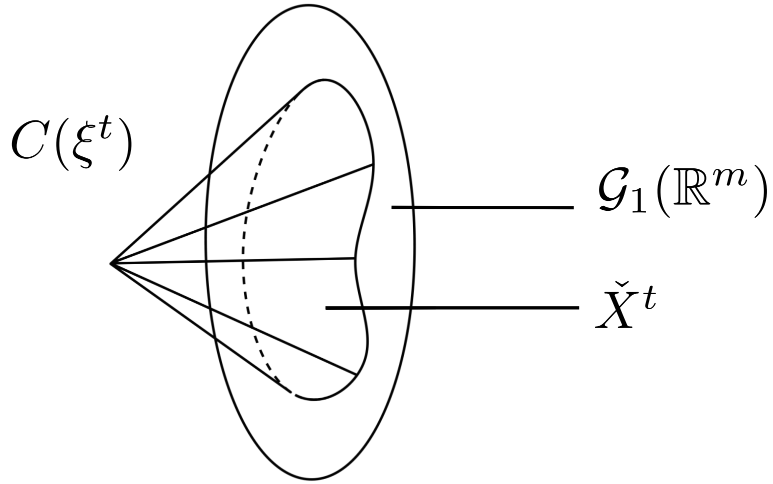

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$\longrightarrow$ Can be computed with the persistence algorithm

Conclusion

- We defined persistent Stiefel-Whitney classes,
- Proved stability and consistency results,
- Proposed an algorithm when $d = 1$. All of this is implemented in the package https://github.com/raphaeltinarrage/velour

Perspectives:

- Ideas could be extended to other characteristic classes (Euler, Chern, Pontrjagin).
- Need for a triangulation of $\mathcal{G}_d(\mathbb{R}^m)$.

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Thank you!