EMAp SUMMER COURSE

4th January ∼ 15th February, 2023

# GENERAL AND COMBINATORIAL TOPOLOGY

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**Abstract.** This course is intended for a 3rd year graduate student with no background on topology. The present document is a collection of notes for each lesson.

**Course webpage.** Various information (schedule, homework) are gathered on `https://raphaeltinarrage.github.io/EMApTopology.html`.

**Homeworks.** Exercises with a vertical segment next to them are your homework. Here is the first one:

**Exercise 0.** Send me an email answering the following questions:

- Do you understand English well?
- Have you ever studied topology?
- Have you ever coded? In which language?
- Any remarks?

Last update: 1st February, 2023

# Contents


## I General topology 3

1 Topological spaces . . . 3
  1.1 Topologies . . . 3
  1.2 Euclidean topology . . . 7
  1.3 Construction of topologies . . . 9
2 Separation and connectedness . . . 13
  2.1 Neighborhoods, interior, closure, boundary . . . 13
  2.2 Separation . . . 15
  2.3 Connectedness . . . 16
3 Continuity and homeomorphisms . . . 18
  3.1 Continuous maps . . . 18
  3.2 Homeomorphisms . . . 20
  3.3 Invariants of homeomorphism classes . . . 22
4 Homotopy equivalence . . . 27
  4.1 Homotopies . . . 27
  4.2 Invariants of homotopy classes . . . 32
  4.3 Algebraic-homotopy invariants . . . 35
5 Metric topology . . . 38
  5.1 Metric spaces . . . 38
  5.2 Examples . . . 42
  5.3 Geodesics . . . 44
6 Limits and completeness . . . 46
  6.1 Limits . . . 46
  6.2 Complete spaces . . . 48
  6.3 Contractions and fixed-points . . . 51
7 Compactness . . . 53
  7.1 Compact spaces . . . 53
  7.2 Compactness and continuity . . . 55
  7.3 Locally compact spaces . . . 56
8 Density . . . 57
  8.1 Dense sets . . . 57
  8.2 Baire spaces . . . 59
9 Functional topology . . . 61
  9.1 Topologies on function spaces . . . 61
  9.2 Hilbert spaces . . . 64

## Bibliography 68

# Chapter I

# General topology

## 1 Topological spaces

In this section, we will introduce the basic vocabulary of topology: topological spaces, open and closed sets. We will study examples of topologies on finite sets, as well as on $\mathbb{R}^n$. In order to build topologies, we will define the notion of generated topologies. This will allow us to build the Euclidean topology, and the product topology. We will also define the subspace and quotient topologies.

In order to prepare this section, I drew inspiration from [1]. We won't introduce some useful notions, such as neighborhoods, initial and final topologies, as well as basis of open sets. The reader may refer to [2] for an extensive presentation.

### 1.1 Topologies

**§1.1.1 Open sets.** Topological spaces are abstractions of the concept of 'shape' or 'geometric object'. We start by defining them via open sets.

**Definition 1.1.** A *topological space* is a pair $(X, \mathscr{T})$ where $X$ is a set and $\mathscr{T}$ is a collection of subsets of $X$ such that:

1. $\emptyset \in \mathscr{T}$ and $X \in \mathscr{T}$,
2. for every (potentially infinite) collection $(O_\alpha)_{\alpha \in A} \subset \mathscr{T}$, we have $\bigcup_{\alpha \in A} O_\alpha \in \mathscr{T}$,
3. for every finite collection $(O_i)_{1 \leq i \leq n} \subset \mathscr{T}$, we have $\bigcap_{1 \leq i \leq n} O_i \in \mathscr{T}$.

The set $\mathscr{T}$ is called a *topology* on $X$. In these notes, we will use the symbol $\mathscr{P}(X)$ to denote the powerset of $X$, that is, the set of subsets of $X$. It follows that a topology on $X$ is an subset of $\mathscr{P}(X)$, i.e., $\mathscr{T} \subset \mathscr{P}(X)$.

The elements of $\mathscr{T}$ are called the *open sets*. With this vocabulary, the previous definition can be reformulated as follows;

1. the empty set is an open set, the set $X$ itself is an open set,
2. an infinite union of open sets is an open set,
3. a finite intersection of open sets is an open set.

In general, in a given topology, an infinite intersection of open sets may not be open. An example is given in Exercise 5. However, when $X$, is finite, this statement is true.

**§1.1.2 Closed sets.** For every open set $O \in \mathscr{T}$, its complementary ${}^cO = \{x \in X \mid x \notin O\}$ is called a *closed set*. In other words, a subset $P \subset X$ is closed if and only if its complementary is open. As a direct consequence of Definition 1.1, one proves the following:

**Proposition 1.2.** *We have:*

1. *the sets $\emptyset$ and $X$ are closed sets,*
2. *for every (potentially infinite) collection $(P_\alpha)\alpha \in A$ of closed set, $\bigcap_{\alpha\in A} P_\alpha$ is a closed set,*
3. *for every finite collection $(P_i)_{1\leq i\leq n}$ of closed sets, $\bigcup_{1\leq i\leq n} P_i$ is a closed set.*

**Proof.** Point 1. The set $\emptyset$ is closed because ${}^c\emptyset = X$ is open, according to Point 1 of Definition 1.1. Same for $X$ since ${}^cX = \emptyset$ is open.

Point 2. If $(P_\alpha)_{\alpha\in A}$ is an infinite collection of closed set, then for every $\alpha \in A$, ${}^cP_\alpha$ is open. Now, we use the relation (known as De Morgan's law)

$$ {}^c\left(\bigcap_{\alpha\in A} P_\alpha\right) = \bigcup_{\alpha\in A} {}^cP_\alpha. $$

This is a union of open sets, hence it is open by Point 2 of Definition 1.1. Hence $\bigcap_{\alpha\in A} P_\alpha$ is closed.

Point 3. Just as previously, if $(P_i)_{1\leq i\leq n}$ is a finite collection of closed set, then each $i \in [\![1,n]\!]$, ${}^cP_i$ is open. We have the relation

$$ {}^c\left(\bigcup_{1\leq i\leq n} P_i\right) = \bigcap_{1\leq i\leq n} {}^cP_i. $$

This is a *finite* intersection of open sets, hence it is open by Point 3 of Definition 1.1. Hence $\bigcup_{1\leq i\leq n} P_i$ is closed. ☐

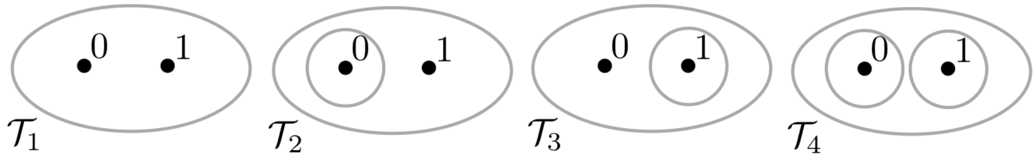
Note that the converse of Proposition 1.2 is true: if $\mathscr{T}'$ is a collection of sets satisfying 1, 2 and 3, then the collection of complementaries $\mathscr{T} = \{{}^cP \mid P \in \mathscr{T}'\}$ satisfies the axioms of Definition 1.1. Therefore, this proposition can serve as an alternative definition for topological spaces. We say that we define a topology via its closed sets.

**Example 1.3.** Let $X = \{0\}$ be a set with one element. There exists only one topology on $X$: $\mathscr{T} = \{\emptyset, \{0\}\}$.

**Example 1.4.** Let $X$ be any set. The subset $\mathscr{T} = \{\emptyset, X\}$ is a topology on $X$, called the *trivial topology*. Likewise, the power set of $X$, denoted $\mathscr{P}(X)$, is a topology on $X$, called the *discrete topology*.

**Example 1.5.** Let $X = \{0,1\}$ be a set with two elements. There exists only four different topologies on $X$:

$$ \mathscr{T}_1 = \{\emptyset, \{0,1\}\}, \quad \mathscr{T}_2 = \{\emptyset, \{0\}, \{0,1\}\}, \quad \mathscr{T}_3 = \{\emptyset, \{1\}, \{0,1\}\}, \quad \mathscr{T}_4 = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}. $$




**Example 1.6.** Let $X = \{0,1,2\}$ be a set with three elements. The set $\mathscr{T} = \{\emptyset\}$ is not a topology on $X$ because the whole set $X = \{1,2,3\}$ does not belong to $\mathscr{T}$. Likewise, the set

$$\mathscr{T} = \{\emptyset, \{0\}, \{1\}, \{0,1,2\}\}$$

is not a topology on $X$ because the finite union $\{0\} \cup \{1\} = \{0,1\}$ does not belong to $\mathscr{T}$.

**Example 1.7.** The set

$$\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{[0,a] \mid a \geq 0\}$$

is not a topology on $\mathbb{R}$. Indeed, the following union of open sets is not an open set:

$$\bigcup_{a \geq 0} [0,a] = [0,+\infty).$$

**§1.1.3 Comparison of topologies.** As illustrated in Example 1.4, any set $X$ of cardinal greater than 1 admits several different topologies. We shall compare them as follows.

**Definition 1.8.** Consider two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on $X$. If $\mathscr{T}_1 \subset \mathscr{T}_2$, we say that $\mathscr{T}_1$ is *coarser* than $\mathscr{T}_2$, and that $\mathscr{T}_2$ is *finer* than $\mathscr{T}_1$.

In other words, $\mathscr{T}_2$ is finer than $\mathscr{T}_1$ if it has 'more open sets'. The relation 'being coarser' is a partial ordering on the set of all topologies on $X$. Using this relation, we can represent the set of all topologies on $X$ as lattice, as drawn below for the case of Example 1.5. It will be called the *lattice of topologies on $X$*. In these notes, when $A$ and $B$ are sets such as $A \subset B$, the map $A \hookrightarrow B$ will denote the inclusion map.

$$\begin{tikzcd}
 & \{\emptyset, \{0\}, \{0,1\}\} \arrow[dr, hook] & \\
\{\emptyset, \{0,1\}\} \arrow[ur, hook] \arrow[dr, hook] & & \{\emptyset, \{0\}, \{1\}, \{0,1\}\} \\
 & \{\emptyset, \{1\}, \{0,1\}\} \arrow[ur, hook] &
\end{tikzcd}$$

Note that the relation 'being coarser' admits a lowest element (that is, an element that is coarser than any other): the trivial topology. Similarly, it admits a greatest element: the discrete topology. In the language of partially ordered sets, we say that this lattice is *bounded*.

**§1.1.4 Intersection and union of topologies.** We now wish to build new topologies, based on an arbitrary collection $\{\mathscr{T}_\alpha\}_{\alpha \in A}$ on $X$. The easiest construction is the intersection $\bigcap_{\alpha \in A} \mathscr{T}_\alpha$.

**Proposition 1.9.** *An arbitrary intersection of topologies on $X$ is a topology.*

**Proof.** It follows directly from Definition 1.1. □

The intersection topology $\bigcap_{\alpha\in A}\mathscr{T}_\alpha$ has the property that is the greatest topology included in all the $\mathscr{T}_\alpha$. In other words, if $\mathscr{T}$ is any topology on $X$ such that $\mathscr{T}\subset\mathscr{T}_a$ for all $\alpha\in A$, then we must have $\mathscr{T}\subset\bigcap_{a\in A}\mathscr{T}_a$. In the language of partially ordered sets, we say that the lattice of topologies has the greatest lower bound property.

As a dual construction, one would be tempted to consider the union $\bigcup_{a\in A}\mathscr{T}_a$. However, it may not be a topology. One should instead consider the following notion.

**Definition 1.10.** Let $S\subset\mathscr{P}(X)$ be any subset. The *topology generated* by $S$ is defined as the intersection of all the topologies on $X$ that contain $S$. It is denoted $\mathscr{T}(S)$.

Using Proposition 1.9, we have that $\mathscr{T}(S)$ is a topology on $X$. Moreover, it is the smallest topology included in all the $\mathscr{T}_\alpha$. That is to say, if $\mathscr{T}$ is any topology on $X$ such that $\mathscr{T}\supset\mathscr{T}_a$ for all $\alpha\in A$, then we must have $\mathscr{T}\supset\mathscr{T}(S)$. We say that the lattice of topologies has the least upper bound property. The following proposition gives an alternative description of the generated topology.

**Proposition 1.11.** *For any $S\subset\mathscr{P}(X)$, the generated topology $\mathscr{T}(S)$ is the collection of arbitrary unions of finite intersections of element of $S$.*

**Proof.** Let $\mathscr{T}'$ denote the collection of arbitrary unions of finite intersections of element of $S$. As a direct consequence of Definition 1.1, one shows that $\mathscr{T}'$ is a topology on $X$. Moreover, since the generated topology $\mathscr{T}$ is a topology, it must contain $\mathscr{T}'$. But since $\mathscr{T}$ is the smallest topology containing $S$, we deduce that $\mathscr{T}'=\mathscr{T}$. □

**Exercise 1** (Enumeration of topologies)**.** Let $X=\{0,1,2\}$ be a set with three elements. How many different topologies does $X$ admit?

*Remark:* Let $t(n)$ be the number of different topologies on a set with $n$ elements. One obtains directly the bound $2\le t(n)\le 2^{2^n}$ for $n\ge 3$. The lower bounds comes from the fact that the trivial and discrete topologies are topologies, and the upper bound from the fact that a topology on $X$ is an element of $\mathscr{P}(\mathscr{P}(X))$. A more involved bound can be found in [3]: $2^n\le t(n)\le 2^{n(n-1)}$.

**Exercise 2.** Let $X$ be a finite set, and $\mathscr{T}$ a topology on $X$ such that all the singletons $\{x\}$, $x\in X$, are closed. Show that $\mathscr{T}$ is the discrete topology.

**Exercise 3** (Hausdorff separability)**.** We say that a topological space $(X,\mathscr{T})$ is Hausdorff (or is a $T_2$-space) if for any $x,y\in X$ such that $x\neq y$,, there exists two open sets $U,V\in\mathscr{T}$ such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. Among the topologies on $X=\{0,1\}$ described in Example 1.5, which ones are Hausdorff?

**Exercise 4.** Show that the following set is a topology on $\mathbb{R}$:

$$\mathscr{T}=\{\emptyset,\mathbb{R}\}\cup\{(-a,a)\mid a>0\}.$$

*Hint:* Remind the least-upper-bound property of the real numbers.

**Exercise 5** (Cofinite topology)**.** Let $\mathbb{Z}$ be the set of integers. Consider the *cofinite topology* $\mathscr{T}$ on $\mathbb{Z}$, defined as follows: a subset $O\subset\mathbb{Z}$ is an open set if and only if $O=\emptyset$ or ${}^cO$ is finite.

1. Show that $\mathscr{T}$ is a topology on $\mathbb{Z}$.
2. Exhibit an sequence of open sets $\{O_n\}_{n\in\mathbb{N}} \subset \mathscr{T}$ such that $\bigcap_{n\in\mathbb{N}} O_n$ is not open.

*Remark:* If the set $X$ is finite, then the cofinite topology is the discrete topology.

**Exercise 6** (Zariski topology)**.** A subset $F \subset \mathbb{R}^n$ is a Zariski-closed set if it can be written as

$$F = \{x \in \mathbb{R}^n \mid \forall \alpha \in A,\ P_\alpha(x) = 0\}$$

where $(P_\alpha)_{\alpha\in A}$ is a (potentially infinite) collection of multivariate polynomials on $\mathbb{R}^n$. Show that the collection of Zariski-closed sets forms the collection of closed sets of a topology on $\mathbb{R}^n$, called Zariski topology.

*Remark:* Actually, as a consequence of Hilbert's Nullstellensatz, any Zariski-closed set can be written as the set of common roots of a *finite* family of polynomials.

**Exercise 7** (Fifth proof of the infinity of primes from [4])**.** For any $a, b \in \mathbb{Z}$, define

$$N_{a,b} = \{a + bn \mid n \in \mathbb{Z}\}.$$

Call a subset $O \subset \mathbb{Z}$ closed if either $O =$ or if to every $a \in Z$ there exists a $b > 0$ such that $N_{a,b} \subset O$. Let $\mathscr{T}$ denote the collection of all open sets.

1. Show that $\mathscr{T}$ is a topology on $\mathbb{Z}$.
2. Show that any nonempty open set is infinite, and that the $N_{a,b}$ are closed sets.
3. Let $\mathbb{P}$ denotes the set of all prime numbers. Show that $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{b\in\mathbb{P}} N_{0,b}$.
4. By contradiction, use 2. and 3. to deduce that $\mathbb{P}$ is infinite.

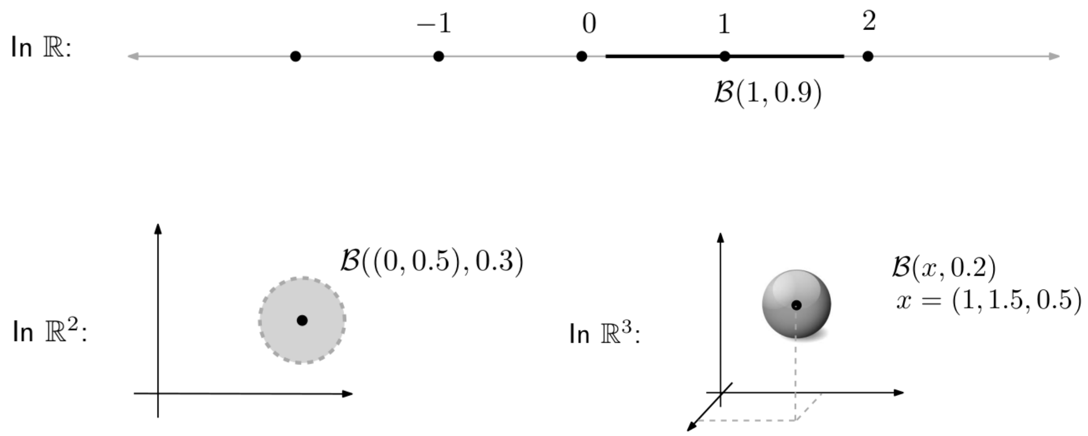
## 1.2 EUCLIDEAN TOPOLOGY

Many topological spaces encountered in practice are subsets of the Euclidean spaces $\mathbb{R}^n$. On $\mathbb{R}^n$, we will mainly consider the *Euclidean topology*. In order to define this topology, we will use open balls. Remind that the Euclidean metric on $\mathbb{R}^n$ is defined for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as:

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

For $x \in \mathbb{R}^n$ and $r > 0$, the *open ball* of center $x$ and radius $r$, denoted $\mathscr{B}(x, r)$, is defined as:

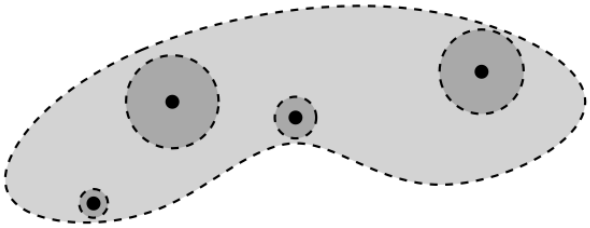
$$\mathscr{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}.$$



**Definition 1.12.** The *Euclidean topology* on $\mathbb{R}^n$, denoted $\mathscr{T}_{\mathbb{R}^n}$, is the topology generated by the balls $\{B(x,t) \mid x \in \mathbb{R}^n,\ t > 0\}$.

According to the discussion of §1.1.3, the Euclidean topology is the smallest topology that contains all the open balls. We now give an alternative definition of it, often more convenient to identify open sets.

**Proposition 1.13.** *A set $A$ is open (for the Euclidean topology) if and only if for every $x \in A$, there exists a $r > 0$ such that $\mathscr{B}(x,r) \subset A$.*



**Proof.** It will be convenient to use the following vocabulary: $A$ is *open around* $x$ if there exists $r > 0$ such that $\mathscr{B}(x,r) \subset A$. Note that the proposition states that $A$ is open if and only if it is open around all of its points. Let us denote by $\mathscr{U}_{\mathbb{R}^n}$ the set of all subsets $A \subset \mathbb{R}^n$ that are open around all of their points, that is,

$$\mathscr{U}_{\mathbb{R}^n} = \{A \subset \mathbb{R}^n \mid \forall x \in A,\ \exists r > 0,\ \mathscr{B}(x,r) \subset A\}.$$

In what follows, we will say that a subset $A \subset \mathbb{R}^n$ is $\mathscr{T}_{\mathbb{R}^n}$-closed (resp. $\mathscr{U}_{\mathbb{R}^n}$-closed) if it belongs to $\mathscr{T}_{\mathbb{R}^n}$ (resp. $\mathscr{U}_{\mathbb{R}^n}$). The proof consists in showing that $\mathscr{U}_{\mathbb{R}^n} \subset \mathscr{T}_{\mathbb{R}^n}$, and that $\mathscr{U}_{\mathbb{R}^n}$ is a topology that contains the open balls. Using the fact that $\mathscr{T}_{\mathbb{R}^n}$ is the smaller topology that contains the open balls, it follows that $\mathscr{T}_{\mathbb{R}^n} = \mathscr{U}_{\mathbb{R}^n}$.

<u>First step: $\mathscr{U}_{\mathbb{R}^n} \subset \mathscr{T}_{\mathbb{R}^n}$.</u> Let $O \in \mathscr{U}_{\mathbb{R}^n}$. For any $x \in O$, let $r_x > 0$ be such that $\mathscr{B}(x,r_x) \subset O$. We have that $O = \bigcup_{x \in O} \mathscr{B}(x,r_x)$. Moreover, by definition, this union of open balls belongs to $\mathscr{T}_{\mathbb{R}^n}$. Hence $O \in \mathscr{T}_{\mathbb{R}^n}$, and we deduce that $\mathscr{U}_{\mathbb{R}^n} \subset \mathscr{T}_{\mathbb{R}^n}$.

<u>Second step: $\mathscr{U}_{\mathbb{R}^n}$ contains the open balls.</u> Let $x \in \mathbb{R}^n$ and $r > 0$. Consider the ball $\mathscr{B}(x,r)$. In order to show that it is $\mathscr{U}_{\mathbb{R}^n}$-open, we must show that it is open around all of its points. Consider $y \in \mathscr{B}(x,r)$, and define $r' = r - \|x-y\|$. We will show that $\mathscr{B}(y,r') \subset \mathscr{B}(x,r)$. To prove so, let $z \in \mathscr{B}(y,r')$. We apply the triangular inequality for the Euclidean norm:

$$\begin{aligned} \|z-x\| &\leq \|z-y\| + \|y-x\| \\ &\leq r' + \|y-x\| = r. \end{aligned}$$

We deduce that $\mathscr{B}(y,r') \subset \mathscr{B}(x,r)$, hence that $\mathscr{B}(x,r)$ belongs to $\mathscr{U}_{\mathbb{R}^n}$.

<u>Third step: $\mathscr{U}_{\mathbb{R}^n}$ is a topology.</u> We shall verify the three axioms of Definition 1.1.

- First axiom. Since $\emptyset$ contains no point, it is open around all of its points, hence belongs to $\mathscr{U}_{\mathbb{R}^n}$. The set $\mathbb{R}^n$ also is open, since for every $x \in \mathbb{R}^n$, the ball $\mathscr{B}(x,1)$ is a subset of $\mathbb{R}^n$.
- Second axiom. Let $\{O_\alpha\}_{\alpha \in A} \subset \mathscr{T}_{\mathbb{R}^n}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha \in A} O_\alpha$. Let $x \in O$. There exists an $\alpha \in A$ such that $x \in O_\alpha$. Since $O_\alpha$ is open, it is open

around $x$, i.e., there exists $r > 0$ such that $\mathscr{B}(x,r) \subset O_\alpha$. We deduce that $\mathscr{B}(x,r) \subset O$, and that $O$ is open around $x$. Since this it true for any $x \in O$, we proved that $O$ is open.

• Third axiom. Consider a finite collection $\{O_i\}_{1\leq i\leq n} \subset \mathscr{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1\leq i\leq n} O_i$. Let $x \in O$. For every $i \in [\![1,n]\!]$, we have $x \in O_i$. Since $O_i$ is open, it is open around $x$, i.e., there exists $r_i > 0$ such that $\mathscr{B}(x,r_i) \subset O_i$. Define $r_{\min} = \min\{r_1,...r_n\}$. For every $i \in [\![1,n]\!]$, we have $\mathscr{B}(x,r_{\min}) \subset O_i$. We deduce that $\mathscr{B}(x,r_{\min}) \subset O$, and that $O$ is open around $x$. Since this it true for any $x \in O$, we have proven that $O$ is open. □

**Example 1.14.** The interval $I = (0,+\infty)$ is an open set for the Euclidean topology on $\mathbb{R}$. Indeed, for any $x \in I$, the open ball $\mathscr{B}(x,x)$ is included in $I$.

**Example 1.15.** The interval $[0,1]$ is a closed set for the Euclidean topology on $\mathbb{R}$. Indeed, its complement ${}^c[0,1] = (-\infty,0)\cup(1,+\infty)$ is open, since it is the union of two open sets.

**Example 1.16.** Let $\mathscr{C} = \{x = (x_1,...,x_n) \in \mathbb{R}^n \mid \|x\|_\infty < 1\}$ be the filled open unit cube of $\mathbb{R}^n$, where $\|x\|_\infty = \max(|x_1|,...,|x_n|)$ is the sup norm. Let $x \in \mathscr{C}$, define $r = 1 - \|x\|_\infty$, and consider the open ball $\mathscr{B}(x,r)$. For any $y \in \mathscr{B}(x,r)$, we have

$$\|y\|_\infty = \max(|y_1|,...,|y_n|) < \max(|x_1|+r,...,|x_n|+r) \leq \max(|x_1|,...,|x_n|)+r \leq 1.$$

Therefore, $\|y\|_\infty < 1$, hence $\mathscr{B}(x,r) \subset \mathscr{C}$. This being true for any $x \in \mathscr{C}$, we deduce that $\mathscr{C}$ is open for the Euclidean topology.

> **Exercise 8.** Consider the real line $\mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?
>
> 1. the interval $[0,1)$,
> 2. the intervals $[x,+\infty)$, $x \in \mathbb{R}$,
> 3. the singletons $\{x\}$, $x \in \mathbb{R}$,
> 4. the rational numbers $\mathbb{Q}$.

**Exercise 9** (Sorgenfrey line)**.** Let $\mathscr{U}$ be the topology on $\mathbb{R}$ generated by the the collection

$$\{[a,b) \mid a,b \in \mathbb{R},\ a \leq b\}.$$

1. Show that $\mathscr{U}$ is finer than the Euclidean topology.
2. Show that for any $x \in \mathbb{R}$, the set $[x,+\infty)$ is open and closed.

## 1.3 Construction of topologies

### §1.3.1 Subspace topology

**Definition 1.17.** Let $(X,\mathscr{T})$ be a topological space, and $Y \subset X$ a subset. We define the *subspace topology on* $Y$ as the following set:

$$\mathscr{T}_{|Y} = \{O \cap Y \mid O \in \mathscr{T}\}.$$

**Proposition 1.18.** *The set* $\mathscr{T}_{|Y}$ *is a topology on* $Y$*.*

**Proof.** We have to check the three axioms of a topological space, as in Definition 1.1.

<u>First axiom.</u> The set $\emptyset$ is clearly open for $\mathscr{T}_{|Y}$ because it can be written as $\emptyset \cap Y$. The set $Y$ also is open for $\mathscr{T}_{|Y}$ because it can be written $X \cap Y$, and $X$ is open for $\mathscr{T}$.

<u>Second axiom.</u> Let $\{O_\alpha\}_{\alpha\in A} \subset \mathscr{T}_{|Y}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha\in A} O_\alpha$. By definition of $\mathscr{T}_{|Y}$, for every $\alpha \in A$, there exists $O'_\alpha$ such that $O_\alpha = O'_\alpha \cap Y$. Define $O' = \bigcup_{\alpha\in A} O'_\alpha$. It is an open set for $\mathscr{T}$. We have

$$O = \bigcup_{\alpha\in A} O_\alpha = \bigcup_{\alpha\in A} O'_\alpha \cap Y = \left(\bigcup_{\alpha\in A} O'_\alpha\right) \cap Y = O' \cap Y.$$

Hence $O \in \mathscr{T}_{|Y}$.

<u>Third axiom.</u> Consider a finite collection $\{O_i\}_{1\leq i\leq n} \subset \mathscr{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1\leq i\leq n} O_i$. Just as before, for every $i \in [\![1,n]\!]$, there exists $O'_i$ such that $O_i = O'_i \cap Y$. Define $O' = \bigcup_{1\leq i\leq n} O'_i$. It is an open set for $\mathscr{T}$. We have

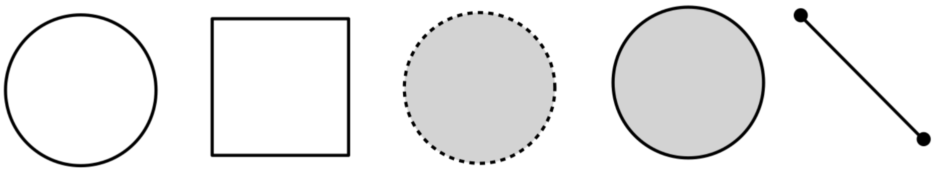
$$O = \bigcap_{1\leq i\leq n} O_\alpha = \bigcap_{1\leq i\leq n} O'_\alpha \cap Y = \left(\bigcap_{1\leq i\leq n} O'_\alpha\right) \cap Y = O' \cap Y.$$

Hence $O \in \mathscr{T}_{|Y}$. $\square$

Thanks to the subspace topology, any subset of $\mathbb{R}^n$ inherits a particular topology. Among the subsets of $\mathbb{R}^n$ that we will consider, let us list:

- the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$,
- the unit cube $\mathscr{C}_{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_\infty = 1\}$ where $\|x\|_\infty = \max(|x_1|, ..., |x_n|)$,
- the open balls $\mathscr{B}(x,r) = \{y \in \mathbb{R}^n \mid \|x-y\| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$,
- the closed balls $\overline{\mathscr{B}}(x,r) = \{y \in \mathbb{R}^n \mid \|x-y\| \leq r\}$ for $x \in \mathbb{R}^n$ and $r \geq 0$,
- the standard simplex

$$\Delta_{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1, ..., x_n \geq 0,\ x_1 + ... + x_n = 1\}.$$



**Exercise 10.** Consider the space $\mathbb{R}^n$ endowed with the Euclidean topology, and its unit sphere $\mathbb{S}^{n-1}$ endowed with the subspace topology. Define the upper hemisphere $\mathbb{S}^{n-1}_+ = \{x \in \mathbb{R}^n \mid \|x\| = 1,\ x_1 > 0\}$. Show that $\mathbb{S}^{n-1}_+$ in open in $\mathbb{S}^{n-1}$, but not in $\mathbb{R}^n$.

**Exercise 11** (Topologist's sine curve)**.** Consider the plane $\mathbb{R}^2$ endowed with the Euclidean topology. Define the set

$$X = \{(x, \sin(1/x) \mid x \in (0,\pi]\} \cup \{(0,0)\}$$

and endow it with the subspace topology. Show that the singleton $\{0\}$ is closed and not open.

**Exercise 12** (Cantor set)**.** Consider the Euclidean line $\mathbb{R}$. Let $C_0 = [0,1]$, $C_1 = [0,1/3]\cup[2/3,1]$, $C_2 = [0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]$, and in general, let $C_{n+1}$ be the union of the $2n+1$ closed intervals, each of length $(1/3)^{n+1}$, obtained by removing the open middle thirds of the $2^n$ closed intervals of $C_n$. We define

$$\mathscr{C} = \bigcap_{n\geq 0} C_n.$$

1. Show that $\mathscr{C}$ is a nonempty closed subset of $\mathbb{R}$.
2. Show that for all $x \in \mathscr{C}$, the singleton $\{x\}$ is open for the subspace topology on $\mathscr{C}$.

### §1.3.2 (FINITE) PRODUCT TOPOLOGY

**Definition 1.19.** Let $((X_\alpha, \mathscr{T}_\alpha))_{\alpha\in A}$ be a collection of topological spaces. We denote by $\Pi_{\alpha\in A}\mathscr{T}_\alpha$ the topology generated by the sets $\Pi_{\alpha\in A}O_\alpha$ where $O_\alpha \in \mathscr{T}_\alpha$ for all $\alpha \in A$. If $A$ is finite, it is called the *product topology*, and when it is infinite, it is called the *box topology*.

**Remark 1.20.** In the context where $A$ is infinite, the term *product topology* is reserved for another topology, that we will study in more details in the section about functional topology.

**Exercise 13.** Let $\mathbb{R}$ be endowed with the Euclidean topology $\mathscr{T}(\mathbb{R})$. Show that the product topology on $\mathbb{R}\times\cdots\times\mathbb{R}$ is equal to the Euclidean topology $\mathscr{T}(\mathbb{R}^n)$ on $\mathbb{R}^n$.

**Exercise 14.** Let $(X,\mathscr{T})$, $(Y,\mathscr{U})$ be two topological spaces. Show that if $A$ is a closed set of $X$ and $B$ a closed set of $Y$, then $A\times B$ is a closed set of the product topology.

**Exercise 15.** Let $(X,\mathscr{T})$ be a topological space, and consider the product topology on $X\times X$. Show that $(X,\mathscr{T})$ is Hausdorff (in the sense of Exercise 3) if and only if the diagonal $\Delta = \{(x,x) \mid x\in X\}$ is closed in $X\times X$.

### §1.3.3 QUOTIENT TOPOLOGY

If $X$ is any set, we remind the reader that an *equivalence relation* on $X$ is a binary relation, denoted $\mathscr{R}$, which satisfies:

$$\begin{aligned}
&\textit{(reflexivity)}\ \forall x\in X,\ x\mathscr{R}x,\\
&\textit{(symmetry)}\ \forall x,y\in X,\ x\mathscr{R}y \iff y\mathscr{R}x,\\
&\textit{(transitivity)}\ \forall x,y,z\in X,\ (x\mathscr{R}y\ \text{ and }\ y\mathscr{R}z) \implies x\mathscr{R}z.
\end{aligned}$$

For any $x\in X$, we define its equivalence class as $\mathscr{O}_x = \{y\in X \mid x\mathscr{R}y\}$. Using the fact that $\mathscr{R}$ is an equivalence relation, we deduce the following fact: $x\mathscr{R}y \iff \mathscr{O}_x = \mathscr{O}_y$. As a consequence, the set of equivalence classes form a partition of $X$. It is denoted $X/\mathscr{R}$, and is called the *quotient set*. We denote the *projection map* as

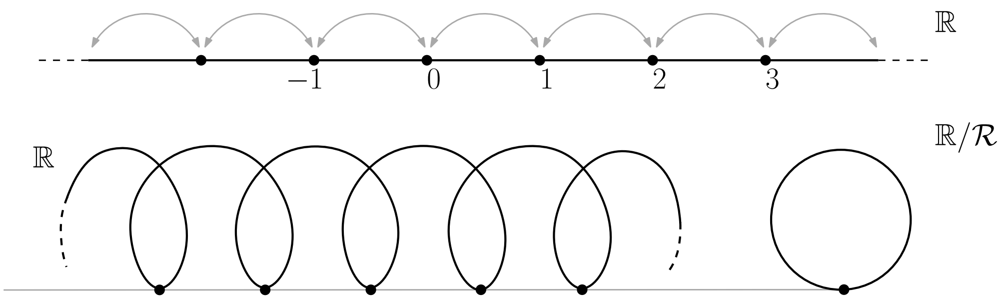
$$\begin{aligned}
\pi\colon X &\longrightarrow X/\mathscr{R}\\
x &\longmapsto \mathscr{O}_x
\end{aligned}$$

**Definition 1.21.** Let $(X,\mathscr{T})$ be topological space and $\mathscr{R}$ an equivalence relation on $X$. The *quotient topology* on $X/\mathscr{R}$ is defined as the topology whose open sets are the subsets $O\subset X/\mathscr{R}$ such that $\pi^{-1}(O)\in\mathscr{T}$.

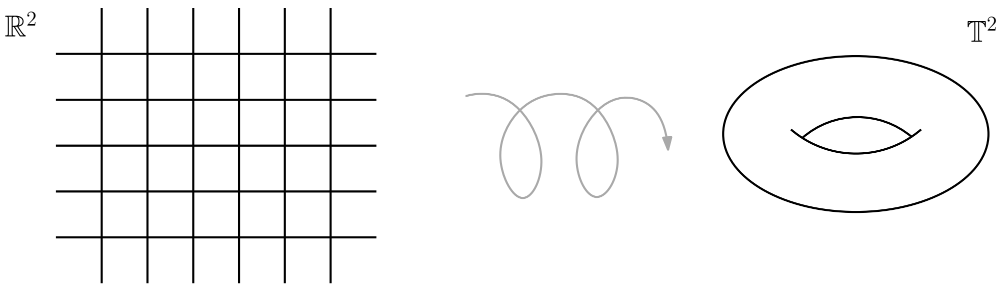
**Remark 1.22.** It is also called the *final topology* with respect to the map $\pi$.

The quotient topology gives a handy way to build new topological spaces. While quotienting the space, we 'merge', or 'identify' points that are in the same equivalence class.

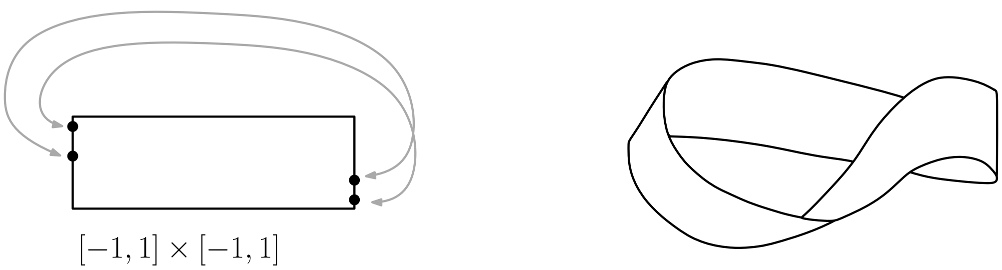
**Example 1.23** (Circle)**.** Let $\mathbb{R}$ be the real line endowed with the Euclidean topology, and consider the relation $x\mathscr{R}y \iff x-y\in\mathbb{Z}$. The equivalence classes are the sets $\mathscr{O}_x=\{x+n \mid n\in\mathbb{Z}\}$, and the quotient space $\mathbb{R}/\mathscr{R}$ can be identified with the interval $[0,1)$. While quotienting $\mathbb{R}$, we 'roll it up on itself'. The quotient topology is the one of a circle. In order to give rigorous sense to this last sentence, we will have to wait until Section 3.




**Example 1.24** (Torii)**.** More generally, the equivalence relation $x\mathscr{R}y \iff \forall i\leq n,\ x_i-y_i\in\mathbb{Z}$ on the Euclidean space $\mathbb{R}^n$ give rise to the *torus* of dimension $n$. It is denoted $\mathbb{T}^n$.



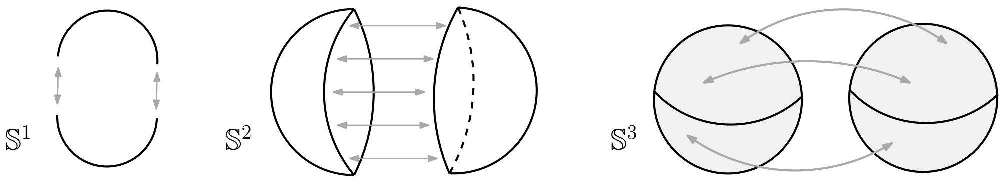

**Example 1.25** (Möbius strip)**.** Let $[-1,1]\times[-1,1]$ be the square of $\mathbb{R}^2$, endowed with the subspace topology. Consider the equivalence relation generated by $(x,y)\mathscr{R}(x',y') \iff |x|=1,\ y=-x,\ x'=-y'$. The quotient topological space is called the *Möbius strip*. The construction consists in gluing the opposite sides of a square, reversing the direction.




**Example 1.26** (Projective spaces)**.** Let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$, endowed with the subspace topology. The *antipodal relation* $x\mathscr{R}y \iff x=-y$ is an equivalence relation on $\mathbb{S}^{n-1}$. The quotient topological space is called the *real projective space* of dimension $n-1$, and is denoted $P^{n-1}\mathbb{R}$. The first projective space $P^1\mathbb{R}$ actually is a circle (make a drawing).

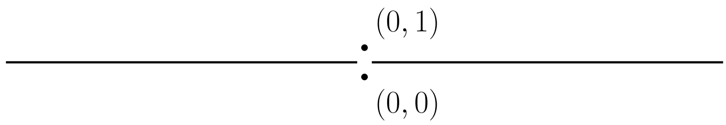
Quotient topology also allows to give a rigorous sense to the idea of 'gluing'. If $(X,\mathscr{T})$ and $(Y,\mathscr{U})$ are two topological spaces, $A \subset X$ a subset and $f\colon A \to Y$ a map, the *gluing of $A$ onto $B$ along $f$* is the quotient of the disjoint union $X \sqcup Y$ by the equivalence relation generated by $x \mathscr{R} f(x)$ for all $x \in A$. It is denoted $(X \sqcup Y)/f$.

**Example 1.27** (Spheres are gluing of disks)**.** Consider two copies $\overline{\mathscr{B}}_1, \overline{\mathscr{B}}_2$ of the unit closed ball $\overline{\mathscr{B}}(0,1)$ of $\mathbb{R}^n$. Let $\partial\overline{\mathscr{B}}_1$ denote the boundary of $\overline{\mathscr{B}}_1$, that is, the sphere. Let $f\colon \partial\overline{\mathscr{B}}_1 \to \overline{\mathscr{B}}_2$ be the inclusion map. Then the gluing $(\overline{\mathscr{B}}_1 \sqcup \overline{\mathscr{B}}_2)/f$ is the sphere $\mathbb{S}^n$.




**Exercise 16** (Double-origin interval)**.** Consider the topological space $X = [-1,1] \times \{0,1\}$, endowed with the subspace topology of $\mathbb{R}^2$. Let $\mathscr{R}$ be the relation on $X$ defined as $(t,a)\mathscr{R}(u,b) \iff (t = u \text{ and } t \neq 0)$ or $(t = u \text{ and } a = b)$.

1. Show that $\mathscr{R}$ is an equivalence relation, and describe its equivalence classes.
2. Show that the quotient topology on $X/\mathscr{R}$ is not Hausdorff (in the sense of Exercise 3).




# 2 Separation and connectedness

In this section, we will continue introducing the basic vocabulary of topological spaces. We will first define the interior, the closure and the boundary of a set. We will then introduce the notion of Hausdorff separability, and finally of connectedness.

## 2.1 Neighborhoods, interior, closure, boundary

**§2.1.1 Neighborhoods.** In what follows, $(X,\mathscr{T})$ denotes a topological space.

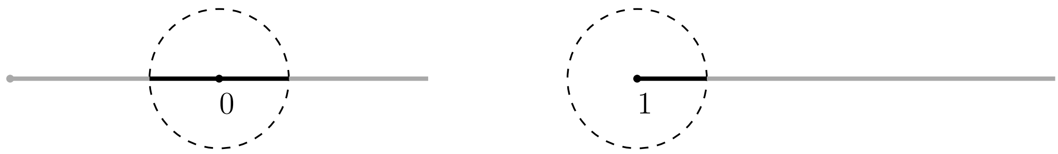
**Definition 2.1.** Let $x \in X$ be a point. We say that a subset $A \subset X$ is a *neighborhood* of $x$ if $A$ contains an open set that contains $x$, that is, if $\exists O \in \mathscr{T}$ such that $O \subset A$ and $x \in O$.

In some textbooks, the set of all neighborhoods of $x$ is denoted $\mathscr{N}(x)$, although we will not use this notation in these notes. Note that an open set is a neighborhood of all of its points. Conversely, a subset $A$ that is a neighborhood of all of its points is open. Indeed, for each point $x \in A$, we can consider an open set $O_x$ that contains $x$, and write $A = \bigcup_{x \in A} O_x$, which is open since it is an union of open sets. However, in general, a neighborhood does not have to be open.

In the case of the Euclidean topology, and as a direct consequence of Proposition 1.13, we get the following characterization:

**Proposition 2.2.** *Let $(\mathbb{R}^n, \mathscr{T}_{\mathbb{R}^n})$ be the Euclidean space, $A \subset \mathbb{R}^n$ a subset and $x \in A$ a point. The set $A$ is a neighborhood of $x$ if and only if there exists a $r > 0$ such that $\mathscr{B}(x, r) \subset A$.*

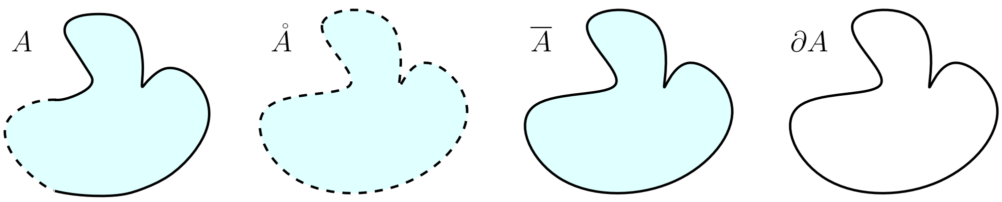
**Example 2.3.** Let $\mathbb{R}$ be the Euclidean line. The set $A = [-1, 1)$ is a neighborhood of $0$, since it contains the open set $(-1, 1)$. However, it is not a neighborhood of $-1$, since it does not contain any open ball of the form $(-1-r, -1+r)$.




### §2.1.2 Interior, closure, boundary.

**Definition 2.4.** Let $A \subset X$ be any subset. We define

- its *interior* $\mathring{A}$, as the set of points for which $A$ is a neighborhood,
- its *closure* $\overline{A}$, as the set of points for which every neighborhood meets $A$,
- its *boundary* as $\partial A = \overline{A} \setminus \mathring{A}$.




**Lemma 2.5.** *For any $A \subset X$, we have ${}^c(\mathring{A}) = \overline{{}^cA}$ and ${}^c(\overline{A}) = \overset{\circ}{\widehat{{}^cA}}$.*

**Proof.** We shall only prove the first equality, since the second one is obtained by taking the complementary of $A$. By definition, $\overline{{}^cA}$ is the set of points for which every neighborhood meets ${}^cA$, that is, the set of points for which no neighborhood is contained in $A$. Consequently, ${}^c(\overline{{}^cA})$ is the set of points for which there exists a neighborhood contained in $A$. In other words, ${}^c(\overline{{}^cA}) = \mathring{A}$, as wanted. $\square$

**Proposition 2.6.** *Let $A \subset X$ be any subset. We have:*

- *$\mathring{A}$ is the union of open sets contained in $A$. As a consequence, it is the largest open set contained in $A$.*
- *$\overline{A}$ is the intersection of closed sets containing $A$. As a consequence, it is the smallest closed set containing $A$.*

**Proof.** The first point is a direct consequence of the definition of the interior. The second point is a consequence of the first point and Lemma 2.5. $\square$

As useful consequences of the previous proposition, we have that a set $A \subset X$ is open of and only if $\mathring{A} = A$, and $A$ is closed if and only if $\overline{A} = A$.

**Example 2.7.** Let $\mathbb{R}$ be the Euclidean line, and $A = [-1,1)$. We have $\mathring{A} = (-1,1)$, $\overline{A} = [-1,1]$ and $\partial A = \{-1,1\}$. In general, in the Euclidean space $\mathbb{R}^n$, the interior of the closed ball is the open ball, and the closure of the open ball is the closed ball. Their boundary is the sphere.

**Proposition 2.8.** *Let $A, B \subset X$. We have:*

- $\overset{\circ}{\widehat{A \cap B}} = \mathring{A} \cap \mathring{B}$ *and* $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
- $\overset{\circ}{\widehat{A \cup B}} \supset \mathring{A} \cup \mathring{B}$ *and* $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$,
- $\partial(A \cup B) \subset \partial A \cup \partial B$.

**Exercise 17.** On the Euclidean line $\mathbb{R}$, give examples of sets $A$ and $B$ for which $\overset{\circ}{\widehat{A \cup B}} \neq \mathring{A} \cup \mathring{B}$, and for which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

**Exercise 18** (Kuratowski axioms)**.** Given a set $X$ and a map $c\colon \mathscr{P}(X) \to \mathscr{P}(X)$, consider the properties

| | |
|---|---|
| **(K1)** $c(\emptyset) = \emptyset$ | **(K2)** $\forall A \subset X,\ A \subset c(A)$ |
| **(K3)** $\forall A \subset X,\ c(c(A)) = c(A)$ | **(K4)** $\forall A, B \subset X,\ c(A \cup B) = c(A) \cup c(B)$ |

Such a map $c$ is called a *closure operator*.

1. Given a topological space $(X, \mathscr{T})$, show that the map $A \mapsto \overline{A}$ is a closure operator on $X$.
2. Given a set $X$ and a closure operator $c\colon \mathscr{P}(X) \to \mathscr{P}(X)$, show that the collection $\{A \subset X \mid c(A) = A\}$ forms the closed set of a topology on $X$.
3. Show that the previous constructions are inverse to one another.

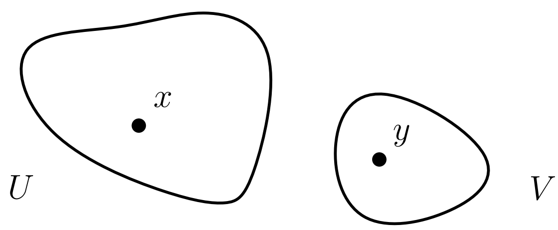
**Exercise 19** (Other formulation of Kuratowski axioms, [5, Exercise 5])**.** Show that the axioms **(K1)**, **(K2)**, **(K3)** and **(K4)** of Exercise 18 are equivalent to

$$\textbf{(K*)} \quad \forall A, B \subset X,\ A \cup c(A) \cup c(c(B)) = c(A \cup B) \setminus c(\emptyset).$$

## 2.2 Separation

The notion of separation captures the idea that any two points can be separated by non-intersecting open sets. Several variations of this notion exist: $T_0$-spaces, $T_1$-spaces, $T_2$-spaces, regular spaces, normal spaces, ... Here, we will only introduce one of them.

**Definition 2.9.** We say that a topological space $(X, \mathscr{T})$ is a *Hausdorff space* (or is a $T_2$-space) if for any $x, y \in X$ such that $x \neq y$, there exists neighborhoods $U, V$ of $x$ and $y$ such that $U \cap V = \emptyset$.



**Example 2.10.** The Euclidean space $(\mathbb{R}^n, \mathscr{T}_{\mathbb{R}^n})$ is Hausdorff. To prove, let $x, y \in X$ be such that $x \neq y$. Let $r = \|x - y\|$ be their distance. The balls $\mathscr{B}\left(x, \frac{r}{2}\right)$ and $\mathscr{B}\left(y, \frac{r}{2}\right)$ are neighborhoods of $x$ and $y$, and we have $\mathscr{B}\left(x, \frac{r}{2}\right) \cap \mathscr{B}\left(y, \frac{r}{2}\right) = \emptyset$.

**Proposition 2.11.** *If $(X, \mathscr{T})$ is a Hausdorff space, then all the singletons $\{x\}$, $x \in X$, are closed.*

**Proof.** Let us show that the complement ${}^c\{x\} = X \setminus \{x\}$ is open. We will show that it is a neighborhood of all of its points. Since $X$ is Hausdorff, for any $y \in X$ such that $y \neq x$, there exists an neighborhood of $y$ that does not contain $x$. Hence $X \setminus \{x\}$ is a neighborhood of $y$. $\square$

**Exercise 20** (Separability of Zariski topology)**.** Show that the Zariski topology on $\mathbb{R}^n$ is not Hausdorff (see Exercise 6).

## 2.3 Connectedness

**§2.3.1 Connected spaces** In a topological space, a set that is both open and closed will be called a *clopen* set.

> **Definition 2.12.** Let $(X, \mathscr{T})$ be a topological space. We say that $X$ is *connected* if the only clopen sets are $\emptyset$ and $X$.

The following proposition shows that a connected topological space cannot be divided into two non-empty disjoint open sets, neither two non-empty disjoint closed sets.

**Proposition 2.13.** *The following assertions are equivalent:*

- *$(X, \mathscr{T})$ is connected,*
- *for every open sets $O, O'$ such that $O \cap O' = \emptyset$ and $X = O \cup O'$, we have $O = \emptyset$ or $O' = \emptyset$,*
- *for every closed sets $P, P'$ such that $P \cap P' = \emptyset$ and $X = P \cup P'$, we have $P = \emptyset$ or $P' = \emptyset$.*

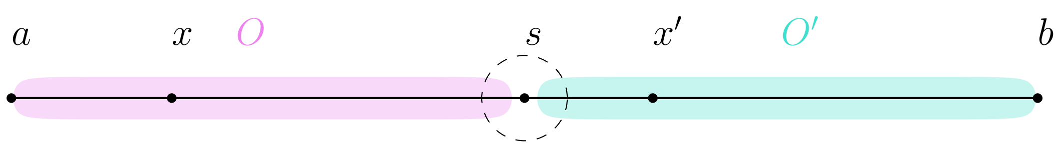
**Proof.** Let us suppose that $X$ is not connected, and let $C$ be a non-trivial clopen set. Then ${}^cC$ also is clopen, and $C \cup {}^cC$ gives the desired partition. $\square$

If $A \subset X$ is a subset, we say that $A$ is *connected* if the topological space $(A, \mathscr{T}_{|A})$ for the subspace topology is connected (see §1.3.1).

**Example 2.14.** The subset $X = [0,1] \cup [2,3]$ of $\mathbb{R}$, endowed with the subspace topology, is not connected. Indeed, $[0,1]$ and $[2,3]$ are closed disjoint non-empty subsets that cover $X$.

**Proposition 2.15.** *Consider $\mathbb{R}$ for the Euclidean topology. For all $a, b \in \mathbb{R}$ such that $a \leq b$, the intervals $(a,b)$, $[a,b)$, $(a,b]$ and $[a,b]$ are connected.*

**Proof.** By contradiction, let us suppose that we can write $(a,b) = O \cup O'$ with $O, O'$ two non-empty disjoint open sets. Let $x \in O$ and $x' \in O'$. Without loss of generality, suppose that $x < x'$. Let $s$ be the supremum of $\{t \in (x, x') \mid (x,t) \subset O\}$. Since $O'$ is open, we have $s < x'$.




By definition of the supremum, $O$ does not contain any open ball around $s$, hence $O$ does not contain $s$, since it is open. Similarly, $O'$ does not contain any open ball around $s$, hence $O'$ does not contain $s$. This is absurd. $\square$

**Proposition 2.16.** *Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$ a connected subset. Then its closure $\overline{A}$ is connected.*

**Proof.** Let $C$ be a clopen set of $\overline{A}$. By definition of the subspace topology, $C \cap A$ is a clopen set for $A$. Since $A$ is connected, $C \cap A$ must be $\emptyset$ of $A$. Without loss of generality, we can suppose that $C \cap A = A$ (otherwise, we replace $C$ with ${}^cC$). The relation $C \cap A = A$ is equivalent to $A \subset C$. Taking the closure, we get $\overline{A} \subset \overline{C} = C$. Moreover, $\overline{C} = C$ since $C$ is closed. Hence $\overline{A} = C$, proving the proposition. □

In the next section, we will introduce the notion of continuous function, and that of *path-connectedness*. This will be a handy tool to prove result about connectedness. In particular, we will show that the balls of $\mathbb{R}^n$ are connected, and more generally, that the convex subsets of $\mathbb{R}^n$ is connected.

> **Exercise 21.** Among the topologies on $X = \{0, 1\}$ (see Example 1.5), which ones yield connected spaces?

**§2.3.2 Connected components** If a space is not connected, we can consider its connected components.

**Definition 2.17.** Let $(X, \mathcal{T})$ be a topological space and $x \in X$. The *connected component* of $x$, denoted $\mathscr{C}(x)$, is defined as the union of all connected subsets $U \subset X$ that contain $x$.

**Proposition 2.18.** *A connected component is connected.*

**Proof.** By contradiction, suppose that $\mathscr{C}(x)$ is not connected, and let $\mathscr{C}(x) = O \cup O'$ be a partition in open sets. Without loss of generality, $x \in O$. Let $A$ be a connected subset of $X$ that contain $x$. We have a partition $A = (O \cap A) \cup (O' \cap A)$ in open sets, hence $A$ by connectedness, we must have $A \subset O$ or $A \subset O'$. Since $x \in A$, we deduce $A \subset O$. This being true for any connected subset $A$ containing $x$, we have $\mathscr{C}(x) = O$, and $O' = \emptyset$. We deduce the result. □

In other words, the connected component $\mathscr{C}(x)$ is the largest connected subspace that contains $x$. As a consequence of Proposition 2.16, every connected component is closed. In general, they may not be open, as shown in Exercise 22. However, this is true in the case of the Euclidean space, and its open subspaces.

Given two points $x, y \in X$, we have $y \in \mathscr{C}(x) \iff \mathscr{C}(x) = \mathscr{C}(y)$. Consequently, the set of connected components of $X$ forms a partition of $X$.

**Proposition 2.19.** *Let $(\mathbb{R}^n, \mathcal{T})$ be the Euclidean space. Let $O \subset \mathbb{R}^n$ be an open set, and consider the topological space $(O, \mathcal{T}_{|O})$ endowed with the subspace topology. Consider a point $x \in O$, and $\mathscr{C}(x)$ its connected component in $(O, \mathcal{T}_{|O})$. Then $\mathscr{C}(x)$ is an open set of $(\mathbb{R}^n, \mathcal{T})$, hence also of $(O, \mathcal{T}_{|O})$.*

**Proof.** Let $y \in \mathscr{C}(x)$. Since $O$ is open in $\mathbb{R}^n$, there exists a ball $\mathscr{B}(y, r)$ included in $O$. By definition of the connected component, we have $\mathscr{B}(y, r) \subset \mathscr{C}(y)$. Using that $\mathscr{C}(x) = \mathscr{C}(y)$, we deduce $\mathscr{B}(y, r) \subset \mathscr{C}(x)$, hence that $\mathscr{C}(x)$ is open in $\mathbb{R}^n$. □

**Remark 2.20.** The previous proposition is actually true for every *locally connected space*.

**Example 2.21.** Consider the subset $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ of $\mathbb{R}$. Each of its subsets $\{i\}$, $i \in X$, are open. They are all non-empty, connected and disjoint. Hence $X$ admits ten connected components.

**Exercise 22** (Connected components of $\mathbb{Q}$)**.** Let $\mathbb{Q}$ be endowed with the subspace Euclidean topology of $\mathbb{R}$.

1. Show that the connected components of $\mathbb{Q}$ are the singletons $\{x\}$, $x \in \mathbb{Q}$.
2. Show that the singletons are not open in $\mathbb{Q}$.

This shows that Proposition 2.19 is not true in general.

*Hint:* Remember that between two distinct rational numbers there exists an irrational number.

# 3 Continuity and homeomorphisms

This chapter in based on [6]. We will (at last!) introduce the notion of continuous map. Often, in textbooks, continuous maps are introduced at the very beginning, allowing to understand topology not as the theory of topological spaces, but as the *category* of topological spaces endowed with continuous maps. In this course, we chose to talk first about topological spaces only, so as to focus on their axioms. By introducing continuous maps, we will be able to define formally what it means to compare topological spaces. We will use, depending on the viewpoint, the relation of homeomorphism, or the relation of homotopy equivalence, defined in the next chapter.

## 3.1 Continuous maps

**§3.1.1 Continuity.** The topologist's point of view allows to define the notion of continuity in great generality. Throughout this section, we will consider two topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$.

**Definition 3.1.** Let $f\colon X \to Y$ be a map. We say that $f$ is *continuous* if for every $O \in \mathscr{U}$, the preimage $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is in $\mathscr{T}$.

In other words, a map is continuous if the preimage of any open set is an open set. As shown in the following example, the continuity of a map depends on the topologies that are given to $X$ and $Y$. Therefore, we should not say '*$f\colon X \to Y$ is continuous*', but '*$f\colon (X, \mathscr{T}) \to (Y, \mathscr{U})$ is continuous*'. However, when it will be clear what topologies we are considering, and when there will be no risk of confusion, we will use the first sentence.

**Example 3.2.** Let $X$ and $Y$ be both $\{0,1\}$, and met $f\colon \{0,1\} \to \{0,1\}$ be the identity map (that is, $f(0) = 0$ and $f(1) = 1$). Consider the trivial and the discrete topology

$$\mathscr{T} = \{\emptyset, \{0,1\}\} \quad \text{and} \quad \mathscr{U} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}.$$

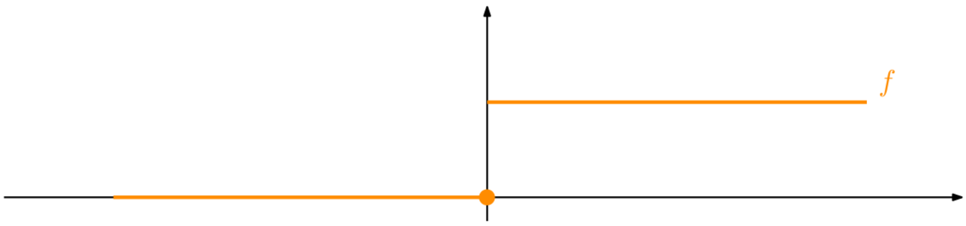
The map $f$, seen as a map between the topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$, is not continuous. Indeed, $\{0\}$ is an open set of $(Y, \mathscr{U})$, but $f^{-1}(\{0\}) = \{0\}$ is not an open set of $(X, \mathscr{T})$. However, seen as a map between the topological spaces $(X, \mathscr{U})$ and $(Y, \mathscr{U})$, $f$ is continuous. For instance the preimage, $f^{-1}(\{0\}) = \{0\}$ is an open set of $(X, \mathscr{U})$.

Continuity can also be stated in terms of closed sets:

**Proposition 3.3.** *A map is continuous if and only if the preimage of closed sets are closed sets.*

**Exercise 23.** Prove Proposition 3.3.
*Hint:* For any subset $A \subset Y$, show that $f^{-1}({}^cA) = {}^c(f^{-1}(A))$.

**Example 3.4.** Let $X = Y = \mathbb{R}$, endowed with the Euclidean topology. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$ for all $x \leq 0$, and $f(x) = 1$ for all $x > 0$. The set $(-1,1)$ is open, but $f^{-1}((-1,1)) = (-\infty, 0]$ is not. Hence $f$ is not continuous.




The following propositions say that the composition of two continuous maps, as well as the restriction of a continuous map, are continuous maps.

**Proposition 3.5.** *Let $(X,\mathscr{T})$, $(Y,\mathscr{U})$ and $(Z,\mathscr{V})$ be three topological spaces, and $f\colon X \to Y$, $g\colon Y \to Z$ two continuous maps. Then the composition $g \circ f\colon X \to Z$ is a continuous map.*

**Proof.** Let $O \in \mathscr{V}$ be an open set of $Z$. We have to show that $(g \circ f)^{-1}(O)$ is in $\mathscr{T}$. First, note that $(g \circ f)^{-1}(O) = f^{-1}\big(g^{-1}(O)\big)$. Since $g$ is continuous, the set $g^{-1}(O)$ is in $\mathscr{U}$, i.e., it is an open set of $Y$. But since $f$ is continuous, its preimage $f^{-1}\big(g^{-1}(O)\big)$ also is an open set (of $X$). Since this is true for any open set $O \in \mathscr{V}$, we deduce that $g \circ f$ is continuous. $\square$

**Proposition 3.6.** *Let $f$ be a continuous map between $(X,\mathscr{T})$ and $(Y,\mathscr{U})$. Consider a subset $A \subset X$, and endow it with the subspace topology $\mathscr{T}_{|A}$. The induced map $f_{|A}\colon (A,\mathscr{T}_{|A}) \to (Y,\mathscr{U})$ is continuous. Moreover, for any subset $B \subset Y$ such that $f(A) \subset B$, the induced map $f_{|A,B}\colon (A,\mathscr{T}_{|A}) \to (B,\mathscr{U}_{|B})$ also is continuous.*

**Proof.** We will only prove the second statement. For every open set $O \in \mathscr{U}_{|B}$, let us show that $(f_{|A,B})^{-1}(O)$ is in $\mathscr{T}_{|A}$. By definition of the subspace topology $\mathscr{U}_{|B}$, there exists $O' \in \mathscr{U}$ such that $O = O' \cap B$. Now, we have

$$f_{|A,B}^{-1}(O) = f_{|A,B}^{-1}(O' \cap B) = f_{|A,B}^{-1}(O') \cap f_{|A,B}^{-1}(B).$$

Because of the assumption $f(A) \subset B$, we have $(f_{|A,B})^{-1}(B) = A$, and we deduce $f_{|A,B}^{-1}(O) = f_{|A,B}^{-1}(O') \cap A$. Since $f$ is continuous, the preimage $f_{|A,B}^{-1}(O')$ is in $\mathscr{T}$, hence the intersection $f_{|A,B}^{-1}(O') \cap A$ is in $\mathscr{T}_{|A}$. $\square$

**Exercise 24** (Trivial and discrete continuity)**.** Consider the maps $f\colon (X,\mathscr{T}) \to (Y,\mathscr{U})$.

1. Show that if $\mathscr{T}$ is the discrete topology, then all the maps are continuous.
2. Show that if $\mathscr{U}$ is the trivial topology, then all the maps are continuous.

**§3.1.2 LINK WITH THE USUAL $\varepsilon$-$\delta$ CALCULUS.** We now investigate what continuity means between the Euclidean spaces. Consider a continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $\varepsilon > 0$. The open ball $\mathscr{B}(f(x),\varepsilon)$ is an open set of $\mathbb{R}^m$. By continuity of $f$, the preimage $f^{-1}(\mathscr{B}(f(x),\varepsilon))$ is an open set. Since $x$ belongs to $f^{-1}(\mathscr{B}(f(x),\varepsilon))$, Proposition 1.13 gives a $\eta > 0$ such that

$$\mathscr{B}(x,\eta) \subset f^{-1}(\mathscr{B}(f(x),\varepsilon)).$$

This is equivalent to

$$\forall y \in \mathscr{B}(x,\eta)\,,\ f(y) \in \mathscr{B}(f(x),\varepsilon)\,.$$

In other words, for all $y \in \mathbb{R}^n$,

$$\|x-y\| < \eta \implies \|f(x)-f(y)\| < \varepsilon.$$

We recognize usual definition of continuity:

**Proposition 3.7.** *A map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if, for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\eta > 0$ such that for all $y \in \mathbb{R}^n$, we have $\|x-y\| < \eta \implies \|f(x)-f(y)\| < \varepsilon$.*

As a consequence, what we already know about continuity between Euclidean spaces still applies in our context.

#### §3.1.3 Connectedness via continuous maps.

Using the notion of continuous maps, we can give an alternative definition of connectedness (see Definition 2.12).

**Proposition 3.8.** *A topological space $(X,\mathscr{T})$ is connected iff every continuous map from $X$ to the discrete space $\{0,1\}$ is constant.*

**Proof.** We prove first the direct implication. Suppose that $X$ is connected, and that $f\colon \{0,1\}$ is continuous. Endowes with the discrete topology, $\{0\}$ is a clopen set of $\{0,1\}$. Hence $f^{-1}(\{0\})$ must be clopen, hence it must be $\emptyset$ or $X$, as $X$ is connected. We deduce that $f$ is respectively constant equal to 1 or to 0.

In order to prove the converse implication, we consider the contraposition. Suppose that $X$ is not connected. Hence $X$ admits a clopen set $A$ such that $\subsetneq A \subsetneq X$. Note that ${}^cA$ also is clopen. We build a map $f\colon \{0,1\}$ by setting $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in {}^cA$. It is a continuous map for the discrete topology on $\{0,1\}$. □

## 3.2 Homeomorphisms

### §3.2.1 Definitions.

**Definition 3.9.** Let $(X,\mathscr{T})$ and $(Y,\mathscr{U})$ be two topological spaces, and $f\colon X \to Y$ a map. We say that $f$ is a *homeomorphism* if

- $f$ is a bijection,
- $f\colon X \to Y$ is continuous,
- $f^{-1}\colon Y \to X$ is continuous.

If there exists such a homeomorphism, we say that the two topological spaces are *homeomorphic*.

**Example 3.10.** In practice, finding the inverse $f^{-1}$ of $f$ consists in finding a map $g\colon Y \to X$ such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. In this case, $g$ is the inverse of $f$. As an example, consider in $\mathbb{R}^2$ the circle and the square, endowed with the subspace topology:

$$\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\} \qquad \text{and} \qquad \mathscr{C} = \{(x_1,x_2) \in \mathbb{R}^2 \mid \max(|x_1|,|x_2|) = 1\}.$$

Let $f\colon \mathbb{S}^1 \to \mathscr{C}$ be the map

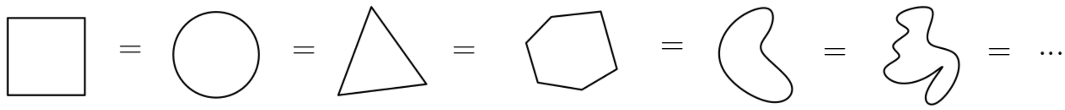
$$f\colon (x_1,x_2) \mapsto \frac{1}{\max(|x_1|,|x_2|)}(x_1,x_2).$$

It is continuous. More over, it admits the following inverse *(check that this is true)*:

$$f^{-1}\colon x \mapsto \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1,x_2).$$

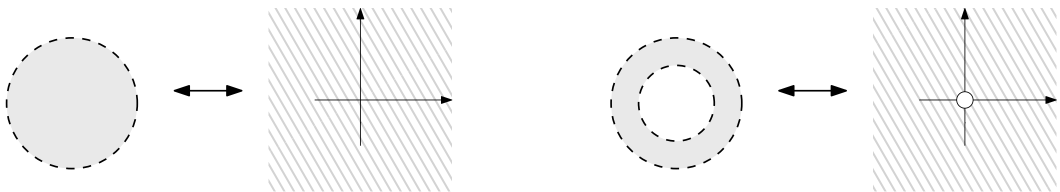
This map is continuous, hence $f$ is a homeomorphism.

More generally, one shows that all the *closed curves* — that is, the images of injective continuous maps $\mathbb{S}^1 \to \mathbb{R}^2$ — are homeomorphic. This illustrates a common way of thinking topology: topological spaces are made of rubber, and we are allowed to deform them.




**Exercise 25.** Show that the topological spaces $\mathbb{R}^n$ and $\mathscr{B}(0,1) \subset \mathbb{R}^n$ are homeomorphic.
*Hint:* Consider the map $f\colon x \mapsto \frac{1}{\|x\|+1}x$.

**Exercise 26.** Show that the punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ and the open annulus $\mathscr{B}(0,2) \setminus \mathscr{B}(0,1) \subset \mathbb{R}^n$ are homeomorphic.



**Example 3.11.** Let $\mathbb{S}^1$ denote the unit circle of $\mathbb{R}^2$, and consider the map

$$\begin{aligned} f\colon [0,2\pi) &\longrightarrow \mathbb{S}^1 \\ \theta &\longmapsto (\cos(\theta),\sin(\theta)) \end{aligned}$$

It is continuous, and admits the following inverse:

$$\begin{aligned} g\colon \mathbb{S}^1 &\longrightarrow [0,2\pi) \\ (x_1,x_2) &\longmapsto \arctan\left(\frac{x_2}{x_1}\right) \end{aligned}$$

This comes from the relation $\theta = \arctan\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$ for all $\theta \in [0,2\pi)$. The map $g$ is **not** continuous. Indeed, $[0,\pi)$ is an open subset of $[0,2\pi)$, but $g^{-1}([0,\pi))$ is not an open subset of $\mathbb{S}^1$ (it is not open around $g^{-1}(0) = (1,0)$).


We will see in Example 3.15 that there exists no homeomorphism between $[0, 2\pi)$ and $\mathbb{S}^1$.

## 3.3 Invariants of homeomorphism classes

**§3.3.1 Homeomorphism classes.** Let us write $X \simeq Y$ if the two topological spaces $X$ and $Y$ are homeomorphic. It is clear that, for any $X$, we have

$$X \simeq X.$$

Moreover, we have:

$$X \simeq Y \iff Y \simeq X.$$

We also have a third property, stated in the following proposition:

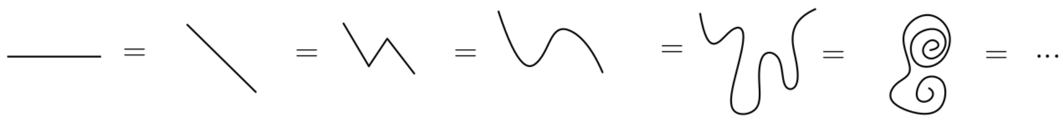
**Proposition 3.12.** *If three topological spaces $X, Y, Z$ are such that $X$ is homeomorphic to $Y$ and $Y$ is homeomorphic to $Z$, then $X$ is homeomorphic to $Z$. In other words,*

$$X \simeq Y \text{ and } Y \simeq Z \implies X \simeq Z.$$

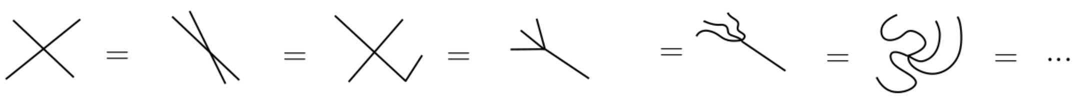
**Proof.** Supppose that $X, Y$ are homeomorphic, and $Y, Z$ too. This means that we have homeomorphisms $f: X \to Y$ and $g: Y \to Z$. Consider the map $g \circ f: X \to Z$. It is continuous (by Proposition 3.5) bijective (composition of bijective maps) and its inverse $f^{-1} \circ g^{-1}: Z \to X$ is also continuous (by Proposition 3.5 too). Hence $g \circ f$ is a homeomorphism, and the spaces $X, Z$ are homeomorphic. $\square$

The three previous properties are *reflexivity*, *symmetry* and *transitivity*, hence **'being homeomorphic' is an equivalence relation**. It allows to classify topological spaces into classes (called *classes of homeomorphism equivalence*):

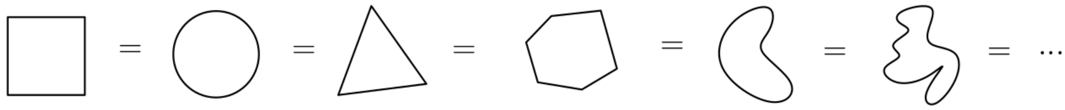
- the class of intervals:



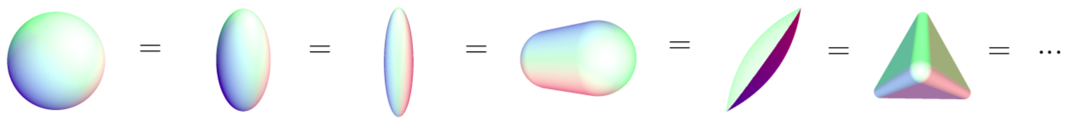
- the class of crosses:



- the class of circles:



- the class of spheres of dimension 2:



- the class of tori, the class of Klein bottles, etc...

**§3.3.2 Connectedness.** The topologists' favorite game is to class topological spaces by homeomorphism equivalence. However, in general, it may be complicated to determine whether two spaces are homeomorphic. To answer this problem, a handy tool is the notion of *invariant*. An invariant is a property, a characteristic, that is shared by all the topological space of a same class. Our first example is connectedness (introduced in §2.3.2).

**Proposition 3.13.** *Let $X$, $Y$ be two topological spaces and $f\colon X \to Y$ a continuous surjective map. Then the number of connected components of $X$ is greater than that of $Y$. In particular, if they are homeomorphic, then they have the same number of connected components.*

**Proof.** This comes from the fact that the image of a connected space is a connected space. □

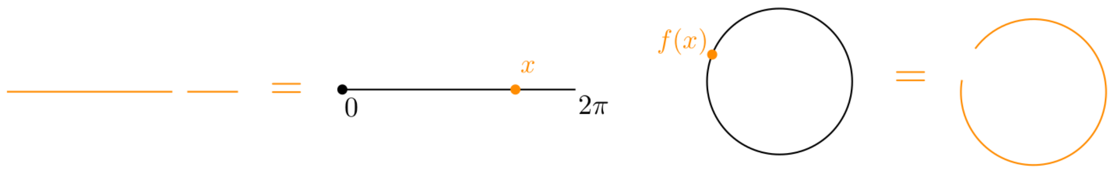
In practice, we use the contraposition of Proposition 3.13 to prove that two spaces are not homeomorphic. Showing that they are is another story.

**Example 3.14.** The subsets $[0,1]$ and $[0,1]\cup[2,3]$ of $\mathbb{R}$ are not homeomorphic. Indeed, the first one has one connected component, and the second one two.


**Example 3.15.** The interval $[0,2\pi)$ and the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ are not homeomorphic. We will prove this by contradiction. Suppose that they are homeomorphic. By definition, this means that there exists a map $f\colon [0,2\pi) \to \mathbb{S}^1$ which is continuous, invertible, and with continuous inverse. Let $x \in [0,2\pi)$ such that $x \neq 0$. Consider the subsets $[0,2\pi)\setminus\{x\} \subset [0,2\pi)$ and $\mathbb{S}^1\setminus\{f(x)\} \subset \mathbb{S}^1$, and the induced map

$$g\colon [0,2\pi)\setminus\{x\} \to \mathbb{S}^1\setminus\{f(x)\}.$$

The map $g$ is a homeomorphism by Proposition 3.6. Moreover, it is clear that $[0,2\pi)\setminus\{x\}$ has two connected components, and $\mathbb{S}^1\setminus\{f(x)\}$ only one. This contradicts Proposition 3.13.




**Example 3.16.** $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Just as before, we will prove this by contradiction. Suppose that there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^2$. Choose any $x \in \mathbb{R}$. The induced map

$$g\colon \mathbb{R}\setminus\{x\} \to \mathbb{R}^2 \setminus \{f(x)\}$$

is still a homeomorphism, but $\mathbb{R}\setminus\{x\}$ has two connected components, while $\mathbb{R}^2\setminus\{f(x)\}$ has only one. This is a contradiction. The same reasoning shows that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic either.

**Remark 3.17.** More generally, the *invariance of domain* is a theorem that says that for every integers $m,n$ such that $m \neq n$, the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic. We will need much more sophisticated tools to prove that (homology of spheres). Although intuitively obvious, We will need much more sophisticated tools to prove that (Brouwer fiwed point theorem, via the homology of spheres or Sperner's lemma). As an example of its non-obviousness, note that there exist continuous surjective maps $\mathbb{R}^n \to \mathbb{R}^m$ for any $n,m > 0$.

> **Exercise 27.** Show that $[0,1)$ and $(0,1)$ are not homeomorphic.
> *Hint:* Use the strategy of Examples 3.15 or 3.16.

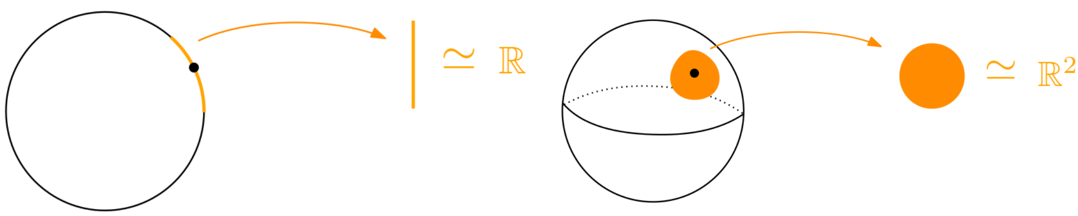
### §3.3.3 Dimension.

We now introduce our second invariant. It is only defined for a particular class of topological spaces.

**Definition 3.18.** Let $(X,\mathscr{T})$ be a Hausdorff topological space (see Subsect. 2.2), and $n \geq 0$. We say that it is a *manifold of dimension n* if for every $x \in X$, there exists an open set $O$ such that $x \in O$, and a homeomorphism from $O$ to an open subset of $\mathbb{R}^n$.

In other words, a manifold of dimension $n$ is a topological space that locally looks like the Euclidean space $\mathbb{R}^n$. Instead of saying 'manifold of dimension $n$', we may say '$n$'-manifold. For instance, one shows that

- the open intervals $(a,b) \subset \mathbb{R}$ are manifolds of dimension 1,
- the circle $\mathbb{S}^1 \subset \mathbb{R}^2$ is a manifold of dimension 1,
- more generally, the spheres $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ are manifolds of dimension $n-1$,
- the open balls $\mathscr{B}(x,r) \subset \mathbb{R}^n$ are manifolds of dimension $n$,
- the Euclidean space $\mathbb{R}^n$ itself is a manifold of dimension $n$.



**Remark 3.19.** For this definition to make sense, we have to make sure that the topological spaces $\mathbb{R}^n$, $n \geq 0$, are all not-homeomorphic. Otherwise, a topological space could have several dimensions. As we said earlier, this result, the *invariance of domain*, will be proved later.

**Proposition 3.20.** *Let $X$, $Y$ be two homeomorphic topological spaces. If $X$ is a manifold of dimension $n$, so is $Y$.*

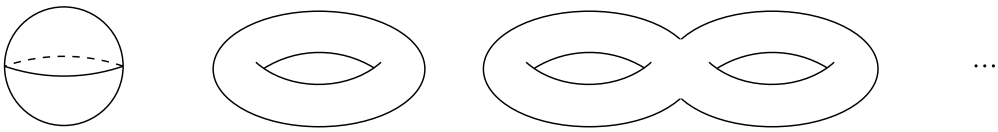
**Proof.** Let $n$ be the dimension of $X$, and consider a homeomorphism $g\colon Y \to X$. Let $y \in Y$, and $x = g(y)$. Since $x$ has dimension $n$, there exists an open set $O$ of $X$ with $x \in O$, and open subset $U \subset \mathbb{R}^n$ and a homeomorphism $h\colon O \to U$. Define $O' = g^{-1}(O)$. It is an open set of $Y$, with $y \in O'$. Moreover, the map $h \circ g\colon O' \to U$ is a homeomorphism. This being true for every $y \in Y$, we deduce that $Y$ has dimension $n$. $\square$

We can read the previous proposition as follows: being a manifold of dimension $n$ is an invariant of homeomorphic spaces. As before, we can use it to show that two spaces are not homeomorphic.

**Example 3.21.** The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and the unit open ball $\mathscr{B}(0,1) \subset \mathbb{R}^3$ are not homeomorphic. Indeed, the first one has dimension 2, and the second one dimension 3.

Given a fixed dimension $n \geq 1$, there exists several manifolds of dimension $n$. Hence, sometimes, the dimension will be not enough to distinguish between spaces. An example is given by the real line $\mathbb{R}$ and the circle $\mathbb{S}^1$: they are both manifolds of dimension 1, though not homeomorphic (this can be proved using the technique of Example 3.15). Actually, they are the only manifolds of dimension 1, up to homeomorphism.

In dimension 2, an interesting example is given by the compact oriented surfaces. These manifolds are indexed by their *genus*, a natural number $g \geq 0$. They are not homeomorphic, however, they have the same dimension (two) and number of connected components (one).



$g = 0$ $\qquad$ $g = 1$ $\qquad$ $g = 2$ $\qquad \dots$

We also have a notion of manifold that allows to have a 'boundary'. Let us denote by $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ the Euclidean half-space, and $\{0\} \times \mathbb{R}^{n-1} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1 = 0\}$.

**Definition 3.22.** Let $(X,\mathscr{T})$ be a Hausdorff topological space and $n \geq 0$. We say that it is a *manifold with boundary of dimension n* if for every $x \in X$, there exists an open set $O$ such that $x \in O$, and a homeomorphism from $O$ to an open subset of $\mathbb{R}^n_+$.

Let $X$ be a manifold with boundary and $x \in X$. As in the definition, let $O \to U$ be an homeomorphism, where $O \subset X$ and $U \subset \mathbb{R}^n_+$ are open. If $U$ contains a point of $\{0\} \times \mathbb{R}^{n-1}$, then it is the case of all the homeomorphisms from a neighborhood of $x$ to an open subset of $\mathbb{R}^n_+$, and we say that $x$ is a *boundary point*. We will denote by $\partial X$ the boundary of $X$, taking care of noting that it is not the same notion of boundary defined in §2.1.2. If $\partial X =$, then $X$ actually is a manifold. As examples, we have that

- the closed intervals $[a,b] \subset \mathbb{R}$ are 1-manifolds with boundary, and $\partial[a,b] = \{a,b\}$,

- the closed balls $\overline{\mathscr{B}}(x,r) \subset \mathbb{R}^n$ are $n$-manifolds with boundary, and $\partial\overline{\mathscr{B}}(x,r) = \mathbb{S}^{n-1}$.

As before, we can show that being a manifold with boundary is a property transferred by homeomorphisms.

**Proposition 3.23.** *Let $X$, $Y$ be two homeomorphic topological spaces. If $X$ is a manifold with boundary of dimension n, then so is $Y$. Moreover, $\partial X$ and $\partial Y$.*

**Proof.** If $f\colon X \to Y$ is a homeomorphism, then the restriction $f_{|\partial X}\colon \partial X \to \partial Y$ still is, by Proposition 3.6. □

**Exercise 28** (Closed Möbius strip)**.** Let $C$ denote the cylinder $M$ denoted the Möbius strip. They are both obtained from the square $[-1,1]\times[-1,1]$, the first one by identifying the points $(-1,t)\sim(1,t)$, the second one by identifying $(-1,t)\sim(1,-t)$, $t\in[0,1]$ (see §1.3.3). Show that they are not homeomorphic.

*Hint:* Show that $\partial C$ and $\partial M$ are not homeomorphic.

§3.3.4 **EMBEDDABILITY.** Let $n \geq 0$. An *embedding* of a topological space $(X,\mathscr{T})$ into $\mathbb{R}^n$ is a continuous injective map $X \to \mathbb{R}^n$. If such an embedding exists, we say that $X$ is *embeddable* into $\mathbb{R}^n$.

**Proposition 3.24.** *Given two homeomorphic topological spaces, if one is embeddable into $\mathbb{R}^n$, then so is the other one.*

**Proof.** Let $f\colon X \to Y$ be a homeomorphism and $g\colon Y \to \mathbb{R}^n$ an embedding of $Y$. Then $g\circ f$ is an embedding of $X$. □

**Example 3.25** (Open Möbius strip)**.** As in Exercise 28, we will show that the cylinder and the Möbius strip are not homeomorphic, but now when considering their open version, that is, seeing them as gluing of the square $[-1,1]\times(-1,1)$. Let us denote them $C$ and $M$. In this case, we have $\partial C = \partial M = \emptyset$, hence we cannot use the same strategy.

1. Show that $C$ is embeddable in $\mathbb{R}^2$.
2. Draw on $M$ two circle that only intersect in one point.
3. Suppose that $M$ is embeddable in $\mathbb{R}^2$. Deduce that we obtain two circles in $\mathbb{R}^2$ that only intersect in one point.
4. Conclude using Jordan's curve theorem.

# 4 Homotopy equivalence

## 4.1 Homotopies

**§4.1.1 Homotopy equivalence between maps.** We will now introduce the homotopy equivalence, another equivalence relation between topological spaces. First, we shall define it at the level of continuous maps.

**Definition 4.1.** Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be two topological spaces, and $f, g\colon X \to Y$ two continuous maps. A *homotopy* between $f$ and $g$ is a map $F\colon X \times [0,1] \to Y$ such that:

- $F(\cdot, 0)$ is equal to $f$,
- $F(\cdot, 1)$ is equal to $g$,
- $F\colon X \times [0,1] \to Y$ is continuous.

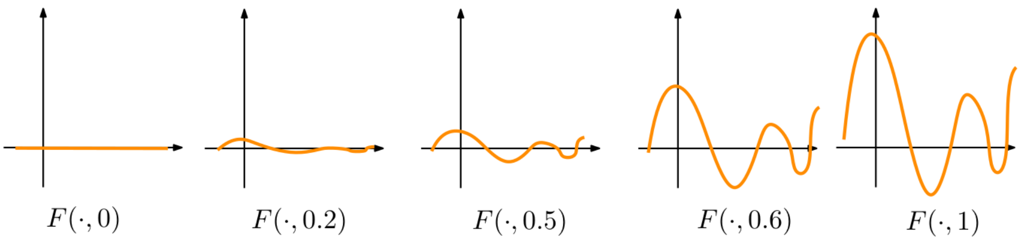
If such a homotopy exists, we say that the maps $f$ and $g$ are *homotopic*.

In this definition, the notation $F(\cdot, t)$ refers to the map

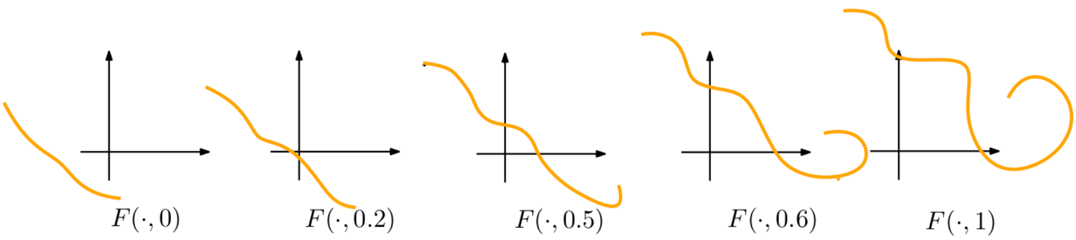
$$\begin{aligned} F(\cdot,t)\colon X &\longrightarrow Y \\ x &\longmapsto F(x,t) \end{aligned}$$

Moreover, before asking for $F\colon X \times [0,1] \to Y$ to be continuous, we have to give $X \times [0,1]$ a topology. The topology we choose is the *product topology* (see §1.3.2). Equivalently, if $X$ is a subset of $\mathbb{R}^n$ and $\mathscr{T}$ is the subspace topology, then the product topology on $X \times [0,1]$ is equal to the subspace topology of the Euclidean space $\mathbb{R}^n \times \mathbb{R}$.

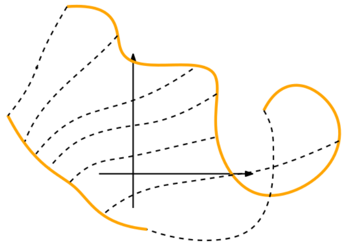
We may represent graphically a homotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ by plotting it for each value of $t \in [0,1]$:




This is an example for $F\colon [0,1] \times [0,1] \to \mathbb{R}^2$:



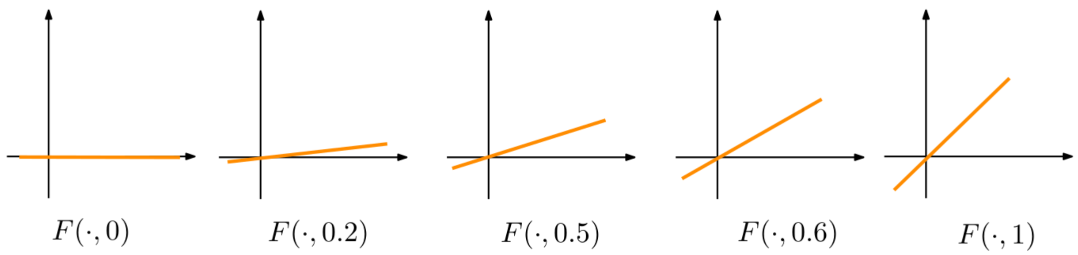

Sometimes we prefer to plot the deformation:



**Example 4.2.** Let $X = Y = [-1,1]$ endowed with the Euclidean topology, and consider the maps $f,g\colon X \to Y$ defined as $f\colon x \mapsto 0$ and $g\colon x \mapsto x$. Let us prove that they are homotopic. Consider the map

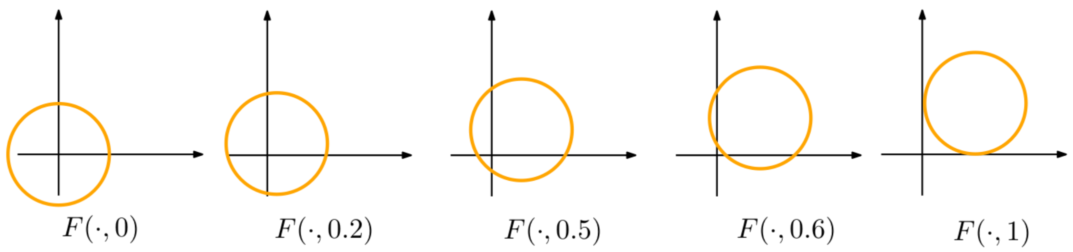
$$\begin{aligned} F\colon X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto tx \end{aligned}$$

We see that $F(\cdot,0)\colon x \mapsto 0$ is equal to $f$, and $F(\cdot,1)\colon x \mapsto x$ is equal to $g$. Moreover, $F$ is continuous. Hence, $F$ is an homotopy between $f$ and $g$. Thus these two maps are homotopic.

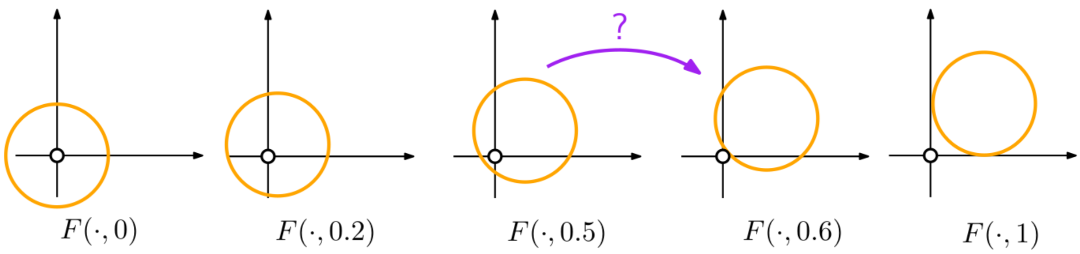



**Example 4.3.** The map $F\colon (x,t) \in \mathbb{S}^1 \times [0,1] \mapsto (\cos(\theta)+2t, \sin(\theta)+2t)$ is a homotopy between

$$f\colon \theta \mapsto (\cos(\theta),\sin(\theta)) \qquad \text{and} \qquad g\colon \theta \mapsto (\cos(\theta)+2,\sin(\theta)+2)$$




**Example 4.4.** In $\mathbb{S}^1$ and $\mathbb{R}^2 \setminus \{(0,0)\}$, the plane without the origin, there is no homotopy between the maps $f$ and $g$ of the previous example. Indeed, the homotopy $F$ would pass through the point $(0,0)$ at some point, which is impossible. In order to prove this result formally, one can use the notion of *degree of a map*.




From a homotopic point a view, a trivial map is a map that is homotopic to a constant map. For instance, the identity map of Example 4.2 is homotopic to the constant map $x \mapsto 0$. More generally, we have:

**Proposition 4.5.** *Let $(X,\mathscr{T})$ be a topological space. Any continuous map $f\colon X \to \mathbb{R}^n$ is homotopic to a constant map.*

**Proof.** Consider the continuous map $F\colon (x,t) \in X \times [0,1] \mapsto tf(x)$. We have that $F(\cdot,1) = f$, and $F(\cdot,0)\colon x \mapsto 0$ is a constant map. □

**Exercise 29.** Let $(X,\mathscr{T})$ be a topological space. Show that any continuous map $f\colon \mathbb{R}^n \to X$ is homotopic to a constant map.

As a consequence, the theory of maps with domain or codomain $\mathbb{R}^n$ is trivial from a homotopy equivalence perspective. However, when the domain and codomain are not Euclidean spaces, as in Example 4.4, many non-homotopic maps may exist.

**Exercise 30** (Maps between the sphere)**.** Let $f\colon \mathbb{S}^1 \to \mathbb{S}^2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map.
*Hint:* Let $x_0 \in \mathbb{S}^2$ be such that $x_0 \notin f(\mathbb{S}^1)$. Find a homotopy between $f$ and the constant map $g\colon x \mapsto -x_0$.
*More complicated question:* Is every continuous map $f\colon \mathbb{S}^1 \to \mathbb{S}^2$ homotopic to a constant map?

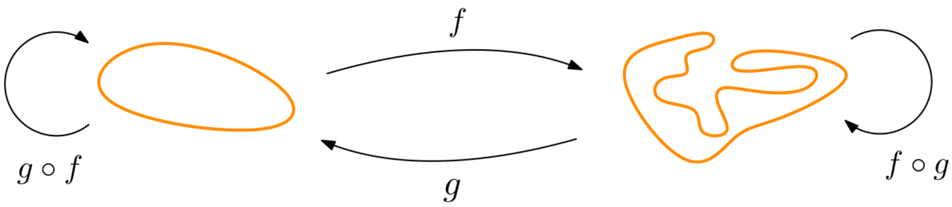
**Exercise 31.** Show that 'being homotopic' is a *transitive* relation between maps: for every triplet of maps $f,g,h\colon X \to Y$, if $f,g$ are homotopic and $g,h$ are homotopic, then $f,h$ are homotopic.

### §4.1.2 Homotopy equivalence

**Definition 4.6.** Let $(X,\mathscr{T})$ and $(Y,\mathscr{U})$ be two topological spaces. A *homotopy equivalence* between $X$ and $Y$ is a pair of continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that:

- $g \circ f\colon X \to X$ is homotopic to the identity map $\mathrm{id}\colon X \to X$,
- $f \circ g\colon Y \to Y$ is homotopic to the identity map $\mathrm{id}\colon Y \to Y$.

If such a homotopy equivalence exists, we say that $X$ and $Y$ are *homotopy equivalent*.




As we shall see in the forthcoming examples, when comparing spaces with the homotopy equivalence, we see them a deformable objects, and we are allowed to *retract* or *flatten* them. The definition of homotopy equivalence, although not obvious to grasp at first sight, should be seen as a relaxation of the definition of homeomorphism. Remind that $X$ and $Y$ are homeomorphic if there exist a continuous and invertible map $f: X \to Y$ such that $f^{-1}$ is continuous. This is equivalent to say that there exists two continuous maps $f$ and $g$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In the case of homotopy equivalence, we do not ask $g \circ f$ and $f \circ g$ to be exactly equal to the identity maps, but 'equal up to homotopy'. Actually, it turns out that homeomorphism equivalence is a stronger notion than homotopy equivalence:

**Proposition 4.7.** *Let $X, Y$ be two topological spaces. If they are homeomorphic, then they are homotopy equivalent.*

**Proof.** Let $f: X \to Y$ be a homeomorphism. Then the pair of maps $(f, f^{-1})$ forms a homotopy equivalence between $X$ and $Y$. □

As a consequence, in order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic. However, this strategy does not always work: some spaces are homotopy equivalent but not homeomorphic. This is the case for $\mathbb{R}^n$ and $\{0\}$ for instance (see Example 4.12).

### §4.1.3 Deformation retractions.

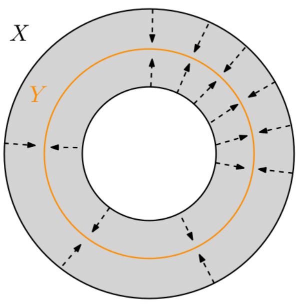
When one is a subset of the other, we have a handy tool to show homotopy equivalence:

**Definition 4.8.** Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $\mathscr{T}_{|Y}$. A *retraction* is a continuous map $r: X \to X$ such that $\forall x \in X, r(x) \in Y$ and $\forall y \in Y, r(y) = y$. A *deformation retraction* is a homotopy $F: X \times [0,1] \to Y$ between the identity map $\mathrm{id}: X \to X$ and a retraction $r: X \to X$.

**Proposition 4.9.** *If a deformation retraction exists, then $X$ and $Y$ are homotopy equivalent.*

**Proof.** Let $r: X \to X$ denote the retraction, and consider the inclusion map $i: Y \to X$. Note that, since $\forall x \in X, r(x) \in Y$, we can see the retraction $r$ as a map $r: X \to Y$. Let us prove that $r, i$ is a homotopy equivalence. First, let us prove that $i \circ r: X \to X$ is homotopic to the identity map $\mathrm{id}: X \to X$. This is clear because $i \circ r = r$, and $r$ is homotopic to the identity by definition of a deformation retraction. Second, let us prove that $r \circ i: Y \to Y$ is homotopic to the identity map $\mathrm{id}: Y \to Y$. This is obvious because $r \circ i = \mathrm{id}$ by definition of a retraction. □

**Example 4.10.** The circle and the annulus are homotopy equivalent. Indeed, the circle can be seen as a subset of the annulus, and we have a deformation retraction:




**Example 4.11.** The letter O and the letter Q are homotopy equivalent. Indeed, O can be seen as a subset of Q, and Q deform retracts on it.

**Example 4.12.** For any $n \geq 1$, the Euclidean space $\mathbb{R}^n$ is homotopy equivalent to the point $\{0\} \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r\colon \mathbb{R}^n &\longrightarrow \{0\} \\ x &\longmapsto 0 \end{aligned}$$

It is homotopic to the identity $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ via the deformation retraction

$$\begin{aligned} F\colon \mathbb{R}^n \times [0,1] &\longrightarrow \mathbb{R}^n \\ x &\longmapsto (1-t)x \end{aligned}$$

Indeed, we have $F(\cdot,0) = \mathrm{id}$ and $F(\cdot,1) = r$.

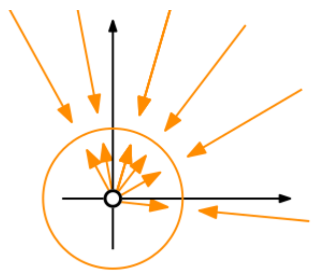

**Example 4.13.** For any $n \geq 1$, the Euclidean space without origin, $\mathbb{R}^n \setminus \{0\}$, is homotopy equivalent to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r\colon \mathbb{R}^n \setminus \{0\} &\longrightarrow \mathbb{S}^{n-1} \\ x &\longmapsto \frac{x}{\|x\|} \end{aligned}$$

It is homotopic to the identity $\mathrm{id}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ via the deformation retraction

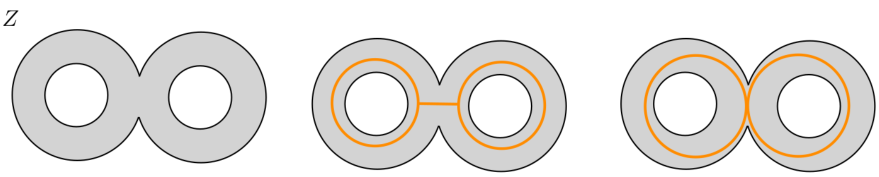
$$\begin{aligned} F\colon (\mathbb{R}^n \setminus \{0\}) \times [0,1] &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ x &\longmapsto \left(1 - t + \frac{t}{\|x\|}\right)x \end{aligned}$$

Indeed, we have $F(\cdot,0) = \mathrm{id}$ and $F(\cdot,1) = r$.

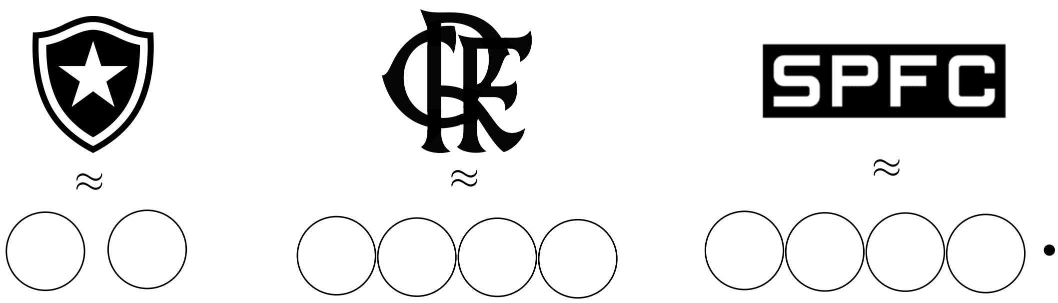


We now give another method to show that two topological spaces $X, Y$ are homotopy equivalent: find a third space $Z$ that contains $X, Y$ and such that there exist a deformation retraction from $Z$ to $X$ and from $Z$ to $Y$. If this is the case, we have that $X$ is homotopy equivalent to $Z$, and that $Y$ is homotopy equivalent to $Z$. Moreover, just as we have seen in Exercise 31, one shows that 'being homotopy equivalent' is transitive. We deduce that $X$ and $Y$ are homotopy equivalent. For instance, consider the two following subspaces of $\mathbb{R}^2$:


They are not included one in another. However, the following space contains them, and we see that it deform retracts on both $X$ and $Y$.




**Example 4.14.** We have the following homotopy equivalences:




**Exercise 32** (Homotopy classes in the alphabet)**.** Classify the letters of the alphabet into homotopy classes.

**Exercise 33.** Show that the Möbius trip and the cylinder are homotopy equivalent (see Exercise 28).

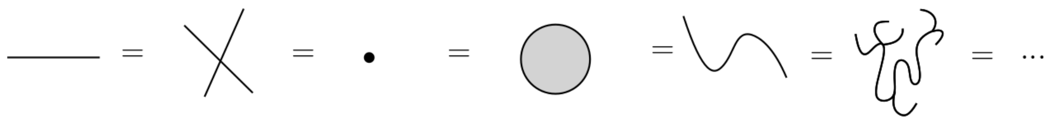
*Hint:* Show that they can be both retracted onto a circle.

**Exercise 34** (Two-out-three property). If $f\colon X \to Y$ and $f\colon Y \to Z$ are maps, then if any two of the maps $f$, $g$ and $g \circ f$ are homotopy equivalences, so is the third map.

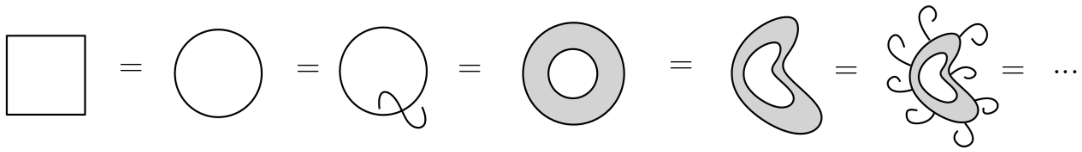
## 4.2 Invariants of homotopy classes

**§4.2.1 Homotopy classes.** Let us denote $X \approx Y$ if the two topological spaces $X$ and $Y$ are homotopy equivalent. Just as for homeomorphic spaces, one shows that 'being homotopy equivalent' is an *equivalence relation*. Consequently, we can classify topological spaces according to this relation, and obtain *classes of homotopy equivalence*:

- the class of points:



- the class of circles:



- the class of spheres, the class of torii, the class of Klein bottles, etc...

At this point, we are not able to prove that the point, the circle and the sphere are not homotopy equivalent. This will come soon, using Brouwer's theorem.

**Exercise 35.** Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).

**§4.2.2 CONNECTEDNESS.** We now investigate how the invariants behave with respect to the homotopy equivalence. The following result should be compared with Proposition 3.13:

**Proposition 4.15.** *Two homotopy equivalent topological spaces admit the same number of connected components.*

**Proof.** Let $f\colon X \to Y$ and $g\colon Y \to X$ be a homotopy equivalence between $X$ and $Y$. Let us denote $H\colon X \times [0,1] \to X$ the homotopy between $g \circ f$ and id. We will show that $f$ induces a correspondence between the connected components of $X$ and $Y$. Let $A \subset X$ be a connected component. The product $A \times [0,1]$ is a connected subset of $X \times [0,1]$. Hence its image $H(A \times [0,1])$ is a connected subset of $X$, therefore it is contained in a connected component of $Y$. Moreover, we have $H(A \times \{1\}) = \mathrm{id}(A) = A$. Hence $H(A \times [0,1])$ is contained in the connected component of $A$. Moreover, $H(A \times \{0\}) = g \circ f(A)$. We deduce that $g \circ f(A) \subset A$. Similarly, one proves that for all connected component $B \subset Y$, one has $f \circ g(B) \subset B$. As a consequence, if $A, A'$ are two distinct connected components of $X$, we have that $f(A)$ and $f(A')$ belongs to distinct connected components of $B$. This proves the result. □

**Example 4.16.** For any $n, m \geq 0$ such that $n \neq m$, the subspaces $\{1, ..., n\}$ and $\{1, ..., m\}$ of $\mathbb{R}$ are not homotopy equivalent. Indeed, the first one admits $n$ connected components, and the second one $m$ components.

**§4.2.3 DIMENSION.** On the other hand, dimension, introduced in §3.3.3, is **not** an invariant of homotopy equivalence. That is, certain homotopy equivalent spaces have different dimensions. This is the case, for instance, with all the Euclidean spaces $\mathbb{R}^n$, $n \geq 0$. They are all homotopy equivalent by Example 4.12, but all with different dimensions ($\mathbb{R}^n$ has dimension $n$).

§4.2.4 **Contractibility.** Let $\{\mathrm{pt}\}$ denote the one-point set, endowed with the trivial topology (it is the only topology it admits). A topological space $(X, \mathscr{T})$ is said to be *contractible* if it is homotopy equivalent to $\{\mathrm{pt}\}$. Equivalently, as a consequence of Definition 4.6, it means that the identity map $\mathrm{id}\colon X \to X$ is homotopic to a constant map. Of course, 'being contractible' is an invariant of homotopy classes. From a topological point of view, we consider that the contractible spaces are the most simple ones.

A large collection of such spaces is given by the convex subsets of $\mathbb{R}^n$. Remind that a subset $X \subset \mathbb{R}^n$ is *convex* if for any $x, y \in X$, the segment $[x,y] = \{(1-t)x + ty \mid t \in [0,1]\}$ is included in $X$.

**Proposition 4.17.** *Let $X \subset \mathbb{R}^n$ be a convex subset of $\mathbb{R}^n$ and endow it with the subspace Euclidean topology. Then it is contractible.*

**Proof.** Let $x \in X$ be any point. We will show that the identity map $\mathrm{id}\colon X \to X$ is homotopic to the constant map $c_x\colon X \to \{x\}$. Consider the map

$$\begin{aligned} H\colon X \times [0,1] &\longrightarrow X \\ (y,t) &\longmapsto (1-t)y + tx \end{aligned}$$

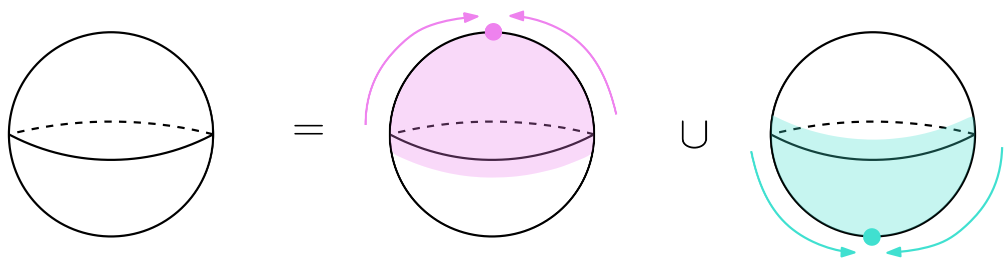
This map is continuous, $H(\cdot, 0) = \mathrm{id}$ and $H(\cdot, 1) = c_x$. Hence it the desired homotopy. $\square$

**Exercise 36.** Show that Proposition 4.17 is still true if $X$ is only *star-shaped*, that is, if there exists $x \in X$ such that for all $y \in X$, the segment $[x,y]$ is included in $X$.

§4.2.5 **Lusternik–Schnirelmann category.** In their study of critical points on manifolds, Lusternik and Schnirelmann introduced an invariant of topological spaces, now known as the LS category [7, 8]. Given a topological space $(X, \mathscr{T})$, we say that an open set $U \subset X$ is *categorical* if the inclusion map $U \hookrightarrow X$ is homotopic to a constant map. Then $\mathrm{cat}(X)$ is defined as the minimal number of categorical open sets needed to cover $X$, minus one. For instance, $X$ has LS category 0 is and only if it is contractible. From this point of view, the LS category can be seen as a generalization of the notion of contractibility.

Note that, for an open subset $U$, 'being categorical' is a weaker property than 'being contractible'. For instance, in a contractible space $X$, any subset is categorical. Even if $U$ is not connected.

As an example, the sphere $\mathbb{S}^n$ has LS category equal to 1. Indeed, a cover in two contractible open sets can be obtained by considering the hemispheres $\mathbb{S}^n_+$ and $\mathbb{S}^n_-$, that we thicken a little bit in order to obtain open sets. They can be contracted onto the north pole and south pole respectively. This gives an upper bound $\mathrm{cat}(\mathbb{S}^n) \leq 1$. The lower bound $\mathrm{cat}(\mathbb{S}^n) \geq 1$ will be proved later using Brouwer's theorem.



As another example given without a proof, the torus $\mathbb{T}^n$, defined as the product $(\mathbb{S}^1)^n$, has LS category equal to $n$.

**Exercise 37** (LS category of the torus)**.** Show that the LS category of the 2-torus is at most 2, by drawing an explicit example of cover.

## 4.3 Algebraic-homotopy invariants

Homotopy is a fundamental notion in topology, allowing to define some of the most important invariants: the fundamental groups, the homotopy groups, the mapping class groups, etc. In this section, we will give a glimpse of these notions.

**§4.3.1 Path-connectedness.** Let $(X, \mathscr{T})$ be a topological space, and $x, y \in X$ two points. We define a *path* from $x$ to $y$ as a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

**Definition 4.18.** We say that $(X, \mathscr{T})$ is *path-connected* if for every $x, y \in X$, there exists a path $\gamma$ from $x$ to $y$.

In other words, a space is path-connected if we can draw a path between any two points. This turns out to be a stronger notion than connectedness, introduced in §2.3.1.


**Proposition 4.19.** *If a topological space $(X, \mathscr{T})$ is path-connected, then it is connected.*

**Proof.** We will prove the contraposition. Suppose that $X$ is not connected. Hence we have a partition $X = U \cup V$ into two disjoint clopen sets. Let $x \in U$ and $y \in V$. Suppose that $\gamma\colon [0,1] \to X$ is a path from $x$ to $y$. Then the preimages $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are clopen subsets of $[0,1]$, and are disjoint since $x \neq y$. This is absurd since $[0,1]$ is connected. $\square$

In practice, it may be easier to prove that a space is path-connected than connected. However, some spaces are connected without being path-connected, as shown in the following.

**Exercise 38** (Adherence of topologist's sine curve)**.** Let $X \subset \mathbb{R}^2$ be the adherence of the topologist's sine curve, defined in Exercise 11. Explicitely, it is

$$X = \{(x, \sin(1/x) \mid x \in (0, \pi]\} \cup \{(0,t) \mid t \in [-1,1]\}$$

Endow $X$ with the subspace topology. Show that $X$ is connected but not path-connected.

**Remark 4.20.** Some partial converses to Proposition 4.19 exist. For instance, if $X$ is an connected *open* subset of $\mathbb{R}^n$, then one shows that it is path-connected.

**§4.3.2 Fundamental groups.** In what follows, we parametrize the circle $\mathbb{S}^1$ by the interval $[0,1)$. Let $(X, \mathscr{T})$ be a path-connected, and $x_0 \in X$ a point. We define a *loop* with base $x_0$ as a continuous map $\gamma\colon \mathbb{S}^1 \to X$ such that $\gamma(0) = x_0$. In other words, it is a path from $x_0$ to $x_0$. Two loops $\gamma, \gamma'$ are *homotopic* if there exists a homotopy $H\colon \mathbb{S}^1 \times [0,1] \to X$ from $\gamma$ to

$\gamma'$. 'Being homotopic' is an equivalence relation on the set of loops on $X$, and we consider the quotient set

$$\pi_1(X, x_0) = \{\text{loops } \mathbb{S}^1 \to X\}/\text{homotopy equivalence}.$$

If $\gamma$ is a loop, we denote by $[\gamma]$ its equivalence class. Given two loops $\gamma, \gamma'$, the *concatenation* $\gamma\gamma'$ is defined as the loop such that $\gamma\gamma'(t) = \gamma(2t)$ if $t \leq 1/2$, and $\gamma\gamma'(t) = \gamma(2t-1)$ if $t \geq 1/2$. As a direct consequence of the definitions, have the following property:

**Proposition 4.21.** *Let $\gamma$ and $\gamma'$ be two loops. If $\eta$ is a loop homotopic to $\gamma$, and $\eta'$ is a loop homotopic to $\gamma'$, then the concatenation $\eta\eta'$ is homotopic to $\gamma\gamma'$.*

Consequently, we can define the concatenation between homotopy classes: for $[\gamma]$ and $[\gamma]$ in $\pi_1(X, x_0)$, $[\gamma\gamma']$ does not depend on the choice of $\gamma$ and $\gamma'$.

**Definition 4.22.** The set $\pi_1(X, x_0)$, endowed with the concatenation operation $[\gamma][\gamma'] = [\gamma\gamma']$, is called the *fundamental group* of $X$ with base $x_0$.

**Proposition 4.23.** *The fundamental group is a group.*

**Proof.** We have to check the three axioms of a group: existence of neutral element, existence of an inverse, and associativity. We only give an idea of the proof. The neutral element is the constant loop. For any loop $\gamma$, its inverse is the reversed loop $t \mapsto \gamma(1-t)$. Last, the associativity is proven by re-parametrizing the loops $(\gamma\gamma')\gamma''$ and $\gamma(\gamma'\gamma'')$. □

If $X$ is path-connected, then $\pi_1(X, x_0)$ does not depend on $x_0$, hence we can talk about the *fundamental group* $\pi_1(X)$.

**Example 4.24.** If $X$ is a contractible topological space, then all the loops are homotopic to a constant map, hence $\pi_1(X) = \{0\}$. For instance, this is the case for the convex subsets of $\mathbb{R}^n$.

As shown by the following proposition, the fundamental group is an invariant of homotopy classes. Hence, as we have seen with the number of connected components, and the Lusternik–Schnirelmann category, it can be used to prove that two spaces are not homotopy equivalent. We will make use of this fact in the next paragraph.

**Proposition 4.25.** *If two path-connected topological spaces $X$ and $Y$ are homotopy equivalent, then the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.*

**Proof.** Let $f\colon X \to Y$ and $g\colon Y \to X$ be a homotopy equivalence. Consider the map

$$\begin{aligned} f\colon \pi_1(X) &\longrightarrow \pi_1(Y) \\ [\gamma] &\longmapsto [f \circ \gamma] \end{aligned}$$

Is is well defined since for any $\gamma' \in [\gamma]$, $f \circ \gamma'$ is included in $[f \circ \gamma]$. □

**§4.3.3 Fundamental group of the circle.** The example of the circle is particularly interesting. One shows that $\pi_1(\mathbb{S}^1)$ is equal to $\mathbb{Z}$, the group of integers. Proofs of this result use the theory of *covering spaces*, that is out of the scope of this course. Instead, we will give some elements of intuition.

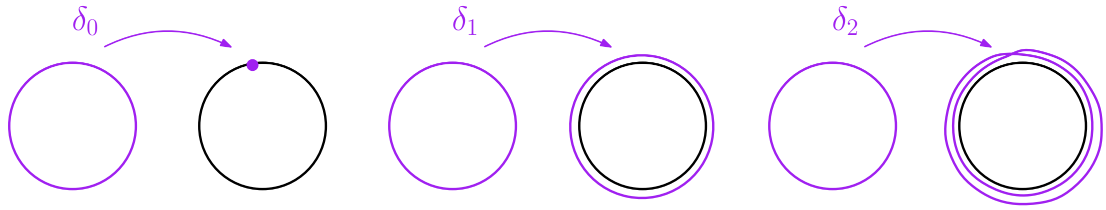
Let us parametrize the circle by an angle $\theta \in [0, 1)$. Let $m \in \mathbb{Z}$. Seeing also $\mathbb{S}^1$ as a subset of the plane $\mathbb{R}^2$, we define a map $\delta_m\colon \mathbb{S}^1 \to \mathbb{S}^1$ by

$$\delta_m\colon \theta \mapsto (\cos(2\pi m\theta), \sin(2\pi m\theta)).$$

The map $\delta_m$ is a loop that *winds m times* around the circle. One shows that the map

$$\begin{aligned} \mathbb{Z} &\to \pi_1(\mathbb{S}^1) \\ m &\mapsto \delta_m \end{aligned}$$

is an isomorphism of groups. In other words, each loop of $\mathbb{S}^1$ is homotopic to a map $\delta$ (surjectivity) and no maps $\delta_m$, $\delta_{m'}$ are homotopic is $m \neq m'$ (injectivity).




As we have seen before, the fundamental group of a contractible space is $\{0\}$. Therefore, as a consequence of Proposition 4.25, the circle is not homotopy equivalent to a contractible space. This is the classical proof that the circle is not contractible.

**§4.3.4 Homotopy groups.** Let $X$ be path-connected space. The notion of fundamental group admits a direct generalization to higher dimensions. Instead of considering maps $\mathbb{S}^1 \to X$, we can study the maps $\mathbb{S}^n \to X$ for any $n \geq 1$. These maps can be compared via homotopy equivalence, and we define the $n^{th}$ *homotopy group* as

$$\pi_n(X) = \{\text{loops } \mathbb{S}^n \to X\}/\text{homotopy equivalence}.$$

As it it the case for $\pi_1$, the $\pi_n$'s can be given a group structure. Moreover, one shows that they are invariant of homotopy classes: if $X$ and $Y$ are homotopy equivalent spaces, then the groups $\pi_n(X)$ and $\pi_n(Y)$ are isomorphic.

The theory of homotopy groups is reputed to be difficult and intriguing. In particular, computing the homotopy groups of spheres is still an open area of research. We give in the following table their first homotopy groups. The notation $\mathbb{Z}/n\mathbb{Z}$ refers to the cyclic group with $n$ elements.

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{S}^1$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbb{S}^2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/12\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $\mathbb{S}^3$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/12\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $\mathbb{S}^4$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}$ |
| $\mathbb{S}^5$ | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |

**§4.3.5 Mapping class groups.** Let $X$ be a topological space. We denote by $\mathrm{Aut}(X)$ the set of *automorphisms* of $X$, that is, the set of homeomorphisms $X \to X$. In this context, we want to compare the automorphisms not up to homotopy, but up to *isotopy*. An isotopy between two automorphisms $g, f \colon X \to X$ is a homotopy $H \colon X \times [0,1] \to X$ between $f$ and $g$ such that for each $t \in [0,1]$, the map $H(\cdot, t)$ is a homeomorphism. 'Being isotopic' is an equivalence relation between homeomorphisms. It is a stronger notion than homotopy. We define the *mapping class group* of $X$ as

$$MCG(X) = \mathrm{Aut}(X)/\text{isotopy equivalence}.$$

One shows that the mapping class group is an invariant of homotopy classes. Let us give some examples, without proofs:

- the circle: $MCG(\mathbb{S}^1) = \mathbb{Z}/2\mathbb{Z}$, corresponding to the maps $\delta_1$ and $\delta_{-1}$ introduced in §4.3.3,
- the sphere: $MCG(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$, corresponding also to orientation-preserving or reversing maps,
- the tori: $MCG(\mathbb{T}^n) = GL(n,\mathbb{Z})$.

# 5 Metric topology

## 5.1 Metric spaces

### §5.1.1 Definition.

**Definition 5.1.** A *metric space* is a pair $(X,d)$ where $X$ is a set and $d\colon X\times X \to [0,+\infty)$ a map such that

$$\begin{aligned}
\textit{(positivity)}\quad & \forall x,y\in X,\ d(x,y)=0 \iff x=y\\
\textit{(symmetry)}\quad & \forall x,y\in X,\ d(x,y)=d(y,x)\\
\textit{(triangle inequality)}\quad & \forall x,y,z\in X,\ d(x,z)\le d(x,y)+d(y,z)
\end{aligned}$$

The map $d$ is called a *metric*, or a *distance*. Two metrics $d,d'$ on $X$ are said *equivalent* if there exists $\alpha,\beta>0$ such that for all $x,y\in X$,

$$\alpha d(x,y) \le d'(x,y) \le \beta d(x,y).$$

On a subset $A\subset X$ of a metric space $(X,d)$, one defines the *induced metric*, or *restricted metric*, as $d_A(x,y) = d(x,y)$ for all $x,y\in A$.

Given two metric spaces $(X,d)$ and $(X',d')$, a map $f\colon X\to X'$ is said *isometric* if $d'(f(x),f(y)) = d(x,y)$ for all $x,y\in X$. Note that an isometric map is necessarily injective. We will also call such a map an *isometric embedding*. More, if $f$ is bijective, we call it an *isometry*, and the spaces $(X,d),(X',d')$ are said *isometric*.

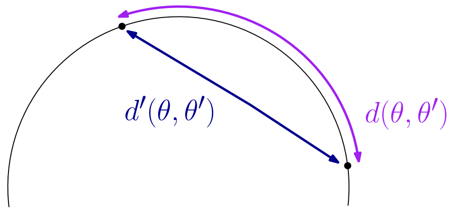
**Example 5.2.** On $\mathbb{R}^2$, one defines:

- The Euclidean distance $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
- The SNCF distance $d_{\mathrm{SNCF}}((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ if $(x_1,y_1)$ and $(x_2,y_2)$ are colinear and $\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}$ otherwise.

They are not equivalent. Indeed, with $a=(1,0)$ and $b=(1,\varepsilon)$, we have $d(a,b)=\varepsilon$, but $d_{\mathrm{SNCF}}(a,b) = 1+\sqrt{1+\varepsilon^2}$. Hence we cannot find a $\beta>0$ such that $d_{\mathrm{SNCF}}\le\beta d$.

**Example 5.3** (Graph distance)**.** Let $G$ be a graph. We define the distance between two vertices $x,y$ as the number of edges in a shortest path connecting them. More generally, suppose that $G$ is a weighted graph, that is, each edge $e$ is endowed with a positive weight $w(e)\in(0,+\infty)$. We define the cost of a path as the sum of the weights of its edges. We then define the distance between two vertices $x,y$ as the smallest cost of a path connecting them.

**Exercise 39.** Let $\mathbb{S}^1$ be the unit circle of $\mathbb{R}^2$. Let $d$ denote the restricted Euclidean distance, and $d'$ the distance defined as $d'(\theta,\theta') = |\theta - \theta'|$. Show that they are equivalent.
*Hint:* First, show that $d(\theta,\theta') = 2|\sin(d'(\theta,\theta')/2)|$.




**Exercise 40** ($p$-adic metric)**.** Let $p > 1$ be a prime number. On $\mathbb{Z}$, we define

$$d(x,y) = \begin{cases} 0 & \text{for } x = y, \\ \frac{1}{p^k} & \text{for } k = \max\{i \geq 0 \mid x - y \equiv 0[p^i].\} \end{cases}$$

1. Show that $d$ is a metric.
2. For $y \in \mathbb{Z}$, show that the translation $x \mapsto x + y$ is an isometry from $(\mathbb{Z}, d)$ to $(\mathbb{Z}, d)$.

**§5.1.2 Topology induced by a metric.** Let $(X,d)$ be a metric space. We define, for all $x \in X$ and $r > 0$,

- the open balls $\mathscr{B}(x,r) = \{y \in \mathbb{R}^n \mid d(x,y) < r\}$,
- the closed balls $\overline{\mathscr{B}}(x,r) = \{y \in \mathbb{R}^n \mid d(x,y) \leq r\}$,
- the spheres $\mathbb{S}(x,r) = \{y \in \mathbb{R}^n \mid d(x,y) = r\}$.

By mimicking the construction of the Euclidean topology, we can endow $X$ with a topology.

**Definition 5.4.** On $X$, the *topology induced by the metric $d$* is defined as the topology generated by the open balls $\mathscr{B}(x,r)$ where $x \in X$ and $r > 0$ (see Definition 1.10). It is denoted $\mathscr{T}_d$.

As it is the case for the Euclidean topology, a set $A \subset X$ is open for the topology induced by the metric $d$ if and only if for every $x \in A$, there exists a $r > 0$ such that $\mathscr{B}(x,r) \subset A$.

**Proposition 5.5.** *Let $(X,d)$ be a metric space and $\mathscr{T}_d$ the topology induced by the distance $d$. The closed balls $\overline{\mathscr{B}}(x,r)$ are closed in $\mathscr{T}_d$.*

**Proposition 5.6.** *If the distances $d, d'$ on $X$ are equivalent, then the induced topologies $\mathscr{T}_d$ and $\mathscr{T}_{d'}$ are equal.*

**Remark 5.7.** It is possible that the topologies $\mathscr{T}_d$ and $\mathscr{T}_{d'}$ coincide but the distances are not equivalent. An example is given by $\mathbb{R}^+$ endowed with the Euclidean metric $d(x,y) = |x-y|$ and the metric $d'(x,y) = |\sqrt{x} - \sqrt{y}|$.

**Exercise 41** (Discrete distance)**.** Let $X$ be a set and $d$ the distance on $X$ defined as $d(x,y) = 1$ if $x = y$ and 0 otherwise. Show that the induced topology $\mathscr{T}_d$ is the discrete topology.

**Exercise 42** (SNCF distance)**.** Let $d$ and $d_{\text{SNCF}}$ denote the the Euclidean and SNCF distance (see Example 5.2). Show that the induced topology $\mathscr{T}_{d_{\text{SNCF}}}$ is strictly finer than $\mathscr{T}_d$.

Consider two metric spaces $(X,d)$, $(X',d')$ and $(X,\mathscr{T}_d)$, $(X',\mathscr{T}_{d'})$ the corresponding topological spaces. By applying the same reasoning as in §3.1.2, one obtains the following.

**Proposition 5.8.** *A a map $f\colon (X,\mathscr{T}_d)\to(X',\mathscr{T}_{d'})$ is continuous if and only if for every $x\in X$ and $\varepsilon>0$, there exists $\eta>0$ such that for all $y\in X$, we have $d(x,y)<\eta \implies d'(f(x),f(y))<\varepsilon$.*

Let $\lambda>0$. A map $f\colon (X,d)\to(X',d')$ is said $\lambda$-*Lipschitz* if for every $x,y\in X$, we have $d'(f(x),f(y))\le\lambda d(x,y)$. Such a map is continuous. As a particular case, if $f\colon (X,d)\to(X',d')$ is an isometric embedding, then it is 1-Lipschitz, hence continuous.

**Exercise 43** (Ultrametric spaces)**.** A distance $d$ on $X$ is said *ultrametric* if $\forall x,y,z\in X$, $d(x,z)\le\max(d(x,y),d(y,z))$. If $d$ is an ultrametric distance, show that

1. $\forall x\in X$, $r>0$ and $y\in\mathscr{B}(x,r)$, $\exists r'>0$ such that $\mathscr{B}(x,r)=\mathscr{B}(y,r')$.
2. $\forall x,y\in X$ and $r,r'>0$, either $\mathscr{B}(x,r)\subset\mathscr{B}(y,r')$ or $\mathscr{B}(x,r)\supset\mathscr{B}(y,r')$, or $\mathscr{B}(x,r)\cap\mathscr{B}(y,r')=\emptyset$.

**§5.1.3 Normed vector spaces.** Several common examples of metric spaces actually have an additional structure: being a vector space. In this case, a particular notion of distance if defined. In what follows, by vector space, we mean $\mathbb{R}$-vector space, although the theory is similar for $\mathbb{C}$.

**Definition 5.9.** A *normed vector space* is a pair $(X,\|\cdot\|)$ where $X$ is a vector space (potentially of infinite dimension) and $\|\cdot\|\colon X\to[0,+\infty)$ a map such that

*(positivity)* $\forall x\in X$, $\|x\|=0\iff x=0$

*(homogeneity)* $\forall x\in X$ and $\lambda\in\mathbb{R}$, $\|\lambda x\|=|\lambda|\|x\|$

*(sub-additivity)* $\forall x,y\in X$, $\|x+y\|\le\|x\|+\|y\|$

The map $\|\cdot\|$ is called a *norm*. Two distances $\|\cdot\|,\|\cdot\|'$ on $X$ are said *equivalent* if there exists $\alpha,\beta>0$ such that for all $x\in X$, we have $\alpha\|x\|\le\|x\|'\le\beta\|x\|$.

**Example 5.10** ($p$-norms on $\mathbb{R}^n$)**.** Let $p\in[1,+\infty)$. We define on $\mathbb{R}^n$ the norm

$$\|(x_1,\dots,x_n)\|_p=\Big(\sum_{k=1}^{n}|x|^p\Big)^{\frac{1}{p}}.$$

For $p=+\infty$, we define $\|(x_1,\dots,x_n)\|_\infty=\max(|x_1|,\dots,|x_n|)$. They are all norms.

**Example 5.11** ($\ell^p$-spaces)**.** Let $p\ge1$. The space of $p$-summable sequences is a vector space, endowed with the $p$-norm

$$\ell^p=\left\{(x_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}\mid\sum_{i=0}^{+\infty}|x_i|^p<+\infty\right\},\qquad \|(x_n)\|_p=\left(\sum_{i=0}^{+\infty}|x_i|^p\right)^{\frac{1}{p}}.$$

The fact that $\|\cdot\|_p$ satisfies the sub-additivity axiom is known as Minkowski inequality.

**Example 5.12** (Lebesgue $L^p$-spaces)**.** Let $(X,\mathscr{F},\mu)$ be a measured space, and $p\ge1$. Just as previously, we define the space of $p$-integrable functions $f\colon\mathbb{R}^n\to\mathbb{R}$ and the $p$-norm as

$$\mathscr{L}^p(\mu)=\left\{f\colon\mathbb{R}^n\to\mathbb{R}\mid\int f(x)\mathrm{d}\mu(x)<+\infty\right\},\qquad \|f\|_p=\left(\int f(x)\mathrm{d}x\right)^{\frac{1}{p}}.$$

However, the operator $\|\cdot\|_p$ may not satisfy the first axiom of a norm (think about a non-negative function with integral zero but that is not zero everywhere). To overcome this issue, let $\mathscr{N} = \{f \in \mathscr{L}^p(\mu) \mid \|f\|_p = 0\}$, and consider the quotient vector space $L^p(\mu) = \mathscr{L}^p(\mu)/\mathscr{N}$. The pair $(L^p(\mu), \|\cdot\|_p)$ now forms a normed vector space, called the *Lebesgue space*.

**Proposition 5.13.** *If $(X, \|\cdot\|)$ is a normed vector space, then the map $d\colon (x,y) \mapsto \|x-y\|$ is a distance on $X$.*

As a consequence of the previous proposition, any normed vector space is associated to a particular distance, hence to a particular topology.

**Exercise 44.** Show that all the norms $\|\cdot\|_p$ on $\mathbb{R}^n$, for $p \in [1, +\infty)$, are equivalent.

*Remark:* As we will see in the chapter about compacity, on a finite-dimensional vector space, all the norms are equivalent.

**Exercise 45.** For any $x \in \mathbb{R}^n$, show that $\lim_{p\to+\infty} \|x\|_p = \|x\|_\infty$.

**Exercise 46** (Projection on the 1-ball)**.** Equip $\mathbb{R}^n$ with the 1-norm. Let $r > 0$ and denote by $\overline{\mathscr{B}}_1(0,r)$ the closed ball. We are interested in the *projection operator* on $\overline{\mathscr{B}}_1(0,r)$, that is, the map $p\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$p(x) = \operatorname{argmin}\{\|x-y\|_2 \mid y \in \overline{\mathscr{B}}_1(0,r)\}.$$

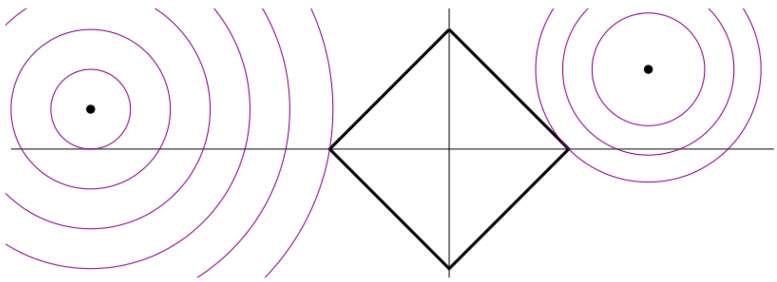
When $\|x\|_1 \leq r$, it is clear that $p(x) = x$, hence we suppose $\|x\| > r$. For any $\lambda \geq 0$, we define

$$S_\lambda(x) = ([x_i(1-\lambda/|x_i|)]_+)_{1\leq i\leq n}$$

where $[\cdot]_+$ denote the positive part.

1. Prove that $S_\lambda(x) \in \operatorname{argmin}\{\|x-y\|_2 + \lambda\|y\|_1 \mid y \in \overline{\mathscr{B}}_1(0,r)\}$
2. Prove that $p(x)$ is equal to $S_\lambda(x)$ where $\lambda$ is such that $\|S_\lambda(x)\|_1 = r$.

This means that projection on the $\ell^1$-ball tends to nullify coordinates. This idea is at the core of the *lasso regression*. An explicit expression for $\lambda$ is given in [9, Exercise 5.5.6].



Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be two normed vector spaces, and let $\mathscr{L}(E,F)$ denote the set of linear maps $E \to F$. When $F = \mathbb{R}$, it is called the *(algebraic) dual space* of $E$. It is a vector space. For any $f \in \mathscr{L}(E,F)$, we define

$$\begin{aligned}\|f\| &= \sup\left\{\frac{\|f(x)\|_F}{\|x\|_E} \mid x \in E\setminus\{0\}\right\}\\ &= \sup\left\{\|f(x)\|_F \mid x \in E,\ \|x\|_E = 1\right\}.\end{aligned} \tag{I.1}$$

**Proposition 5.14.** *The linear map $f \in \mathscr{L}(E,F)$ is continuous if and only if $\|f\|$ is finite.*

Let $\mathscr{L}_c(E,F)$ denote the space of *continuous* linear map maps $E \to F$. When $F = \mathbb{R}$, it is called the *topological dual space* of $E$. Endowed with the norm $\|\cdot\|$ defined Equation (I.1), $\mathscr{L}_c(E,F)$ is a normed vector space. When $E$ has finite dimension, we will see later that $\mathscr{L}_c(E,F) = \mathscr{L}_c(E,F)$, that is, all linear maps are continuous.

**Exercise 47.** Give an example of a linear map between two normed spaces that is not continuous.

## 5.2 Examples

**§5.2.1 Matrix subspaces** Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices. It is an algebra for the sum and product of matrices. We will denote by $A = (a_{i,j})_{1 \leq i,j \leq n}$ its elements. A first family of norms on $M_n(\mathbb{R})$ is given by the *entry-wise p-norms*:

$$\|A\|_p = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|^p \right)^{\frac{1}{p}}.$$

In the particular case $p = 2$, it is called the *Frobenius norm*, and is denoted $\|\cdot\|_{\mathrm{F}}$. It can also be written as $\|A\|_{\mathrm{F}} = \mathrm{tr}(A^\top A)$, where $A^\top$ denotes the transpose and tr the trace. The Frobenius norm is particular: it comes from a scalar product, and gives $M_n(\mathbb{R})$ the structure of a *Euclidean vector space*.

Another family is given by the norms *induced by vector norms*. If $\|\cdot\|_p$ denotes the $p$-norm on $\mathbb{R}^n$, we define

$$\|A\|_{(p)} = \sup\{\|Ax\|_p \mid x \in \mathbb{R}^n,\ \|x\| = 1\}.$$

They are sub-multiplicative: $\|AB\|_{(p)} \leq \|A\|_{(p)}\|B\|_{(p)}$. This property is particularly handy in the context of error analysis in linear computing, or to study iterates of linear maps.

For any matrix $A \in M_n(\mathbb{R})$, its *singular values* are defined as the square roots of the eigenvalues of the matrix $A^\top A$ (which is symmetric, hence admits non-negative eigenvalues). We list them in decreasing order $\sigma_1, \ldots, \sigma_n$. One shows that the induced 2-norm defined previously satisfies $\|A\|_{(2)} = \sigma_1$. Moreover, the entrywise 2-norm satisfies $\|A\|_2 = \sqrt{\sum_{i=1}^n \sigma_i^2}$. By mimicking this formula, for any $p \geq 1$, we define the *Schatten norm*

$$\|A\|_{((p))} = \left( \sum_{i=1}^{n} \sigma_i^p \right)^{\frac{1}{p}}.$$

**Exercise 48** (Rank distance)**.** Let $d \colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{N}$ be the map $d(A,B) = \mathrm{rank}(A - B)$.

1. Verify that $d$ is a distance.
2. Show that $d$ induces the discrete topology on $M_n(\mathbb{R})$.

*Remark:* This distance has been used in the context of phylogenetics in [10].

**Exercise 49** (Spectral radius)**.** Let $A \in M_n(\mathbb{R})$. We define its spectral radius $\rho(A)$ as the maximum modulus of its complex eigenvalues. Let $\|\cdot\|$ be any norm $M_n(\mathbb{R})$.

1. Show that $\rho(A) < 1 \implies \lim_{r\to+\infty} \|A^k\| = 0$ and $\rho(A) > 1 \implies \lim_{r\to+\infty} \|A^k\| = +\infty$.
2. Show that $\lim_{r\to+\infty} \|A^k\|^{\frac{1}{k}} = \rho(A)$.

*Hint:* Write $A$ in Jordan normal form.

§5.2.2 **Hausdorff distance.** Let $(X,d)$ be a metric space and $\mathscr{P}_{\mathrm{c}}(X)$ the set of non-empty bounded and closed subsets of $X$. For any $K \in \mathscr{P}_{\mathrm{c}}(X)$, we define the *distance to* $K$ as the map $d_K : X \to [0,+\infty)$ defined as

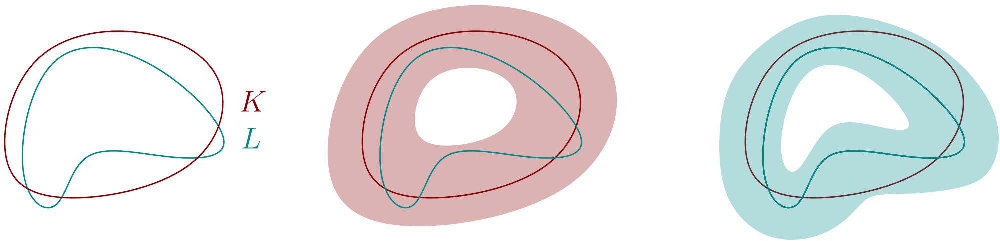
$$d_K(x) = \inf\{d(x,y) \mid y \in K\}.$$

The *Hausdorff distance* between $K,L \in \mathscr{P}_{\mathrm{x}}(X)$ is defined as

$$d_{\mathrm{H}}(K,L) = \max\left\{\sup_{x\in K} d_L(x),\ \sup_{x\in L} d_K(x)\right\}.$$

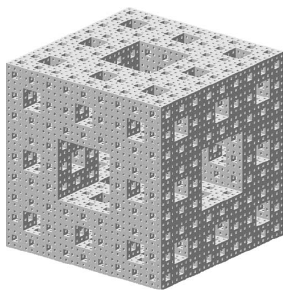
For any $\varepsilon > 0$, define the thickening $K^{\varepsilon} = \{x \in X \mid d_K(x) \leq \varepsilon\}$. One shows that

$$d_{\mathrm{H}}(K,L) = \inf\{\varepsilon > 0 \mid K \subset L^{\varepsilon},\ L \subset K^{\varepsilon}\}.$$




**Proposition 5.15.** *The Hausdorff distance is a metric on* $\mathscr{P}_c(X)$.

The Hausdorff distance allows to construct fractal objects, such as the Menger sponge [11].



§5.2.3 **Probability spaces** For an extensive review of metrics on probability spaces, see [12]. Let $(\Omega,\mathscr{F})$ be a probability space, and $\mu,\nu$ two probability measure. We define the *total variation* distance as

$$\delta(\mu,\nu) = \sup\{|\mu(A) - \nu(A)| \mid A \in \mathscr{F}\}$$

If $\mu$ and $\nu$ are absolutely continuous with respect to another measure $\lambda$, with densities $p$ and $q$, we define the Hellinger distance

$$H(\mu,\nu) = \sqrt{\frac{1}{2}\int (\sqrt{p(x)} - \sqrt{q(x)})^2 \mathrm{d}\lambda(x)}.$$

It is independent of the dominating measure $\lambda$. They satisfy the classical inequalities $H^2(\mu,\nu) \leq \delta(\mu,\nu) \leq \sqrt{2}H(\mu,\nu)$. But they are not equivalent.

Let $d$ be any metric on $\Omega$, $p \geq 1$ a real number, and $\mathscr{P}_p(\Omega)$ the set of $p$-integrable measures. The *p-Wasserstein metric* between $\mu,\nu \in \mathscr{P}_p(\Omega)$ is

$$W(\mu,\nu) = \left(\inf\left\{\mathbb{E}[d(X,Y)]^p \mid X \sim \mu, Y \sim \nu\right\}\right)^{\frac{1}{p}}$$

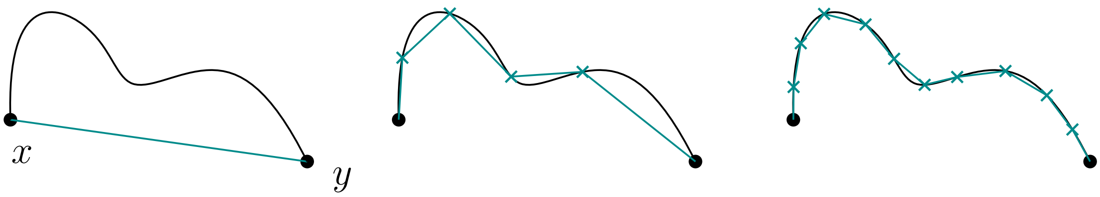
where the infimum is taken over all joint distributions $(X,Y)$ with marginals $\mu$ and $\nu$. The case where $p=2$ and $d$ is the Euclidean 2-norm on $\mathbb{R}^n$ is particularly studied. If the domain $\Omega$ is not bounded, the Wasserstein metric is not equivalent to the others.

## 5.3 Geodesics

**§5.3.1 Length spaces.** Let $(X,d)$ be a metric space, and $x,y \in X$ two points. Remind that a path between $x$ and $y$ is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0)=x$ and $\gamma(1)=y$. If $\gamma$ is a path, we define its *length* as

$$\operatorname{len}(\gamma) = \sup\left\{\sum_{i=1}^{n-1} d(\gamma(t_{i+1}),\gamma(t_i)) \mid 0=t_1<\cdots<t_n=1\right\}$$

where the supremum is taken over all subdivisions of $[0,1]$ and all $n\in\mathbb{N}$. In particular, we have $\operatorname{len}(\gamma) \geq d(\gamma(0),\gamma(1))$.




In what follows, we will suppose that $(X,d)$ is *path-connected by rectifiable curves*, meaning that there exists a finite-length path between any two points. We define the *intrinsic metric* for all $x,y\in X$ as

$$d_{\mathrm{i}}(x,y) = \inf\{\operatorname{len}(\gamma) \mid \gamma \text{ path from } x \text{ to } y\}. \tag{I.2}$$

**Definition 5.16.** A *length space* is a path-connected by rectifiable curves metric space $(X,d)$ such that $d=d_{\mathrm{i}}$.

**Proposition 5.17.** *If $(X,d)$ is path-connected by rectifiable curves, then $d_i$ is a metric on $X$.*

**Example 5.18** (Intrinsic metric on $\mathbb{S}^1$)**.** We continue the example of Exercise 39. Let $\mathbb{S}^1$ be the unit circle of $\mathbb{R}^2$, endowed with the restricted Euclidean distance. The intrinsic distance is $d_{\mathrm{i}}(x,y) = 2\arcsin(\|x-y\|/2)$.

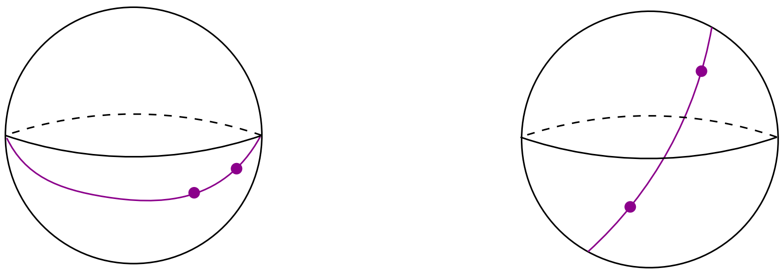
**§5.3.2 Geodesics.** Suppose that $(X,d)$ is path-connected by rectifiable curves, and let $x,y\in X$.

**Definition 5.19.** A path $\gamma$ from $x$ to $y$ is a *minimizing geodesic* if it attains the minimum of Equation (I.2). A length space $(X,d)$ is a *geodesic space* if it there exists a minimizing geodesic between each pair of points.

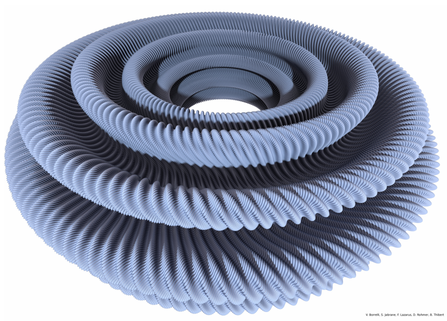
The punctured Euclidean space $\mathbb{R}^2\setminus\{0\}$ is an example of length space that is not a geodesic space: there is no minimizing geodesic between $(1,0)$ and $(-1,0)$. As we will see later, the generalized Hopf–Rinow theorem states that any locally compact complete length space is a geodesic space.

**Remark 5.20.** A minimizing geodesic satisfies $d_{\mathrm{i}}(\gamma(s),\gamma(t)) = |s-t|d_{\mathrm{i}}(x,y)$ for all $s,t \in [0,1]$. That is to say, for all $s,t \in [0,1]$ such that $s<t$, the curve $\eta : t \in [0,1] \mapsto \gamma((1-u)s+ut)$ is a minimizing geodesic from $\gamma(s)$ to $\gamma(t)$. More generally, in Riemannian geometry, a *geodesic* is understood as a path which satisfy the previous condition locally. A geodesic may not be a minimizing geodesic if it does not minimize the length globally.


**Example 5.21.** Let $\mathbb{S}^2 \subset \mathbb{R}^2$ be endowed with the intrinsic metric induced by the Euclidean metric on $\mathbb{R}^2$. It is a geodesic space. Moreover, the geodesic between any two points must lie in a great circle, that is, the intersection of the sphere and a plane passing through the origin.



**Example 5.22** (Flat torus)**.** Let $(X,d)$ be a metric space, and $\mathbb{R}^n$ endowed with the Euclidean distance. An *isometric embedding* of $X$ into $\mathbb{R}^n$ is an injective continuous map $f : X \to \mathbb{R}^n$ such that the intrinsic Euclidean distance on $f(X)$ corresponds to $d$. It is a well-studied problem in the context of Riemannian geometry. As an example, consider the *flat torus* $X$, obtained by gluing the opposite sides of the square $[0,1]\times[0,1]$. It is given the quotient metric $d(x,y) = \min\{\|x-y+(m,n)\| \mid m,n \in \mathbb{Z}\}$. Recently has been built an explicit isometric embedding of $(X,d)$ in $\mathbb{R}^3$ [13].



**Example 5.23** (Wasserstein space)**.** Let $X \subset \mathbb{R}^n$ be closed, bounded and convex, $(X,\mathscr{F})$ the probability space of the Borel algebra, $p \geq 1$ a real number and $\mathscr{P}_p(X)$ the $p$-integrable measure endowed with the Wasserstein distance $W$ (see §5.2.3). Given two measure $\mu,\nu \in \mathscr{P}_{\surd}(\mathscr{X})$, let $\Pi(\mu,\nu)$ denote an optimal *transport plan*, that is, a measure with marginals $\mu$ and $\nu$ such that $W(\mu,\nu) = \left(\{\mathbb{E}[d(X,Y)]^p \mid (X,Y)\sim\Pi(\mu,\nu)\}\right)^{\frac{1}{p}}$. For all $t\in[0,1]$, define the linear interpolation

$\pi_t : X \times X \to X$ as $\pi_t(x,y) = (A-t)x + ty$, and define the family of measures

$$\gamma_t = (\pi_y)_\sharp \Pi(\mu,\nu).$$

One shows that the path $t \mapsto \gamma_t$ is a geodesic from $\mu$ to $\nu$.

**Exercise 50** (SNCF distance)**.** Show that the plane $\mathbb{R}^2$ endowed with the SNCF distance (defined in Example 5.2) is a geodesic space.

# 6 Limits and completeness

## 6.1 Limits

The notion of limit of a map between topogical spaces is defined in greater generality using the notion of *filter* [14]. In these notes, we will give a particular version of this definition, already quite general.

**§6.1.1 Topological definition.** In what follows, $(X,\mathscr{T})$ and $(Y,\mathscr{U})$ are two topological spaces.

**Definition 6.1.** Let $A \subset X$, $f : A \to Y$, $a \in \overline{A}$ and $l \in Y$. We say that $f$ *converges* to $l$ at $a$ in $A$ if for every neighborhood $V \subset Y$ of $l$, there exists a neighborhood $U \subset X$ of $a$ such that $f(U \cap A) \subset V$. In this case, we write $\lim_a f = l$, $\lim_{x\to a} f(x) = l$, or $f(x) \underset{x\to a}{\longrightarrow} l$.

We say that $f$ admits a limit at $a$ if there exists a $l \in Y$ such that $f$ converges to $l$ at $a$.

**Remark 6.2.** In the case where $A = X$, we recognize the definition of continuity of $f$ at $l$. That is, $f$ admits a limit at $a$ iff it is continuous, and we have $\lim_{x\to a} f(x) = a$.

**Proposition 6.3** (Unicity of limits)**.** *Suppose that* $\lim_a f = l$*. If* $Y$ *is Hausdorff, then the limit is unique.*

**Proof.** By contradiction, let us suppose that $f$ admits two limits $l, l'$ at $a$. Let $V$ and $V'$ be non-intersecting neighborhoods for $l$ and $l'$, and let $U, U'$ be the neighborhoods of $a$ given by the definition. We have $f(U \cap A) \subset V$ and $f(U' \cap A) \subset V'$. Next, note that $U \cap U'$ is a neighborhood of $a$, and since $a \in \overline{A}$, the intersection $U \cap U' \cap A$ is non-empty. Hence $f(U \cap U' \cap A) \subset V \cap V'$ is non-empty, which is absurd since $f(U \cap U' \cap A) \subset V \cap V'$ and $V \cap V'$ is empty. $\square$

**Proposition 6.4** (Composition of limits)**.** *Let* $X, Y, Z$ *be three topological spaces, and* $f : X \to Y$, $g : Y \to Z$ *such that* $\lim_x f = y$ *and* $\lim_y g = z$*. Then* $\lim_x g \circ f = z$*.*

**Example 6.5** (One-sided limits)**.** Let $X = \mathbb{R}$, $a \in X$ and $A = (-\infty, a)$ (resp. $A = (a, +\infty)$). If $f$ converges to $l$ at $a$ in $A$, then we say that $l$ is the *left-sided limit* (resp. *left-sided limit*) of $f$ at $a$.

**Example 6.6** (Limits in metric spaces)**.** As a direct consequence of the definition of the topology induced by metrics, we obtain that a map $f : (X,d) \to (Y,d')$ between metric spaces admits a limit $l \in Y$ at $a \in X$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X, d(x,a) < \delta \implies d(f(x),l) < \varepsilon.$$

That is, $f$ converges to $l$ iff the map $x \mapsto d(f(x),l)$ converges to 0 when $x$ tends to $a$.

**Proposition 6.7.** *Suppose that $f$ converges to $l$ at $a$ in $A$. Then $l \in \overline{f(A)}$.*

**Corollary 6.8.** *Let $X = \mathbb{R}$ and $Y = \mathbb{R}$ endowed with the Euclidean topology, and $a, l \in \mathbb{R}$. If $\lim_a f = l$, and $f \geq 0$ in a neighborhood of $a$, then $l \geq 0$.*

**§6.1.2 LIMITS AT INFINITY.** In order to define limits of function $f : \mathbb{R} \to Y$ at infinity, we can consider $A = \mathbb{R}$ and $X = \mathbb{R} \cup \{+\infty\}$. The topology we choose on $\mathbb{R} \cup \{+\infty\}$ will be the one generated by the open sets of the Euclidean topology and the sets $(x, +\infty)$ for $x \in \mathbb{R}$. This is the topology is known as the *extended real line*. Let $l \in Y$. With respect to this topology, we have:

$$\lim_{+\infty} f = l \iff \text{for every neighborhood } V \subset Y \text{ of } l,\ \exists L > 0,\ \forall x > L,\ f(x) \in V.$$

In particular, if $Y = \mathbb{R}$, we get the classical definition

$$\lim_{+\infty} f = l \iff \forall \varepsilon > 0, \exists L > 0, \forall x \in X, x > L \implies \|f(x) - l\| < \varepsilon.$$

We can mimick this construction to define limits of sequences. Let $X = \mathbb{N} \cup \{+\infty\}$ and $A = \mathbb{N}$. A map $A \to \mathbb{R}$ is a real sequence $(x_n)_{n \in N}$. Let $l \in Y$. We have

$$\lim_{+\infty} x_n = l \iff \text{for every neighborhood } V \subset Y \text{ of } l, \exists N > 0, \forall n > N, x_n \in V. \tag{I.3}$$

In particular, if $Y = \mathbb{R}$, we obtain

$$\lim_{+\infty} x_n = l \iff \forall \varepsilon > 0, \exists N \geq 0, \forall n \geq N, |x_n - l| < \varepsilon.$$

**Example 6.9.** Let $\mathscr{C}^0([0,1])$ the space of continuous maps from $[0,1]$ to $\mathbb{R}$, endowed with the sup norm. The sequence $f_n : x \mapsto \sin(x/n)$ converges to the zero map.

**Example 6.10.** Let $\mathscr{P}_c(\mathbb{R}^n)$ be endowed with the Hausdorff distance $d_{\mathrm{H}}$ (see §5.2.2). The sequence $x_n = \overline{\mathscr{B}}(0, 1/n)$ converges to $\{0\}$.

**Exercise 51.** Let $X = \mathbb{N}$ and $Y = \mathscr{C}^0([0,1])$ the space of continuous maps from $[0,1]$ to $\mathbb{R}$. On $Y$, consider the norms

$$\|f\|_\infty = \sup\{f(x) \mid x \in [0,1]\} \qquad \text{and} \qquad \|f\|_1 = \int_0^1 f(x)\mathrm{d}x.$$

Exhibit a sequence of functions $(f_n)_{n \in \mathbb{N}}$ that admits a limit for $\|\cdot\|_1$ but not for $\|\cdot\|_\infty$.

**Remark 6.11.** In general, in a topological space, it is not true that two topologies are equal iff they admit the same converging sequences. For instance, on $\mathbb{R}$, the trivial and the cofinite topologies are different, but for both of them, any sequence converges to any point. However, when the topologies come from metrics, the result is true. Indeed, in a metric space $X$, one shows that a subset $F \subset X$ is closed if and only if for all sequence $(x_n)_{n \in \mathbb{N}}$ of $F$ that admits a limit $x \in X$, we have $x \in F$. In other words, we can define the notion of closeness using only the notion of limits of sequences. In general, such a topological space is called a *sequential space*.

**§6.1.3 ACCUMULATION POINTS.**

**Definition 6.12.** Let $A \subset X$, $f : A \to Y$, $a \in \overline{A}$ and $l \in Y$. We say that $l$ is an *accumulation point* of $f$ at $a$ in $A$ if for all neighborhood $V$ of $l$, for all neighborhood $U$ of $a$ such that $f(U \cap A) \cap V$ is non-empty.

**Remark 6.13.** In the case where $A = X$, we see that a limit for $f$ is an accumulation point for $f$ (indeed, we actually have $f(U \cap A) \subset V$). Therefore, in a Hausdorff space, if $f$ admits a limit at $a$, it is the only accumulation point it admits at $a$.

**Example 6.14.** Consider the Euclidean space $\mathbb{R}$, define $X = \mathbb{R}$, $Y = \mathbb{R}$, $A = \mathbb{R} \setminus \{0\}$ and $f\colon A \to \mathbb{R}$ defined by $f(x) = \arctan(1/x)$. The map $f$ admits two accumulation points at 0: $\pi/2$ and $-\pi/2$.

**Proposition 6.15.** *Let $f\colon (X,d) \to (Y,d')$ be a map between metric spaces, $a \in X$ and $l \in Y$. Then $l$ is an* accumulation point *of $f$ at $a$ iff*

$$\forall \varepsilon > 0, \forall \delta > 0, \exists x \in X \text{ such that } d(x,a) < \delta \text{ and } d'(f(x),l) < \varepsilon.$$

**Example 6.16.** The accumulations points at $+\infty$ of a real sequence $(x_n)_{n\in\mathbb{N}}$ and a map $f\colon [0,+\infty) \to \mathbb{R}$ are equal to

$$\bigcap_{N\in\mathbb{N}} \overline{\{x_n \mid n \geq N\}} \qquad \text{and} \qquad \bigcap_{x\geq 0} \overline{f([0,+\infty))}.$$

In particular, via the process of extraction, a sequence $(x_n)_{n\in\mathbb{N}}$ admits an accumulation point $l$ if and only if there exists a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ that converges to $l$.

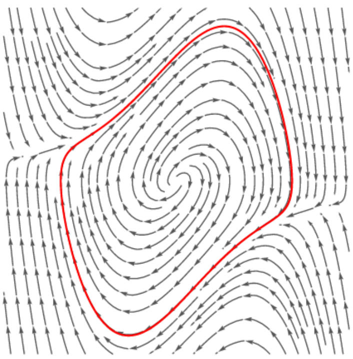
**Example 6.17.** The accumulation points of the sequence $x_n = (-1)^n \frac{n}{n+1}$ are 1 and $-1$.

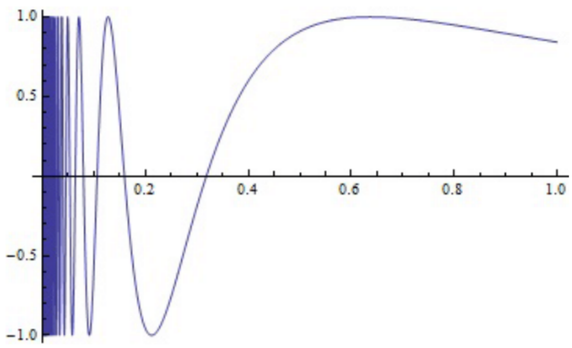
**Example 6.18** (Van der Pol Oscillator)**.** Consider the differential equation

$$x'' - \mu(1-x^2)x' + x = 0$$

where $\mu > 0$. Let $x\colon [0,+\infty) \to \mathbb{R}$ be a solution, and consider the moment map $f\colon t \mapsto (x(t), x'(t))$. The accumulation points of $f$ at $+\infty$ is a closed curve.

> **Exercise 52.** Show that the accumulation points at 0 of the topologist's sine curve $x \in (0,1] \mapsto \sin(1/x)$ is the set $[0,1]$.






## 6.2 Complete spaces

We will only define completeness in the context of **metric spaces**, although this notion can be formulated more generally for spaces endowed with a uniform structure.

### §6.2.1 Cauchy sequences.

In what follows, $(X,d)$ denotes a metric space.

**Definition 6.19.** We say that a sequence $(x_n)_{n\in N} \in X^{\mathbb{N}}$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n > N, d(x_m, x_n) < \varepsilon.$$

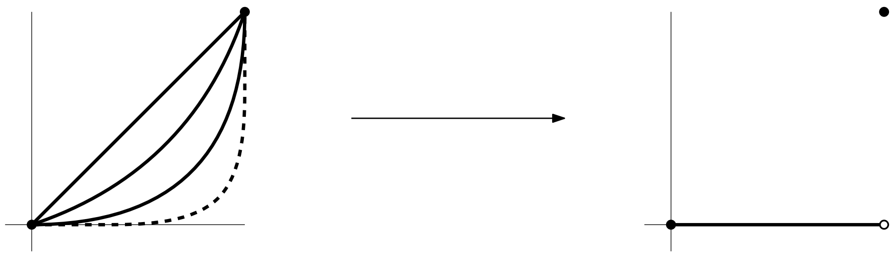
**Proposition 6.20.** *If a sequence converges, then it is a Cauchy sequence.*

The converse of this proposition is false in general. This is the whole point of the notion of completeness. Roughly speaking, if a Cauchy sequence does not converge, it means that its 'virtual limit' goes out of the space. This idea can be made rigorous via the notion of *completion* of a space.

**Example 6.21.** Let $\mathbb{R}\setminus\{0\}$ be endowed with the Euclidean norm. The sequence $x_n = 1/n$ is Cauchy but does not converge.

**Example 6.22.** Let $(0,+\infty)$ be endowed with the distance $d(x,y) = |1/x - 1/y|$. The sequence $x_n = n$ is Cauchy but does not converge.

**Example 6.23.** Let $\mathscr{C}^0([0,1])$ be endowed with the norm $\|\cdot\|_1$ (see Exercise 51). The sequence $f_n \colon x \mapsto x^n$ is Cauchy but does not converge.



**Proposition 6.24.** *Let $d'$ be another metric on $X$, equivalent to $d$. Then a Cauchy sequence for $d$ also is a Cauchy sequence for $d'$.*

### §6.2.2 Completeness.

**Definition 6.25.** A metric space $(X,d)$ is *complete* said is all of its Cauchy sequences converge.

Following our interpretation, a space is complete if it is 'without holes'. A particularly handy feature of a complete space is the following: in order to show that a sequence converges, we only have to show that it is Cauchy, without having to compute the limit explicitly.

**Exercise 53.** Endow $X$ with the discrete metric $d(x,y) = 1$ if $x = y$ and 0 otherwise. Show that it is complete.

**Remark 6.26.** We stress out that completeness is defined for a metric space, and not a topological space. Observe that the spaces $\mathbb{R}$ and $(0,1]$, endowed with the Euclidean distance, are homeomorphic, but only the first one is complete.

**Remark 6.27.** Given a metric space $(X,d)$, it is possible to build a canonical complete metric space $(X',d')$, such that $X$ injects isometrically in $X'$ as a dense subset. This space is called the *completion* of $X$, and can be defined via the Cauchy sequences. This is one of the possible constructions of the real numbers.

In order to prove the following proposition, we only use the fact that $\mathbb{R}$ admits the least upper bound property.

**Proposition 6.28.** *The real line* $(\mathbb{R}, |\cdot|)$ *is complete.*

**Proof.** Let $(x_n)_{n\in\mathbb{N}}$ be a real Cauchy sequence. Let us show that it admits a limit. This will be a consequence of the following three facts, interesting on their own.

First fact: A Cauchy sequence is bounded. Use the definition of a Cauchy sequence with $\varepsilon = 1$, we have a $n \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x_N) < 1$, hence $|x_n| \leq |x_N| + 1$. Besides, denote $A = \max\{|x_n| \mid n \leq N\}$. We obtain that for all $n \in \mathbb{N}$, $|x_n| \leq \max(|x_N| + 1, A)$. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, we can extract from it an increasing subsequence, that we still denote $(x_n)_{n\in\mathbb{N}}$.

Second fact: A real bounded and increasing sequence (resp. decreasing) converges to its supremum (resp. infimum). Let $l$ denote its supremum. This is a direct consequence of the definition of a limit, see §6.1.2.

Third fact: A Cauchy sequence admitting an accumulation point converges. We will show that $(x_n)$ converges to $l$. By the definition of a Cauchy sequence, let $\varepsilon > 0$, and $N$ such that $|x_m - x_n| < \varepsilon/2$ for all $m, n \geq N$. Also definition of an accumulation point, consider also $N' \leq N$ be such that $|x_n - l|\varepsilon < 2$. We deduce that for any $n \geq N$,

$$|x_n - l| \leq |x_n - x'_N| + |x_{N'} - l| \leq \varepsilon.$$

Hence the sequence converges to $l$. $\square$

**Corollary 6.29.** *For all* $k \geq 1$*, the space* $\mathbb{R}^k$ *endowed with the Euclidean norm is complete.*

**Proof.** If $(x_n = (x_n^1, \dots, x_n^k))_{n\in\mathbb{N}}$ is a real Cauchy sequence, then it is easy to prove that each $(x_n^k)_{n\in\mathbb{N}}$ also are. Therefore, we can build a limit coordinate-wise. $\square$

**Corollary 6.30.** *The space of closed bounded subsets* $\mathscr{P}_c(\mathbb{R}^n)$ *endowed with the Hausdorff distance is complete.*

**Proof.** We only give an idea of the proof. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. By defining a new sequence $A'_n = \overline{\bigcup_{k\geq n} A_n}$, we can suppose that $A_n$ is descreasing. Next, one shows that $A_n$ converges to $\bigcup_{n\geq 0} A_k$, called the *limsup* of the sequence. $\square$

### §6.2.3 BANACH SPACES.

**Definition 6.31.** We say that a normed vector space $(X, \|\cdot\|)$ is a *Banach space* if it is complete.

For instance,

- all the normed vector spaces $\mathbb{R}^n$ and $M_n(\mathbb{R})$ are Banach, and this holds for any norm, since all norms are equivalent in finite dimension.
- Moreover, for any $p \in [1, +\infty)$ the spaces $\ell^p$ and $\ell^\infty$ (see Examples 5.11 and 5.11) are complete.
- As illustrated by Example 6.23, the space $\mathscr{C}^0([0,1])$ of continuous functions on $[0,1]$ is complete for $\|\cdot\|_\infty$ but not for $\|\cdot\|_1$.

## 6.3 CONTRACTIONS AND FIXED-POINTS

**§6.3.1 BANACH FIXED-POINT THEOREM.** We now give some applications of the notion of completeness. Let $(X,d)$ be a metric space and $f\colon X \to X$ a map. We say that $f$ is a *contraction map* if there exists a $c \in [0,1)$ such that $d(f(x),f(y)) \leq d(x,y)$ for all $x,y \in X$. In particular, $f$ is continuous. A *fixed point* for $f$ is a point $x \in X$ such that $f(x) = x$.

**Theorem 6.32** (Banach fixed-point)**.** *Suppose that $(X,d)$ is a complete metric space. If $f$ is a contraction map, then it admits a unique fixed point $x^*$. Moreover, for any $x \in X$, the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ converges to $x^*$.*

**Proof.** Using the contraction property of $f$, we get, for all $n,m \in \mathbb{N}$ such that $n \geq n$,

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m+1}) + d(x_m,x_{m+1}) + \cdots + d(x_{n-1},x_n) \\ &\leq c^m d(x_0,x_1) + c^{m+1} d(x_0,x_1) \cdots + c^{n+1} d(x_0,x_1) \\ &\leq \frac{c^n}{c+1} d(x_0,x_1). \end{aligned}$$

We deduce that the sequence is Cauchy, hence admits a limit, denoted $x^*$. It must be a fixed point of $x$, since, by continuity of $f$,

$$x^* = \lim x_n = \lim f(x_{n+1}) = f(\lim x_n) = f(x^*).$$

Last, this fixed point is unique since $f$ is contracting. Indeed, if $y^*$ is another fixed point, we have

$$d(x^*,y^*) = d(f(x^*),f(y^*)) \leq c d(x^*,y^*),$$

hence $x^* = y^*$ since $c < 1$. □

Let us give an application of Banach fixed-point theorem in the context of partial differential equations.

**Theorem 6.33** (Cauchy-Lipschitz or Picard–Lindelöf theorem)**.** *Let $\Omega \times I \subset \mathbb{R}^n \times \mathbb{R}$ be a closed rectangle, $F\colon \Omega \times I \to \mathbb{R}^n$ a continuous map such that is Lipschitz in the first variable, that is, $\exists \lambda > 0$ such that $F((x',t),(x,t)) \leq \lambda |x - x'|$ for all $x,x' \in \Omega$ and $t \in I$. Let $(x_0,t_0) \in D$ and consider the differential equation*

$$x'(t) = F(x(t),t), \quad x(t_0) = x_0$$

*where the solutions are differentiable maps $x\colon I \to \Omega$. Then there exists a $\varepsilon > 0$ and a solution of the equation for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.*

**Proof.** We only give an idea of the proof. Let us consider the complete metric space $(\mathscr{C}^0(I,\Omega), \|\cdot\|_\infty)$ of continuous functions $I \to \Omega$. We prove the theorem by applying the Banach fixed point theorem to the contracting map $f\colon \mathscr{C}^0(I,\Omega) \to \mathscr{C}^0(I,\Omega)$ defined as

$$f(x)(t) = f(t_0) + \int_{t_0}^{t} F(x,s)\mathrm{d}s.$$

On shows that $x$ is a fixed point of this operator if and only if it is a solution of the differential equation. □

§6.3.2 **FRACTALS.** Self-similar objects in $\mathbb{R}^2$ are often obtained via repeated applications of an operator. It will be convenient to work in the space $\mathscr{P}_c(\mathbb{R}^2)$ of bounded closed subsets.

**Proposition 6.34.** *Let $f\colon X \to X$ be a c-contraction map on a metric space. Then the induced map $F\colon \mathscr{P}_c(X) \to \mathscr{P}_c(X)$ is also c-contracting.*

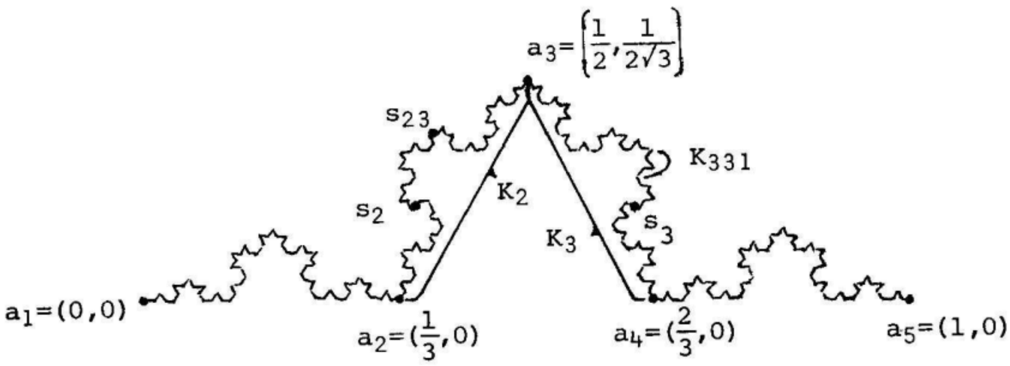
More generally, if $f_1, \dots, f_n$ is a collection of contracting maps, the operator

$$F\colon A \mapsto f_1(A) \cup \cdots \cup f_n(A)$$

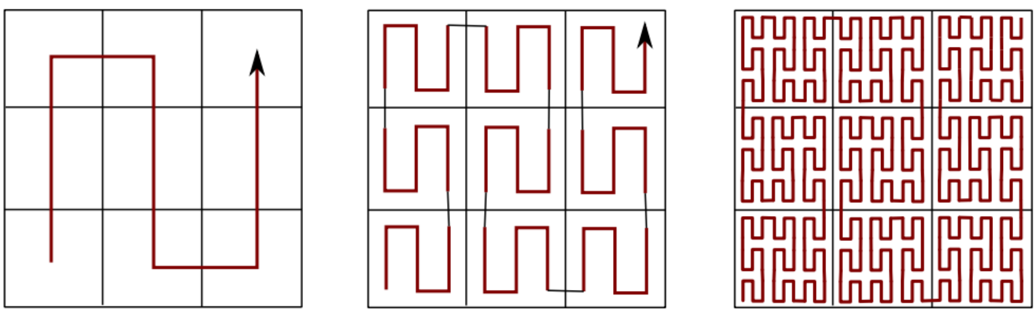
is a contracting map, with constant $\max\{c_1, \dots, c_n\}$. Its unique fixed point, given by the Banach fixed-point theorem, is called a $F$-fractal.

**Example 6.35** (Cantor set)**.** It is obtained from the previous construction, starting from the interval $[0,1] \subset \mathbb{R}$ and applying the transformations $f_1(x) = 1/3x$ and $f_2(x) = 1/3x + 2/3$.

**Example 6.36** (Koch snowflake)**.** It is also obtained from the previous construction, starting from the interval $[0,1] \times \{0\} \subset \mathbb{R}^2$ and applying the five transformations described in [15, §3.3].




§6.3.3 **PEANO SPACE-FILLING CURVES.** Consider the metric space $\mathscr{C}^0([0,1],[0,1]^2)$ of functions from $[0,1]$ to $[0,1]$ endowed with the sup norm $\|\cdot\|_\infty$. It is a complete space. Let $f_n$ be the sequence of functions defined as in the following figure: we obtain $f_{n+1}$ from $f_n$ by replacing $f_n$ by a copy of $f_0$ in each of the squares of length $1/3^n$, and connecting the boundary points.



One shows that it is a Cauchy sequence, hence it admits a limit in $\mathscr{C}^0([0,1],[0,1]^2)$, called the *Peano curve*. It is a continuous surjective map from $[0,1]$ to $[0,1]^2$.

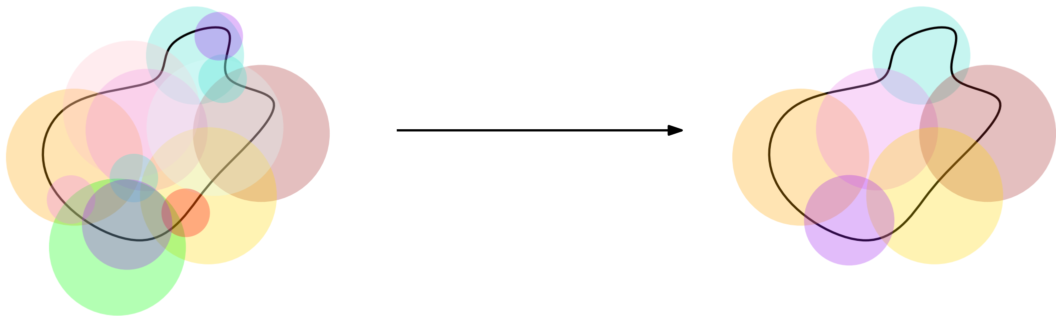
# 7 Compactness

## 7.1 Compact spaces

**§7.1.1 Topological formulation.** In what follows, $(X,\mathscr{T})$ is a topological space. A *cover* of $X$ is a family (potentially infinite) of subsets $(A_i)_{i\in I}$ of $X$ such that $\bigcup_{i\in I} A_i = X$. It is called an *open cover* (resp. *open cover*) is all the $A_i$'s are open (resp. closed). A *subcover* of a cover $(A_i)_{i\in I}$ is a family $(A_j)_{j\in J}$ for a $J\subset I$ such that we still have $\bigcup_{j\in J} A_j = X$.

**Definition 7.1.** We say that a topological space $(X,\mathscr{T})$ is *compact* if it Hausdorff and if all open covers of $X$ admit a finite subcover.

If $A\subset X$ is a subset, we say that it is *compact* if the topological subspace $(A,\mathscr{T}_{|A})$ is compact. This is equivalent to the condition that every cover of $A$ by open sets of $X$ (i.e., such that $\bigcup_{i\in I} A_i \supset A$) admits a finite subcover.



**Example 7.2.** A discrete space is compact iff it is finite. Indeed, each singleton $\{x\}$ is open, hence the family $\{\{x\}\mid x\in X\}$ is an open cover of $X$. It admits a finite subcover iff $X$ finite.

**Example 7.3.** The interval $(0,1)$, for the Euclidean topology, is not compact. Indeed, the cover by the open sets $(0,1-1/n)$, where $n\geq 1$, does not admit a finite subcover.

As a direct consequence of the definition, we get:

**Proposition 7.4.** *Let $(X,\mathscr{T})$ be a compact topological space and $A\subset X$ a closed subset. Then $A$ is compact. Moreover, a finite union of compact subsets is compact.*

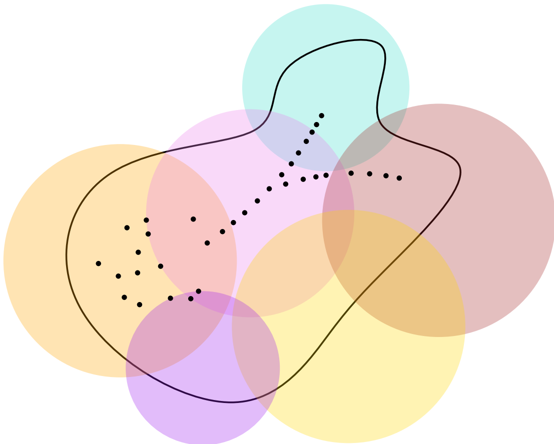
**Proposition 7.5.** *Let $(X,\mathscr{T})$ be a topological space and $A\subset X$ a compact. Then $A$ is closed.*

**Proof.** Let us show that $X\setminus A$ is open. Consider $x\in X\setminus A$, and let us find an open neighborhood of $x$ in $X\setminus A$. For every $y\in A$, let $U_y$ and $V_y$ be two disjoint open sets such that $y\in U_y$ an $x\in V_y$, given by the Hausdorff separability. The family $(U_y)_{y\in A}$ is an open cover of $A$, hence, by compacity, we can extract a finite subcover $(U_{y_1},\dots,U_{y_n})$. In other words, we have $A\subset \bigcup_{1\leq i\leq n} U_{y_i}$. Note that $\bigcap_{1\leq i\leq n} V_{y_i}$ is an open subset of $X\setminus A$ that contains $x$, as wanted. □

§7.1.2 **Sequential formulation.** This formulation only holds in metric spaces. If $(X,d)$ is a metric space, we say that it is *complete* if the topology induced by the metric is. We remind the reader that a sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ admits an accumulation point $l \in X$ iff we can extract a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$, with $\phi : \mathbb{N} \to \mathbb{N}$ a strictly increasing map, such that $(x_{\phi(n)})_{n\in\mathbb{N}}$ converges to $l$.

**Theorem 7.6** (Bolzano-Weierstrass)**.** *Let $(X,d)$ be a metric space. It is compact iff any sequence admits an accumulation point.*

In particular, a compact metric space is complete. This is a consequence of the fact that a Cauchy sequence admitting an accumulation point converges.



**Proof.** We first show the direct implication. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ of $X$. According to Example 6.16, the accumulation points of this sequence are

$$A = \bigcap_{N\in\mathbb{N}} \overline{\{x_n \mid n \geq N\}}$$

We shall show that this set is non-empty. We suppose that it is. Hence $X = {}^cA$ is covered by the open sets $({}^c\overline{\{x_n \mid n \geq N\}})_{N\in\mathbb{N}}$. By compacity of $X$, we extract a finite subcover

$$X = {}^c\overline{\{x_n \mid n \geq N_1\}} \cup \cdots \cup {}^c\overline{\{x_n \mid n \geq N_k\}}.$$

If $N_k$ is the higher index, we deduce that $X = {}^c\overline{\{x_n \mid n \geq N_k\}}$, i.e., $\emptyset = \overline{\{x_n \mid n \geq N_k\}}$, which is clearly absurd.

In order to prove the converse, we need the following proposition. $\square$

Let $\varepsilon > 0$. We say that a subset $A \subset X$ is $\varepsilon$-dense if for any $x \in X$, there exists $a \in A$ such that $d(x,a) < \varepsilon$.

**Proposition 7.7.** *Let $(X,d)$ be a metric space. It is compact iff it is complete and for all $\varepsilon > 0$ there exists a finite $\varepsilon$-dense subset.*

As a consequence, a compact metric space is bounded, that is, there exists a $l > 0$ such that $d(x,y) \leq l$ for all $x,y \in X$.

**Corollary 7.8.** *The compact subsets of $\mathbb{R}$ are the non-empty closed bounded subsets.*

**Example 7.9.** Let $(X,d)$ be a metric space and $\mathscr{P}_c(X)$ the set of non-empty closed subsets of $X$, endowed with the Hausdorff distance $d_H$ (as in §5.2.2). It is a compact metric space.

**Remark 7.10.** For $X$, the property 'any sequence admits an accumulation point' is also called *sequential compactness*. The previous theorem states that a metric space is compact iff it is sequentially compact. This is not true in general for topological spaces. For instance the *Stone–Čech compactification of* $\mathbb{N}$ is compact but not sequentially compact, and the *long line* is sequentially compact but not compact.

**Remark 7.11.** A finite product of compact spaces is compact. This is still true for the product topology of an infinite collection of compact spaces, using the axiom of choice.

## 7.2 Compactness and continuity

### §7.2.1 Extrema.

**Proposition 7.12.** *Let $(X, \mathscr{T})$ be a compact topological space, $(Y, \mathscr{U})$ a Hausdorff space and $f\colon X \to Y$ a continuous map. Then $f(X) \subset Y$ is compact. Moreover, if $f$ is bijective, then it is a homeomorphism.*

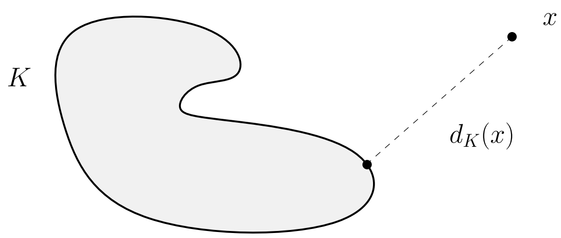
**Proof.** One shows that $f(X)$ is compact by pulling back to $X$ covers of $Y$. Now we suppose that $f$ is bijective. In order to show that $f$ is a homeomorphism, we have to show that $f^{-1}$ is continuous. It is enough to show that the image of closed sets of $Y$ are closed sets of $X$. Since $Y = f(Y)$ is compact, the closed sets $A$ of $Y$ are compact, hence their images are compact, by the first point of the proposition, hence closed. $\square$

If $(X, \mathscr{T})$ is a topological space and $f\colon X \to \mathbb{R}$ a map, we say that $f$ is *bounded* if $\sup f < +\infty$, where $\sup f = \sup\{|f(x)| \mid x \in X\}$. We say that $f$ *attains* its extremum (resp. minimum) if there exists a $x \in X$ such that $f(x) = \sup f$ (resp. $f(y) = \inf f$). It attains its extrema if it attains both its maximum and minimum.

**Corollary 7.13.** *Let $(X, \mathscr{T})$ be a compact topological space and $f\colon X \to \mathbb{R}$ a continuous map. Then $f$ is bounded and attains its extrema.*

**Proof.** By the previous proposition, $f(X)$ is a compact subset of $\mathbb{R}$, hence is closed and bounded, by Corollary 7.8. $\square$

**Example 7.14.** Let $(X,d)$ be a metric space and $K \subset X$ a compact subspace. Let $d_K\colon X \to [0, +\infty)$ the distance to $K$, defined in §5.2.2. For any $x \in X$, this distance is attained, in the sense that there exists a $y \in K$ such that $d_H(x) = d(x,y)$.




**Corollary 7.15.** *If $(X, \|\cdot\|)$ is a finite dimensional normed vector space, then*

- *all the norms are equivalent,*
- *$X$ is complete,*
- *the compact subsets of $X$ are the bounded and closed subsets.*

**Example 7.16.** In the matrix space $M_n(\mathbb{R})$, the orthogonal group $O(n) = \{M \in M_n(\mathbb{R}) | OO^\top = I\}$ is a compact subset. Indeed, it is bounded, and closed since it is the preimage of $I$ by the map $M \mapsto OO^\top$.

**Corollary 7.17** (Heine theorem)**.** *Let $(X,d)$ be a compact topological space, $(X',d')$ a metric space and $f\colon X \to X'$ a continuous map. Then $f$ is* equicontinuous*, that is,*

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x, y \in X, d(x,y) < \eta \implies d(f(x), f(y)) < \varepsilon.$$

**§7.2.2 Proper maps.** A map $f\colon X \to Y$ between Hausdorff spaces is said *proper* if the preimage of a closed set is a closed set.

**Proposition 7.18.** *If $f$ is proper, then the preimage of a compact subset is compact.*

**Corollary 7.19.** *If $f\colon E \to F$ is a map between two normed vector spaces of finite dimension, then $f$ is proper iff for all sequence $(x_n)$ such that $\|x_n\| \to +\infty$, we have $\|f(x_n)\| \to +\infty$.*

**Proposition 7.20.** *A continuous, bijective and proper map between Hausdorff spaces is a homeomorphism.*

## 7.3 Locally compact spaces

**§7.3.1 Definition.**

> **Definition 7.21.** We say that a topological space $(X, \mathscr{T})$ is *locally compact* if it Hausdorff and all its points admit a compact neighborhood.

As a consequence of Theorem 7.6, $\mathbb{R}$ is locally compact. We also obtain the sequential characterization:

**Proposition 7.22.** *A metric space is locally compact iff any bounded sequence admits an accumulation point.*

**§7.3.2 Normed vector spaces.** Let $(X, \|\cdot\|)$ be normed vector space.

**Proposition 7.23.** *The space $(X, \|\cdot\|)$ is locally compact iff its unit closed ball is compact.*

**Proof.** Only the reverse direction is not trivial. It the unit closed ball is compact, then it it the case for any closed ball. Using that any point admits a closed ball as a neighborhood, we obtain the result. $\square$

**Theorem 7.24** (Riesz's lemma)**.** *A normed vector space is locally compact iff it has finite dimension.*

**§7.3.3 In function spaces.** A subset $A \subset X$ is said *relatively compact* is its adherence $\overline{A}$ is compact. According to the previous results, in a finite-dimensional vector space, the relatively compact subsets are the bounded subset. In infinite dimension, the situation is more complicated. The following theorem gives a characterization of relatively compact subsets in the space of continuous maps.

Let $(X, \mathscr{T})$ be a compact space and $\mathscr{C}^0(X)$ the set of continuous maps $X \to \mathbb{R}$, endowed with the sup norm $\|\cdot\|_\infty$. A subset $\mathscr{F} \subset \mathscr{C}^0(X)$ is said

- *equicontinuous* if $\forall x \in X$, $\forall \varepsilon > 0$, $\exists U$ neighborhood of $x$ such that $\forall f \in \mathscr{F}$, $\forall y \in U$, $|f(y) - f(x)| < \varepsilon$,
- *pointwise bounded* if for all $x \in X$, $\sup\{f(x) \mid f \in \mathscr{F}\} < +\infty$.

**Theorem 7.25** (Arzelà–Ascoli theorem)**.** *Let $(X, \mathscr{T})$ be a compact space. The compact subsets of the metric space $(\mathscr{C}^0(X), \|\cdot\|_\infty$ are the equicontinuous and pointwise bounded families of functions.*

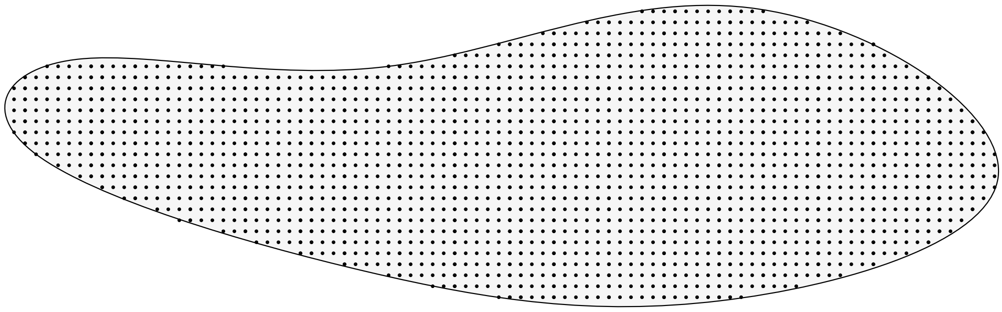
# 8 Density

## 8.1 Dense sets

### §8.1.1 Definition.

**Definition 8.1.** In a topological space $(X, \mathscr{T})$, a subset $A \subset X$ is *dense* if its adherence $\overline{A}$ is equal to $X$.

If $A, B$ are subsets of $X$ such that $A \subset B$, we say that $A$ is *dense in $B$* if it is when seeing $B$ as a topological space endowed with the subspace topology. Equivalently, this means that $\overline{A} \supset B$, where the adherence is taken with respect to the topology on $X$.



**Example 8.2.** The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$.

**Example 8.3.** If a set $X$ is endowed with the discrete topology, the only dense subset is $X$.

**Proposition 8.4.** *A subset $A \subset X$ is dense if and only if it intersects every open subset.*

**Proof.** Suppose that $A$ is not dense. This adherence $\overline{A}$ is closed, hence its complement $X \setminus \overline{A}$ is an open set, that does not intersect $A$. □

**Exercise 54** (Density on the circle)**.** Let $a, b \in \mathbb{R}$. Show that $a\mathbb{Z} + b\mathbb{Z}$ is dense in $\mathbb{R}$ if and only if $a$ and $b$ are incommensurables (i.e., there is no $\alpha \in \mathbb{Q}$ such that $a = \alpha b$ or $b = \alpha a$).

**Exercise 55** (Subgroups of $\mathbb{R}$)**.** Show that a subgroup $H$ of $(\mathbb{R}, +)$ is either dense or equal to $a\mathbb{Z}$ for some $a \geq 0$.

*Hint:* If it is not dense, define $a = \inf H \cap (0, +\infty)$.

**§8.1.2 In metric spaces.** If the topology on $X$ is given by a metric $d$, we have another formulation of density. Remind that the notion of $\varepsilon$-dense subset has been defined in §7.1.2.

**Proposition 8.5.** *A subset $A$ is dense in $(X,d)$ is and only if it is $\varepsilon$-dense for all $\varepsilon > 0$.*

**Proof.** It is a direct consequence of Proposition 8.4, using the open sets $\mathscr{B}(x,\varepsilon)$. $\square$

We now give example of dense subsets in the spaces of $n \times n$ matrices $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$. In what follows, $\chi_A$ will denote the characteristic polynomial of a matrix $A \in M_n(\mathbb{R})$ or $M_n(\mathbb{C})$. It is defined as $\chi_A(\lambda) = \det(A - \lambda I)$.

**Proposition 8.6.** *The set of invertible matrices $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$.*

**Proof.** Remind that a matrix $A$ is invertible iff 0 is not an eigenvalue, and consequently, iff 0 is not a root of $\chi_A$. Now, for any matrix $A$, consider the matrix $A + \varepsilon I$, $\varepsilon \in \mathbb{R}$. Its eigenvalues are exactly the eigenvalues of $A$ plus $\varepsilon$. Hence, for any $\varepsilon > 0$ small enough, 0 is not an eigenvalue of $A + \varepsilon I$, hence it is invertible. We conclude using Proposition 8.5. $\square$

**Proposition 8.7.** *The space of diagonalizable matrices is dense in $M_n(\mathbb{C})$.*

**Proof.** In the same vein as the previous proposition, we will prove the statement by perturbing the triangularization of matrices. Let $A$ be any matrix of $M_n(\mathbb{C})$. We know, by Schur triangularization, that $A$ is conjugate to an upper triangular matrix. Its eigenvalues are read on the diagonal, but their multiplicity may not match de number of times they appear. Applying a small diagonal deformation to $A$, we can make all the diagonal terms different. This implies that it is diagonalizable. $\square$

**Exercise 56.** Show that the set of diagonalizable matrices is not dense in $M_n(\mathbb{R})$, $n \geq 2$.
*Hint:* In $M_2(\mathbb{R})$, using the continuity of the characteristic polynomial, show that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ admits a neighborhood made of non-diagonalizable matrices.

**§8.1.3 In function spaces.** Let $(X,d)$ be a compact metric space, and $\mathscr{C}(X,\mathbb{R})$ the set of continuous functions $f \colon X \to \mathbb{R}$, endowed with the sup norm. As we have seen in §6.2.3, $(\mathscr{C}(X,\mathbb{R}), \|\cdot\|_\infty)$ is a complete metric space. However, it is not compact, nor locally compact (see Theorem 7.25).

For the next theorem, by *separating subalgebra* of $\mathscr{C}(X,\mathbb{R})$ we mean a subset $A$ that is stable by the algebra operations on the functions (addition and multiplication) and such that for every $x,y \in X$ such that $x \neq y$, there exists a $f \in A$ such that $f(x) \neq f(y)$. We omit the proof.

**Theorem 8.8** (Stone-Weierstrass theorem)**.** *Let $(X,d)$ be compact and $A \subset \mathscr{C}(X,\mathbb{R})$ be a separating subalgebra containing a constant map. Then it is dense.*

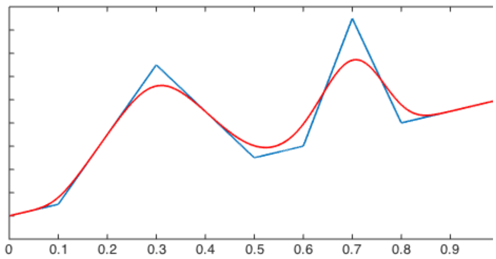
As a direct consequence, we obtain:

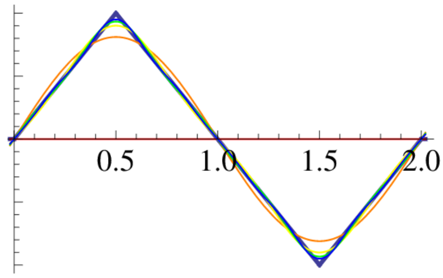
**Corollary 8.9** (Weierstrass theorem)**.** *For any $I \subset \mathbb{R}$ compact, the polynomials are dense in $\mathscr{C}(I,\mathbb{R})$.*

Let us give another formulation of density, valid in metric spaces. A subset $A \subset X$ is dense if any point $x$ of $X$ is either in $A$, or is an *accumulation point* of $A$, that is, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim x_n = x$. In some contexts, we do not only need a dense subset, but also an explicit way to approximate a point, that is, to write it as an accumulation point.

**Example 8.10** (Berstein polynomials)**.** In order to explicitly write a continuous function $f\colon [0,1] \to \mathbb{R}$ as a limit of polynomials, as stated in Corollary 8.9, we can use the approximation by Berstein polynomials.

**Example 8.11** (Fourier series)**.** We can also apply Stone-Weierstrass theorem to show that the trigonometric polynomials (polynomials in the variable $\theta \mapsto \exp(i\theta)$) are dense in $\mathscr{C}(\mathbb{S}^1,\mathbb{C})$. Explicit approximations of continuous functions are obtained via their Fourier series. More precisely, uniform convergence is given by Fejér's theorem.




**§8.1.4 Density and continuous maps.** A useful application of the notion of density is the following: if a continuous map is constant on a dense subset, then it is constant on the whole space.

**Proposition 8.12.** *For all $A,B \in M_n(\mathbb{R})$, we have $\chi_{AB} = \chi_{BA}$.*

**Proof.** According the the previous observation, it is enough to prove the result on a dense subset of $M_n(\mathbb{R})$. This will be $GL_n(\mathbb{R})$. If $A,B \in GL_n(\mathbb{R})$, we have that $AB$ and $BA$ are conjugate, indeed, $A^{-1}(AB)A = BA$. It is then direct to see that $\chi_{AB} = \chi_{BA}$. □

**Exercise 57** (Cayley-Hamilton theorem)**.** For any $A \in M_n(\mathbb{C})$, show that $\chi_A(A) = 0$.
*Hint:* Show it first for the diagonalizable matrices.

**Exercise 58.** Let $f\colon [0,1] \to \mathbb{R}$ be a continuous map such that $\int_0^1 f(x)x^n \mathrm{d}x = 0$ for all $n \in \mathbb{N}$. Show that $f = 0$.
*Hint:* Use the density of the polynomials in $C([0,1],\mathbb{R})$.

## 8.2 Baire spaces

**§8.2.1 Baire theorem.** In a topologcal space, a finite intersection of *open and dense* subsets is dense (and open). in general, this is not the case for infinite intersections, as shown by the rational numbers $\mathbb{Q}$ for the subspace topology of $\mathbb{R}$.

**Theorem 8.13** (Baire category theorem)**.** *In a complete metric space $(X,d)$, a countable intersection of open and dense subsets is dense. It is also true if $X$ is locally compact.*

**Proof.** We shall only prove the result for $X$ complete. Let $(O_i)_{i\in\mathbb{N}}$ be a collection of dense and open sets, and $U \subset X$ an open set. By density of $O_0$, let $x_0 \in X$ and $r_0 \in [0,1)$ such that $\mathscr{B}(x_0,r_0) \subset O_0 \cap U$. By recurrence, we build a sequence $(x_i)_{i\in\mathbb{N}}$ and $(r_i)_{i\in\mathbb{N}}$ such that $\mathscr{B}(x_i,r_i) \subset \mathscr{B}(x_{i-1},r_{i-1}) \cap U$ for all $n \geq 0$. We can choose the sequence of radii decreasing, so as to make $(x_i)_{i\in\mathbb{N}}$ a Cauchy sequence. Let $x$ be an accumulation point. We have that $x \in \bigcap_{i\geq 0} O_i$ by construction, and $x \in O$ since $x \in \mathscr{B}(x_0,r_0) \subset O$. □

More generally, we define:

**Definition 8.14.** A topological space is a Baire space if every countable intersection of open and dense subsets is dense.

By Baire theorem, any complete metric space, or locally compact space, is a Baire space.

**Example 8.15.** An example of a non-complete Baire space is $\mathbb{R} \setminus \mathbb{Q}$. Indeed, if $(O_n)_{n\in\mathbb{N}}$ is a collection of open dense subsets of $\mathbb{R} \setminus \mathbb{Q}$, the denote by $O$ their intersection. By definition of the subspace topology, the collection $(O_n \cup \mathbb{Q})_{n\in\mathbb{N}}$ is made of open and dense subsets of $\mathbb{R}$. Next, the collection $\{O_n \cup \mathbb{Q} \mid n \in \mathbb{N}\} \cup \{\mathbb{R} \setminus \{q\} \mid q \in \mathbb{Q}\}$ is a countable set of open dense subsets of $\mathbb{R}$, whose intersection is $O$. It is dense in $\mathbb{R}$ by Baire category theorem.

### §8.2.2 Meagre and comeagre sets.

As a direct corollary of Baire theorem, we obtain:

**Corollary 8.16.** *In a complete metric space, or locally compact space, a countable union of closed subsets with empty interior has empty interior.*

A subset $A \subset X$ of a topological space is called *meagre* (Bourbaki's terminology) or *first category set* (Baire's terminology) if it is included in a countable union of closed subsets with empty interior. As set with empty interior will also called *nowhere dense*. The previous corollary then reads as follows: a meagre set is nowhere dense.

A subset is said *comeagre* if its complement is meagre. In other words, it contains a countable intersection of open dense sets. We can think of comeagre set as the equivalent of 'almost everywhere' in the context of Baire theory. However, a comeagre subset need not have nonzero Lebesgue measure, and a full Lebesgue measure subset need not be comeagre.

**Proposition 8.17.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{C}(\mathbb{R},\mathbb{R})$ admitting a pointwise limit $f$. Then $f$ is continuous on a comeagre set.*

**Proof.** Define

$$F_{n,p,q} = \{x \in \mathbb{R} \mid \|f_p(x) - f_q(x) \leq 1/(n+1)\|\} \qquad \text{and} \qquad F_{n,p} = \bigcap_{q=p}^{+\infty} F_{n,p,q}.$$

For each $x$, the sequence $(f_p(x))_{n\in\mathbb{N}}$ converges to $f(x)$ hence is a Cauchy sequence. We deduce that $\mathbb{R} = \bigcup_{p\in\mathbb{N}} F_{n,p}$. Let $A_n$ denote the union of the interiors of the $F_{n,p}$, $p \in \mathbb{N}$. It is open. Let us show that it is a dense subset of $\mathbb{R}$. Let $U \subset \mathbb{R}$ be and open subset. Note that $U$ is a Baire space. By writing $U = \cup_{p\in\mathbb{N}} F_{n,p} \cap U$, we can apply Corollary 8.16 to get that the interior of $F_{n,p} \cap U$ must be nonempty for some $p \in \mathbb{N}$. Since $U$ is open, we get that $U \subset F_{n,p}$.

Let us consider the comeagre set $A = \bigcap_{n\in\mathbb{N}} A_n$. Let us show that $f$ is continuous on $A$. Note that, for every $y \in F_{n,p}$, we have $\|f_p(y) - f(y)\| \leq 1/(n+1)$. Let $x_0 \in A$, $n \in \mathbb{N}$, and $p$ such that $x_0$ belongs to the interior of $F_{n,p}$. For $x$ close enough to $x_0$, we have $x \in F_{n,p}$, and $\|f_p(x) - f_p(x_0)\| \leq 1/(n+1)$ by continuity. We use the triangular inequality:

$$\|f(x) - f(x_0)\| \leq \|f(x) - f_p(x)\| + \|f_p(x) - f_p(x_0)\| + \|f_p(x_0) - f(x_0)\| \leq 3/(n+1).$$

We deduce that $f$ is continuous at $x_0$. □

**Corollary 8.18.** *The derivative of a differentiable map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on a comeagre set.*

**Proof.** We apply the previous proposition on the sequence $f_n \colon x \mapsto n(f(x+1/n) - f(x))$. $\square$

Similar applications of the corollary of Baire theorem include:

- There exists a continuous nowhere differentiable map from $\mathbb{R}$ to $\mathbb{R}$.
- There exists a continuous map from $\mathbb{S}^1$ to $\mathbb{C}$ such that its Fourier series diverges at 0.

**§8.2.3 ALGEBRAIC BASES IN BANACH SPACES.** Let $V$ be any vector space. Using axiom of choice, one proves that it admits a basis, that is, a subset $B \subset V$ such that any element $x \in V$ can be written uniquely as a linear combination of elements in $B$. We denote $\mathrm{span}(B) = V$. Note that, by linear combination, we mean linear combination of a **finite number** of elements. The cardinal of such a basis is called the dimension of $V$. This notion is sometimes called *algebraic basis*, to distinguish it from a *Hilbert basis*. A Hilbert basis, that is only defined in Hilbert spaces, satisfies the weaker condition that $\overline{\mathrm{span}(B)} = V$.

We remind the reader that Banach spaces have been defined in §6.2.3.

**Proposition 8.19.** *A Banach space has dimension either finite or uncountable.*

**Proof.** By contradiction, suppose that it has countable dimension. Let $B = (e_n)_{n\in\mathbb{N}}$ denote a basis. The linear subspace $L_n = \mathrm{span}(e_1, \ldots, e_n)$ is meagre. By Corollary 8.16, their union is nowhere dense. In particular, it cannot be equal to the whole space. $\square$

Consequently, the space of polynomials $\mathbb{R}[X]$ is not complete, and the spaces of $p$-integrable maps $L^p(\mathbb{R})$ have uncountable dimension.

# 9 FUNCTIONAL TOPOLOGY

## 9.1 TOPOLOGIES ON FUNCTION SPACES

**§9.1.1 UNIFORM CONVERGENCE.** Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be two topological spaces. The set of maps from $X$ to $Y$ will be denoted $\mathscr{F}(X,Y)$. From a set-theoretic point of view, it can be seen as $Y^X$, the $X$-fold product of $Y$. We will also denote by $\mathscr{C}(X,Y)$ the set of continuous functions.

In what follow, we will suppose that $Y$ is a metric space: its topology $\mathscr{T}$ is induced by a metric $d$. For any $f, g \in \mathscr{C}(X,Y)$, we define the quantity

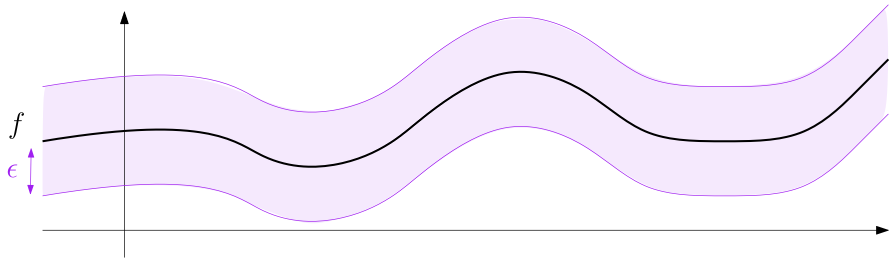
$$d_\infty(f,g) = \min\{1, \sup\{d(f(x), g(x)) \mid x \in X\}\}. \tag{I.4}$$

It is metric on $\mathscr{F}(X,Y)$, hence also on $\mathscr{C}(X,Y)$. It induces a topology called the *topology of uniform convergence*. When a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{F}(X,Y)$ converges to a map $f$ with respect to this topology, we say that it converges *uniformly* to $f$.

When $X$ is compact, the supremum of Equation (I.4) is finite, and we can also consider the metric

$$d_\infty(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

They are not equivalent, but induce the same topology (see Remark 9.11).




If $Y$ is complete, then the metric space $(\mathscr{C}(X,Y),d_\infty)$ is complete. Consequently, $\mathscr{C}(X,Y)$ is a closed subset of $\mathscr{F}(X,Y)$. This last statement is true in general, even when $Y$ is not complete.

**Proposition 9.1.** *Let $X$ be a topological space, $Y$ a metric space, and let $\mathscr{F}(X,Y)$ be endowed with the metric $d_\infty$. The subset $\mathscr{C}(X,Y) \subset \mathscr{F}(X,Y)$ is closed. In other words, a uniform limit of continuous maps is continuous.*

Besides, we have seen in Theorem 7.25 conditions under which subsets are compact, and in Theorem 8.8 conditions under which subsets are dense.

**§9.1.2 POINTWISE CONVERGENCE.** We come back to the idea of product topology, discussed in §1.3.2. Consider the set of maps $\mathscr{F}(X,Y)$. We can define on it two topologies:

- the **box topology**, generated by the $\prod_{x\in X} O_x$ where the $(O_x)_{x\in X}$ are open sets of $Y$
- the **product topology**, generated by the $\prod_{x\in X} O_x$ where the $(O_x)_{x\in X}$ are open sets of $Y$ such that only finitely many of them are not equal to $Y$.

The box topology is finer than the product topology, and is often too fine in practice, as illustrated by the following proposition.

**Proposition 9.2.** *Suppose that $Y$ is a Hausdorff space. Let $(f_n)$ be a sequence of $\mathscr{F}(X,Y)$ and $f \in \mathscr{F}(X,Y)$.*

- *$(f_n)$ converges to $f$ for the product topology iff $f_n(x)$ converges to $f(x)$ for all $x \in X$*
- *$(f_n)$ converges to $f$ for the box topology iff $f_n(x)$ converges to $f(x)$ for all $x \in X$ and if there is a finite subset $S \subset X$ and $N \in \mathbb{N}$ such that $f_n(x) = f(x)$ for $x \in S$ and $n \geq N$.*

**Proof.** We will apply the definition of convergence stated in Equation (I.3).

*First point:* Suppose that $\lim f_n = f$ in the product topology. For $x_0 \in X$, denote $l = f(x_0)$, and choose any neighbor $V \subset Y$ of $l$. By considering the open set $\prod_{x\in X} O_x$ where $O_x = V$ if $x = x_0$ and $Y$ otherwise, we get that $f_n(x_0)$ tends to $f(x_0)$. The converse is proven the same way.

*Second point:* Similarly, suppose that $\lim f_n = f$ in the box topology. We shall only prove the result in the simpler case of $X = \mathbb{N}$ and $Y = \mathbb{R}$, which already contains the idea of the proof. By contradiction, we suppose that for infinitely many values $x \in \mathbb{N}$, $\{n \in \mathbb{N} \mid f_n(x) \neq f(x)\}$ is infinite. Let these values be sorted in increasing order $x_0, x_1, \ldots$. Let $n_0$ be such that $f_{n_0}(x_0) \neq f(x_0)$, and denote $\varepsilon_0 = (f_{n_0}(x_0) - f(x_0))/2$. Recursively, we define $n_{i+1}$ such that $n_{i+1} > n_i$, $f_{n_{i+1}}(x_{i+1}) \neq f(x_{i+1})$ and we denote $\varepsilon_{i+1} = (f_{n_{i+1}}(x_{i+1}) - f(x_{i+1}))/2$. Define the set $O = \prod_{x\in X} O_x$ where $O_x = (f(x_i) - \varepsilon_i, f(x_i) + \varepsilon_i)$ if $x = x_i$ for some $i \in \mathbb{N}$, or $O_x = X$ otherwise. It is an open set of the box topology, but we see, by a diagonal argument, that we do not have $f_n \in O$ for any $n \in \mathbb{N}$. $\square$

We often prefer the product topology, called in this context the *topology of pointwise convergence*. Using the axiom of choice, one shows the following important result.

**Theorem 9.3** (Tychonoff's theorem)**.** *The product of any collection of compact topological spaces is compact with respect to the product topology.*

**Example 9.4.** The space of sequences $[0,1]^{\mathbb{N}}$ (resp. of functions $[0,1]^{\mathbb{R}}$) is compact for the product topology. That is, to any sequence of sequences (resp. of functions $\mathbb{R} \mapsto [0,1]$), we can extract a pointwise converging subsequence.

**Example 9.5** (Hilbert's cube)**.** The Hilbert cube is a set defined as $\mathscr{C} = \prod_{n\in\mathbb{N}}[0,1/(n+1)]$. Equivalently, it can be seen as the set of sequences

$$\{(x_n) \in \mathbb{R}^{\mathbb{N}} \mid \forall n \in \mathbb{N},\ 0 \le x_n \le 1/(n+1)\}.$$

Let $\mathscr{T}_1$ be the product topology (pointwise convergence), and $\mathscr{T}_2$ the topology induced by the sup norm.

1. Show that $\mathscr{T}_1 = \mathscr{T}_2$.
2. Show that the result is false if we consider $\mathscr{C} = \prod_{n\in\mathbb{N}}[0,1]$ instead.

**§9.1.3 Link between uniform and pointwise convergence.** If a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{C}(X,Y)$ converges uniformly to a continuous map $f$, then we have $\lim f_n(x) = f(x)$ for all $x \in X$. That is, uniform convergence implies pointwise convergence. The converse is false, as shown by the maps $f_n\colon x \mapsto x^n$ in $\mathscr{C}([0,1],\mathbb{R})$. Indeed, the pointwise limit is not continuous, which would contradict Proposition 9.1. However, we have a partial converse between uniform and pointwise convergence:

**Theorem 9.6** (Dini's theorem)**.** *Suppose that $X$ is compact. If a sequence of maps $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{C}(X,\mathbb{R})$ is monotone converges pointwise to a continuous map $f$, then the convergence is uniform.*

**Proof.** We suppose that $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$. Let $g_n = f - f_n$. Let $\varepsilon > 0$ and denote $F_n = \{x \in X \mid g_n(x) \ge \varepsilon\}$. The $F_n$ form a decreasing sequence of closed set, and $\bigcap_{n\ge 0} F_n = \emptyset$ since $g_n$ converges pointwise to 0. Since $X$ is compact, we deduce that $F_n = \emptyset$ for $n$ large enough (their complementary form an open cover). ☐

**§9.1.4 Compact-open topology.** On the space of functions $\mathscr{F}(X,Y)$ and $\mathscr{C}(X,Y)$, the pointwise convergence may be too weak, and the uniform convergence too strong, in particular if $X$ is not compact.

**Example 9.7.** In $\mathscr{C}(\mathbb{R},\mathbb{R})$ endowed with the uniform convergence (Equation (I.4)), if a sequence of polynomials admits a limit, then the limit must be a polynomial. As a consequence, non-polynomial power series do not converge in the uniform topology.

In order to circumvent this issue, one may use the following topology.

**Definition 9.8.** The *compact-open* topology on $\mathscr{F}(X,Y)$ is defined as the topology generated by the sets $O(K,U)$ for all compacts $K \subset X$ and open sets $U \subset Y$, where

$$O(K,U) = \{f\colon X \to Y \mid f(K) \subset U\}.$$

If $Y$ is a metric space, then we have the following characterization of convergence for the compact-open topology. It is also called the *compact convergence*.

**Proposition 9.9.** *Suppose that the topology on $Y$ is induced by a metric. A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to $f$ in $\mathscr{F}(X,Y)$ iff for any compact $K\subset X$, the restrictions $(f_{n|K})_{n\in\mathbb{N}}$ converge uniformly to $f_{|K}$.*

When $Y$ is a metric space, we have

$$\mathscr{T}_{\text{pointwise convergence}} \subset \mathscr{T}_{\text{compact-open}} \subset \mathscr{T}_{\text{uniform convergence}} \subset \mathscr{T}_{\text{box topology}}.$$

As a corollary of Proposition 9.9, we also have:

**Proposition 9.10.** *When $X$ is compact, we have $\mathscr{T}_{\textit{compact-open}} \subset \mathscr{T}_{\textit{uniform convergence}}$*

**Remark 9.11.** We remind the reader that it is not true that two topologies are equal iff they admit the same converging sequences. However, when the topologies come from metrics, the result is true (see Remark 6.11). Consequently, Proposition 9.10, in the particular case where $X$ is a compact subset of $\mathbb{R}^n$, can be seen as a consequence of Proposition 9.9 and the following proposition.

**Proposition 9.12.** *If $X\subset\mathbb{R}^n$ is a compact and $Y$ is a metric space, then the compact-open topology on $\mathscr{F}(X,Y)$ is* metrizable*, that is, there exists a metric on $\mathscr{F}(X,Y)$ that induces this topology.*

## 9.2 Hilbert spaces

**§9.2.1 Definitions.** Let $V$ be a $\mathbb{R}$-vector space. We remind that an *inner product* on $V$ is a map $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{R}$ such that

*(bilinearity)* $\forall x,y,z\in V$, $\forall\lambda\in\mathbb{R}$, $\langle x+\lambda y,z\rangle=\langle x,z\rangle+\lambda\langle y,z\rangle$ and $\langle x,y+\lambda z\rangle=\langle x,y\rangle+\lambda\langle x,z\rangle$,

*(symmetry)* $\forall x,y\in V$, $\langle x,y\rangle=\langle y,x\rangle$,

*(positive-definiteness)* $\forall x\in V$, $\langle x,x\rangle\geq 0$, with equality iff $x=0$.

When endowed with an inner product, $V$ is called an *inner product space*. In this case, we say that two vector $u,v\in V$ are *orthogonal*, and we denote $u\perp v$, if $\langle u,v\rangle=0$.

**Proposition 9.13** (Cauchy-Schwarz inequality)**.** *For all $u,v\in V$, we have $|\langle u,v\rangle|\leq\sqrt{\langle u,u\rangle}\sqrt{\langle v,v\rangle}$.*

**Proof.** Let us suppose that $\langle u,v\rangle\neq 0$. Consider the map $P\colon\lambda\in\mathbb{R}\mapsto\langle u+\lambda v,u+\lambda v\rangle$. By bilinearity and symmetry, we can expand the expression as

$$P(\lambda)=\langle u,u\rangle+\lambda^2\langle v,v\rangle+2\lambda\langle u,v\rangle.$$

The discriminant of this polynomial is

$$4\langle u,v\rangle^2-4\langle u,u\rangle\langle v,v\rangle.$$

Moreover, $P$ is a non-negative polynomial. Therefore, its discriminant is non-positive, proving the result. □

As a consequence, one shows that the map $\|\cdot\|\colon V\to\mathbb{R}$ defined as $\|x\|=\sqrt{\langle x,x\rangle}$ is a norm on the vector space $V$. In particular, it satisfies the triangle inequality $\|x+y\|\leq\|x\|+\|y\|$. Therefore, an inner product space has a natural metric, hence topology. Here are a few properties satisfied by this norm. For all $u,v,w\in V$,

- *Parallelogram law:* $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$,
- *Polarization identity:* $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle$
- *Pythagorean theorem:* if $u \perp v$ then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$
- *Ptolemy's inequality:* $\|u-v\|\|w\| + \|v-w\|\|u\| \geq \|u-w\|\|v\|$.

**Definition 9.14.** An inner product space $(H, \langle\cdot,\cdot\rangle)$ is a *Hilbert space* if the topology induced by the inner product is complete.

**Example 9.15.** The norm $\|\cdot\|_2$ on $\mathbb{R}^n$ (see Example 5.10) is induced by the usual inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

Hence it is an Hilbert space.

**Example 9.16.** The spaces $\ell^2$ and $\mathscr{L}^2(\mathbb{R})$ (see Examples 5.11 and 5.12) are Hilbert spaces, and their norms come from the inner products

$$\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\rangle = \sum_{i=0}^{+\infty} x_i y_i \qquad \text{and} \qquad \langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{d}x.$$

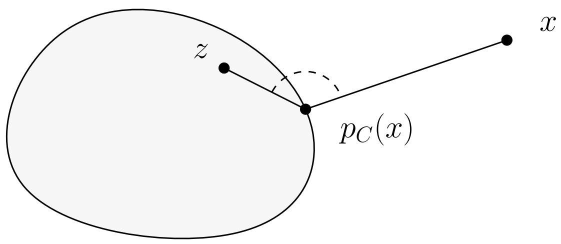
The spaces $\ell^p$ and $\mathscr{L}^p(\mathbb{R})$ for $p \in (1,+\infty)] \setminus \{0\}$ are not inner product spaces. Indeed, one shows that the $p$-norm do not satisfy the parallelogram law.

### §9.2.2 Projection on closed convex subsets.

**Theorem 9.17** (Projection on closed convex set)**.** *Let $H$ be a Hilbert space, and $C \subset H$ a closed and convex subset. For all $x \in H$, there exists a unique $y = p_C(x)$ such that*

$$\|x-y\| = \min_{z\in C} \|x-z\|.$$

*Moreover, the map $p_C \colon H \to C$ is continuous and 1-Lipschitz. Last, if $C$ is a closed linear subspace, then $p_C$ is linear, and $y = p_C(x)$ is the unique element of $C$ such that $\langle x-y, z-y\rangle \geq 0$ for all $z \in C$.*




We now give an important consequence of this result. If $E \subset H$ is a linear subspace, we define its *orthogonal* as

$$E^{\perp} = \{x \in H \mid \forall y \in E,\ x \perp y\}.$$

Moreover, we say that two linear subspaces $E, F$ are in *direct sum* if $E \cap F = \{0\}$ and $E + F = H$. In this case, $F$ is called a *complement* of $E$.

**Corollary 9.18.** *If $E \subset H$ is a closed linear subspace, then $E^{\perp}$ is a closed linear subspace, and is a complement of $E$. It is called its* orthogonal complement*.*

**Proof.** We have $E \cap E^{\perp} = \{0\}$ by positive-definiteness of the inner product. Moreover, we get $E + E^{\perp} = H$ by decomposing any element $x \in H$ as $x = p_C(x) + (x - p_C(x))$. Last, $E^{\perp}$ is closed since it is the preimage of 0 by the continuous linear map $p_C$. □

**Corollary 9.19.** *If $E \subset H$ is any linear subspace, then $E$ is dense iff $E^{\perp} = \{0\}$.*

**Proof.** By continuity of the inner product, any linear subspace satisfy $E^{\perp} = \overline{E}^{\perp}$. We conclude using the previous corollary. □

> **Exercise 59.** Let $E \subset H$ be a linear subspace of a Hilbert space. Show that $(E^{\perp})^{\perp} = E$ iff $E$ is closed.

**Remark 9.20.** The fact that any closed linear subspace admits a closed complement is a property satisfied only by Hilbert spaces, that is, it is false in a Banach space whose topology does not come from an inner product.

### §9.2.3 HILBERT BASES.

**Definition 9.21.** A sequence $(e_n)_{n \in \mathbb{N}}$ of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a *Hilbert basis* (also called *orthognonal basis*) if

*(normalization)* $\forall n \in \mathbb{N}$, $\|e_n\| = 1$,

*(orthogonality)* $\forall m, n \in \mathbb{N}$ such that $m \neq n$, $e_m \perp e_n$,

*(completeness)* the span of $(e_n)_{n \in \mathbb{N}}$ is dense in $H$.

We remind that, as explained in §8.2.3, the span of an infinite family is the set of linear combination of a **finite number** of elements. By Gram Schmidt orthonormalization process, we have:

**Proposition 9.22.** *If $H$ admits a countable family $(x_n)_{n \in \mathbb{N}}$ such that its span is dense, then $H$ admits a Hilbert basis.*

Hilbert basis allows to work with infinite dimensional using the tools of finite dimensional ones. For instance, we have:

**Proposition 9.23** (Parseval's identity)**.** *Let $H$ be a Hilbert space and $(e_n)_{n \in \mathbb{N}}$ a Hilbert basis. Let $u \in H$, and for all $n \in \mathbb{N}$, define $u_n$ as the projection of $u$ on the line spanned by $e_n$ (it is a closed linear subspace). Then the series $\sum_{n=0}^{\infty} \|u_n\|^2$ and $\sum_{n=0}^{\infty} u_n$ are convergent, and we have*

$$\sum_{n=0}^{\infty} u_n = u \quad \text{and} \quad \sum_{n=0}^{\infty} \|u_n\|^2 = \|u\|^2.$$

**Example 9.24** (Fourier series)**.** Let $\mathscr{L}^2(\mathbb{S}^1, \mathbb{C})$ be the space of functions from $\mathbb{S}^1$ to $\mathbb{C}$, endowed with the inner product $\langle f, g \rangle = \int_{\mathbb{S}^1} f(x) g(x) \mathrm{d}x$. It is a Hilbert space, and a Hilbert basis is given by the $e_n \colon x \mapsto \exp(inx)$, $n \in \mathbb{N}$. Given a map $f \in \mathscr{L}^2(\mathbb{S}^1, \mathbb{C})$, its projection on the line spanned by $e_n$ is equal to $c_n(f) e_n$, where $c_n(f)$ is the $n^{\text{th}}$ *Fourier coefficient*

$$c_n(f) = \int_{\mathbb{S}^1} f(x) \exp(-inx) \mathrm{d}x.$$

**Example 9.25** (Orthogonal polynomials)**.** Let $\rho\colon \mathbb{R} \to [0,+\infty)$ be a measurable map such that $\int_{\mathbb{R}} |x^n|\rho(x)\mathrm{d}x < +\infty$ for all $n \in \mathbb{N}$. It is called a *weight function*. We define the Hilbert space

$$\mathscr{L}^2(\mathbb{R},\rho) = \{f\colon \mathbb{R} \mapsto \mathbb{R} \text{ measurable} \mid \int_{\mathbb{R}} |f(x)|^2\rho(x)\mathrm{d}x < +\infty\} \quad \text{and} \quad \langle f,g\rangle = \int_{\mathbb{R}} f(x)g(x)\rho(x)\mathrm{d}x.$$

One shows that there exists a unique family of unitary polynomials $(P_n)_{n\in\mathbb{N}}$ pairwise orthogonal such that $\deg(P_n) = n$. Moreover, if there exists a $\alpha > 0$ such that $\int_{\mathbb{R}} \exp(\alpha x)\rho(x)\mathrm{d}x < +\infty$, then $(P_n)_{n\in\mathbb{N}}$ is a Hilbert basis. This basis depends on the choice of the weight function. For instance, with $\rho(x) = \exp(-x^2)$, we obtain the Hermite polynomials.

# Bibliography


[1] Frédéric Paulin. *Topologie, analyse et calcul différentiel*. 2008. https://www.imo.universite-paris-saclay.fr/~frederic.paulin/notescours/cours_analyseI.pdf.

[2] James R Munkres. *Topology*, volume 2. Prentice Hall Upper Saddle River, 2000.

[3] Roland E Larson and Susan J Andima. The lattice of topologies: a survey. *The Rocky Mountain Journal of Mathematics*, 5(2):177–198, 1975.

[4] Martin Aigner and Günter M Ziegler. Proofs from the book. *Berlin. Germany*, 1, 1999.

[5] William J Pervin. *Foundations of general topology*. Academic Press, 2014.

[6] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2000.

[7] L Lusternik, L Schnirelmann, and J Kravtchenko. Méthodes topologiques dans les problèmes variationnels. première partie: Espaces à un nombre fini de dimensions. *Revue de Métaphysique et de Morale*, 42(1), 1935.

[8] Octavian Cornea, Gregory Lupton, John Oprea, Daniel Tanré, et al. *Lusternik-Schnirelmann category*. Number 103. American Mathematical Soc., 2003.

[9] Christophe Giraud. *Introduction to high-dimensional statistics*. Chapman and Hall/CRC, 2021.

[10] Zanetti JPP, Biller P, and Meidanis J. Median approximations for genomes modeled as matrices. *Bulletin of mathematical biology*, 78(4):786–814, 2016.

[11] Benoit Mandelbrot. *Les objets fractals: forme, hasard et dimension*, volume 17. Flammarion Paris, 1975.

[12] Alison L Gibbs and Francis Edward Su. On choosing and bounding probability metrics. *International statistical review*, 70(3):419–435, 2002.

[13] Vincent Borrelli, Saïd Jabrane, Francis Lazarus, and Boris Thibert. Flat tori in three-dimensional space and convex integration. *Proceedings of the National Academy of Sciences*, 109(19):7218–7223, 2012.

[14] Nicolas Bourbaki. *General Topology: Chapters 1–4*, volume 18. Springer Science & Business Media, 2013.

[15] John E Hutchinson. Fractals and self similarity. *Indiana University Mathematics Journal*, 30(5):713–747, 1981.