EMAp Summer Course

Topological Data Analysis with Persistent Homology

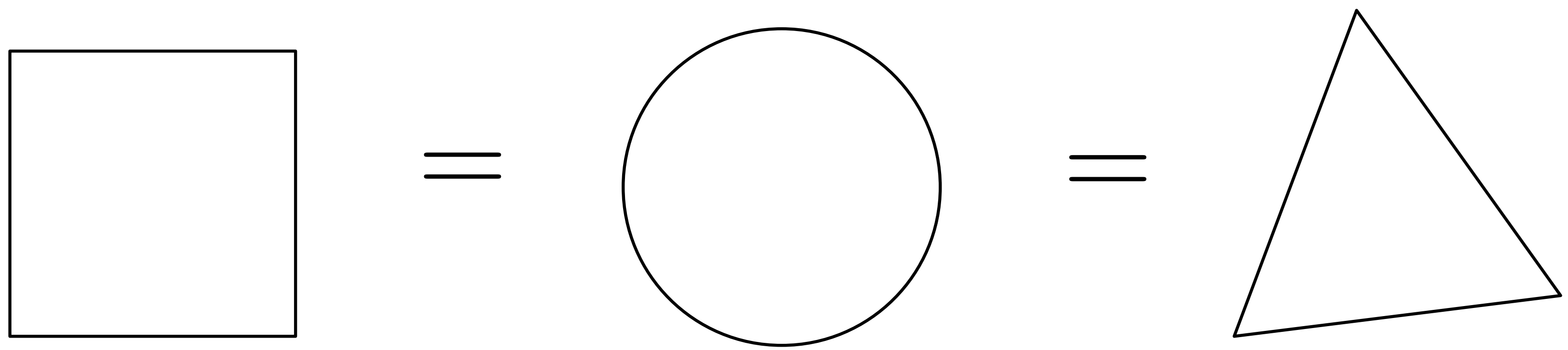
https://raphaeltinarrage.github.io/EMAp.html

Lesson 2: Homeomorphisms

Last update: January 17, 2021

Introduction

In topology, we are allowed to deform shapes.



I - Homeomorphic topological spaces

II - Connected components

III - Connectedness as an invariant

VI - Dimension

Definition

Definition: Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces, and $f\colon X \to Y$ a map.
We say that f is a *homeomorphism* if

- $f\colon X \to Y$ is continuous,
- f is a bijection,
- $f^{-1}\colon Y \to X$ is continuous.

If there exist such a homeomorphism, we say that the two topological spaces are homeomorphic.

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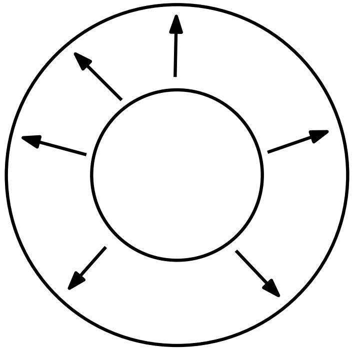
Example: Consider the following circles of $\mathbb{R}^2$: $\mathbb{S}\,(0,1) = \{x \in \mathbb{R}^2, \|x\| = 1\}$,
$\mathbb{S}\,(0,2) = \{x \in \mathbb{R}^2, \|x\| = 2\}$.

and the map
$$\begin{aligned} f\colon \mathbb{S}\,(0,1) &\longrightarrow \mathbb{S}\,(0,2) \\ x &\longmapsto 2x \end{aligned}$$



It is continuous, bijective, and its inverse $f^{-1}\colon x \mapsto \frac{1}{2}x$ also is continuous. Hence f is a homeomorphism.

Hence these two circles are homeomorphic.

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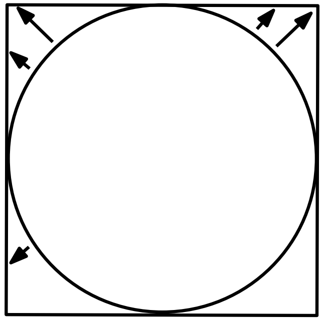
Example: Consider a circle and a square $\mathbb{S}(0,1) = \{x \in \mathbb{R}^2, \|X\| = 1\}$,

$$\mathcal{C} = \left\{(x_1,x_2) \in \mathbb{R}^2,\ \max(|x_1|,|x_2|) = 1\right\}.$$

and the map $f\colon \mathbb{S}(0,1) \longrightarrow \mathcal{C}$

$$(x_1,x_2) \longmapsto \frac{1}{\max(|x_1|,|x_2|)}(x_1,x_2)$$



It is continuous, bijective, and its inverse $f^{-1}\colon x \mapsto \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1,x_2)$ also is continuous.

Hence f is a homeomorphism.

Hence the circle and the square are homeomorphic.

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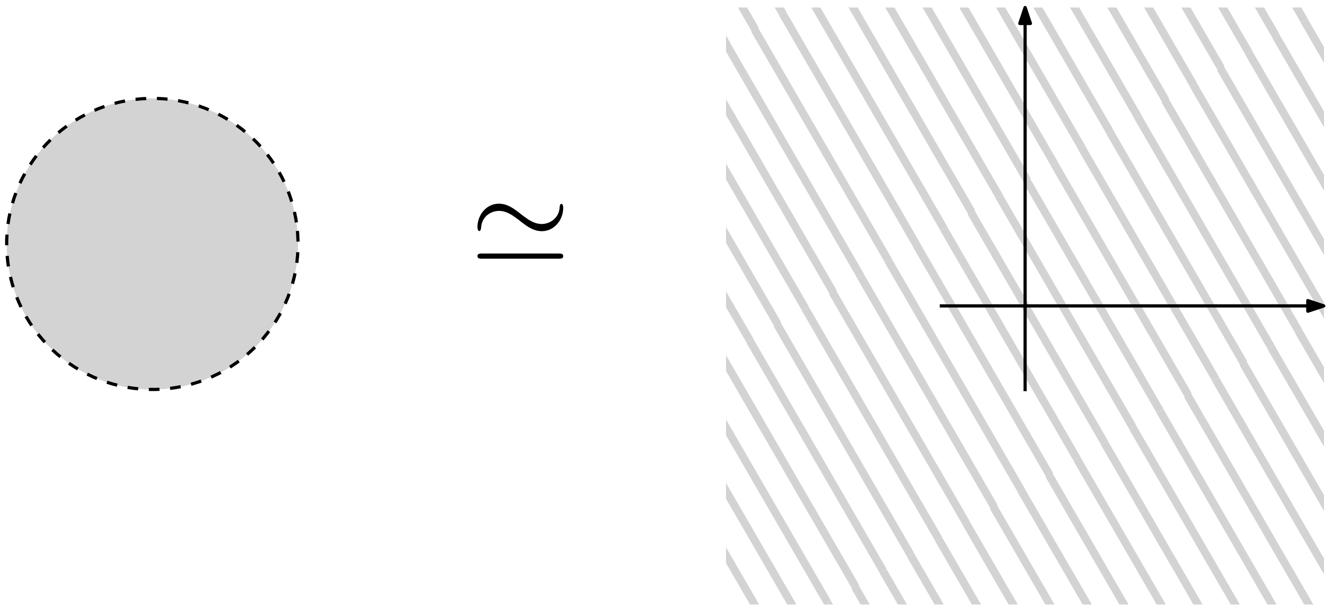
Exercise: The topological spaces $\mathcal{B}(0,1) \subset \mathbb{R}^n$ and $\mathbb{R}^n$ are homeomorphic.



Non-homeomorphic spaces

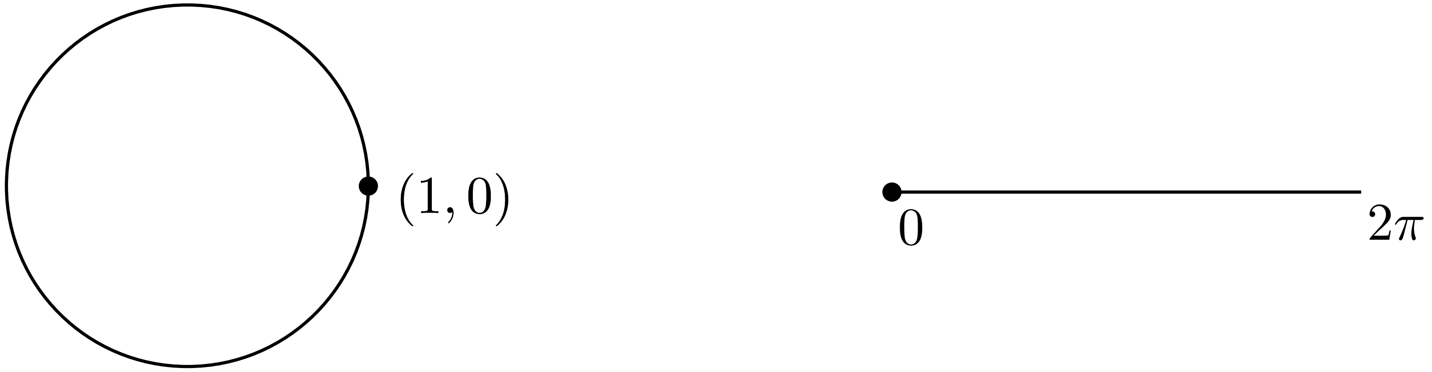
Non-example: Consider the interval $[0, 1)$ and the circle $\mathbb{S}(0, 1) \subset \mathbb{R}^2$.

Define the map
$$\begin{aligned} f \colon [0, 2\pi) &\longrightarrow \mathbb{S}(0, 1) \\ \theta &\longmapsto (\cos(\theta), \sin(\theta)) \end{aligned}$$

It is continuous, and admits the following inverse:

$$\begin{aligned} g \colon \mathbb{S}(0, 1) &\longrightarrow [0, 2\pi) \\ (x_1, x_2) &\longmapsto \arctan\left(\frac{x_2}{x_1}\right) \end{aligned}$$

The map g is **not** continuous.



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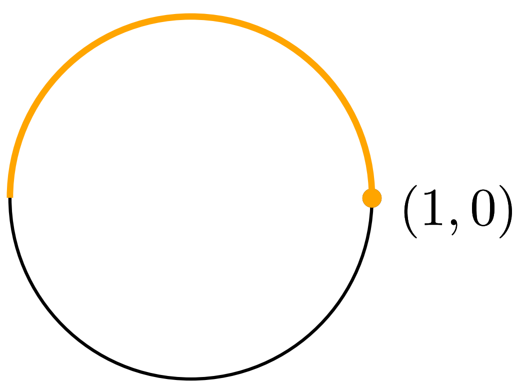

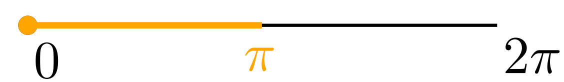

Indeed, $[0,\pi)$ is an open subset of $[0,2\pi)$, but $g^{-1}([0,\pi))$ is not an open subset of $\mathbb{S}(0,1)$ (it is not open around $g^{-1}(0) = (1,0)$).

Let us write $X \simeq Y$ if the two topological spaces X and Y are homeomorphic, i.e., if there exists a homeomorphism $f\colon X \to Y$.

For any X, we have

$$X \simeq X.$$

Proof: Consider the identity map $\mathrm{id}\colon X \to X$, $x \mapsto x$.
It is a homeomorphism between X and X.

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Moreover, we have:

$$X \simeq Y \iff Y \simeq X.$$

Proof: Suppose that X and Y are homeomorphic: $f\colon X \to Y$.
Then $f^{-1}\colon Y \to X$ is a homeomorphism between Y and X.

Homeomorphism equivalence relation

Let us write $X \simeq Y$ if the two topological spaces X and Y are homeomorphic, i.e., if there exists a homeomorphism $f\colon X \to Y$.

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We also have a third property:

$$X \simeq Y \text{ and } Y \simeq Z \implies X \simeq Z.$$

Proof: Suppose that we have two homeomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$. Then $g \circ f\colon X \to Z$ is a homeomorphism between X and Z.

Let us write $X \simeq Y$ if the two topological spaces X and Y are homeomorphic, i.e., if there exists a homeomorphism $f\colon X \to Y$.

For any X, we have

$$X \simeq X. \qquad \text{reflexivity}$$

Moreover, we have:

$$X \simeq Y \iff Y \simeq X. \qquad \text{symmetry}$$

We also have a third property:

$$X \simeq Y \text{ and } Y \simeq Z \implies X \simeq Z. \qquad \text{transitivity}$$

Conclusion: *Being homeomorphic* is an *equivalence relation*.

It allows to classify topological spaces in classes (called *classes of homeomorphism equivalence*):

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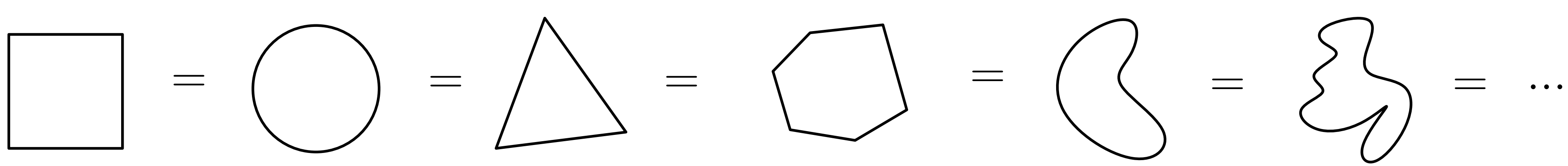

the class of circles

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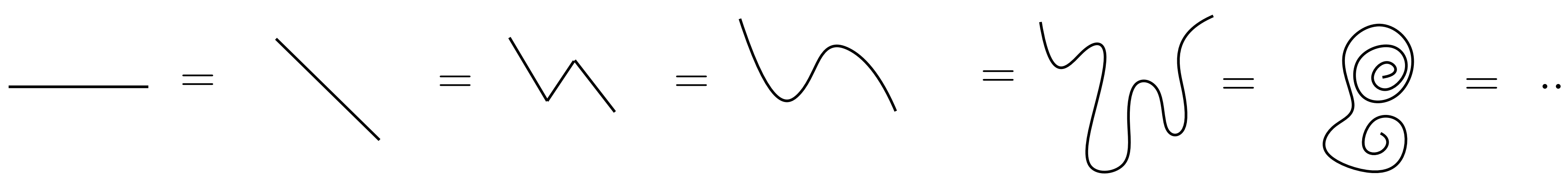

the class of intervals

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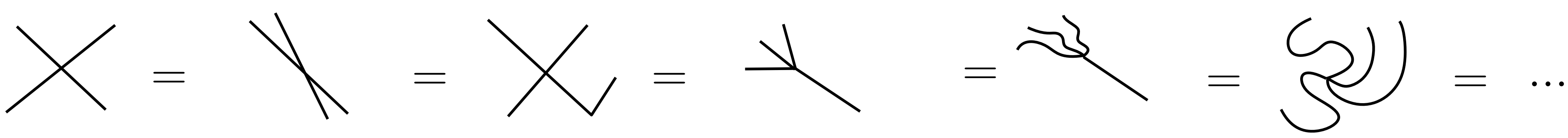

the class of crosses

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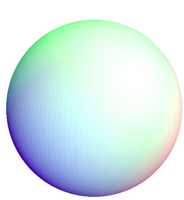
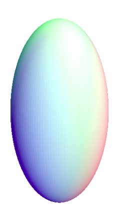
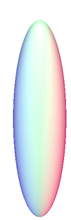
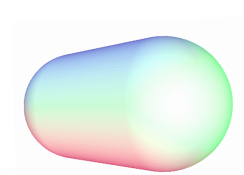
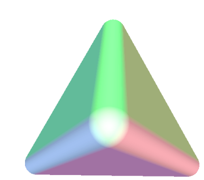
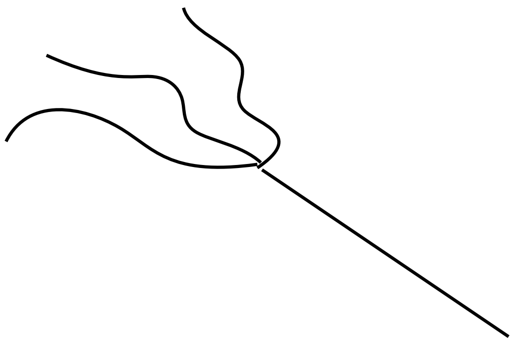
 =  =  =  =  =  = ...

the class of spheres

Homeomorphism problem

In general, it may be complicated to determine whether two spaces are homeomorphic.



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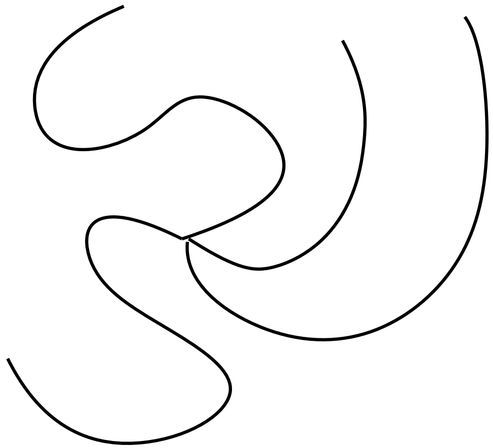

To answer this problem, we will use the notion of *invariant*.

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Connectedness

Definition: Let $(X, \mathcal{T})$ be a topological space. We say that X is *connected* if for every open sets $O, O' \in \mathcal{T}$ such that $O \cap O' = \emptyset$, we have

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

In other words, a connected topological space cannot be divided into two non-empty disjoint open sets.

connected space

non-connected space

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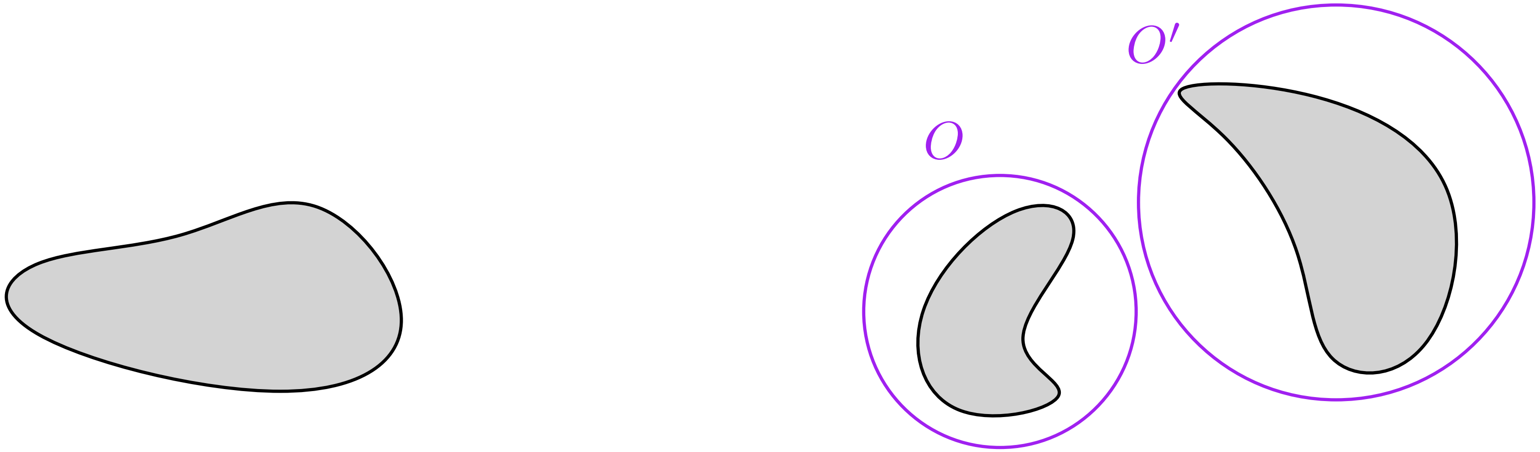

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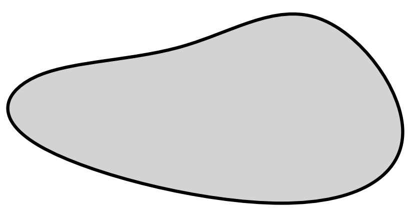
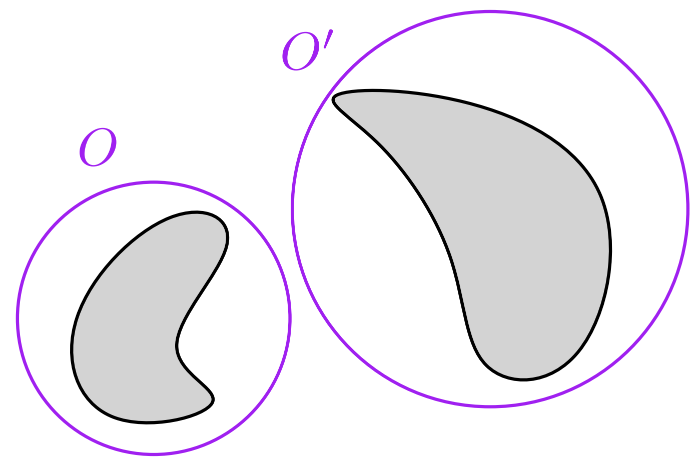


connected space



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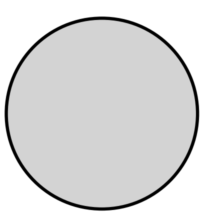
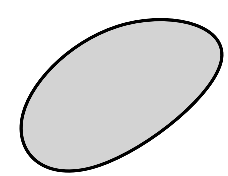


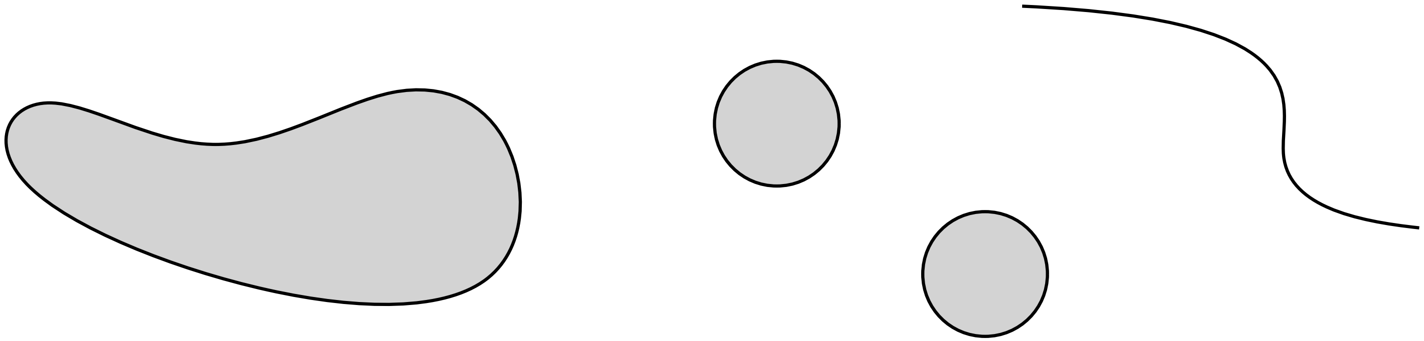
Proposition: The balls of $\mathbb{R}^n$ are connected.
More generally, any convex set is connected.



Connected components

If a space is not connected, we can consider its connected components.
Let $x \in X$. The connected component of x is defined as the largest subset of X that is connected.

X :



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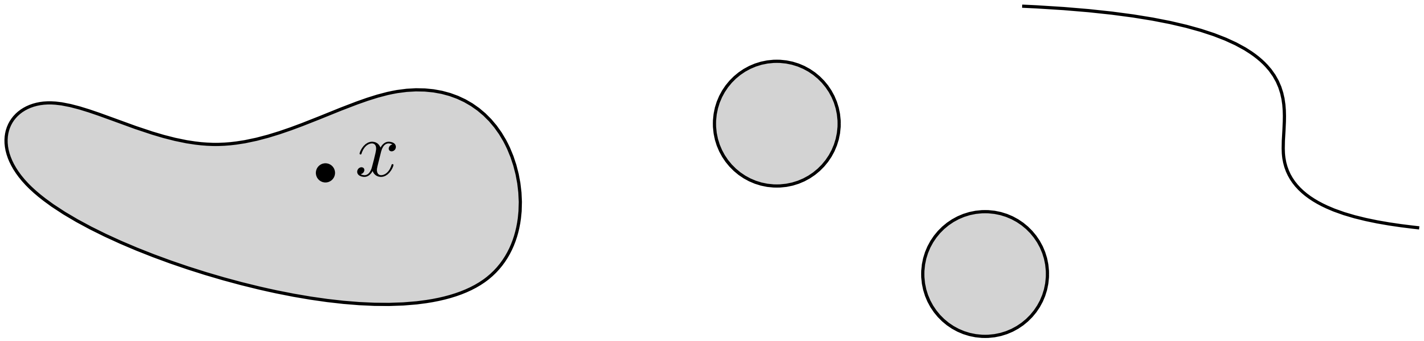

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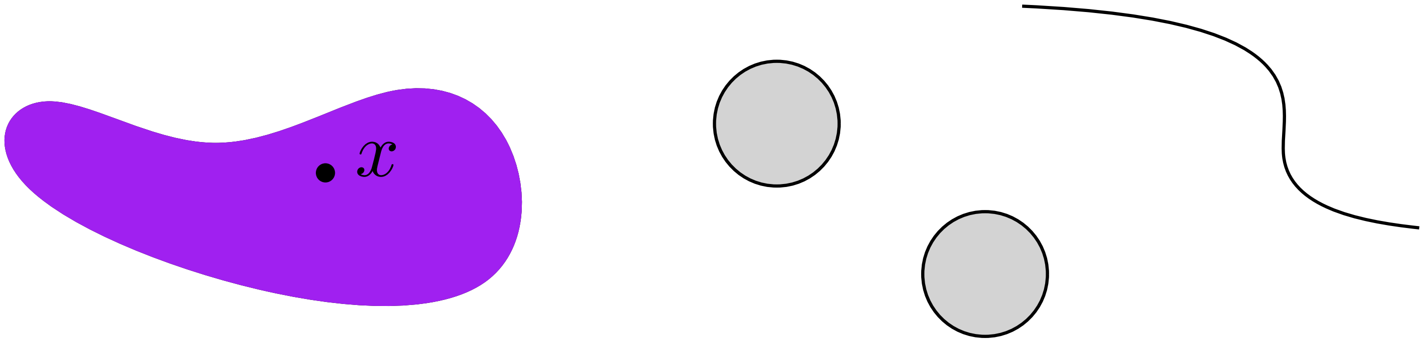

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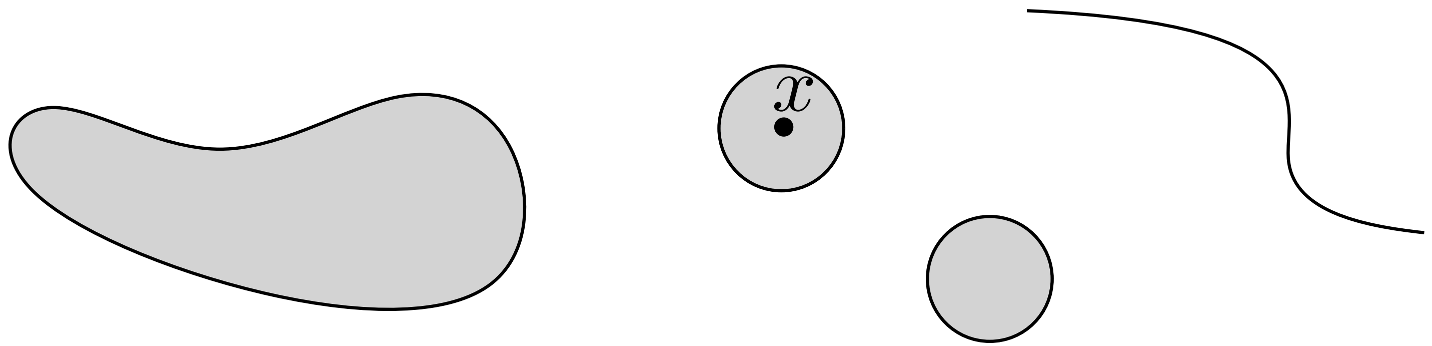

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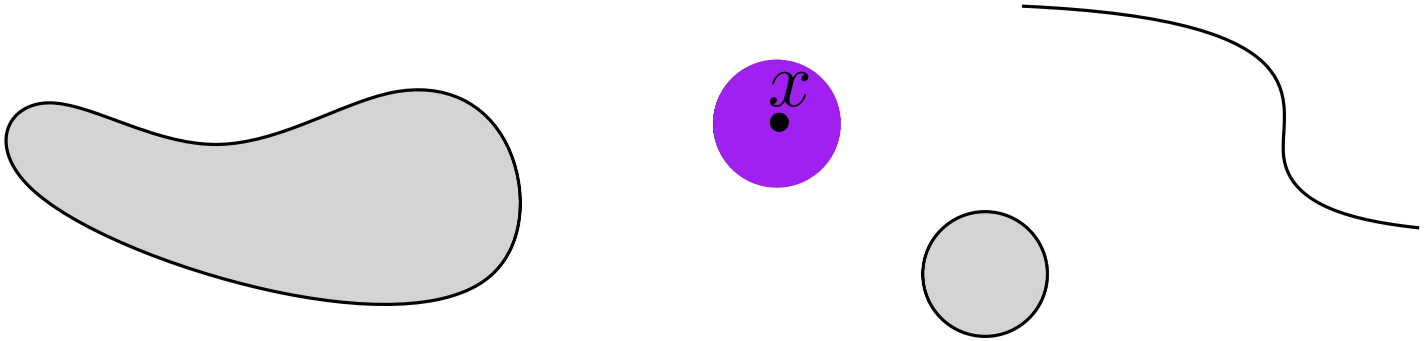

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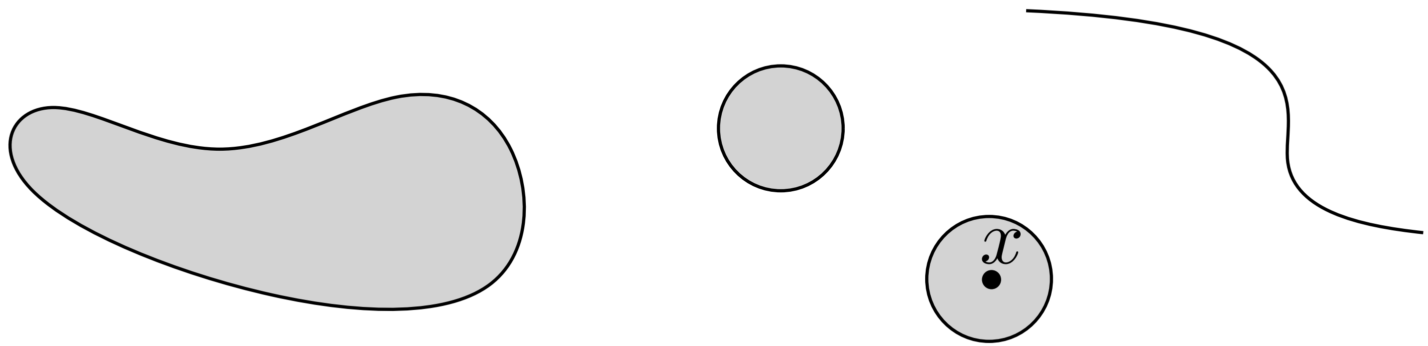

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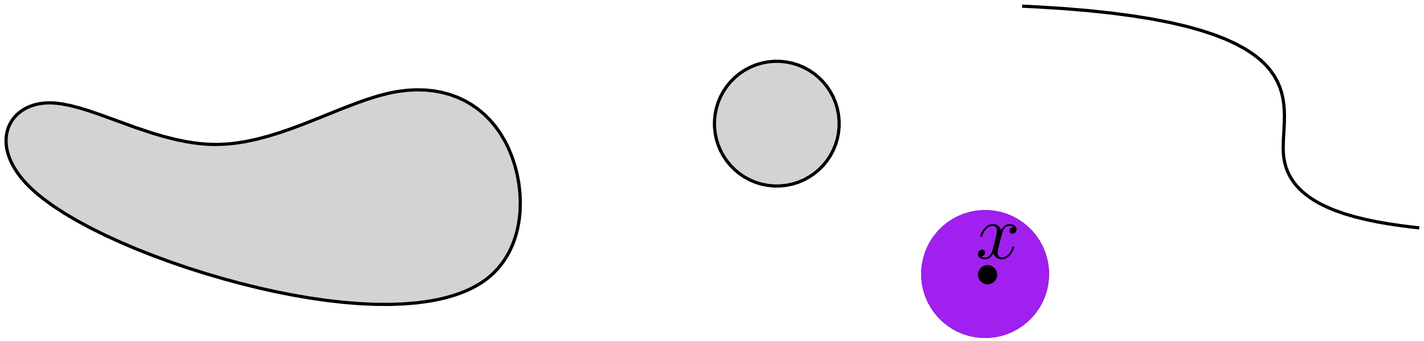

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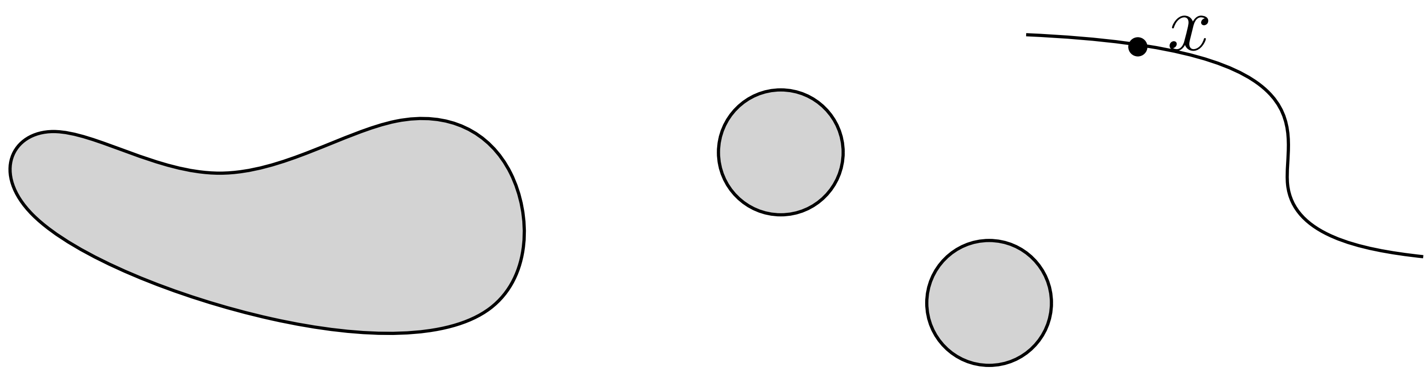

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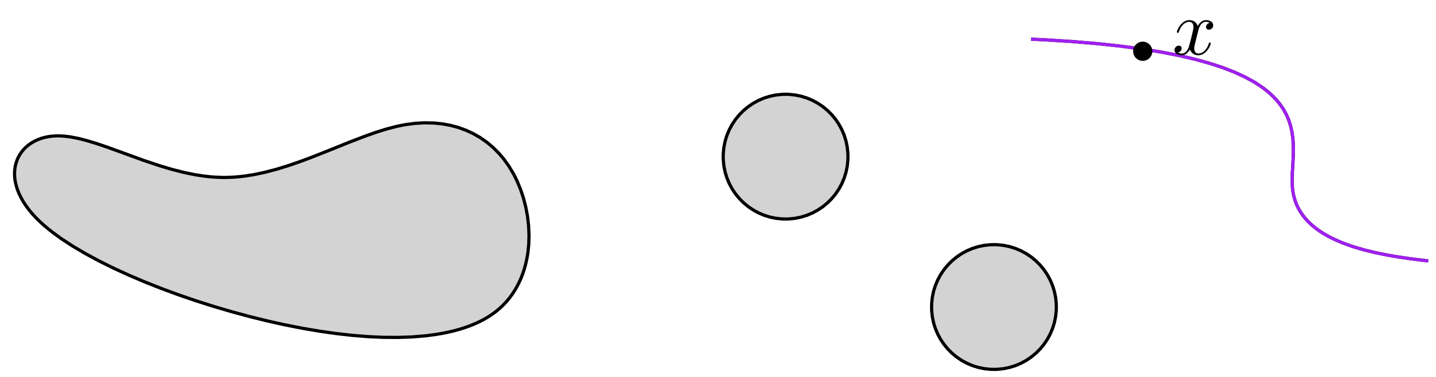

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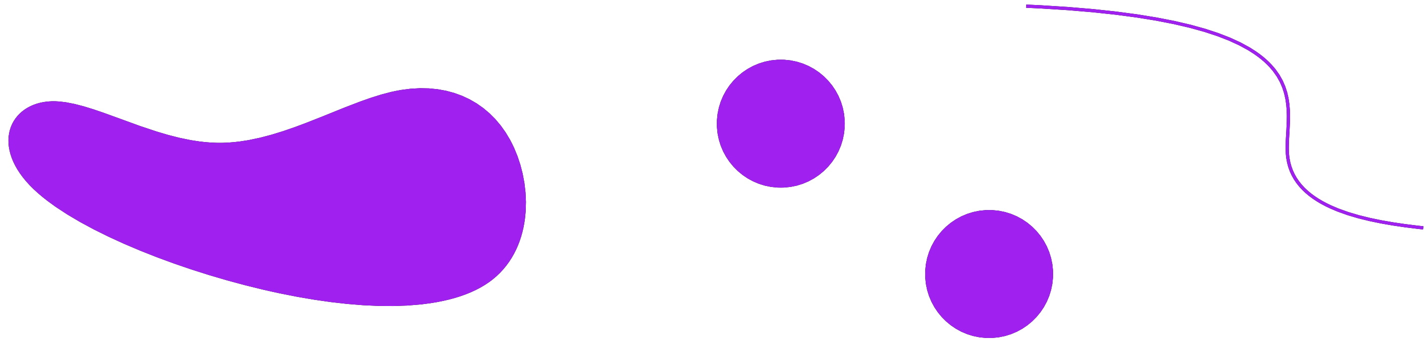

The set of connected components of X forms a partition of X into open sets.

Definition: Let $(X, \mathcal{T})$ be a topological space. Suppose that there exists a collection of n **non-empty**, **disjoint** and **connected** sets $(O_1, ..., O_n)$ such that

$$\bigcup_{1 \leq i \leq n} O_i = X.$$

Then we say that X admits n connected components.

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Invariant property

Proposition Two homeomorphic topological spaces admit the same number of connected components.

Proof: Let $f\colon X \to Y$ be a homeomorphism. Let n be the number of connected components of Y, and m the number of X. Let us show that $m = n$.

Suppose that Y admits n connected components. We can write $Y = \bigcup\limits_{1 \leq i \leq n} O_i$ where the O_i are disjoint non-empty connected sets. Also, we have seen that the O_i are open.

For all $i \in [\![1, n]\!]$, define $O'_i = f^{-1}(O_i)$. We have:

- for all $i \in [\![1, n]\!]$ $O'_i = f^{-1}(O_i)$ is open (because f is continuous),
- $X = \bigcup\limits_{1 \leq i \leq n} O'_i$ (because f is a map)
- for all $i, j \in [\![1, n]\!]$ with $i \neq j$, $O'_i \cap O'_j = f^{-1}(O_i) \cap f^{-1}(O_j) = f^{-1}(O_i \cap O_j) = \emptyset$
- for all $i \in [\![1, n]\!]$, $O'_i = f^{-1}(O_i) \neq \emptyset$ (because f is a bijection).

Hence X can be covered by n disjoint non-empty open sets. We deduce that X admits at least n connected components.

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Hence X can be covered by n disjoint non-empty open sets. We deduce that X admits at least n connected components.

Now, suppose that X admits m connected components. Using the same reasoning, one shows that Y admits at least m connected components. Hence we have $n \ge m \ge n$, that is, $n = m$.

Invariant property

Proposition Two homeomorphic topological spaces admit the same number of connected components.

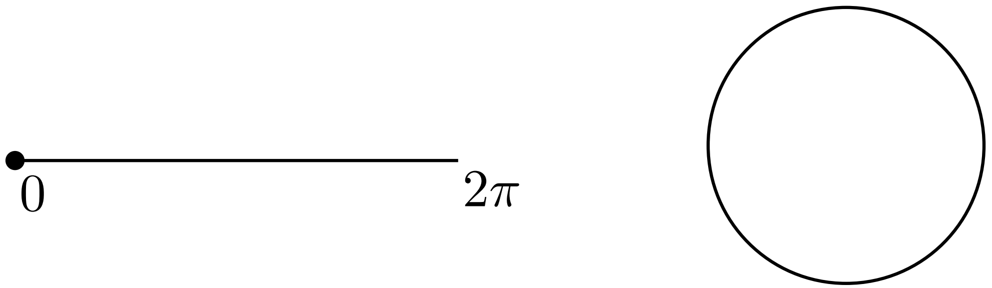
Example: The subsets $[0, 1]$ and $[0, 1] \cup [2, 3]$ of $\mathbb{R}$ are not homeomorphic. Indeed, the first one has one connected component, and the second one two.

Invariant property

Proposition Two homeomorphic topological spaces admit the same number of connected components.

Example: The interval $[0, 2\pi)$ and the circle $\mathbb{S}(0,1) \subset \mathbb{R}^2$ are not homeomorphic.

We will prove this by contradiction. Suppose that they are homeomorphic. By definition, this means that there exists a map $f\colon [0, 2\pi) \to \mathbb{S}(0,1)$ which is continuous, inversible, and with continuous inverse.



Invariant property

Proposition Two homeomorphic topological spaces admit the same number of connected components.

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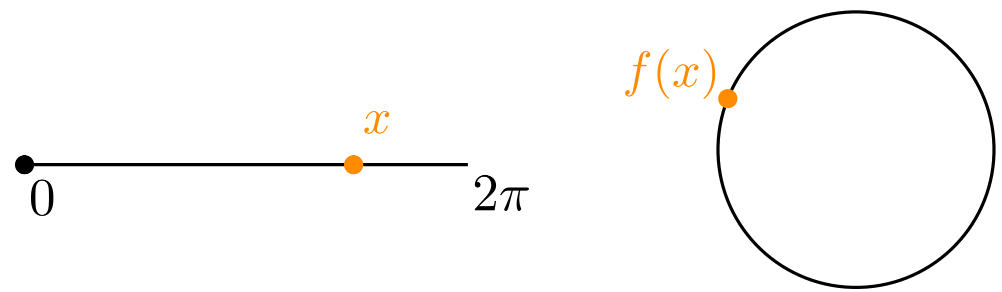

Let $x \in [0, 2\pi)$ such that $x \neq 0$. Consider the subsets $[0, 2\pi) \setminus \{x\} \subset [0, 2\pi)$ and $\mathbb{S}(0, 1) \setminus \{f(x)\} \subset \mathbb{S}(0, 1)$, and the induced map

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The map g is a homeomorphism.

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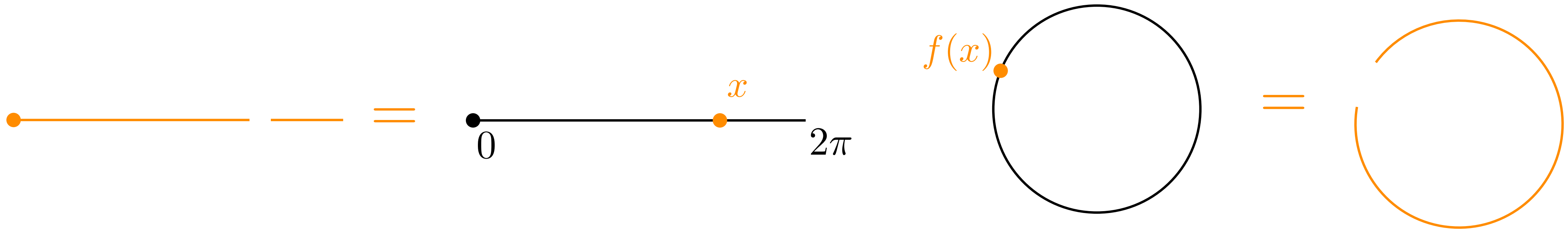

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Moreover, $[0, 2\pi) \setminus \{x\}$ has two connected components, and $\mathbb{S}(0,1) \setminus \{f(x)\}$ only one. This is absurd.

I - Homeomorphic topological spaces

II - Connected components

III - Connectedness as an invariant

VI - Dimension

Invariance of domain

Theorem: If $m \neq n$, the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

We will have to wait a little bit before proving this result.

However, we can prove some particular cases.

Example: $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.
Just as before, we will prove this by contradiction. Suppose that there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^2$. Choose any $x \in \mathbb{R}$. The induced map

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is still a homeomorphism, but $\mathbb{R} \setminus \{x\}$ has two connected components, while $\mathbb{R}^2 \setminus \{f(x)\}$ has one. This is a contradiction.

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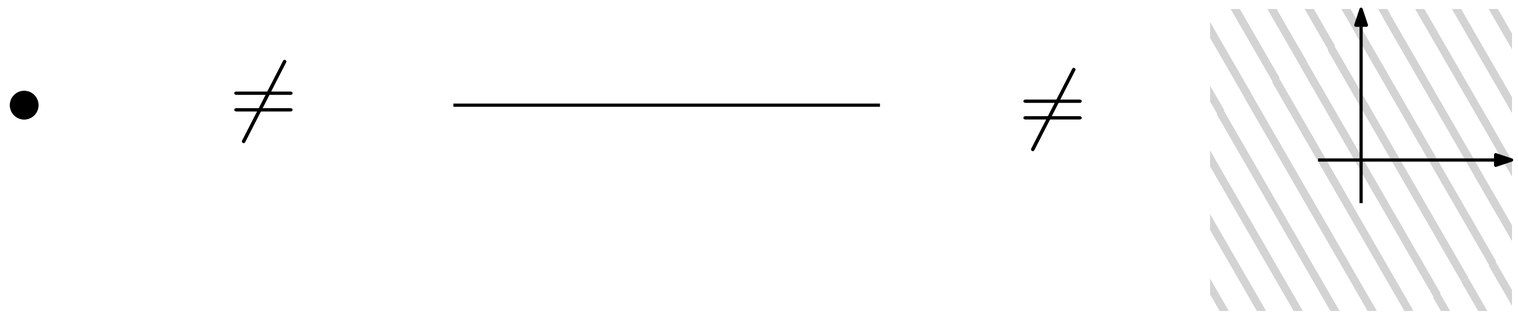

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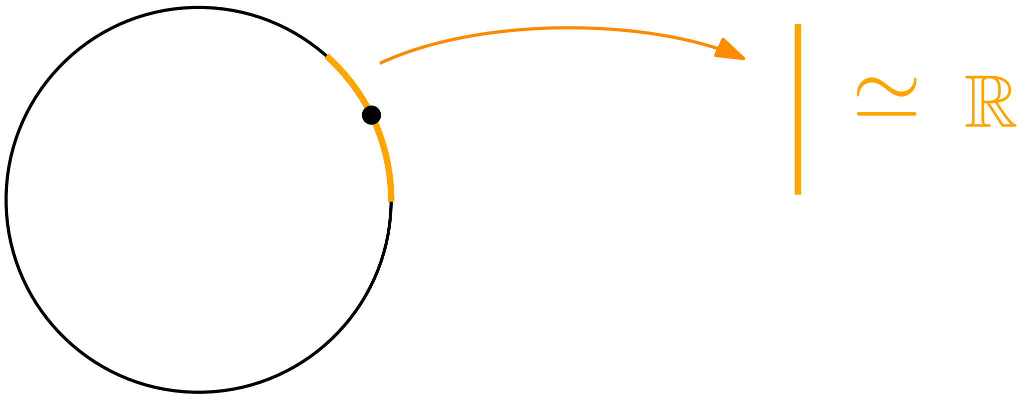
The same reasoning shows that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic either.

Dimension

Definition: Let $(X, \mathcal{T})$ be a topological space, and $n \geq 0$. We say that it *has dimension* n if the following is true: for every $x \in X$, there exists an open set O such that $x \in O$, and a homeomorphism $O \to \mathbb{R}^n$.

Example: The circle has dimension 1.



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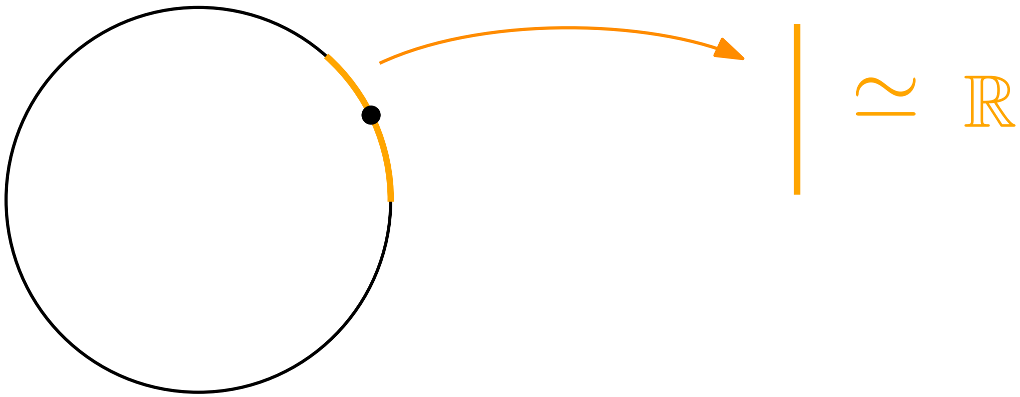
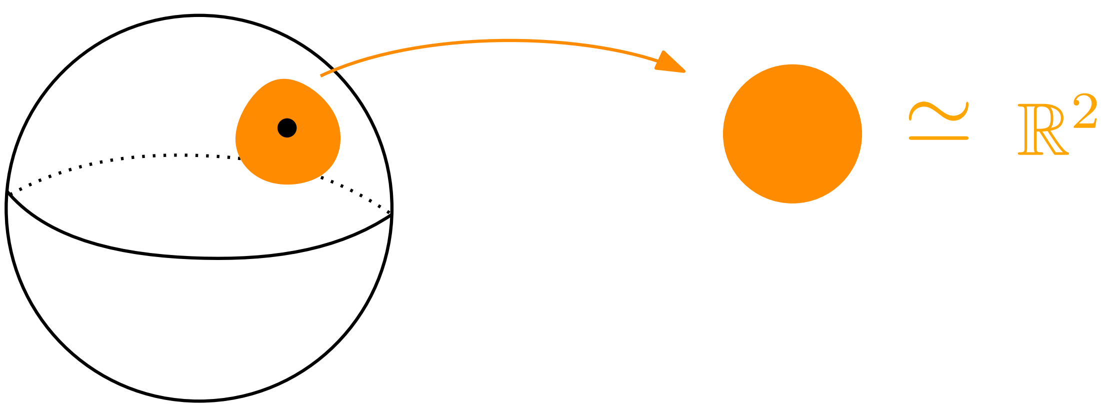


Example: The sphere has dimension 2.



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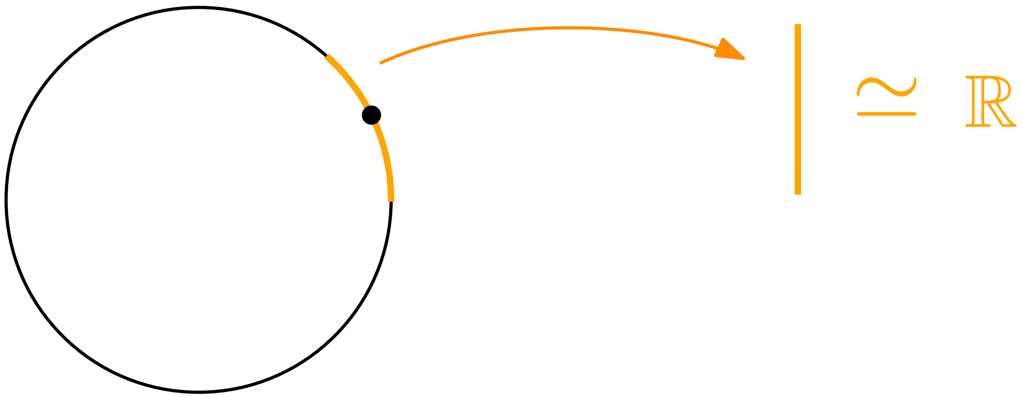

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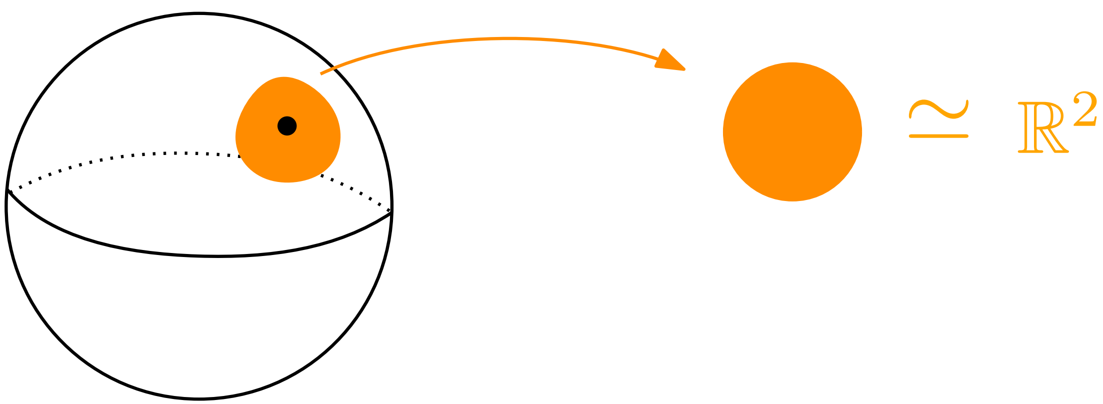

Interpretation: a topological space of dimension n is a topological space that locally looks like the Euclidean space $\mathbb{R}^n$.

Dimension invariant

Theorem: Let X, Y be two homeomorphic topological spaces. If X has dimension n, then Y also has dimension n.

In other words, **dimension is an invariant.**

We can use it to show that two spaces are not homeomorphic.

Example: The unit circle $\mathbb{S}_1 \subset \mathbb{R}^2$ and the unit sphere $\mathbb{S}_2 \subset \mathbb{R}^3$ are not homeomorphic. Indeed, the first one has dimension 1, and the second one dimension 2.

Dimension invariant

Theorem: Let X, Y be two homeomorphic topological spaces. If X has dimension n, then Y also has dimension n.

Proof: Let n be the dimension of X, and consider a homeomorphism $g\colon Y \to X$.

Let $y \in Y$, and $x = g(y)$. Since x has dimension n, there exists an open set O of X, with $x \in O$, and a homeomorphism $h\colon O \to \mathbb{R}^n$.

Define $O' = g^{-1}(O)$. It is an open set of Y, with $y \in O'$. Moreover, the map $h \circ g\colon O' \to \mathbb{R}^n$ is a homeomorphism.

This being true for every $y \in Y$, we deduce that Y has dimension n.

Conclusion

We learnt to look at topological spaces from a homeomorphic-equivalence perspective.

We study two invariants: number of connected components and dimension. This allows to understand whether two topological spaces are homeomorphic or not.

Homework for tomorrow: Exercise 8 and 11

Facultative exercise: Exercise 10

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Obrigado!