Erlangen AI Hub Seminar – 11/11/2025

# Linear orbits of compact Lie groups and machine learning

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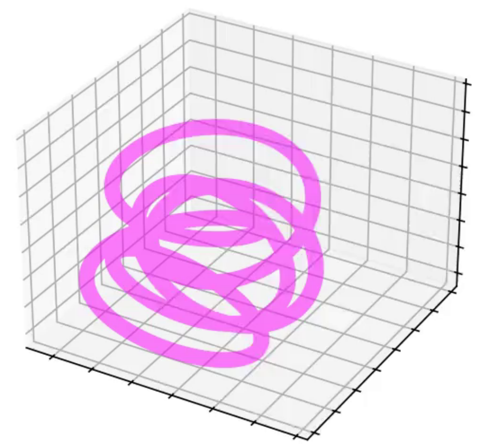
# Pixel permutations

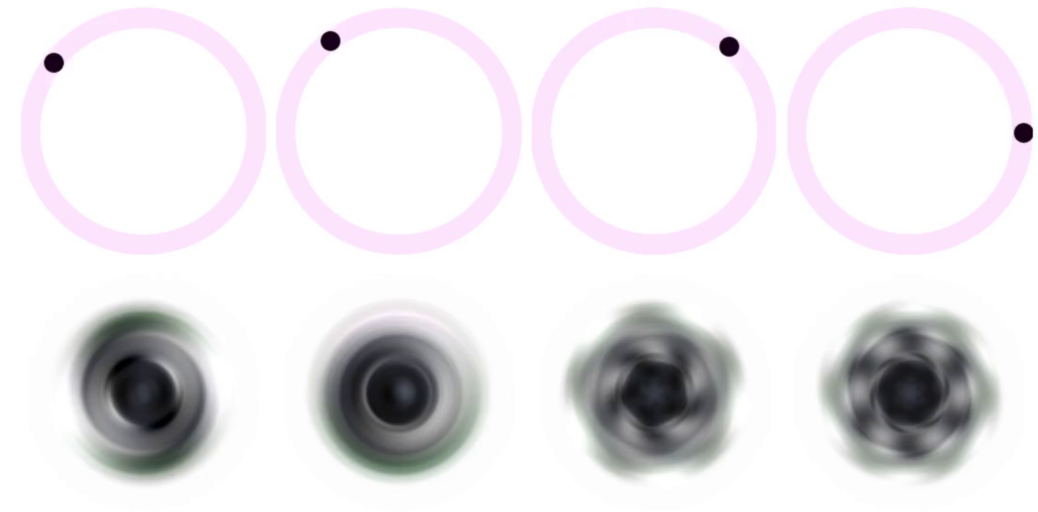


Rotations of $m \times m$ RGB image

Embedding in $\mathbb{R}^{m\times m\times 3}$



Projection in eigenplanes



Eigenvalues of the point cloud's covariance matrix:

$$311.2, \quad 311.2, \quad 221.3, \quad 221.3, \quad 82.3, \quad 82.3, \quad 79.4, \quad 79.4, \quad \ldots$$

In these eigenplanes, the orbit is close to

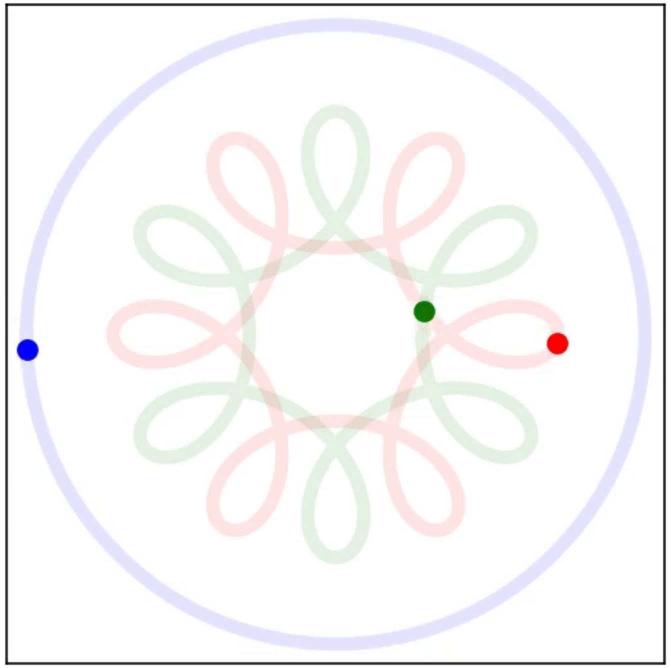
$$\theta \longmapsto \begin{pmatrix} \mu_1 \cos\omega_1\theta \\ \mu_1 \sin\omega_1\theta \\ \mu_2 \cos\omega_2\theta \\ \mu_2 \sin\omega_2\theta \\ \vdots \\ \mu_k \cos\omega_k\theta \\ \mu_k \sin\omega_k\theta \end{pmatrix} = \begin{pmatrix} \cos\omega_1\theta & -\sin\omega_1\theta & & & & & \\ \sin\omega_1\theta & \cos\omega_1\theta & & & & & \\ & & \cos\omega_2\theta & -\sin\omega_2\theta & & & \\ & & \sin\omega_2\theta & \cos\omega_2\theta & & & \\ & & & & \ddots & & \\ & & & & & \cos\omega_k\theta & -\sin\omega_k\theta \\ & & & & & \sin\omega_k\theta & \cos\omega_k\theta \end{pmatrix} \begin{pmatrix} \mu_1 \\ 0 \\ \mu_2 \\ 0 \\ \vdots \\ \mu_k \\ 0 \end{pmatrix}$$

In 1975, Roger Broucke found several periodic orbits.

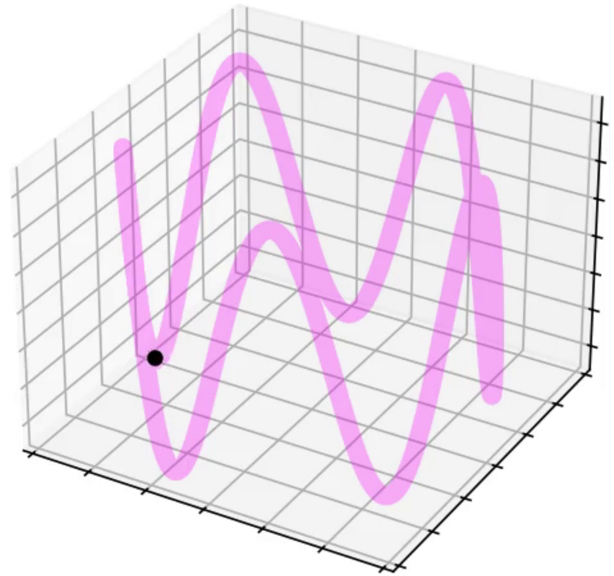
Let $x_1(t)$, $x_2(t)$, $x_3(t)$ be the three bodies, and define $z(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^6$.

<u>Orbit `A3`</u>

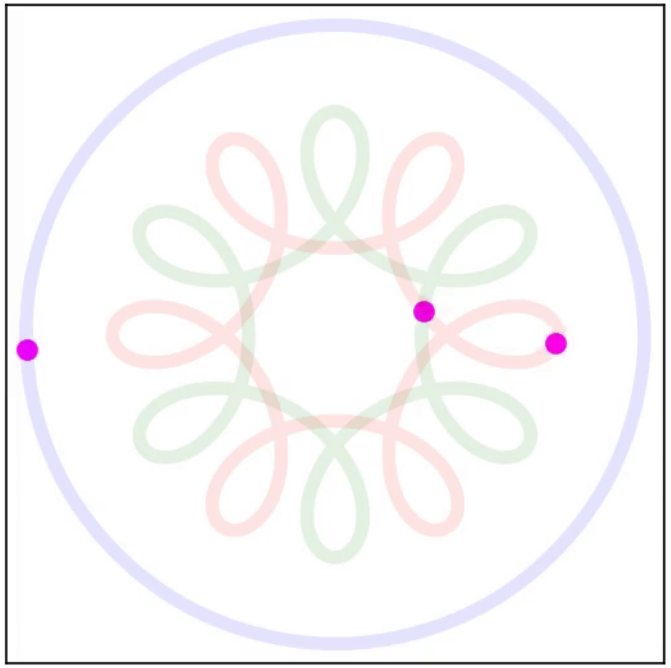
Trajectory of $x_1, x_2, x_3$
(found by integration)



Trajectory of $z$



Reconstructed orbit of SO(2)

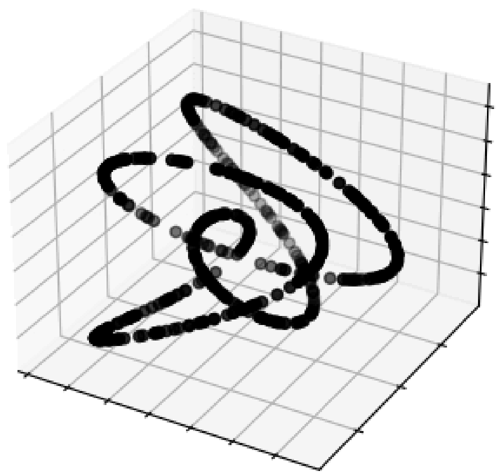


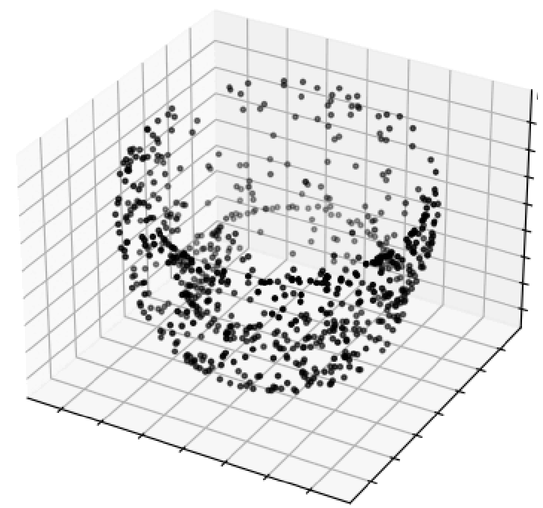
**Input:** A point cloud $X = \{x_1 \ldots, x_N\} \subset \mathbb{R}^n$.

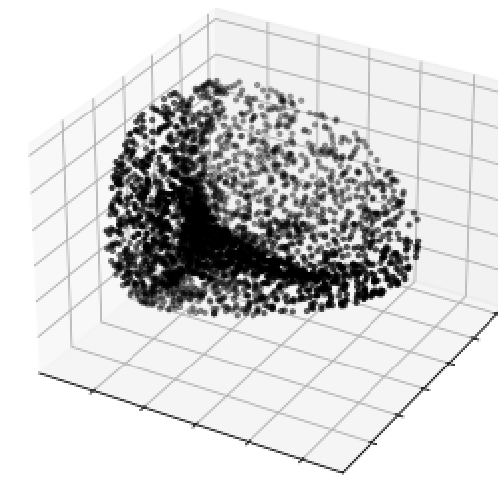
**Output:** A compact Lie group $G$, a representation $\phi$ in $\mathbb{R}^n$, and an orbit $\mathcal{O}$ close to $X$.

Orbit of SO(2) in $\mathbb{R}^6$ | Orbit of $T^2$ in $\mathbb{R}^6$ | Orbit of SO(3) in $\mathbb{R}^9$

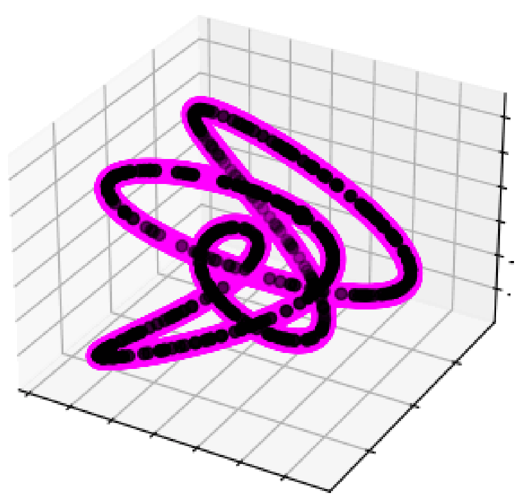
**Input:**

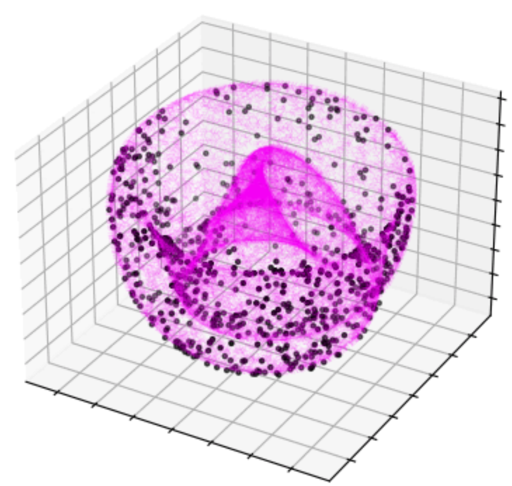


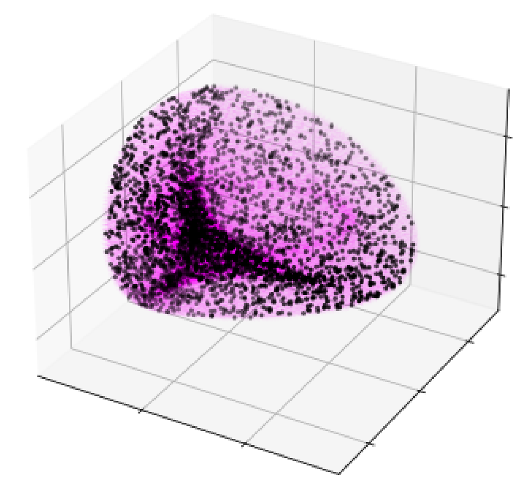




**Output:**







A **Lie group** is a smooth manifold endowed with a group operation, and such that $(g, h) \mapsto gh^{-1}$ is smooth.

- $\mathrm{GL}_n(\mathbb{R})$ general linear group: the $n \times n$ invertible matrices.
- $\mathrm{O}(n)$ orthogonal group: the $n \times n$ orthogonal matrices ($A^\top = A^{-1}$).
- $\mathrm{SO}(n)$ special orthogonal group: the $n \times n$ orthogonal matrices with determinant $+1$.
- $\mathrm{U}(n)$ unitary group: $n \times n$ (complex) unitary matrices ($A^* = A^{-1}$).
- $\mathrm{SU}(n)$ special unitary group: $n \times n$ (complex) unitary matrices with determinant $+1$.
- $T^d$ $d$-torus: the product $\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)$.

A (real) **representation** of dimension $n$ is a smooth homomorphism $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$.

- If $G$ is a matrix group, the natural embedding $G \to \mathrm{GL}_n(\mathbb{R})$ is a representation.
- For $G = \mathrm{SO}(2)$ and $\omega \in \mathbb{Z}$, one has $\theta \mapsto \begin{pmatrix} \cos \omega\theta & -\sin \omega\theta \\ \sin \omega\theta & \cos \omega\theta \end{pmatrix}$.

A representation $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$ is **irreducible** (irrep) if no non-trivial subspace $V \subset \mathbb{R}^n$ is stabilized.

**Fact:** Every representation $\phi$ is equivalent to a sum of irreps. That is, one has a decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$, irreps $\phi_i\colon G \to V_i$ and a change of basis $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$A\phi A^{-1} = \phi_1 \oplus \cdots \oplus \phi_k.$$

Irreps can be explicitly enumerated:

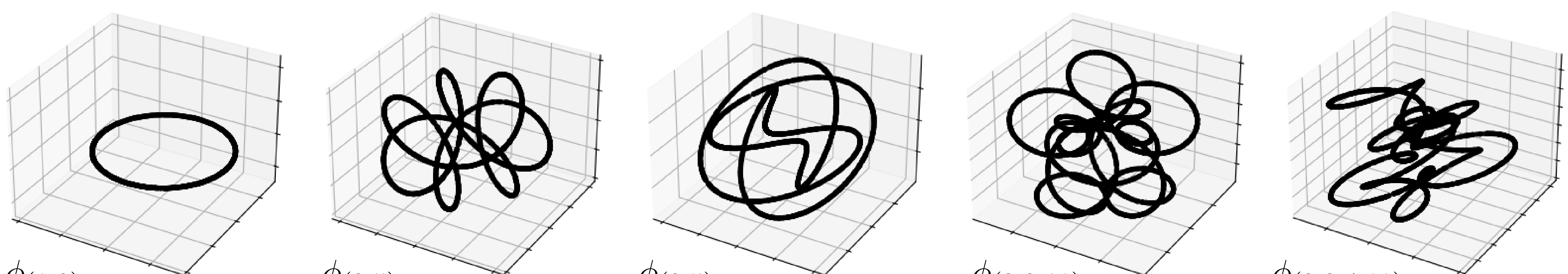
- SO(2) the $\theta \mapsto R(\omega\theta)$ for $\omega \in \mathbb{Z} \setminus \{0\}$, where $R(\theta) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
- $T^d$ the $(\theta_i)_{i=1}^d \mapsto R(\sum_{i=1}^d \omega_i\theta_i)$ for $(\omega_i)_{i=1}^d \in \mathbb{Z}^d \setminus \{0\}$.
- SO(3) one irrep in $\mathbb{R}^n$ for $n$ odd.
- SU(2) one irrep in $\mathbb{R}^n$ for $n$ odd or $n \equiv 0 \mod 4$.

The **orbit** of $x_0 \in \mathbb{R}^n$ under a representation $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$ is $\mathcal{O} = \{\phi(g)x_0 \mid g \in G\}$.

**Example:** Orbits of $\mathrm{SO}(2)$ in $\mathbb{R}^{2k}$.

Let us write $\phi \simeq \phi_{\omega_1} \oplus \cdots \oplus \phi_{\omega_k}$. The orbit is made of the points

$$\phi(\theta)x_0 = \begin{pmatrix} \cos\omega_1\theta & -\sin\omega_1\theta & & & & & \\ \sin\omega_1\theta & \cos\omega_1\theta & & & & & \\ & & \cos\omega_2\theta & -\sin\omega_2\theta & & & \\ & & \sin\omega_2\theta & \cos\omega_2\theta & & & \\ & & & & \ddots & & \\ & & & & & \cos\omega_k\theta & -\sin\omega_k\theta \\ & & & & & \sin\omega_k\theta & \cos\omega_k\theta \end{pmatrix} x_0$$



$\phi_{(1,0)}$ $\phi_{(2,5)}$ $\phi_{(3,5)}$ $\phi_{(2,3,11)}$ $\phi_{(2,3,4,11)}$

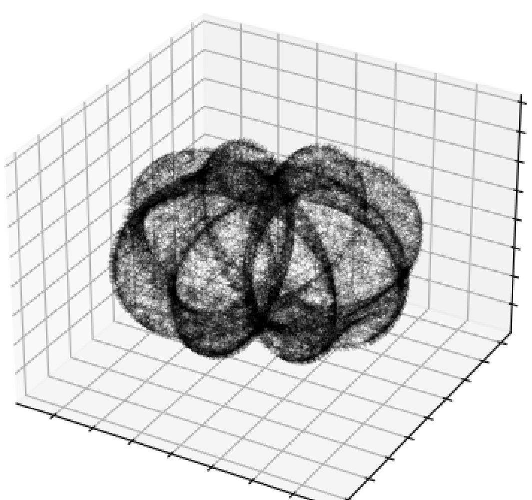
The **orbit** of $x_0 \in \mathbb{R}^n$ under a representation $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$ is $\mathcal{O} = \{\phi(g)x_0 \mid g \in G\}$.

**Example:** Orbits of $T^2$ in $\mathbb{R}^{2k}$.

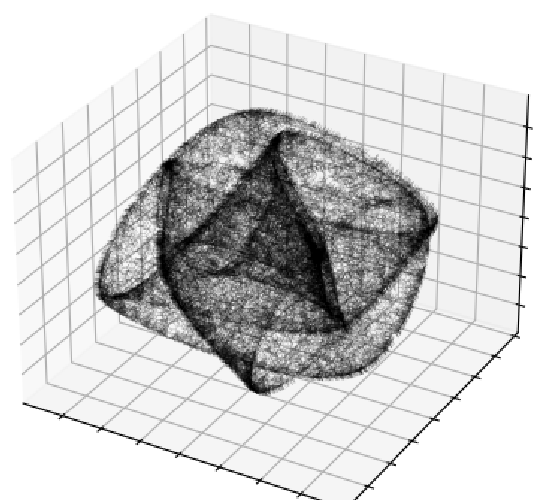
Let us write

$$\phi \simeq \phi_{\begin{pmatrix} \omega_1^{(1)} \\ \omega_1^{(2)} \end{pmatrix}} \oplus \cdots \oplus \phi_{\begin{pmatrix} \omega_k^{(1)} \\ \omega_k^{(2)} \end{pmatrix}}$$

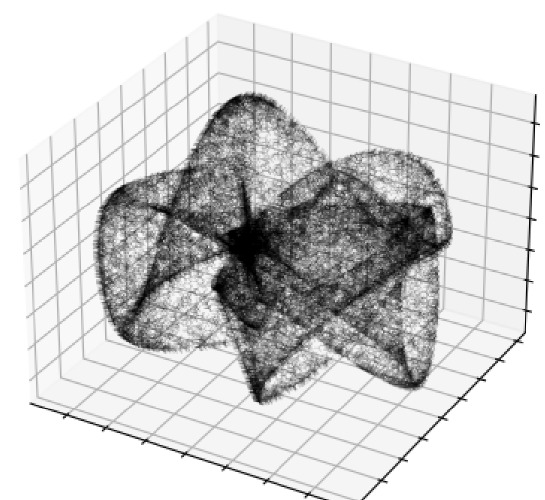
One builds the integer matrix of weights $\begin{pmatrix} \omega_1^{(1)} & \cdots & \omega_k^{(1)} \\ \omega_1^{(2)} & \cdots & \omega_k^{(2)} \end{pmatrix}$.



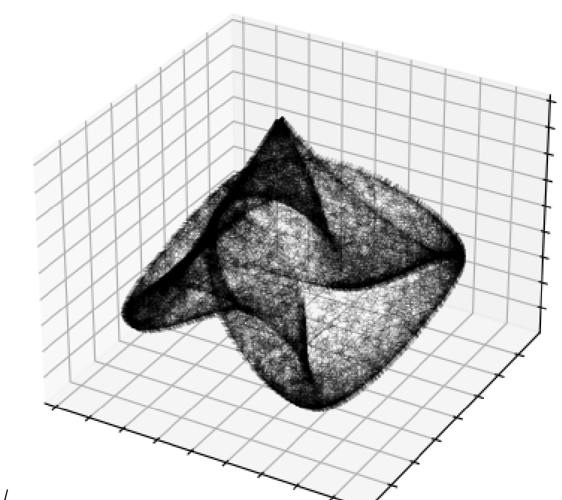
$\phi_{\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}}$



$\phi_{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}}$



$\phi_{\begin{pmatrix} -2 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \end{pmatrix}}$



$\phi_{\begin{pmatrix} 2 & -2 & 0 & 2 \\ 1 & 1 & -1 & 2 \end{pmatrix}}$

The **orbit** of $x_0 \in \mathbb{R}^n$ under a representation $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$ is $\mathcal{O} = \{\phi(g)x_0 \mid g \in G\}$.

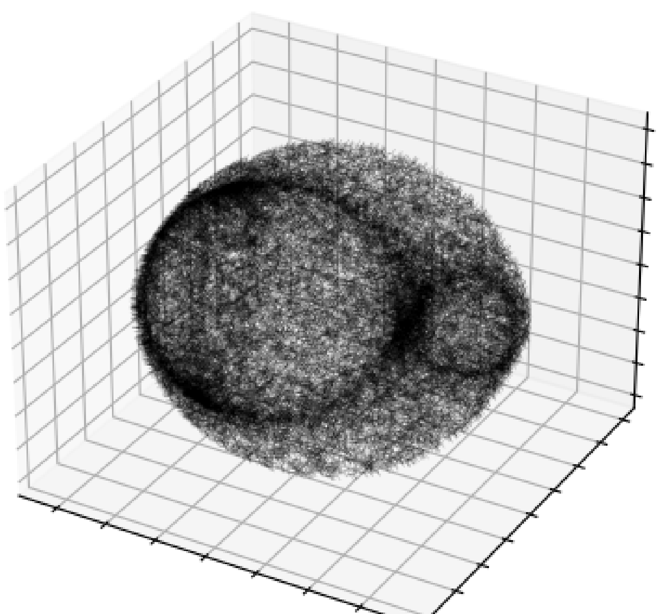
**Example:** Orbits of SO(3) and SU(2) in $\mathbb{R}^n$.

SO(3) has a finite number of (equivalence classes) of representations in $\mathbb{R}^n$: one for each decomposition

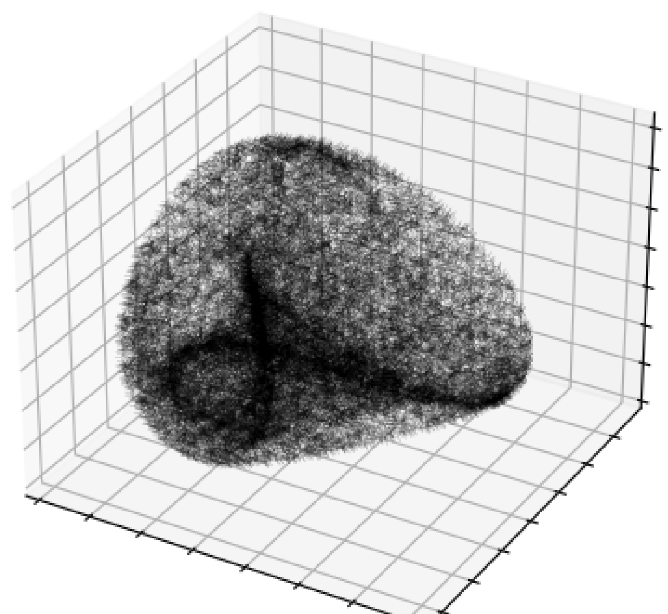
$$n = \omega_1 + \cdots + \omega_k$$

where the $\omega_i$ are odd.

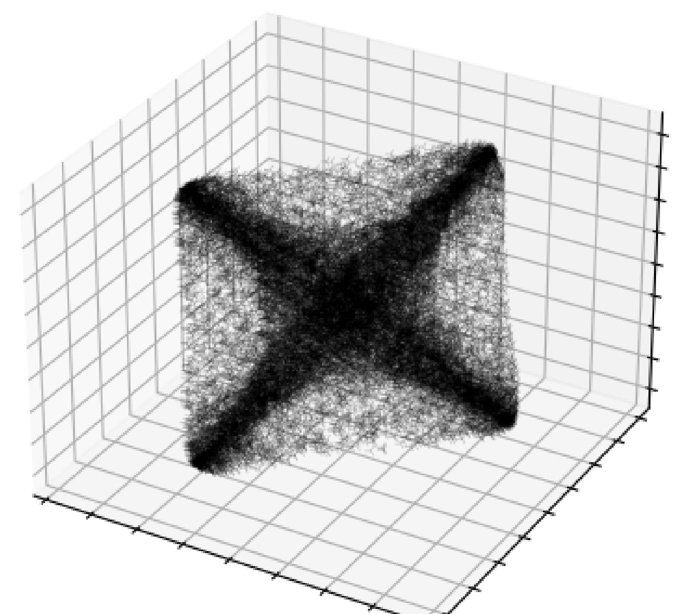
For SU(2), one can also use multiples of 4.



$\phi_{(5)}$ in $\mathbb{R}^5$



$\phi_{(3,4)}$ in $\mathbb{R}^7$



$\phi_{(8)}$ in $\mathbb{R}^8$

Say we observe an orbit $\mathcal{O}$ of a representation $\phi_1 : G \to \mathrm{GL}_n(\mathbb{R})$, and we want to find $\phi_1$.

Identifiability problem: another representation $\phi_2$ may generate $\mathcal{O}$.

$$\phi_1 : \theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \phi_2 : \theta \mapsto \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

The representations are said **orbit-equivalent** if there exists a $A \in \mathrm{GL}_n(\mathbb{R})$ such that for all $x_0 \in \mathbb{R}^n$,

$$\underbrace{\{A\phi_1(g)A^{-1}x_0 \mid g \in G\}}_{\text{orbit of } x_0 \text{ under } A\phi_1 A^{-1}} = \underbrace{\{\phi_2(g)x_0 \mid g \in G\}}_{\text{orbit of } x_0 \text{ under } \phi_2}.$$

The orbit-equivalence classes of representations of:

- $\mathrm{SO}(2)$ in $\mathbb{R}^{2k}$ are the (non-negative, non-decreasing) primitive $k$-tuples of integers.
- $T^d$ in $\mathbb{R}^{2k}$ are the primitive lattices in $\mathbb{Z}^k$ of rank $\leq d$.

For $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$, equivalence and orbit-equivalence coincide.

A Lie group $G$ admits a **Lie algebra**, denoted $\mathfrak{g}$. It is a vector space endowed with a **Lie bracket** $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

For $G \subset \mathrm{M}_n(\mathbb{R})$, $\mathfrak{g}$ is the tangent space of $G$ at identity, and the bracket is the commutator $[A, B] = AB - BA$.

- $\mathrm{GL}_n(\mathbb{R})$ $\mathfrak{gl}(n)$ is the set of $n \times n$ matrices.
- $\mathrm{SO}(n)$ $\mathfrak{so}(n)$ is the set of $n \times n$ skew-symmetric matrices.
- $T^n$ $t^n$ is the set of $2n \times 2n$ skew-symmetric matrices that are $2 \times 2$ block-diagonal.

One has an **exponential map** $\exp\colon \mathfrak{g} \to G$. Is it surjective when $G$ is connected and compact.

Given a representation $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$, one builds the **derived homomorphism** $\mathrm{d}\phi$:

$$\begin{tikzcd} G \arrow[r, "\phi"] & \mathrm{GL}_n(\mathbb{R}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{d}\phi"] & \mathfrak{gl}(n) \arrow[u, "\exp"'] \end{tikzcd}$$

We call $\mathrm{d}\phi(\mathfrak{g})$ the **pushforward** Lie algebra. It is a subalgebra of $\mathfrak{gl}(n)$.

**Fact:** Two representations $\phi_1, \phi_2$ are orbit-equivalent iff there exists $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$A\mathrm{d}\phi_1(\mathfrak{g})A^{-1} = \mathrm{d}\phi_2(\mathfrak{g}).$$

Moduli space of Lie algebras: This is an invitation to work in

$$\mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n)) \big/ \mathrm{GL}_n(\mathbb{R})$$

where $\mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n))$ is the Grassmannian of $d$-dimensional Lie subalgebras of $\mathfrak{gl}(n)$, acted upon by $\mathrm{GL}_n(\mathbb{R})$.

Denote $\mathfrak{h} = \mathrm{d}\phi(\mathfrak{g})$. There exists a intermediate space between $\mathrm{d}\phi(\mathfrak{g}) \subset \mathfrak{gl}(n)$.

$$\begin{tikzcd}
G \arrow[r, "\phi"] & \phi(G) & \subset & \mathrm{Sym}(\mathcal{O}) & \subset & \mathrm{GL}_n(\mathbb{R}) \\
\mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{d}\phi"] & \mathrm{d}\phi(\mathfrak{g}) \arrow[u, "\exp"] & \subset & \mathfrak{sym}(\mathcal{O}) \arrow[u, "\exp"] & \subset & \mathfrak{gl}(n) \arrow[u, "\exp"]
\end{tikzcd}$$

**Symmetry group:**

$$\mathrm{Sym}(\mathcal{O}) = \{P \in \mathrm{GL}_n(\mathbb{R}) \mid P\mathcal{O} = \mathcal{O}\}$$

**Symmetry algebra:**

$$\mathfrak{sym}(\mathcal{O}) = \{P \in \mathfrak{gl}(n) \mid \exp(P) \in \mathrm{Sym}(\mathcal{O})\}$$

Good news: $\mathfrak{sym}(\mathcal{O})$ can be estimated from a finite sample of $\mathcal{O}$.

[Cahill, Mixon, Parshall, Lie PCA: Density estimation for symmetric manifolds, 2023]

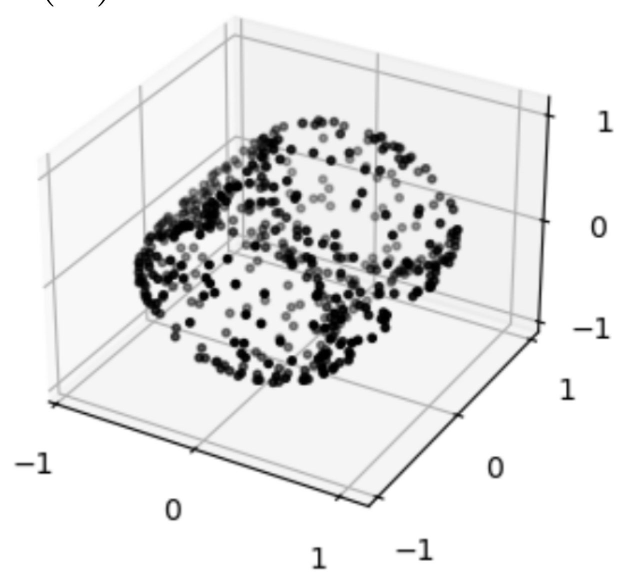
Temporary hypothesis: We will suppose that $\mathrm{d}\phi(\mathfrak{g}) = \mathfrak{sym}(\mathcal{O})$.

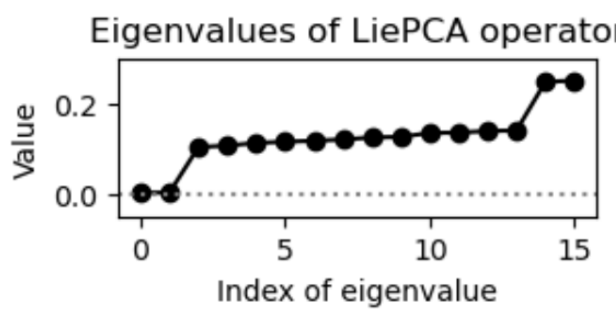
`LiePCA` operator: Say we observe $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$, assumed close to $\mathcal{O}$.

Define $\Lambda \colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ as $$\Lambda(A) = \frac{1}{N} \sum_{1 \leq i \leq N} \widehat{\Pi}\big[\mathrm{N}_{x_i} X\big] \cdot A \cdot \Pi\big[\langle x_i \rangle\big]$$

where
- $\widehat{\Pi}\big[\mathrm{N}_{x_i} X\big]$ are estimations of projection matrices onto the normal spaces $\mathrm{N}_{x_i}\mathcal{O}$,
- $\Pi\big[\langle x_i \rangle\big]$ the are projection matrices on the lines $\langle x_i \rangle$.

**Lemma:** $\ker \Lambda \approx \mathfrak{sym}(\mathcal{O})$.

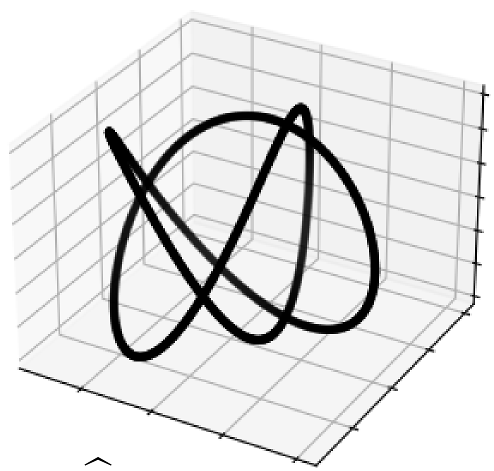


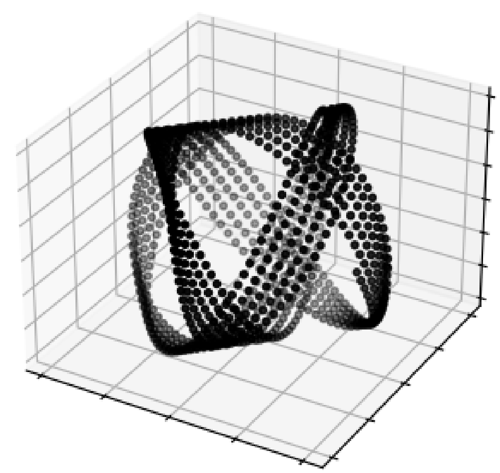

Define $\widehat{\mathfrak{h}}$ as the subspace of $\mathfrak{gl}(n)$ spanned by the $d$ bottom eigenvectors of $\Lambda$.

What can go wrong: $\widehat{\mathfrak{h}}$ is estimated as if it were a vector subspace.

- It may not be a Lie algebra ($A, B \in \widehat{\mathfrak{h}} \implies AB - BA \in \widehat{\mathfrak{h}}$).
- It may not come from a compact Lie group.
- We still do not know what is the representation.



exact $\widehat{\mathfrak{h}}$



inexact $\widehat{\mathfrak{h}}$

We wish to find the Lie algebra closest to $\widehat{\mathfrak{h}}$. The problem reads

$$\min \left\{ \mathrm{d}(\widehat{\mathfrak{h}}, V) \mid V \in \mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n)) \right\}.$$

Remember that $\mathrm{d}\phi(\mathfrak{g}) \subset \mathfrak{sym}(\mathcal{O})$ and $\ker \Lambda \approx \mathfrak{sym}(\mathcal{O})$ (`LiePCA` operator).

Case $\mathrm{d}\phi(\mathfrak{g}) = \mathfrak{sym}(\mathcal{O})$: We compute the span $\widehat{\mathfrak{h}}$ of bottom eigenvectors of $\Lambda$, and solve

$$\min\left\{\mathrm{d}(\widehat{\mathfrak{h}}, V) \mid V \in \mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n))\right\}.$$

Case $\mathrm{d}\phi(\mathfrak{g}) \subset \mathfrak{sym}(\mathcal{O})$: We consider instead

$$\min\left\{\sum_{i=1}^{d} \|\Lambda(A_i)\|^2 \mid \langle A_1, \ldots, A_d \rangle = V \in \mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n))\right\}$$

**Tentative implementation:** Let us embed $\mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n)) \hookrightarrow \mathrm{M}_{n^2}(\mathbb{R})$, the $n^2 \times n^2$ matrices, via $V \mapsto \mathrm{proj}[V]$.

$$\min \operatorname{tr}(\Lambda^2 P) \quad \text{such that} \quad \begin{cases} P \text{ is a } n^2 \times n^2 \text{ matrix}, \\ P^2 = P, \\ P^\top = P, \\ \operatorname{rank}(P) = d, \\ \forall i, j \in [1 \ldots, d],\ P(Pe_i \cdot Pe_j - Pe_j \cdot Pe_i) = Pe_i \cdot Pe_j - Pe_j \cdot Pe_i. \end{cases}$$

Fix $G$ and let $\mathcal{G}(G, \mathfrak{gl}(n))$ be the $d$-dimensional Lie subalgebras of $\mathfrak{gl}(n)$ that are pushforward of $\mathfrak{g}$.

The set ${}^{\mathcal{G}(G,\mathfrak{gl}(n))}\!\diagup_{\mathrm{GL}_n(\mathbb{R})}$ is in correspondence with the orbit-equivalence classes of reps of $G$ in $\mathbb{R}^n$.

$\boxed{\text{Let } \mathfrak{orb}(G,n) \text{ denote a choice of representatives.}}$

**Lemma:** The optimization problem is equivalent to

$$\min \sum_{i=1}^{d} \left\| \Lambda\big(A \mathrm{diag}(B_i^k)_{k=1}^p A^{-1}\big) \right\|^2 \quad \text{such that } \begin{cases} (B^1, \ldots, B^p) \in \mathfrak{orb}(G,n), \\ A \in \mathrm{GL}_n(\mathbb{R}), \end{cases}$$

Any representation $\phi \colon G \to \mathrm{GL}_n(\mathbb{R})$, up to a change of basis, decomposes as $\phi = \phi_1 \oplus \cdots \oplus \phi_p$.

By denoting $B^i = \mathrm{d}\phi_i(\mathfrak{g})$, the element $\mathrm{d}\phi(\mathfrak{g})$ of $\mathfrak{orb}(G,n)$ is associated to $(B^1, \ldots, B^p)$.

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$A \in \mathrm{O}(n)$

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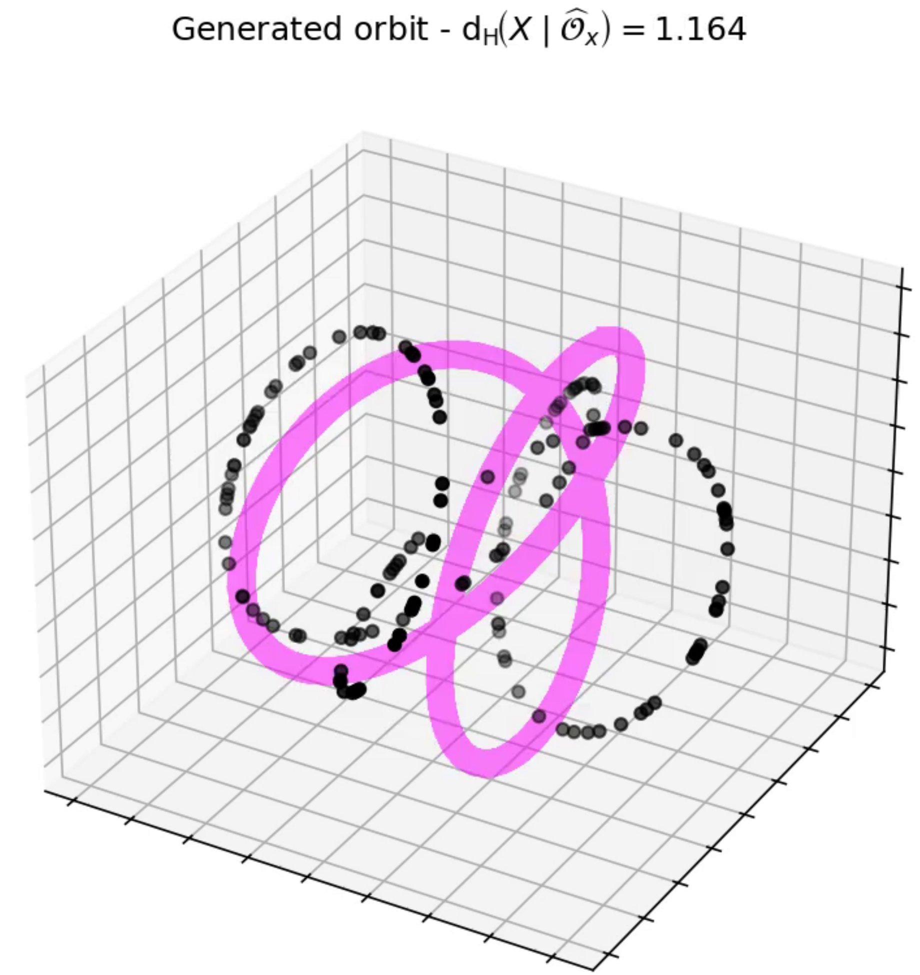
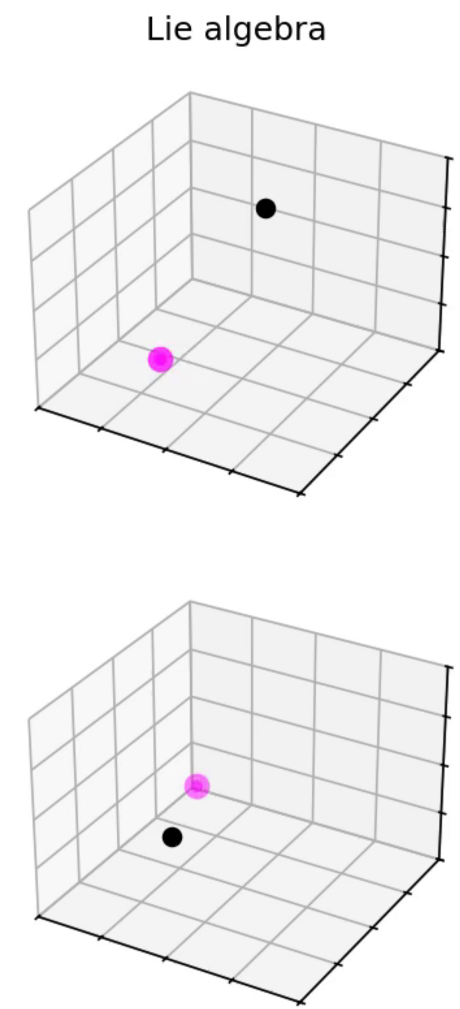
**Orthonormalization trick:** After a pre-processing step, we can reduce the program to $A \in \mathrm{O}(n)$.

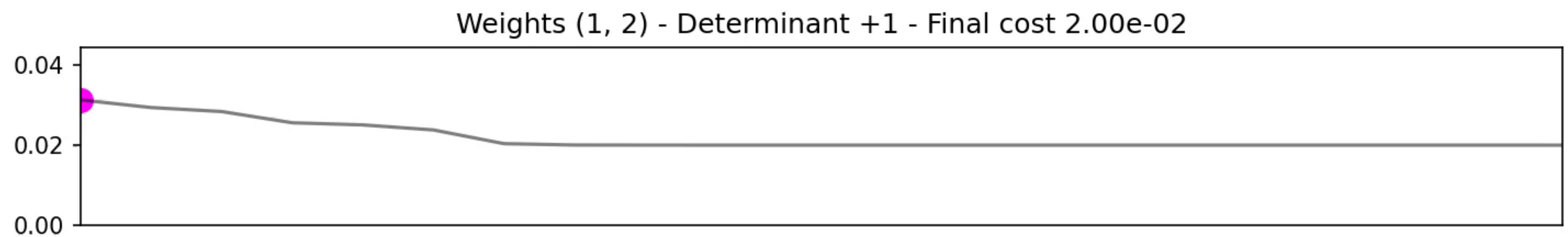
# Closest Lie algebra

SO(2)-orbit in $\mathbb{R}^4$

| Rep | Score |
|---|---|
| (1,2) | $+1 : 0.020$<br>$-1 : 0.001$ |
| (1,3) | $+1 : 0.017$<br>$-1 : 1 \times 10^{-5}$ |
| (1,4) | $+1 : 0.014$<br>$-1 : 4 \times 10^{-4}$ |
| (2,3) | $+1 : 0.020$<br>$-1 : 0.004$ |
| (3,4) | $+1 : 0.022$<br>$-1 : 0.005$ |



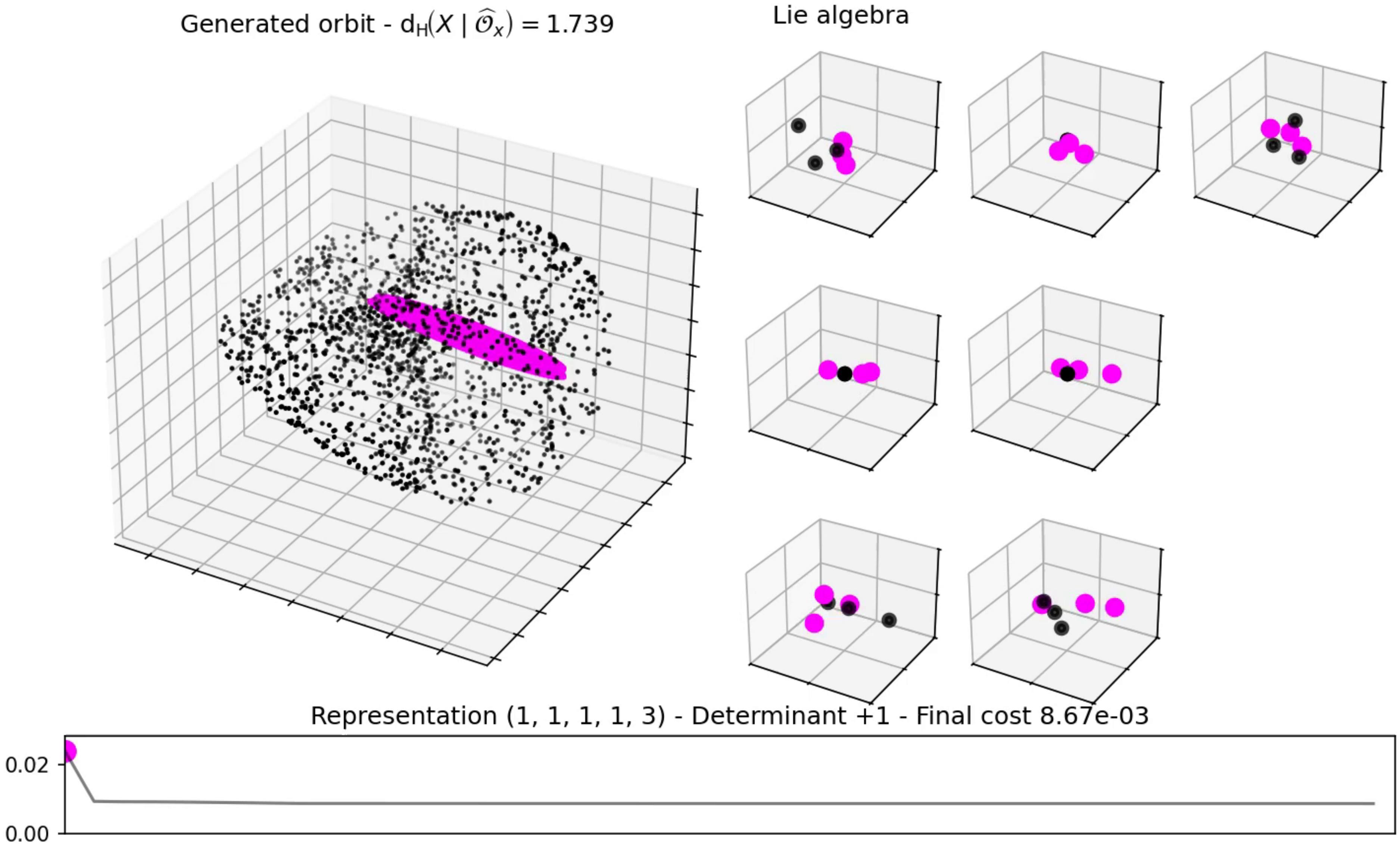



# Closest Lie algebra

SU(2)-orbit in $\mathbb{R}^7$

| Rep | Score |
|---|---|
| (3) | $+1 : 0.008$ |
| (4) | $+1 : 0.013$ |
| (5) | $+1 : 0.003$ |
| (3,3) | $+1 : 0.003$ |
| (3,4) | $+1 : 3 \times 10^{-5}$ |
| (7) | $+1 : 0.005$ |



**Reformulation of the optimization program:**

$$\min \left\{ \mathrm{d}(\widehat{\mathfrak{h}}, V) \mid V \in \mathcal{G}^{\mathrm{Lie}}(d, \mathfrak{gl}(n)) \right\}.$$

reduces to:

- SO(2) two-sided orthogonal Procrustes problem $\longrightarrow$ reduction of skew-symmetric matrix
- $T^d$ simultaneous reduction of $d$ skew-symmetric matrices $\longrightarrow$ optimization over $\mathrm{O}(n)$
- SO(3), SU(2) no reduction found

**Fact:** If $G$ is compact, for every representation $G \to \mathrm{GL}_n(\mathbb{R})$, there exists $M$ positive-definite such that

$$\forall g \in G,\ M\phi(g)M^{-1} \in \mathrm{O}(n).$$

Given an orbit $\mathcal{O} = G \cdot x_0$, consider the Haar measure $\mu_G$, and define the **covariance matrix**

$$\Sigma[\mathcal{O}] = \int_G \big(\phi(g)x_0\big)\big(\phi(g)x_0\big)^\top \mathrm{d}\mu_G(g).$$

$M$ is found as the square root of the Moore-Penrose pseudo-inverse:

$$M[\mathcal{O}] = \sqrt{\Sigma[\mathcal{O}]^+}.$$

Given a sample $X$, we build $\Sigma[X] = \frac{1}{N}\sum_{i=1}^N x_i x_i^\top$ and $M[X] = \sqrt{\Sigma[X]^+}$.

**Example:** With $M = \frac{1}{\sqrt{2}}\mathrm{diag}\big(1, 1/2, 1, 1\big)$,

$$\phi\colon t \mapsto \mathrm{diag}\left(\begin{pmatrix}\cos t & -(1/2)\sin t \\ 2\sin t & \cos t\end{pmatrix}, \begin{pmatrix}\cos 4t & -\sin 4t \\ \sin 4t & \cos 4t\end{pmatrix}\right), \qquad \mathcal{O} = \big\{(\cos t, 2\sin t, \cos 4t, \sin 4t) \mid t \in [0, 2\pi)\big\}.$$

$$M\phi M^{-1}\colon t \mapsto \mathrm{diag}\left(\begin{pmatrix}\cos t & \sin t \\ \sin t & \cos t\end{pmatrix}, \begin{pmatrix}\cos 4t & -\sin 4t \\ \sin 4t & \cos 4t\end{pmatrix}\right), \qquad M\mathcal{O} = \big\{\tfrac{1}{\sqrt{2}}(\cos t, \sin t, \cos 4t, \sin 4t) \mid t \in [0, 2\pi)\big\}.$$

**Input:** A point cloud $X = \{x_1 \dots, x_N\} \subset \mathbb{R}^n$ and a candidate Lie group $G$.

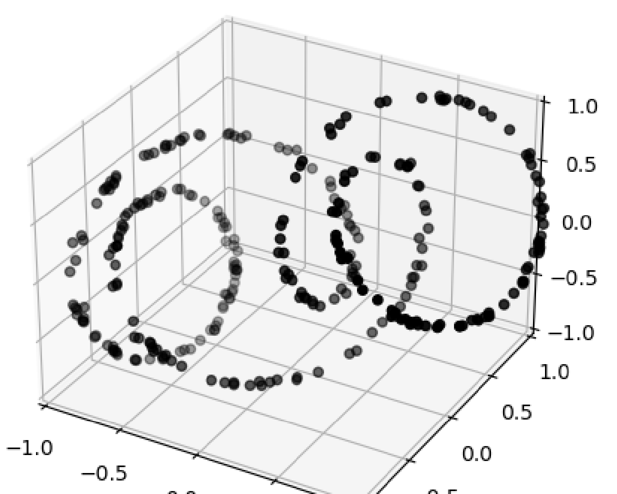
**Output:** A representation $\phi$ of $G$ in $\mathbb{R}^n$, and an orbit $\mathcal{O}$ close to $X$.

**Step 1 (Orthonormalization):** Reduce the dimension and orthonormalize the orbit.

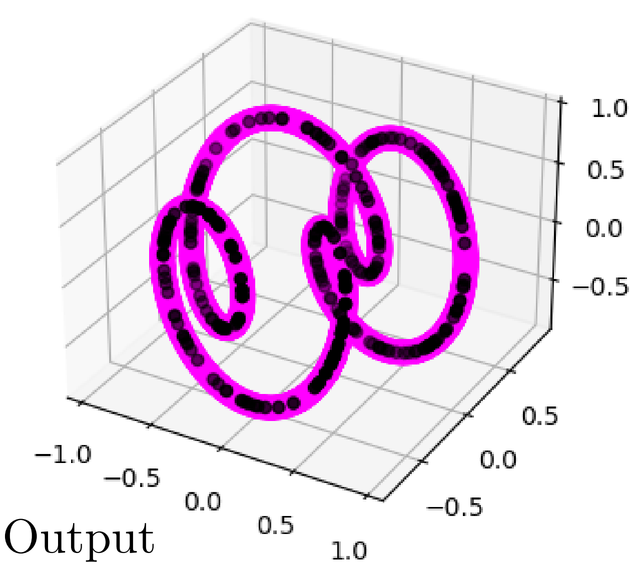
**Step 2 (LiePCA):** Diagonalize the operator $\Lambda\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$.

**Step 3 (Closest Lie algebra):** Estimate $\widehat{\mathfrak{h}}$ through an optimization over $\mathrm{O}(n)$.

**Step 4 (Distance to orbit):** Choose a $x \in X$, generate $\widehat{\mathcal{O}}_x = \exp(\widehat{\mathfrak{h}}) \cdot x$ and verify that it is close to $X$.




Input




Output

In **Step 4**, we compute the (non-symmetric) Hausdorff distance $\mathrm{d_H}\big(X|\widehat{\mathcal{O}}_x\big)$.

**Input:** A point cloud $X = \{x_1 \ldots, x_N\} \subset \mathbb{R}^n$ and a candidate Lie group $G$.

**Output:** A representation $\phi$ of $G$ in $\mathbb{R}^n$, and an orbit $\mathcal{O}$ close to $X$.

**Step 1 (Orthonormalization):** Reduce the dimension and orthonormalize the orbit.

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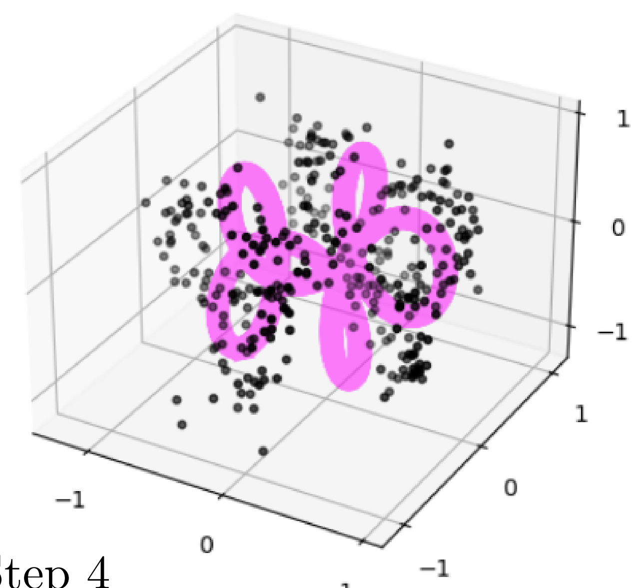
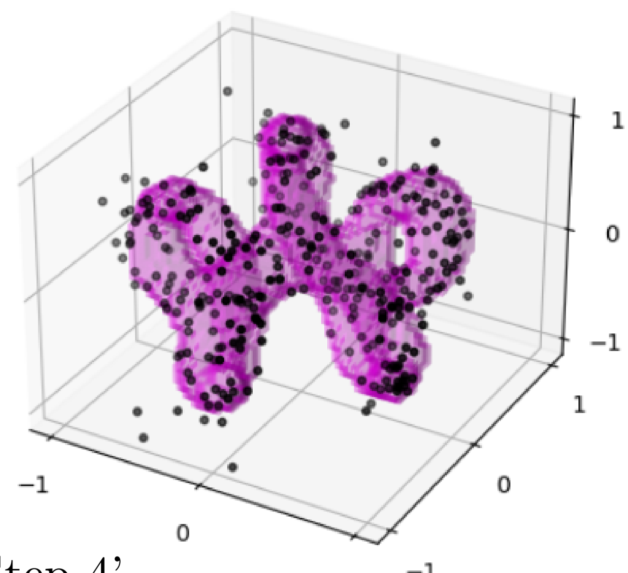
**Step 3 (Closest Lie algebra):** Estimate $\widehat{\mathfrak{h}}$ through an optimization over $\mathrm{O}(n)$.

**Step 4 (Distance to orbit):** Choose a $x \in X$, generate $\widehat{\mathcal{O}}_x = \exp(\widehat{\mathfrak{h}}) \cdot x$ and verify that it is close to $X$.

**Step 4' (Distance to noisy orbit):** Build the measure $\mu_{\widehat{\mathcal{O}}} = \frac{1}{N}\sum_{x\in X} \mu_{\widehat{\mathcal{O}}_x}$ and verify that it is close to $\mu_X$.




Step 4




Step 4'

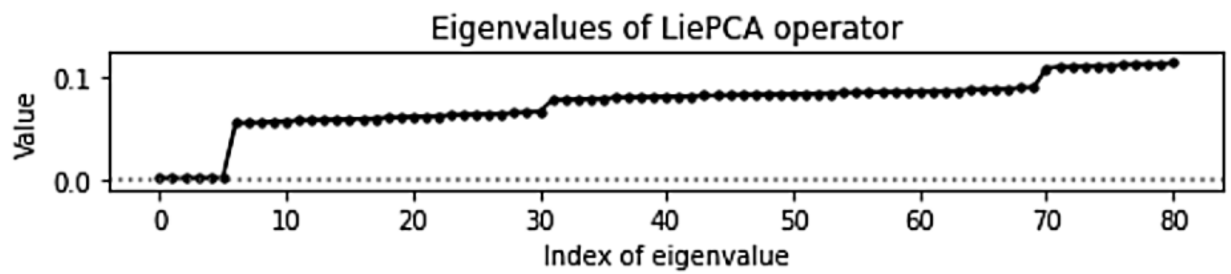
In **Step 4**, we compute the (non-symmetric) Hausdorff distance $\mathrm{d_H}\big(X|\widehat{\mathcal{O}}_x\big)$.

In **Step 4'**, we compute the Wasserstein distance $\mathrm{W}_2\big(\mu_X, \mu_{\widehat{\mathcal{O}}}\big)$.

**Example:** Embed $\mathrm{SO}(3) \hookrightarrow \mathbb{R}^9$ and sample 3000 points on it.

`LiePCA` shows a kernel of dimension 6.

This is consistent with $\mathrm{Isom}(\mathrm{SO}(3)) \simeq \mathrm{SO}(3) \rtimes \mathrm{SO}(3) \times \{\pm 1\}$




We look for an action of $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$. **Step 3** yields

| | | | | | | |
|---|---|---|---|---|---|---|
| Representation | $(3,5)$ | $(3,3,3)$ | $(4,5)$ | $(8)$ | $(5)$ | $(7)$ |
| Cost | $\mathbf{2\times 10^{-5}}$ | $\mathbf{4\times 10^{-5}}$ | 0.001 | 0.001 | 0.03 | 0.004 |
| Representation | $(9)$ | $(3,3)$ | $(3,4)$ | $(4,4)$ | $(3)$ | $(4)$ |
| Cost | 0.004 | 0.006 | 0.007 | 0.009 | 0.011 | 0.013 |

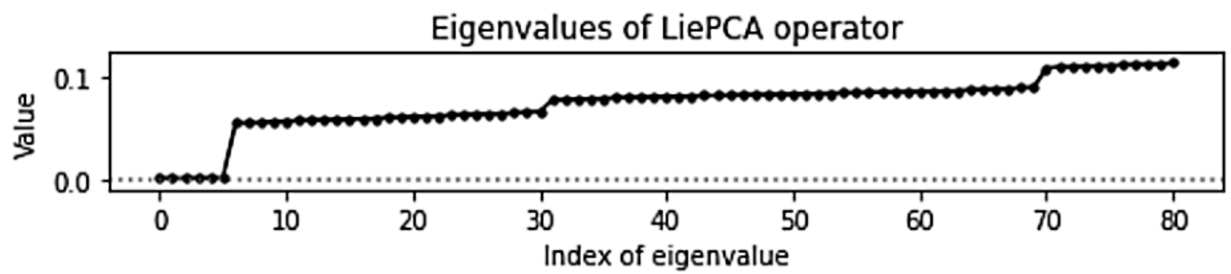
Representation $(3,5)$: $\mathrm{d_H}\left(X|\widehat{\mathcal{O}}_x\right) \approx 2.658$. However, $\mathrm{d_H}\left(\widehat{\mathcal{O}}_x|X\right) \approx 0.543$.

Representation $(3,3,3)$: $\mathrm{d_H}\left(X|\widehat{\mathcal{O}}_x\right) \approx 0.061$.

**Example:** Embed $\mathrm{SO}(3) \hookrightarrow \mathbb{R}^9$ and sample 3000 points on it.

`LiePCA` shows a kernel of dimension 6.

This is consistent with $\mathrm{Isom}(\mathrm{SO}(3)) \simeq \mathrm{SO}(3) \rtimes \mathrm{SO}(3) \times \{\pm 1\}$

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| | | | | | | |
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Representation $(3,5)$: $\mathrm{d_H}\left(X|\widehat{\mathcal{O}}_x\right) \approx 2.658$. However, $\mathrm{d_H}\left(\widehat{\mathcal{O}}_x|X\right) \approx 0.543$.

action $\mathrm{SO}(3) \curvearrowright \mathrm{SO}(3)$ by conjugation (not transitive)

Representation $(3,3,3)$: $\mathrm{d_H}\left(X|\widehat{\mathcal{O}}_x\right) \approx 0.061$.

action $\mathrm{SO}(3) \curvearrowright \mathrm{SO}(3)$ by translation (transitive)

# Robustness

**Input:** $X = \{x_1 \ldots, x_N\} \subset \mathbb{R}^n$ and $G$ compact.

**Model:** $X$ sampled close to an orbit $\mathcal{O}$ of a representation $\phi\colon G \to \mathbb{R}^n$

**Step 1:** Orthonormalization via
$X \leftarrow \sqrt{\Sigma[X]^+} \cdot \Pi^{\geq\epsilon}_{\Sigma[X]} \cdot X$

**Step 2:** Diagonalize the operator
$\Lambda\colon A \mapsto \frac{1}{N}\sum_{i=1}^N \widehat{\Pi}\big[\mathrm{N}_{x_i}X\big] \cdot A \cdot \Pi\big[\langle x_i\rangle\big]$

**Step 3:** Solve $\arg\min \sum_{i=1}^d \|\Lambda(A_i)\|^2$
with $(A_i)_{i=1}^d \in \mathcal{V}^{\mathrm{Lie}}(G, \mathfrak{so}(n))$

**Step 4:** Output $\widehat{\mathcal{O}}_x = \big\{\exp(A)x \mid A \in \widehat{\mathfrak{h}}\big\}$

**Goal:** Show that $\widehat{\mathcal{O}}_x$ is close to $\mathcal{O}$

**Input:** $X = \{x_1 \ldots, x_N\} \subset \mathbb{R}^n$ and $G$ compact.

$\mu$ measure on $\mathbb{R}^n$. E.g., $\mu_X$ empirical measure on $X$

**Model:** $X$ sampled close to an orbit $\mathcal{O}$ of a representation $\phi\colon G \to \mathbb{R}^n$

$\mu_{\mathcal{O}}$ uniform measure on $\mathcal{O}$

**Step 1:** Orthonormalization via
$X \leftarrow \sqrt{\Sigma[X]^+} \cdot \Pi^{\geq \epsilon}_{\Sigma[X]} \cdot X$

$\mu \leftarrow \sqrt{\Sigma[\mu]^+} \cdot \Pi^{\geq \epsilon}_{\Sigma[\mu]} \cdot \mu$

**Step 2:** Diagonalize the operator
$\Lambda\colon A \mapsto \frac{1}{N} \sum_{i=1}^N \widehat{\Pi}\big[\mathrm{N}_{x_i} X\big] \cdot A \cdot \Pi\big[\langle x_i \rangle\big]$

$\Lambda[\mu]\colon A \mapsto \int_{i=1}^N \widehat{\Pi}\big[\mathrm{N}_{x_i} X\big] \cdot A \cdot \Pi\big[\langle x_i \rangle\big] \mathrm{d}\mu$

**Step 3:** Solve $\arg\min \sum_{i=1}^d \|\Lambda(A_i)\|^2$
with $(A_i)_{i=1}^d \in \mathcal{V}^{\mathrm{Lie}}(G, \mathfrak{so}(n))$

$\arg\min \sum_{i=1}^d \|\Lambda[\mu](A_i)\|^2$
with $(A_i)_{i=1}^d \in \mathcal{V}^{\mathrm{Lie}}(G, \mathfrak{so}(n))$

**Step 4:** Output $\widehat{\mathcal{O}}_x = \big\{ \exp(A)x \mid A \in \widehat{\mathfrak{h}} \big\}$

$\mu_{\widehat{\mathcal{O}}_x} = \exp(\widehat{\mathfrak{h}}) \cdot \mu$

**Goal:** Show that $\widehat{\mathcal{O}}_x$ is close to $\mathcal{O}$

Show that $\mathrm{W}_2\big(\mu_{\widehat{\mathcal{O}}_x}, \mu_{\mathcal{O}}\big)$ is small

**Theorem:** Under technical assumptions (sufficiently small $\mathrm{W}_2(\mu_X, \mu_{\mathcal{O}})$), for a certain choice of parameters, the algorithm outputs a representation $\widehat{\phi}$ that is orbit-equivalent to $\phi$.

Let $l = \dim \mathcal{O}$. The output measure $\mu_{\widehat{\mathcal{O}}}$ satisfies

$$\mathrm{W}_2\big(\mu_{\widehat{\mathcal{O}}}, \mu_{\mathcal{O}}\big) \leq \mathrm{const} \cdot \mathrm{W}_2\big(\mu_X, \mu_{\mathcal{O}}\big)^{1/4(l+3)}.$$

In addition, for all $x \in X$, the output orbit $\widehat{\mathcal{O}}_x$ satisfies

$$\mathrm{d_H}\big(\widehat{\mathcal{O}}_x, \mathcal{O}\big) \leq \mathrm{const} \cdot \mathrm{d}(x, \mathcal{O}) + \mathrm{const} \cdot \mathrm{W}_2\big(\mu_X, \mu_{\mathcal{O}}\big)^{1/4(l+3)}.$$

**Theorem:** Let $G$ be a compact Lie group of dimension $d$, $\mathcal{O}$ an orbit of an almost-faithful representation $\phi\colon G \to \mathbb{R}^n$, potentially non-orthogonal, and $l$ its dimension. Let $\mu_{\mathcal{O}}$ be the uniform measure on $\mathcal{O}$, and $\mu_{\widetilde{\mathcal{O}}}$ that on the orthonormalized orbit. Let $X \subset \mathbb{R}^n$ be a finite point cloud and $\mu_X$ its empirical measure.

Let $\widehat{\phi}$, $\widehat{\mathfrak{h}}$, $\widehat{\mathcal{O}}_x$, $\mu_{\widehat{\mathcal{O}}}$ be the output of the algorithm. Under technical assumptions, $\widehat{\phi}$ is orbit-equivalent to $\phi$, and

$$\|\Pi[\widehat{h}] - \Pi[\mathfrak{sym}(\mathcal{O})]\|_{\mathrm{F}} \leq 9d\frac{\rho}{\lambda}\left(r + 4\left(\frac{\widetilde{\omega}}{r^{l+1}}\right)^{1/2}\right)$$

$$\mathrm{d}_{\mathrm{H}}(\widehat{\mathcal{O}}_x, \mathcal{O}) \leq \sqrt{2}\frac{\mathrm{d}(x,\mathcal{O})}{\sigma_{\min}} + 3\sqrt{dn}\left(\frac{\rho}{\lambda}\right)^{1/2}\left(r + 4\left(\frac{\widetilde{\omega}}{r^{l+1}}\right)^{1/2}\right)^{1/2}$$

$$\mathrm{W}_2(\mu_{\widehat{\mathcal{O}}}, \mu_{\widetilde{\mathcal{O}}}) \leq \frac{1}{\sqrt{2}}\frac{\mathrm{W}_2(\mu_X, \mu_{\mathcal{O}})}{\sigma_{\min}} + 3\sqrt{dn}\left(\frac{\rho}{\lambda}\right)^{1/2}\left(r + 4\left(\frac{\widetilde{\omega}}{r^{l+1}}\right)^{1/2}\right)^{1/2}$$

where

- $\rho = 16l(l+2)6^l \max(\mathrm{vol}(\widetilde{\mathcal{O}}), \mathrm{vol}(\widetilde{\mathcal{O}})^{-1}) / \min(1, \mathrm{reach}(\widetilde{\mathcal{O}}))$,
- $\sigma_{\max}^2$, $\sigma_{\min}^2$ the top and bottom nonzero eigenvalues of the covariance matrix $\Sigma[\mu_{\mathcal{O}}]$,
- $\widetilde{\omega} = 4(n+1)^{3/2}\left(\frac{\sigma_{\max}^3}{\sigma_{\min}^3}\right)\left(\omega(\upsilon+\omega)\right)^{1/2}$ with $\omega = \frac{\mathrm{W}_2(\mu_{\mathcal{O}}, \mu_X)}{\sigma_{\min}}$ and $\upsilon = \left(\frac{\mathbb{V}\left[\|\mu_{\mathcal{O}}\|\right]}{\sigma_{\min}^2}\right)^{1/2}$,
- $r$ is the radius of local PCA (estimation of tangent spaces),
- $\lambda$ the bottom nonzero eigenvalue of the ideal Lie-PCA operator $\Lambda_{\mathcal{O}}$.

**Technical assumptions:** Define the quantities

$$\omega = \frac{\mathrm{W}_2(\mu_{\mathcal{O}}, \mu_X)}{\sigma_{\min}}, \qquad \upsilon = \left(\frac{\mathbb{V}\big[\|\mu_{\mathcal{O}}\|\big]}{\sigma_{\min}^2}\right)^{1/2},$$

$$\widetilde{\omega} = 4(n+1)^{3/2}\left(\frac{\sigma_{\max}^3}{\sigma_{\min}^3}\right)\Big(\omega(\upsilon+\omega)\Big)^{1/2}, \qquad \rho = \Big(16l(l+2)6^l\Big)\frac{\max(\mathrm{vol}(\widetilde{\mathcal{O}}), \mathrm{vol}(\widetilde{\mathcal{O}})^{-1})}{\min(1, \mathrm{reach}(\widetilde{\mathcal{O}}))},$$

$$\gamma = \left(4(2d+1)\sqrt{2}\right)^{-1} \cdot \lambda \cdot \Gamma(G, n, \omega_{\max}) \quad \text{(rigidity constant of Lie subalgebras)}$$

Suppose that $\omega$ is small enough, so as to satisfy

$$\omega < \left(\left(\upsilon^2 + \frac{1}{2}\right)^{1/2} - \upsilon\right)\Big/\left(3(n+1)\frac{\sigma_{\max}^2}{\sigma_{\min}^2}\right), \qquad \widetilde{\omega} \leq \min\left\{\left(\frac{1}{6\rho}\right)^{3(l+1)}, \frac{\gamma^{l+3}}{16}, \left(\frac{\gamma}{(6\rho)^2}\right)^{l+1}\right\}.$$

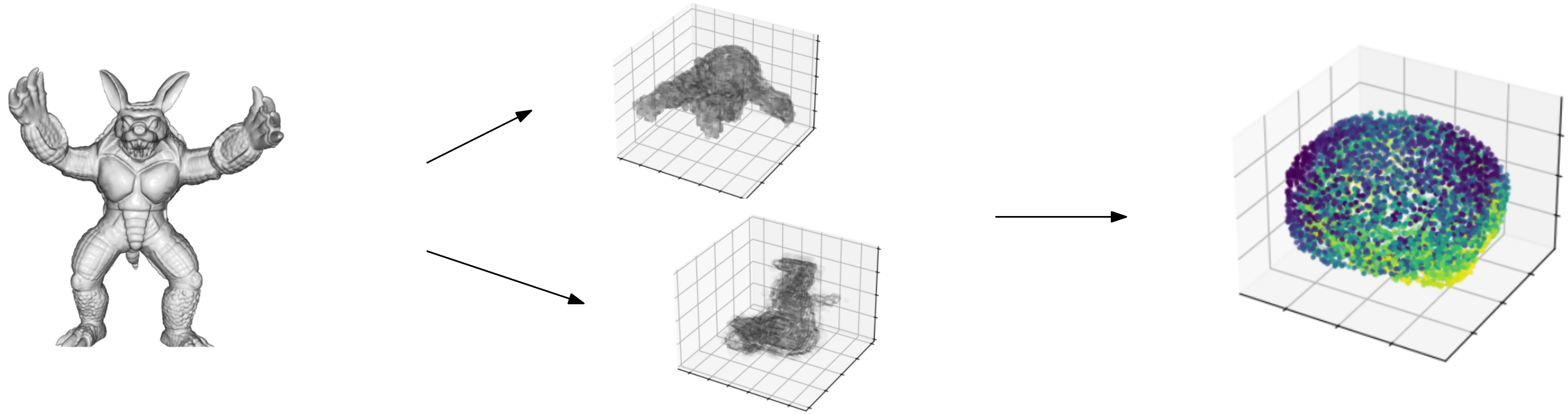
Choose two parameters $\epsilon$ and $r$ in the following nonempty sets:

$$\epsilon \in \left((2\upsilon+\omega)\omega\sigma_{\min}^2,\ \frac{1}{2}\sigma_{\min}^2\right], \qquad r \in \left[\left(6\rho\right)^2 \cdot \widetilde{\omega}^{1/(l+1)},\ \left(6\rho\right)^{-1}\right] \cap \left[\left(4/\gamma\right)^{2/(l+1)} \cdot \widetilde{\omega}^{1/(l+1)},\ \gamma\right].$$

Moreover, we suppose that

- the minimization problems are computed exactly,
- $\mathfrak{sym}(\mathcal{O})$ is spanned by matrices whose spectra come from primitive vectors of coordinates at most $\omega_{\max}$,
- The candidate Lie group is $\simeq \mathrm{Sym}(\mathcal{O})$.

# Orientation estimation



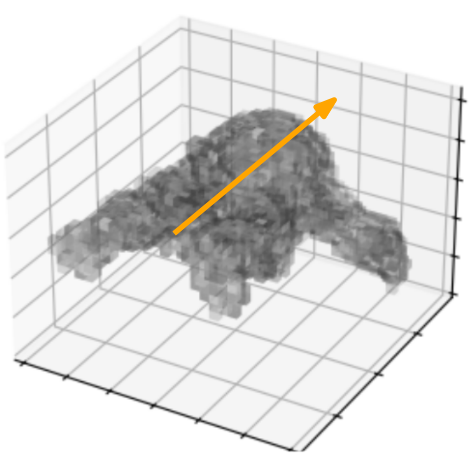
**(1)** Take a $m \times m \times m$ image.

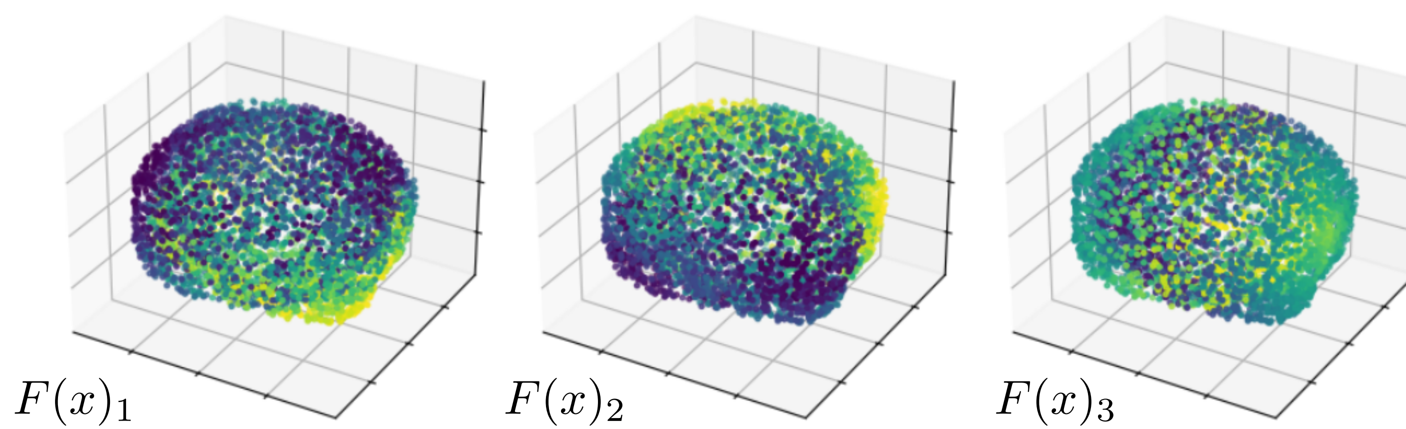
**(2)** Generate several rotations to get a point cloud $X \subset \mathbb{R}^{m \times m \times m}$.

**(3)** Project $X$ in $\mathbb{R}^n$ via PCA.

**Problem:** given $x \in X$, estimate the unit vector $F(x) \in \mathbb{R}^3$ that points toward the armadillo's head.

We define train/test sets of 90%/10%.





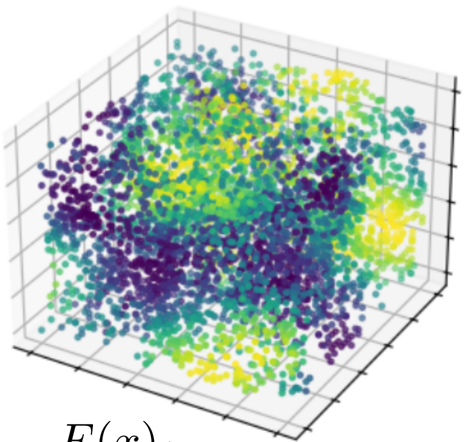
**Conventional solution:** Train a SVM.

**Orthogonal coordinates:** Our algorithm detect a SO(3)-orbit in $\mathbb{R}^8$ that is close to $X$: $\mathrm{d_H}\big(X, \mathcal{O}\big) \simeq 0.1909$.

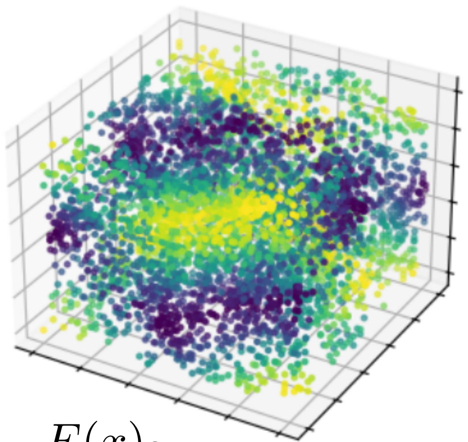
$$\begin{tikzcd} \mathrm{SO}(3) \arrow[r, "\phi"] & \mathrm{GL}_n(\mathbb{R}) \\ \mathfrak{so}(3) \arrow[u, "\exp"] \arrow[r, "\mathrm{d}\phi"] & \mathfrak{gl}(n) \arrow[u, "\exp"'] \end{tikzcd}$$

The orbit is $\mathcal{O} = \{\phi(g) \cdot x_0 \mid g \in G\}$. Every $x \in X$ can be pulled back to $\mathfrak{so}(3)$ via

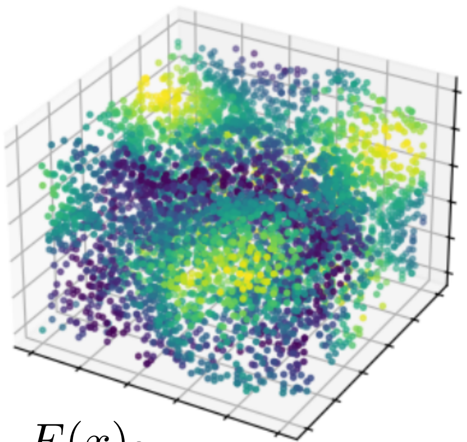
$$\min_{c \in \mathfrak{so}(3)} \|x - \phi(\exp(c)) \cdot x_0\|.$$



$F(x)_1$



$F(x)_2$



$F(x)_3$

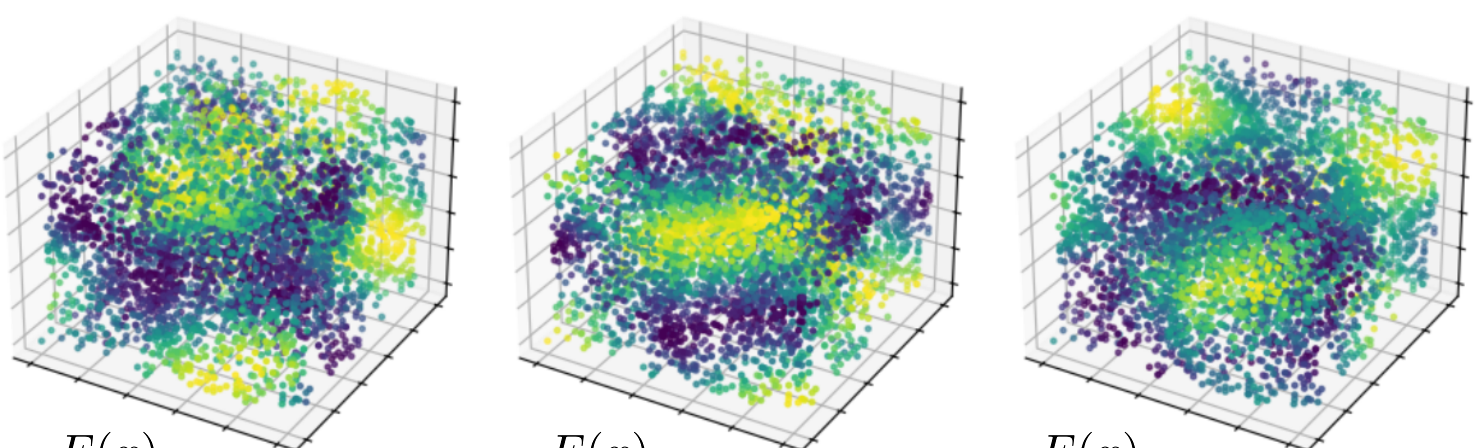
**Conventional solution:** Train a SVM.

**Orthogonal coordinates:** Our algorithm detect a SO(3)-orbit in $\mathbb{R}^8$ that is close to $X$: $\mathrm{d_H}\big(X, \mathcal{O}\big) \simeq 0.1909$.

$$\begin{array}{ccc} \mathrm{SO}(3) & \xrightarrow{\phi} & \mathrm{GL}_n(\mathbb{R}) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{so}(3) & \xrightarrow{\mathrm{d}\phi} & \mathfrak{gl}(n) \end{array}$$

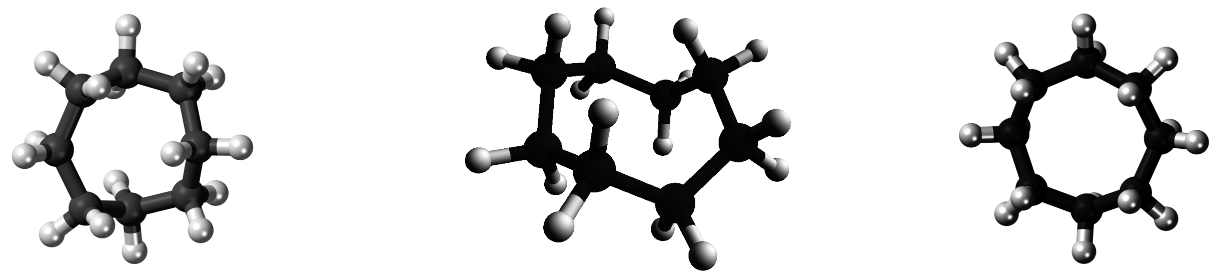
The orbit is $\mathcal{O} = \{\phi(g) \cdot x_0 \mid g \in G\}$. Every $x \in X$ can be pulled back to $\mathfrak{so}(3)$ via

$$\min_{c \in \mathfrak{so}(3)} \|x - \phi(\exp(c)) \cdot x_0\|.$$



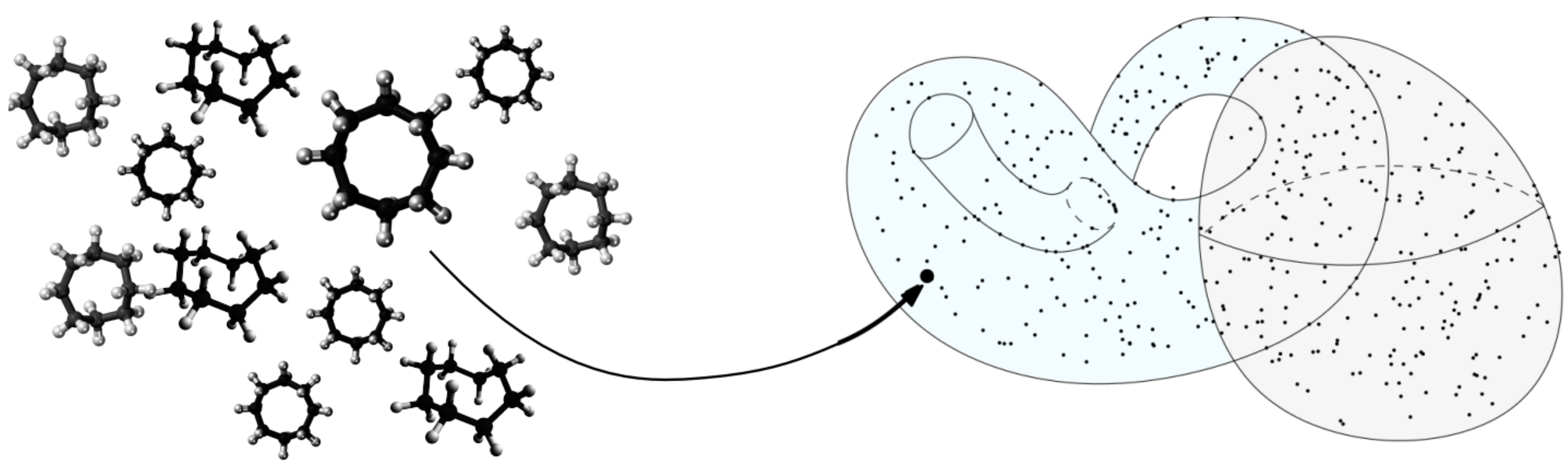
$F(x)_1$ $F(x)_2$ $F(x)_3$

| Model | MSE on test data |
|---|---|
| SVM in dimension 3 | 0.4003 |
| SVM in dimension 4 | 0.2496 |
| SVM in dimension 5 | 0.1295 |
| SVM in dimension 6 | 0.0380 |
| SVM in dimension 7 | 0.0148 |
| SVM in dimension 8 | 0.0119 |
| SVM in dimension 9 | 0.0114 |
| SVM in dimension 10 | 0.0122 |
| SVM on orthogonal coordinates | **0.0066** |



A conformer of cyclooctane can be seen as a point in $\mathbb{R}^{72}$ ($3 \times 24 = 72$).

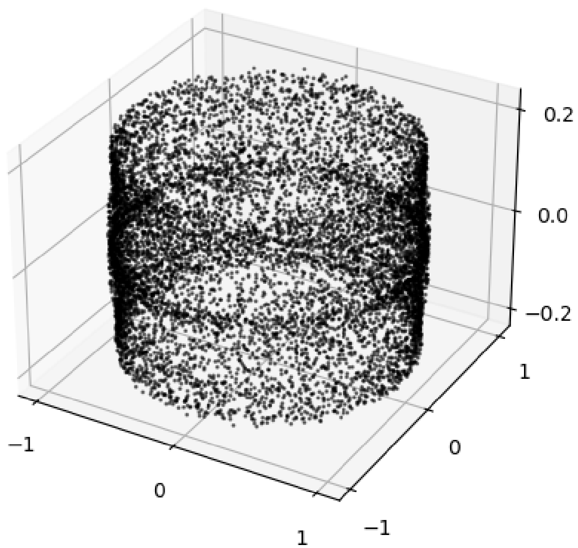
A collection of conformers yield a point cloud $X \subset \mathbb{R}^{72}$.



[Martin, Thompson, Coutsias, Watson, Topology of cyclo-octane energy landscape, 2010]

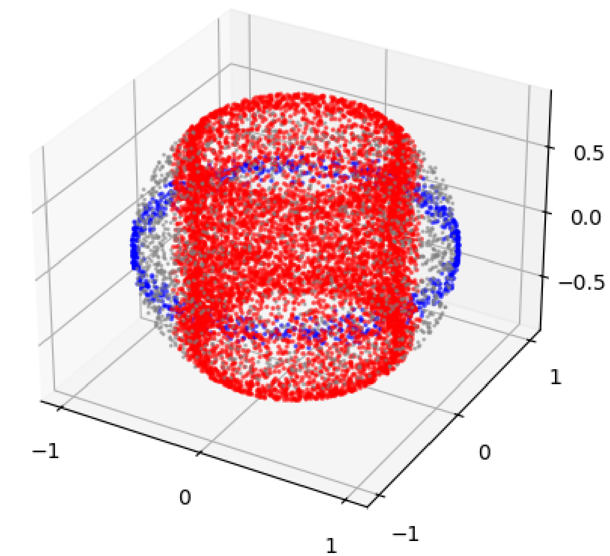
**Idea:** check whether $X$ lies close to a linear orbit of a Lie group.

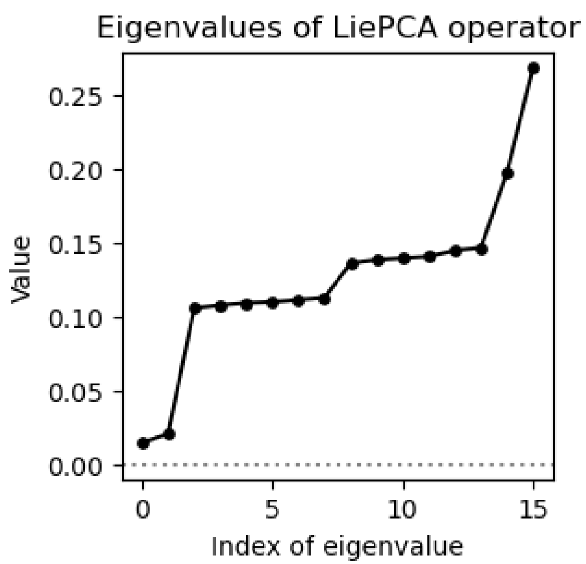
**Unaligned confomers:** We generate 10,000 cyclooctane confomers without aligning them.




We see a cylinder surrounded by a circle.

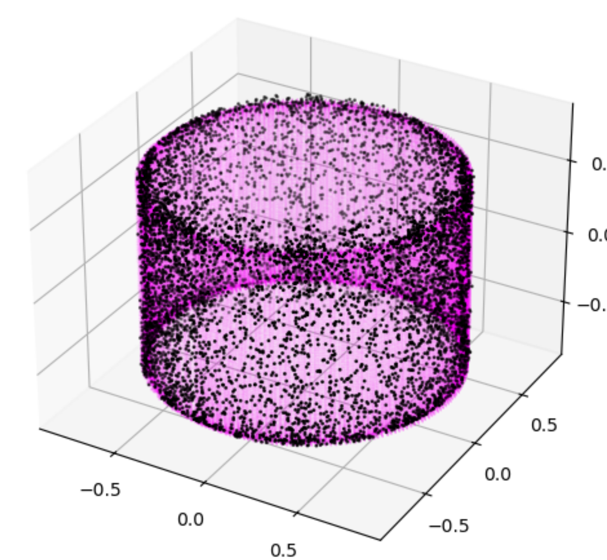
$X$ is projected onto $\mathbb{R}^4$ and orthonormalized. After discarding 15% of the outliers (gray), two clusters appear. We take the red one.




`LiePCA` has two small eigenvalues, suggesting a symmetry group of dim 2.

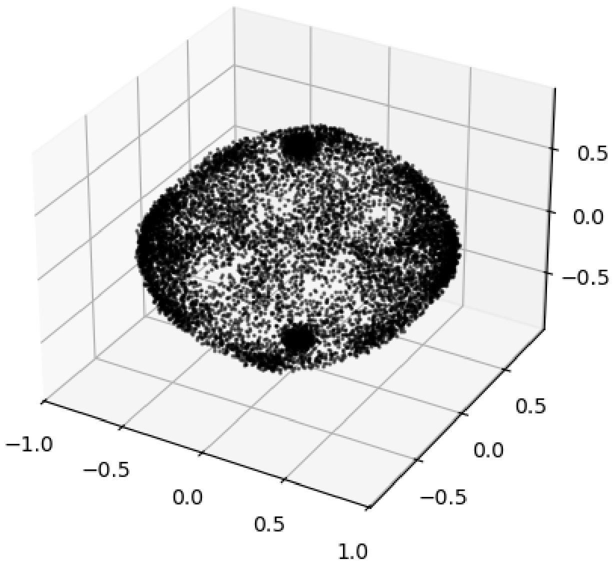
We find a $T^2$-orbit in $\mathbb{R}^4$ close $X$: $\mathrm{d_H}\big(X, \mathcal{O}\big) \simeq 0.2$.



# Conformational space of cyclooctane $C_8H_{16}$

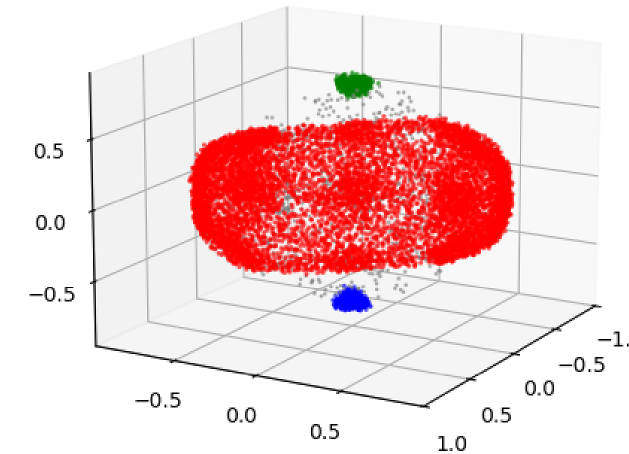


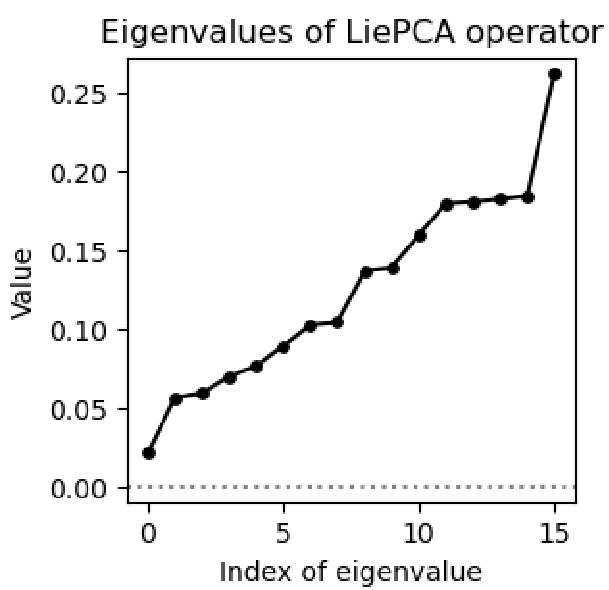
**Aligned confomers:** We now generate 10,000 aligned confomers (`AlignMolConformers` in `RDKit`).



We see three components:
a surface and two clusters.

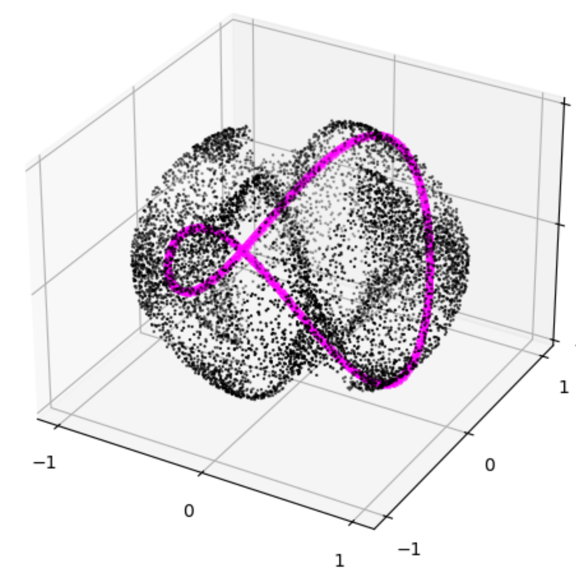
After discarding 10% of the outliers (gray), the points are grouped into three classes. We keep the red class.






`LiePCA` has one small eigenvalue, suggesting a symmetry group of dim 1.

We find a SO(2)-action that stabilizes $X$.
Average distance: $\mathrm{d_H}\big(\widehat{\mathcal{O}}_x|X\big) \simeq 0.1$.



Consider a neural network $$V = V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{p-1}} V_p = W.$$

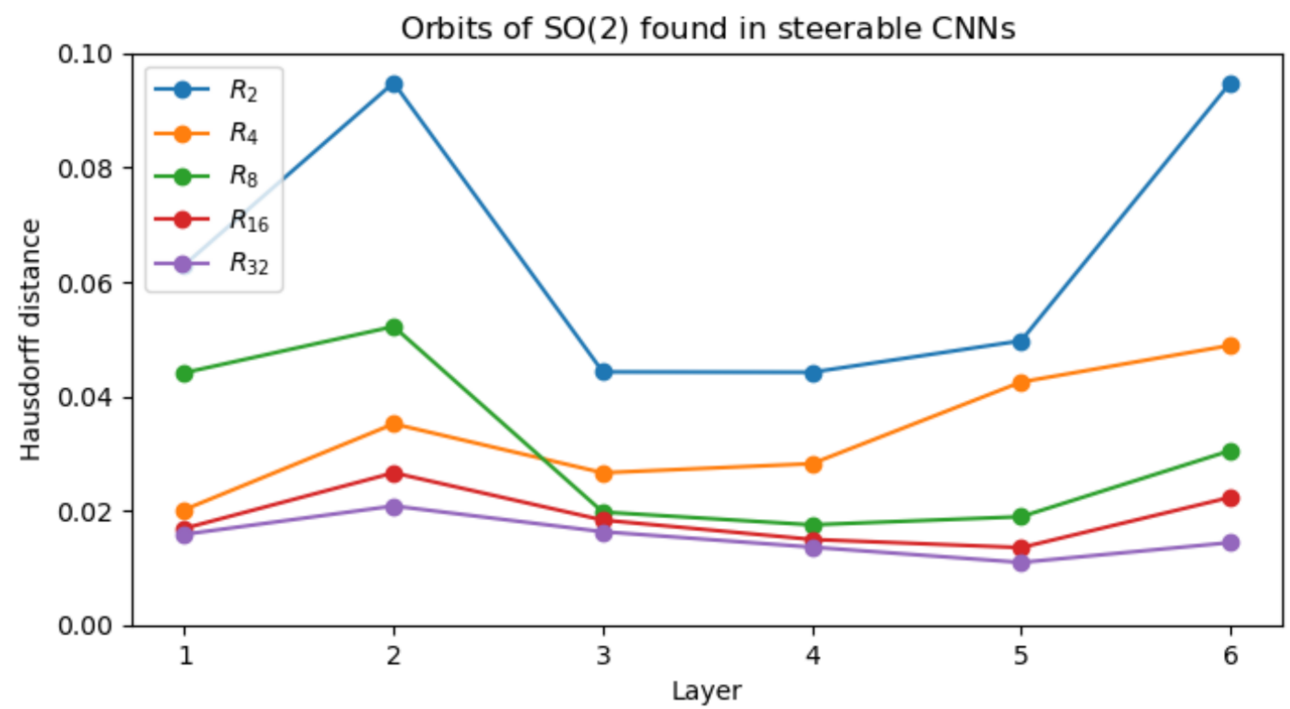
Denote $\mathcal{F}_i = f_1 \cdots f_i$ and $\mathcal{F} = \mathcal{F}_{p-1}$.

Say $G$ acts linearly on $V$, via $\phi \colon G \to \mathrm{GL}(V)$.

The network is **equivariant** if there exists representations $\phi_i \colon G \to \mathrm{GL}(V_i)$ such that $\forall x \in V$, $\forall g \in G$,

$$\mathcal{F}_i(\phi(g)x) = \phi_i(g)\mathcal{F}_i(x).$$

**Experiment:** Consider **steerable CNNs** for several rotation groups $R_n$. We pick an image, and generate 500 rotations. In each of the layers, we apply our algorithm to find a linear-orbit of $\mathrm{SO}(2)$.



LieDetect: Detection of representation orbits of compact Lie groups from point clouds
Foundations of Computational Mathematics (2025)
DOI:10.1007/s10208-025-09728-4

`https://github.com/HLovisiEnnes/LieDetect`

**Thanks!**

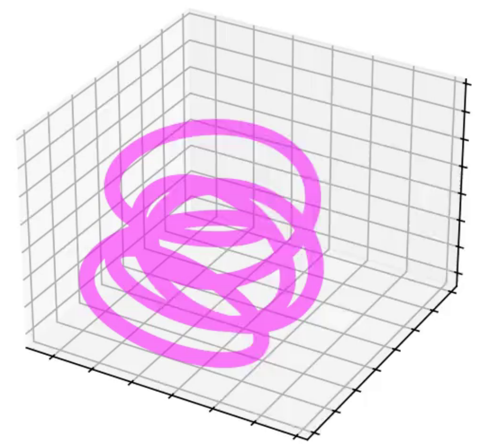
Detection of **actions** via the induced representation on space of vector fields

$$\begin{tikzcd} G \arrow[r, "\phi"] & \mathrm{Diff}(\mathcal{M}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{d}\phi"] & \mathscr{X}(\mathcal{M}) \arrow[u, "\exp"'] \end{tikzcd}$$

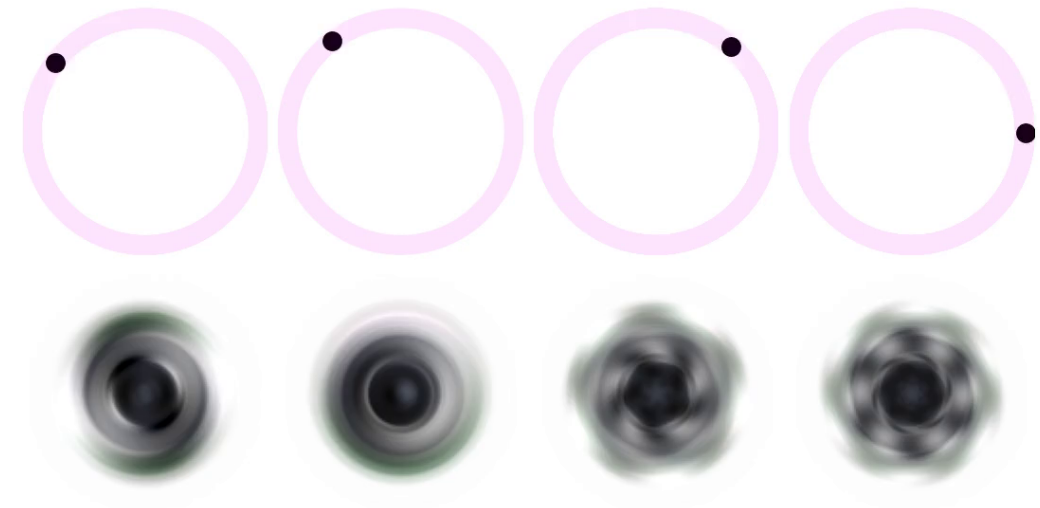
**Statistical** guarantees to test the linear-orbit hypothesis.

Rotations of $m \times m$ RGB image

Embedding in $\mathbb{R}^{m\times m\times 3}$



Projection in eigenplanes



Eigenvalues of the point cloud's covariance matrix:

$$0.155,\quad 0.155,\quad 0.11,\quad 0.11,\quad 0.041,\quad 0.041,\quad 0.04,\quad 0.04,\quad 0.038,\quad 0.038,\quad 0.026,\quad 0.026,\quad \ldots$$

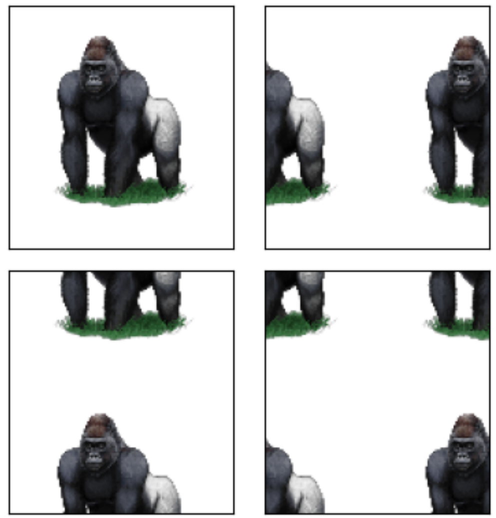
In these eigenplanes, the orbit is close to

$$\theta \longmapsto \begin{pmatrix} \mu_1 \cos \omega_1\theta \\ \mu_1 \sin \omega_1\theta \\ \mu_2 \cos \omega_2\theta \\ \mu_2 \sin \omega_2\theta \\ \vdots \\ \mu_k \cos \omega_k\theta \\ \mu_k \sin \omega_k\theta \end{pmatrix} = \begin{pmatrix} \cos \omega_1\theta & -\sin \omega_1\theta & & & & & \\ \sin \omega_1\theta & \cos \omega_1\theta & & & & & \\ & & \cos \omega_2\theta & -\sin \omega_2\theta & & & \\ & & \sin \omega_2\theta & \cos \omega_2\theta & & & \\ & & & & \ddots & & \\ & & & & & \cos \omega_k\theta & -\sin \omega_k\theta \\ & & & & & \sin \omega_k\theta & \cos \omega_k\theta \end{pmatrix} \begin{pmatrix} \mu_1 \\ 0 \\ \mu_2 \\ 0 \\ \vdots \\ \mu_k \\ 0 \end{pmatrix}$$

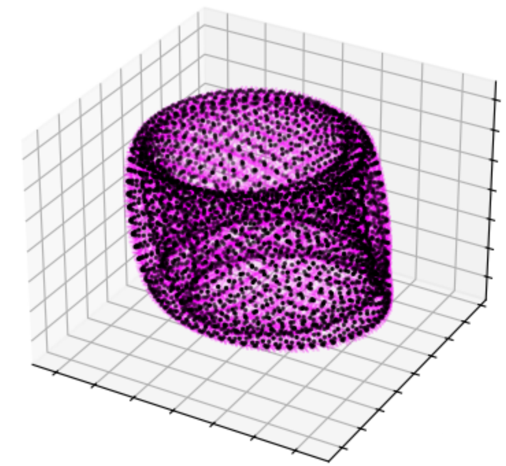
# Additional experiments: Pixel permutations



Translations of $m \times m$ RGB image



Embedding in $\mathbb{R}^{m\times m\times 3}$



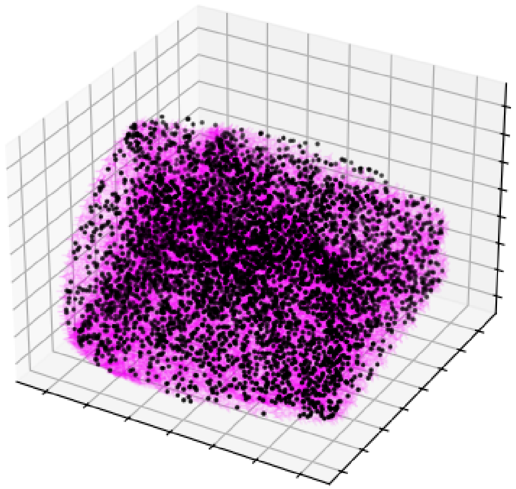
Covariance matrix eigenvalues: 0.228, 0.228, 0.142, 0.142, 0.108, 0.108, 0.022, 0.022, ...

In these eigenplanes, the orbit is close to

$$\theta^{(1)},\theta^{(2)} \longmapsto \begin{pmatrix} \mu_1 \cos\left(\omega_1^{(1)}\theta^{(1)} + \omega_1^{(2)}\theta^{(2)}\right) \\ \mu_1 \sin\left(\omega_1^{(1)}\theta^{(1)} + \omega_1^{(2)}\theta^{(2)}\right) \\ \mu_2 \cos\left(\omega_2^{(1)}\theta^{(1)} + \omega_2^{(2)}\theta^{(2)}\right) \\ \mu_2 \cos\left(\omega_2^{(1)}\theta^{(1)} + \omega_2^{(2)}\theta^{(2)}\right) \\ \vdots \\ \mu_k \cos\left(\omega_k^{(1)}\theta^{(1)} + \omega_k^{(2)}\theta^{(2)}\right) \\ \mu_k \cos\left(\omega_k^{(1)}\theta^{(1)} + \omega_k^{(2)}\theta^{(2)}\right) \end{pmatrix} = \text{ linear action of } T^2 \text{ on } \begin{pmatrix} \mu_1 \\ 0 \\ \mu_2 \\ 0 \\ \vdots \\ \mu_k \\ 0 \end{pmatrix}$$

Rotations of $m \times m \times m$ greyscale object

Embedding in $\mathbb{R}^{m \times m \times m}$



Covariance matrix eigenvalues: 0.246, 0.239, 0.234, 0.058, 0.057, 0.056, 0.055, 0.054 ...

In these eigenplanes, the orbit is close to

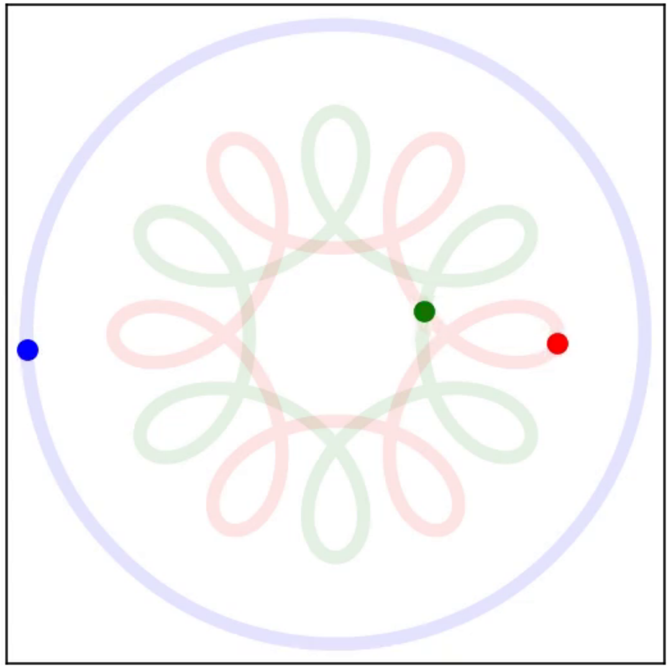
$$\theta^{(1)}, \theta^{(2)}, \theta^{(3)} \longmapsto \text{ linear action of } \mathrm{SO}(3) \text{ on } \begin{pmatrix} \mu_1 \\ 0 \\ \vdots \\ \mu_2 \\ 0 \\ \vdots \end{pmatrix}$$

In 1975, Roger Broucke found several periodic orbits.

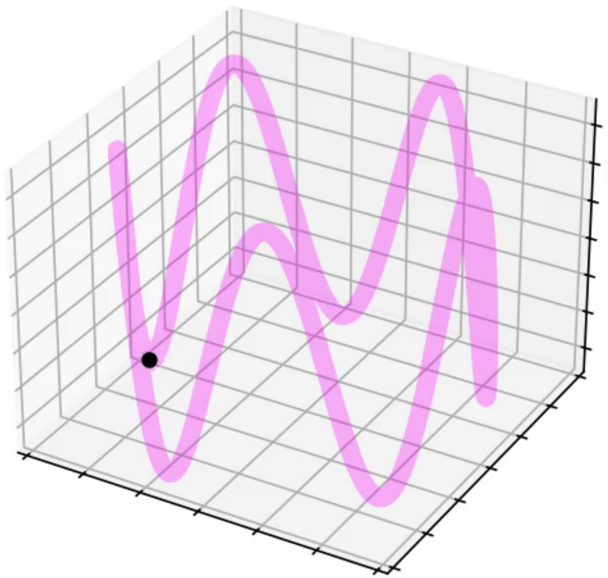
Let $x_1(t)$, $x_2(t)$, $x_3(t)$ be the three bodies, and define $z(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^6$.

Orbit `A3`

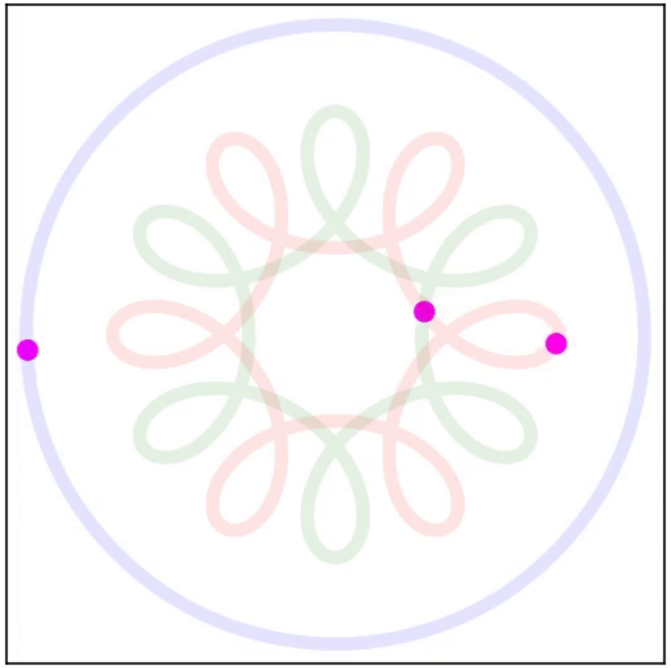
Trajectory of $x_1, x_2, x_3$
(found by integration)



Trajectory of $z$



Reconstructed orbit of SO(2)

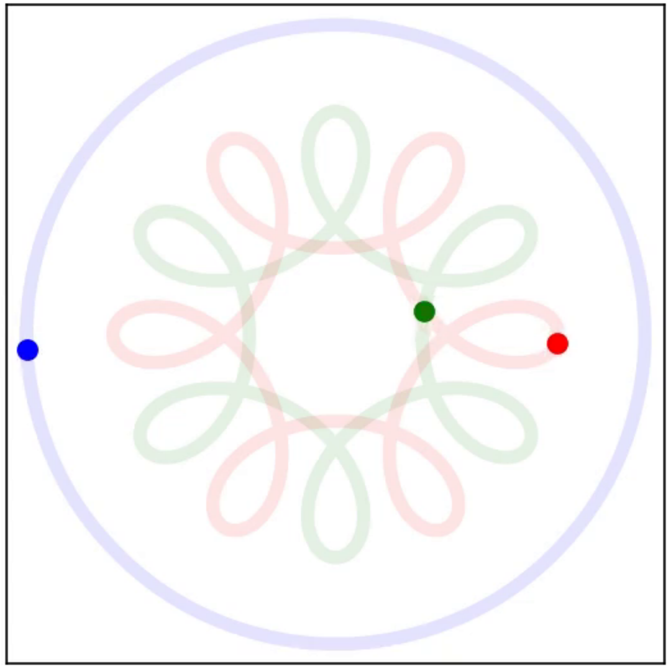


In 1975, Roger Broucke found several periodic orbits.

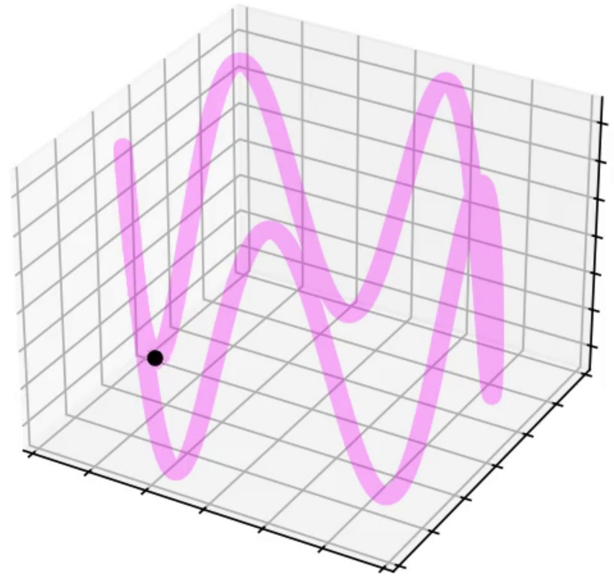
Let $x_1(t)$, $x_2(t)$, $x_3(t)$ be the three bodies, and define $z(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^6$.

<u>Orbit `R2`</u>

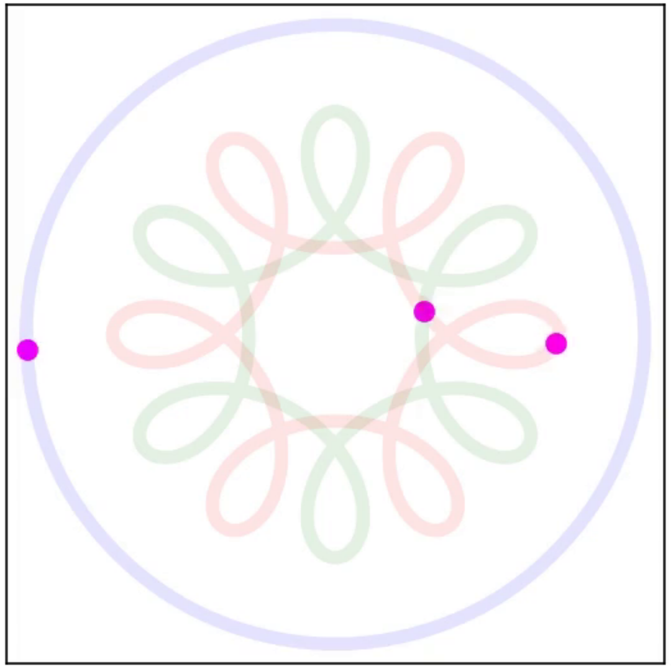
Trajectory of $x_1, x_2, x_3$ (found by integration)



Trajectory of $z$



Reconstructed orbit of SO(2)



# Step 3 - Case of SO(2)

Let $G = \mathrm{SO}(2)$, whose dimension is $d = 1$. The output $\widehat{h}$ of `LiePCA` is a skew symmetric $n \times n$ matrix $A$.
Suppose that $n$ is even. The representations of $\mathrm{SO}(2)$ in $\mathbb{R}^n$ take the form

$$\phi_{(\omega_1,\dots,\omega_{n/2})}(\theta) = \begin{pmatrix} R(\omega_1\theta) & & \\ & \ddots & \\ & & R(\omega_{n/2}\theta) \end{pmatrix} \qquad \text{where} \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and where $(\omega_1, \dots, \omega_{n/2}) \in \mathbb{Z}^{n/2}$. In practice, we fix a maximal frequence $\omega_{\max} \in \mathbb{N}$.
The corresponding pushforward Lie algebra is spanned by the matrix

$$B_{(\omega_1,\dots,\omega_{n/2})} = \begin{pmatrix} L(\omega_1) & & \\ & \ddots & \\ & & L(\omega_{n/2}) \end{pmatrix} \qquad \text{where} \qquad L(\omega) = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

In this context, the minimization problem reads

$$\min \left\| \mathrm{proj}[A] - \mathrm{proj}[O B_{(\omega_1,\dots,\omega_{n/2})} O^\top] \right\| \quad \text{s.t.} \quad \begin{cases} (\omega_1, \dots, \omega_{n/2}) \in \mathbb{Z}^{n/2}, \\ O \in \mathrm{O}(n). \end{cases}$$

This is equivalent to

$$\min \left\| A \pm O B_{(\omega_1,\dots,\omega_{n/2})} O^\top \right\| \quad \text{s.t.} \quad \begin{cases} (\omega_1, \dots, \omega_{n/2}) \in \mathbb{Z}^{n/2}, \\ O \in \mathrm{O}(n). \end{cases}$$

We recognize a **two-sided orthogonal Procrustes problem with one transformation**.

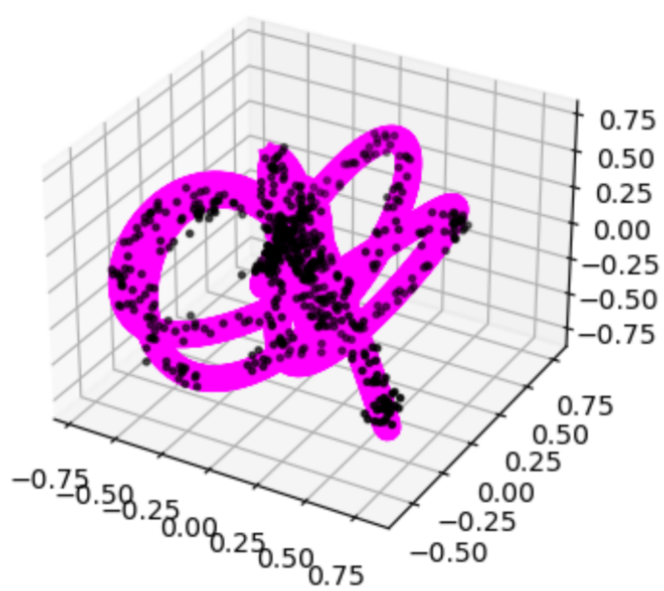
**Example:** We consider a representation of SO(2) in $\mathbb{R}^{10}$ with frequencies $(2, 4, 5, 7, 8)$ and sample 600 points on one of its orbits, that we corrupt with a Gaussian additive noise of deviation $\sigma = 0.03$.

We perform the minimization over all representations of SO(2) in $\mathbb{R}^{10}$, with parameter $\omega_{\max} = 10$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Representation | $(2,4,5,7,8)$ | $(2,5,6,8,9)$ | $(3,5,7,9,10)$ | $(3,6,7,9,10)$ | $(3,5,6,8,9)$ | $(2,4,5,6,7)$ |
| Cost | **0.028** | 0.032 | 0.037 | 0.037 | 0.038 | 0.044 |
| Representation | $(3,5,6,9,10)$ | $(2,5,7,9,10)$ | $(2,3,4,5,6)$ | $(2,5,6,9,10)$ | $(2,6,7,9,10)$ | $(3,5,6,8,10)$ |
| Cost | 0.046 | 0.055 | 0.057 | 0.058 | 0.058 | 0.058 |

The correct representation is found.

The Hausdorff distance between the point cloud and the estimated orbit is $\mathrm{d_H}\big(X \mid \widehat{\mathcal{O}}\big) \approx 0.231$.



Let $G = T^d$ the torus of dim $d$. The output of `LiePCA` is a $d$-tuple $(A_1, \ldots, A_d)$ of skew symmetric matrices.

The representations of $T^d$ in $\mathbb{R}^n$ take the form

$$\phi_{(\omega_i^j)}(\theta_1, \ldots, \theta_d) = \textstyle\sum_{j=1}^d \phi_{(\omega_1^j, \ldots, \omega_{n/2}^j)}(\theta_j)$$

where $(\omega_i^j)_{1 \leq i \leq n/2}^{1 \leq j \leq d}$ is a $n/2 \times d$ matrix with integer coefficients.

The push-forward Lie algebra is spanned by

$$B_{(\omega_1^1, \ldots, \omega_{n/2}^1)}, \quad B_{(\omega_1^2, \ldots, \omega_{n/2}^2)}, \quad \ldots, \quad B_{(\omega_1^d, \ldots, \omega_{n/2}^d)}.$$

In this context, the minimization problem reads

$$\min \left\| \mathrm{proj}\left[\langle A_i \rangle_{j=1}^d\right] - \mathrm{proj}\left[\langle O B_{(\omega_1^j, \ldots, \omega_{n/2}^j)} O^\top \right] \rangle_{j=1}^d) \right\| \quad \text{s.t.} \quad \begin{cases} (\omega_i^j)_{1 \leq i \leq n/2}^{1 \leq j \leq d} \in \mathbb{Z}^{n/2 \times d}, \\ O \in \mathrm{O}(n). \end{cases}$$

This is linked to the **simultaneous reduction of a tuple of skew-symmetric matrices**.

**Lemma:** Denote by $(\rho_i)_{i=1}^d$ the coefficients of an optimal simultaneous reduction of the matrices $(A_i)_{i=1}^d$ in normal form. Then the problem is equivalent to

$$\min_{(\omega_i^j)} \sum_{k=1}^d f\left( (\rho_i^k)_{i=1}^{n/2}, \ (\omega_i^k)_{i=1}^{n/2} \right) \qquad \text{where} \qquad f(x, y) = \left\| x/\|x\| - y/\|y\| \right\|^2.$$

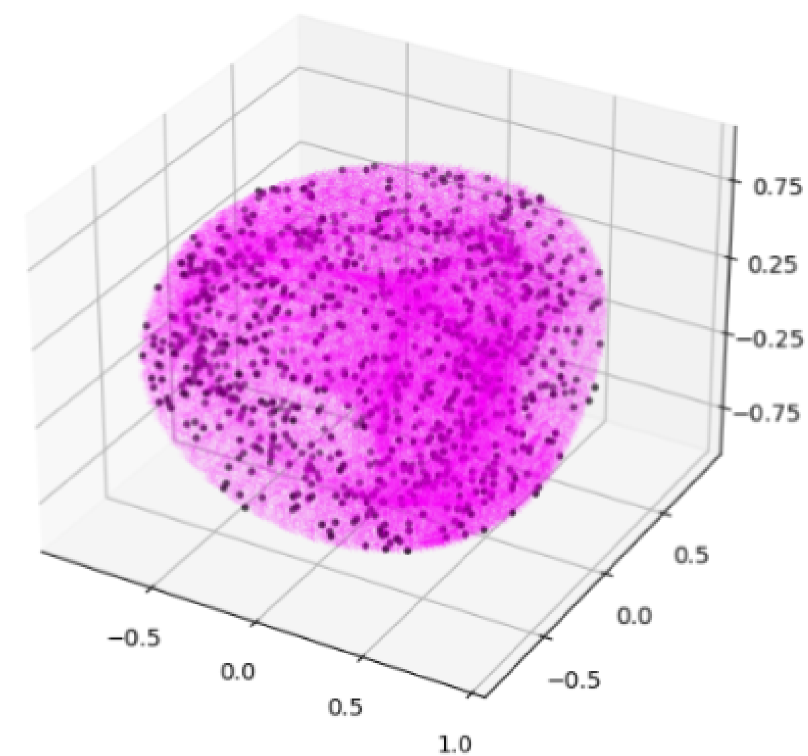
**Example:** Let $X$ be a uniform 750-sample of an orbit of the representation $\phi_{\left(\begin{smallmatrix}1&1&2\\1&2&1\end{smallmatrix}\right)}$ of the torus $\mathrm{T}^2$ in $\mathbb{R}^6$.

We apply the algorithm with $G = T^2$ restrict to representations with frequencies at most $\omega_{\max} = 2$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Representation | $\begin{pmatrix}0&1&1\\2&-2&1\end{pmatrix}$ | $\begin{pmatrix}1&1&2\\-2&2&-1\end{pmatrix}$ | $\begin{pmatrix}0&1&2\\2&-2&-1\end{pmatrix}$ | $\begin{pmatrix}0&1&1\\1&-2&0\end{pmatrix}$ | $\begin{pmatrix}0&1&1\\1&-2&-1\end{pmatrix}$ | $\begin{pmatrix}0&1&2\\2&-2&1\end{pmatrix}$ |
| Cost | **0.036** | 0.136 | 0.198 | 0.233 | 0.244 | 0.312 |
| Representation | $\begin{pmatrix}0&1&2\\1&-2&-2\end{pmatrix}$ | $\begin{pmatrix}0&1&2\\1&-2&-1\end{pmatrix}$ | $\begin{pmatrix}1&2&2\\-2&-2&1\end{pmatrix}$ | $\begin{pmatrix}1&1&1\\-2&-1&2\end{pmatrix}$ | $\begin{pmatrix}0&1&2\\1&-2&0\end{pmatrix}$ | $\begin{pmatrix}0&1&1\\1&-2&1\end{pmatrix}$ |
| Cost | 0.331 | 0.348 | 0.388 | 0.447 | 0.457 | 0.472 |

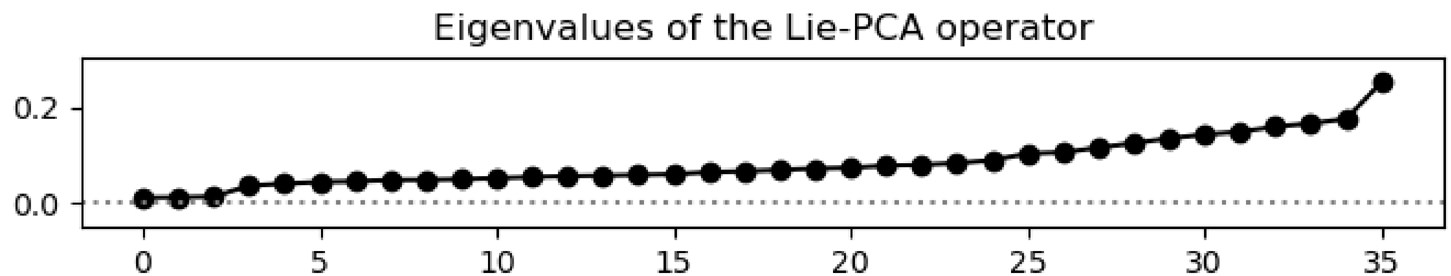
The algorithm's output is $\left(\begin{smallmatrix}0&1&1\\2&-2&1\end{smallmatrix}\right)$. It is equivalent to $\phi_{\left(\begin{smallmatrix}1&1&2\\1&2&1\end{smallmatrix}\right)}$.

Moreover, the Hausdorff distance is $\mathrm{d_H}\big(X|\widehat{\mathcal{O}}\big) \approx 0.071$.



When the underlying group is unknown, we can guess it from `LiePCA` or test several candidates.

**Example:** Let $X$ be a 1500-sample of an orbit of the representation $(1,5)$ of $\mathrm{SU}(2)$ in $\mathbb{R}^6$.




We see a Lie algebra of dimension 3. One expects the torus $T^3$, $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$.

| Representation of $\mathrm{SU}(2)$ | $(1,5)$ | $(1,1,1,3)$ | $(1,1,4)$ | $(3,3)$ |
|---|---|---|---|---|
| Cost | $\mathbf{8.6 \times 10^{-5}}$ | 0.007 | 0.008 | 0.015 |

| Representation of $T^3$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
|---|---|
| Cost | **0.014** |

Representation $(1,5)$: we get the (non-symmetric) Hausdorff distance $\mathrm{d_H}(X|\widehat{\mathcal{O}}) \approx 0.062$.

Representation $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: we get the (non-symmetric) Hausdorff distance $\mathrm{d_H}(X|\widehat{\mathcal{O}}) \approx 0.751$.

# Proof of robustness - Orthonormalization



**Ideal covariance matrix:** Suppose that $\mathcal{O}$ is an orbit of the representation $\phi\colon G \to \mathrm{M}_n(\mathbb{R})$, and $\mu_{\mathcal{O}}$ the uniform measure on it. With $x_0 \in \mathcal{O}$ an arbitrary point, the covariance matrix can be written

$$\Sigma[\mu_{\mathcal{O}}] = \int \big(\phi(g)x_0\big) \cdot \big(\phi(g)x_0\big)^\top \mathrm{d}\mu_G(g).$$

Now, let $\mathbb{R}^n = \bigoplus_{i=1}^m V_i$ be the decomposition of $\phi$ into irreps, and denote as $(\Pi\big[V_i\big])_{i=1}^m$ the projection matrices on these subspaces. We can decompose

$$\Sigma[\mu_{\mathcal{O}}] = \sum_{i=1}^m C_i \quad \text{where} \quad C_i = \int \phi_i(g)\Big(\Pi\big[V_i\big](x_0) \cdot \Pi\big[V_i\big](x_0)^\top\Big)\phi_i(g)^\top \mathrm{d}\mu_G(g).$$

If $\phi$ is orthogonal, then by Schur's lemma, the $C_i$ are homotheties:

$$\Sigma[\mu_{\mathcal{O}}] = \sum_{i=1}^m \sigma_i^2 \Pi\big[V_i\big] \qquad \text{where} \qquad \sigma_i^2 = \frac{\big\|\Pi\big[V_i\big](x_0)\big\|^2}{\dim(V_i)}.$$

This shows that, in general, important quantities are:

- The variance $\mathbb{V}[\|\mu_{\mathcal{O}}\|]$, a measure of *deviation from orthogonality* of $\mathcal{O}$
- The ratio $\sigma_{\max}^2/\sigma_{\min}^2$, a measure of *homogeneity* of $\mathcal{O}$.

**Proposition:** Let $\mathcal{O} \subset \mathbb{R}^n$ be the orbit of a representation, potentially non-orthogonal, $\mu_{\mathcal{O}}$ its uniform measure, $\Pi\big[\langle\mathcal{O}\rangle\big]$ the projection on its span, and $\sigma^2_{\max}, \sigma^2_{\min}$ the top and bottom nonzero eigenvalues of $\Sigma[\mu_{\mathcal{O}}]$.

Besides, let $\nu$ be a measure, $\Sigma[\nu]$ its covariance matrix, $\epsilon > 0$ and $\Pi^{\geq\epsilon}_{\Sigma[\nu]}$ the projection on the subspace spanned by eigenvectors with eigenvalue at least $\epsilon$.

If $\mathrm{W}_2(\mu_{\mathcal{O}}, \nu)$ is small enough, we have the following bound between the pushforward measures after Step 1:

$$\begin{aligned}
&\mathrm{W}_2\left(\sqrt{\Sigma[\mu_{\mathcal{O}}]^{+}}\Pi\big[\langle\mathcal{O}\rangle\big]\mu_{\mathcal{O}},\ \sqrt{\Sigma[\nu]^{+}}\Pi^{\geq\epsilon}_{\Sigma[\nu]}\nu\right) \\
&\leq 8(n+1)^{3/2}\left(\frac{\sigma^3_{\max}}{\sigma^3_{\min}}\right)\left(\frac{\mathrm{W}_2(\mu_{\mathcal{O}}, \nu)}{\sigma_{\min}}\right)^{1/2}\left(\left(\frac{\mathbb{V}\big[\|\mu_{\mathcal{O}}\|\big]}{\sigma^2_{\min}}\right)^{1/2} + \frac{\mathrm{W}_2(\mu_{\mathcal{O}}, \nu)}{\sigma_{\min}}\right)^{1/2}.
\end{aligned}$$

**Proof:** Consequence of Davis-Kahan theorem, together with

$$\big\|\Sigma[\mu_{\mathcal{O}}]^{-1/2} - \Sigma[\nu]^{-1/2}\big\|_{\mathrm{op}} \leq \frac{\sqrt{2}}{\sigma^2_{\min}} \cdot \left(2\mathbb{V}\big[\|\mu_{\mathcal{O}}\|\big]^{1/2} + \mathrm{W}_2(\mu_{\mathcal{O}}, \nu)\right)^{1/2} \cdot \mathrm{W}_2(\mu_{\mathcal{O}}, \nu)^{1/2}.$$

`LiePCA` operator: Say we observe $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$, assumed close to $\mathcal{O}$.

Define $\Lambda\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ as $$\Lambda(A) = \frac{1}{N} \sum_{1 \leq i \leq N} \widehat{\Pi}\big[\mathrm{N}_{x_i} X\big] \cdot A \cdot \Pi\big[\langle x_i \rangle\big]$$

where

- $\widehat{\Pi}\big[\mathrm{N}_{x_i} X\big]$ are estimations of projection matrices onto the normal spaces $\mathrm{N}_{x_i}\mathcal{O}$,
- $\Pi\big[\langle x_i \rangle\big]$ the are projection matrices on the lines $\langle x_i \rangle$.

Explanation: On the one hand, $\mathfrak{sym}(\mathcal{O}) = \big\{A \in \mathrm{M}_n(\mathbb{R}) \mid \forall x \in \mathcal{O}, Ax \in \mathrm{T}_x\mathcal{O}\big\}$. Thus,

$$\mathfrak{sym}(\mathcal{O}) = \bigcap_{x \in \mathcal{O}} S_x\mathcal{O} \qquad \text{where} \qquad S_x\mathcal{O} = \{A \in \mathrm{M}_n(\mathbb{R}) \mid Ax \in \mathrm{T}_x\mathcal{O}\}.$$

On the other hand, considering only $X$, one has

$$\bigcap_{i=1}^{N} S_{x_i}\mathcal{O} \approx \ker\left(\sum_{i=1}^{N} \Pi\big[(S_{x_i}\mathcal{O})^\perp\big]\right),$$

Last, the authors showed that $\Pi\big[(S_{x_i}\mathcal{O})^\perp\big](A) = \Pi\big[\mathrm{N}_{x_i}\mathcal{O}\big] \cdot A \cdot \Pi\big[\langle x_i \rangle\big]$.

**Ideal Lie-PCA:** Suppose that $\mathcal{O}$ is an orbit of the representation $\phi\colon G \to \mathrm{M}_n(\mathbb{R})$, and $\mu_{\mathcal{O}}$ its uniform measure. We define

$$\Lambda_{\mathcal{O}}(A) = \int \Pi\big[\mathrm{N}_x\mathcal{O}\big] \cdot A \cdot \Pi\big[\langle x\rangle\big] \mathrm{d}\mu_{\mathcal{O}}(x).$$

**Proposition:** Its kernel is equal to $\mathfrak{sym}(\mathcal{O})$. Moreover, when $\mathcal{O} = S^{n-1}$, its nonzero eigenvalues are exactly $\delta_n$ and $\delta_n'$ where

$$\delta_n = \frac{2(n-1)}{n(n(n+1)-2)} \qquad \text{and} \qquad \delta_n' = \frac{1}{n}.$$

**Proof:** Show that $\Lambda_{\mathcal{O}}$ is equivariant with respect to the action of $G$ by conjugation:

$$\phi(g)\Lambda(A)\phi(g)^{-1} = \Lambda\Big(\phi(g)A\phi(g)^{-1}\Big)$$

Then use Schur's lemma.

**Empirical observation:** More generally, the nonzero eigenvalues of $\Lambda_{\mathcal{O}}$ belong to $[1/n^2, 1/n]$ when $\mathcal{O}$ is *homogenous*, i.e., $\sigma^2_{\max}/\sigma^2_{\min} = 1$.

**Stability:** Comparing

$$\Lambda(A) = \sum_{1\leq i\leq N} \widehat{\Pi}\big[\mathrm{N}_{x_i} X\big] \cdot A \cdot \Pi\big[\langle x_i\rangle\big] \qquad \text{and} \qquad \Lambda_{\mathcal{O}}(A) = \int \Pi\big[\mathrm{N}_x \mathcal{O}\big] \cdot A \cdot \Pi\big[\langle x\rangle\big] \mathrm{d}\mu_{\mathcal{O}}(x).$$

amounts to quantifying the quality of normal space estimation. We use local PCA:

$$\widehat{\Pi}\big[\mathrm{N}_{x_i} X\big] = I - \Pi^{l,r}_{x_i}[X],$$

where $\Pi^{l,r}_{x_i}[X]$ is the projection matrix on any $l$ top eigenvectors of the *local covariance matrix* $\Sigma^r_{x_i}[X]$ centered at $x_i$ and at scale $r$, itself defined as

$$\Sigma^r_{x_i}[X] = \frac{1}{|Y|} \sum_{y\in Y} (y - x_i)(y - x_i)^\top,$$

where $Y = \{y \in X \mid \|y - x_i\| \leq r\}$, the set input points at distance at most $r$ from $x_i$.

**Measure-theoretic formulation:** If $\mu$ is a measure on $\mathbb{R}^n$, we define its *local covariance matrix* centered at $x$ at scale $r$ as

$$\Sigma^r_x[\mu] = \int_{\mathcal{B}(x,r)} (y - x)(y - x)^\top \frac{d\mu(x)}{\mu(\mathcal{B}(x, r))}.$$

**Bias-variance tradeoff:** Let $\mu_{\mathcal{M}}$ be measure on a submanifold $\mathcal{M} \subset \mathbb{R}^n$ of dimension $l$, $x \in \mathcal{M}$, $\nu$ a measure on $\mathbb{R}^n$ and $y \in \operatorname{supp}(\nu)$. We decompose

$$\left\| \frac{1}{l+2}\Pi\left[\mathrm{T}_x\mathcal{M}\right] - \frac{1}{r^2}\Sigma_y^r[\nu] \right\|_{\mathrm{F}} \leq$$

$$\underbrace{\left\| \frac{1}{l+2}\Pi\left[\mathrm{T}_x\mathcal{M}\right] - \frac{1}{r^2}\Sigma_x^r[\mu_{\mathcal{M}}] \right\|_{\mathrm{F}}}_{\text{consistency}} + \underbrace{\left\| \frac{1}{r^2}\Sigma_x^r[\mu_{\mathcal{M}}] - \frac{1}{r^2}\Sigma_y^r[\mu_{\mathcal{M}}] \right\|_{\mathrm{F}}}_{\text{spatial stability}} + \underbrace{\left\| \frac{1}{r^2}\Sigma_y^r[\mu_{\mathcal{M}}] - \frac{1}{r^2}\Sigma_y^r[\nu] \right\|_{\mathrm{F}}}_{\text{measure stability}}$$

**Lemma:** If the parameters are chosen correctly, this is

$$\lesssim r + \|x - y\| + \left( \frac{\mathrm{W}_2(\mu, \nu)}{r^{l+1}} \right)^{\frac{1}{2}}.$$

**Corollary:** We deduce a bound between Lie-PCA operators:

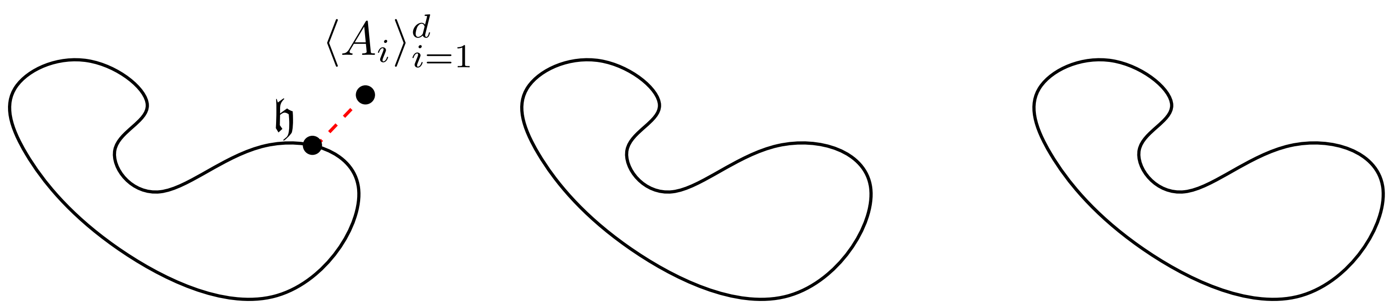
$$\|\Lambda_{\mathcal{O}} - \Lambda\|_{\mathrm{op}} \leq \sqrt{2}\rho\left( r + 4\left( \frac{\mathrm{W}_2(\mu_{\mathcal{O}}, \mu_X)}{r^{l+1}} \right)^{1/2} \right).$$

In Step 3, we consider the bottom eigenvectors $A_1, \ldots, A_d$ of Lie-PCA, and solve

$$\min \sum_{i=1}^{d} \|\Lambda(A_i)\|^2 \quad \text{s.t.} \quad \langle A_1, \ldots, A_d \rangle \in \mathcal{G}^{\mathrm{Lie}}(G, \mathfrak{gl}(n)).$$

with $\mathcal{G}(G, \mathfrak{so}(n))$ the subspace of $\mathfrak{so}(n)$ consisting of the Lie subalgebras pushforward of $G$ by a representation.

The set $\mathcal{G}(G, \mathfrak{so}(n))$ has many connected components, one for each *orbit-equivalence* class of representations. We want to make sure that the minimizer belongs to the correct connected component.

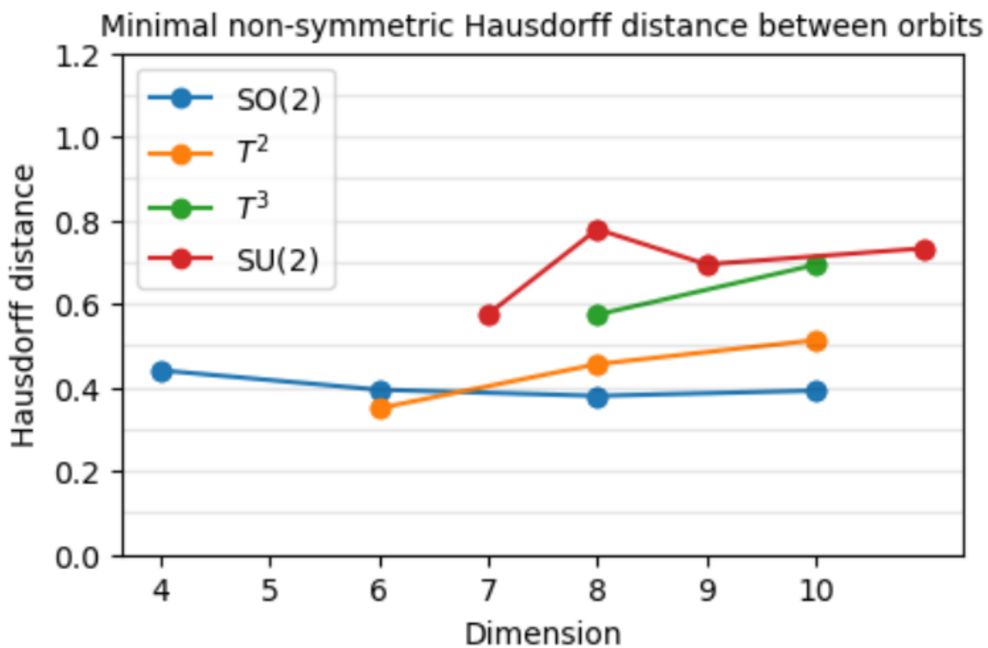


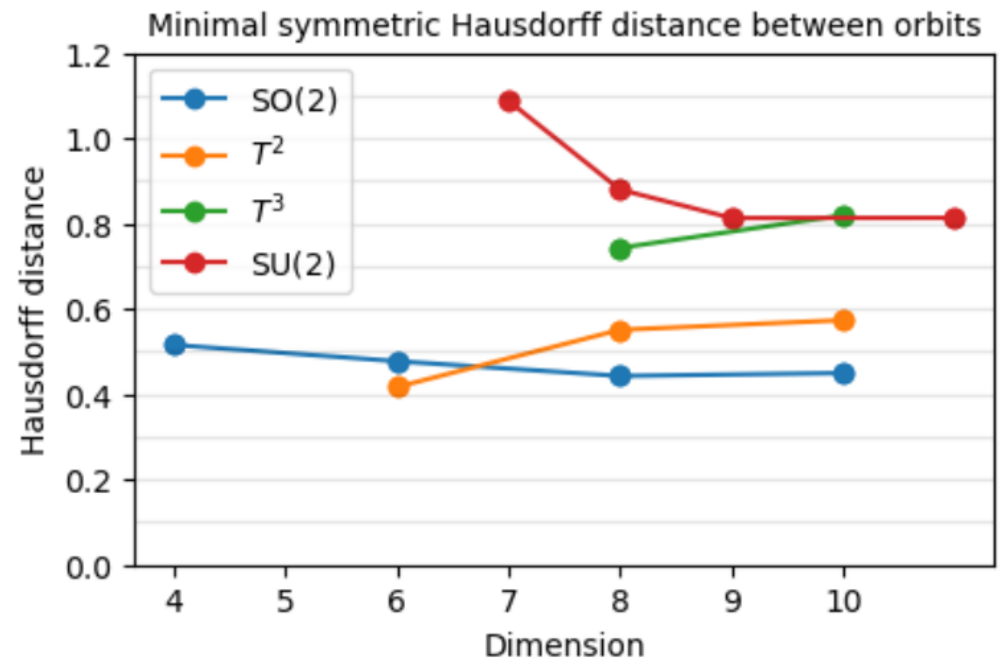

The distance from $\langle A_i \rangle_{i=1}^d$ to $\mathfrak{h}$ must be lower than the *reach* of $\mathcal{G}(G, \mathfrak{so}(n))$. In this context, it is called *rigidity*:

$$\Gamma(G, n) = \inf \|\Pi[\mathfrak{h}]\Pi[\mathfrak{s}^\perp]\|^2 \quad \text{s.t.} \quad \mathfrak{h} \in \mathcal{G}^{\mathrm{Lie}}(G, \mathfrak{gl}(n)), \mathfrak{s} \in \mathcal{G}^{\mathrm{Lie}}(H, \mathfrak{gl}(n)), s \not\simeq \mathfrak{h}.$$

**Lemma:** Consider the subset of $\mathcal{G}(G, \mathfrak{so}(n))$ with weights at most $\omega_{\max}$. Then

$$\Gamma(G, n, \omega_{\max}) \geq 4/(n\omega_{\max}^2).$$




**Left:** empirical estimation of the minimal non-symmetric Hausdorff distance $\mathrm{d_H}\big(\widehat{\mathcal{O}}_x^1|\widehat{\mathcal{O}}_x^2\big)$ between two orbits of a same initial point $x$ for two non-orbit equivalent representations $\phi_1, \phi_2$ of a compact Lie group $G$ in $\mathbb{R}^n$. The minimal value is approximately 0.35.

**Right:** same for the symmetric Hausdorff distance $\mathrm{d_H}\big(\widehat{\mathcal{O}}_x^1, \widehat{\mathcal{O}}_x^2\big)$. The minimal value is 0.42.

Running time (in seconds or minutes) and success rate (percentage) of full execution of `LieDetect`, as a function of the input group, and the dimension of the ambient Euclidean space. The input of the algorithm is a point cloud sampled from the uniform measure on an orbit chosen randomly.

For the Abelian groups SO(2), $T^2$, and $T^3$, the representations are considered up to a maximal frequency, 100 runs of the algorithm are performed, and the results are averaged. For SU(2), 10 runs have been performed.

| Dimension | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| Running time | 0.04s | 0.05s | 0.08s | 0.14s |
| Success | 100.0% | 100.0% | 100.0% | 100.0% |

(a) SO(2)

| Dimension | 6 | 8 | 10 |
|---|---|---|---|
| Running time | 0.24s | 0.63s | 4.03s |
| Success | 82.0% | 100.0% | 98.0% |

(b) $T^2$

| Dimension | 8 | 10 |
|---|---|---|
| Running time | 1.44s | 5.98s |
| Success | 100.0% | 100.0% |

(c) $T^3$

| Dimension | 4 | 5 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|
| Running time | 0.6s | 5.04s | 4min 21s | 13min 7s | 16min 9s | 10min 53s |
| Success | 100.0% | 100.0% | 90.0% | 100.0% | 100.0% | 100.0% |

(d) SU(2)