

EMAp Summer Course

Topological Data Analysis with Persistent Homology

<https://raphaeltinarrage.github.io/EMAp.html>

Lesson 4: Simplicial complexes

I - Combinatorial simplicial complexes

II - Topology

III - Euler characteristic

(VI - Tutorial)

Standard simplices

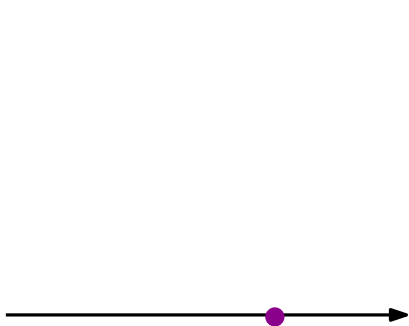
4/13

In order to describe topological spaces, we will decompose them into simpler pieces.

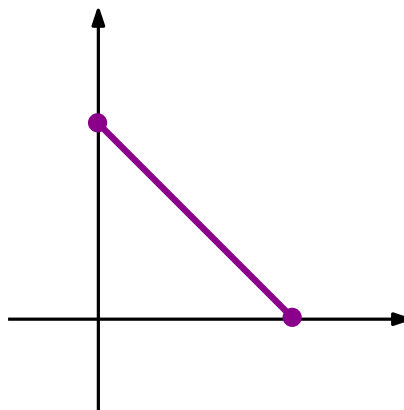
The pieces we shall consider are the standard simplices.

The *standard simplex of dimension n* is the following subset of \mathbb{R}^{n+1}

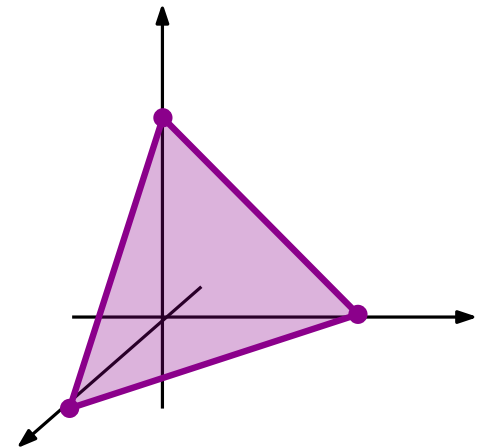
$$\Delta_n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}, x_1, \dots, x_{n+1} \geq 0 \text{ and } x_1 + \dots + x_{n+1} = 1\}$$



Δ_0



Δ_1



Δ_2

First, a purely combinatorial definition (without geometry):

Definition: Let V be a set (called the set of *vertices*). A *simplicial complex* over V is a set K of subsets of V (called the *simplices*) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$.

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If $\sigma \in K$ is a simplex, its non-empty subsets $\tau \subset \sigma$ are called *faces* of σ .

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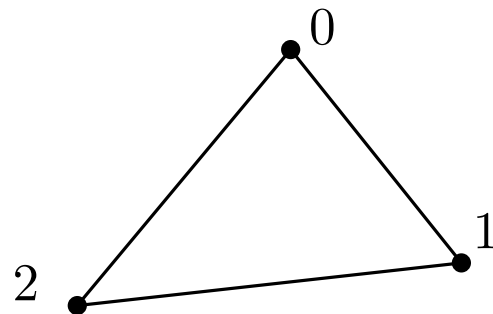
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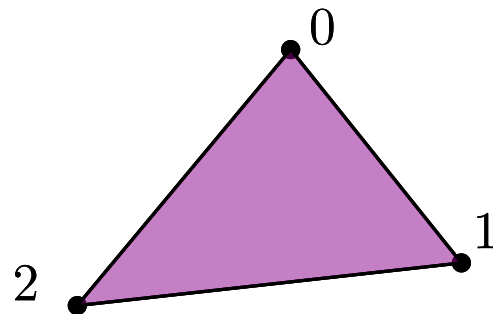
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Simplicial complexes

5/13 (6/7)

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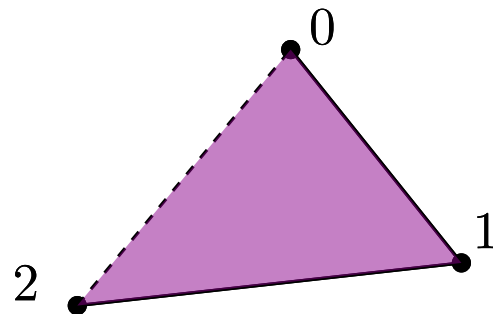
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Example: Let $V = \{0, 1, 2\}$ and

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This is not a simplicial complex.

Indeed, the simplex $[0, 1, 2]$ admits a face $[2, 0]$ that is not included in V .



Simplicial complexes

5/13 (7/7)

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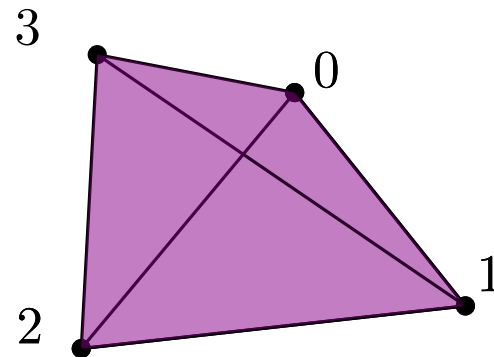
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If σ is a simplex, its dimension is defined as $|\sigma| - 1$ (cardinality of σ minus 1). If K is a simplicial complex, its dimension is defined as the maximal dimension of its simplices.

Example: Let $V = \{0, 1, 2, 3\}$ and

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}$$

It a simplicial complex of dimension 2.



I - Combinatorial simplicial complexes

II - Topology

III - Euler characteristic

(VI - Tutorial)

Let us give simplicial complexes a topology.

Definition: Let K be a simplicial complex, with vertex $V = \{1, \dots, n\}$.

In \mathbb{R}^n , consider, for every $i \in \llbracket 1, n \rrbracket$, the vector $e_i = (0, \dots, 1, 0, \dots, 0)$ (i^{th} coordinate 1, the other ones 0).

Let $|K|$ be the subset of \mathbb{R}^{n+1} defined as:

$$|K| = \bigcup_{\sigma \in K} \text{conv}(\{e_j, j \in \sigma\})$$

where conv represent the convex hull of points.

Endowed with the subspace topology, $(|K|, \mathcal{T}_{||K|})$ is a topological space, that we call the *topological realization of K* .

If $x_1, \dots, x_k \in \mathbb{R}^n$, the convex hull is defined as:

$$\text{conv}(\{x_1 \dots x_k\}) = \left\{ \sum_{1 \leq i \leq k} t_i x_i, \quad t_1 + \dots + t_k = 1, \quad t_1, \dots, t_k \geq 0 \right\}.$$

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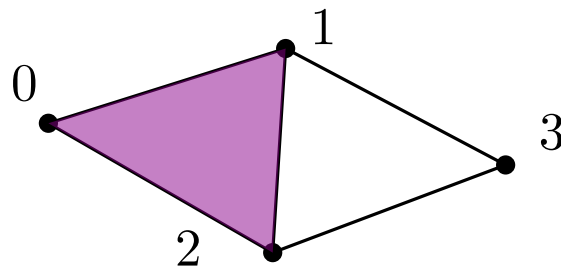
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Remark: If the simplicial complex can be drawn in the plane (or space) without crossing itself, then its topological realization simply is the subspace topology.

Example: $K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 0], [1, 3], [2, 3], [0, 1, 2]\}$.

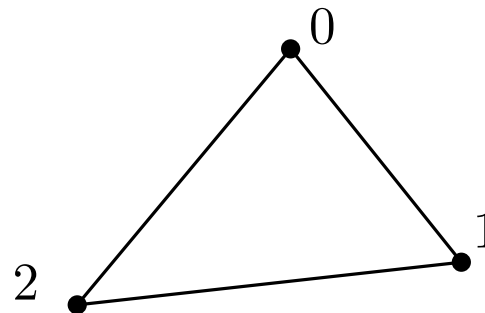


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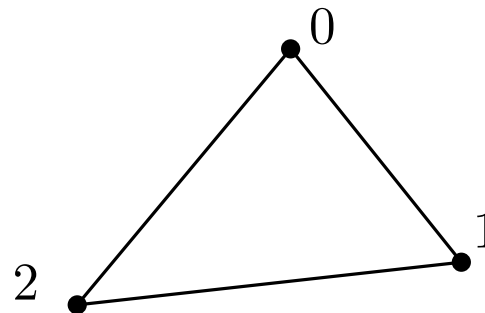
Triangulations

8/13 (3/4)

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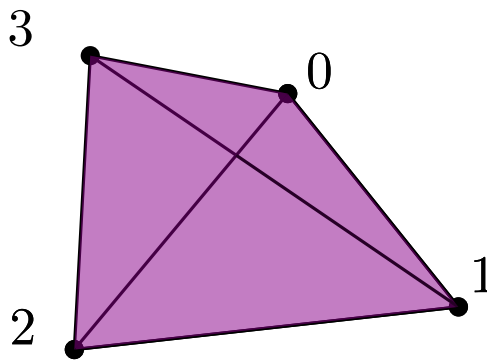
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Example: The following simplicial complex is a triangulation of the sphere:

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}.$$

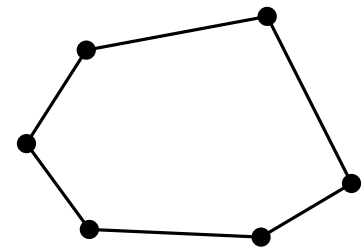
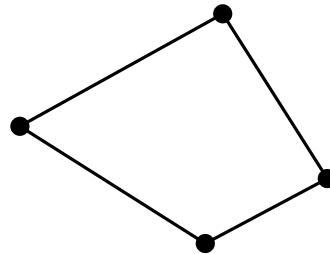
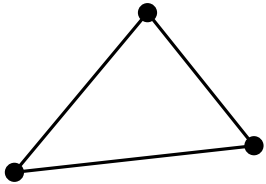


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Given a topological space, it is not always possible to triangulate it. However, when it is, there exists many different triangulations.



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(VI - Tutorial)

Definition: Let K be a simplicial complex of dimension n . Its *Euler characteristic* is the integer

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Euler characteristic

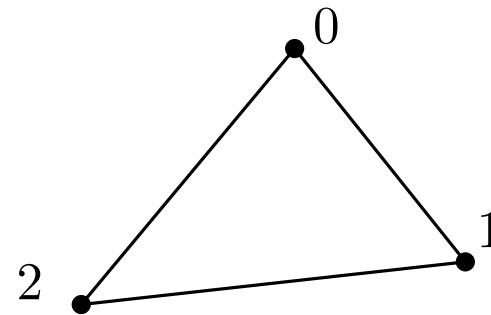
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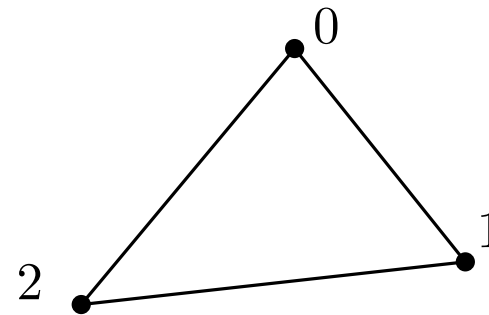
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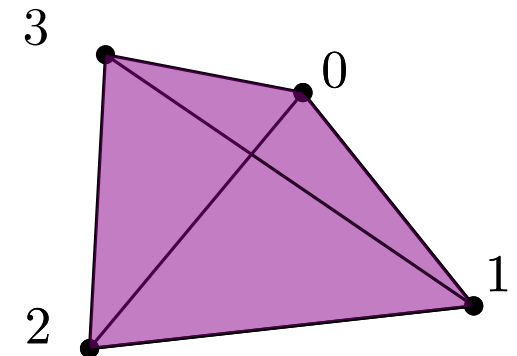


Example: The simplicial complex

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$$\chi(K) = 4 - 6 + 4 = 2$$



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Definition: Let X be a topological space. Its Euler characteristic is defined as the Euler characteristic of any triangulation of it.

Two issues:

- X must admit a triangulation
- we have to make sure the the triangulations of X all have the same Euler characteristic. In other words, if K and K' are two triangulations of X , we must have $\chi(K) = \chi(K')$.

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→ this is true!

but we won't be able to prove it in this summer course...

Euler characteristic

10/13 (7/7)

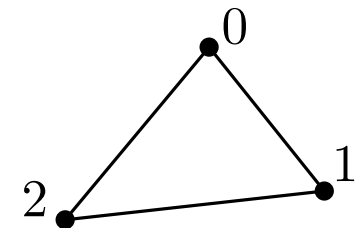
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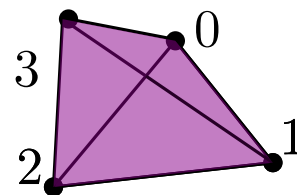
Example: The circle has Euler characteristic 0 because it admits a triangulation

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Example: The sphere has Euler characteristic 2 because it admits a triangulation

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Euler characteristic is an invariant

11/13 (1/2)

Proposition: If X and Y are two homotopy equivalent topological spaces, then $\chi(X) = \chi(Y)$.

Therefore, the Euler characteristic is an *invariant* of homotopy equivalence classes.

We can use this information to prove that two spaces are not homotopy equivalent.

Euler characteristic is an invariant

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Example: The circle has Euler characteristic 0, and the sphere Euler characteristic 2. Therefore, they are not homotopy equivalent.

Exercise (21): Show that \mathbb{R}^3 and \mathbb{R}^4 are not homotopy equivalent.

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Conclusion

We learnt how to represent topological spaces on a computer.

We defined a new topological invariant.

Homeworks for next week: Exercises 20, 23, 24

Facultative exercise: Exercises 21, 26***

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Obrigado!