XXII Encontro Brasileiro de Topologia - 31/07/2024

Classifying spaces in Topological Data Analysis

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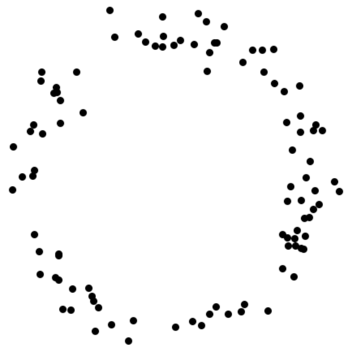
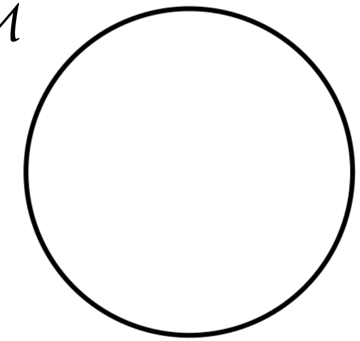
https://raphaeltinarrage.github.io

We observe a point cloud $X \subset \mathbb{R}^n$, that we suppose close to a submanifold $\mathcal{M}$.

X



$\mathcal{M}$



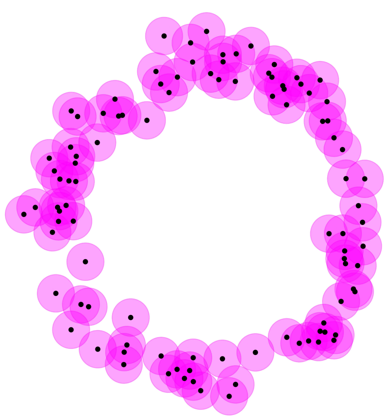
We wish to estimate, from X, the **singular (co)homology** of $\mathcal{M}$ with coefficients $\mathbb{Z}/p\mathbb{Z}$.

(1) Define the ***t*-thickening** of X as: $X^t = \{y \in \mathbb{R}^n \mid \exists x \in X, \|x-y\| \leq t\}$

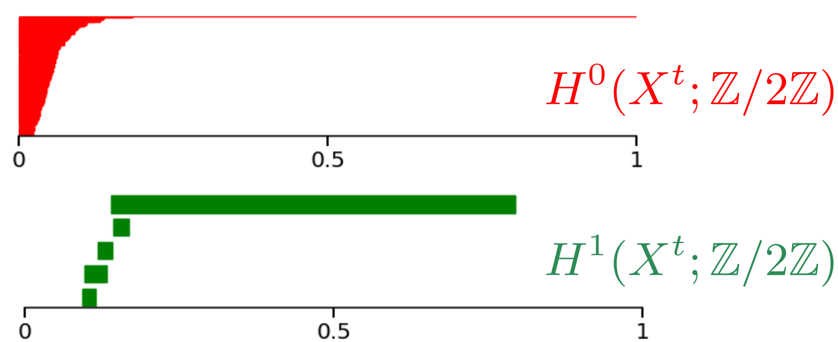
and the **Čech filtration** as: $V[X] = \left(X^t\right)_{t \geq 0}$

(2) Consider the **persistence module**: $\mathbb{V}[X] = \left(H^i(X^t)\right)_{t \geq 0}$

(3) Compute its **barcode**: $\mathrm{Barcode}[X]$



$\longrightarrow \mathbb{V} \longrightarrow$



Filtration $V[X]$ Persistence module $\mathbb{V}[X]$ $\mathrm{Barcode}[X]$

Roadmap to classifying spaces in TDA

[Persistent Cohomology and Circular Coordinates, de Silva & Vejdemo-Johansson, DCG, 2010]

Let X be a topological space (that has the homotopy type of a CW complex). Then

first integral singular cohomology group

$$\boxed{H^1(X;\mathbb{Z}) \simeq [X, \mathbb{S}^1]}$$

set of homotopy classes of continuous map $X \to \mathbb{S}^1$

[Coordinatizing Data With Lens Spaces and Persistent Cohomology, Polanco & Perea, CCCG, 2019]

[Sparse Circular Coordinates via Principal $\mathbb{Z}$-Bundles, Perea, Abel symposium, 2020]

Let G be a finitely generated abelian group and $K(G,1)$ an **Eilenberg-MacLane space**.

$$\begin{tikzcd} H^1(X;G) \arrow[dr, bend right] & \simeq & [X, K(G,1)] \\ & \mathrm{Prin}_G(X) \arrow[ur, bend right] & \end{tikzcd}$$

set of isomorphism classes of principal G-bundles over X

[Multiscale projective coordinates via persistent cohomology of sparse filtrations, Perea, DCG, 2018]

[Computing persistent Stiefel–Whitney classes of line bundles, T., JACT, 2021]

[Approximate and discrete Euclidean vector bundles, Scoccola & Perea, Sigma, 2023]

Let G be a topological group, and $\mathcal{B}G$ a **classifying space**. Then $\mathrm{Prin}_G(X) \simeq [X, \mathcal{B}G]$.

real vector bundles on X

$$\begin{array}{ccc} \mathrm{Vect}_d^{\mathbb{R}}(X) & \simeq & [X, \mathcal{B}\mathrm{O}(d)] \\ & & \downarrow \\ & & H^*(X;\mathbb{Z}/2\mathbb{Z}) \end{array}$$

with $G = \mathrm{O}(d)$, Stiefel-Whitney classes

$$\begin{array}{ccc} \mathrm{Vect}_d^{\mathbb{C}}(X) & \simeq & [X, \mathcal{B}\mathrm{U}(d)] \\ & & \downarrow \\ & & H^*(X;\mathbb{Z}) \end{array}$$

with $G = \mathrm{U}(d)$, Chern classes

I. Classifying spaces of discrete groups

- I.1. Circular coordinates
- I.2. Eilenberg-MacLane coordinates

II. Classifying spaces of vector bundles

- II.1. Vector bundles
- II.2. Persistent characteristic classes

III. Triangulations of the Grassmannian

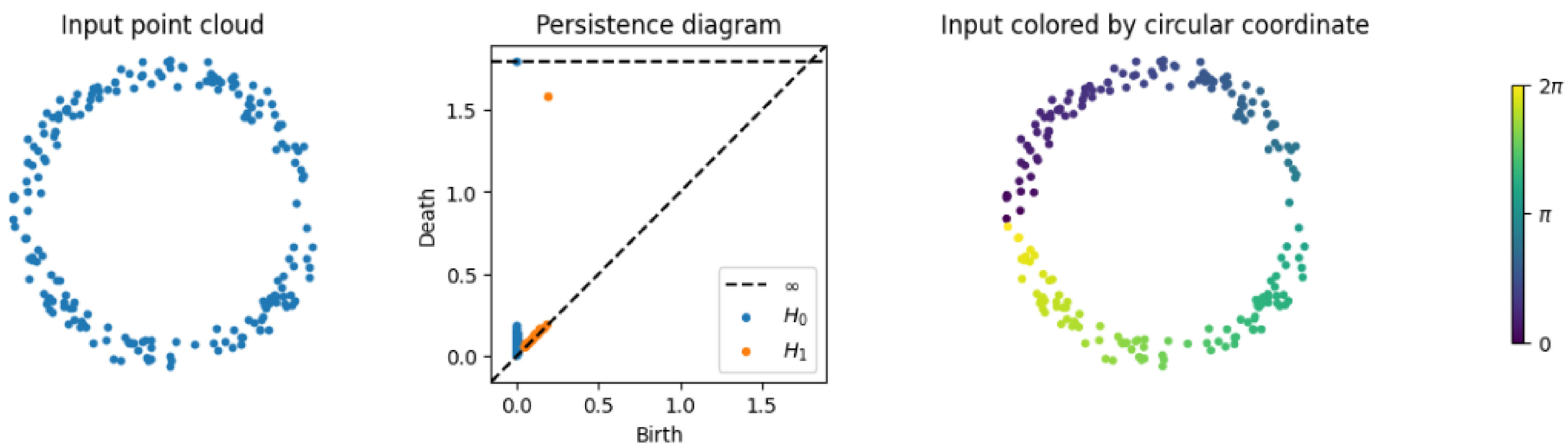
- III.1. Simplicial approximation in practice
- III.2. Simplicial approximation to CW complexes

Naive construction of circular coordinates

[Persistent Cohomology and Circular Coordinates, de Silva & Vejdemo-Johansson, DCG, 2010]
Implemented in `DREiMac` by Perea, Scoccola & Tralie (https://github.com/scikit-tda/DREiMac)

Input: A finite point cloud $X \subset \mathbb{R}^n$.

Output: A map $X \to \mathbb{S}^1$.



(1) Build a simplicial complex K and identify a cohomology class $c \in H^1(K; \mathbb{Z}/p\mathbb{Z})$.

(2) Lift c to an integer cocyle $c' \in H^1(K; \mathbb{Z})$.

(3) Use the identity $H^1(K; \mathbb{Z}) \simeq [K, \mathbb{S}^1]$ to find $\theta \colon K \to \mathbb{S}^1$.

(1) Build a simplicial complex K and identify a cohomology class $c \in H^1(K; \mathbb{Z}/p\mathbb{Z})$.

For instance, choose K as the Čech or Vietoris-Rips complex on X, and identify c in its persistence diagram.

(2) Lift c to an integer cocyle $c' \in H^1(K; \mathbb{Z})$.

Consider $c\colon K^{(1)} \to \{0 \dots q-1\}$. Most of the time, the following should work:

$$c'(\sigma) = \begin{cases} c(\sigma) & \text{if } c(\sigma) \leq (p-1)/2, \\ c(\sigma) - p & \text{if } c(\sigma) > (p-1)/2. \end{cases}$$

More precisely, if $H^2(K; \mathbb{Z})$ has no p-torsion, then $H^1(K; \mathbb{Z}) \to H^1(K; \mathbb{Z}/p\mathbb{Z})$ is surjective.

(3) Use the identity $H^1(K; \mathbb{Z}) \simeq [K, \mathbb{S}^1]$ to find $\theta\colon K \to \mathbb{S}^1$.

The map θ can be defined as:

- on vertices $x \in K^{(0)}$ as $\theta(x) = 0$;
- on edges $[xy] \in K^{(1)}$ as turning $c'([xy])$ times around $\mathbb{S}^1$;
- on triangles $\sigma \in K^{(2)}$: θ can be extended to σ since $\theta\colon \partial\sigma \to \mathbb{S}^1$ has degree zero. Indeed, c' is a cocycle;
- on higher dimensional simplices $\sigma \in K^{(i)}$, $i \geq 3$: no obstructions, since $\pi_{i-1}(\mathbb{S}^1) = 0$.

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problem: θ is trivial on X

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- on edges $[xy] \in K^{(1)}$ as turning $c'([xy])$ times around $\mathbb{S}^1$;
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(2) Lift c to an integer cocyle $c' \in H^1(K; \mathbb{Z})$.

~~**(3)** Use the identity $H^1(X; \mathbb{Z}) \simeq [X, \mathbb{S}^1]$ to find $\theta \colon K \to \mathbb{S}^1$.~~

(3) Find a "smooth" cocycle $c'' \in C^1(K; \mathbb{R})$ homologous to c'.

Through the minimization

$$c'' \in \operatorname{argmin}\left\{ \|c' + d_0 f\| \mid f \colon K^{(0)} \to \mathbb{R} \right\}$$

where $\|\alpha\|^2 = \sum_{[xy] \in K^{(1)}} \alpha([xy])^2$. This yields a *harmonic cocycle* ($\Delta c'' = 0$).

(4) Integrate $c'' \in C^1(K; \mathbb{R})$ to $\theta \colon K \to \mathbb{S}^1$.

The map θ can be defined as follows.

- vertices $x \in K^{(0)}$ are sent to $\theta(x) = f(x) \mod \mathbb{Z}$;
- edges $[x, y] \in K^{(1)}$ are sent to intervals of length $c''([xy])$.
  Indeed, $\theta(y) - \theta(x) = c''([xy]) + c'([xy]) = c''([xy]) \mod \mathbb{Z}$;
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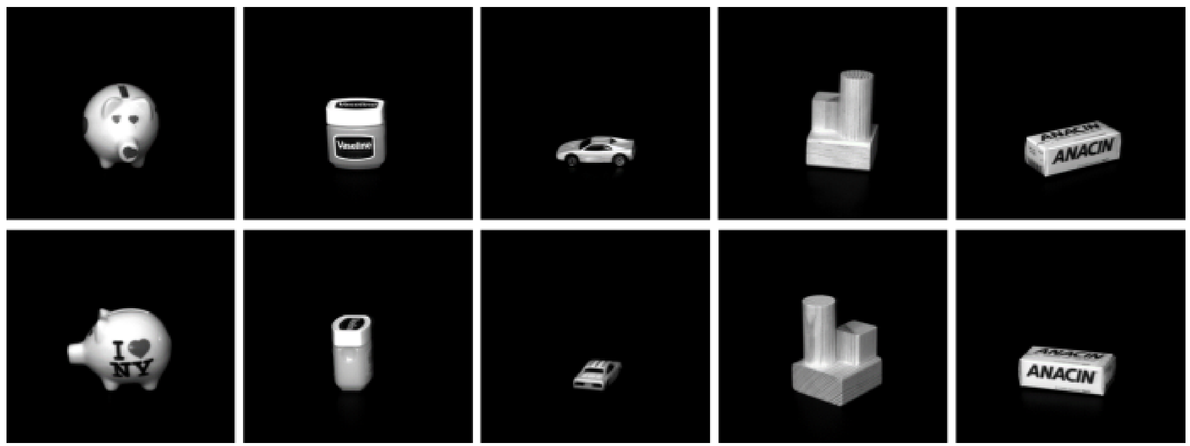
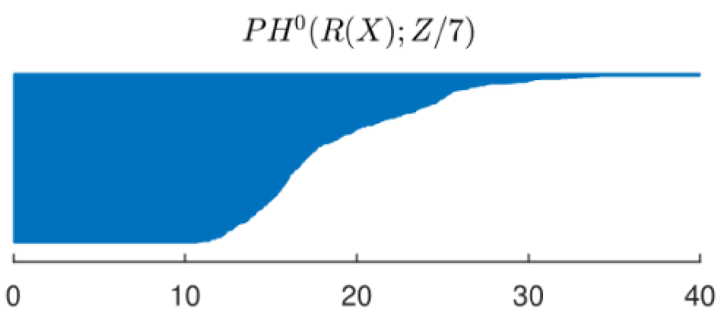
Examples

[Sparse Circular Coordinates via Principal $\mathbb{Z}$-Bundles, Perea, Abel symposium, 2020]

COIL-20 dataset: images of shape 448×416 pixels, representing 5 objects, each rotated 72 times. This yields a point cloud in $\mathbb{R}^{448\times 416}$.



$H^0(X^t; \mathbb{Z}/7\mathbb{Z})$



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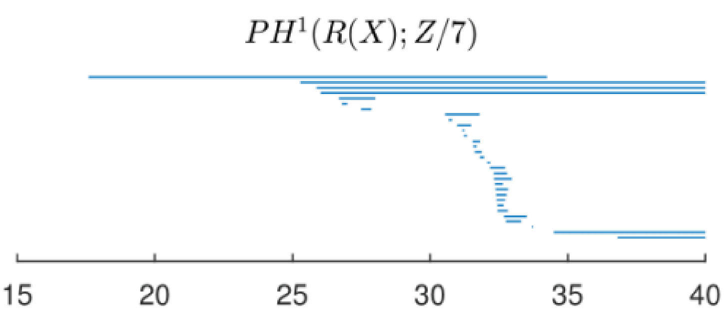

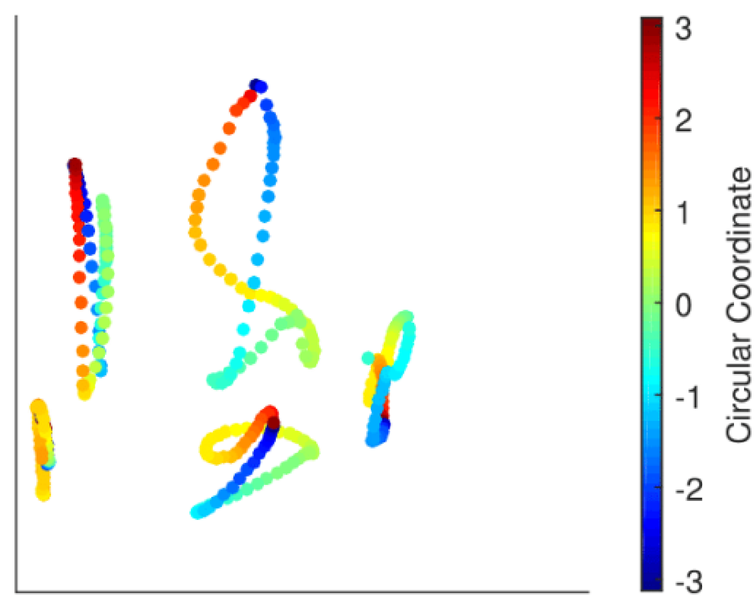

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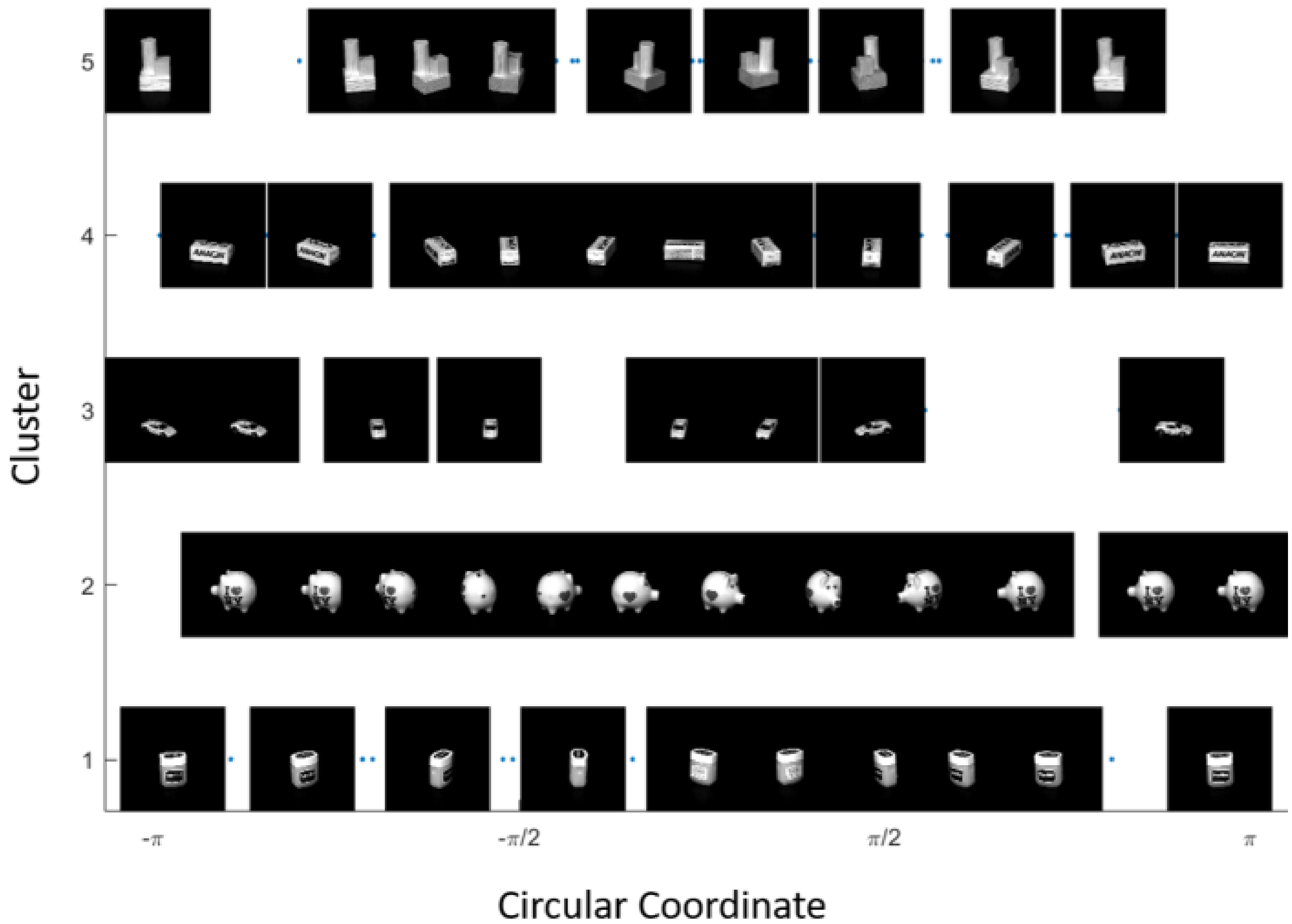

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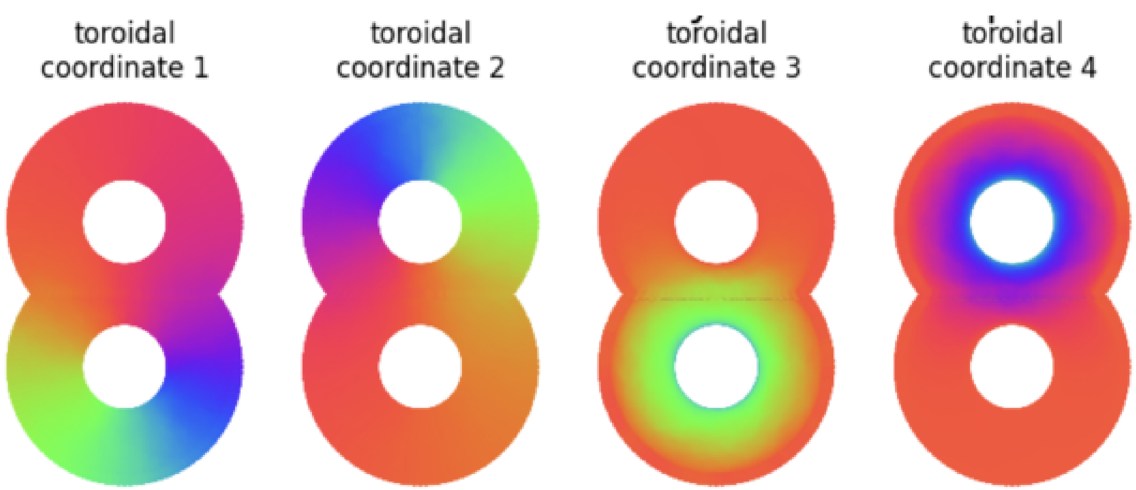
[Toroidal Coordinates: Decorrelating Circular Coordinates with Lattice Reduction, Scoccola, Gakhar, Bush, Schonsheck, Rask, Zhou & Perea, 2022]

Several $c_i \in H^1(X; \mathbb{Z}/p\mathbb{Z})$ yield several $\theta_i \colon X \to \mathbb{S}^1$, but potentially "geometrically correlated".



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Let G be a group and $n \geq 1$. An **Eilenberg-MacLane space** $K(G, n)$ is a connected topological space such that

$$\pi_i(K(G,n)) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

A few facts:

- Each group admits a $K(G, 1)$, and each abelian group a $K(G, n)$, $n \geq 2$.
- With G and n fixed, all the $K(G, n)$ are homotopy equivalent.
- If G has an element of finite order, then $K(G, n)$ has infinite dimension.
- The singular homology of $K(G, n)$ may be complicated!
- $K(G, 1)$ are particular cases of classifying spaces.

Examples:

G	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$
$K(G,1)$	$\mathbb{S}^1$	$\mathbb{R}P^\infty$	$L(\infty, p)$
$K(G,2)$	$\mathbb{C}P^\infty$		

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$$\pi_i(K(G,n)) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem: Given G abelian and X paracompact, one has a bijection

$$\begin{aligned} [X, K(G,n)] &\longrightarrow H^n(X;G) \\ f &\longmapsto f^*(\alpha) \end{aligned}$$

for a certain class $\alpha \in H^n(K(G,n);G)$.

Remind that we take $[X, K(G,n)]$ to be the set of homotopy classes of maps $X \to K(G,n)$.

Proof: Let $K_n = K(G,n)$. Show $h^n(X) = [X, K(G,n)]$ is a reduced cohomology theory.

One must provide natural isomorphisms $h^n(X) \simeq h^n(\Sigma X)$, where ΣX is the suspension.

Use Eckmann-Hilton duality: $[\Sigma X, K_n] \simeq [X, \Omega K_n]$, where ΩK_n is the loop space.

One deduces that $(K_n)_{n \geq 1}$ must be a Ω-spectrum.

Classifying spaces of discrete groups

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Problem: this construction is not explicit.

A detour through principal bundles

To make the correspondance more explicit, one may use the intermediary space

$$H^1(X;G) \simeq \mathrm{Prin}_G(X) \simeq [X, K(G,1)]\,.$$

Čech cocycles

classifying maps

principal G-bundles over X

Definition: A **fiber bundle** with fiber F over a topological space X is a surjection $\xi\colon E \to X$, with E a topological space, such that for all $x \in X$, there exists a neigborhood $U \subset X$ and a homeomorphism $h\colon U \times F \to \xi^{-1}(U)$ such that for all $(y, f) \in U \times F$, one has $h(y, f) = y$.

In addition, $\xi\colon E \to X$ is a **principal G-bundle** if it comes equipped with a right G-action on E that is transitive on the fibers $\xi^{-1}(\{x\})$.

A few facts:

- The fibers are homeomorphic to G.
- A **section** is a map $s\colon X \to E$ s.t. $\xi \circ s = \mathrm{id}$. It exists, then the bundle is trivial.

A detour through principal bundles

From $H^1(X;G)$ to $\mathrm{Prin}_G(X)$: Given explicitly via Čech cohomology $\check{H}^1(X;\mathcal{C}_G)$.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. A Čech 1-cocycle is a family of maps

$$\{\rho_{i,j} \colon U_i \cap U_j \to G\}_{i,j\in I}$$

that verify the cocycle condition: $\forall i,j,k \in I$, $\forall x \in U_i \cap U_j \cap U_k$,

$$\rho_{i,j}(x)\rho_{j,k}(x) = \rho_{i,k}(x).$$

From such a cocyle, one builds a principal G-bundle with total space

$$E = \left(\bigcup_{i\in I} U_i \times \{i\} \times G \right) / \sim$$

where $(x,i,g) \sim (x,j,g\rho_{i,j}(x))$ for $x \in U_i \cap U_j$, and projection

$$\xi \colon [x,i,g] \mapsto x.$$

From $\mathrm{Prin}_G(X)$ to $[X, K(G,1)]$: Depends on the group.

$G = \mathbb{Z}/2\mathbb{Z}$ [Coordinatizing Data With Lens Spaces and Persistent Cohomology, Polanco & Perea, CCCG, 2019]

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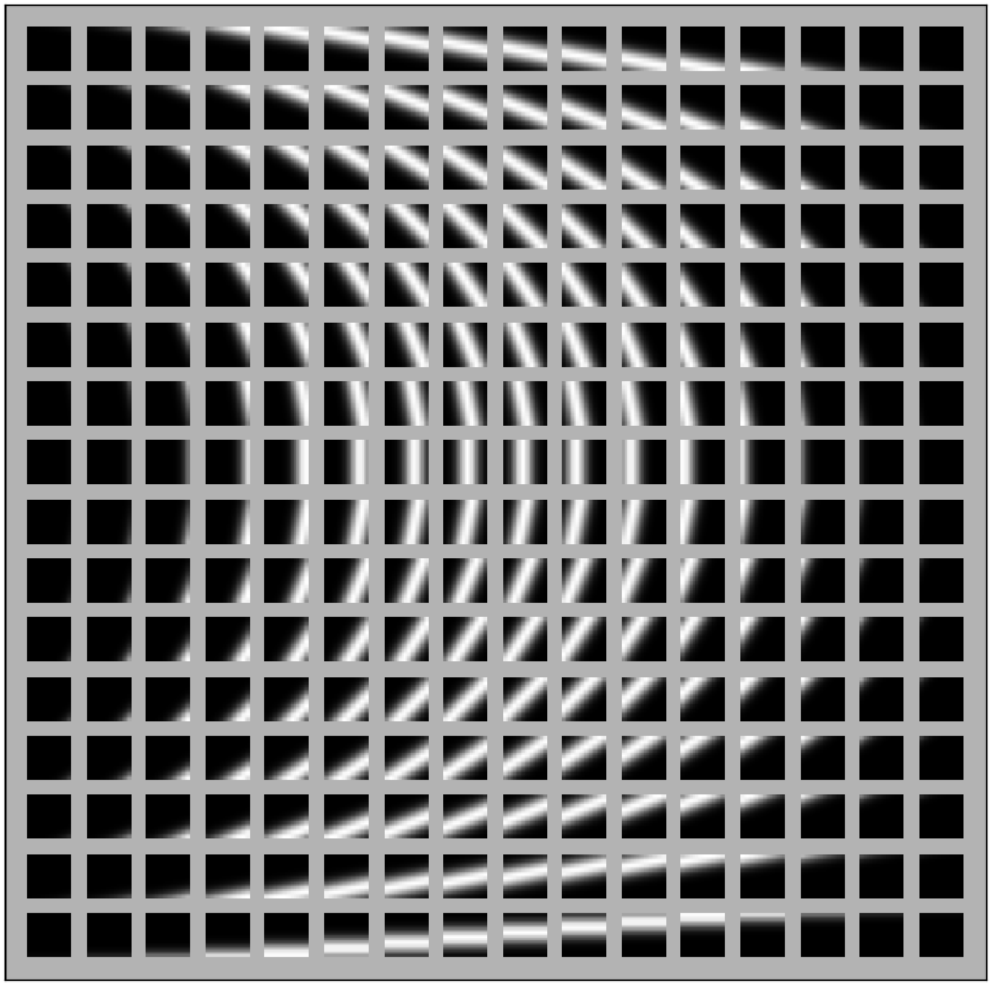
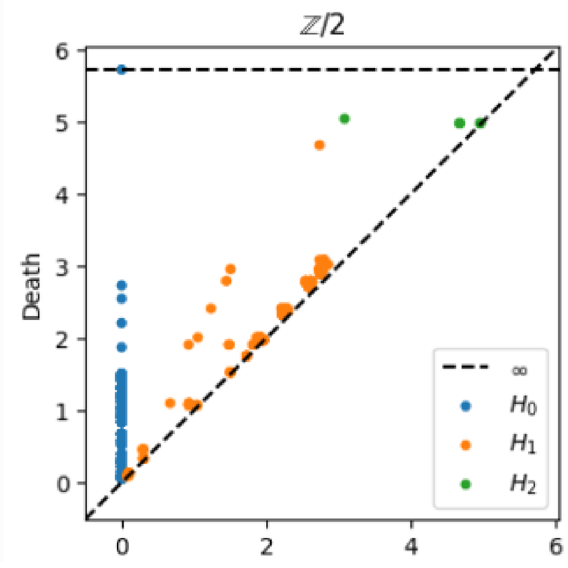
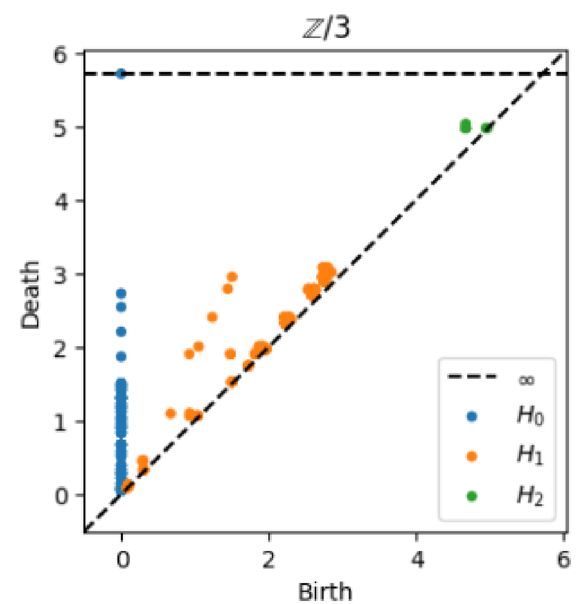
Examples

Projective coordinates with $H^1(X;\mathbb{Z}/2\mathbb{Z}) \simeq [X, \mathbb{R}P^\infty]$.

Consider a collection of images of shape $m \times m$ pixels. This yields a point cloud $X \subset \mathbb{R}^{m\times m}$.







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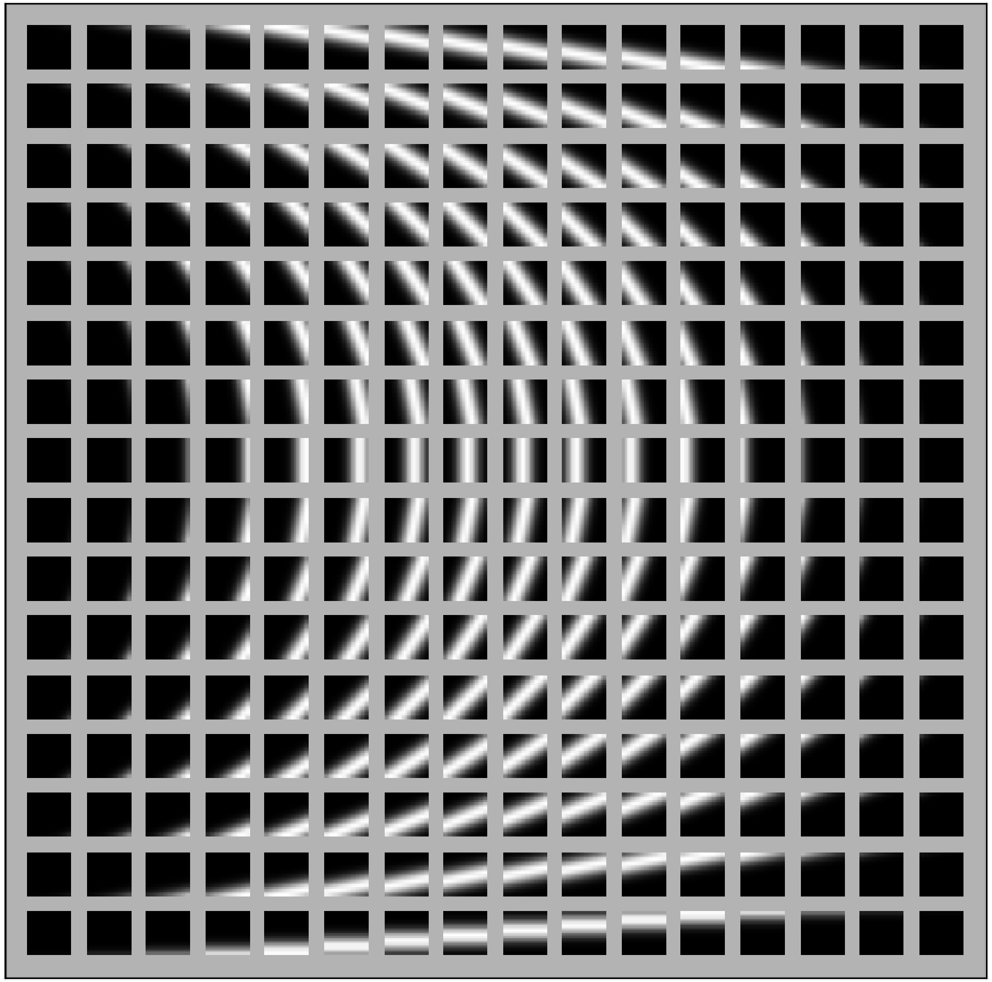
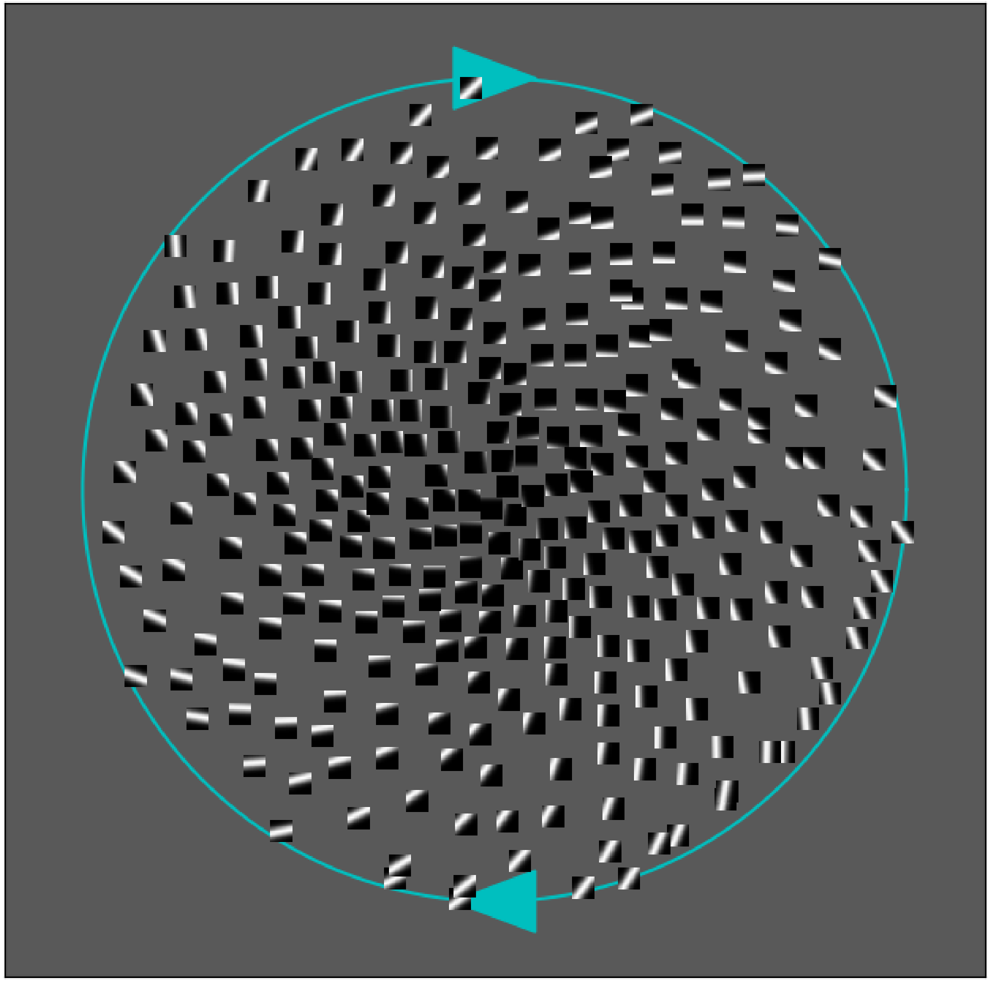
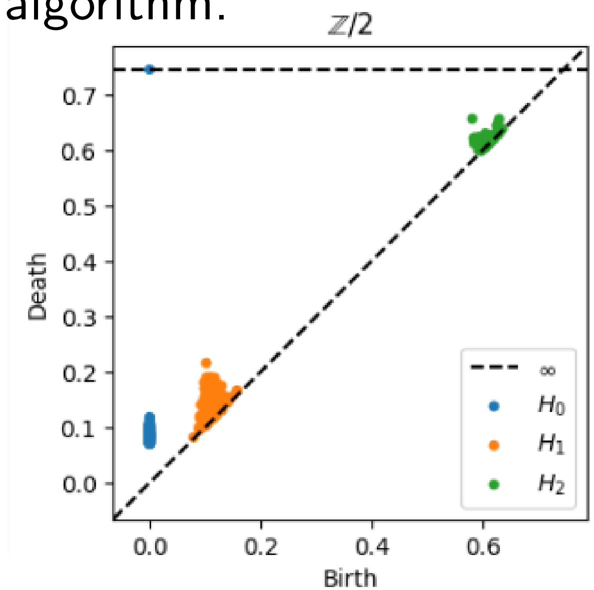


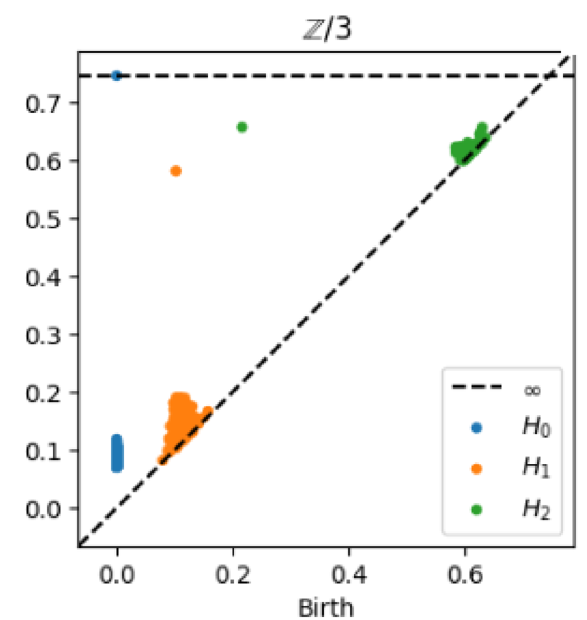


coordinatization $X \to \mathbb{R}P^2$

Lens coordinates with $H^1(X;\mathbb{Z}/3\mathbb{Z}) \simeq [X, \mathbb{S}^\infty/(\mathbb{Z}/3\mathbb{Z})]$.

Points are sampled on the Moore space $M(1,\mathbb{Z}/3\mathbb{Z})$. The point cloud X is represented in $\mathbb{R}^2$, but the intrindic distance is considered for the algorithm.





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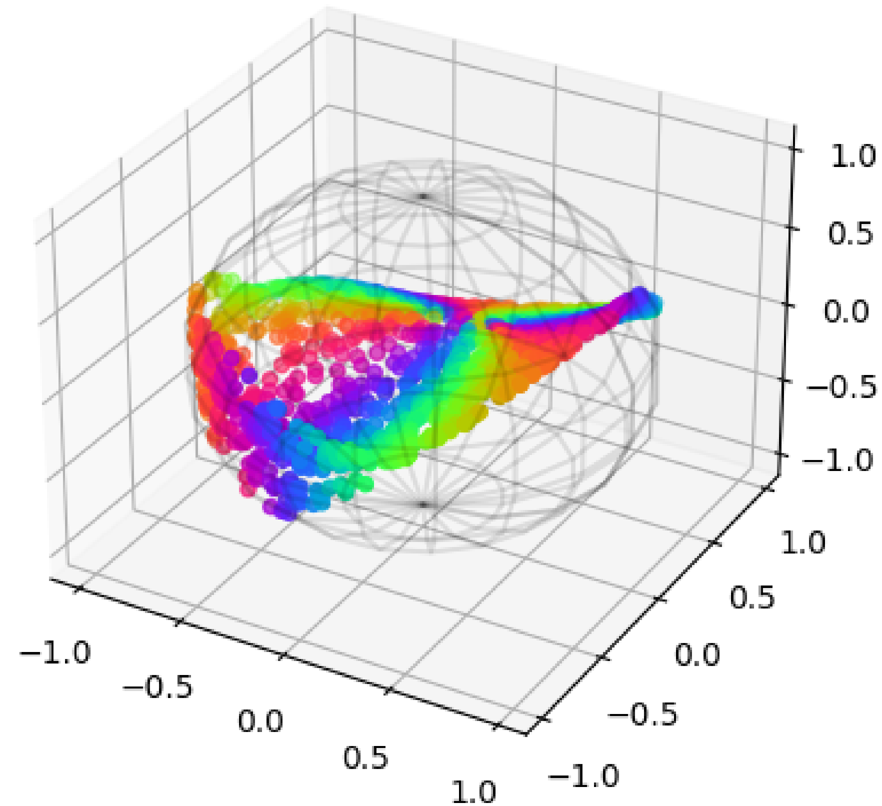


coordinatization $X \to L_3^2$

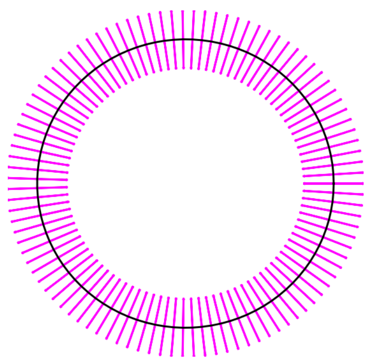
where $L_3^2 = \mathbb{S}^3/(\mathbb{Z}/3\mathbb{Z})$

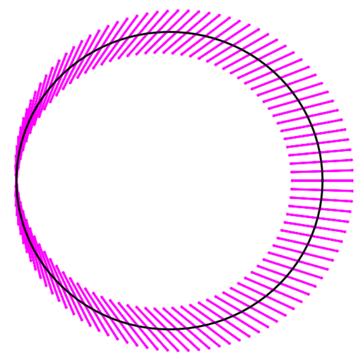
Definition: A **vector bundle** of dimension d over a topological space X is a surjection $\xi\colon E \to X$, with E a topological space, such that:

- the fibers $\xi^{-1}(\{x\}), x \in X$, are vector spaces of dimension d,
- **local triviality condition:** for all $x \in X$, there exists a neigborhood $U \subset X$ and a homeomorphism $h\colon U \times \mathbb{R}^d \to \xi^{-1}(U)$ such that for all $y \in U$, $h(y, \cdot)$ is an isomorphism of vector spaces.

Examples:

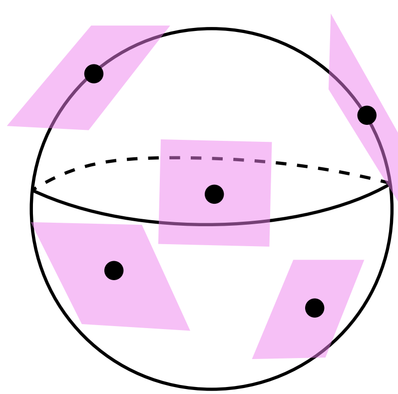
Normal bundle
of the circle

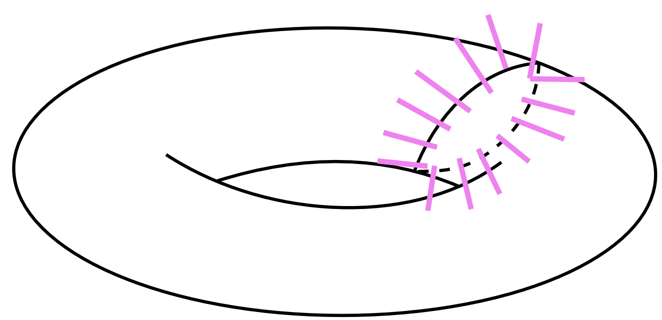




Möbius strip
on the circle

Tangent bundle
of the sphere





Normal bundle
of the torus

Vector bundles

Definition: A **vector bundle** of dimension d over a topological space X is a surjection $\xi\colon E \to X$, with E a topological space, such that:

- the fibers $\xi^{-1}(\{x\}), x \in X$, are vector spaces of dimension d,
- **local triviality condition:** for all $x \in X$, there exists a neigborhood $U \subset X$ and a homeomorphism $h\colon U \times \mathbb{R}^d \to \xi^{-1}(U)$ such that for all $y \in U$, $h(y, \cdot)$ is an isomorphism of vector spaces.

A few facts:

- Every vector bundle admits a section $s\colon X \to E$ (choose $s = 0$).
  It may not admit a nowhere-zero section.
- A vector bundle is trivial iff it admits d sections $s_1, \ldots, s_d$ everywhere independent.
- Line bundles ($d = 1$) on X are classified, up to isomorphism, by $H^1(X; \mathbb{Z}/2\mathbb{Z})$.

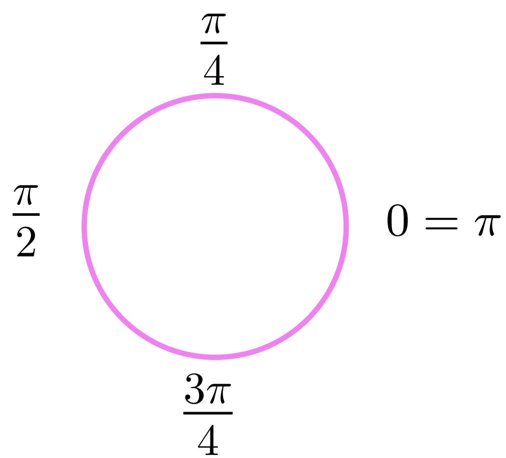
Definition: If X' is another space and $f\colon X' \to X$ a map, one defines the **pullback** (or induced) bundle, denoted $f^*\xi\colon E' \to X'$, with

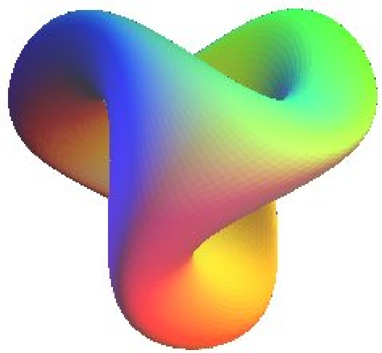
$$E' = \big\{(e, x) \in E \times X' \mid \xi(e) = f(x)\big\}.$$

Let $n \geq d \geq 1$. The **Grassmannian** $\mathcal{G}(d, \mathbb{R}^n)$ is the set of d-dimensional linear subspaces of $\mathbb{R}^n$. It can be endowed with a manifold structure, of dimension $d(n-d)$.

Example: $\mathcal{G}(1, \mathbb{R}^{n+1})$ is the projective space $\mathbb{R}P^n$.

$\mathcal{G}(1, \mathbb{R}^2)$





$\mathcal{G}(1, \mathbb{R}^3) \simeq \mathcal{G}(2, \mathbb{R}^3)$

Let $\mathbb{R}^\infty$ denotes the space of sequences of real numbers with a finite number of nonzero terms. We can also define the **infinite Grassmannian** $\mathcal{G}(d, \mathbb{R}^\infty)$.

Theorem: The infinite Grassmannian has $\mathbb{Z}/2\mathbb{Z}$-cohomology

$$H^*(\mathcal{G}(d, \mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, ..., w_d]$$

where w_i has degree i.

In particular, $H^*(\mathcal{G}(1, \mathbb{R}^\infty)) = \mathbb{Z}/2\mathbb{Z}[w_1]$.

One has a d-dimensional vector bundle $\gamma_d^n\colon E(\gamma_d^n) \to \mathcal{G}(d, \mathbb{R}^n)$, where

$$E(\gamma_d^n) = \big\{(T, x) \mid T \in \mathcal{G}(d, \mathbb{R}^n),\ x \in T\big\},$$

as well as the d-dimensional **universal bundle** $\gamma_d^\infty\colon E(\gamma_d^\infty) \to \mathcal{G}(d, \mathbb{R}^\infty)$.

Theorem: For any d-dimensional vector bundle $\xi\colon E \to X$ with X paracompact, there exists $f\colon X \to \mathcal{G}(d, \mathbb{R}^\infty)$ such that

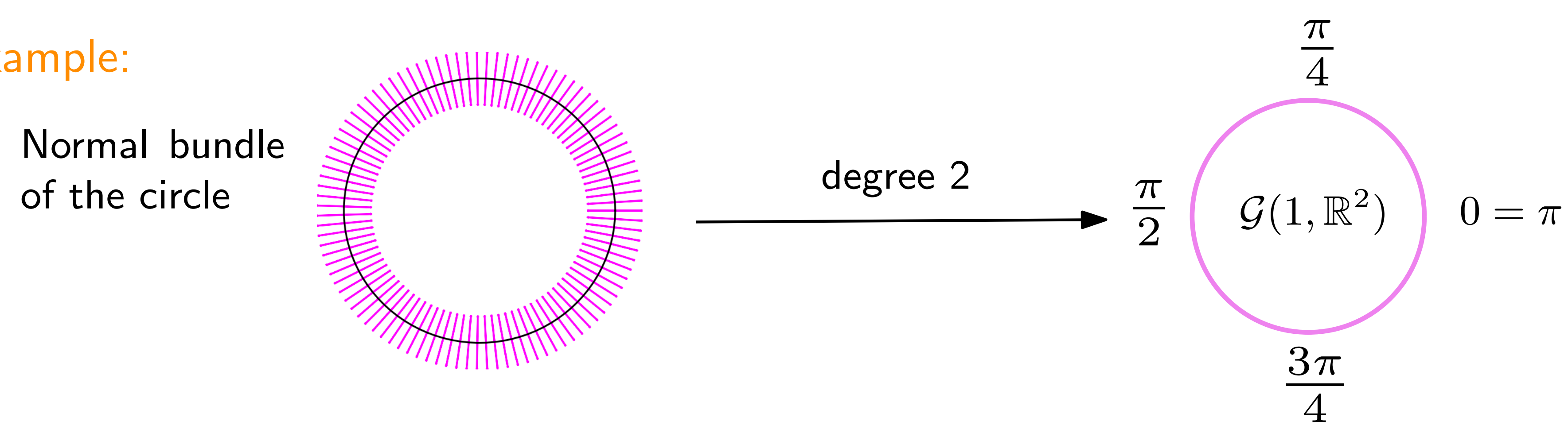
$$\xi = f^*\gamma_d^\infty.$$

Moreover, up to isomorphism and homotopy, one has

$$\mathrm{Vect}_d^{\mathbb{R}}(X) \ \simeq\ [X, \mathcal{G}(d, \mathbb{R}^\infty)]\,.$$

Remark: When X is compact, we can choose $f\colon X \to \mathcal{G}(d, \mathbb{R}^m)$ for m large enough.

Example:



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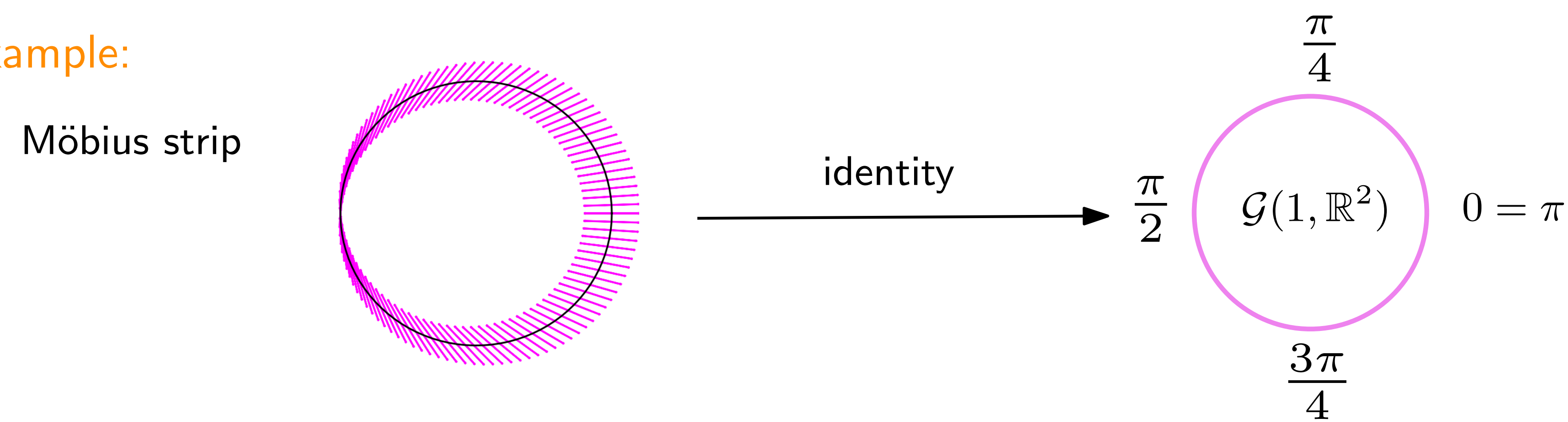

Definition: Each topological group G can be associated a weakly contractible space $\mathcal{E}G$ endowed with a free action of G. The quotient $\mathcal{B}G = \mathcal{E}G/G$ is called a **classifying space**.

In particular, $\mathcal{E}G \to \mathcal{B}G$ is a G-principal bundle.

Theorem: Given a topological group G and a paracompact space X, one has a bijection

$$\begin{aligned}[X, \mathcal{B}G] &\longrightarrow \mathrm{Prin}_G(X) \\ f &\longmapsto f^*\mathcal{E}G\end{aligned}$$

When G is discrete, $\mathcal{B}G$ is a $K(G,1)$. We obtain

$$[X, K(G,1)] \;\simeq\; \mathrm{Prin}_G(X).$$

When $G = \mathrm{O}(d)$, $\mathcal{B}\mathrm{O}(d)$ can be taken as the Grassmannian $\mathcal{G}(d, \mathbb{R}^\infty)$.

On the other hand, $\mathrm{Vect}_d^{\mathbb{R}}(X) \simeq \mathrm{Prin}_{\mathrm{O}(d)}(X)$ (via the bundle of orthonormal frames).

We deduce, again,

$$[X, \mathcal{G}(d, \mathbb{R}^\infty)] \;\simeq\; \mathrm{Vect}_d^{\mathbb{R}}(X).$$

When $G = \mathrm{U}(d)$, we get, similarly,

$$[X, \mathcal{G}(d, \mathbb{C}^\infty)] \;\simeq\; \mathrm{Vect}_d^{\mathbb{C}}(X).$$

Stiefel-Whitney classes

For every vector bundle $\xi\colon E \to X$, there exists a sequence of cohomology classes

$$(w_i(\xi))_{i\geq 0} \quad \text{where} \quad w_i(\xi) \in H^i(X;\mathbb{Z}/2\mathbb{Z}),$$

that satisfy the following axioms:

- **Axiom 1:** $w_0(\xi) = 1$, and if ξ is of dimension d then $w_i(\xi) = 0$ for $i > d$.
- **Axiom 2:** if $f\colon \xi \to \eta$ is a morphism of vector bundles $\xi\colon E_\xi \to X$ and $\eta\colon E_\eta \to Y$, then $w_i(\xi) = f^*(w_i(\eta))$, where $f^*\colon H^*(X) \leftarrow H^*(Y)$ induced map in cohomology.
- **Axiom 3:** if ξ, η are vector bundles over the same base space X, then for all $k \in \mathbb{N}$, $w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta)$ (cup product).
- **Axiom 4:** $w_1(\gamma_1^1) \neq 0$, where γ_1^1 denotes the Möbius strip bundle over the circle.

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Basic facts:

- If two bundles are isomorphic, then their Stiefel-Whitney classes are equal.
- If ξ admits a nowhere vanishing section, then $w_d(\xi) = 0$, where ξ has dimension d.
- If ξ admits k independent nowhere vanishing sections, then $w_i(\xi) = 0$ for $i > d - k$.

Topological facts:

- If τ is the tangent bundle of a manifold $\mathcal{M}$, then $\mathcal{M}$ is orientable iff $w_1(\tau) = 0$.
- If a manifold $\mathcal{M}$ of dimension n is immersible in $\mathbb{R}^{n+k}$, then $\overline{w}_i(\tau) = 0$ for $i > k$.

Construction of Stiefel-Whitney classes: Let $\xi\colon X \to \mathcal{G}(d,\mathbb{R}^\infty)$ be a vector bundle, and $\xi^*\colon H^*(X;\mathbb{Z}/2\mathbb{Z}) \leftarrow H^*(\mathcal{G}(d,\mathbb{R}^\infty);\mathbb{Z}/2\mathbb{Z})$ the map induced in cohomology.

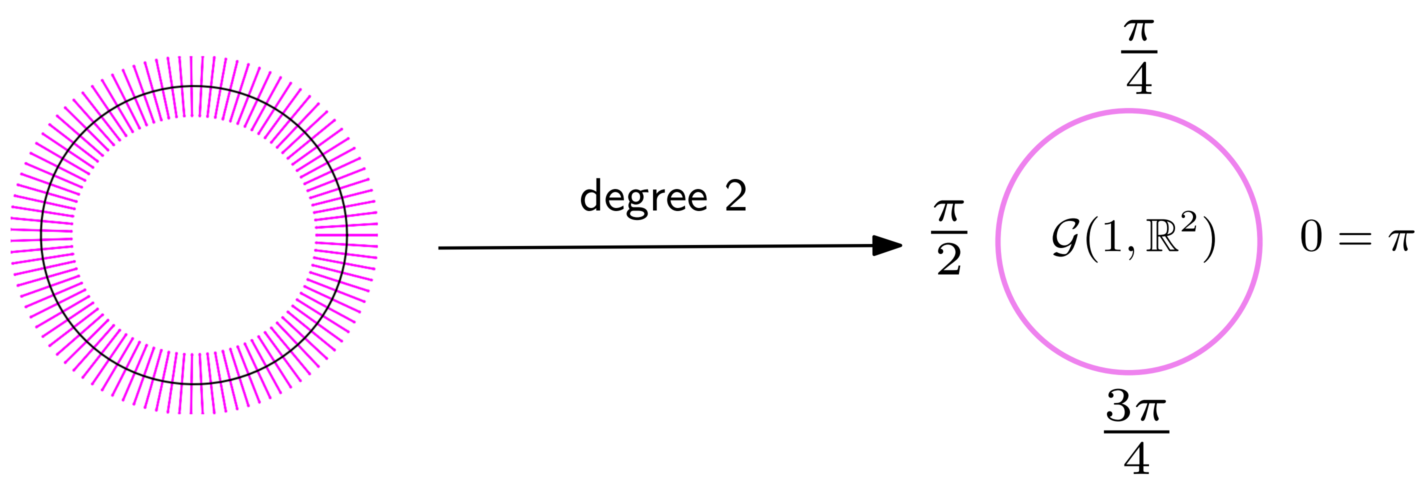
Recall that $H^*(\mathcal{G}(d,\mathbb{R}^\infty)) = \mathbb{Z}/2\mathbb{Z}[w_1,\ldots,w_d]$.

The Stiefel-Whitney classes can be defined as

$$w_i(\xi) = \xi^*(\omega_i) \in H^i(X;\mathbb{Z}/2\mathbb{Z}).$$

Example:

Normal bundle of the circle



$$\begin{array}{ccc} H^1(\mathbb{S}^1;\mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\quad 0 \quad} & H^1(\mathcal{G}(1,\mathbb{R}^2);\mathbb{Z}/2\mathbb{Z}) \\ w_1(\xi)=0 & \longleftarrow\!\!\shortmid & w_1 \end{array}$$

Stiefel-Whitney classes

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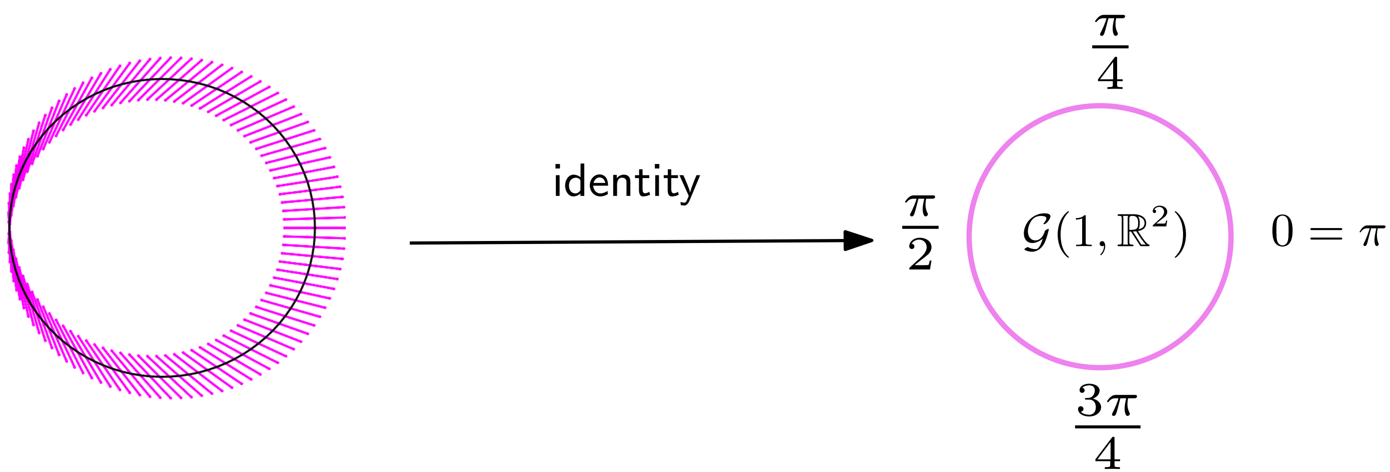
Example:

Möbius strip



$$H^1(\mathbb{S}^1;\mathbb{Z}/2\mathbb{Z}) \xleftarrow{\text{identity}} H^1(\mathcal{G}(1,\mathbb{R}^2);\mathbb{Z}/2\mathbb{Z})$$

$$w_1(\xi) = 1 \longleftarrow\!\mid \quad w_1$$

I. Classifying spaces of discrete groups

I.1. Circular coordinates

I.2. Eilenberg-MacLane coordinates

II. Classifying spaces of vector bundles

II.1. Vector bundles

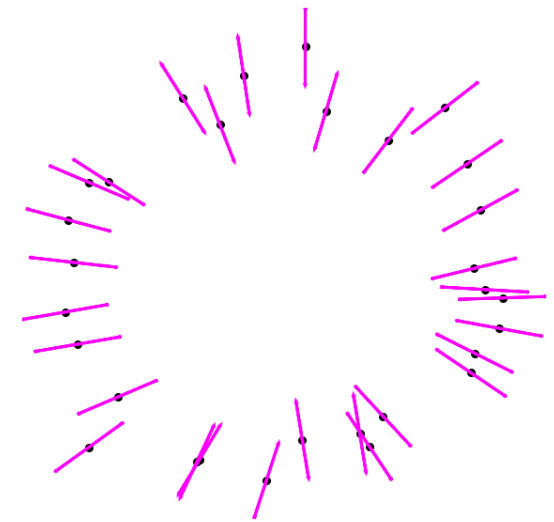
II.2. Persistent characteristic classes

III. Triangulations of the Grassmannian

III.1. Simplicial approximation in practice

III.2. Simplicial approximation to CW complexes

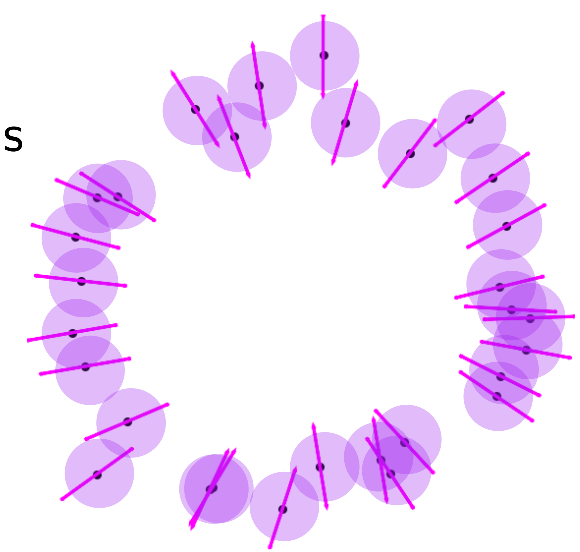
Sampling model for vector bundles:



Let $n, m, d > 0$ with $d \leq m$.

We observe $\left| \begin{array}{l} \text{a point cloud } X \subset \mathbb{R}^n \\ \textbf{and} \text{ a map } \xi \colon X \to \mathcal{G}(d, \mathbb{R}^m). \end{array} \right.$

Defining a vector bundle filtration (first attempt):



Let $(X^t)_{t \geq 0}$ be the Čech filtration of X, where the t-thickening is

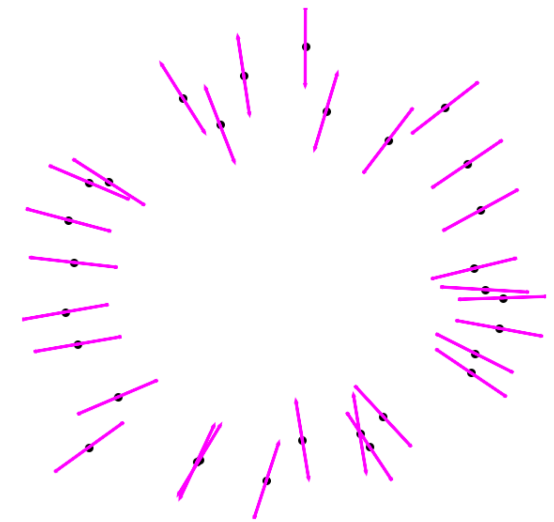
$$X^t = \left\{ y \in \mathbb{R}^n \mid \exists x \in X, \|x - y\| \leq t \right\}.$$

We want to define extended maps $\xi^t \colon X^t \to \mathcal{G}(d, \mathbb{R}^m)$.

For instance, $\xi^t(x) = \xi(\mathrm{proj}\,(x, X))$.

Problem: $x \mapsto \mathrm{proj}\,(x, X)$ is ill-defined on X^t

Sampling model for vector bundles:



Let $n, m, d > 0$ with $d \leq m$.

We observe | a point cloud $X \subset \mathbb{R}^n$
and a map $\xi \colon X \to \mathcal{G}(d, \mathbb{R}^m)$.

or, equivalently, a point cloud $\check{X} \subset \mathbb{R}^n \times \mathcal{G}(d, \mathbb{R}^m)$, through

$$\check{X} = \{(x, \xi(x)) \mid x \in X\}.$$

Defining a vector bundle filtration (second attempt):

By embedding $\mathcal{G}(d, \mathbb{R}^m) \hookrightarrow \mathcal{M}(\mathbb{R}^m)$ in matrix space, we see $\check{X}$ as a subset of $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$.

Let $(\check{X}^t)_{t \geq 0}$ be the Čech filtration of $\check{X}$ in the ambient space $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$, endowed with the metric $\|(x, A)\| = \sqrt{\|x\|_2^2 + \|A\|_{\mathrm{F}}^2}$.

We can define extended maps as
$$\begin{aligned} \xi^t \colon \check{X}^t &\longrightarrow \mathcal{G}(d, \mathbb{R}^m) \\ (x, A) &\longmapsto \operatorname{proj}\left(A, \mathcal{G}(d, \mathbb{R}^m)\right). \end{aligned}$$

The data of $(\check{X}^t)_{t \geq 0}$ and $(\xi^t \colon \check{X}^t \to \mathcal{G}(d, \mathbb{R}^m))_{t \in [0, \sqrt{2}/2)}$ is called the **Čech bundle filtration** of $\check{X}$.

Lemma: The reach of the Grassmannian $\mathcal{G}(d, \mathbb{R}^m)$, seen as a subset of $\mathcal{M}(\mathbb{R}^m)$, is $\frac{\sqrt{2}}{2}$.

That is to say, if $A \in \mathcal{M}(\mathbb{R}^m)$ is such that $\mathrm{dist}\left(A, \mathcal{G}(d, \mathbb{R}^m)\right) < \frac{\sqrt{2}}{2}$, then it admits a unique projection on $\mathcal{G}(d, \mathbb{R}^m)$.

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Persistent Stiefel-Whitney classes

Let $\check{X} \subset \mathbb{R}^n \times \mathcal{G}(d, \mathbb{R}^m)$ and $(\check{X}^t)_{t \geq 0}, (\xi^t)_{t \in [0, \sqrt{2}/2)}$ its Čech bundle filtration.

For every $t \geq 0$, $(\check{X}^t, \xi^t)$ is a vector bundle. Its i^{th} Stiefel-Whitney class can be defined as

$$w_i(\xi^t) = (\xi^t)^*(w_i),$$

where $(\xi^t)^* \colon H^*(\check{X}^t; \mathbb{Z}/2\mathbb{Z}) \leftarrow H^*(\mathcal{G}(d, \mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$.

Definition: The $\boldsymbol{i}^{\text{th}}$ **persistent Stiefel-Whitney class** is $w_i(\check{X}) = (w_i(\xi^t))_{t \leq \frac{\sqrt{2}}{2}}$.

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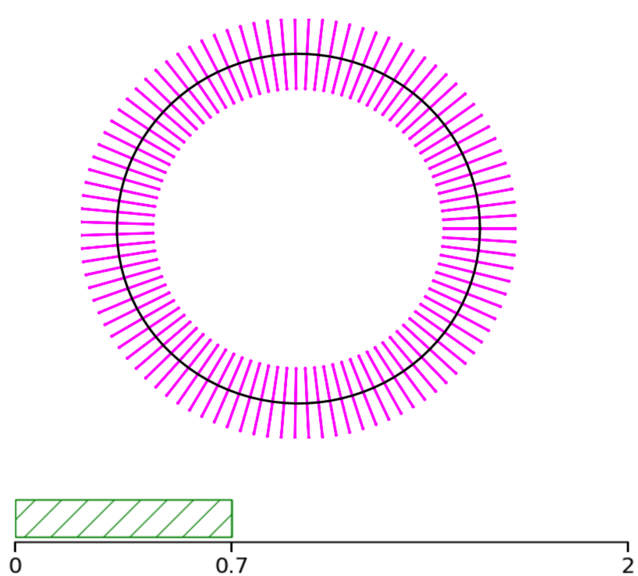
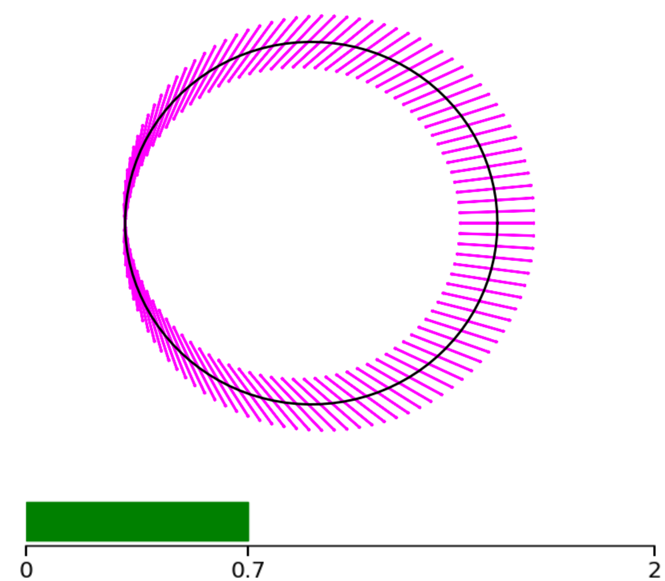
Definition: The $\boldsymbol{i}^{\text{th}}$ **persistent Stiefel-Whitney class** is $w_i(\check{X}) = (w_i(\xi^t))_{t \leq \frac{\sqrt{2}}{2}}$.

Definition: The **lifebar** of $w_i(\check{X})$ is the set $\left\{ t \in [0, \frac{\sqrt{2}}{2}) \mid w_i(\xi^t) \neq 0 \right\}$.

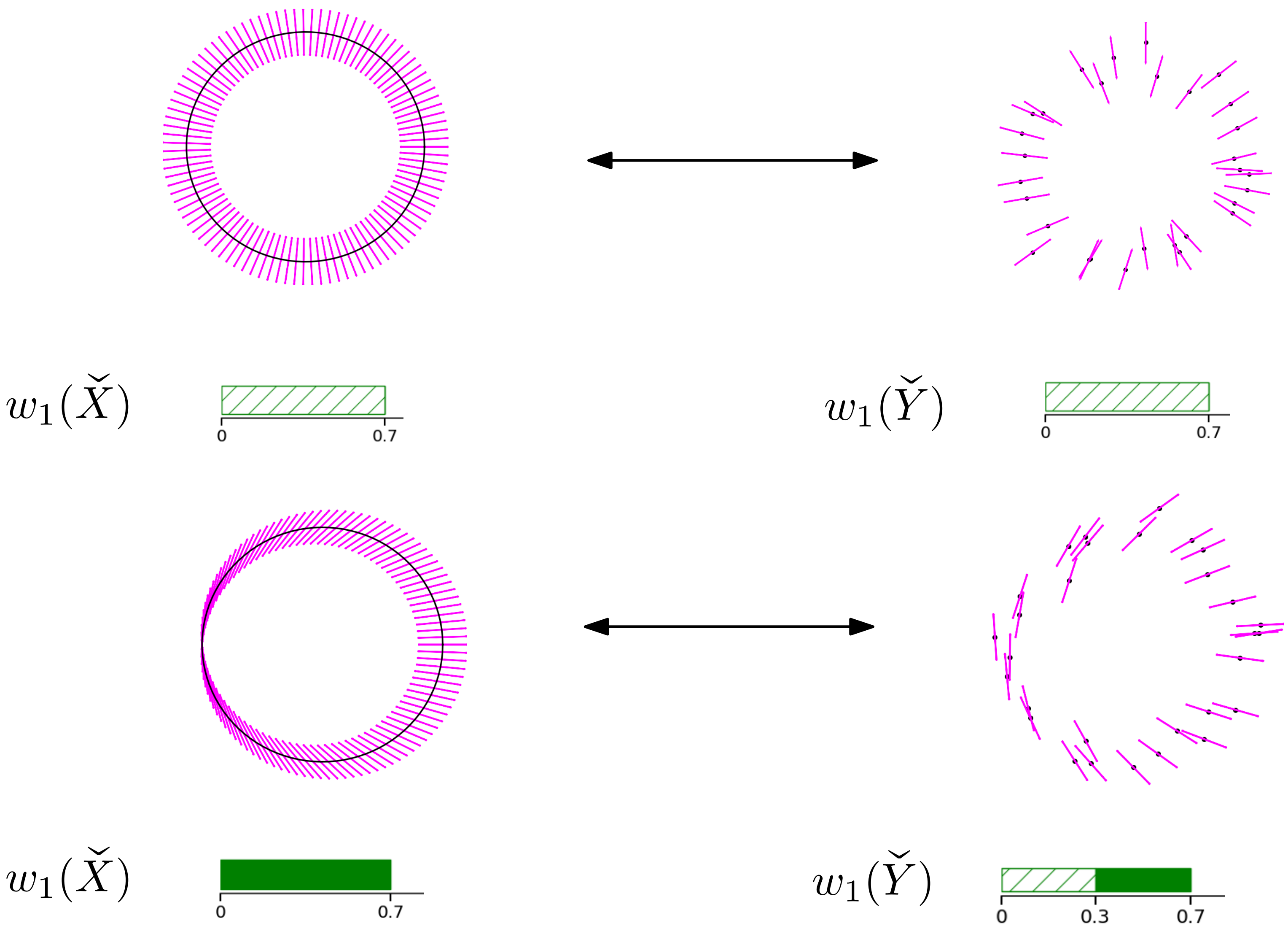
Example: Lifebars of first persistent Stiefel-Whitney classes





Stability

If two subsets $\check{X}, \check{Y} \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$ satisfies $\mathrm{d_H}(\check{X}, \check{Y}) \leq \epsilon$ (Haudorff distance), then for all $i \geq 0$, the lifebars of their i^{th} Stiefel-Whitney classes are ϵ-close.



Persistent Stiefel-Whitney classes

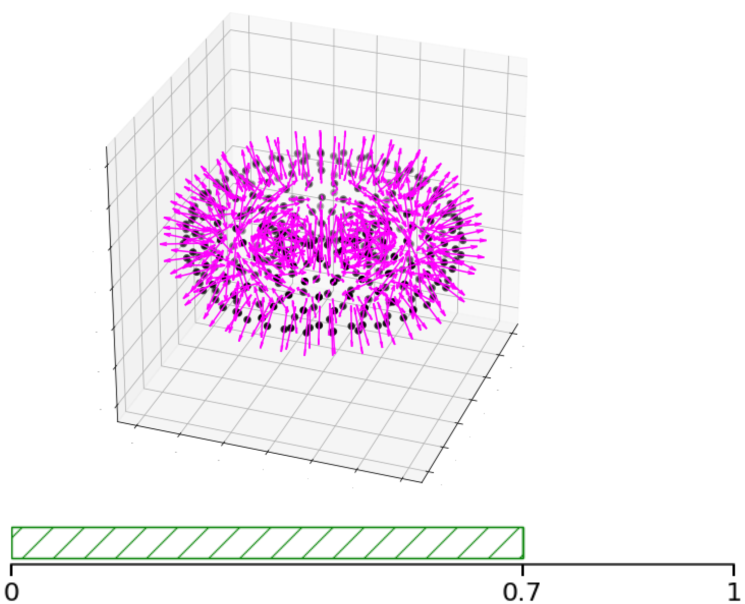
If $u\colon \mathcal{M}_0 \to \mathcal{M} \subset \mathbb{R}^n$ is an immersion and $\xi\colon \mathcal{M}_0 \to \mathcal{G}(d, \mathbb{R}^m)$ a vector bundle, consider

$$\widecheck{\mathcal{M}} = \left\{ \big(u(x_0), \xi(x_0)\big) \mid x_0 \in \mathcal{M}_0 \right\} \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m).$$

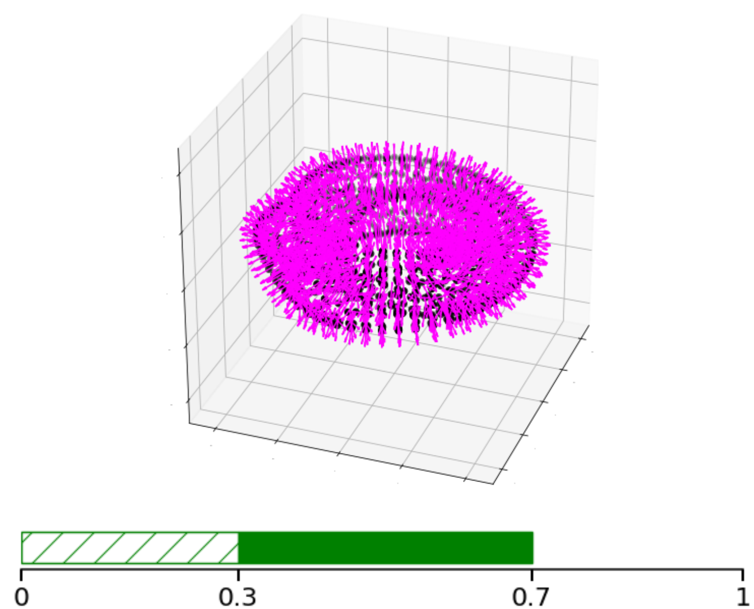
Consistency

Let $\check{X} \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$ be any subset such that $\mathrm{d_H}(X, \widecheck{\mathcal{M}}) \leq \epsilon$. Then for every $t \in [4\epsilon, \mathrm{reach}(\widecheck{\mathcal{M}}) - 3\epsilon)$, the composition of inclusions $\mathcal{M}_0 \hookrightarrow \widecheck{\mathcal{M}} \hookrightarrow X^t$ induces an isomorphism $H^*(\mathcal{M}_0) \leftarrow H^*(X^t)$ which sends the i^{th} persistent Stiefel-Whitney class $w_i^t(\check{X})$ of $\check{X}$ to the i^{th} Stiefel-Whitney class of $(\mathcal{M}_0, \xi)$.

Normal bundle of the torus

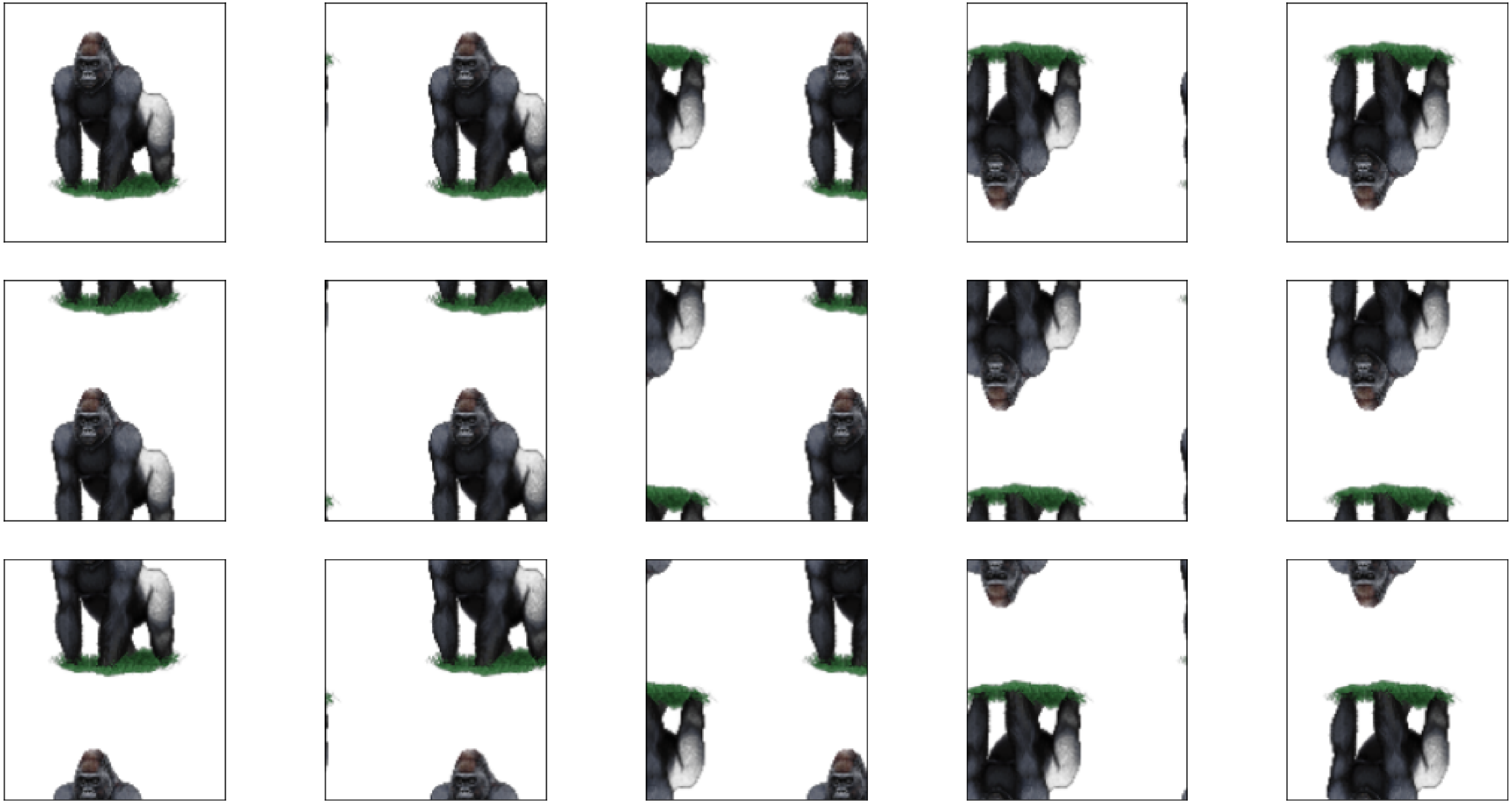


Normal bundle of the Klein bottle



Examples

We consider a collection of images of shape 130×120 pixels, representing a gorilla moving forward and downwards on a Klein bottle. This yields a point cloud in $\mathbb{R}^{130\times 120}$.



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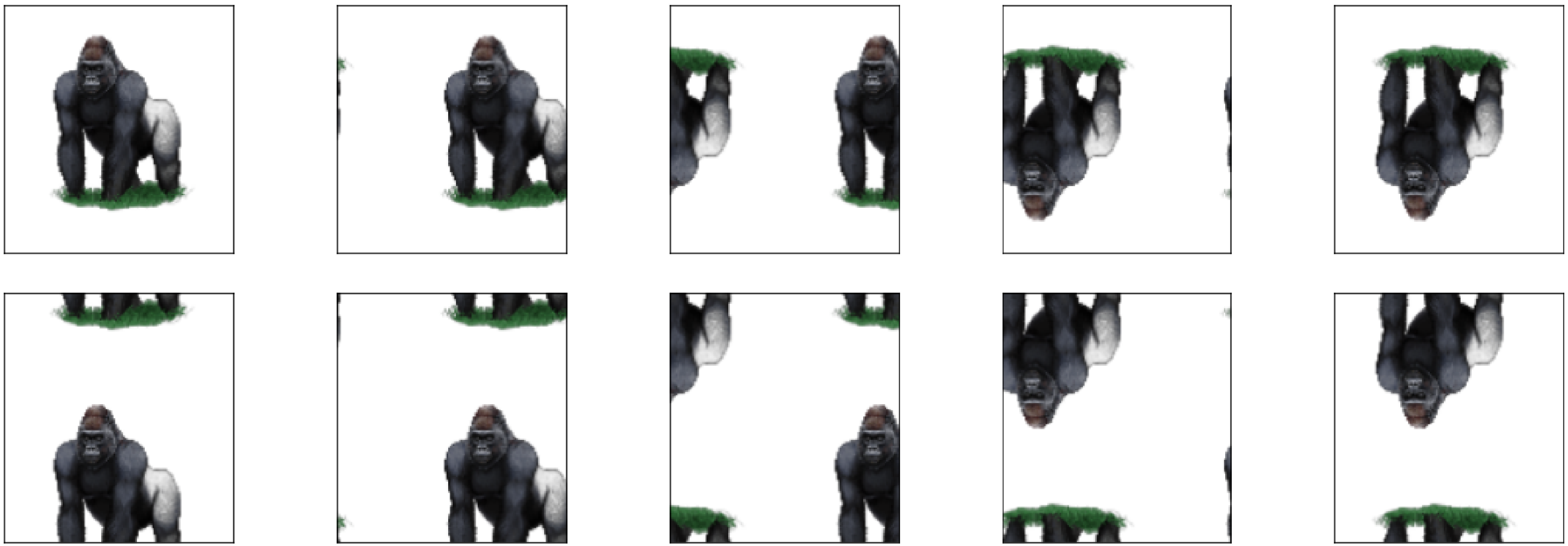
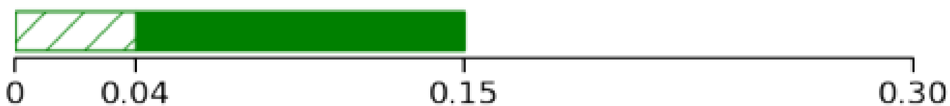
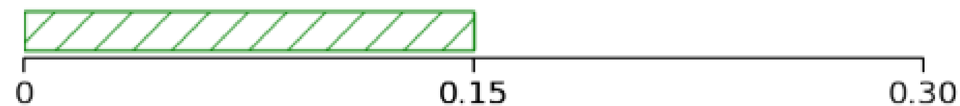


Consider the horizontal line bundle $\check{X}_{\mathrm{hor}}$ and the vertical line bundle $\check{X}_{\mathrm{vert}}$. We compute their persistent w_1.



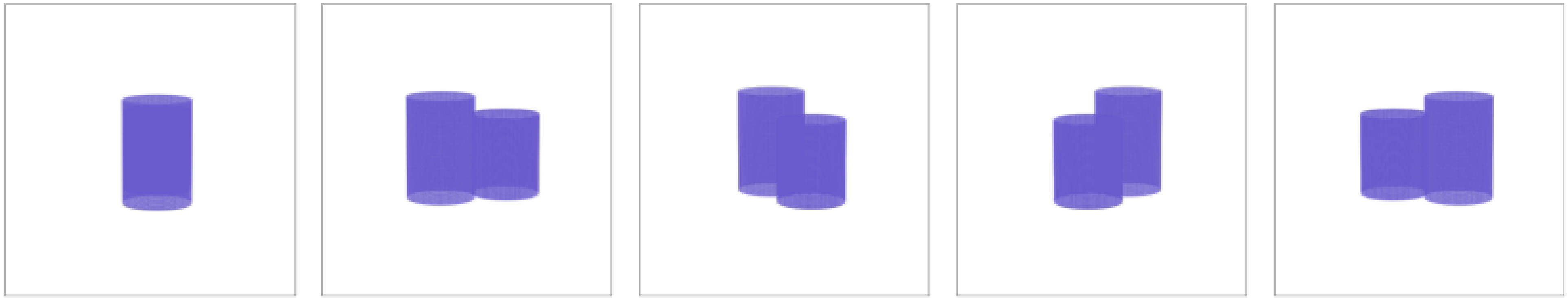
lifebar of $\check{X}_{\mathrm{hor}}$



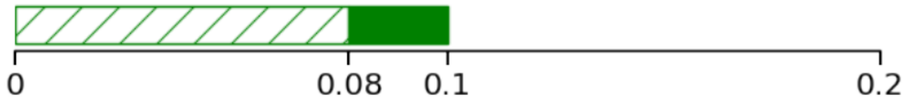
lifebar of $\check{X}_{\mathrm{vert}}$

Examples

We consider a collection of images of shape 500×500 pixels, consisting in different views of two cylinders in $\mathbb{R}^3$. This yields a point cloud in $\mathbb{R}^{500\times500}$.



Consider the tangent line bundle $\check{X}$. We compute its persistent w_1.



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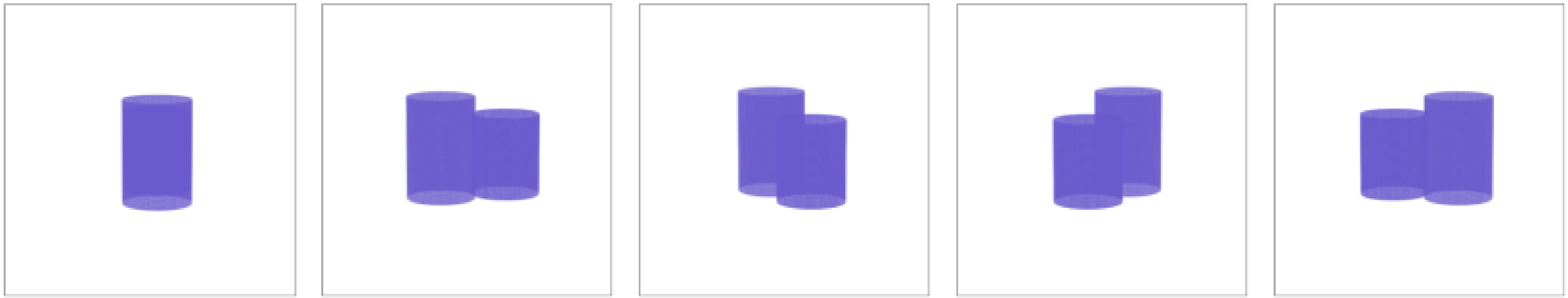

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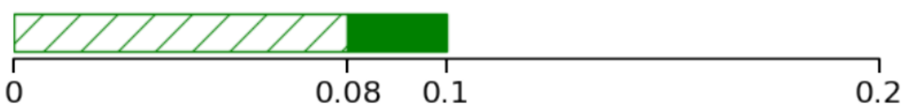
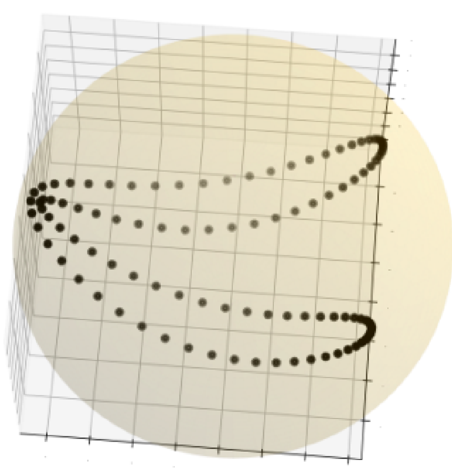
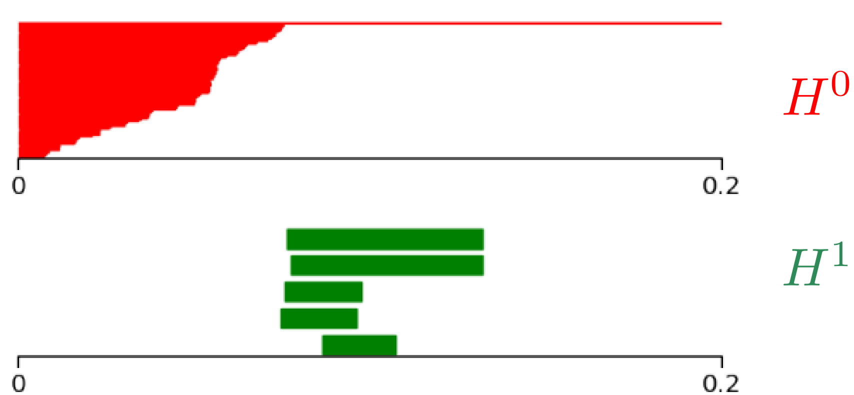


embedding in $\mathbb{R}^{500\times 500}$



(traditional) persistent homology of Čech filtration



Simplicial approximation

Let
- $\check{X} \subset \mathbb{R}^n \times \mathcal{G}(d, \mathbb{R}^m)$ or $\check{X} \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$,
- $(\check{X}^t)_t, (\xi^t)_t$ its Čech bundle filtration,
- $(w_i(\xi^t))_t$ its i^{th} persistent Stiefel-Whitney class.

$$
\begin{array}{rccc}
\xi^t: & \check{X}^t & \longrightarrow & \mathcal{G}(d, \mathbb{R}^m) \\
(\xi^t)^*: & H^*(\check{X}^t; \mathbb{Z}/2\mathbb{Z}) & \longleftarrow & H^*(\mathcal{G}(d, \mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \\
 & w_i(\xi^t) & \longleftarrow\!\shortmid & w_i
\end{array}
$$

Problem: Calculate $w_i(\xi^t)$ on a computer.

Suppose that we have triangulations K^t of $\check{X}^t$ and L of $\mathcal{G}(d, \mathbb{R}^m)$.

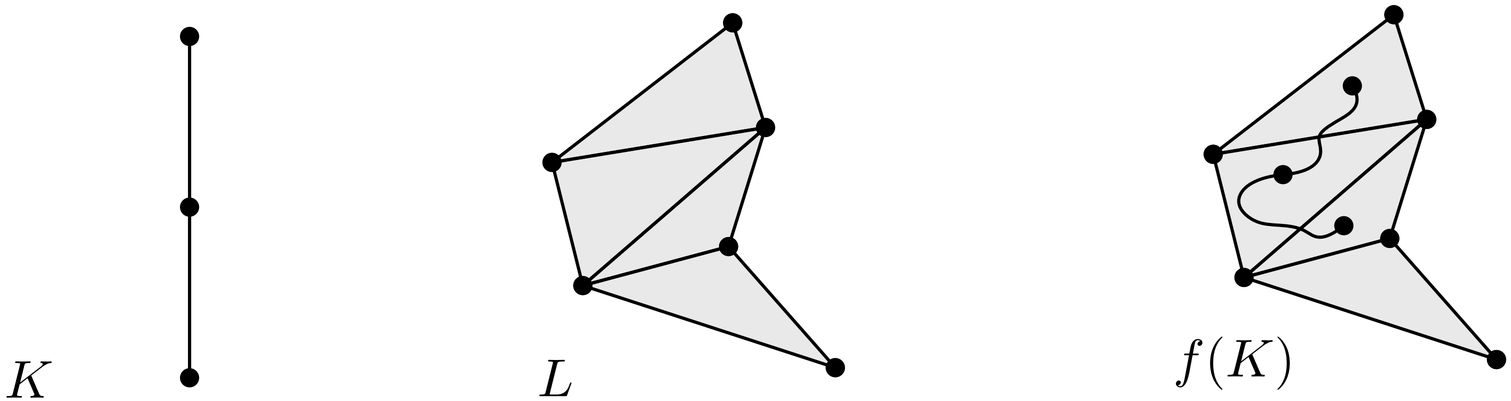
$$
\begin{tikzcd}
\check{X}^t \arrow[r, "\xi^t"] & \mathcal{G}(d, \mathbb{R}^m) \\
{|K^t|} \arrow[u, "\wr"] \arrow[r, dashed] & {|L|} \arrow[u, "\wr"]
\end{tikzcd}
$$

We look for a simplicial map $\xi^t_{\text{simp}}: K^t \to L$ that "corresponds to" ξ^t.

Input: Two simplicial complexes K, L and a continuous map $f\colon |K| \to |L|$.

Output: A simplicial map $g\colon K \to L$ such that $|g|\colon |K| \to |L|$ is homotopic to f.

Notations:
- $|K|$ denotes the geometric realization of K (it is a topological space),
- $|g|$ denotes the geometric realization of g (it is a continuous map),
- $K^{(0)}$ denotes the vertex set of K.



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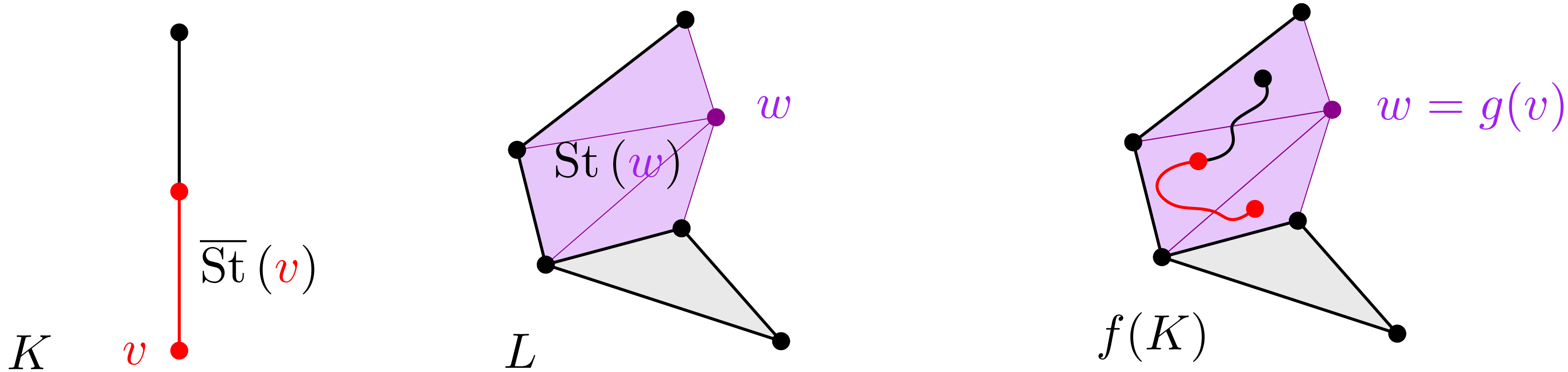


Define, for all vertex $v \in K^{(0)}$, its *open star* and its *closed star*

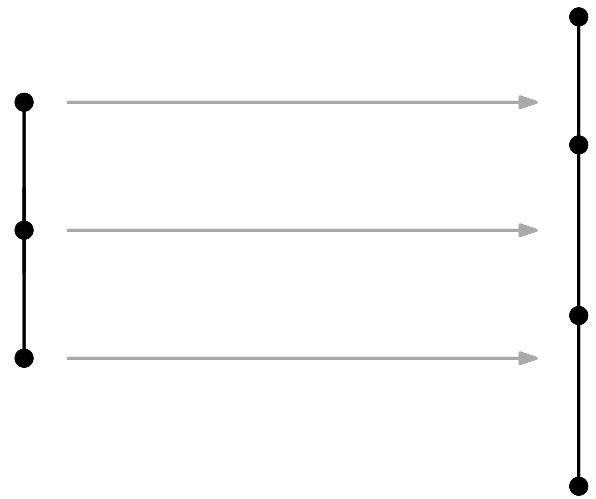
$$\mathrm{St}\,(v) = \{\sigma \in K \mid v \in \sigma\} \qquad \overline{\mathrm{St}}\,(v) = \{\tau \in K \mid \exists \sigma \in \mathrm{St}\,(v)\,,\ \tau \subset \sigma\}$$

The map f satisfies the **star condition** if $\forall v \in K^{(0)}$, $\exists w \in L^{(0)}$ s. t. $f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(w)\,|$.

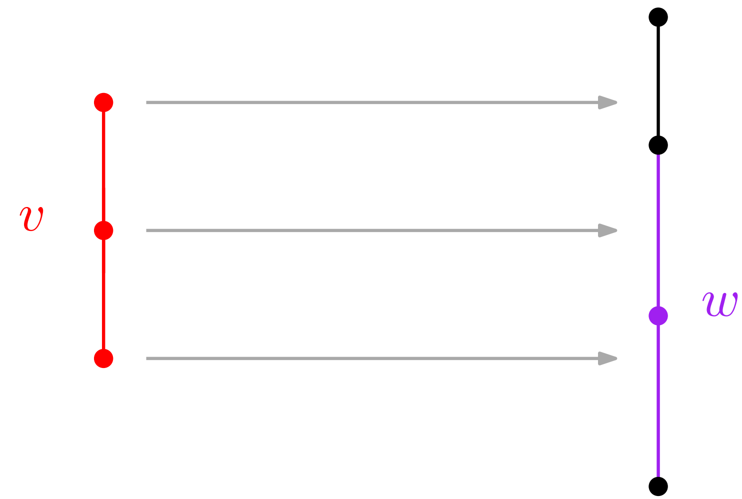
If this is the case, let $g\colon K^{(0)} \to L^{(0)}$ be such that $\forall v \in K^{(0)}$, $f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(g(v))\,|$.

Such g is called a **simplicial approximation** to f. It is simplicial and homotopic to f.

What if the map f does not satisfy the star condition?

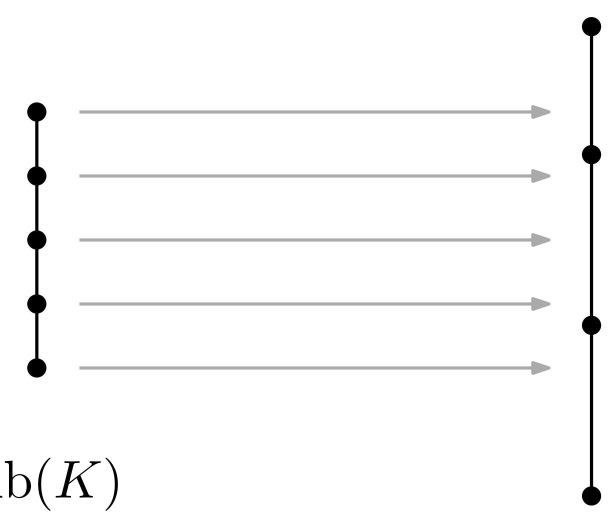


K L

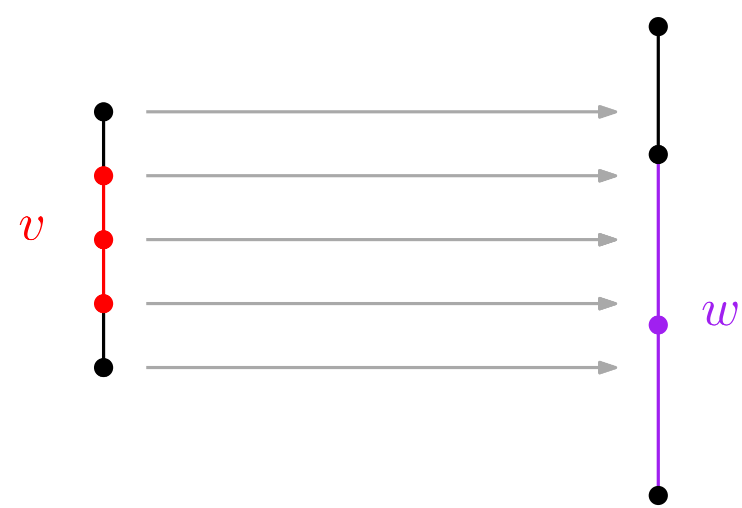


$\overline{\mathrm{St}}\,(v)$ $\mathrm{St}\,(w)$

One can refine K via *barycentric subdivision*.

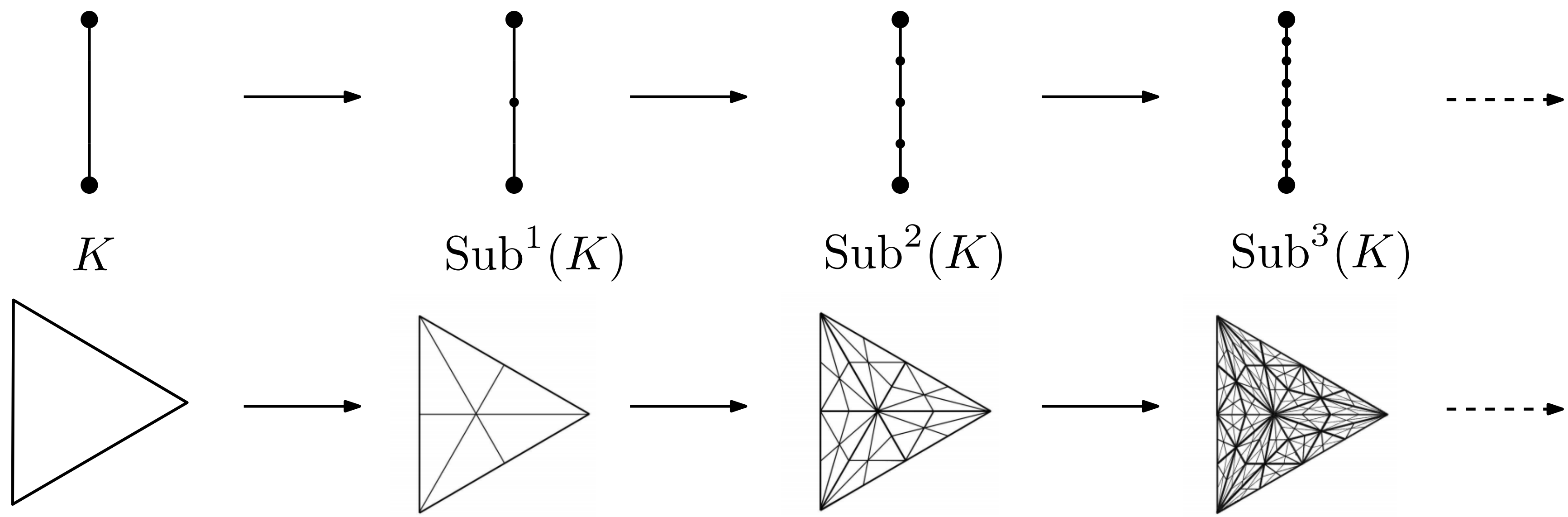


$\mathrm{Sub}(K)$



Simplicial approximation

Simplicial approximation theorem: By repeating barycentric subdivisions on K, the map $f\colon |\mathrm{Sub}^k(K)| \to |L|$ satisfies the star condition at some point.



Proof: Endow K with a metric, and by denote $\mathcal{U}$ the cover $\left\{ f^{-1}(|\mathrm{St}\,(w)\,|) \mid w \in L \right\}$ of $|K|$. A **Lebesgue number** for $\mathcal{U}$ is a $\epsilon > 0$ such that

$$\forall x \in |K|,\ \exists U \in \mathcal{U},\ \mathcal{B}(x, \epsilon) \subset U \quad \text{(open ball of radius } \epsilon\text{)}.$$

Hence, f satisfies the star condition if for every $v \in K^{(0)}$, $\mathrm{Diameter}\left(|\overline{\mathrm{St}}\,(v)\,|\right) < \epsilon$.

But barycentric subdivision reduces the diameter of a d-simplex by a factor $\frac{d}{d+1}$. Hence each simplex is small enough at some point.

Algorithm for $w_i(\xi^t)$

Let $\check{X} \subset \mathbb{R}^n \times \mathcal{G}(d,\mathbb{R}^m)$ or $\check{X} \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$,
$(\check{X}^t)_t, (\xi^t)_t$ its Čech bundle filtration,
$(w_i(\xi^t))_t$ its i^{th} persistent Stiefel-Whitney class.

$$\begin{array}{rlcl}
\xi^t\colon & \check{X}^t & \longrightarrow & \mathcal{G}(d,\mathbb{R}^m) \\
(\xi^t)^*\colon & H^*(\check{X}^t;\mathbb{Z}/2\mathbb{Z}) & \longleftarrow & H^*(\mathcal{G}(d,\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z}) \\
 & w_i(\xi^t) & \longleftarrow\!\shortmid & w_i
\end{array}$$

Problem: Calculate $w_i(\xi^t)$ on a computer.

Suppose that we have triangulations K^t of $\check{X}^t$ and L of $\mathcal{G}(d,\mathbb{R}^m)$.

(1) Compute a simplicial approximation $\xi^t_{\text{simp}}\colon K^t \to L$ to ξ^t.

(2) Identify the class w_i in $H^i(\mathcal{G}(d,\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$.

(3) Calculate the image $(\xi^t_{\text{simp}})^*(w_i)$.

Algorithm for $w_i(\xi^t)$

Let $\left| \begin{array}{l} \check{X} \subset \mathbb{R}^n \times \mathcal{G}(d, \mathbb{R}^m) \text{ or } \check{X} \subset \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m), \\ (\check{X}^t)_t, (\xi^t)_t \text{ its Čech bundle filtration,} \\ (w_i(\xi^t))_t \text{ its } i^{\text{th}} \text{ persistent Stiefel-Whitney class.} \end{array} \right.$

$$\begin{array}{rccc} \xi^t : & \check{X}^t & \longrightarrow & \mathcal{G}(d, \mathbb{R}^m) \\ (\xi^t)^* : & H^*(\check{X}^t; \mathbb{Z}/2\mathbb{Z}) & \longleftarrow & H^*(\mathcal{G}(d, \mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \\ & w_i(\xi^t) & \longleftarrow\!\shortmid & w_i \end{array}$$

Problem: Calculate $w_i(\xi^t)$ on a computer.

Suppose that we have triangulations K^t of $\check{X}^t$ and L of $\mathcal{G}(d, \mathbb{R}^m)$.

(1) Compute a simplicial approximation $\xi^t_{\text{simp}} : K^t \to L$ to ξ^t.

(2) Identify the class w_i in $H^i(\mathcal{G}(d, \mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$.

(3) Calculate the image $(\xi^t_{\text{simp}})^*(w_i)$.

$\longrightarrow$ this computes $w_i(\xi^t)$, t fixed.

Algorithm for $w_1(\breve{X})$

Recall that the lifebar of $w_i(\check{X})$ is the set $\left\{t \in [0, \frac{\sqrt{2}}{2}) \mid w_i(\xi^t) \neq 0\right\}$.

Reminder: $H^1(\mathcal{G}(d, \mathbb{R}^m); \mathbb{Z}/\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. That is,

$$w_1(\xi^t) \neq 0 \iff \operatorname{rank}\Big((\xi^t)^* \colon H^1(\check{X}^t) \leftarrow H^1(\mathcal{G}(d, \mathbb{R}^m)) \Big) > 0.$$

Let $C(\xi^t_{\mathrm{simp}})$ be the **simplicial mapping cone** of $\xi^t_{\mathrm{simp}} \colon K^t \to L$.

We have a long exact sequence

$$\ldots \longrightarrow H^k(K^t) \longrightarrow H^{k+1}(C(\xi^t_{\mathrm{simp}})) \longrightarrow H^{k+1}(L) \longrightarrow H^{k+1}(K^t) \longrightarrow \ldots$$

We deduce that

$$\operatorname{rank}((\xi^t)^*) = \sum_{k=1}^{+\infty} (-1)^k \Big(\dim H^k(K^t) - \dim H^{k+1}(C(\xi^t_{\mathrm{simp}})) + \dim H^{k+1}(L) \Big).$$

$\longrightarrow$ can be computed with the persistence algorithm

Triangulation K^t of $\breve{X}^t$

Since $\breve{X}^t$ is a thickening, it can be written as

$$\breve{X}^t = \bigcup_{(x,A)\in\breve{X}} \overline{\mathcal{B}}\left((x,A),t\right)$$

where the closed ball is taken in $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$ with metric $\|(x,A)\| = \sqrt{\|x\|_2^2 + \|A\|_{\mathrm{F}}^2}$.

It is a cover of $\breve{X}^t$ by closed convex sets, which can be triangulated via the nerve theorem.

Triangulation K^t of $\check{X}^t$

Since $\check{X}^t$ is a thickening, it can be written as

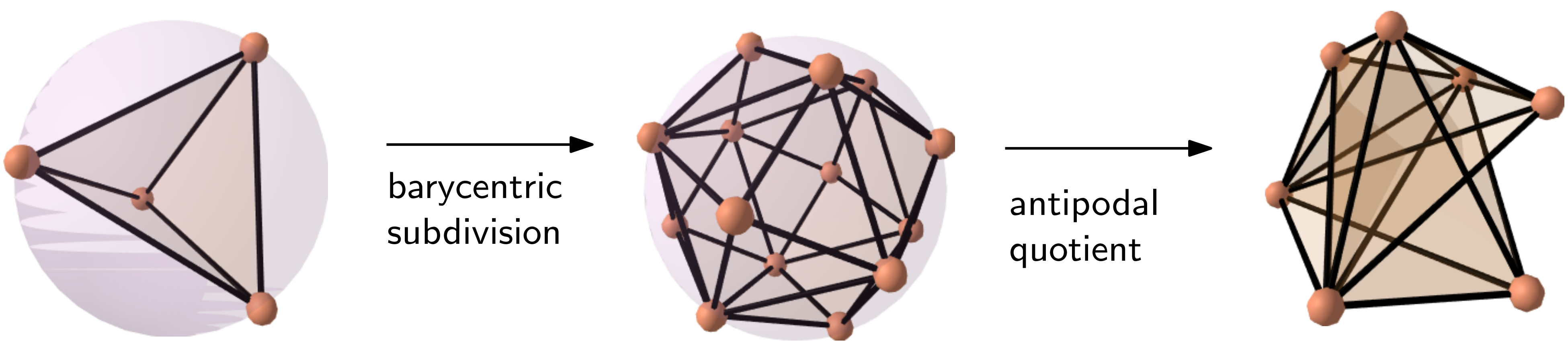
$$\check{X}^t = \bigcup_{(x,A)\in\check{X}} \overline{\mathcal{B}}\left((x,A),t\right)$$

where the closed ball is taken in $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^m)$ with metric $\|(x,A)\| = \sqrt{\|x\|_2^2 + \|A\|_{\mathrm{F}}^2}$.

It is a cover of $\check{X}^t$ by closed convex sets, which can be triangulated via the nerve theorem.

Triangulation L of $\mathcal{G}(d,\mathbb{R}^n)$

<u>What is known:</u> triangulations of the projective space $\mathcal{G}(1,\mathbb{R}^n) = \mathbb{R}P^{n-1}$. [Kühnel, 1987]



boundary of n-simplex

$\mathbb{R}P^{n-1}$

<u>What is not known:</u> triangulations of $\mathcal{G}(d,\mathbb{R}^n)$ for $1 < d < n-1$.

Simplicial methods in early algebraic topology

H. Poincaré
1854 - 1912

A. Whitehead
1861 - 1947

L. E. J. Brouwer
1881 - 1966

S. Lefschetz
1884 - 1972

J. W. Alexander
1888 - 1971

W. Hurewicz
1904 - 1956

Before Poincaré: Combinatorial relations in topological problems
Euler characteristic (1758): in a planar graph, $\#\text{Faces} - \#\text{Edges} + \#\text{Vertices} = 2$.
Riemann's inequality (1857) about algebraic curves. Betti's order of connections (1871).

1895: Poincaré's Analysis Situs
Defines fundamental group (for manifolds) and simplicial homology (for triangulated manifolds).
Conjectures: existence of triangulations, hauptvermutung, topological invariance of homology.

1910: Brouwer's simplicial approximation
Defines the degree of maps. Proves the invariance of domain ($\mathbb{R}^n \simeq \mathbb{R}^m \iff n = m$), fixed point theorem (for maps $f\colon B^n \to B^n$), Jordan–Brouwer separation (for $f\colon S^{n-1} \to \mathbb{R}^n$).

1915: Alexander generalizes the simplicial approximation
For any continuous map between simplicial complexes. Proves invariance of homology.

Hauptvermutung (equivalence of triangulations)

1920, Radó: true for manifolds of dimension 2

1950, Moise: true for manifolds of dimension 3

1961, Milnor: false for a simplicial complex of dimension ≥ 6

1969, Kirby and Siebenmann: false for certain manifolds of dimension ≥ 5

Triangulation conjecture (existence of triangulations)

1935, Cairns: true for smooth manifolds

1990, Casson: false for a topological manifold of dimension 4

2013, Manolescu: false for certain topological manifolds of dimension ≥ 5

From combinatorial topology to homotopy theory

1925: Singular homology, without triangulations
By Princeton topologists (Veblen, Alexander, Lefschetz) and Eilenberg.

1935: Hurewicz generalizes the homotopy groups
Defines homotopy equivalence. Isomorphism theorem $\pi_n(X) \to H_n(X)$ if $(n-1)$-connected.

1938: Whitehead's CW complexes
Generalization of simplicial complexes. Cellular homology.

Libraries for simplicial complexes

- algebra-oriented: `Magma`, `CHomP`, `GAP`, `Kenzo`...
- 3-dimensional manifolds: `regina`, `SnapPy`, `Twister`, ...
- Topological Data Analysis: `gudhi`, `TTK`, `ripser`, ...

Known explicit triangulations

- the surfaces,
- $\mathbb{S}^n$, the spheres, for any $n \geq 1$,
- $\mathbb{R}P^n$ and $\mathbb{C}P^n$, the real and complex projective space, for any $n \geq 1$
- $\mathrm{SO}(n)$, the special orthogonal group, only when $n \leq 4$,
- $\mathcal{V}(d, n)$, the Stiefel manifold of d-frames in $\mathbb{R}^n$, only when $n \leq 3$, when $(d, n) = (3, 4)$, or $d = 1$ (these cases correspond to the spheres),
- $\mathcal{G}(d, n)$, the Grassmann manifolds of d-planes in $\mathbb{R}^n$, only when $d = 1$ or $n - 1$ (these cases correspond to the real projective spaces).

Any top. manifold is homotopy equivalent to a CW complex [Kirby, Siebenmann, 1969]

simplicial approximation to CW complexes

Any top. manifold is homotopy equivalent to a simplicial complex

CW structures

- $\mathbb{S}^n$: one 0-cell, and one n-cell,
- $\mathbb{R}P^n$: one cell per dimension,
- $\mathcal{G}(d, \mathbb{R}^n)$: $p(k)$ cells of dimension k, where $p(k)$ is the number of partitions of k into at most d integers each of which is $\leq n - d$ (in total, $\binom{n}{d}$ cells),
- $L_p(q_1, \ldots, q_n)$ (lens space): one cell per dimension.

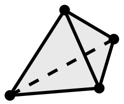
Triangulations

- $\mathbb{S}^n$: at least $n+1$ vertices, $n+1$ facets, $2^n - 1$ simplices.
- $\mathbb{R}P^n$: at least $\frac{(n+1)(n+2)}{2}$ vertices. [Arnoux, Marin, 1991]
  We know a triangulation with $2^{n+1} - 1$ vertices. [Kühnel, 1987]
- $\mathcal{G}(d, \mathbb{R}^n)$: at least $\frac{n(n+1)}{2}$ vertices and $\big(n(n-1) - 2d(n-d)\big)2^{d(n-d)+1} - 1$ simplices.
  [Govc, Marzantowicz, Pavešić, 2020]

Edgewise sudivisions

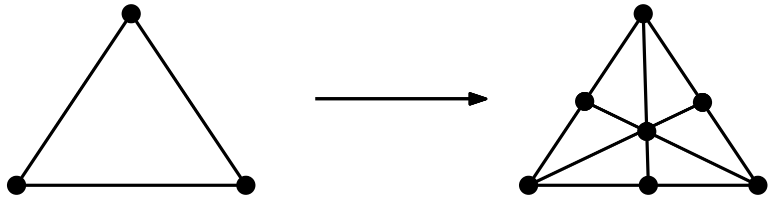
Barycentric subdivisions increase the number of simplices drastically:
a d-simplex turns into a simplicial complex with $(d+1)!$ simplices and $2^{d+1}-1$ vertices.

Example: Triangulation of the unit sphere $\mathbb{S}^2$, starting from the boundary of the 3-simplex



	K	$\mathrm{Sub}^1(K)$	$\mathrm{Sub}^2(K)$	$\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	$\mathrm{Sub}^5(K)$
vertices:	4	14	74	434	2594	15554
simplices:	14	74	434	2594	15554	93314
max diameter:	1.63	1.15	0.66	0.42	0.25	0.15

Barycentric subdivision:



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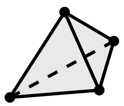

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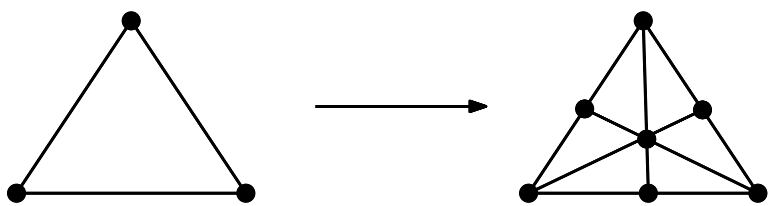
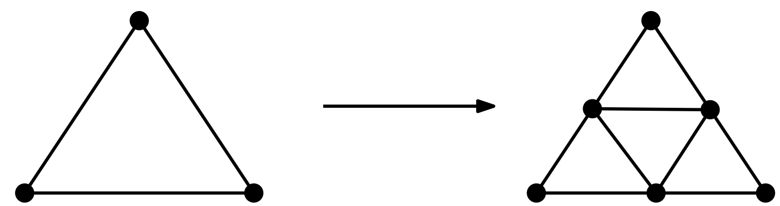
Barycentric subdivision:

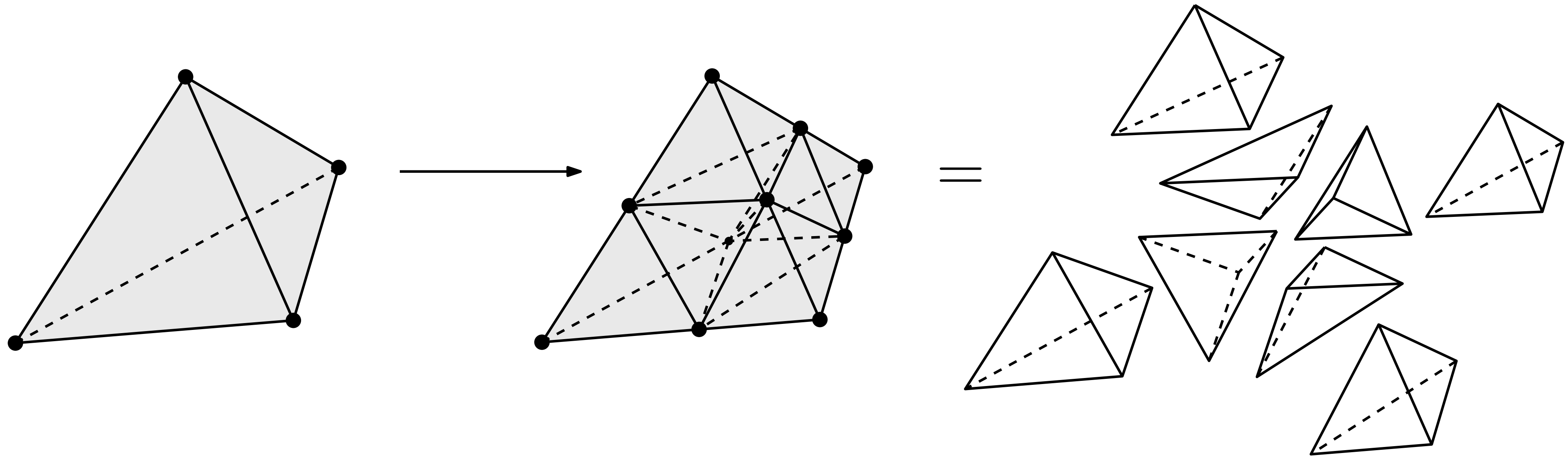


We can do better with **edgewise subdivisions**:



	K	$\mathrm{Sub}^1(K)$	$\mathrm{Sub}^2(K)$	$\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	$\mathrm{Sub}^5(K)$
vertices:	4	10	39	130	514	2050
simplices:	14	50	194	770	3074	12290
max diameter:	1.63	1.41	1	0.58	0.30	0.15

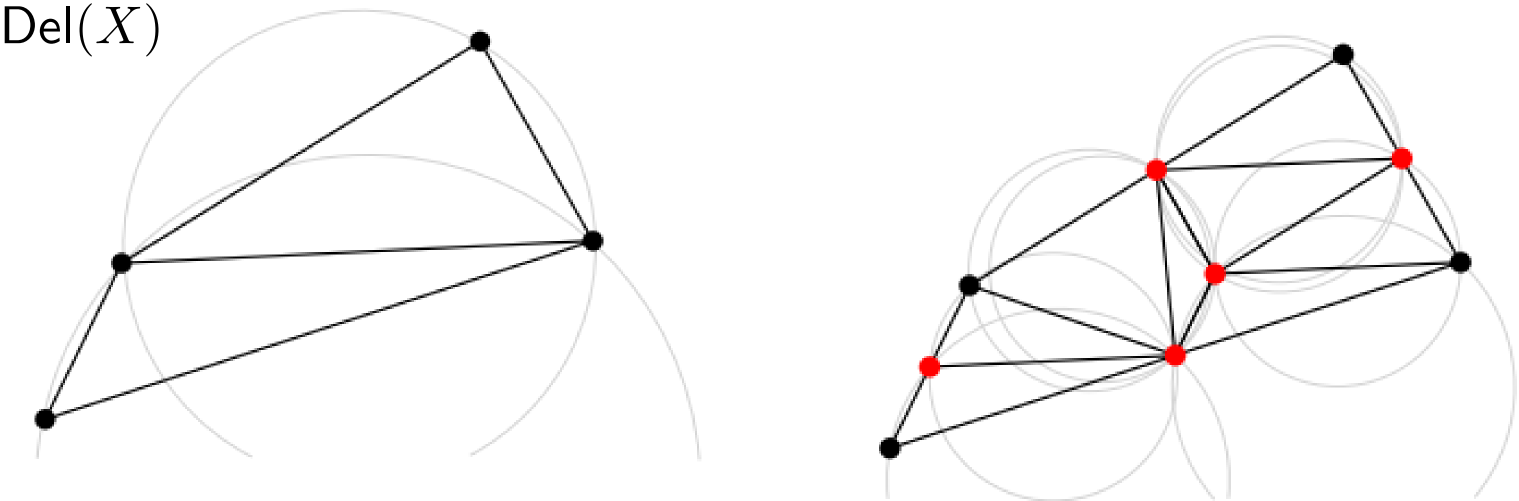
Edgewise (Coxeter-Freudenthal-Kuhn) subdivision can also be defined for n-simplices:



Example: Triangulation of the sphere $\mathbb{S}^3$, starting from the boundary of the 4-simplex.

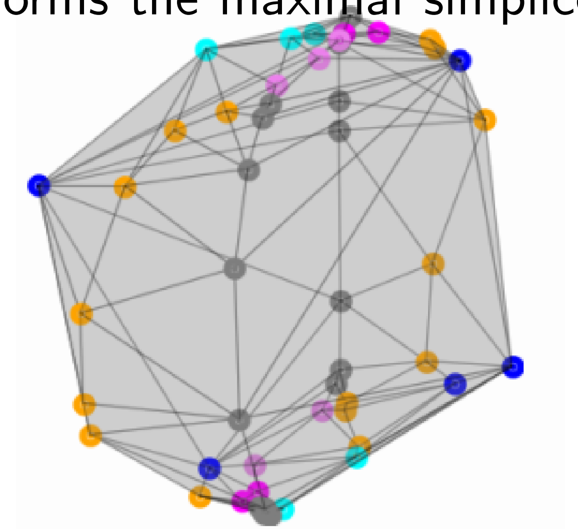
	barycentric: $\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	edgewise: $\mathrm{Sub}^5(K)$	$\mathrm{Sub}^6(K)$
vertices:	$12'600$	$301'680$	$27'440$	$218'720$
simplices:	$301'680$	$7'238'880$	$710'240$	$5'680'320$
max diameter:	0.54	0.36	0.47	0.29

Delaunay triangulation in Euclidean space. Let $X \subset \mathbb{R}^n$ finite.
A subset of $n+1$ points of X has the *empty circle property* if it has a circumscribing open ball empty of points of X. Their collection forms the maximal simplices of the *Delaunay complex* $\mathrm{Del}(X)$



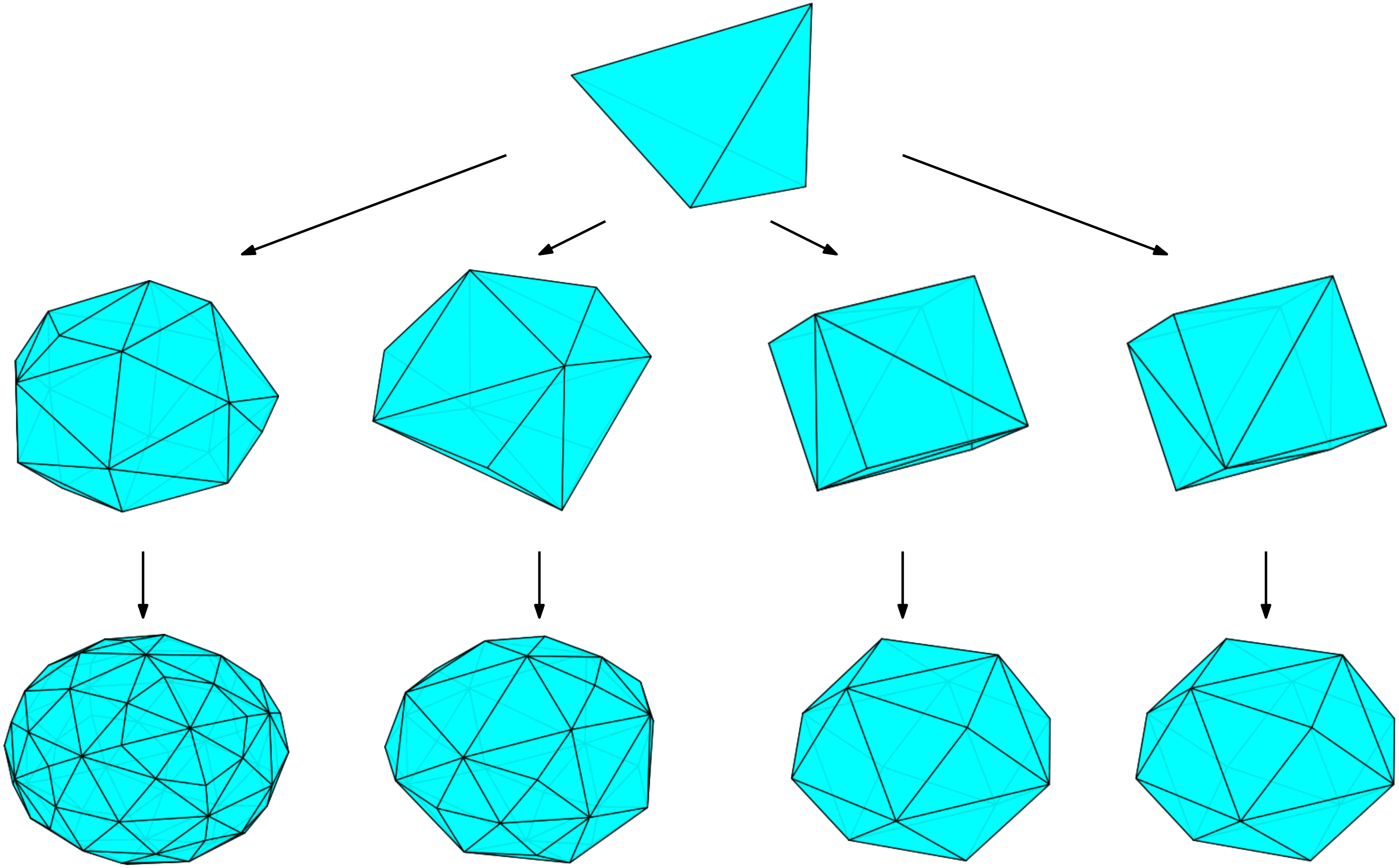
Delaunay triangulation in sphere. Let $X \subset \mathbb{S}^n$ finite.
A subset of $n+1$ points of X has the *empty circle property* if it has a circumscribing open *geodesic* ball empty of points of X. Their collection forms the maximal simplices of the *spherical Delaunay complex* $\mathrm{Del}(X)$

Proposition: On the sphere, $\mathrm{Del}(X)$ is the convex hull of X.



Delaunay subdivisions

One can "subdivide" Delaunay triangulations by adding new vertices.



add **barycenters** of all simplices

add **midpoints** of edges

add **minicenters** (center of smallest enclosing spheres)

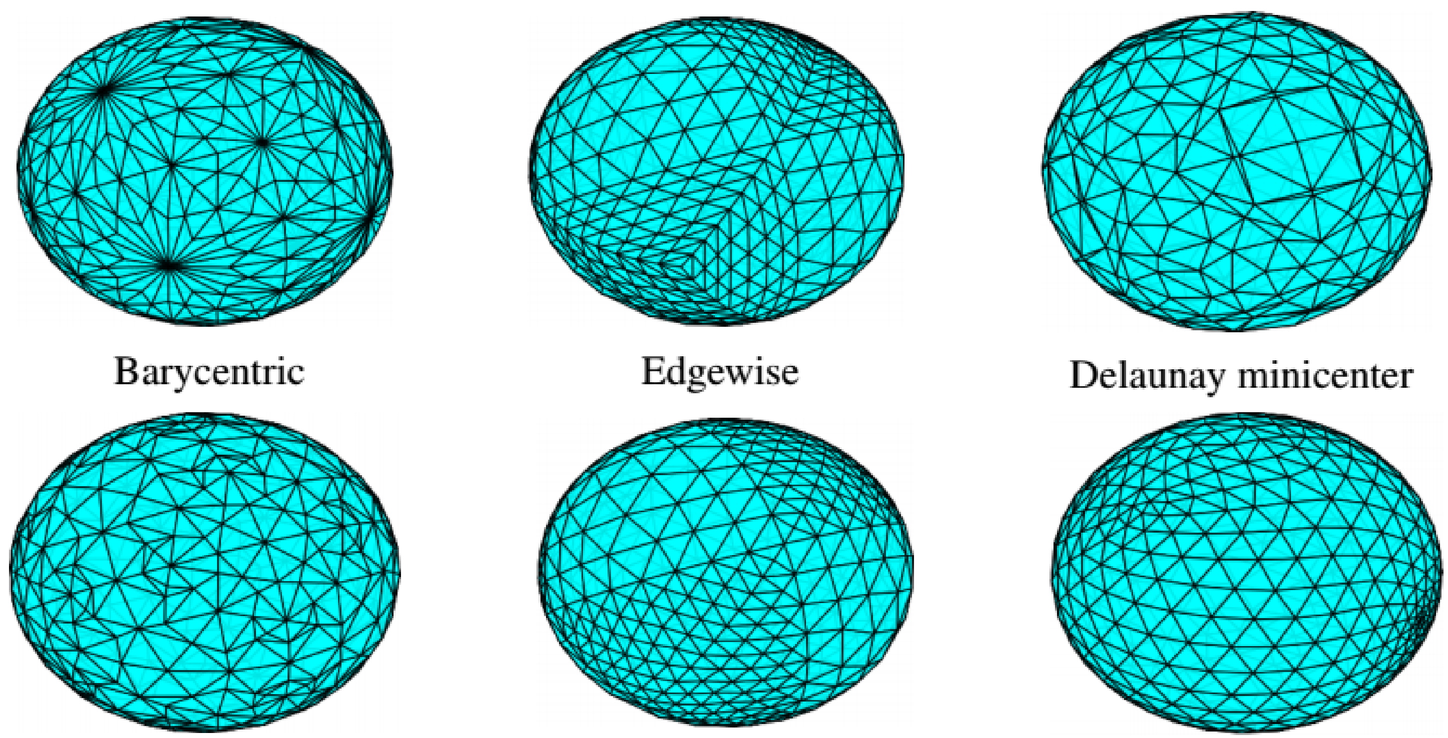
add **centroid** of maximal simplices

Delaunay subdivisions

Number of subdivisions (and vertices) for simplicial approximation to $|\partial\Delta^3| \to |\mathrm{sub}^n(\partial\Delta^3)|$

	$\partial\Delta^3$	$\mathrm{sub}^1(\partial\Delta^3)$	$\mathrm{sub}^2(\partial\Delta^3)$	$\mathrm{sub}^3(\partial\Delta^3)$
Barycentric	2 (74)	3 (434)	4 (2594)	5 (15554)
Edgewise	2 (34)	4 (514)	5 (2050)	7 (32770)
Delaunay barycentric	2 (74)	3 (434)	5 (15554)	6 (93314)
Delaunay edgewise	2 (34)	4 (514)	5 (2050)	7 (32770)
Delaunay minicenter	3 (38)	5 (326)	7 (2918)	9 (26246)
Delaunay centroid	3 (56)	5 (488)	6 (1460)	8 (13124)

Resulting complexes for $|\partial\Delta^3| \to |\mathrm{sub}^1(\partial\Delta^3)|$.

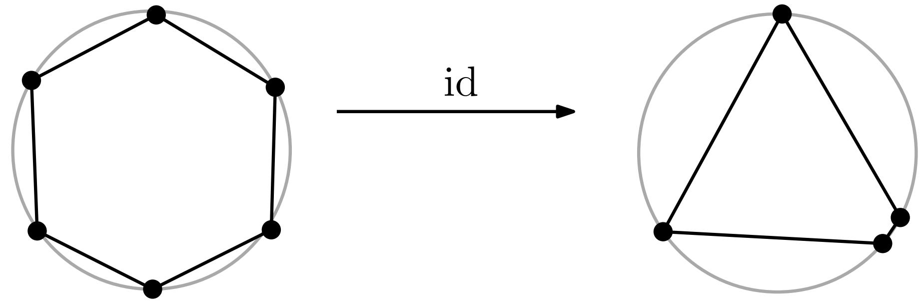


Barycentric | Edgewise | Delaunay minicenter

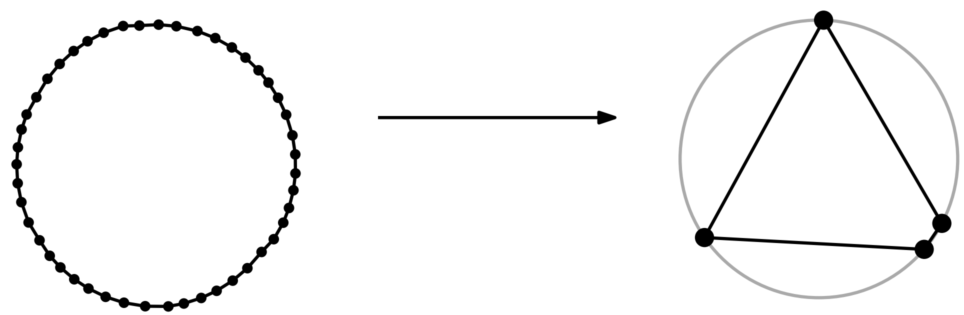
Delaunay barycentric | Delaunay edgewise | Delaunay centroid

Three tricks for simplicial approximation

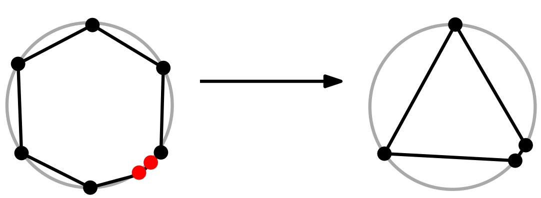
Suppose we want to find a simplicial approximation to the identity on $\mathbb{S}^1$, with the triangulations



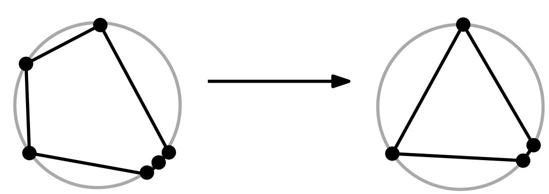
We must subdivise the domain many times to ensure that id satisfies the star condition.



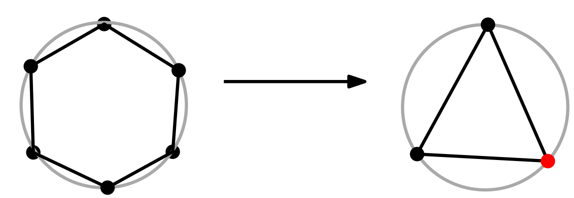
We could have done simpler:



generalized subdivision



Delaunay simplifications



edge contractions

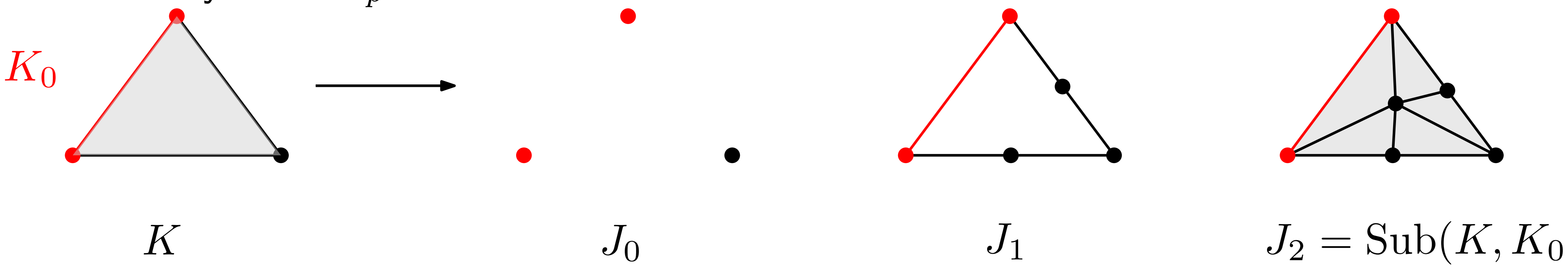
Three tricks for simplicial approximation

Generalized subdivisions

Let K be a simplicial complex, and $K_0 \subset K$ a sub-complex.

The **barycentric subdivision of K holding K_0 fixed**, denoted $\mathrm{Sub}(K, K_0)$, is defined by induction. We define simplicial complexes of increasing dimension $J_0, \dots, J_d$ as follows:

- Start with J_0 the 0-skeleton of K
- From J_{p-1}, build J_p by adding all the p-simplices of K_0
- Moreover, for any p-simplex of K not in K_0, add a point $\hat{\sigma}$, and cone it to the boundary $\partial\sigma \in J_p$



To find a simplicial approximation to $f\colon |K| \to |L|$, it is enough to execute repeated barycentric subdivisions of K holding K_0 fixed, where K_0 is the subcomplex *on which f satisfies the star condition*.

This process can be adapted to edgewise and Delaunay subdivisions.

Delaunay simplifications

When working with Delaunay complexes, we can define a post-processing step.
Given a simplicial map $g\colon \mathrm{Del}\,(X) \to L$, we say a vertex $v \in X$ satisfies the **simplex condition** if

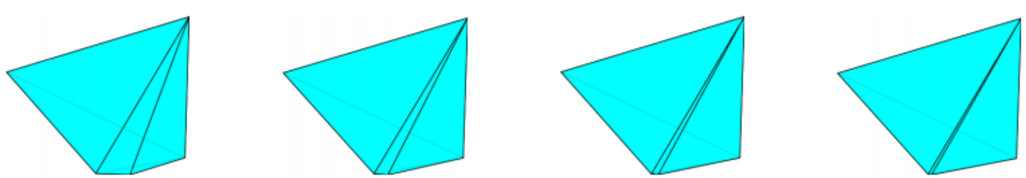
$$g\left(\overline{\mathrm{St}}\,(v)^{\,0}\right) \in L.$$

We can define a map between vertex sets $g'\colon \mathrm{Del}\,(X \setminus \{v\})^{\,0} \to L^{\,0}$ by restricting g to $X \setminus \{v\}$.

Lemma: If v satisfies the simplex condition, then g' is a simplicial map, homotopic to g.

Proof: Removing a vertex v from $\mathrm{Del}\,(X)$ only changes its structure in the open star $\mathrm{St}\,(v)$.

Therefore, we can remove the vertices incrementally. We indicate the number of vertices after Delaunay simplifications (before between parenthesis).



	$L_{0.25}$	$L_{0.1}$	$L_{0.05}$	$L_{0.01}$
Delaunay barycentric	10 (162)	8 (201)	12 (256)	11 (315)
Delaunay edgewise	7 (85)	8 (94)	9 (123)	8 (142)
Delaunay minicenter	8 (82)	8 (105)	14 (141)	11 (163)
Delaunay centroid	8 (102)	10 (119)	8 (140)	11 (152)

Edge contractions

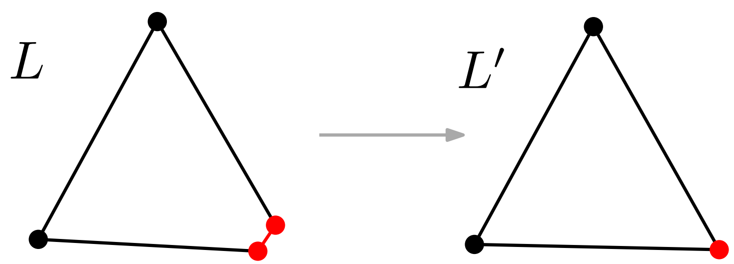
Let $[a, b]$ be an edge of L, and $c \notin L^{(0)}$ a new vertex. Define the quotient map as

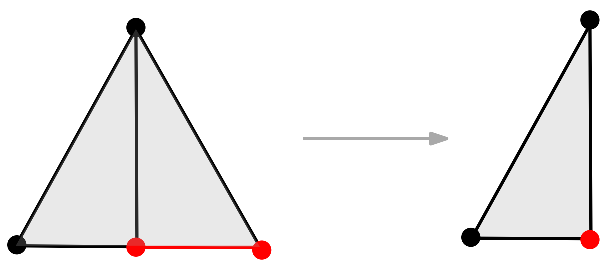
$$p\colon L^{(0)} \longrightarrow \left(L^{(0)} \setminus \{a, b\}\right) \sqcup \{c\}$$
$$x \longmapsto \begin{cases} c & \text{if } x = a \text{ or } b \\ x & \text{else.} \end{cases}$$

The **contracted complex** is defined as

$$L' = \{p(\sigma),\ \sigma \in L\}.$$

We have a surjective simplicial map $p\colon L \to L'$.

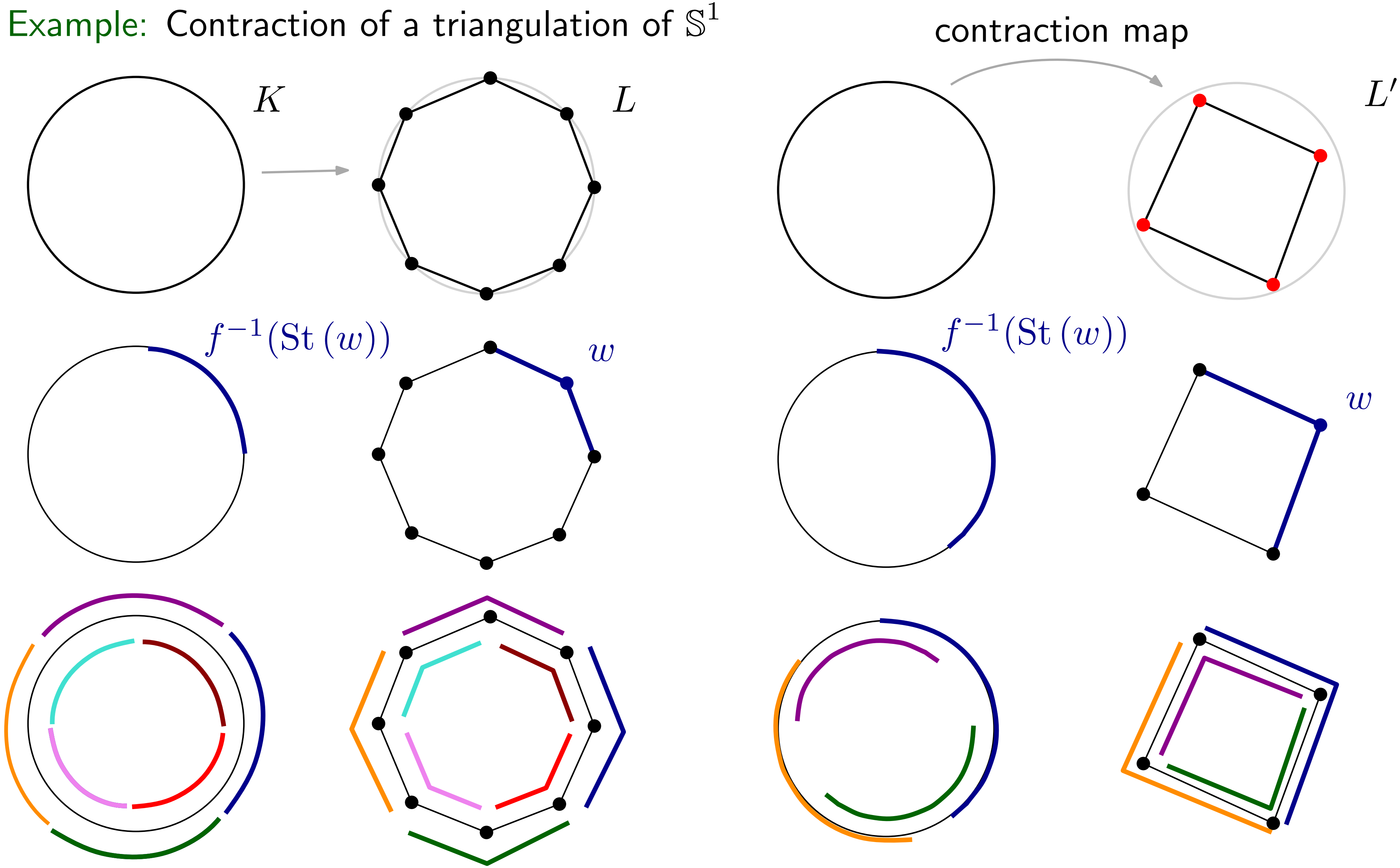




Theorem [Dey, Edelsbrunner, Guha, Nekhayev, 1998], [Attali, Lieutier, Salinas, 2012]:
The map p is a homotopy equivalence when the edge $[a, b]$ satisfies the *link condition*, that is, $\mathrm{Lk}(ab) = \mathrm{Lk}(a) \cap \mathrm{Lk}(b)$.

Contracting L can help the simplicial approximation to $f\colon |K| \to |L|$.

Example: Contraction of a triangulation of $\mathbb{S}^1$



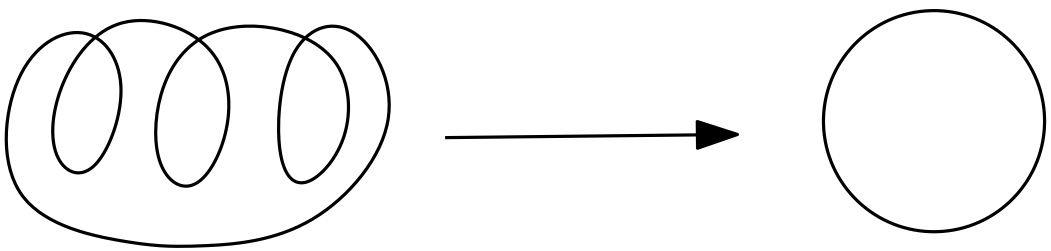
Experiment

Gromov showed in the 70's that for a continuous map $f\colon \mathbb{S}^d \to \mathbb{S}^d$, the degree and Lipschitz constant satisfies

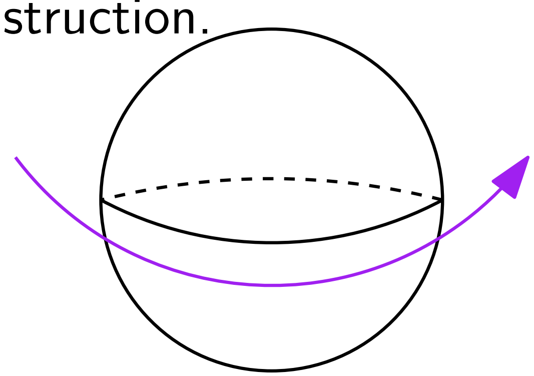
$$\mathsf{Deg}(f) \leq \mathsf{Lip}(f)^d,$$

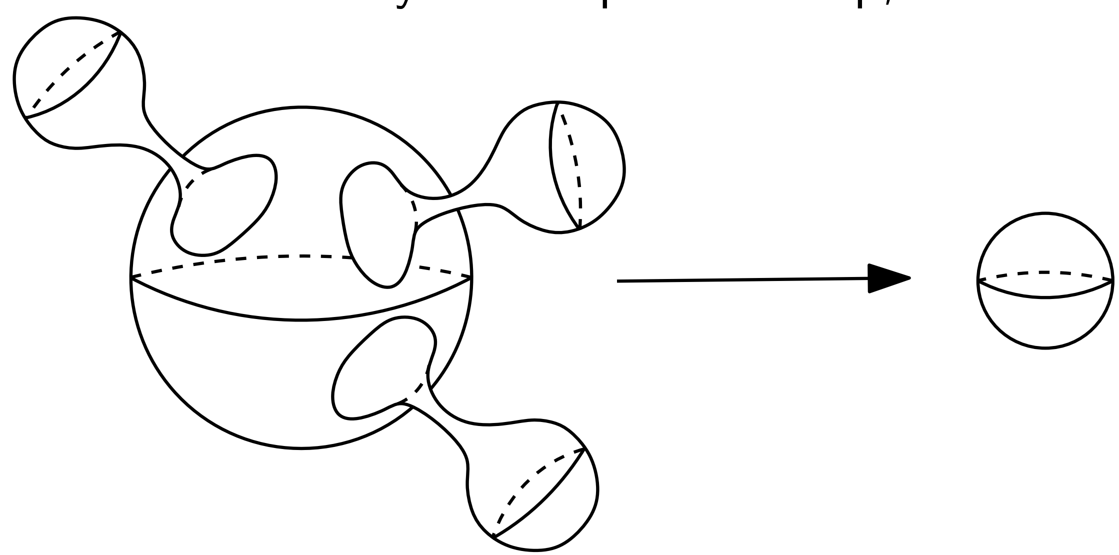
and the bound is sharp.

In dimension 1, $\mathsf{Deg}(f) \simeq \mathsf{Lip}(f)$ is obtained by turning around the circle:



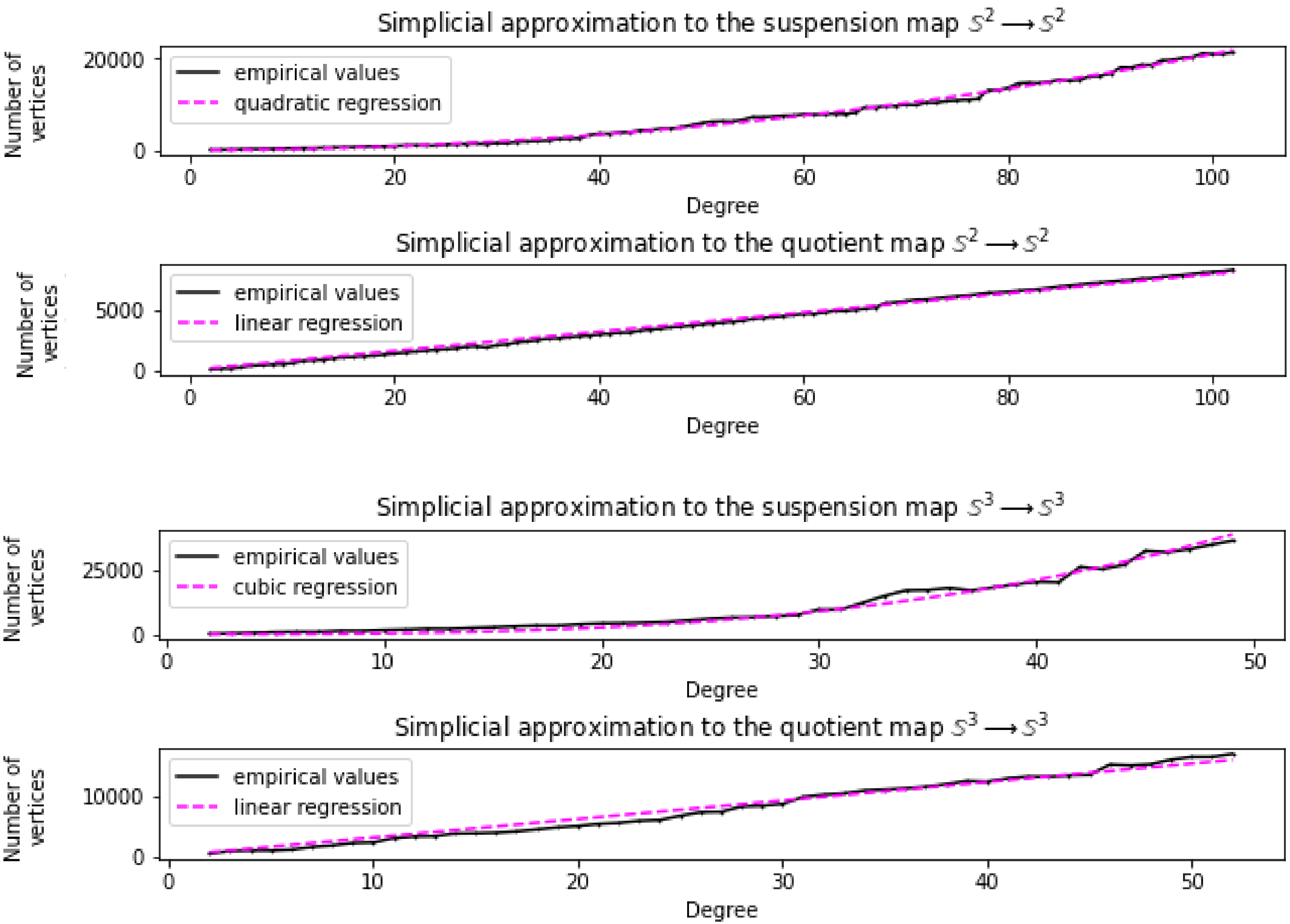
In dimension 2, $\mathsf{Deg}(f) \simeq \mathsf{Lip}(f)^2$ is not obtained by the suspension map, but another construction.





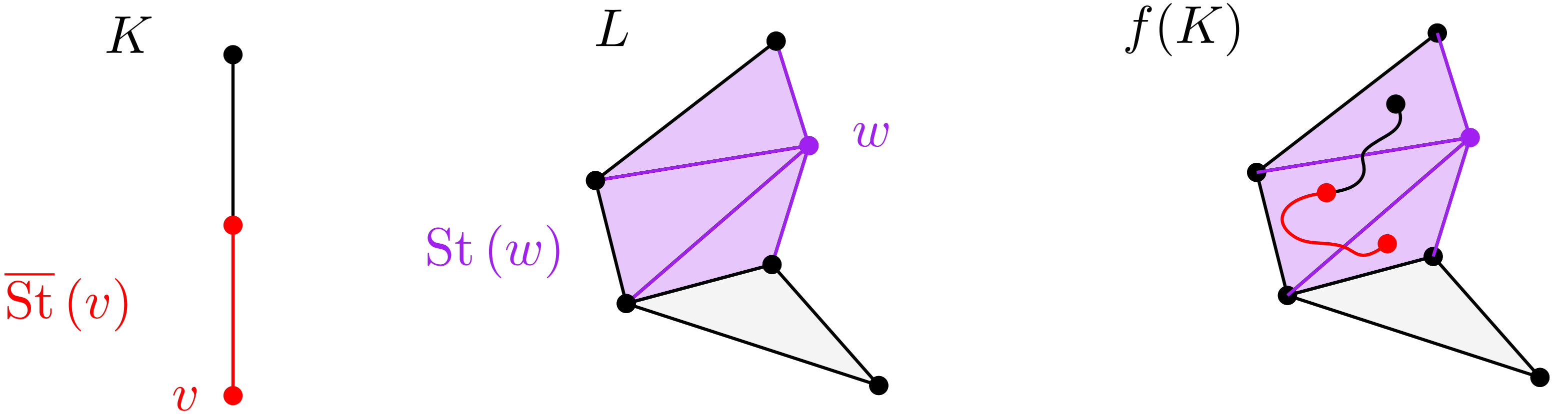
After subdividing K to get a simplicial approximation, we expect the number of vertices $n \simeq \mathsf{Lip}(f)^d$. So, at least $n \gtrsim \mathsf{Deg}(f)$.

Experiment



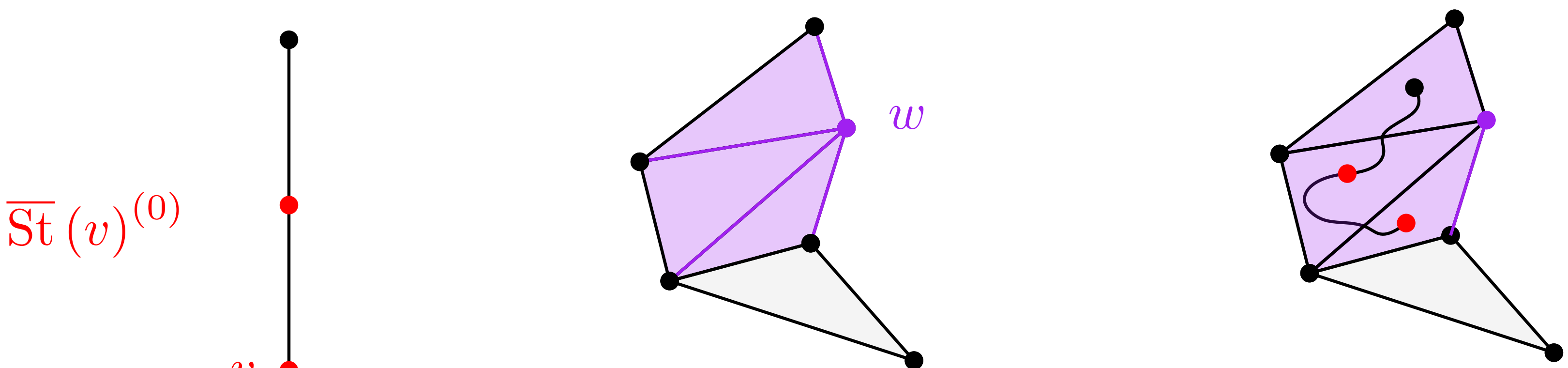
After subdividing K to get a simplicial approximation, we expect the number of vertices $n \simeq \mathrm{Lip}(f)^d$. So, at least $n \gtrsim \mathrm{Deg}(f)$.

In practice, we cannot check whether a simplicial map $f\colon |K| \to |L|$ satisfies the star condition.



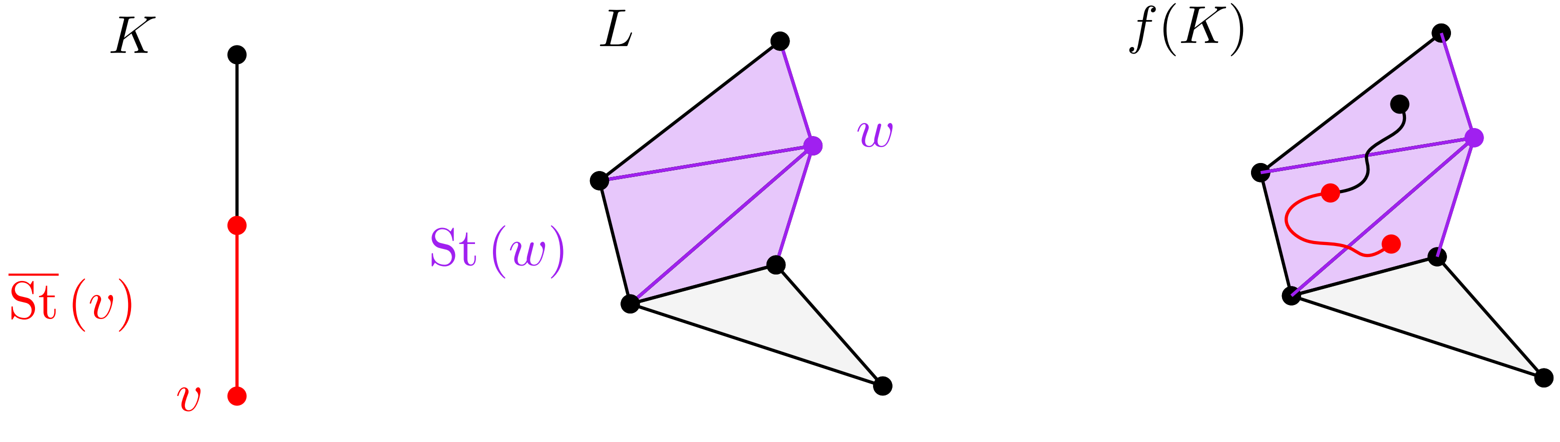
$$f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(w)\,|?$$

Definition: The map f satisfies the **weak star condition** if for every vertex $v \in K^{(0)}$, there exists a $w \in L^{(0)}$ such that $f\left(\overline{\mathrm{St}}\,(v)^{(0)}\right) \subseteq |\mathrm{St}\,(w)\,|$.



Weak star condition

In practice, we cannot check whether a simplicial map $f\colon |K| \to |L|$ satisfies the star condition.



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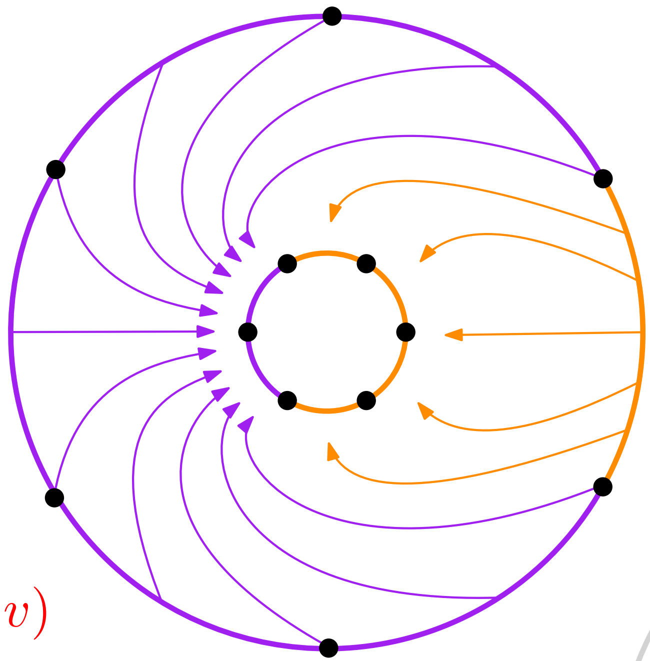
If this is the case, let $g\colon K^{(0)} \to L^{(0)}$ be any map such that $f\left((\overline{\mathrm{St}}\,(v))^{(0)}\right) \subseteq |\mathrm{St}\,(g(v))\,|$ for all $v \in K^{(0)}$.

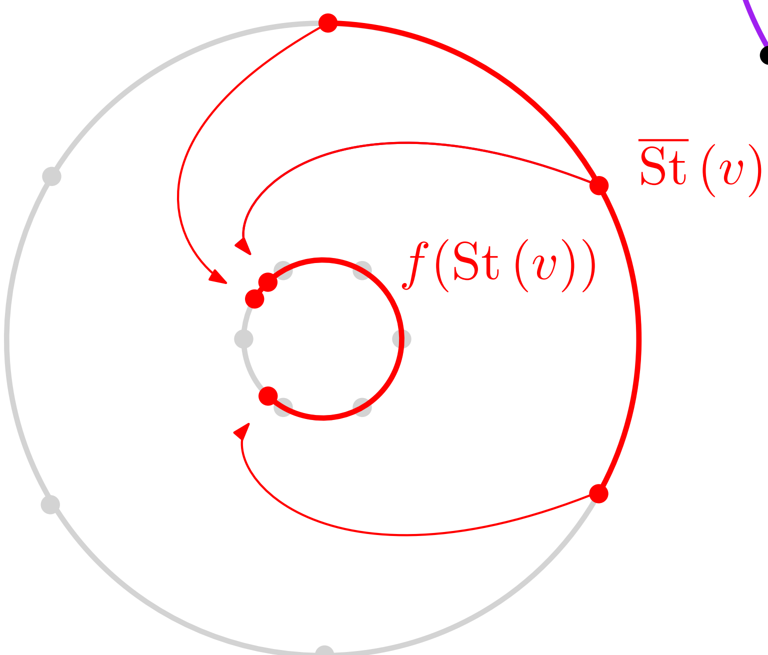
Such a map g is called a **weak simplicial approximation** to f. It is a simplicial map.

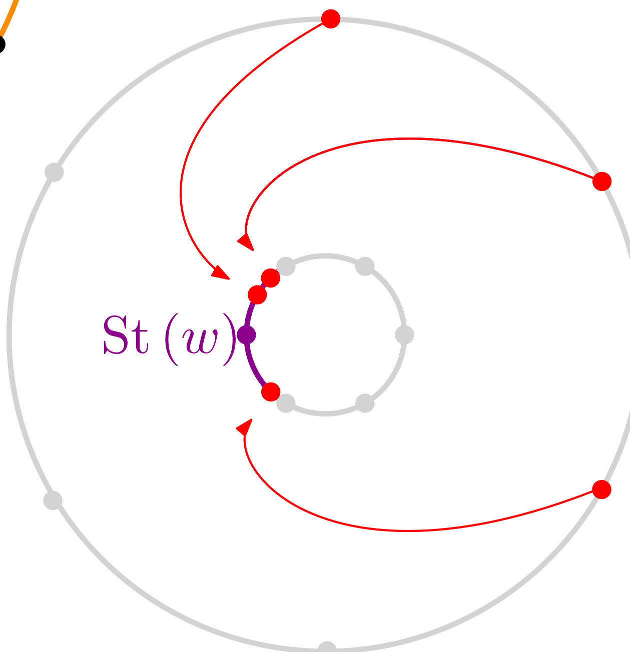
Fact: If K is subdivised enough, then any weak simplicial approximation is a simplicial approximation.

Inconsistency of weak simplicial approximation

Consider the map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$:







f does not satisfy the star condition at v

f satisfies the star weak condition at v

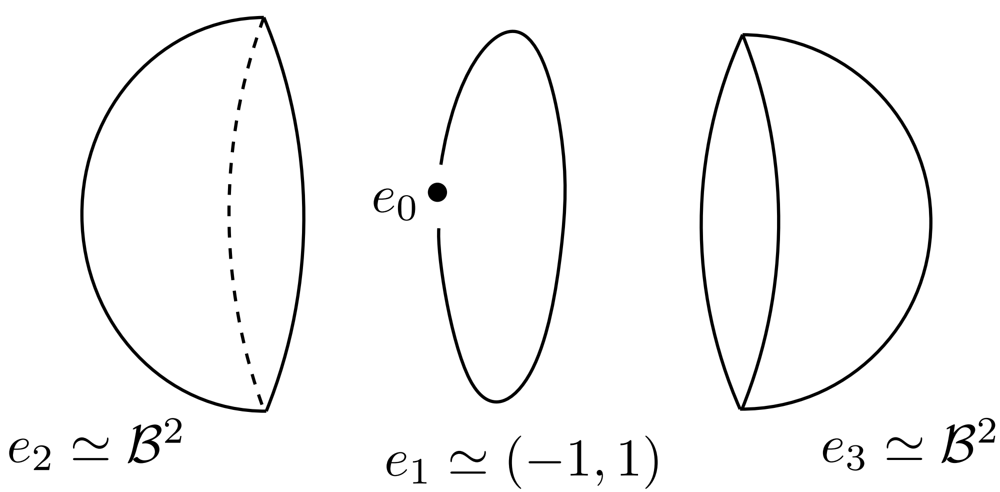
$\longrightarrow$ One finds a constant weak simplicial approximation to f

Definition of CW-complexes

Definition: A **CW-complex** is a topological Hausdorff space X together with a finite partition $\{e_i\}_i$ of X (the cells) such that:

- For each e_i, there exists an integer $n(i)$ and a homeomorphism $\mathcal{B}^{n(i)} \to e_i$, where $\mathcal{B}^{n(i)}$ is the open ball of $\mathbb{R}^{n(i)}$.
- Moreover, this homeomorphism extends to a continuous map, $f_i \colon \overline{\mathcal{B}}^{\,n(i)} \to X$, where $\overline{\mathcal{B}}^{\,n(i)}$ is the closed ball.
  Its restriction to the sphere, denoted $\phi_i \colon \partial\overline{\mathcal{B}}^{\,n(i)} \to X$, is called the **gluing map**.
- Each point $x \in \overline{e}_i \setminus e_i$ must lie in a cell e_j of lower dimension.

Example: The sphere $\mathbb{S}^2$ admits a CW-structure with one cell of dimension 0, one cell of dimension 1, two cells of dimension 2.



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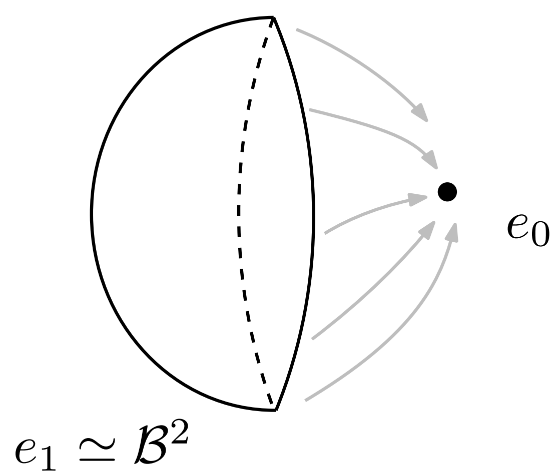
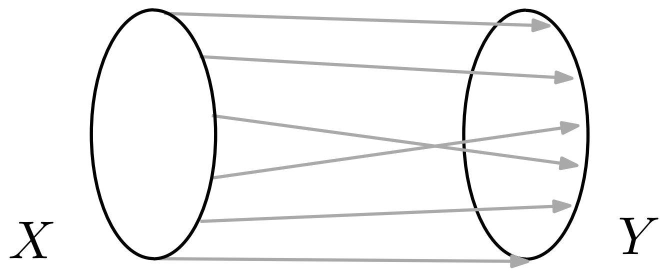
Example: The sphere $\mathbb{S}^2$ admits a CW-structure with one cell of dimension 0, **one** cell of dimension 2.

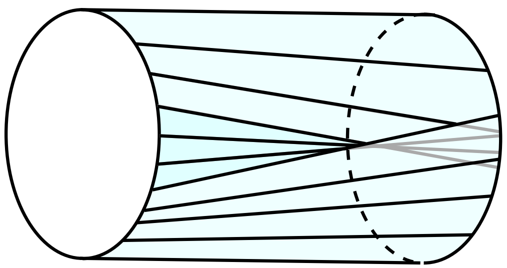


Mapping cones

Let $f\colon X \to Y$ be a continuous map. The **mapping cylinder** is the quotient space

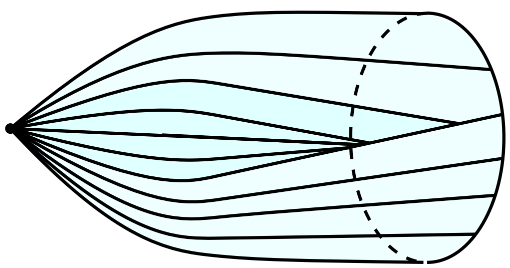
$$\mathrm{MapCyl}(f) = X \times [0,1] \sqcup Y \ / \ (x,1) \sim f(x)$$





The **mapping cone** is obtained by identifying the upper part of the cylinder

$$\mathrm{MapCone}(f) = \mathrm{MapCyl}(f) \ / \ (x,0) \sim \mathrm{point}$$



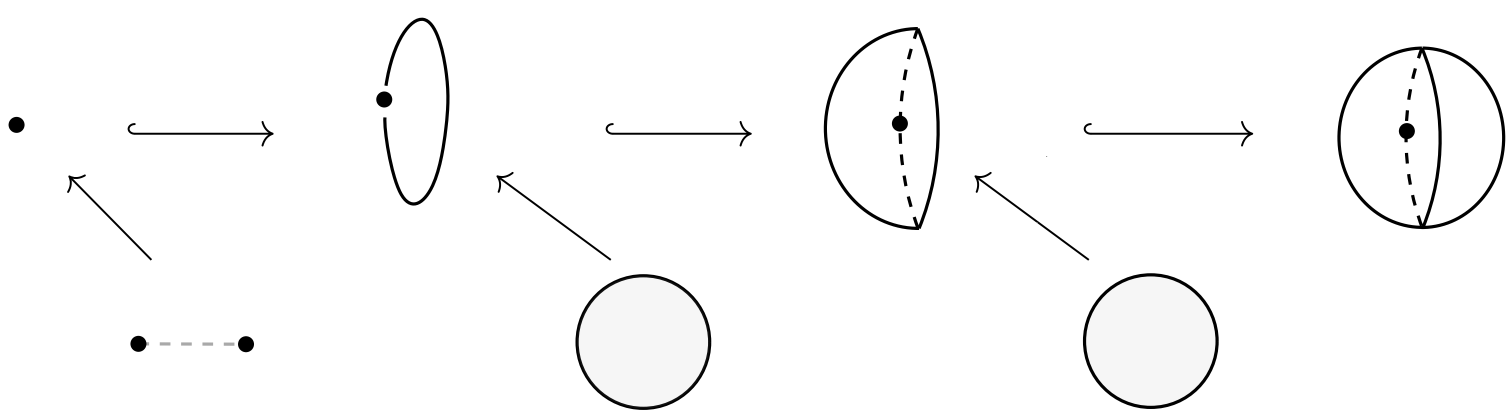
Lemma: If $f,g\colon X \to Y$ are homotopic, then so are $\mathrm{MapCone}(f)$ and $\mathrm{MapCone}(g)$.

Mapping cones

Let X be a CW-complex, with cells $\{e_i\}_i$, and gluing maps $\phi_i\colon \partial\mathcal{B}^{n(i)} \to X$.

One shows that X is homeomorphic to the sequence of mapping cones

$$
\begin{tikzcd}
e_0 \arrow[d, "\simeq"] & & e_0 \cup e_1 \arrow[d, "\simeq"] & & e_0 \cup e_1 \cup e_2 \arrow[d, "\simeq"] & & e_0 \cup e_1 \cup e_2 \cup e_3 \arrow[d, "\simeq"] \\
\{\mathrm{point}\} \arrow[rr, hook] & & \mathrm{MapCone}(\phi_1) \arrow[rr, hook] & & \mathrm{MapCone}(\phi_2) \arrow[rr, hook] & & \mathrm{MapCone}(\phi_3) \arrow[r, hook] & \ldots \\
& \partial\mathcal{B}^{n(1)} \arrow[ul, "\phi_1"] & & \partial\mathcal{B}^{n(2)} \arrow[ul, "\phi_2"] & & \partial\mathcal{B}^{n(3)} \arrow[ul, "\phi_3"] & &
\end{tikzcd}
$$



One builds X by induction: $X^i = \mathrm{MapCone}(\phi_i\colon \partial\mathcal{B}^{n(i)} \to X^{i-1})$.

Let X be a CW-complex, with cells $\{e_i\}_i$, and gluing maps $\phi_i \colon \partial\mathcal{B}^{n(i)} \to X$.

One shows that X is homeomorphic to the sequence of mapping cones

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e_0 \arrow[d, "\simeq"] & & e_0 \cup e_1 \arrow[d, "\simeq"] & & e_0 \cup e_1 \cup e_2 \arrow[d, "\simeq"] & & e_0 \cup e_1 \cup e_2 \cup e_3 \arrow[d, "\simeq"] & \\
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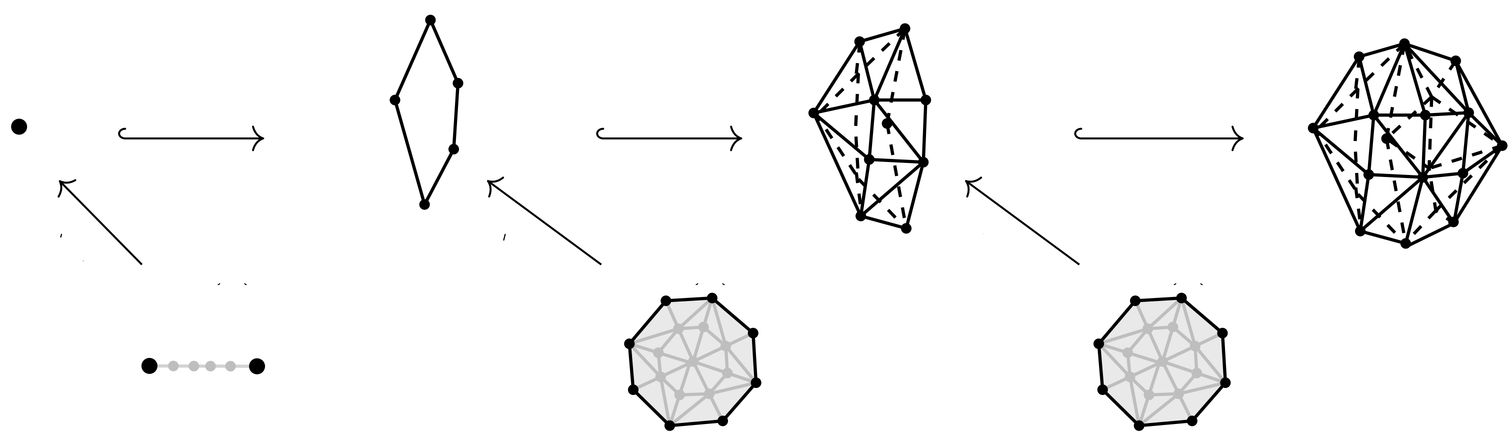
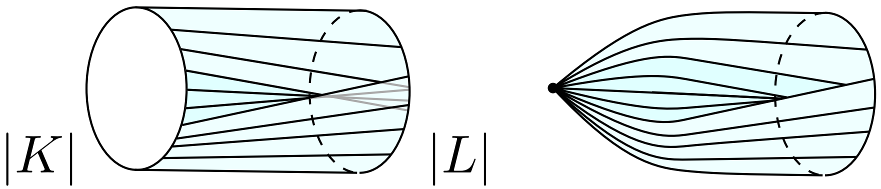
\end{tikzcd}$$



In order to triangulate X, it is enough to triangulate mapping cones.

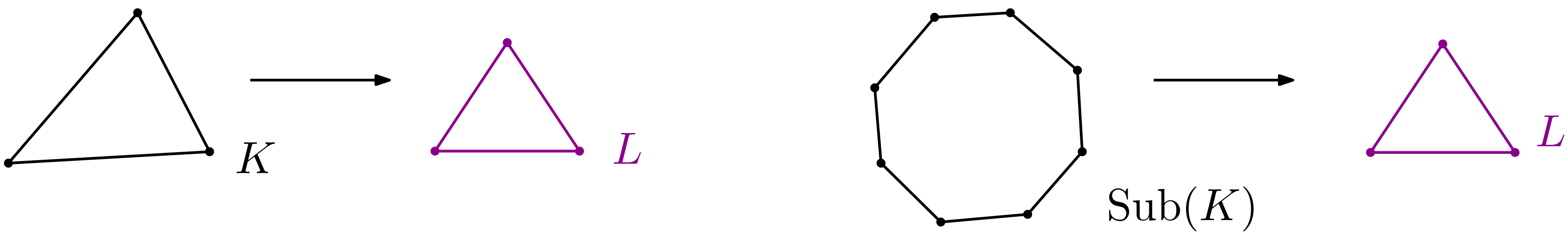
How to triangulate the mapping cone of $f\colon |K| \to |L|$?



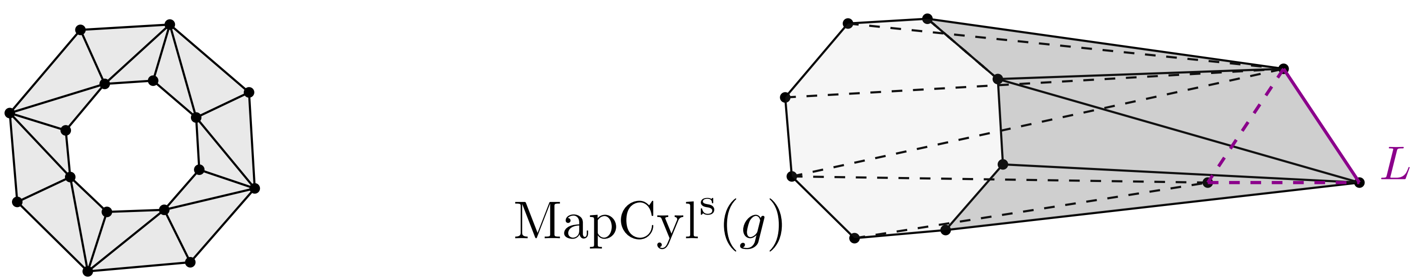
$$\mathrm{MapCyl}(f) = X \times [0,1] \sqcup Y \;/\; (x,1) \sim f(x)$$

$$\mathrm{MapCone}(f) = \mathrm{MapCyl}(f) \;/\; (x,0) \sim \text{point}$$

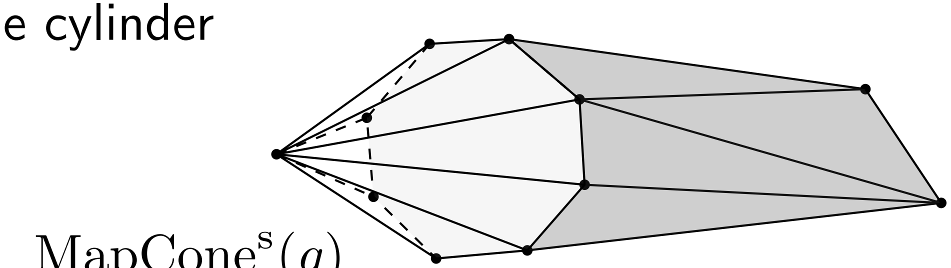
1. Find a simplicial approximation $g\colon K \to L$ to $f\colon |K| \to |L|$



2. Triangulate $|K| \times [0,1]$ and glue L at the end



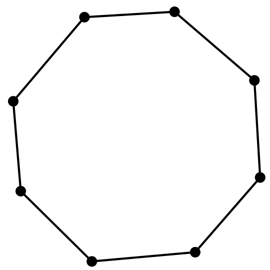
3. Cone the upper part of the cylinder



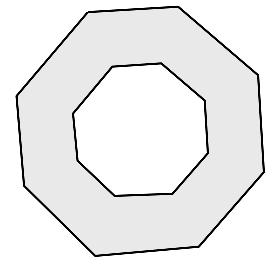
Triangulation of mapping cones

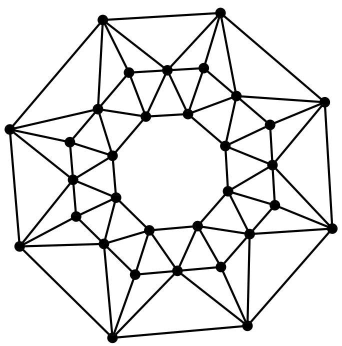
2. Let K be a simplicial complex. How to triangulate $|K| \times [0,1]$?

K

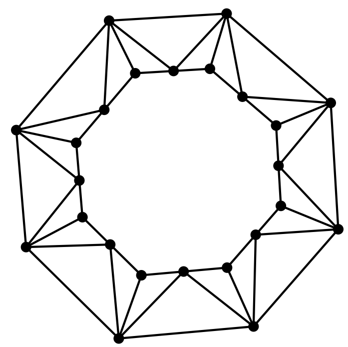


$K \times [0,1]$

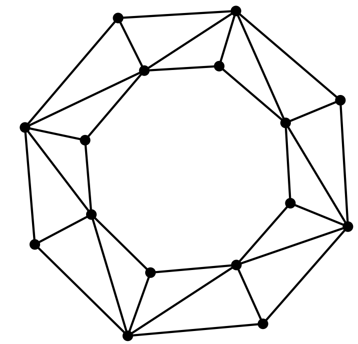




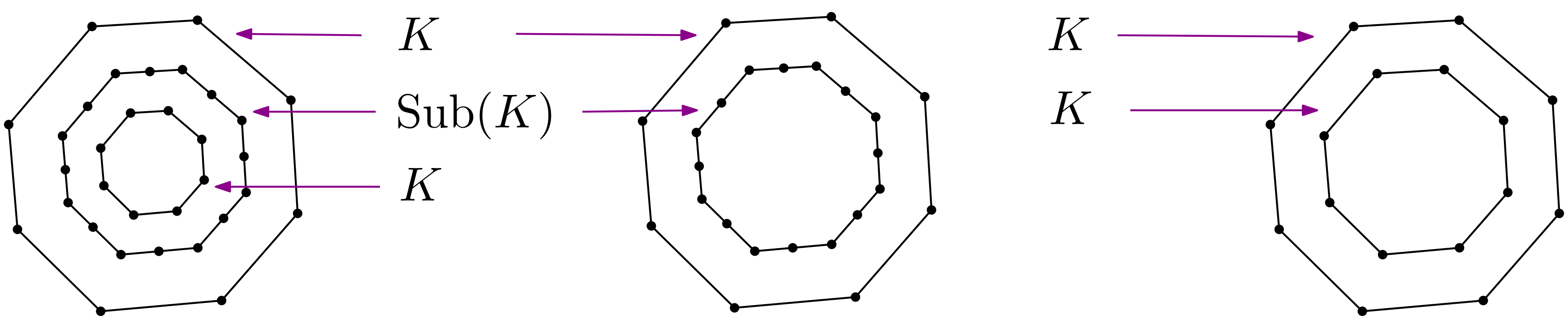
[Whitehead, *Simplicial spaces, nuclei and m-groups*, 1939]



[Cohen, *Simplicial structures and transverse cellularity*, 1967]

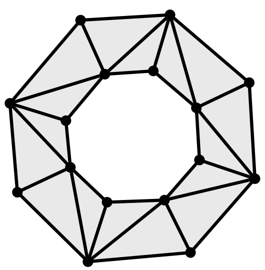


Simplicial set product

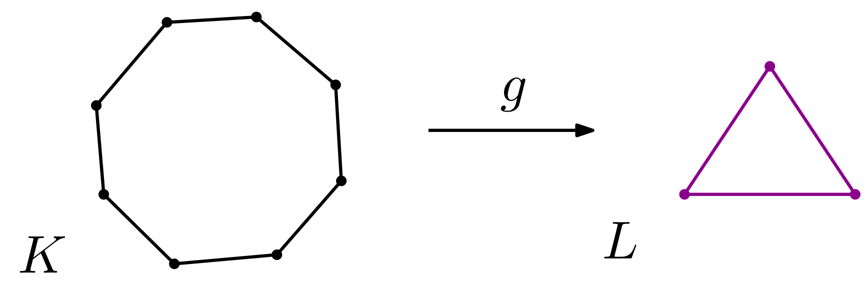


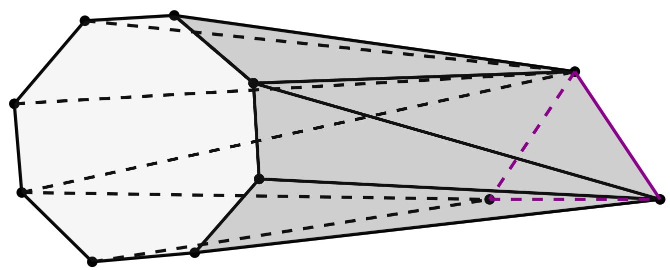
Once $K \times [0,1]$ is triangulated, we glue it to L via $g\colon K \to L$ as follows:

$$\mathrm{MapCyl}^{\mathrm{s}}(g) = \left\{ \sigma_0 \sqcup g(\sigma_1),\ \sigma \in K \times [0,1],\ \sigma = \sigma_0 \,\middle|\, \begin{array}{l} \sqcup\, \sigma_1,\ \ \sigma_0 \in K \times \{0\} \\ \sigma_1 \in K \times \{1\} \end{array} \right\}$$



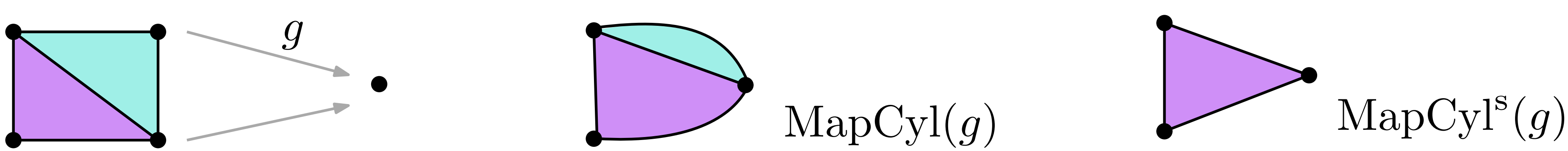
$K \times [0,1]$





$\mathrm{MapCyl}^{\mathrm{s}}(g)$

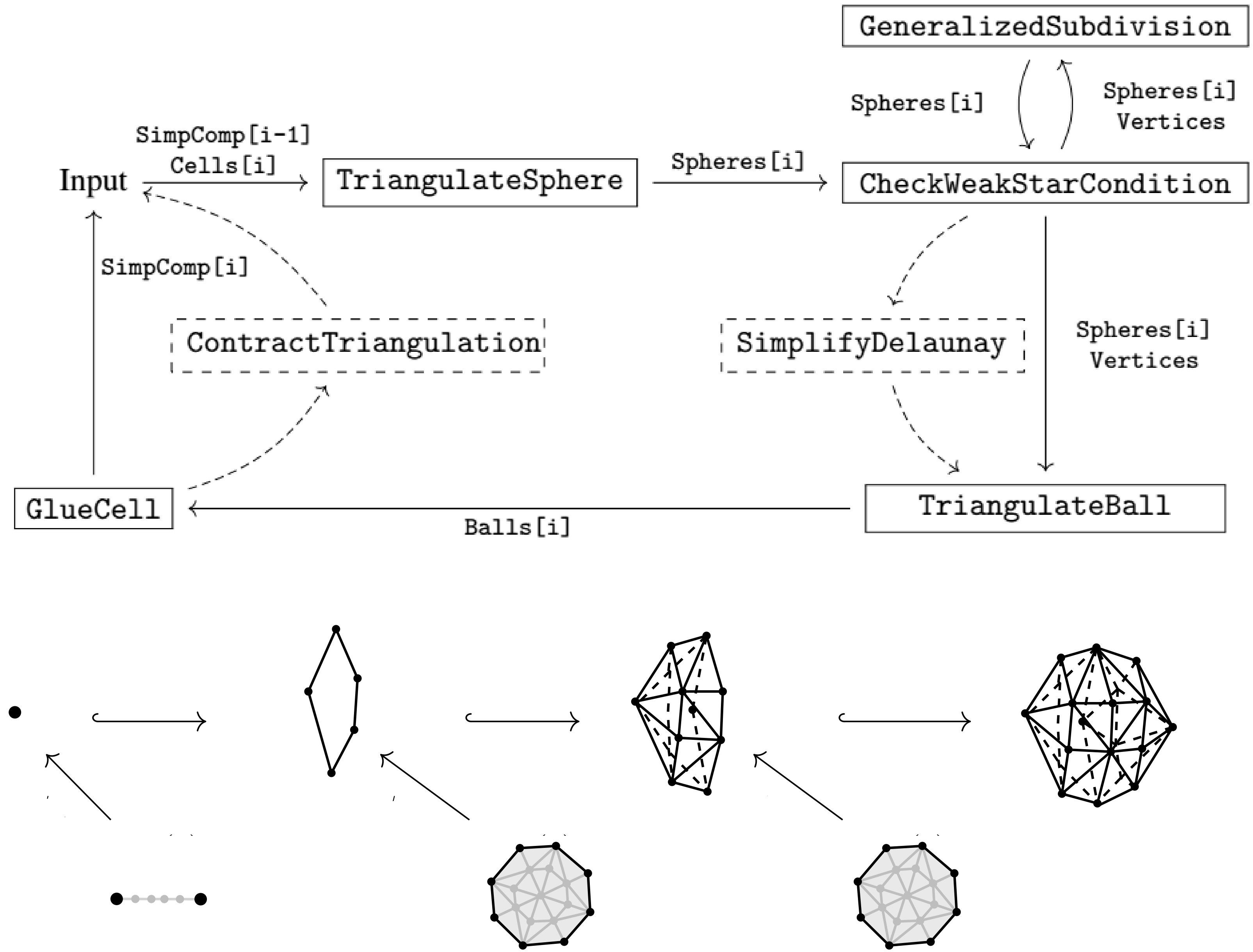
The map $\mathrm{MapCyl}(g) \to \mathrm{MapCyl}^{\mathrm{s}}(g)$ may not be a homeomorphism!



Proposition: It is a homotopy equivalence.

Sketch of algorithm

One can glue the cells one by one. The only information needed is that of the characteristic maps.



Cell structure

Consider a plane $T \in \mathcal{G}(2, \mathbb{R}^4)$ and its matrix in reduced echelon form

$$\begin{pmatrix} v_1' & v_2' & v_3' & v_4' \\ w_1' & w_2 & w_3' & w_4' \end{pmatrix}$$

Let i (resp. j) be the index of the last nonzero coordinate of v' (resp. w').

The pair (i, j) is called the **Schubert symbol** of the plane T.

There are 6 potential Schubert symbols for $\mathcal{G}(2, \mathbb{R}^4)$:

dim 0 $\quad (1,2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\qquad (1,3) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$ dim 1

dim 2 $\quad (1,4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix}$ $\qquad (2,3) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$ dim 2

dim 3 $\quad (2,4) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}$ $\qquad (1,2) \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$ dim 4

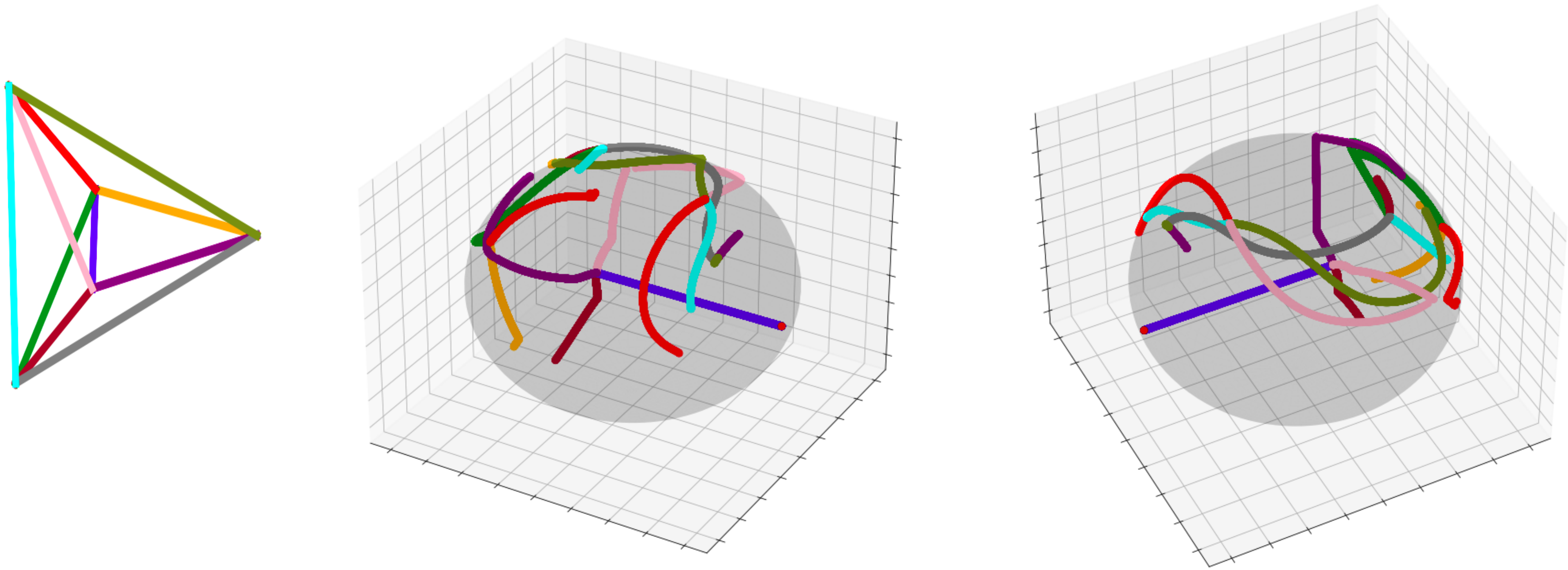
Proposition: $\mathcal{G}(2, \mathbb{R}^4)$ admits a CW-structure with 6 cells. Each cell corresponds to a pair (i, j), and contains all the planes T with Schubert symbol (i, j).

Simplicial approximation to $\mathcal{G}(2, \mathbb{R}^4)$

We apply the algorithm for the Delaunay edgewise subdivision, performing edge contractions and Delaunay simplification steps.

The following table gathers the number of vertices of the complexes at each step, (in parenthesis the number before edge contractions).

Schubert symbol σ	$(1,2)$	$(1,3)$	$(2,3)$	$(1,4)$	$(2,4)$	$(3,4)$
Number of vertices	1	$3\ (4)$	$6\ (10)$	$10\ (13)$	$22\ (93)$	$825\ (3450)$

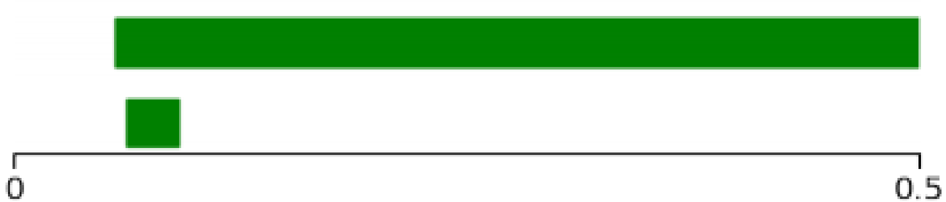


Embedding of the 4-simplex, seen as a triangulation of the 3-sphere, in the 3-cell of $\mathcal{G}(2, \mathbb{R}^4)$.

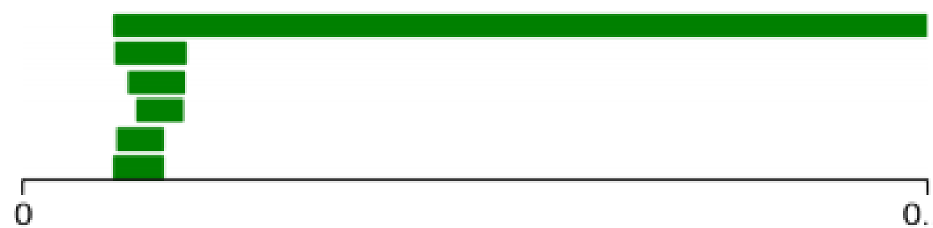
Simplicial approximation to $\mathcal{G}(2, \mathbb{R}^4)$

Example: We sample the point clouds X and Y on

$$\left\{ \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ t \\ 0 \end{pmatrix} \middle| \begin{matrix} \theta \in [0, 2\pi) \\ t \in [-1, 1] \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ t\cos(\theta/2) \\ t\sin(\theta/2) \end{pmatrix} \middle| \begin{matrix} \theta \in [0, 2\pi) \\ t \in [-1, 1] \end{matrix} \right\}.$$

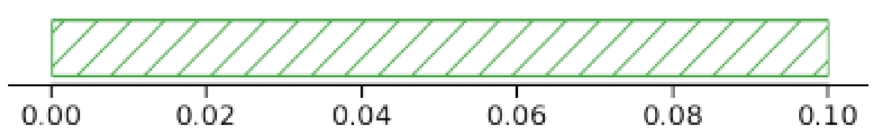


$H^1(X^t; \mathbb{Z}/2\mathbb{Z})$

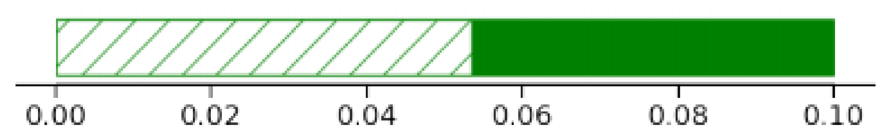


$H^1(Y^t; \mathbb{Z}/2\mathbb{Z})$

We estimate their tangent bundles $X, Y \to \mathcal{G}(2, \mathbb{R}^4)$, and compute their persistent w_1.



$w_1(\check{X}^t)$



$w_1(\check{Y}^t)$

Thanks!