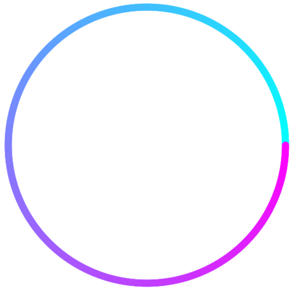
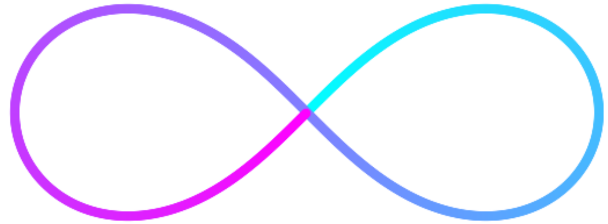
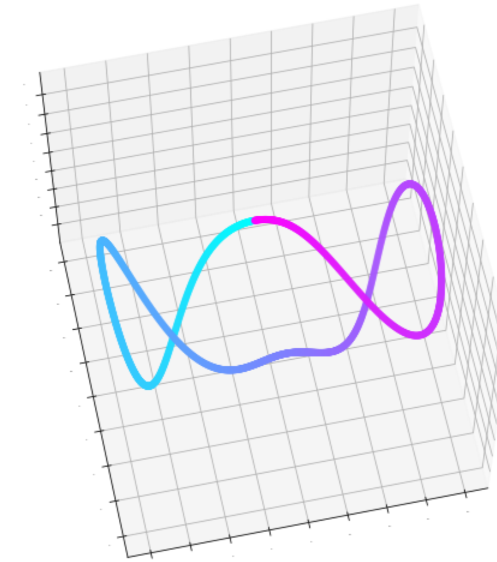
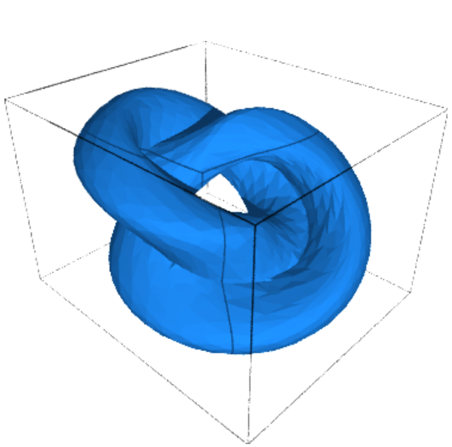
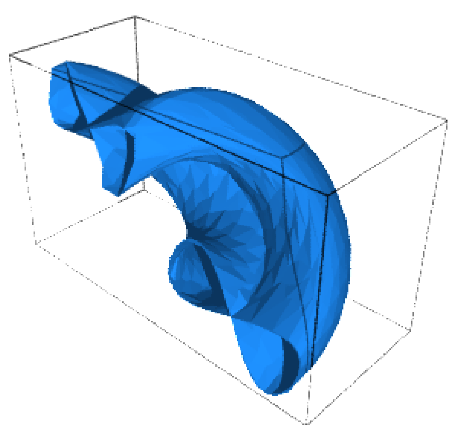
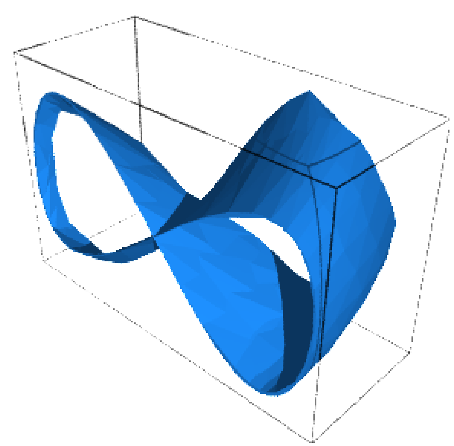
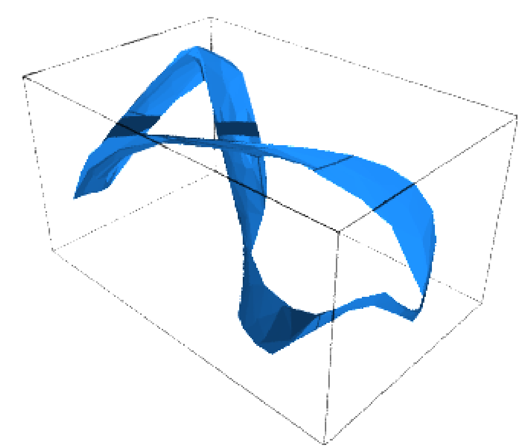




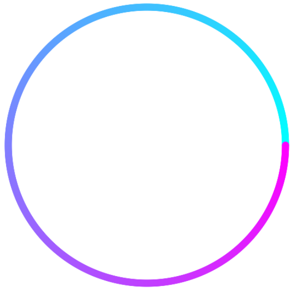
Recovering the homology of immersed manifolds

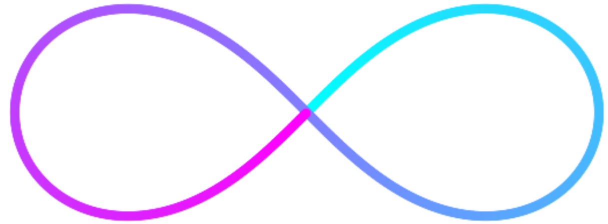
Raphaël Tinarrage

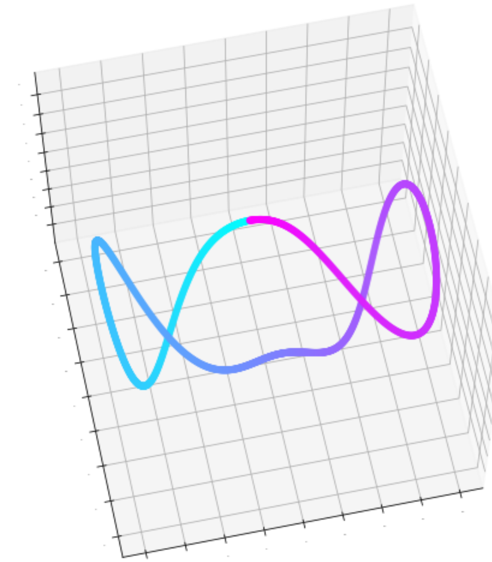
arxiv.org/abs/1912.03033

embedded manifold





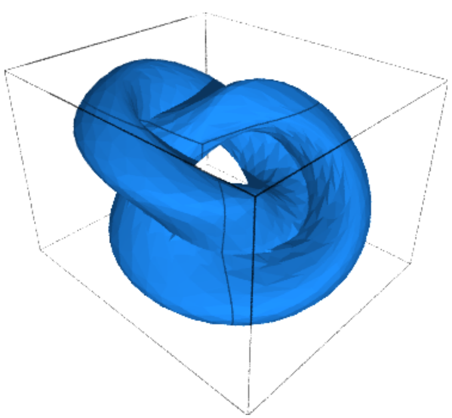


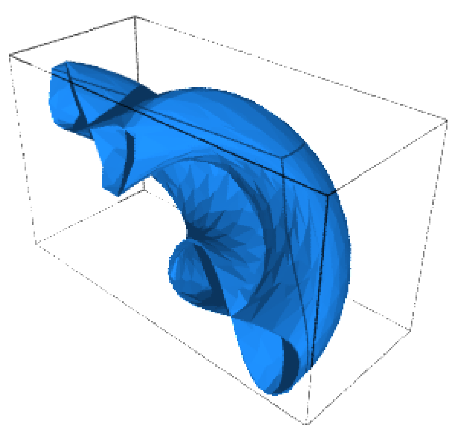
Recovering the homology of immersed manifolds

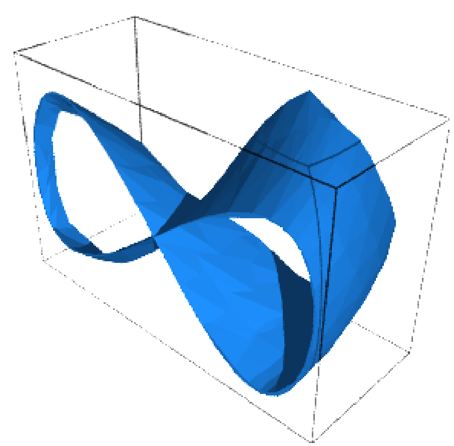
(may self-intersect)

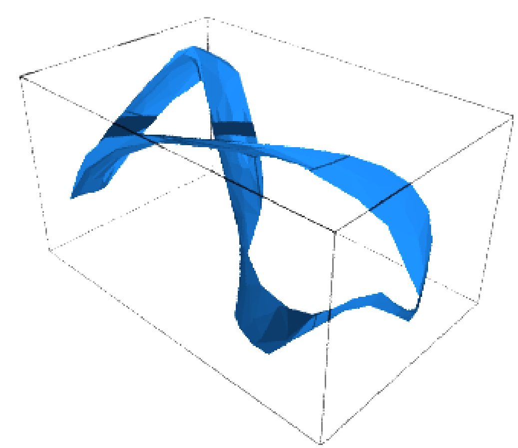
Raphaël Tinarrage

arxiv.org/abs/1912.03033









Statement of the problem

We are observing an immersed manifold $\mathcal{M} \subset \mathbb{R}^n$.

Abstract manifold	Immersion	Immersed manifold
$\mathcal{M}_0$	$\xrightarrow{u}$	$\mathcal{M} = u(\mathcal{M}_0) \quad \subset \mathbb{R}^n$

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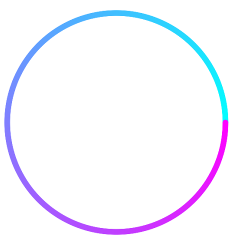
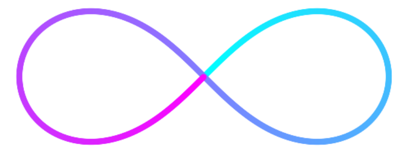
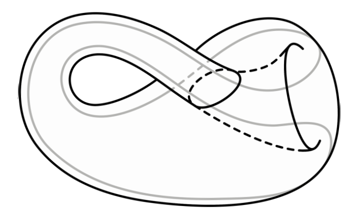
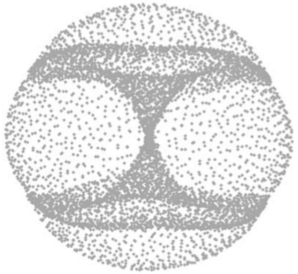
	$\longrightarrow$	
Klein bottle	$\longrightarrow$	
	$\longrightarrow$	
Klein bottle $\cup$ sphere	$\longrightarrow$	

[Martin, Coutsias, Thompson, Topology of Cyclooctane Energy Landscape]

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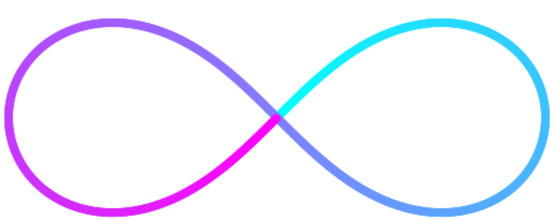
Abstract manifold	Immersion	Immersed manifold

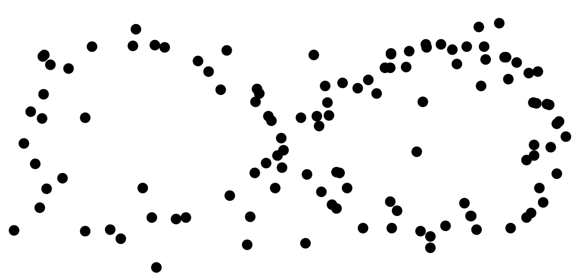
$$\mathcal{M}_0 \xrightarrow{\quad u \quad} \mathcal{M} = u(\mathcal{M}_0) \quad \subset \mathbb{R}^n$$

Question 1:	Given $\mathcal{M}$, compute the (singular) homology groups of $\mathcal{M}_0$.

Question 2:	Given $X \subset \mathbb{R}^n$ close to $\mathcal{M}$, compute the homology groups of $\mathcal{M}_0$.



$\mathcal{M}$



X

$$H_0 = \mathbb{Z}/2\mathbb{Z}$$

$$H_1 = \mathbb{Z}/2\mathbb{Z}$$

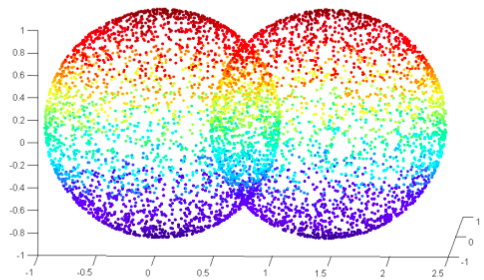
Statement of the problem

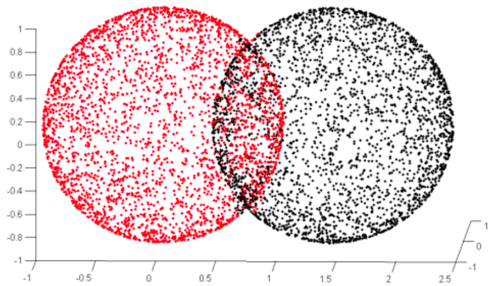
We are observing an immersed manifold $\mathcal{M} \subset \mathbb{R}^n$.

Abstract manifold	Immersion	Immersed manifold
$\mathcal{M}_0$	$\xrightarrow{\quad u \quad}$	$\mathcal{M} = u(\mathcal{M}_0) \quad \subset \mathbb{R}^n$

A bit of context

- [Arias-Castro, Ery, Gilad Lerman and Teng Zhang. Spectral clustering based on local PCA.]



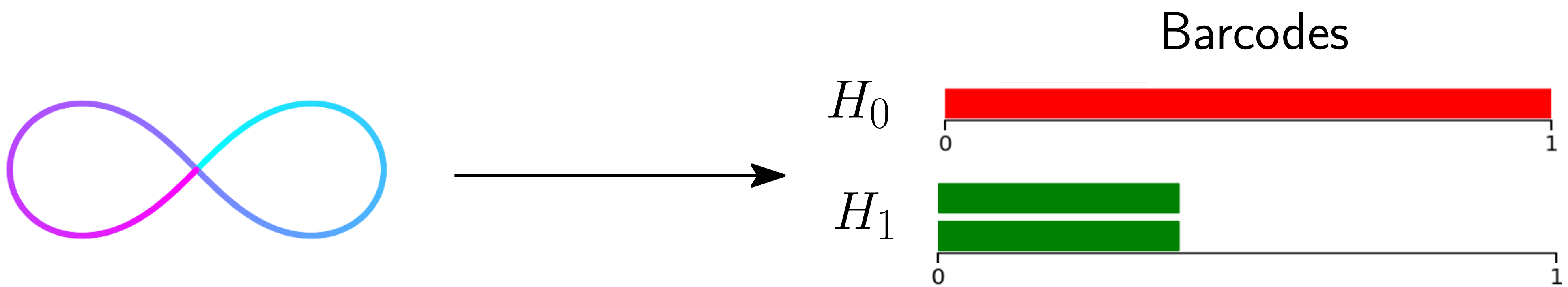


- [Memoli, Smith and Wan. The Wasserstein transform.]
- [Díaz Martínez, Mémoli and Mio. The shape of data and probability measures.]

Our method

We will use persistent homology.

Unfortunately, the persistent homology of the Čech filtration of $\mathcal{M}$ does not reveal the homology of $\mathcal{M}_0$.



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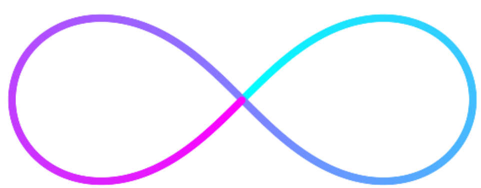
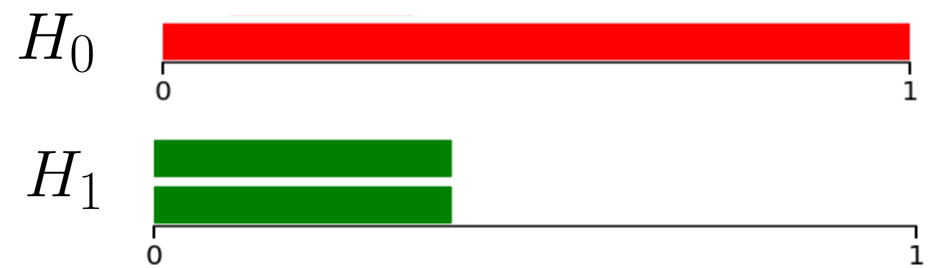
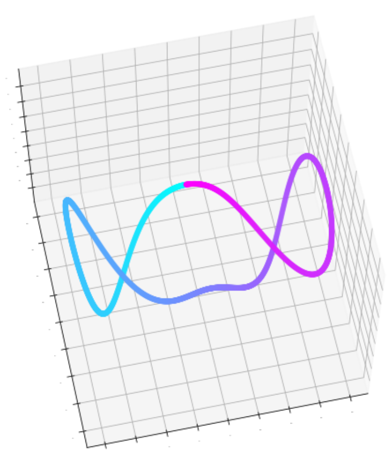


Barcodes



We will try to lift $\mathcal{M}$ in a higher dimensional space, where the Čech filtration reveals a circle.



Our method

How to lift $\mathcal{M}$?

$$\begin{array}{ccc}
\mathcal{M}_0 & \xrightarrow{u} & \mathcal{M} \subset \mathbb{R}^n \\
x_0 & \longmapsto & u(x_0)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{M}_0 & \xrightarrow{\check{u}} & \check{\mathcal{M}} \subset \mathbb{R}^n \times \mathbb{R}^m \\
x_0 & \longmapsto & (u(x_0), f(x_0))
\end{array}$$

Choose f such that $\check{u}$ is an embedding.

.

Our method

How to lift $\mathcal{M}$?

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{\quad u \quad} & \mathcal{M} \subset \mathbb{R}^n \\ x_0 & \longmapsto & u(x_0) \end{array}$$

lifted manifold
lift space

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{\quad \check{u} \quad} & \check{\mathcal{M}} \subset \mathbb{R}^n \times \mathbb{R}^m \\ x_0 & \longmapsto & (u(x_0), f(x_0)) \end{array}$$

Choose f such that $\check{u}$ is an embedding.

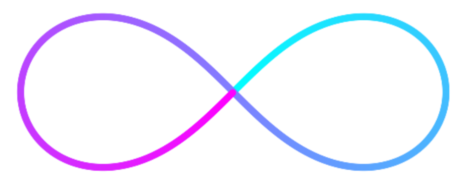
Our choice is $\boxed{f : x_0 \longmapsto T_{x_0}\mathcal{M}_0}$ (tangent space of $\mathcal{M}_0$ at x_0)

- $\check{u}$ is an embedding under a reasonable assumption
- we are actually estimating the tangent bundle of $\mathcal{M}_0$
  [T., Computing Stiefel-Whitney classes of line bundles]

Our method

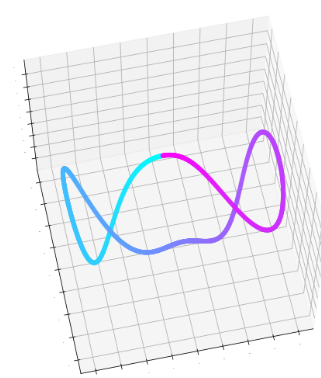
Notations:

- $u\colon \mathcal{M}_0 \to \mathcal{M} \subset \mathbb{R}^n$ is an immersion
- For $x_0 \in \mathcal{M}_0$, $x = u(x_0)$
- For $x_0 \in \mathcal{M}_0$, $T_x\mathcal{M}$ denotes the tangent space of $\mathcal{M}_0$ seen in $\mathbb{R}^n$
- $M(\mathbb{R}^n)$ denotes the space of $n \times n$ matrices
- $p_{T_x\mathcal{M}} \in M(\mathbb{R}^n)$ denotes the orthogonal projection matrix on $T_x\mathcal{M}$
- *Lift space*: $\mathbb{R}^n \times M(\mathbb{R}^n)$
- *Lifted manifold*: $\check{\mathcal{M}} = \left\{\left(x, p_{T_x\mathcal{M}}\right), x_0 \in \mathcal{M}_0\right\} \subset \mathbb{R}^n \times M(\mathbb{R}^n)$
- *Lifting map*: $\check{u}\colon \mathcal{M}_0 \to \check{\mathcal{M}}$



$$\begin{tikzcd}
 & \mathcal{M} \subset \mathbb{R}^n \\
\mathcal{M}_0 \arrow[ur, "u"] \arrow[dr, "\check{u}"'] & \\
 & \check{\mathcal{M}} \subset \mathbb{R}^n \times M(\mathbb{R}^n)
\end{tikzcd}$$

$$\check{u}\colon x_0 \longmapsto \left(x, p_{T_x\mathcal{M}}\right)$$



(PCA dimension reduction)

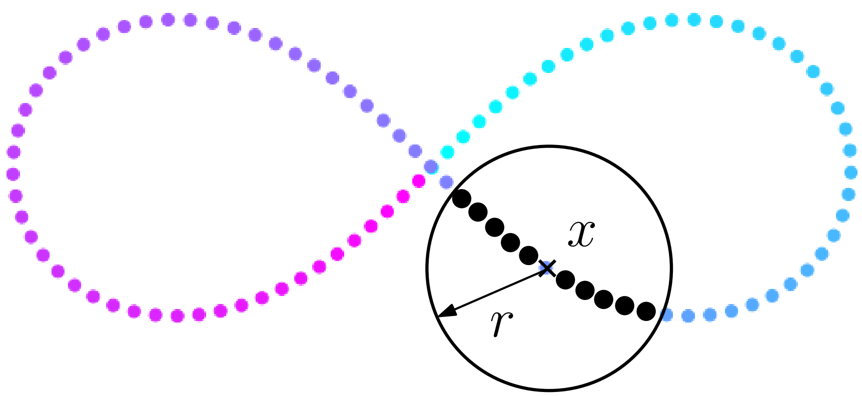
Recipe in practice

- We observe a point cloud $X \subset \mathbb{R}^n$ close to $\mathcal{M}$.

- Let $r > 0$ be a parameter.
  For every $x \in X$, compute a *local covariance matrix*

$$\Sigma_X(x, r) = \frac{1}{|A|} \sum_{y \in A} (x - y)^{\otimes 2} \qquad \in M(\mathbb{R}^n)$$

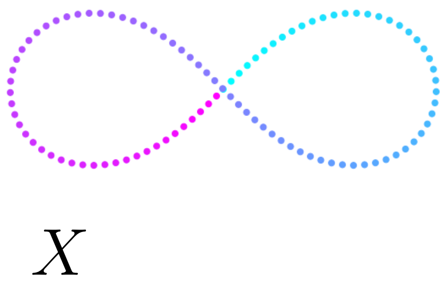
where $A = \{y \in X, \|x - y\| \leq r\}$.

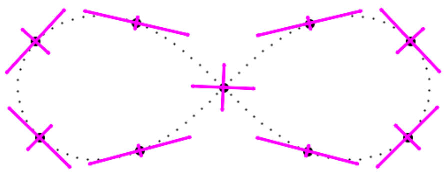


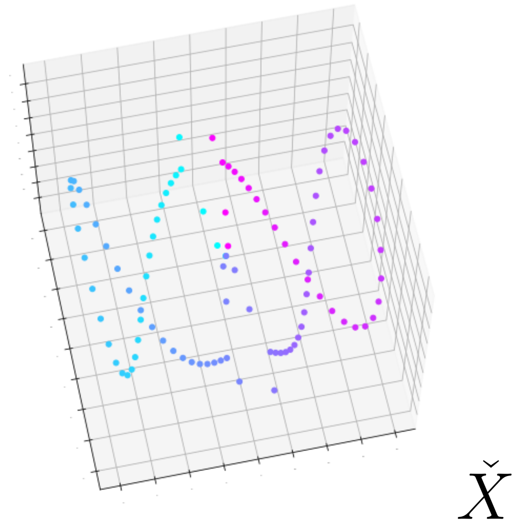
- Consider the set

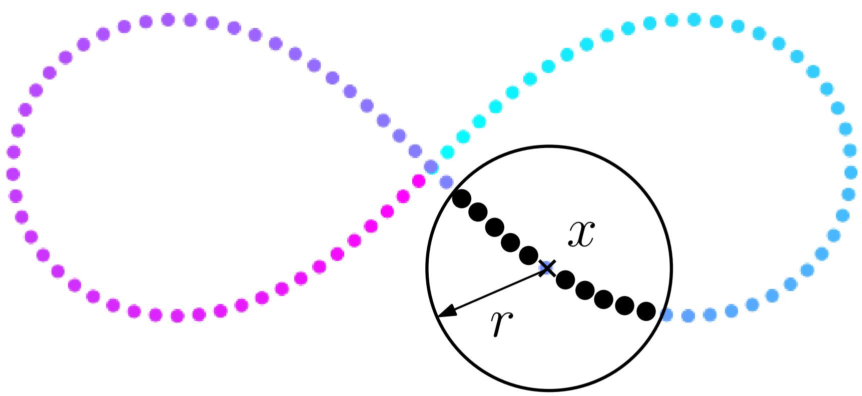
$$\check{X} = \{(x, \Sigma_X(x, r)), x \in X\} \subset \mathbb{R}^n \times M(\mathbb{R}^n).$$

Recipe in practice





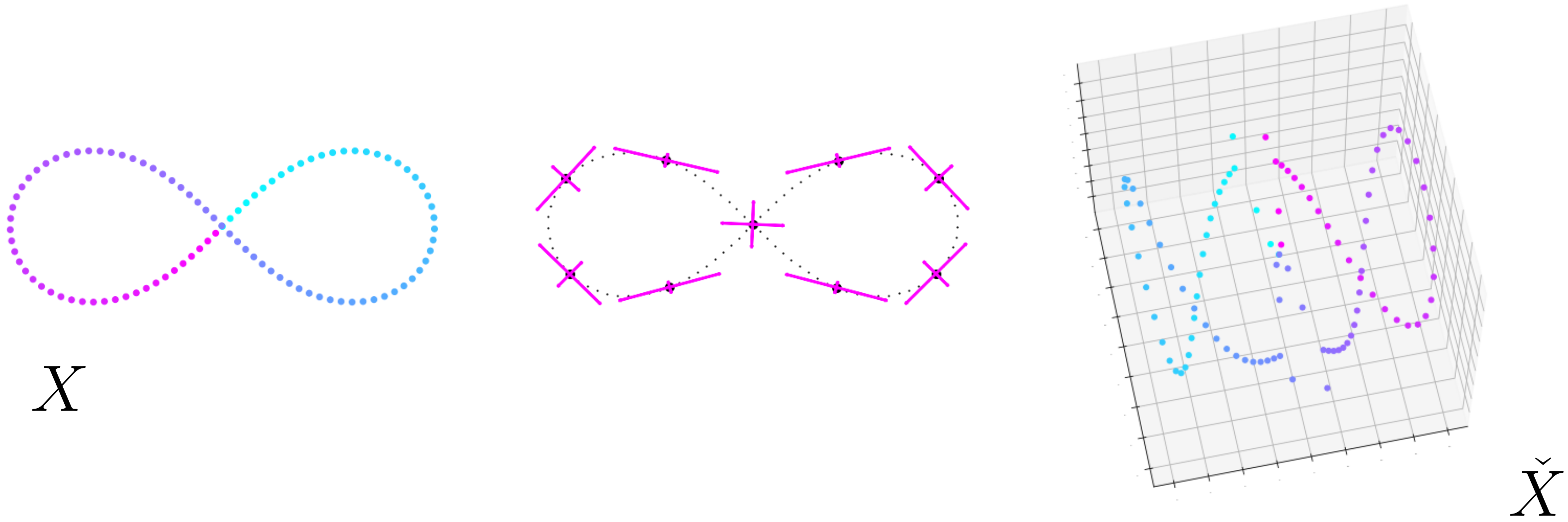


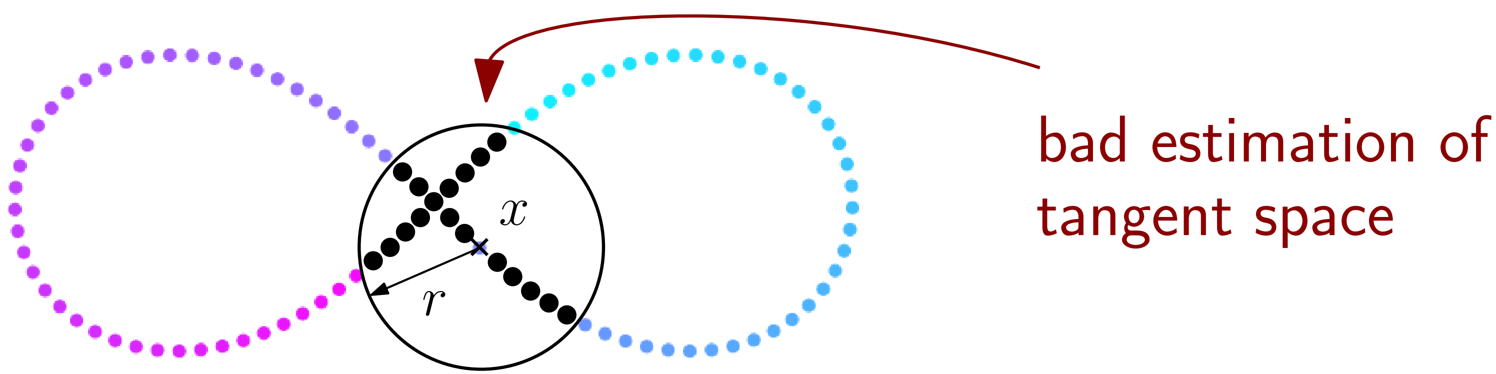


- Consider the set

$$\check{X} = \left\{ \left(x, \Sigma_X(x, r)\right), x \in X \right\} \subset \mathbb{R}^n \times M(\mathbb{R}^n).$$

Recipe in practice

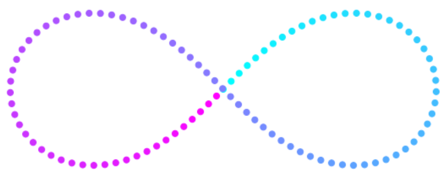




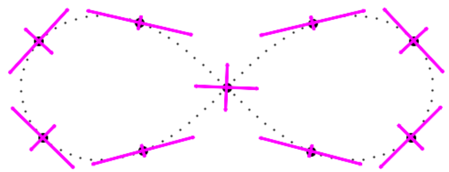
- Consider the set

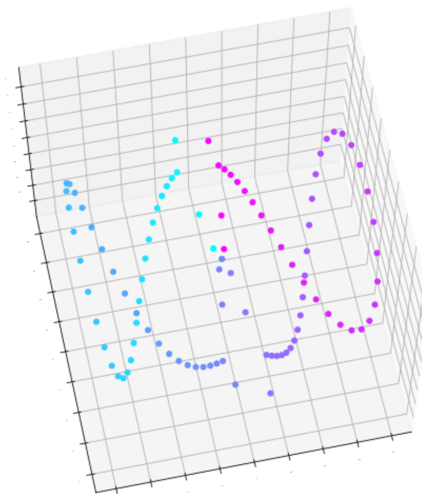
$$\check{X} = \{(x, \Sigma_X(x,r)), x \in X\} \subset \mathbb{R}^n \times M(\mathbb{R}^n).$$

Recipe (that does not work) in practice



X

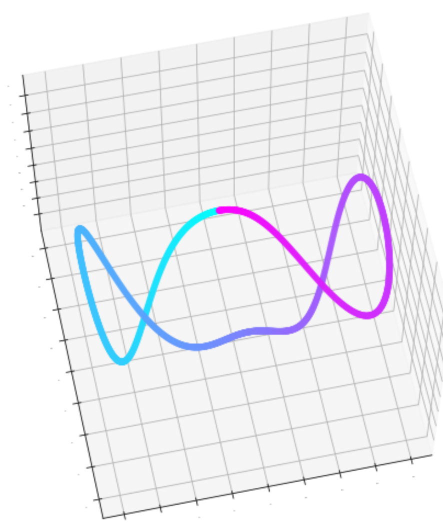




$$\check{X} = \{(x, \Sigma_X(x, r)), x \in \mathcal{M}\}$$

$$\check{\mathcal{M}} = \{(x, p_{T_x\mathcal{M}}), x_0 \in \mathcal{M}_0\}$$

$\check{\mathcal{M}}$ and $\check{X}$ are not close in Hausdorff distance :(

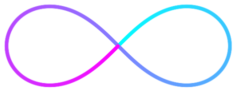


A measure-theoretic setting

$$\mathcal{M}_0 \xrightarrow{\quad u \quad} \mathcal{M} \subset \mathbb{R}^n$$

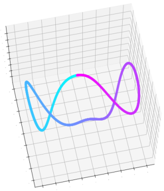
μ_0 measure on $\mathcal{M}_0$

μ push-forward measure on $\mathcal{M}$



$$\mathcal{M}_0 \xrightarrow{\quad \check{u} \quad} \check{\mathcal{M}} \subset \mathbb{R}^n \times M(\mathbb{R}^m)$$

$\check{\mu}_0$ push-forward measure on $\check{\mathcal{M}}$



where $\check{u} \colon x_0 \longmapsto \left(x, \frac{1}{d+2} p_{T_x\mathcal{M}}\right)$

$\check{\mu}_0$ can be defined as follows: for every test function $\phi \colon \mathbb{R}^n \times M(\mathbb{R}^n) \to \mathbb{R}$,

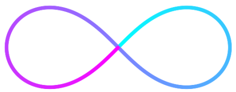
$$\int \phi(x, A) \cdot \mathrm{d}\check{\mu}_0(x, A) = \int \phi\left(x, \frac{1}{d+2} p_{T_x\mathcal{M}}\right) \cdot \mathrm{d}\mu_0(x_0).$$

A measure-theoretic setting

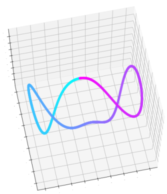
$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, "u"] \arrow[dr, "\check{u}"'] & \mathcal{M} \subset \mathbb{R}^n \\ & \check{\mathcal{M}} \subset \mathbb{R}^n \times M(\mathbb{R}^m) \end{tikzcd}$$

μ_0 measure on $\mathcal{M}_0$

μ push-forward measure on $\mathcal{M}$



$\check{\mu}_0$ push-forward measure on $\check{\mathcal{M}}$



where $\check{u}\colon x_0 \longmapsto \left(x, \frac{1}{d+2} p_{T_x\mathcal{M}}\right)$

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Now, we are observing a measure ν close to μ

Define $\check{\nu}$ as follows: for every test function $\phi\colon \mathbb{R}^n \times M(\mathbb{R}^n) \to \mathbb{R}$,

$$\int \phi(x, A) \cdot \mathrm{d}\check{\nu}(x, A) = \int \phi\left(x, \frac{1}{r^2} \Sigma_\nu(x, r)\right) \cdot \mathrm{d}\nu(x),$$

where $\Sigma_\nu(x, r)$ is the local covariance matrix.

A measure-theoretic setting

Theorem:
Let $r > 0$. Suppose that $W_1(\mu, \nu)$ is small enough. Under several assumptions on $\mathcal{M}_0$ and μ_0, we have

$$W_p(\check{\nu}, \check{\mu}_0) \leq \text{constant} \cdot r^{\frac{1}{p}}$$

where W_p denote the p-Wasserstein distance.

$\check{\mu}_0$ can be defined as follows: for every test function $\phi\colon \mathbb{R}^n \times M(\mathbb{R}^n) \to \mathbb{R}$,

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where $\Sigma_\nu(x, r)$ is the local covariance matrix.

Persistent homology for measures

[Anai, Chazal, Glisse, Ike, Inakoshi, T., Umeda. DTM-based filtrations]

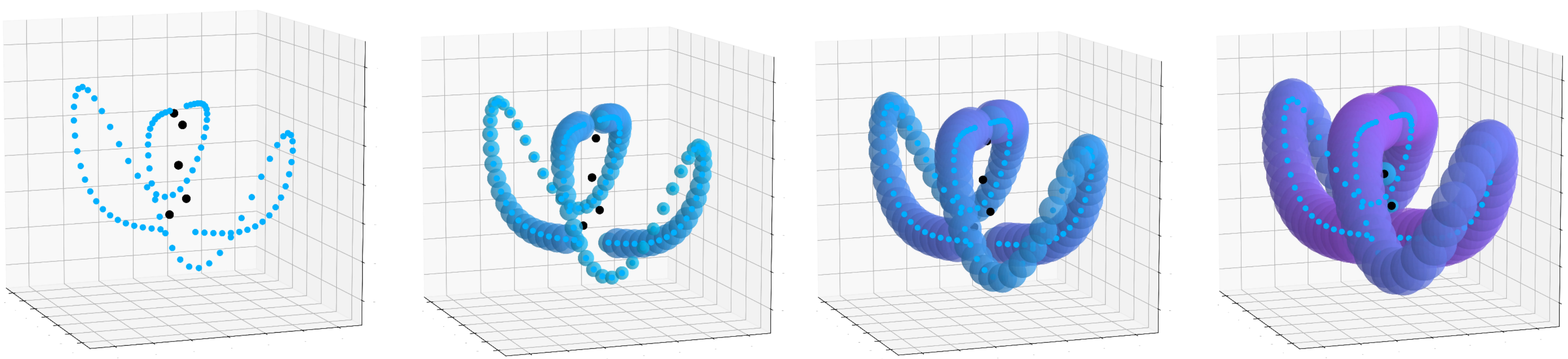
- Usual Čech filtration: with $X \subset \mathbb{R}^k$,

$$X^t = \bigcup_{x \in X} \overline{\mathcal{B}}(x, t)$$

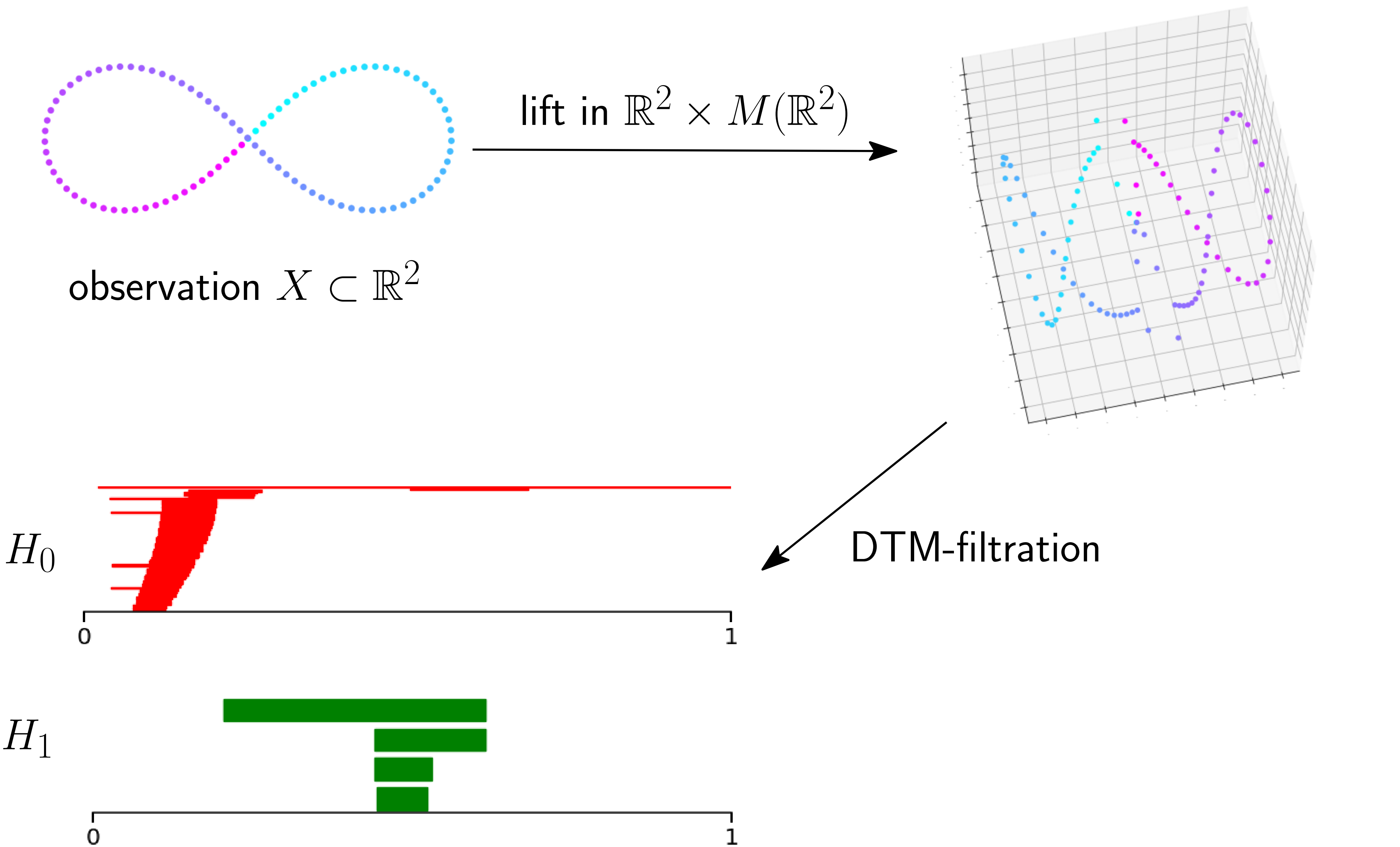
- DTM-filtration: with μ measure on $\mathbb{R}^k$,

$$V^t = \bigcup_{x \in \mathrm{supp}(\mu)} \overline{\mathcal{B}}(x, t - d_\mu(x))$$

where d_μ is the distance-to-measure (DTM) associated to μ



Persistent homology for measures



A last illustration