EMAp Summer Course

Topological Data Analysis with Persistent Homology

https://raphaeltinarrage.github.io/EMAp.html

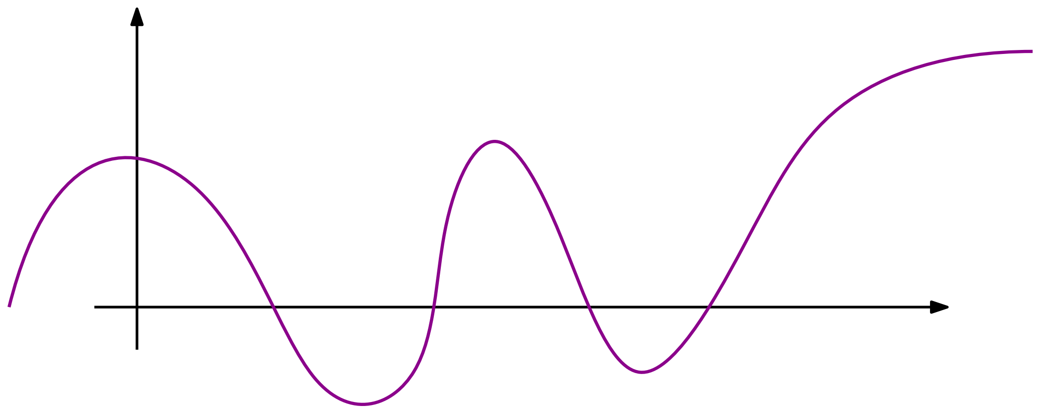
Lesson 1: Topological spaces

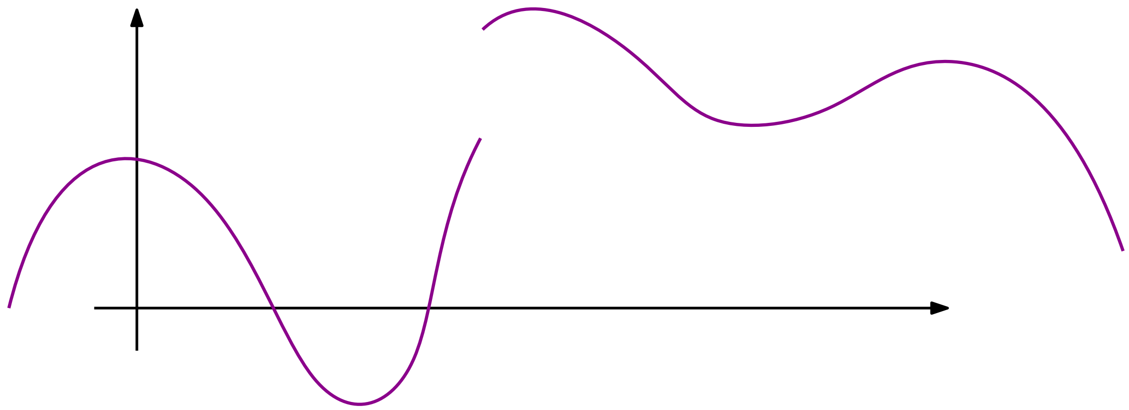
Last update: January 17, 2021

Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a map. Remember that f is continuous if

$$\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists \eta > 0, \forall y \in \mathbb{R}, \|x - y\| < \eta \implies \|f(x) - f(y)\| < \epsilon.$$





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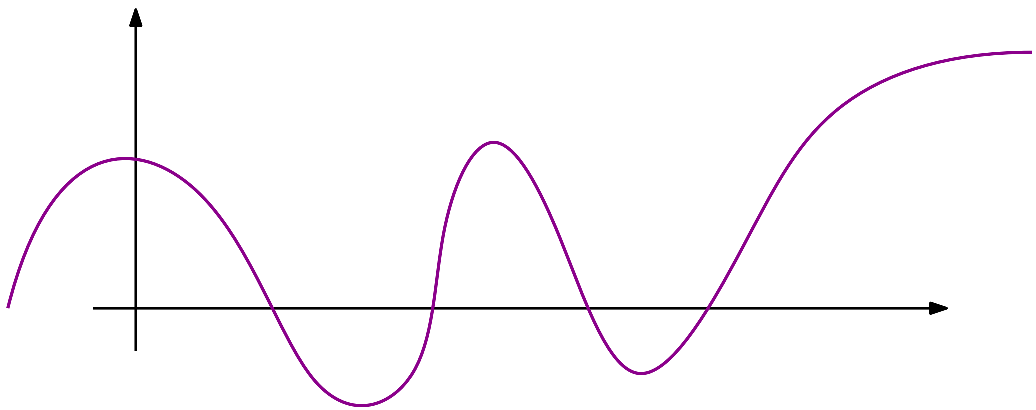

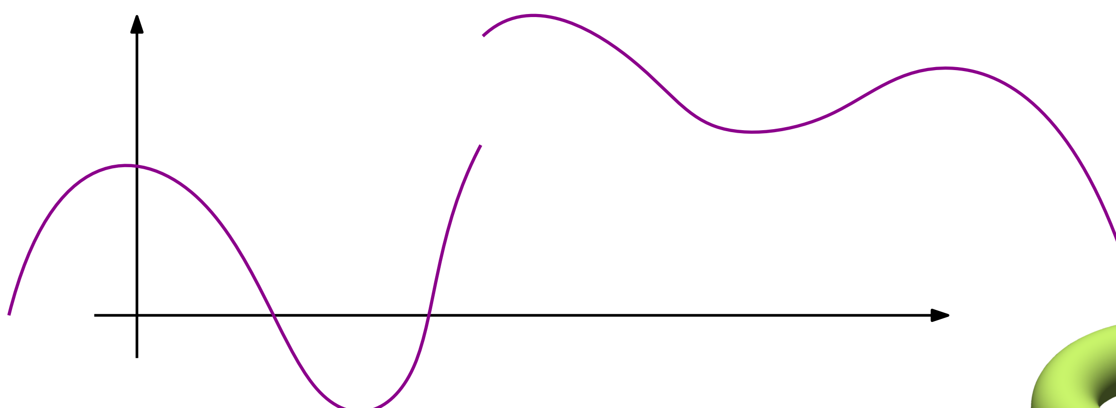

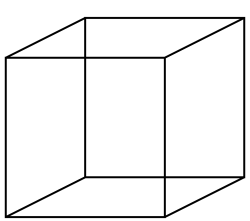

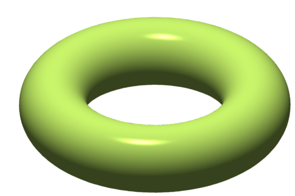

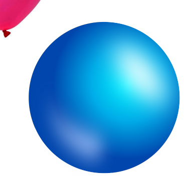

Aim of this lesson: generalize the notion of continuity to more general spaces

I - Topological spaces

II - Topology of $\mathbb{R}^n$

III - Topology of subsets of $\mathbb{R}^n$

VI - Continuous maps

Topological spaces

Topological spaces are abstractions of the concept of 'shape' or 'geometric object'.

Definition: A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
- for every infinite collection $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, we have $\bigcup\limits_{\alpha \in A} O_\alpha \in \mathcal{T}$,
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In other words,

- the empty set is an open set, the set X itself is an open set,
- an infinite union of open sets is an open set,
- a finite intersection of open sets is an open set.

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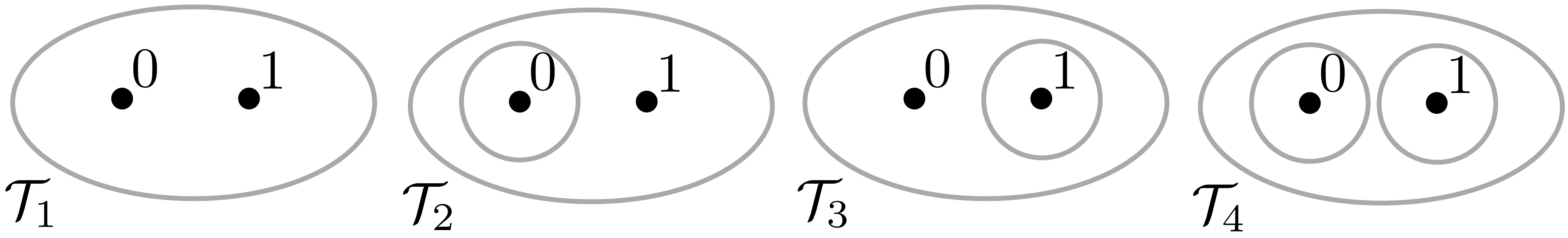
Example: Let $X = \{0, 1\}$ be a set with two elements. There exists four different topologies on X:

- $\mathcal{T}_1 = \{\emptyset, \{0, 1\}\}$,
- $\mathcal{T}_2 = \{\emptyset, \{0\}, \{0, 1\}\}$,
- $\mathcal{T}_3 = \{\emptyset, \{1\}, \{0, 1\}\}$,
- $\mathcal{T}_4 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.



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Example: Let $X = \{0, 1, 2\}$ be a set with three elements.
The following is a topology on X:

$$\mathcal{T} = \{\emptyset, \{0\}, \{0, 1, 2\}\}$$

But the following are not:

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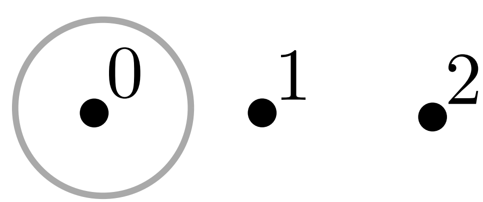

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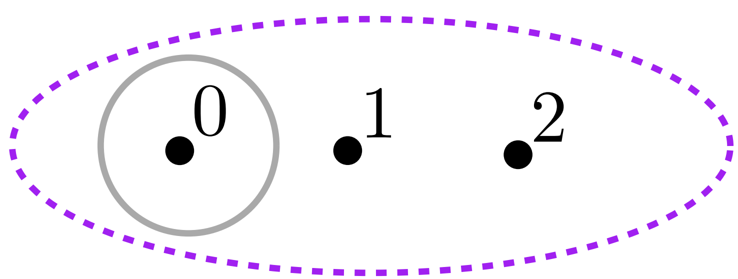

$X = \{0, 1, 2\}$ is missing

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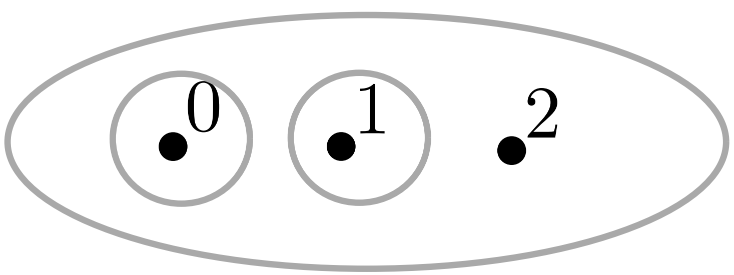

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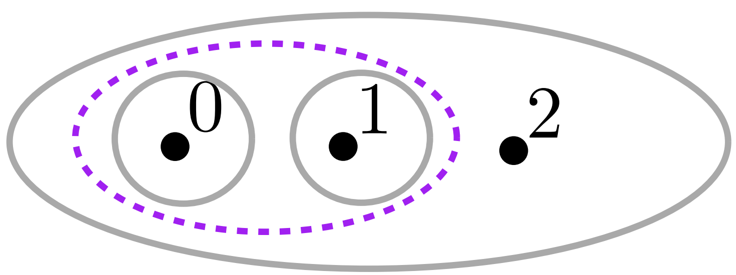

$\{0, 1\} = \{0\} \cup \{1\}$ is missing

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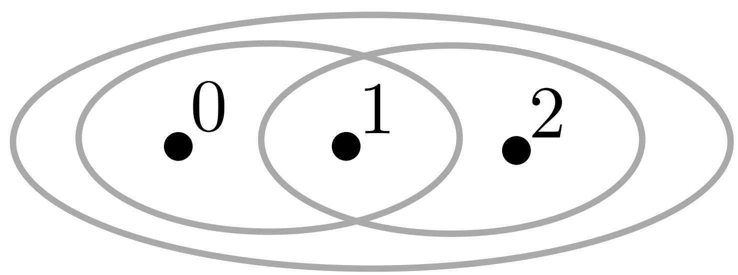

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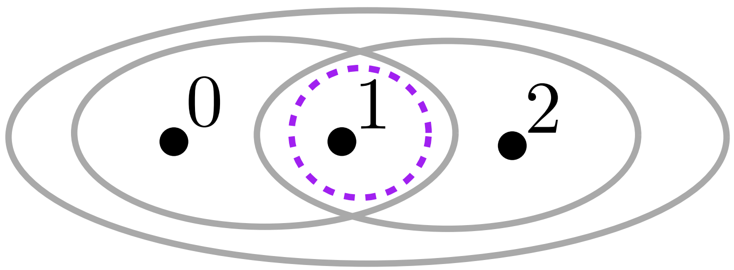

$\{1\} = \{0, 1\} \cap \{1, 2\}$ is missing

Definition via closed sets

Let $(X, \mathcal{T})$ be a topological space. For every open set $O \in \mathcal{T}$, its complementary ${}^cO = \{x \in X, x \notin O\}$ is called a closed set.

In other words, a set $A \subset X$ is closed iff ${}^cA$ is open.

Proposition: We have:

- the sets $\emptyset$ and X are closed sets,
- for every infinite collection $\{P_\alpha\}_{\alpha \in A}$ of closed set, $\bigcap_{\alpha \in A} P_\alpha$ is a closed set,
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Proof of first point: The set $\emptyset$ is closed because ${}^c\emptyset = X$ is open. The set X is closed because ${}^cX = \emptyset$ is open.

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Proof of second point: If $\{P_\alpha\}_{\alpha \in A}$ is an infinite collection of closed set, then for every $\alpha \in A$, ${}^cP_\alpha$ is open. Now, we use the relation

$${}^c\left(\bigcap_{\alpha \in A} P_\alpha\right) = \bigcup_{\alpha \in A} {}^cP_\alpha.$$

This is a union of open sets, hence it is open. Hence $\bigcap\limits_{\alpha \in A} P_\alpha$ is closed.

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Proof of third point: If $\{P_i\}_{1 \leq i \leq n}$ is a finite collection of closed set, then for every $1 \leq i \leq n$, ${}^cP_i$ is open. Now, we use the relation

$${}^c\left(\bigcup_{1 \leq i \leq n} P_i\right) = \bigcap_{1 \leq i \leq n} {}^cP_i.$$

This is a *finite* intersection of open sets, hence it is open. Hence $\bigcup\limits_{1 \leq i \leq n} P_i$ is closed.

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II - Topology of $\mathbb{R}^n$

III - Topology of subsets of $\mathbb{R}^n$

VI - Continuous maps

Open balls of $\mathbb{R}^n$

We want to give $\mathbb{R}^n$ a topology.

The Euclidean metric on $\mathbb{R}^n$ is defined for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ as:

$$\|x\| = \sqrt{x_1^2 + ... + x_n^2}.$$

Definition: Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center x and radius r, denoted $\mathcal{B}(x, r)$, is defined as:

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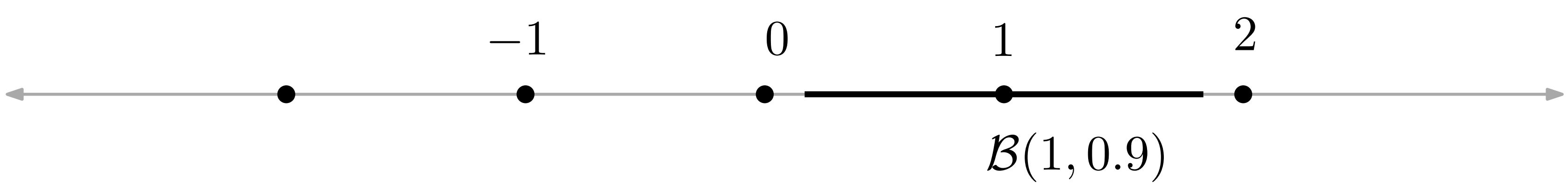
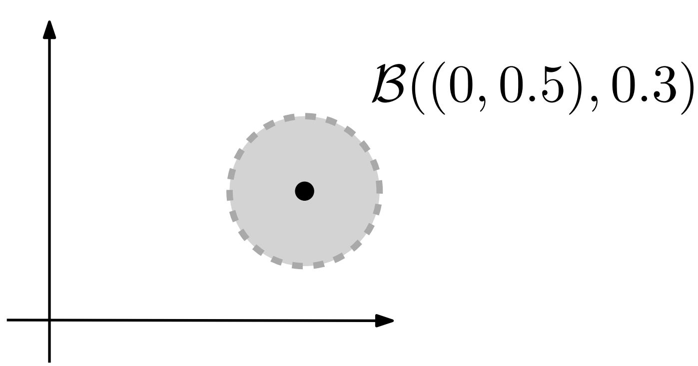
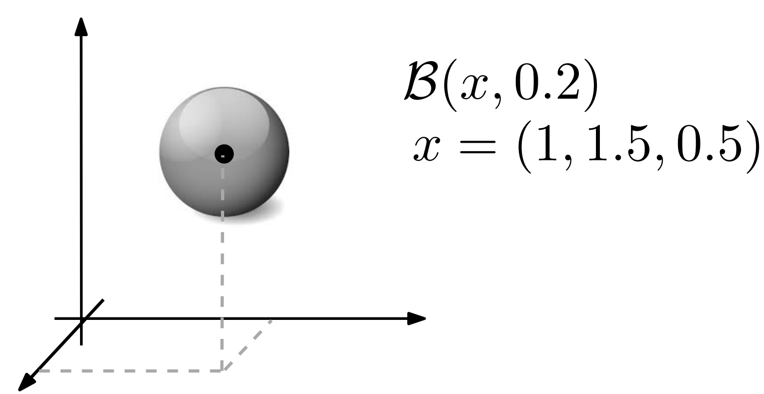


In $\mathbb{R}^2$:



In $\mathbb{R}^3$:



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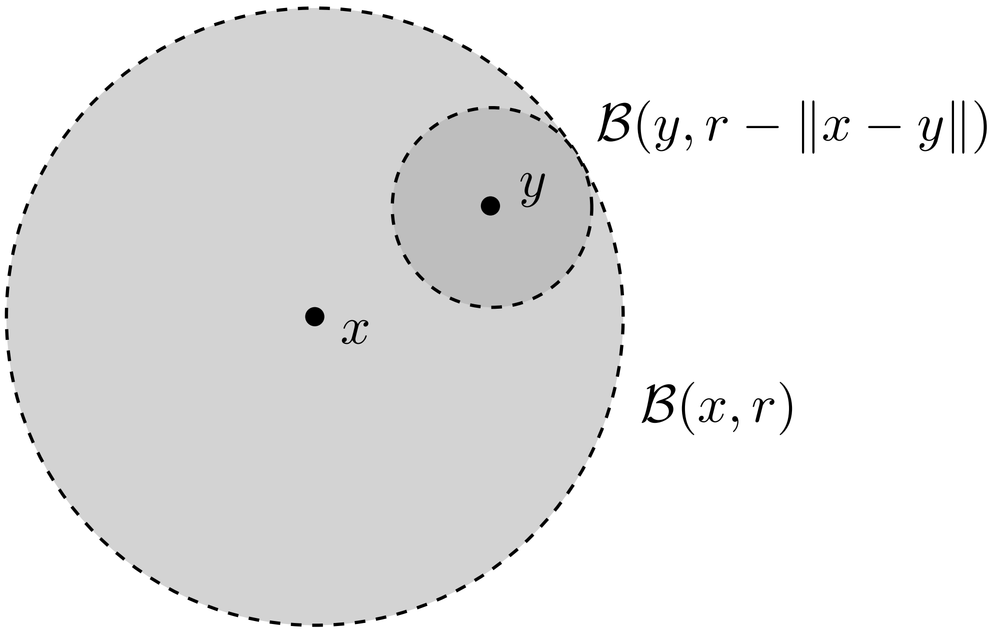
Proposition: Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x,r)$ We have

$$\mathcal{B}(y, r - \|x-y\|) \subset \mathcal{B}(x,r).$$



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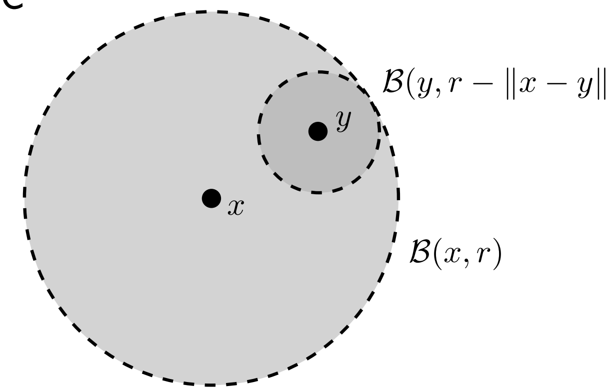


Proof:
By definition,

$$\mathcal{B}(x, r) = \{z \in \mathbb{R}^n, \|x - z\| < r\}$$

$$\mathcal{B}(y, r - \|x - y\|) = \{z \in \mathbb{R}^n, \|y - z\| < r - \|x - y\|\}$$

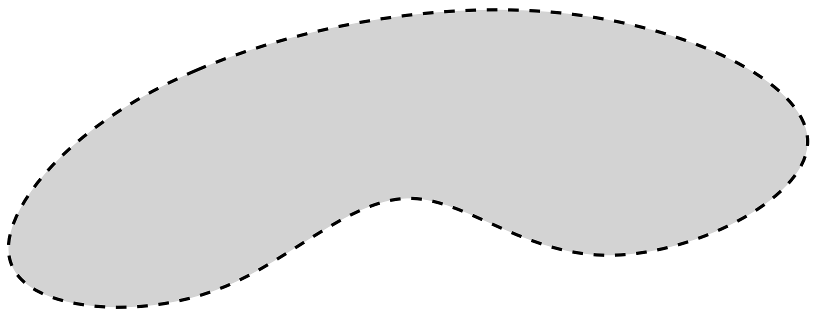
Let $z \in \mathcal{B}(y, r - \|x - y\|)$.
We have to show that $\|x - z\| < r$. But

$$\begin{aligned} \|x - z\| &\leq \|x - y\| + \|y - z\| && \text{(triangle inequality)} \\ &< \|x - y\| + (\|x - y\| - r) && \text{(definition of } z\text{)} \\ &= r \end{aligned}$$

Hence $z \in \mathcal{B}(x, r)$. $\square$

Definition: Let $A \subset \mathbb{R}$ be a subset. Let $x \in A$.
We say that A is *open around* x if there exists $\epsilon > 0$ such that $\mathcal{B}(x, r) \subset A$.
We say that A is *open* if for every $x \in A$, A is open around x.

We denote the set of such open sets by $\mathcal{T}_{\mathbb{R}^n}$, the Euclidean topology on $\mathbb{R}^n$.



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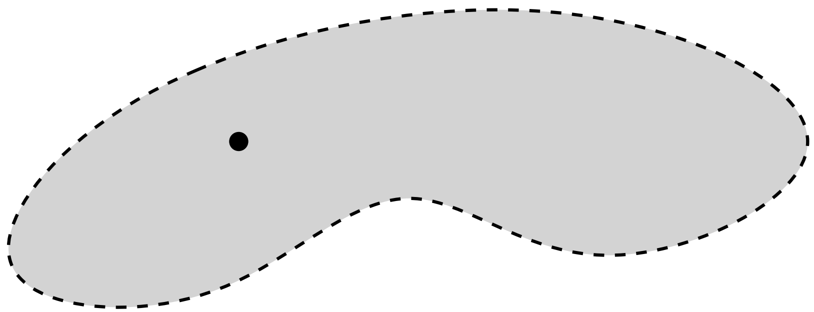

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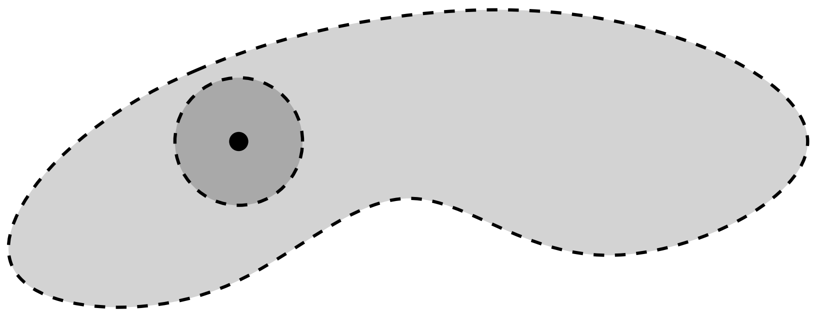

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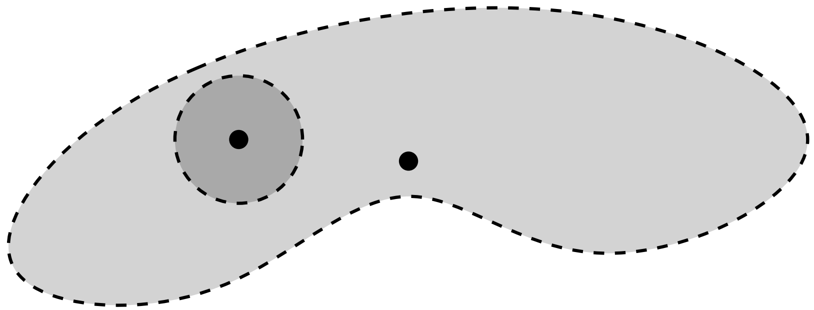

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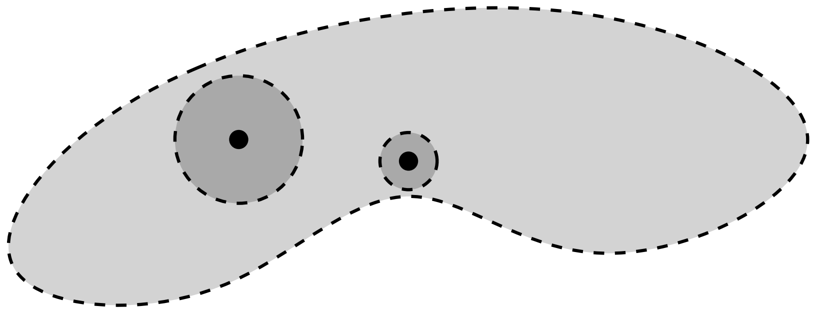
We say that A is *open around* x if there exists $\epsilon > 0$ such that $\mathcal{B}(x, r) \subset A$.
We say that A is *open* if for every $x \in A$, A is open around x.

We denote the set of such open sets by $\mathcal{T}_{\mathbb{R}^n}$, the Euclidean topology on $\mathbb{R}^n$.



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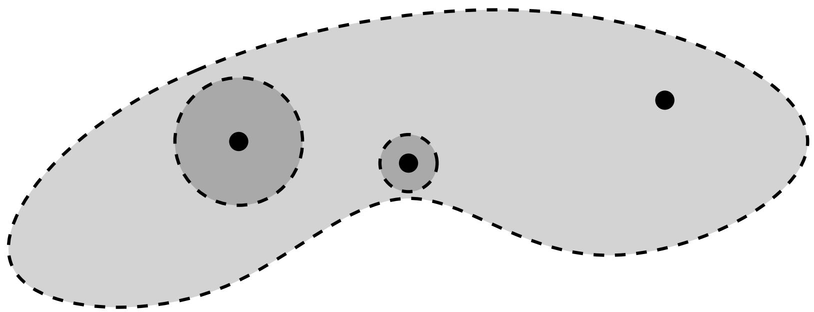

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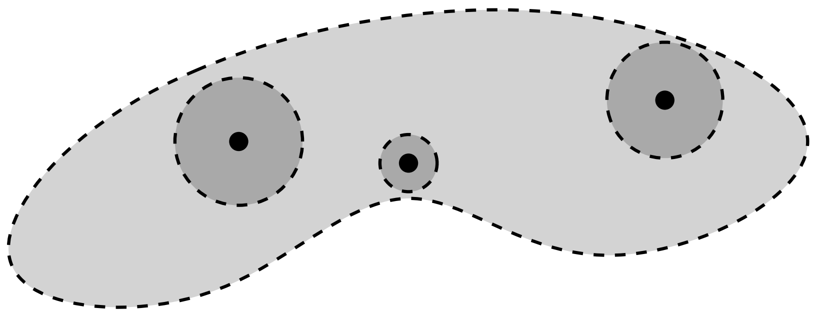

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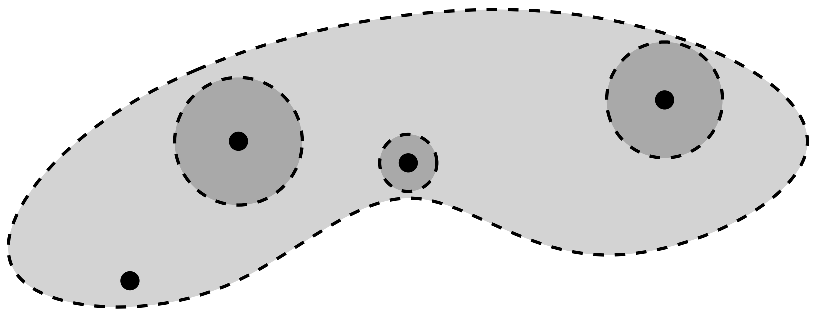

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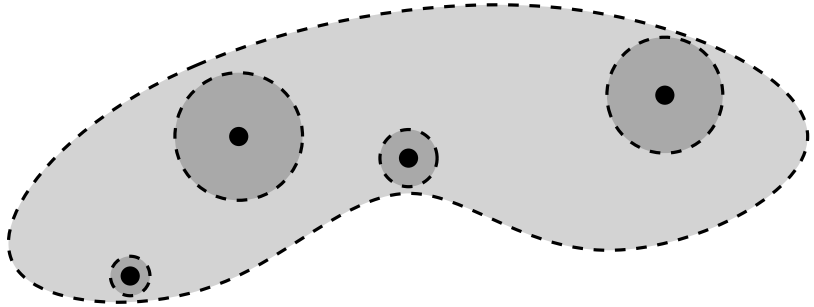

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First axiom (the empty set and the set X are open sets).

The set $\emptyset$ is clearly open according to the definition of $\mathcal{T}_{\mathbb{R}^n}$ (indeed, $\emptyset$ contains no point.)

The set $\mathbb{R}^n$ also is open: for every $x \in \mathbb{R}^n$, the ball $\mathcal{B}(x, 1)$ is a subset of $\mathbb{R}^n$.

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Proposition: $\mathcal{T}_{\mathbb{R}^n}$ is a topology on $\mathbb{R}^n$.

Proof:
Second axiom (an infinite union of open sets is an open set).

Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}_{\mathbb{R}^n}$ be a infinite collection of open sets, and define

$$O = \bigcup_{\alpha \in A} O_\alpha.$$

Let $x \in O$. There exists an $\alpha \in A$ such that $x \in O_\alpha$. Since O_α is open, it is open around x, i.e. there exists $r > 0$ such that $\mathcal{B}(x, r) \subset O_\alpha$.
We deduce that $\mathcal{B}(x, r) \subset O$, and that O is open around x.

Since this it true for any $x \in O$, we proved that O is open.

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Third axiom (a finite intersection of open sets is an open set).
Consider a finite collection $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}_{\mathbb{R}^n}$, and define

$$O = \bigcap_{1 \leq i \leq n} O_i.$$

Let $x \in O$. For every $i \in [\![1, n]\!]$, we have $x \in O_i$. Since O_i is open, it is open around x, i.e. there exists $r_i > 0$ such that $\mathcal{B}(x, r_i) \subset O_i$.

Define $r_{\mathsf{min}} = \min\{r_1, ... r_n\}$. For every $i \in [\![1, n]\!]$, we have $\mathcal{B}(x, r_{\mathsf{min}}) \subset O_i$.

We deduce that $\mathcal{B}(x, r_{\mathsf{min}}) \subset O$, and that O is open around x.

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Euclidean topology - case of $\mathbb{R}$

Proposition: In $(\mathbb{R}^n, \mathcal{T}_{\mathbb{R}^n})$, the open balls $\mathcal{B}(x, r)$ are open sets.

In particular, in $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$, the open intervals (a, b) are open sets.

Exercise:
Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?

- $[0, 1]$,
- $[0, 1)$,
- $(-\infty, 1)$,
- the singletons $\{x\}$, $x \in \mathbb{R}$,
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Not open:
with $x = 0$, there exist no $r > 0$ such that $\mathcal{B}(x,r) = (x-r, x+r) \subset [0,1]$

Closed:
its complementary is ${}^c[0,1] = (-\infty, 0) \cup (1, +\infty)$. It is the union of two open sets.

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This set is not open around 1.

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It is an interval

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It is not open around x.

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I - Topological spaces

II - Topology of $\mathbb{R}^n$

III - Topology of subsets of $\mathbb{R}^n$

VI - Continuous maps

Subspace topology

Definition: Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$ a subset. We define the *subspace topology on* Y as the following set:

$$\mathcal{T}_{|Y} = \{O \cap Y, O \in \mathcal{T}\}.$$

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Proposition: The set $\mathcal{T}_{|Y}$ is a topology on Y.

Proof: We have to check the three axioms of a topological space.

First axiom (the empty set and the set X are open sets).

The set $\emptyset$ is clearly open for $\mathcal{T}_{|Y}$ because it can be written $\emptyset \cap Y$. The set Y also is open for $\mathcal{T}_{|Y}$ because it can be written $X \cap Y$, and X is open for $\mathcal{T}$.

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Second axiom (an infinite union of open sets is an open set).

Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}_{|Y}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha \in A} O_\alpha$.

By definition of $\mathcal{T}_{|Y}$, for every $\alpha \in A$, there exists O'_α such that $O_\alpha = O'_\alpha \cap Y$.

Define $O' = \bigcup_{\alpha \in A} O'_\alpha$. It is an open set for $\mathcal{T}$. We have

$$O = \bigcup_{\alpha \in A} O_\alpha = \bigcup_{\alpha \in A} O'_\alpha \cap Y = \left(\bigcup_{\alpha \in A} O'_\alpha \right) \cap Y = O' \cap Y.$$

Hence $O \in \mathcal{T}_{|Y}$.

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Consider a finite collection $\{O_i\}_{1\leq i\leq n} \subset \mathcal{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1\leq i\leq n} O_i$.

Just as before, for every $i \in [\![1, n]\!]$, there exists O'_i such that $O_i = O'_i \cap Y$.

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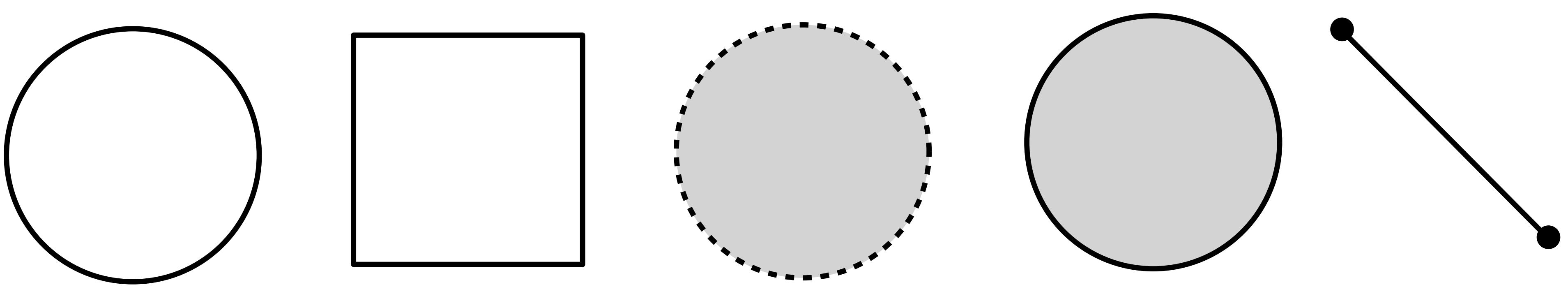
Among the subsets of $\mathbb{R}^n$ that we will consider, let us list:

- the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, \|x\| = 1\}$
- the unit cube $\mathcal{C}_{n-1} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n, \max(|x_1|, ..., |x_n|) = 1\}$
- the open balls $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}$
- the closed balls $\overline{\mathcal{B}}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| \leq r\}$
- the standard simplex

$$\Delta_{n-1} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n, x_1, ..., x_n \geq 0 \text{ and } x_1 + ... + x_n = 1\}$$



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Continuous maps

The topologist's point of view allows to define the notion of continuity in great generality.
Let us consider two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$.

Definition: Let $f\colon X \to Y$ be a map. We say that f is *continuous* if for every $O \in \mathcal{U}$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in $\mathcal{T}$.

In other words, a map is continuous if **the preimage of any open set is an open set**.

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Proposition: A map is continuous if and only if the preimage of closed sets are closed sets.

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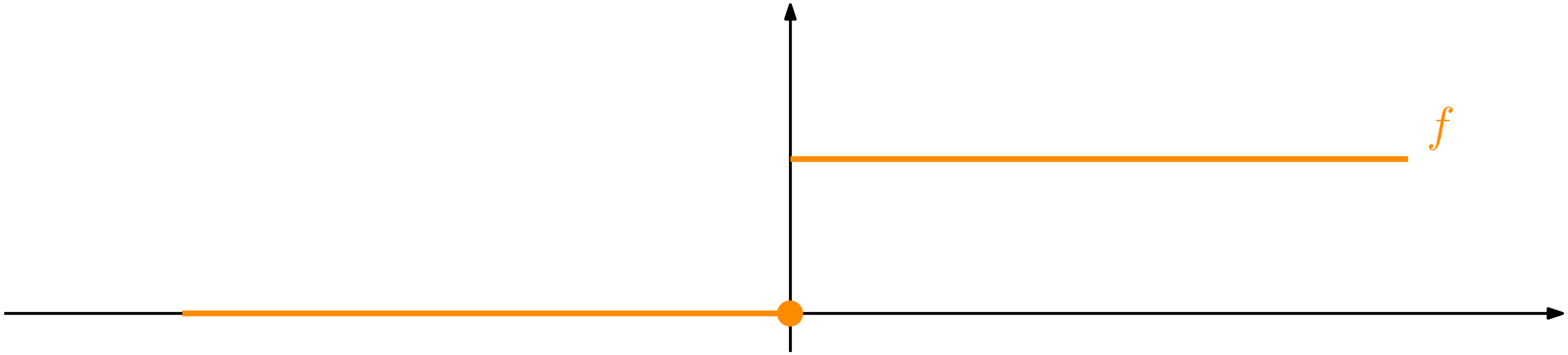
Example: Let $X = Y = \mathbb{R}$, endowed with the Euclidean topology.
Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$ for all $x \leq 0$, and $f(x) = 1$ for all $x > 0$.
The set $\{0\}$ is closed, but $f^{-1}(\{0\}) = (-\infty, 0)$ is not. Hence f is not continuous.



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Proposition: Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be three topological spaces, and $f\colon X \to Y$, $g\colon Y \to Z$ two continuous maps. The composition $g \circ f$, defined as

$$\begin{aligned} g \circ f\colon X &\longrightarrow Z \\ x &\longmapsto g(f(x)) \end{aligned}$$

is a continuous map.

Proof: Let $O \in \mathcal{V}$ be an open set of Z. We have to show that $(g \circ f)^{-1}(O)$ is in $\mathcal{T}$. First, note that $(g \circ f)^{-1}(O) = f^{-1}\big(g^{-1}(O)\big)$.

Since g is continuous, the set $g^{-1}(O)$ is in $\mathcal{U}$, i.e., it is an open set of Y.

But since f is continuous, its preimage $f^{-1}\big(g^{-1}(O)\big)$ also is an open set (of X).

Since this is true for any open set $O \in \mathcal{V}$, we deduce that $g \circ f$ is continuous.

Link with ϵ-δ calculus

We now investigate what continuity means between the Euclidean spaces $\mathbb{R}^n$.

Consider a continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $\epsilon > 0$.

We have seen that the open ball $\mathcal{B}(f(x), \epsilon)$ is an open set of $\mathbb{R}^m$. By continuity of f, the preimage $f^{-1}(\mathcal{B}(f(x), \epsilon))$ is an open set.

Note that x belongs to $f^{-1}(\mathcal{B}(f(x), \epsilon))$. By definition of the Euclidean topology, we have that:

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We deduce that, for all $y \in \mathbb{R}^n$,

$$\|x - y\| < \eta \implies \|f(x) - f(y)\| < \epsilon.$$

We recognize **the usual definition of continuity**.

Link with ϵ-δ calculus

Proposition: A map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if, for every $x \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in \mathbb{R}^n$,

$$\|x - y\| < \eta \implies \|f(x) - f(y)\| < \epsilon.$$

As a consequence, what you already know about continuity still applies here.

Conclusion

We have generalized the notion of continuity (from ϵ-δ calculus) to topological spaces.

This will allow us to define more general concepts (connectedness, triangulations, topological functoriality, ...)

Homework for tomorrow: Exercise 4 and 5

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Thank you!