EMAp Seminar — 25/04/2024

Introduction to triangulation of manifolds and simplicial approximation in practice

https://raphaeltinarrage.github.io

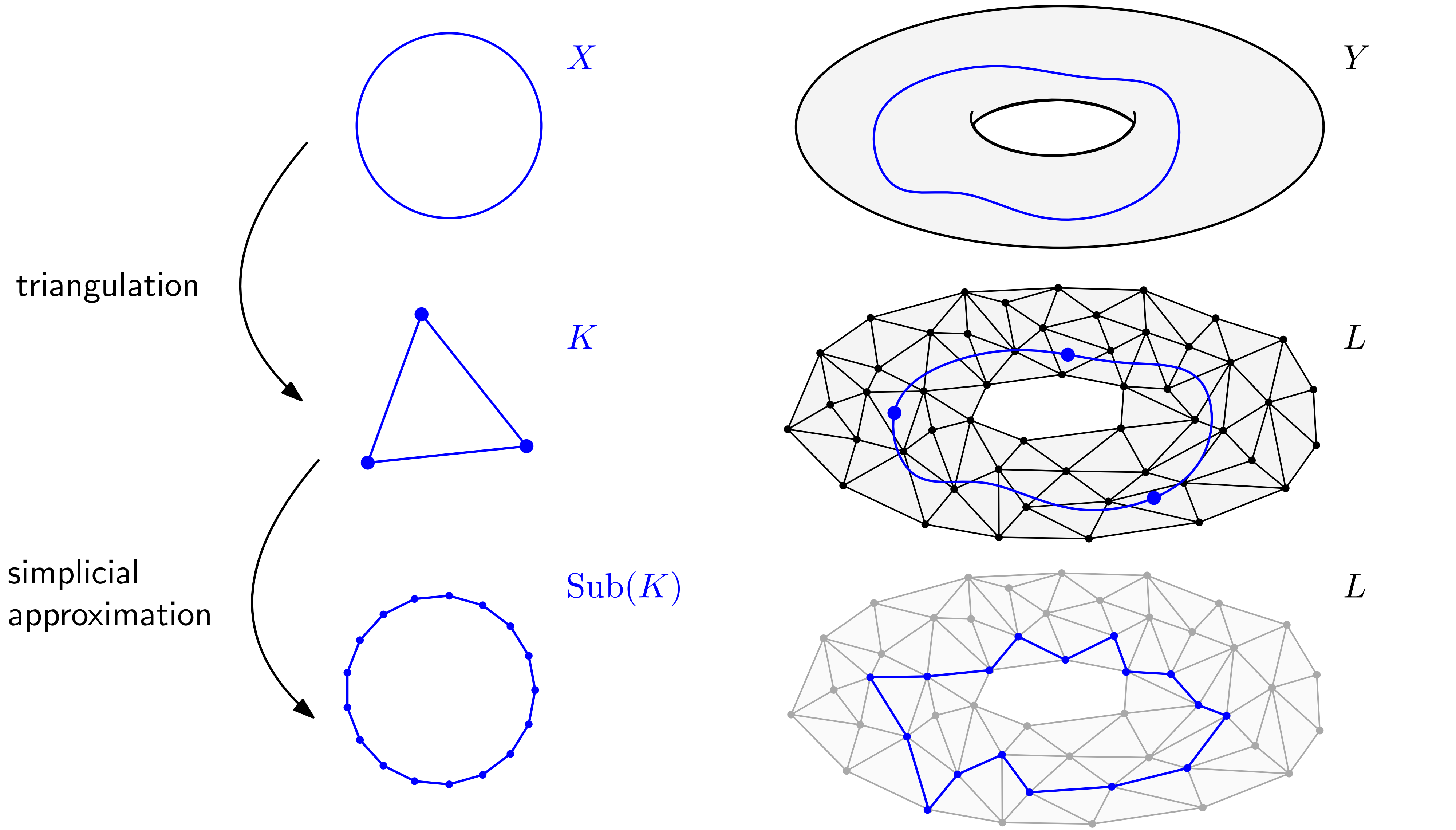
Last update: April 25, 2024

Simplicial approximation

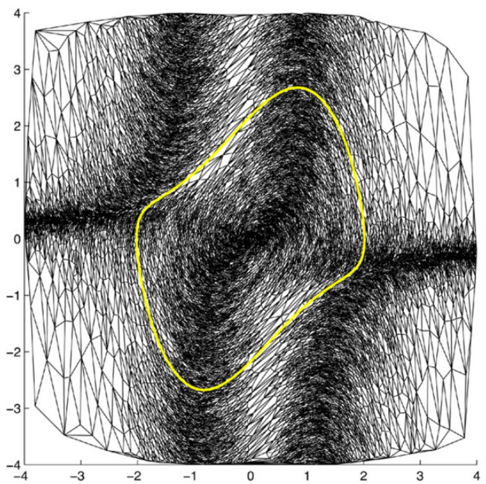
A topological space X is a n-**manifold** if each point admits a neighborhood homeomorphic to $\mathbb{R}^n$.

A **simplicial complex** over a set V is a subset $K \subset \mathcal{P}(V)$ such that $\sigma \in K$ and $\nu \subset \sigma \implies \nu \in K$.

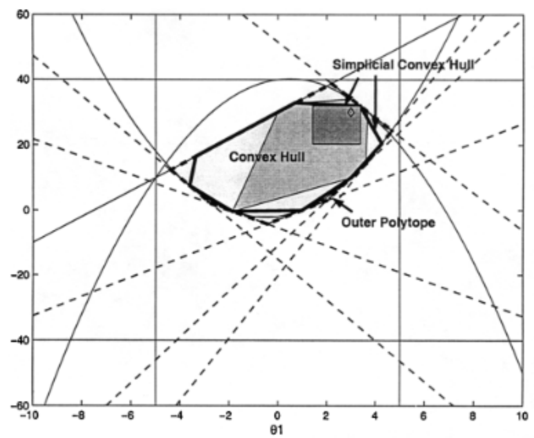
A map $f\colon K \to L$ between simplicial complexes is a **simplicial map** if $\sigma \in K \implies f(\sigma) \in L$.



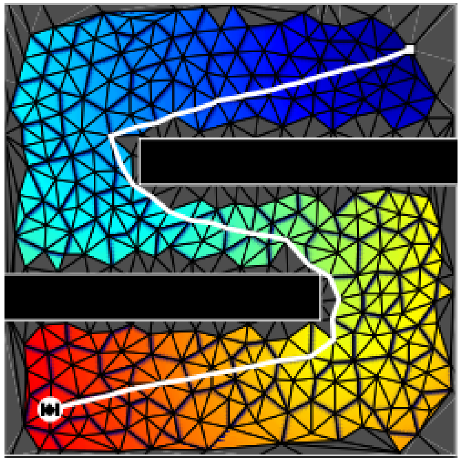
Applications of simplicial approximation:



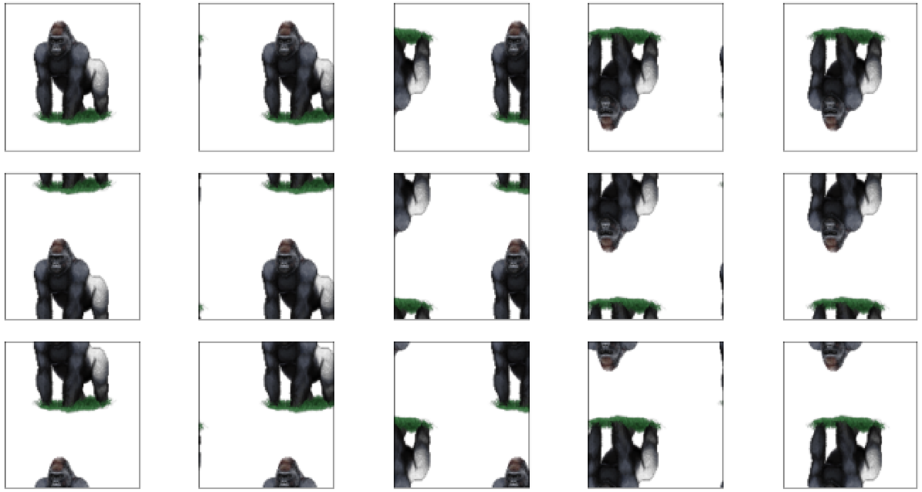
[Polygonal approximation of flows, Boczko, Kalies, Mischaikow, 2006]



[Stochastic MINLP optimization using simplicial approximation, Goyal, Lerapetritou, 2007]



[A Randomized Greedy Algorithm for Piecewise Linear Motion Planning, Ortiz, Lara, González, Borat, 2021]



[Computing persistent Stiefel-Whitney classes of line bundles, T, 2021]

Simplicial methods in early algebraic topology


H. Poincaré
1854 - 1912


A. Whitehead
1861 - 1947


L. E. J. Brouwer
1881 - 1966


S. Lefschetz
1884 - 1972


J. W. Alexander
1888 - 1971


W. Hurewicz
1904 - 1956

Before Poincaré: Combinatorial relations in topological problems
Euler characteristic (1758): in a planar graph, $\#\text{Faces} - \#\text{Edges} + \#\text{Vertices} = 2$.
Riemann's inequality (1857) about algebraic curves. Betti's order of connections (1871).

1895: Poincaré's Analysis Situs
Defines fundamental group (for manifolds) and simplicial homology (for triangulated manifolds).
Conjectures: existence of triangulations, hauptvermutung, topological invariance of homology.

1910: Brouwer's simplicial approximation
Defines the degree of maps. Proves the invariance of domain ($\mathbb{R}^n \simeq \mathbb{R}^m \iff n = m$), fixed point theorem (for maps $f\colon B^n \to B^n$), Jordan–Brouwer separation (for $f\colon S^{n-1} \to \mathbb{R}^n$).

1915: Alexander generalizes the simplicial approximation
For any continuous map between simplicial complexes. Proves invariance of homology.

Hauptvermutung (equivalence of triangulations)

1920, Radó: true for manifolds of dimension 2

1950, Moise: true for manifolds of dimension 3

1961, Milnor: false for a simplicial complex of dimension ≥ 6

1969, Kirby and Siebenmann: false for certain manifolds of dimension ≥ 5

Triangulation conjecture (existence of triangulations)

1935, Cairns: true for smooth manifolds

1990, Casson: false for a topological manifold of dimension 4

2013, Manolescu: false for certain topological manifolds of dimension ≥ 5

From combinatorial topology to homotopy theory

1925: Singular homology, without triangulations
By Princeton topologists (Veblen, Alexander, Lefschetz) and Eilenberg.

1935: Hurewicz generalizes the homotopy groups
Defines homotopy equivalence. Isomorphism theorem $\pi_n(X) \to H_n(X)$ if $(n-1)$-connected.

1938: Whitehead's CW complexes
Generalization of simplicial complexes, via mapping cones. Cellular homology.

Libraries for topology:

- algebra-oriented: `Magma`, `CHomP`, `GAP`
- 3-dimensional manifolds: `regina`, `SnapPy`, `Twister`
- data analysis: `gudhi`, `TTK`, `ripser`

Having acess to a triangulation of a space allows to:

- compute algorithmically certain of its **topological invariants**: (co)homology groups over $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}$, fundamental group, Stiefel-Whitney classes, first Pontryagin class.
- study its **combinatorial complexity**: size of minimal triangulations, Lusternik–Schnirelmann category.
- study **quantitative homotopy**
- algorithms in **Topological Data Analysis**:

  Persistent Stiefel-Whitney classes from $X \to \mathcal{G}(d, \mathbb{R}^n)$ (Grassmannian) [T., Computing Persistent Stiefel-Whitney classes of line bundles, 2021]

  Lens-PCA dimensionality reduction from $X \to \mathbb{S}^\infty/(\mathbb{Z}/p\mathbb{Z})$ (infinite Lens space) [Luis Polanco, Jose A. Perea, Coordinatizing Data With Lens Spaces and Persistent Cohomology , 2019]

Computational aspects

Known explicit triangulations:

- the surfaces,
- $\mathbb{S}^n$, the spheres, for any $n \geq 1$,
- $\mathbb{R}P^n$ and $\mathbb{C}P^n$, the real and complex projective space, for any $n \geq 1$
- $SO(n)$, the special orthogonal group, only when $n \leq 4$,
- $\mathcal{V}(d, n)$, the Stiefel manifold of d-frames in $\mathbb{R}^n$, only when $n \leq 3$, when $(d, n) = (3, 4)$, or $d = 1$ (these cases correspond to the spheres),
- $\mathcal{G}(d, n)$, the Grassmann manifolds of d-planes in $\mathbb{R}^n$, only when $d = 1$ or $n - 1$ (these cases correspond to the real projective spaces).

How to find triangulations:

- creating families of triangulations “by hand”,
- computational geometry point of view: embedding the manifold in some Euclidean space, generating a finite sample of it, and building a triangulation on top of these points,
- by exhaustive enumeration of all combinatorial manifolds,
- via simplicial approximation to CW complexes.

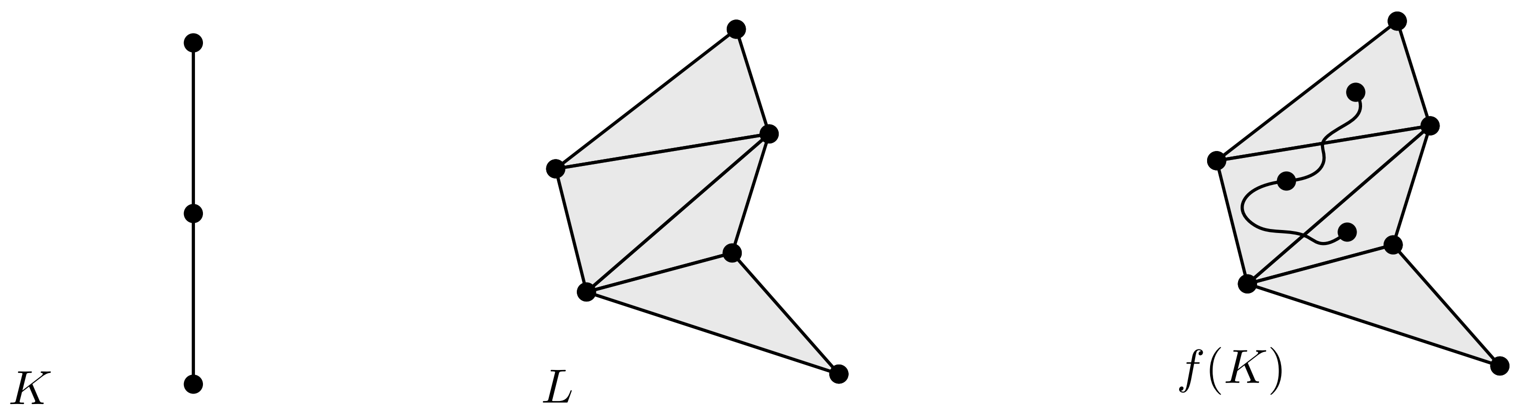
1. Simplicial approximation
2. Computational improvements
3. Simplicial approximation to CW complexes
4. Weak simplicial approximation

Input: Two simplicial complexes K, L and a continuous map $f\colon |K| \to |L|$.

Output: A simplicial map $g\colon K \to L$ such that $|g|\colon |K| \to |L|$ is homotopic to f.

Notations:

- $|K|$ denotes the geometric realization of K (it is a topological space),
- $|g|$ denotes the topological realization of g (it is a continuous map),
- $K^{(0)}$ denotes the vertex set of K.



Simplicial approximation

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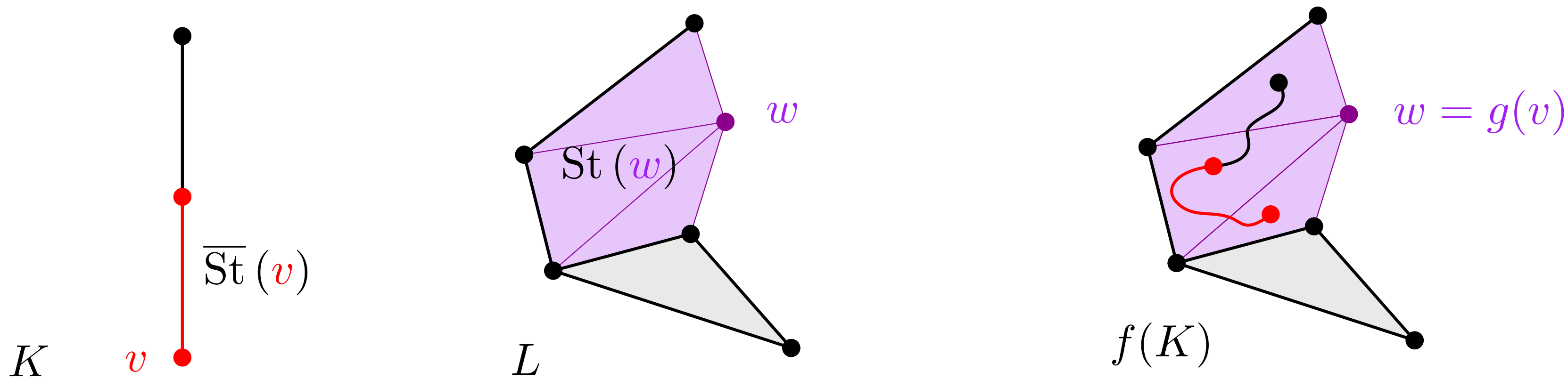
- $|K|$ denotes the geometric realization of K (it is a topological space),
- $|g|$ denotes the topological realization of g (it is a continuous map),
- $K^{(0)}$ denotes the vertex set of K.

Define, for all vertex $v \in K^{(0)}$, its *open star* and its *closed star*

$$\mathrm{St}\,(v) = \{\sigma \in K \mid v \in \sigma\} \qquad \overline{\mathrm{St}}\,(v) = \{\tau \in K \mid \exists \sigma \in \mathrm{St}\,(v)\,,\ \tau \subset \sigma\}$$

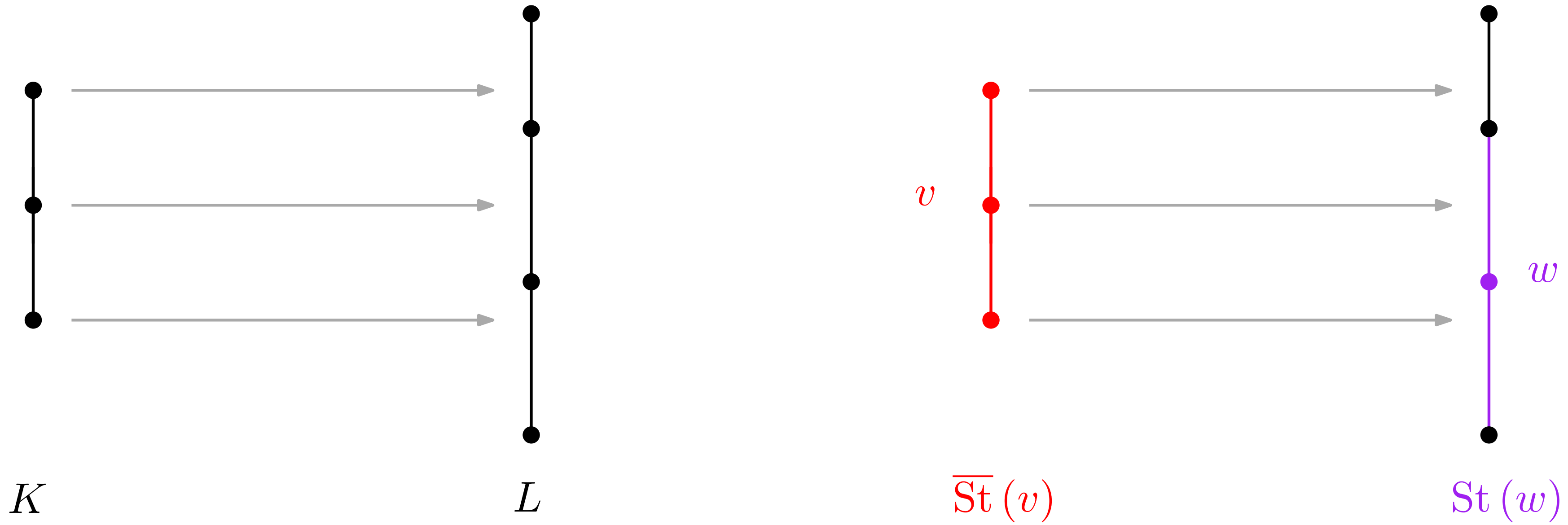
The map f satisfies the **star condition** if $\forall v \in K^{(0)}$, $\exists w \in L^{(0)}$ such that $f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(w)\,|$.



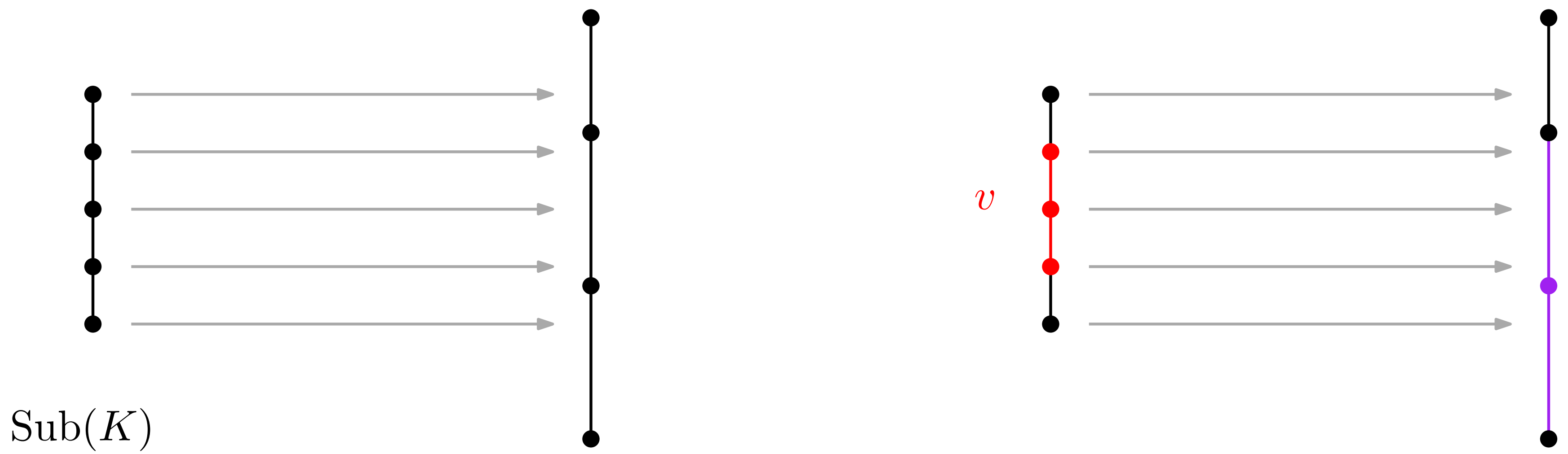
If this is the case, let $g\colon K^{(0)} \to L^{(0)}$ be such that $\forall v \in K^{(0)}$, $f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(g(v))\,|$.

Such a map g is called a **simplicial approximation** to f. It is a simplicial, and homotopic to f.

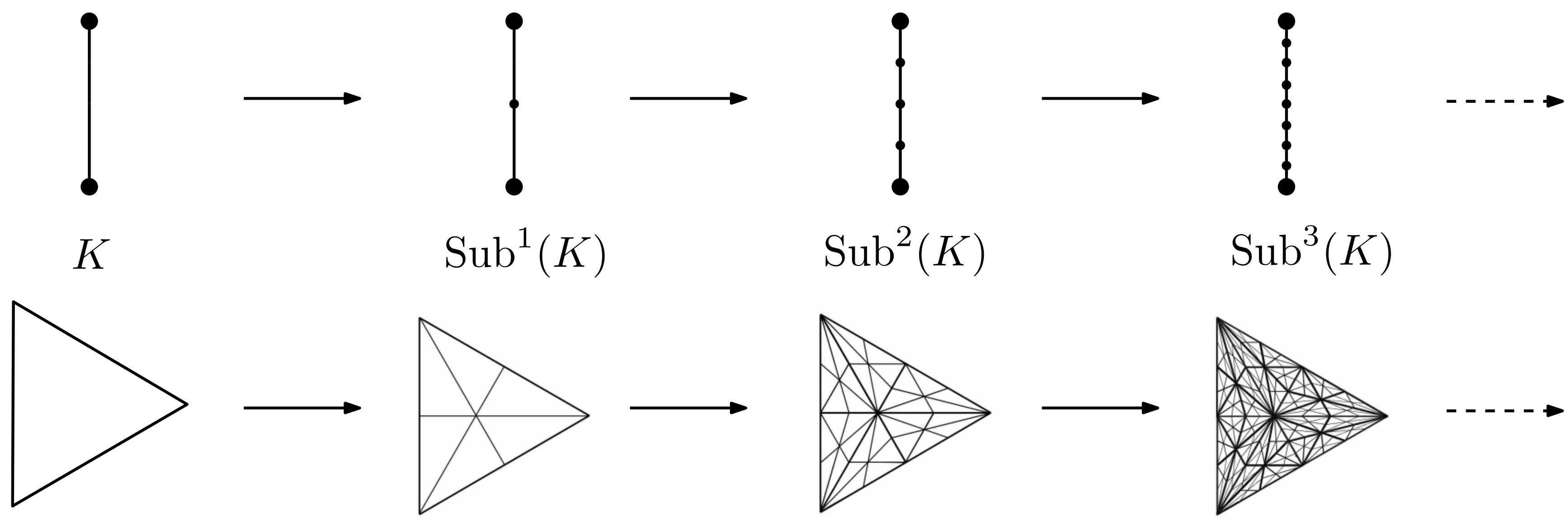
What if the map f does not satisfy the star condition?



One can refine K via *barycentric subdivision*.



Simplicial approximation theorem: By repeating barycentric subdivisions on K, the map $f\colon |\mathrm{Sub}^k(K)| \to |L|$ satisfies the star condition at some point.



Proof: Endow K with a metric, and by denote $\mathcal{U}$ the cover $\left\{ f^{-1}(|\mathrm{St}\,(w)\,|) \mid w \in L \right\}$ of $|K|$.
A **Lebesgue number** for $\mathcal{U}$ is a $\epsilon > 0$ such that

$$\forall x \in |K|,\ \exists U \in \mathcal{U},\ \mathcal{B}(x, \epsilon) \subset U \quad \text{(open ball of radius } \epsilon\text{).}$$

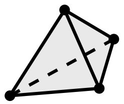
Hence, f satisfies the star condition if for every $v \in K^{(0)}$, $\mathrm{Diameter}\left(|\overline{\mathrm{St}}\,(v)\,|\right) < \epsilon$.

But barycentric subdivision reduces the diameter of a d-simplex by a factor $\frac{d}{d+1}$. Hence each simplex is small enough at some point.

Edgewise sudivisions

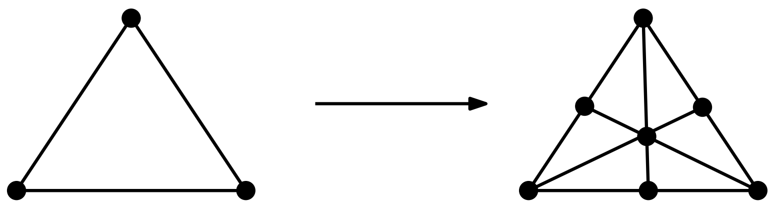
Barycentric subdivisions increase the number of simplices drastically:
a d-simplex turns into a simplicial complex with $(d+1)!$ simplices and $2^{d+1}-1$ vertices.

Example: Triangulation of the unit sphere $\mathbb{S}^2$, starting from the boundary of the 3-simplex



	K	$\mathrm{Sub}^1(K)$	$\mathrm{Sub}^2(K)$	$\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	$\mathrm{Sub}^5(K)$
vertices:	4	14	74	434	2594	15554
simplices:	14	74	434	2594	15554	93314
max diameter:	1.63	1.15	0.66	0.42	0.25	0.15

Barycentric subdivision:



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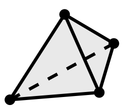

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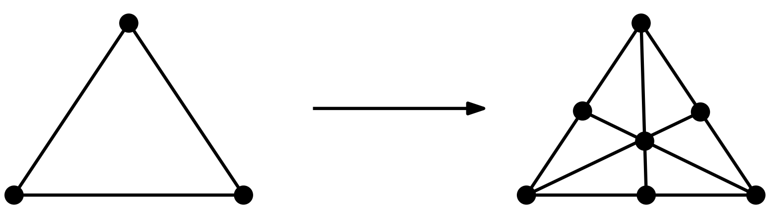
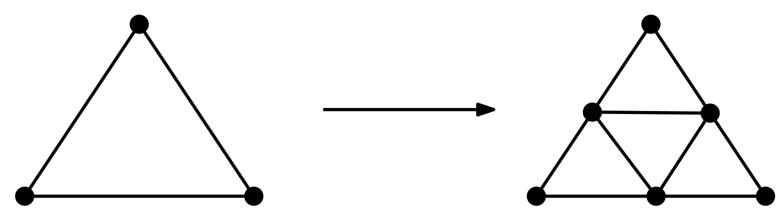


We can do better with **edgewise subdivisions**:

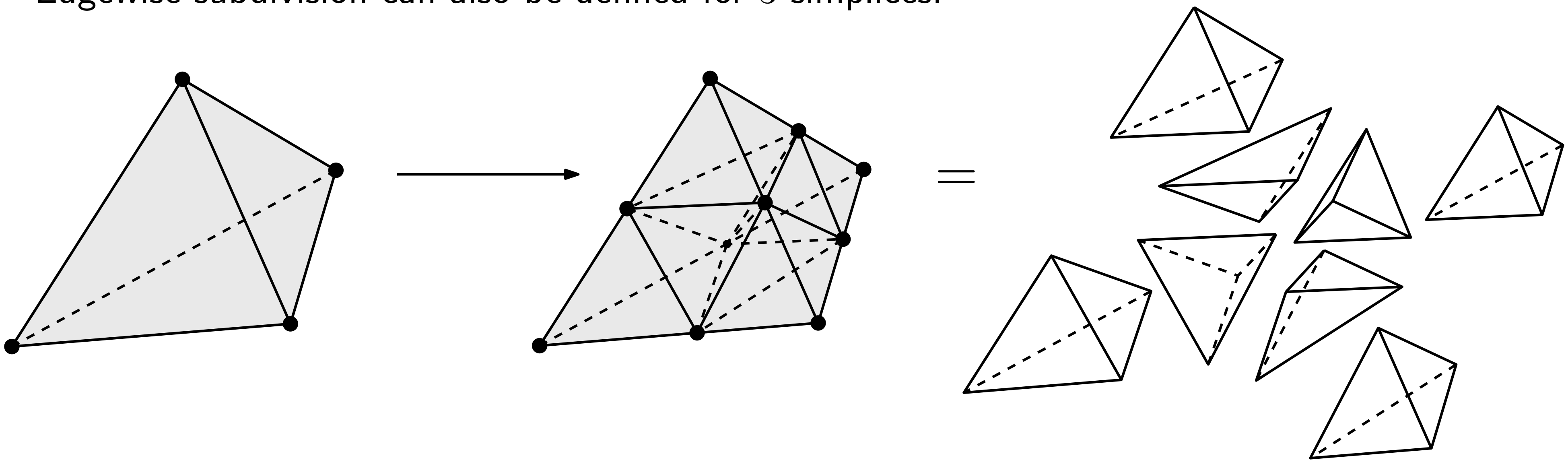


	K	$\mathrm{Sub}^1(K)$	$\mathrm{Sub}^2(K)$	$\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	$\mathrm{Sub}^5(K)$
vertices:	4	10	39	130	514	2050
simplices:	14	50	194	770	3074	12290
max diameter:	1.63	1.41	1	0.58	0.30	0.15

Edgewise sudivisions

[Edgewise subdivision of a simplex, Edelsbrunner & Grayson, 2000]

Edgewise subdivision can also be defined for 3-simplices:

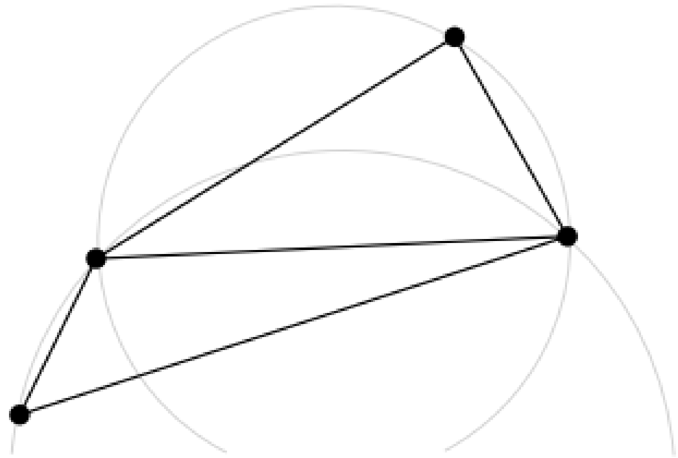


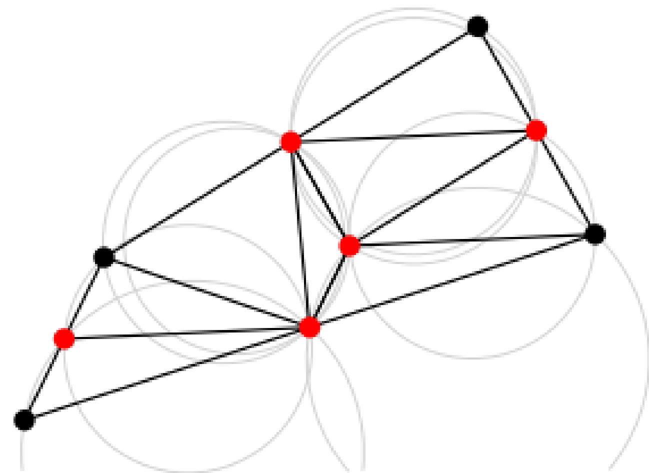
Example: Triangulation of the sphere $\mathbb{S}^3$, starting from the boundary of the 4-simplex.

	barycentric: $\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	edgewise: $\mathrm{Sub}^5(K)$	$\mathrm{Sub}^6(K)$
vertices:	$12'600$	$301'680$	$27'440$	$218'720$
simplices:	$301'680$	$7'238'880$	$710'240$	$5'680'320$
max diameter:	0.54	0.36	0.47	0.29

Delaunay subdivisions

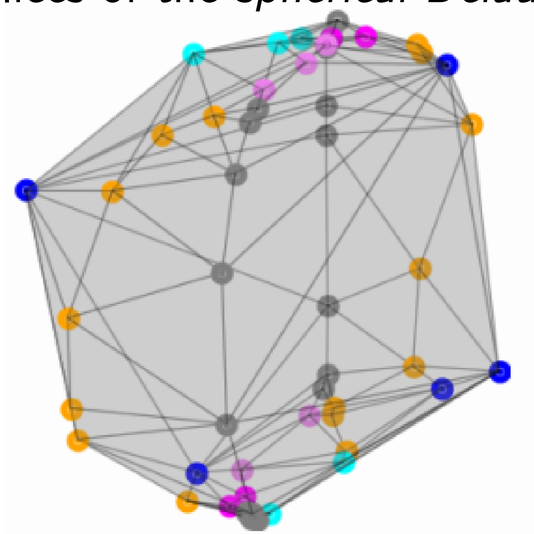
Delaunay triangulation in Euclidean space. Let $X \subset \mathbb{R}^n$ finite.
A subset of $n+1$ points of X has the *empty circle property* if it has a circumscribing open ball empty of points of X. Their collection forms the maximal simplices of the *Delaunay complex* Del(X)



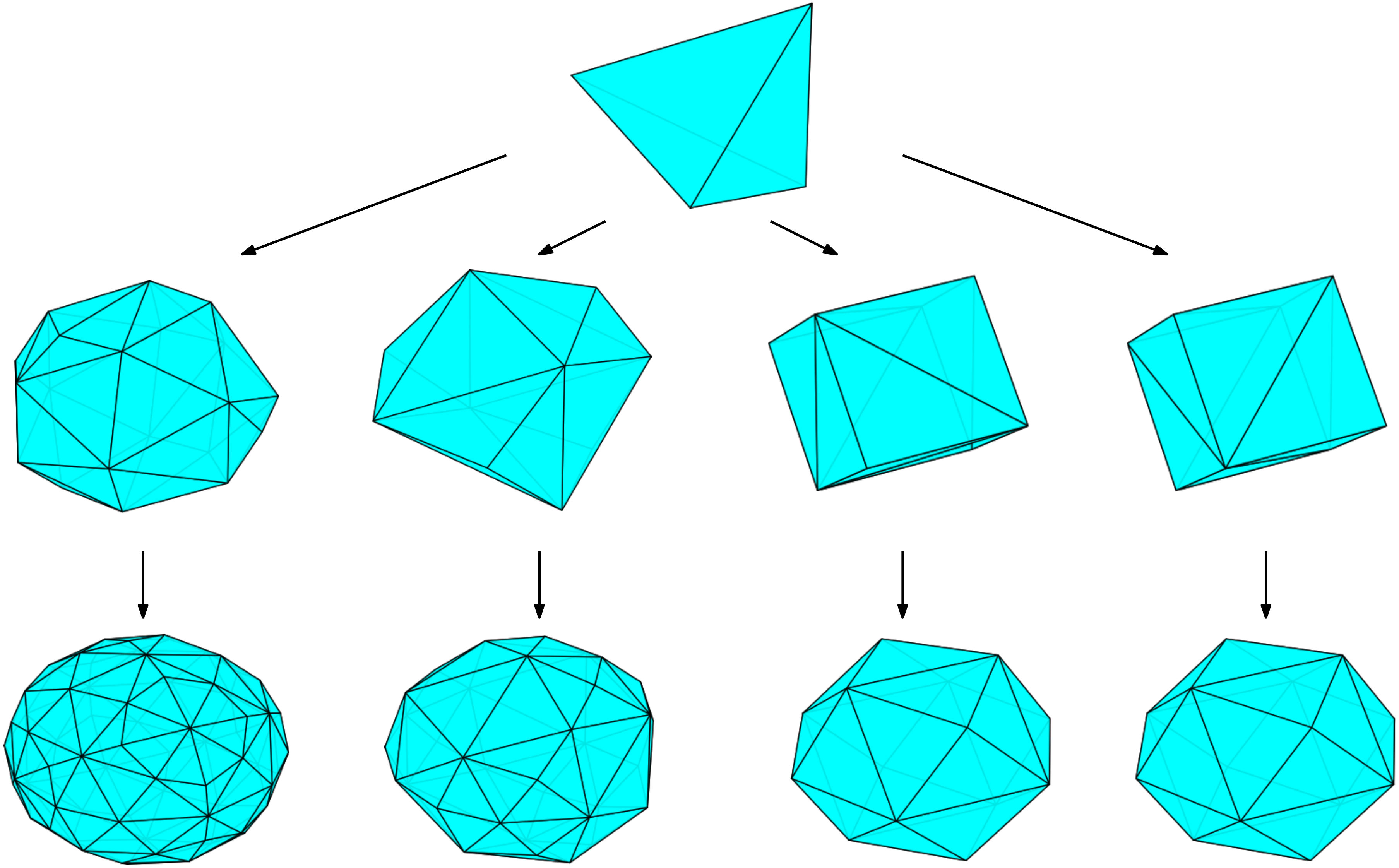


Delaunay triangulation in sphere. Let $X \subset \mathbb{S}^n$ finite.
A subset of $n+1$ points of X has the *empty circle property* if it has a circumscribing open *geodesic* ball empty of points of X. Their collection forms the maximal simplices of the *spherical Delaunay complex* Del(X)

Proposition: On the sphere, Del(X) is the convex hull of X.



One can "subdivide" Delaunay triangulations by adding new vertices.



add **barycenters** of all simplices

add **midpoints** of edges

add **minicenters** (center of smallest enclosing spheres)

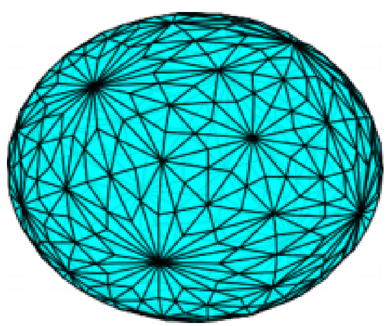
add **centroid** of maximal simplices

Example: simplicial approximation to identity

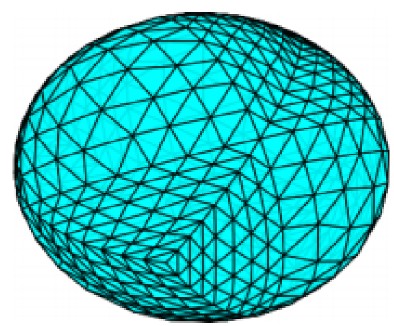
Number of subdivisions (and vertices) for simplicial approximation to $|\partial\Delta^3| \to |\mathrm{sub}^n\left(\partial\Delta^3\right)|$.

	$\partial\Delta^3$	$\mathrm{sub}^1\left(\partial\Delta^3\right)$	$\mathrm{sub}^2\left(\partial\Delta^3\right)$	$\mathrm{sub}^3\left(\partial\Delta^3\right)$
Barycentric	2 (74)	3 (434)	4 (2594)	5 (15554)
Edgewise	2 (34)	4 (514)	5 (2050)	7 (32770)
Delaunay barycentric	2 (74)	3 (434)	5 (15554)	6 (93314)
Delaunay edgewise	2 (34)	4 (514)	5 (2050)	7 (32770)
Delaunay minicenter	3 (38)	5 (326)	7 (2918)	9 (26246)
Delaunay centroid	3 (56)	5 (488)	6 (1460)	8 (13124)

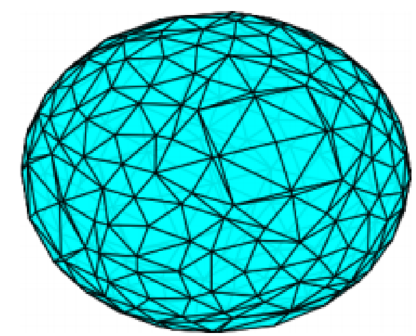
Resulting complexes for $|\partial\Delta^3| \to |\mathrm{sub}^1\left(\partial\Delta^3\right)|$.



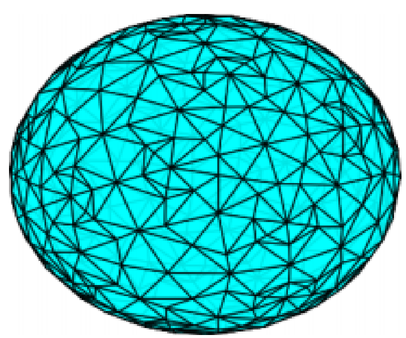
Barycentric



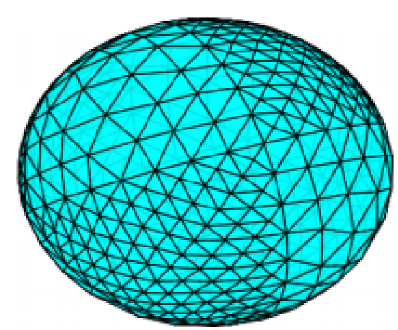
Edgewise



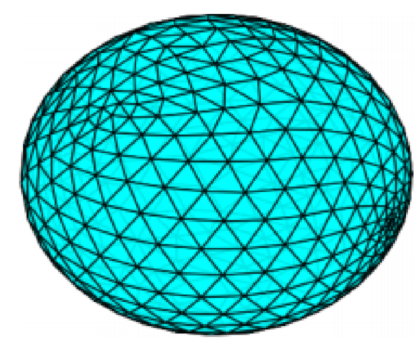
Delaunay minicenter



Delaunay barycentric



Delaunay edgewise

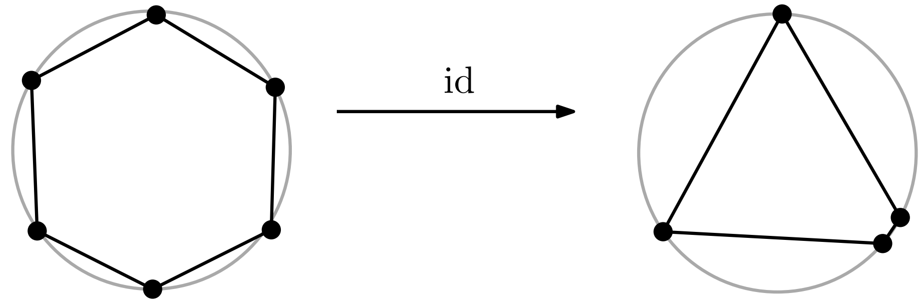


Delaunay centroid

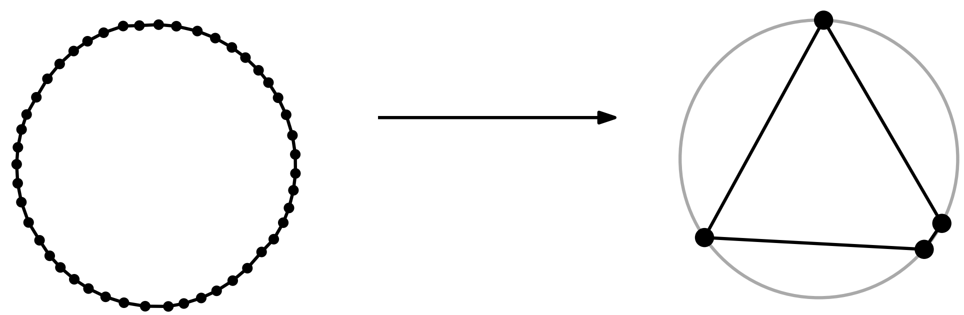
1. Simplicial approximation
2. Computational improvements
3. Simplicial approximation to CW complexes
4. Weak simplicial approximation

Three tricks for simplicial approximation

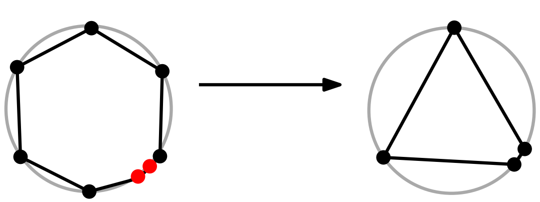
Suppose we want to find a simplicial approximation to the identity on $\mathbb{S}^1$, with the triangulations



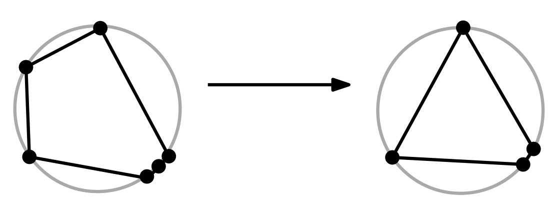
We must subdivise the domain many times to ensure that id satisfies the star condition.



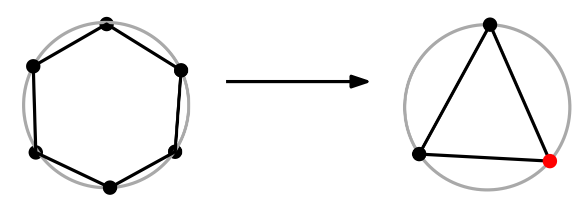
We could have done simpler:



generalized subdivision



Delaunay simplifications



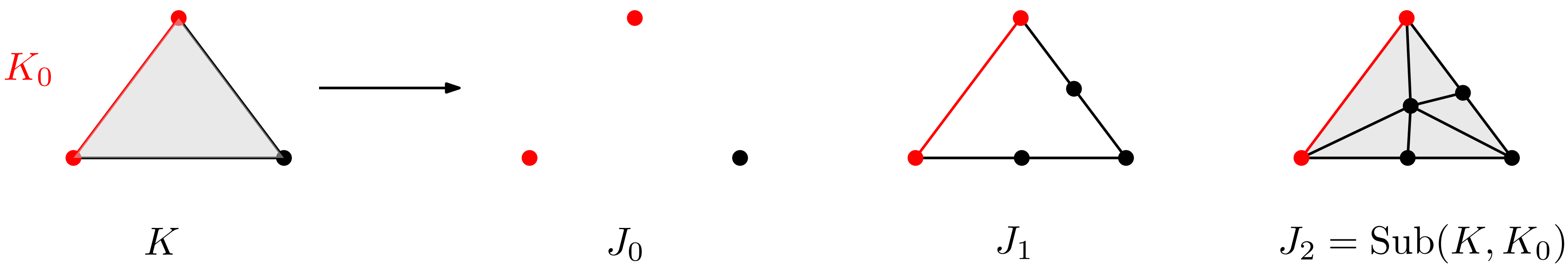
edge contractions

Generalized subdivisions

Let K be a simplicial complex, and $K_0 \subset K$ a sub-complex.

The **barycentric subdivision of** K **holding** K_0 **fixed**, denoted $\mathrm{Sub}(K, K_0)$, is defined by induction. We define simplicial complexes of increasing dimension $J_0, \ldots, J_d$ as follows:

- Start with J_0 the 0-skeleton of K
- From J_{p-1}, build J_p by adding all the p-simplices of K_0
- Moreover, for any p-simplex of K not in K_0, add a point $\hat{\sigma}$, and cone it to the boundary $\partial\sigma \in J_p$

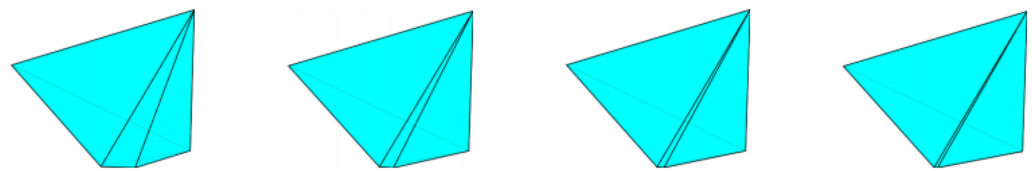


To find a simplicial approximation to $f\colon |K| \to |L|$, it is enough to execute repeated barycentric subdivisions of K holding K_0 fixed, where K_0 is the subcomplex *on which f satisfies the star condition*.

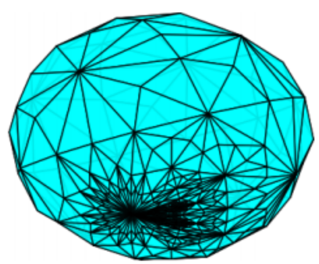
This process can be adapted to edgewise and Delaunay subdivisions.

Generalized subdivisions

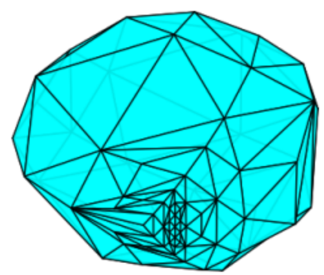
Example: Let L_ϵ be a triangulation of the sphere with an edge of length ϵ. We look for a simplicial approximation to the identity map $|\partial\Delta^3| \to |L_\epsilon|$. We indicate the resulting number of vertices



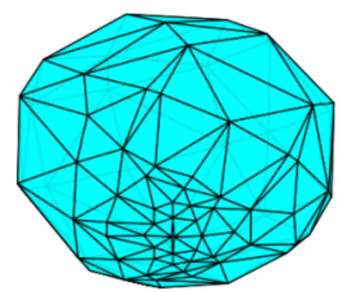
	$L_{0.25}$	$L_{0.1}$	$L_{0.05}$	$L_{0.01}$
Barycentric	262	858	1721	5524
Edgewise	75	90	105	124
Delaunay barycentric	162	201	256	315
Delaunay edgewise	85	94	123	142
Delaunay minicenter	82	105	141	163
Delaunay centroid	102	119	140	152



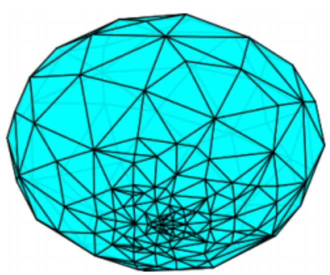
Barycentric



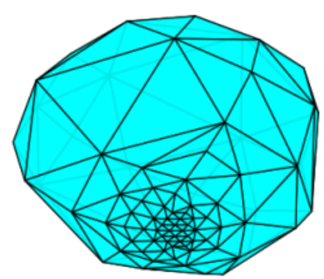
Edgewise



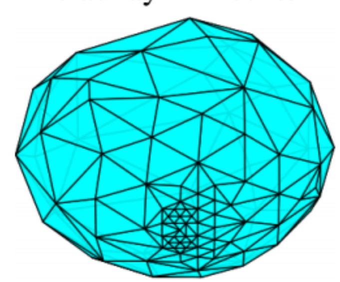
Delaunay minicenter



Delaunay barycentric



Delaunay edgewise



Delaunay centroid

Delaunay simplifications

When working with Delaunay complexes, we can define a post-processing step to reduce the vertices.

Given a simplicial map $g\colon \mathrm{Del}\,(X) \to L$, we say a vertex $v \in X$ satisfies the **simplex condition** if

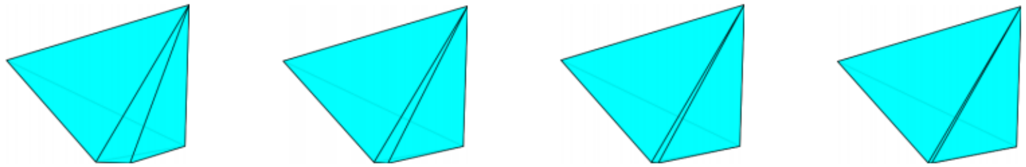
$$g\left(\overline{\mathrm{St}}\,(v)^{\,0}\right) \in L.$$

We can define a map between vertex sets $g'\colon \mathrm{Del}\,(X \setminus \{v\})^{\,0} \to L^{\,0}$ by restricting g to $X \setminus \{v\}$.

Lemma: If v satisfies the simplex condition, then g' is a simplicial map, homotopic to g.

Proof: Because removing a vertex v from $\mathrm{Del}\,(X)$ only changes its structure in the open star $\mathrm{St}\,(v)$.

Therefore, we can remove the vertices incrementally. We indicate the number of vertices after Delaunay simplifications (before between parenthesis).



	$L_{0.25}$	$L_{0.1}$	$L_{0.05}$	$L_{0.01}$
Delaunay barycentric	10 (162)	8 (201)	12 (256)	11 (315)
Delaunay edgewise	7 (85)	8 (94)	9 (123)	8 (142)
Delaunay minicenter	8 (82)	8 (105)	14 (141)	11 (163)
Delaunay centroid	8 (102)	10 (119)	8 (140)	11 (152)

Edge contractions

It is a usual tool to simplify a simplicial complex.

Let $[a, b]$ be an edge of L, and $c \notin L^{(0)}$ a new vertex. Define the quotient map as

$$\begin{aligned} p\colon L^{(0)} &\longrightarrow \left(L^{(0)} \setminus \{a,b\}\right) \sqcup \{c\} \\ x &\longmapsto c \text{ if } x = a \text{ or } b \\ & \qquad x \text{ else.} \end{aligned}$$

The **contracted complex** is defined as

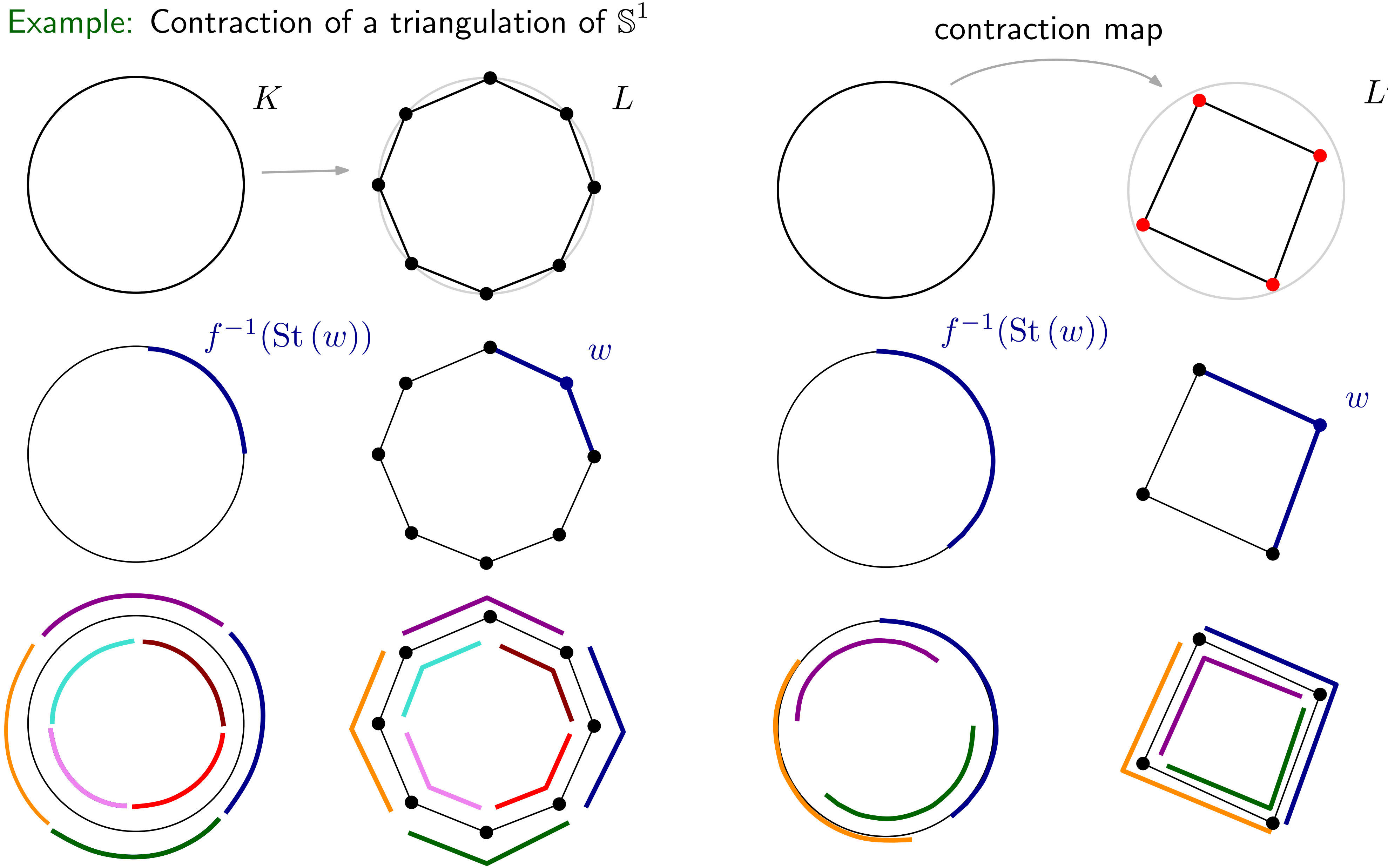
$$L' = \{p(\sigma),\ \sigma \in L\}.$$

We have a surjective simplicial map $p\colon L \to L'$.

Theorem [Dey, Edelsbrunner, Guha, Nekhayev, 1998], [Attali, Lieutier, Salinas, 2012]:
The map p is a homotopy equivalence when the edge $[a, b]$ satisfies the *link condition*, that is, $\mathrm{Lk}(ab) = \mathrm{Lk}(a) \cap \mathrm{Lk}(b)$.

Contracting L can help the simplicial approximation to $f\colon |K| \to |L|$.

Example: Contraction of a triangulation of $\mathbb{S}^1$



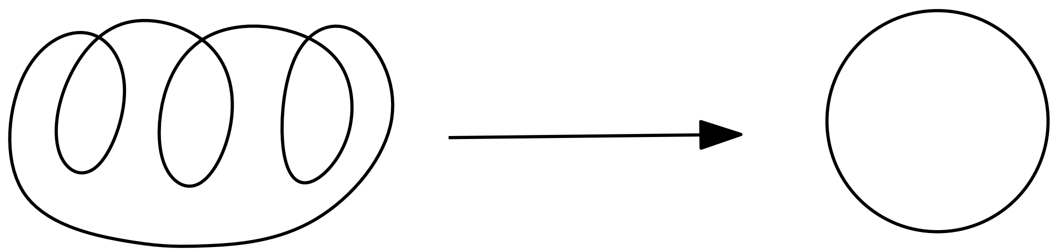
Experiment: quantitative homotopy

Gromov showed in the 70's that for a continuous map $f\colon \mathbb{S}^d \to \mathbb{S}^d$, the degree and Lipschitz constant satisfies

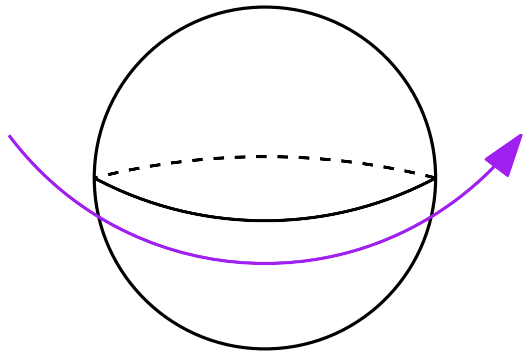
$$\mathsf{Deg}(f) \leq \mathsf{Lip}(f)^d,$$

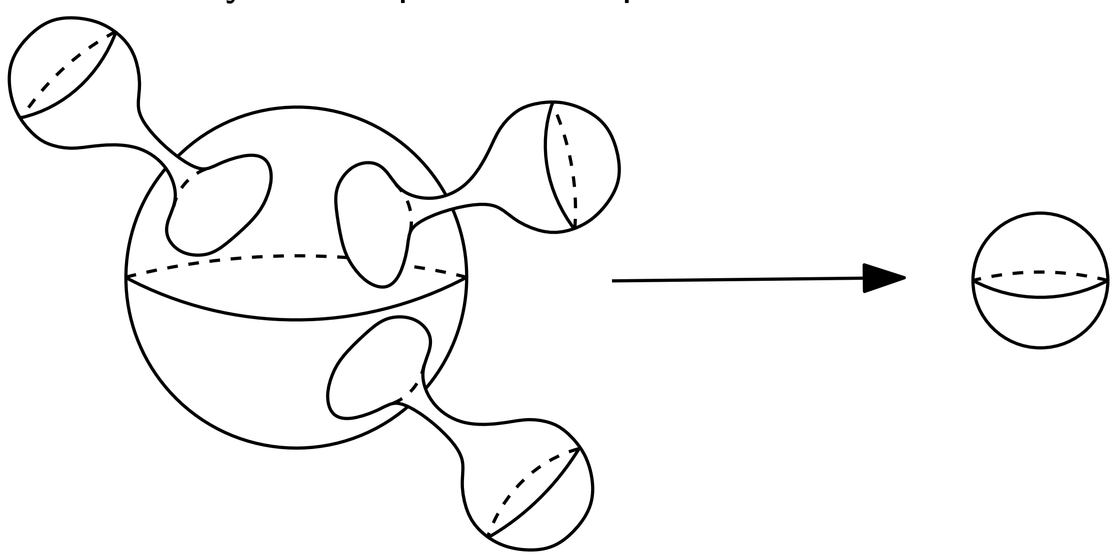
and the bound is sharp.

In dimension 1, $\mathsf{Deg}(f) \simeq \mathsf{Lip}(f)$ is obtained by turning around the circle:



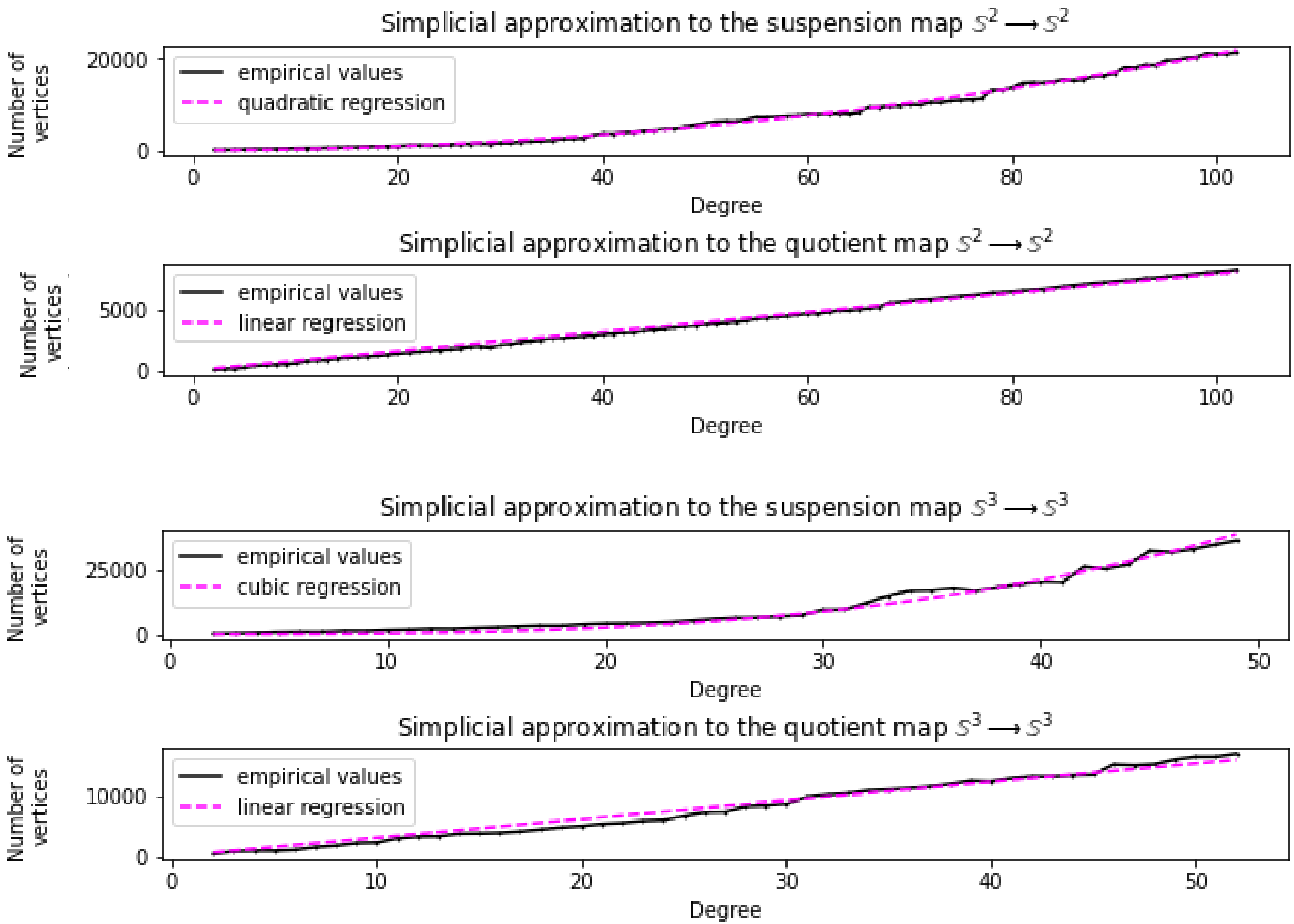
In dimension 2, $\mathsf{Deg}(f) \simeq \mathsf{Lip}(f)^2$ is not obtained by the suspension map, but another construction.





After subdividing K to get a simplicial approximation, we expect the number of vertices $n \simeq \mathsf{Lip}(f)^d$.
So, at least $n \gtrsim \mathsf{Deg}(f)$.

Experiment: quantitative homotopy



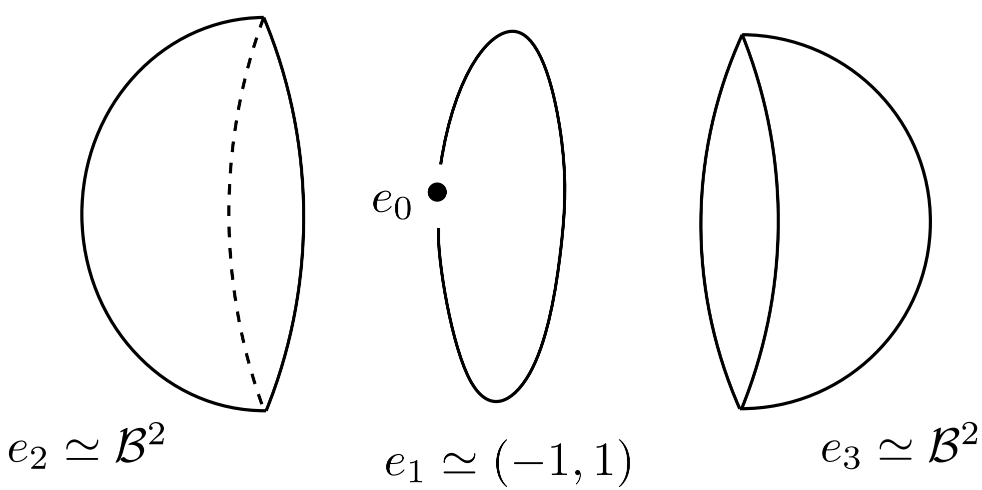
After subdivising K to get a simplicial approximation, we expect the number of vertices $n \simeq \text{Lip}(f)^d$.
So, at least $n \gtrsim \text{Deg}(f)$.

Definition of CW-complexes

Definition: A **CW-complex** is a topological Hausdorff space X together with a finite partition $\{e_i\}_i$ of X (the cells) such that:

- For each e_i, there exists an integer $n(i)$ and a homeomorphism $\mathcal{B}^{n(i)} \to e_i$, where $\mathcal{B}^{n(i)}$ is the open ball of $\mathbb{R}^{n(i)}$.
- Moreover, this homeomorphism extends to a continuous map, $f_i \colon \overline{\mathcal{B}}^{\,n(i)} \to X$, where $\overline{\mathcal{B}}^{\,n(i)}$ is the closed ball.
  Its restriction to the sphere, denoted $\phi_i \colon \partial\overline{\mathcal{B}}^{\,n(i)} \to X$, is called the **gluing map**.
- Each point $x \in \overline{e}_i \setminus e_i$ must lie in a cell e_j of lower dimension.

Example: The sphere $\mathbb{S}^2$ admits a CW-structure with
one cell of dimension 0,
one cell of dimension 1,
two cells of dimension 2.



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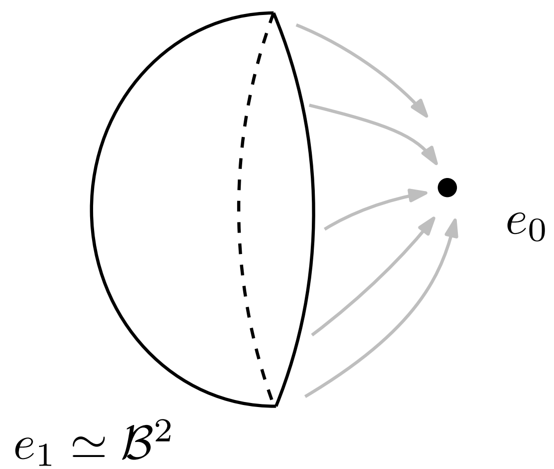
Example: The sphere $\mathbb{S}^2$ admits a CW-structure with one cell of dimension 0, **one** cell of dimension 2.



CW-complex structures

CW-complex structures provide concise descriptions of spaces:

- The **sphere** of dimension n: one 0-cell, and one n-cell,
- The **projective space** of dimension n: one cell per dimension,
- The **real Grassmannian** $\mathcal{G}(d, \mathbb{R}^n)$ (dimension $d(n-d)$): $p(k)$ cells of dimension k, where $p(k)$ is the number of partitions of k into at most d integers each of which is $\leq n-d$ (in total, $\binom{n}{d}$ cells),
- The **lens space** $L_p(q_1, \ldots, q_n)$ (dimension $2n-1$): one cell per dimension.

Triangulations are more complicated:

- The **sphere** of dimension n: at least $n+1$ vertices, $n+1$ facets, $2^n - 1$ simplices,
- The **projective space** of dimension n: must have $\geq \frac{(n+1)(n+2)}{2}$ vertices [Arnoux, Marin, 1991]. We know a triangulation with $2^{n+1}-1$ vertices [Kühnel, 1987],
- The **real Grassmannian** $\mathcal{G}(d, \mathbb{R}^n)$: a triangulation has at least $\frac{n(n+1)}{2}$ vertices and $\big(n(n-1) - 2d(n-d)\big)2^{d(n-d)+1} - 1$ simplices [Govc, Marzantowicz, Pavešić, 2020].

Cellularisation:

- Any topological manifold of dimension $d \neq 4$ is homeomorphic to a CW-complex [Kirby, Siebenmann] [Quinn]
- Any topological manifold is homotopy equivalent to a CW complex [Kirby, Siebenmann, 1969]

Triangulations:

- Any smooth manifold admits a PL-structure hence is triangulable [Cairns, 1935] [Whitehead, 1940]
- Any topological manifold is homotopy equivalent to a simplicial complex

CW-complex structures

Cellularisation:

- Any topological manifold of dimension $d \neq 4$ is homeomorphic to a CW-complex [Kirby, Siebenmann] [Quinn]
- Any topological manifold is homotopy equivalent to a CW complex [Kirby, Siebenmann, 1969]

Triangulations:

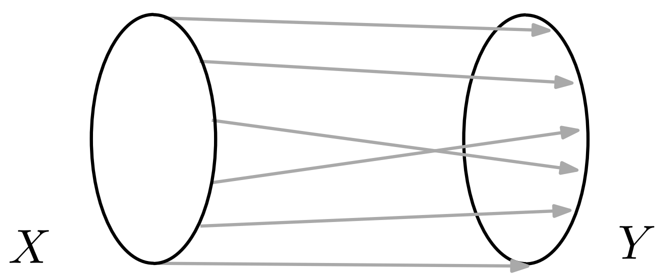
- Any smooth manifold admits a PL-structure hence is triangulable [Cairns, 1935] [Whitehead, 1940]
- Any topological manifold is homotopy equivalent to a simplicial complex

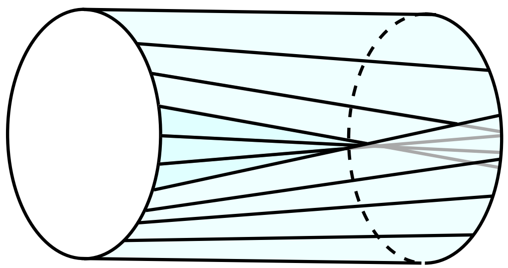
simplicial approximation to CW complexes

Mapping cones

Let $f\colon X \to Y$ be a continuous map. The **mapping cylinder** is the quotient space

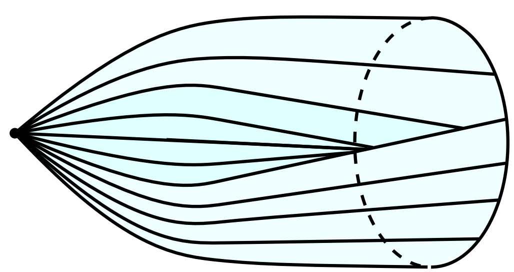
$$\mathrm{MapCyl}(f) = X \times [0,1] \sqcup Y \ / \ (x,1) \sim f(x)$$





The **mapping cone** is obtained by identifying the upper part of the cylinder

$$\mathrm{MapCone}(f) = \mathrm{MapCyl}(f) \ / \ (x,0) \sim \mathrm{point}$$



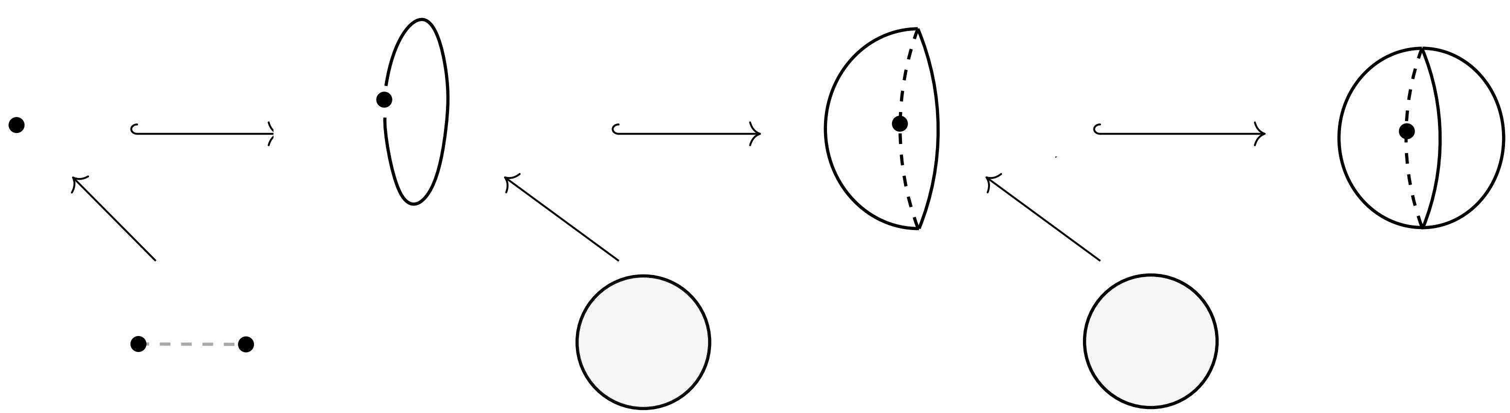
Lemma: If $f, g\colon X \to Y$ are homotopic, then so are $\mathrm{MapCone}(f)$ and $\mathrm{MapCone}(g)$.

Mapping cones

Let X be a CW-complex, with cells $\{e_i\}_i$, and gluing maps $\phi_i \colon \partial\mathcal{B}^{n(i)} \to X$.

One shows that X is homeomorphic to the sequence of mapping cones

$$
\begin{tikzcd}
e_0 \arrow[d, "\simeq"] & & e_0 \cup e_1 \arrow[d, "\simeq"] & & e_0 \cup e_1 \cup e_2 \arrow[d, "\simeq"] & & e_0 \cup e_1 \cup e_2 \cup e_3 \arrow[d, "\simeq"] & \\
\{\text{point}\} \arrow[rr, hook] & & \mathrm{MapCone}(\phi_1) \arrow[rr, hook] & & \mathrm{MapCone}(\phi_2) \arrow[rr, hook] & & \mathrm{MapCone}(\phi_3) \arrow[r, hook] & \ldots \\
 & \partial\mathcal{B}^{n(1)} \arrow[ul, "\phi_1"] & & \partial\mathcal{B}^{n(2)} \arrow[ul, "\phi_2"] & & \partial\mathcal{B}^{n(3)} \arrow[ul, "\phi_3"] & &
\end{tikzcd}
$$



One builds X by induction: $X^i = \mathrm{MapCone}(\phi_i \colon \partial\mathcal{B}^{n(i)} \to X^{i-1})$.

Mapping cones

Let X be a CW-complex, with cells $\{e_i\}_i$, and gluing maps $\phi_i \colon \partial\mathcal{B}^{n(i)} \to X$.

One shows that X is homeomorphic to the sequence of mapping cones

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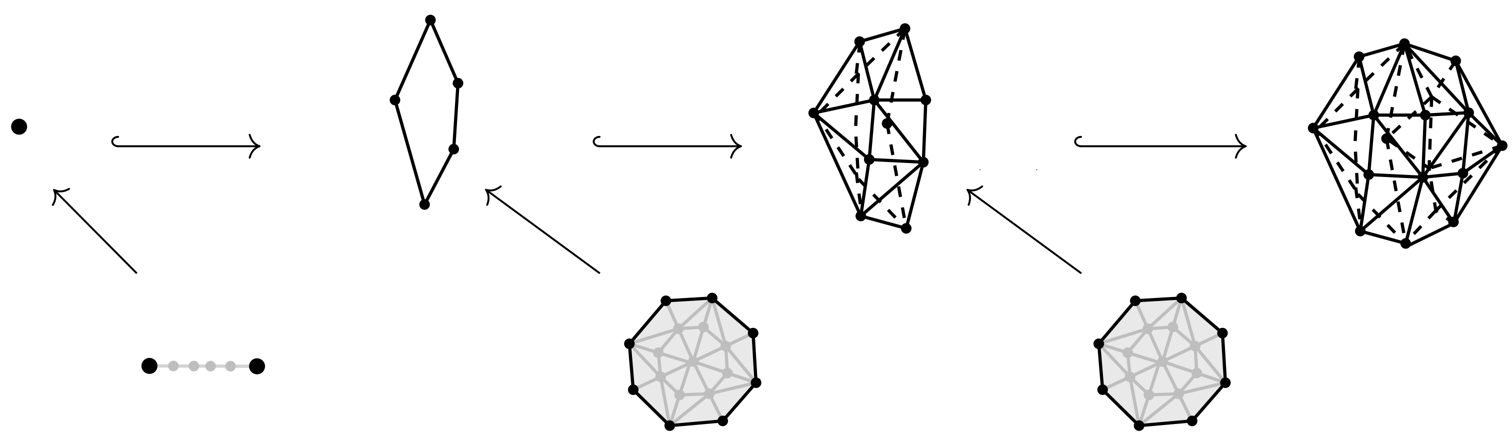
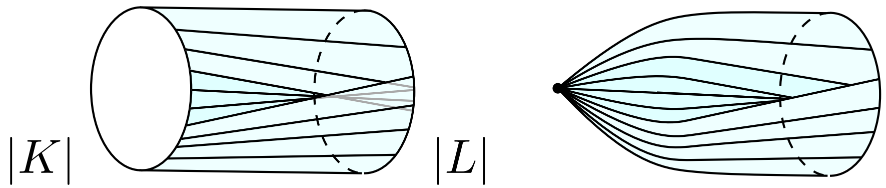
\{\mathrm{point}\} \arrow[r, hook] & \mathrm{MapCone}(\phi_1) \arrow[r, hook] & \mathrm{MapCone}(\phi_2) \arrow[r, hook] & \mathrm{MapCone}(\phi_3) \arrow[r, hook] & \dots \\
\partial\mathcal{B}^{n(1)} \arrow[u, "\phi_1"] & \partial\mathcal{B}^{n(2)} \arrow[u, "\phi_2"] & \partial\mathcal{B}^{n(3)} \arrow[u, "\phi_3"] & &
\end{tikzcd}
$$



In order to triangulate X, it is enough to triangulate mapping cones.

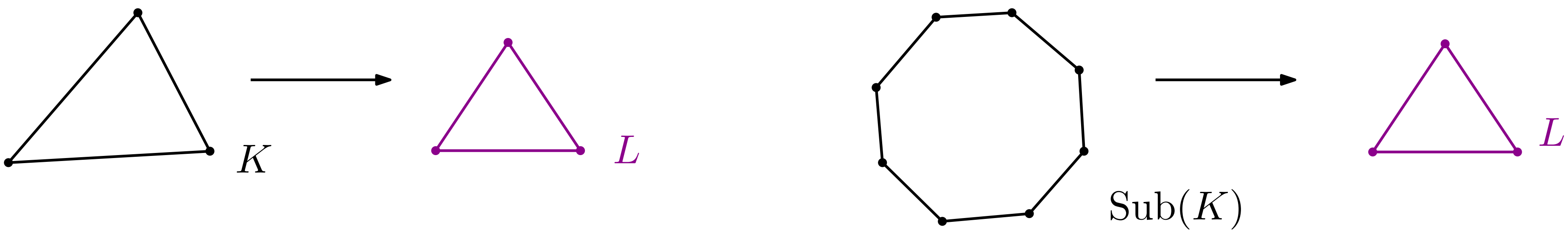
How to triangulate the mapping cone of $f\colon |K| \to |L|$?



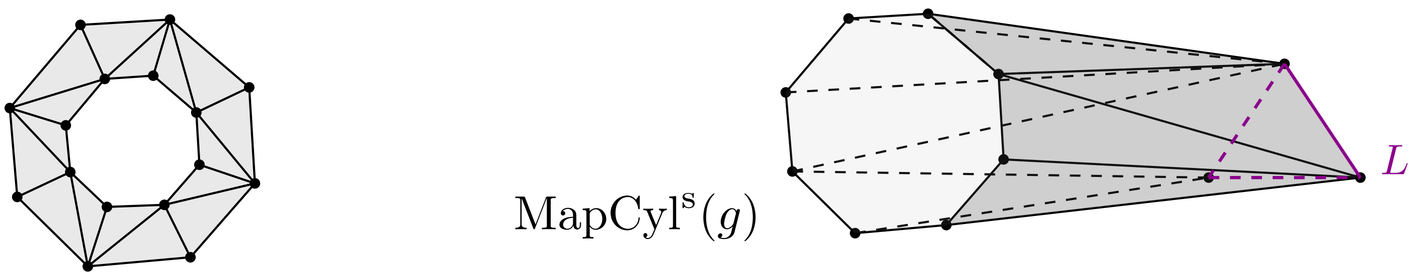
$$\mathrm{MapCyl}(f) = X \times [0,1] \sqcup Y \;/\; (x,1) \sim f(x)$$

$$\mathrm{MapCone}(f) = \mathrm{MapCyl}(f) \;/\; (x,0) \sim \text{point}$$

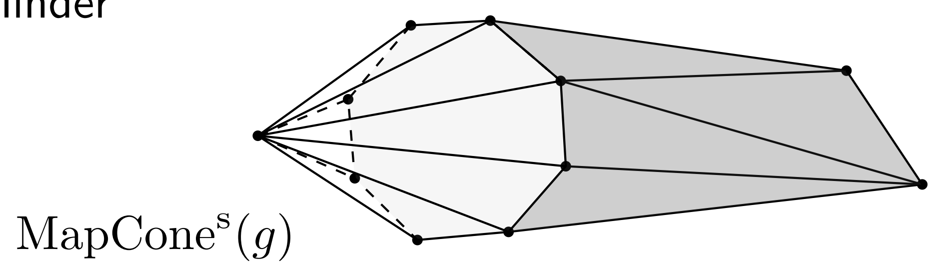
1. Find a simplicial approximation $g\colon K \to L$ to $f\colon |K| \to |L|$



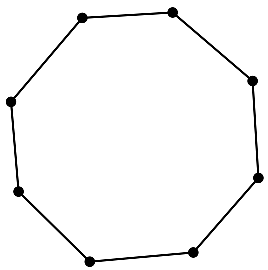
2. Triangulate $|K| \times [0,1]$ and glue L at the end

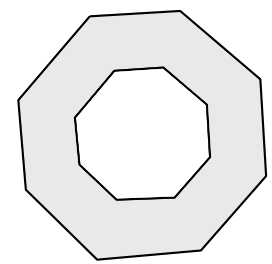


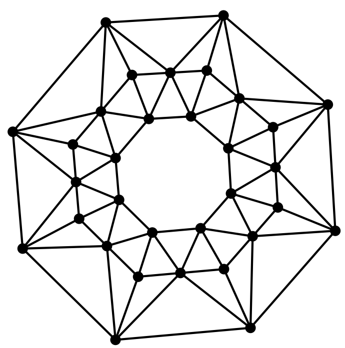
3. Cone the upper part of the cylinder



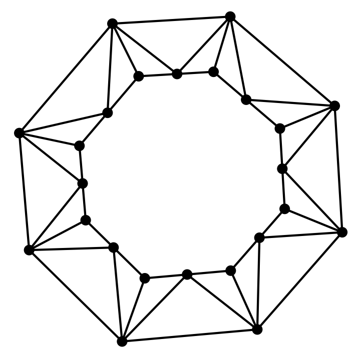
2. Let K be a simplicial complex. How to triangulate $|K| \times [0,1]$?

K 

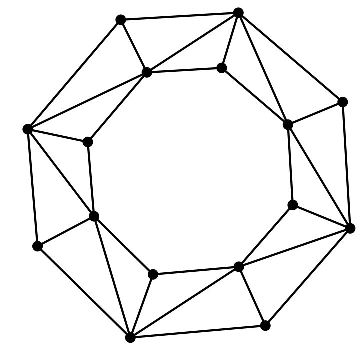
$K \times [0,1]$ 



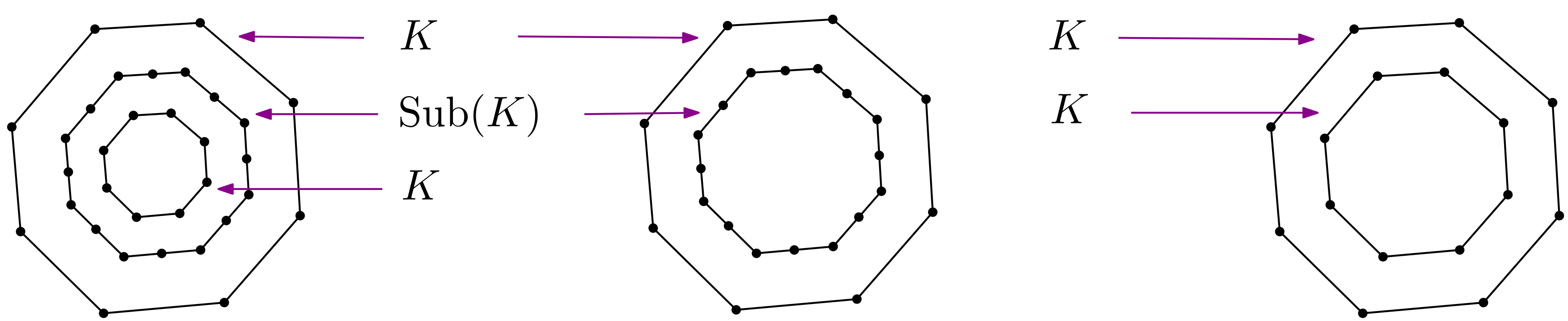
[Whitehead, *Simplicial spaces, nuclei and m-groups*, 1939]



[Cohen, *Simplicial structures and transverse cellularity*, 1967]

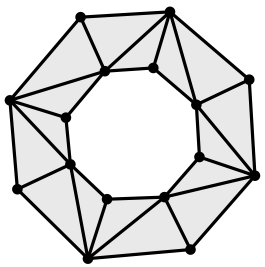


Simplicial set product

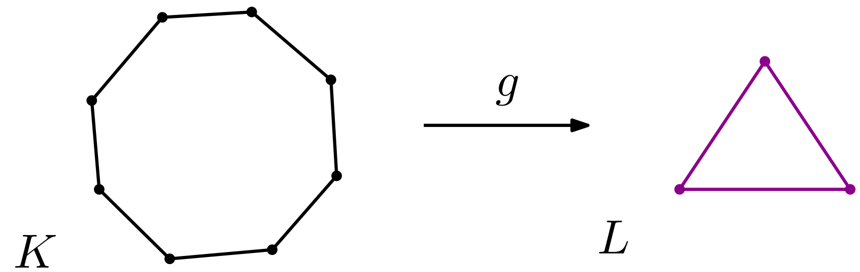


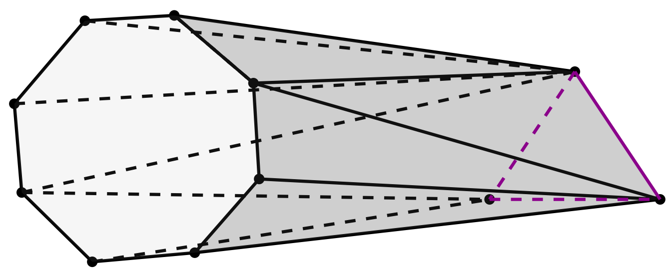
Once $K \times [0,1]$ is triangulated, we glue it to L via $g\colon K \to L$ as follows:

$$\mathrm{MapCyl}^{\mathrm{s}}(g) = \left\{ \sigma_0 \sqcup g(\sigma_1),\ \sigma \in K \times [0,1],\ \sigma = \sigma_0 \sqcup \sigma_1,\ \middle|\ \begin{array}{l} \sigma_0 \in K \times \{0\} \\ \sigma_1 \in K \times \{1\} \end{array} \right\}$$



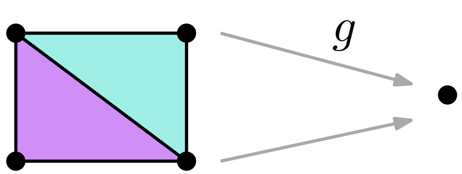
$K \times [0,1]$

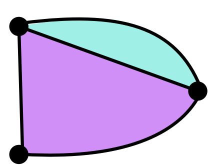




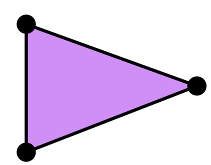
$\mathrm{MapCyl}^{\mathrm{s}}(g)$

The map $\mathrm{MapCyl}(g) \to \mathrm{MapCyl}^{\mathrm{s}}(g)$ may not be a homeomorphism!





$\mathrm{MapCyl}(g)$

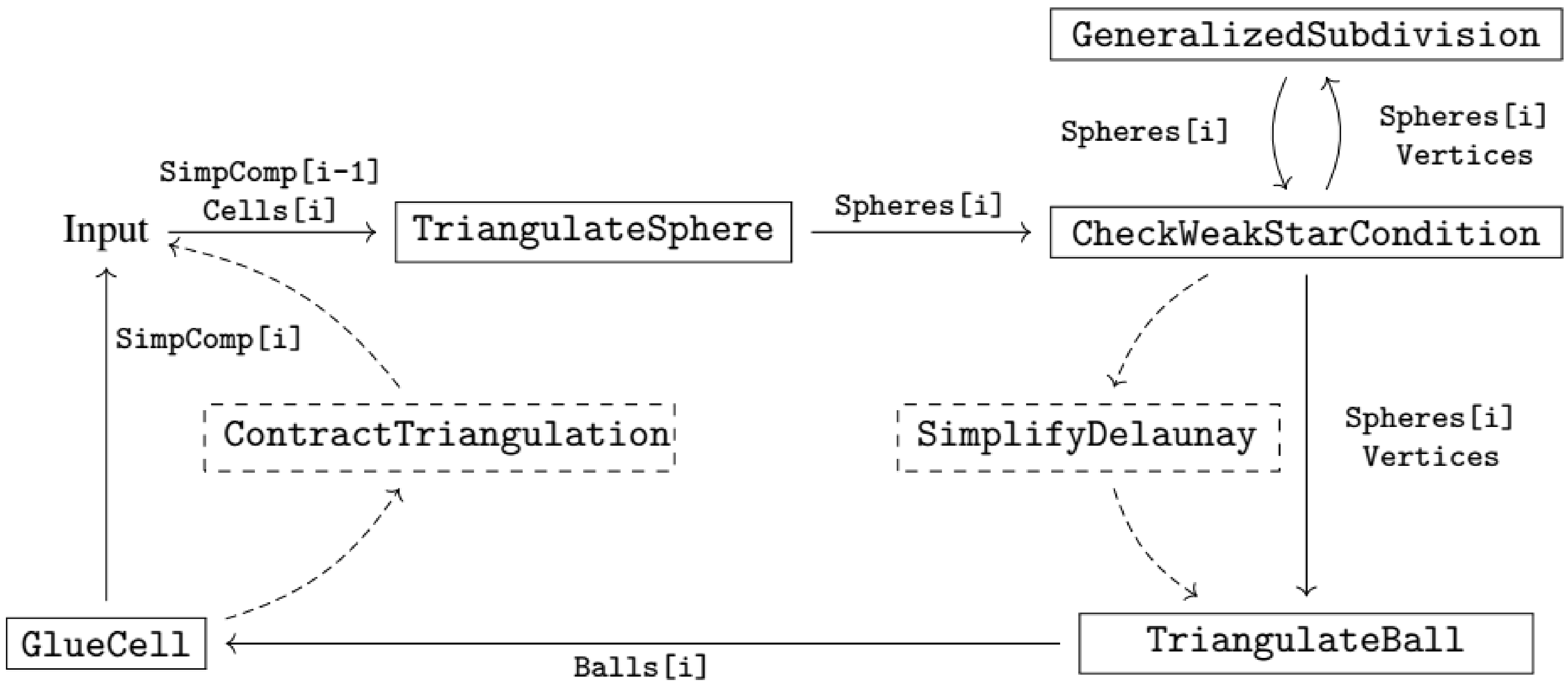


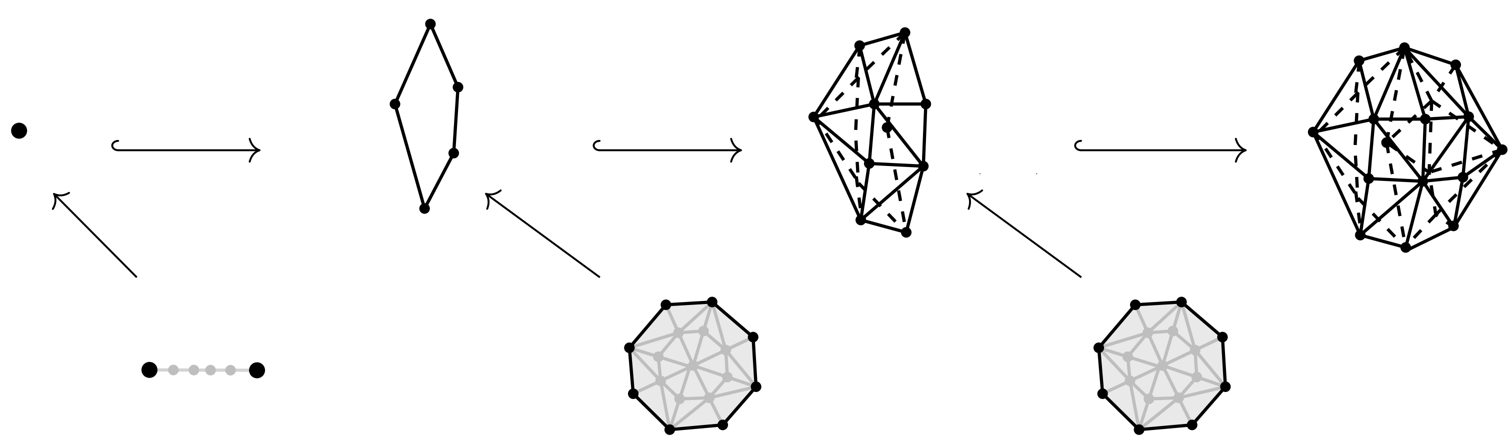
$\mathrm{MapCyl}^{\mathrm{s}}(g)$

Proposition: It is a homotopy equivalence.

Sketch of algorithm

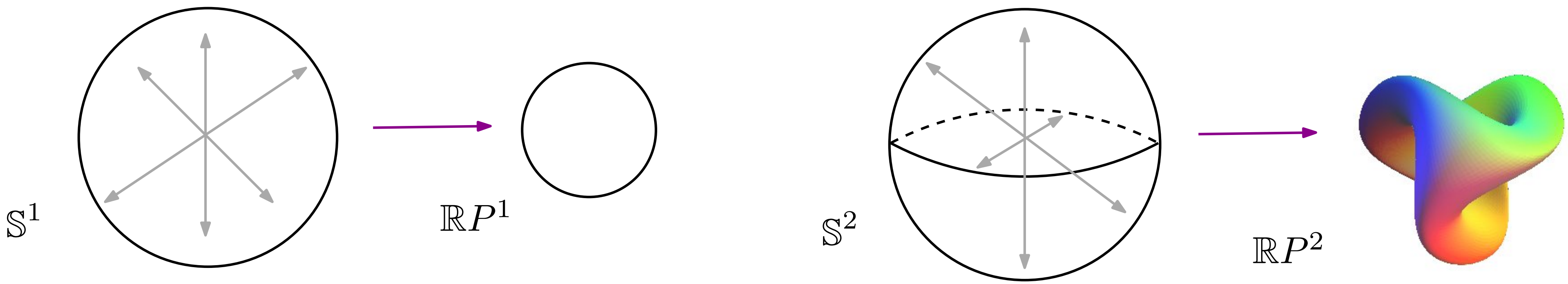
One can glue the cells one by one. The only information needed is that of the characteristic maps.



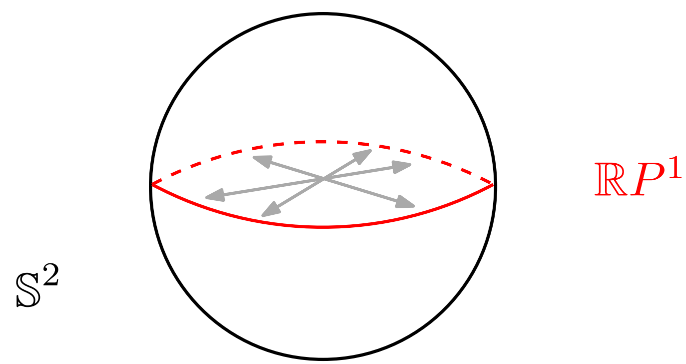


Triangulating the projective spaces

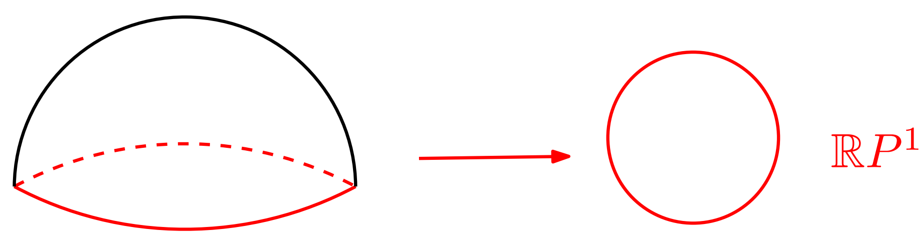
The n-dimensional projective space $\mathbb{R}P^n$ can be defined as the quotient of the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by the antipodal relation $x \sim -x$.



The projective space $\mathbb{R}P^n$ contains $\mathbb{R}P^{n-1}$ as its equator.



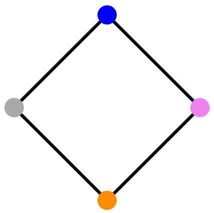
By induction, we get a CW-structure for $\mathbb{R}P^n$: take a hemisphere of $\mathbb{S}^n$, and glue its equator on $\mathbb{R}P^{n-1}$.



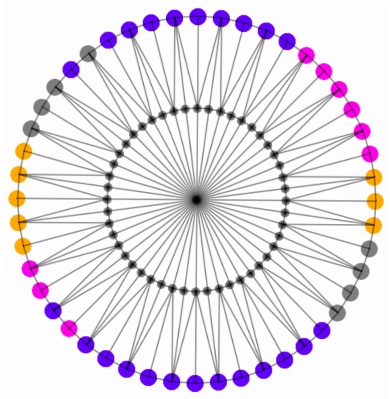
We apply the algorithm with **global barycentric sudivisions**:

$\mathbb{R}P^0$: 1 vertex

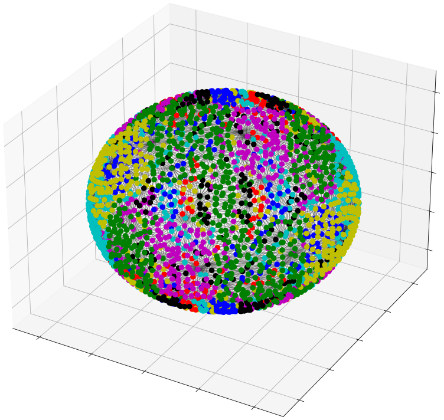
$\mathbb{R}P^1$: 4 vertices



$\mathbb{R}P^2$: 53 vertices
(4 subdivisions)



$\mathbb{R}P^3$: $560'024$ vertices
(7 subdivisions)



Triangulating the projective spaces

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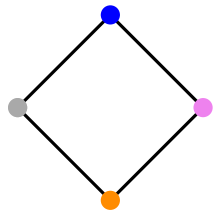
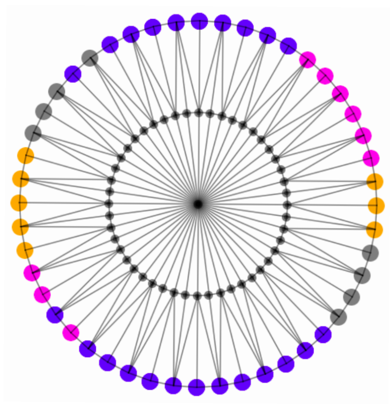
$\mathbb{R}P^0$: 1 vertex

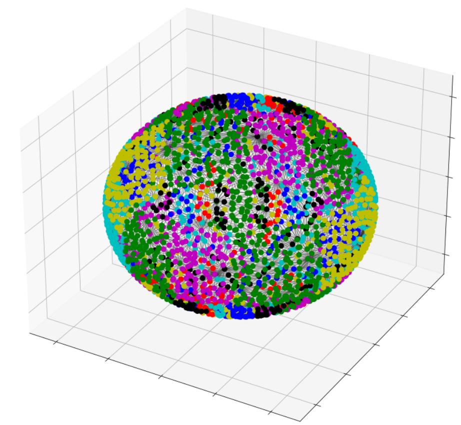
$\mathbb{R}P^1$: 4 vertices



$\mathbb{R}P^2$: 53 vertices
(4 subdivisions)



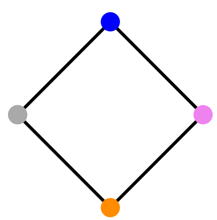
$\mathbb{R}P^3$: $560'024$ vertices
(7 subdivisions)



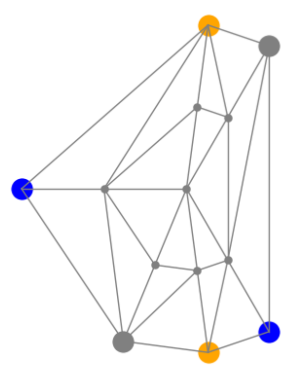
Now with **generalized Delaunay centroid subdivisions**, **simplifications** and **edge contractions**:

$\mathbb{R}P^0$: 1 vertex

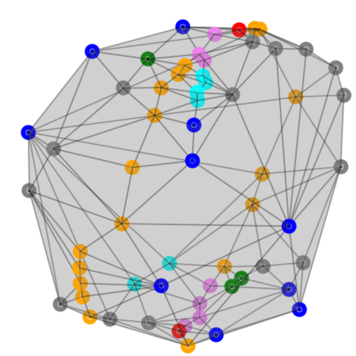
$\mathbb{R}P^1$: 4 vertices
(contract to 3 vertices)



$\mathbb{R}P^2$: 32 vertices
(contract to 7 vertices)



$\mathbb{R}P^3$: 62 vertices
(contract to 14 vertices)



Triangulating the projective spaces

Number of vertices of the output of the algorithm (in parenthesis, the number before edge contractions).

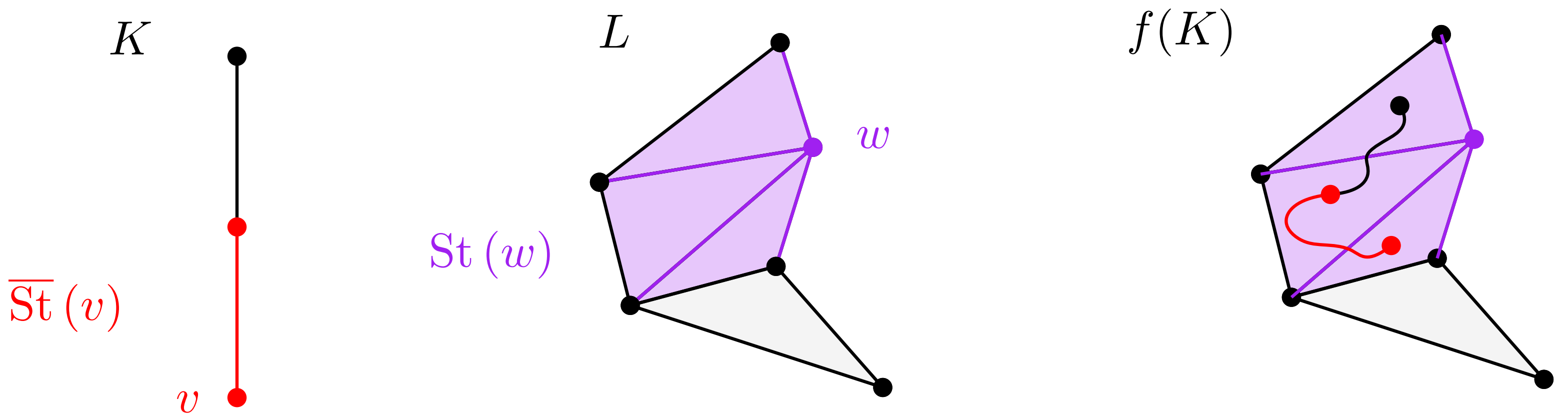
	$\mathbb{R}P^1$	$\mathbb{R}P^2$	$\mathbb{R}P^3$	$\mathbb{R}P^4$
Barycentric	3 (4)	7 (32)	739 (7498)	$\times$
Edgewise	3 (4)	7 (32)	46 (1328)	$\times$
Delaunay barycentric	3 (4)	7 (10)	16 (56)	577 (1923)
Delaunay edgewise	3 (4)	6 (11)	15 (73)	708 (2664)
Delaunay minicenter	3 (4)	7 (11)	12 (11)	505 (1622)
Delaunay centroid	3 (4)	6 (11)	12 (66)	533 (2104)

The minimal number of vertices of a triangulation of $\mathbb{R}P^2$ is 6, $\mathbb{R}P^3$ is 11, and $\mathbb{R}P^4$ is 16.

1. Simplicial approximation
2. Computational improvements
3. Simplicial approximation to CW complexes
4. Weak simplicial approximation

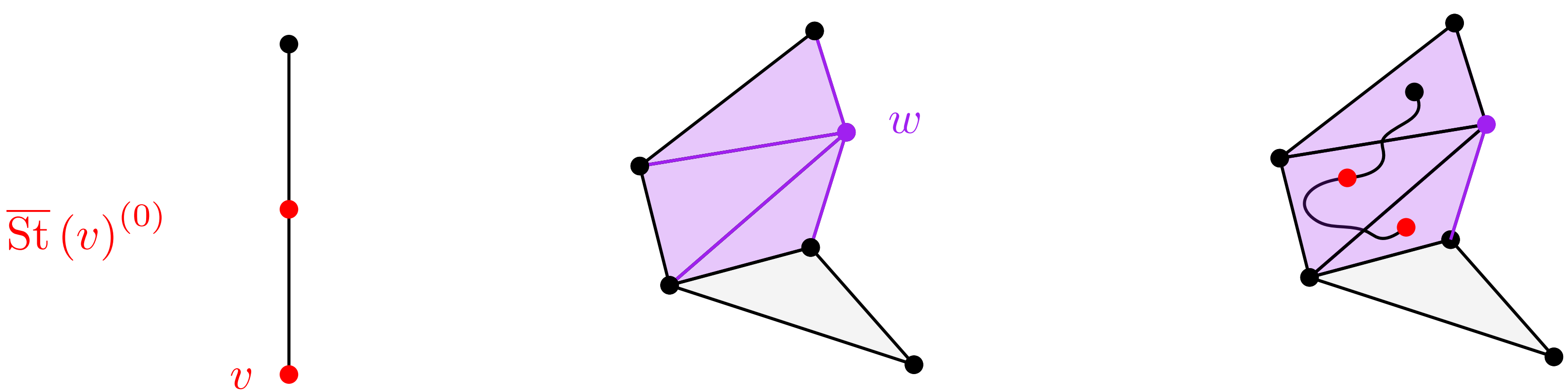
Weak star condition

In practice, we cannot check whether a simplicial map $f\colon |K| \to |L|$ satisfies the star condition...

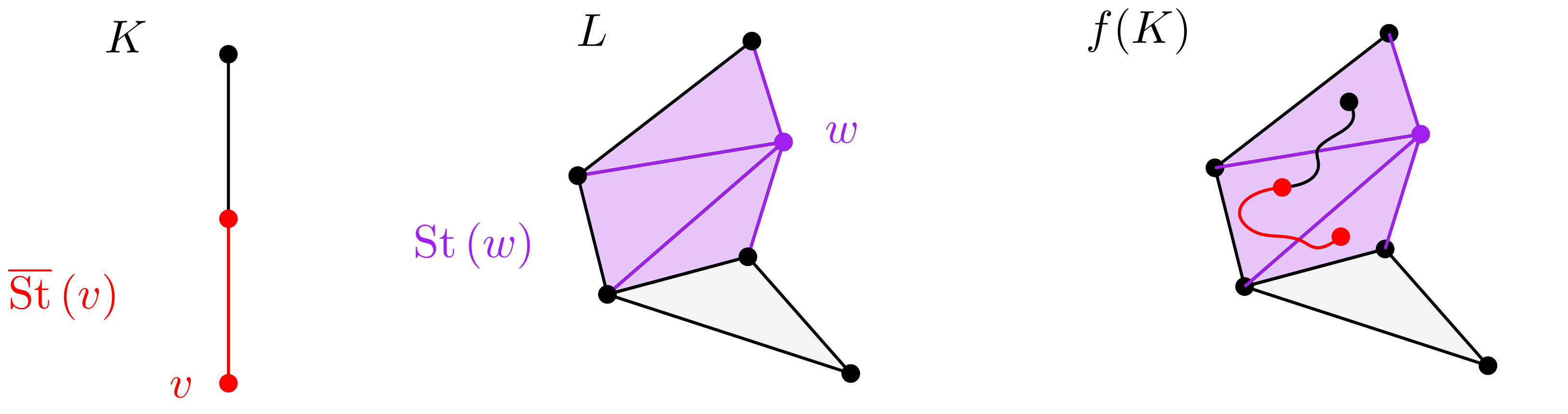


$$f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(w)\,|?$$

Definition: The map f satisfies the **weak star condition** if for every vertex $v \in K^{(0)}$, there exists a $w \in L^{(0)}$ such that $f\left(\overline{\mathrm{St}}\,(v)^{(0)}\right) \subseteq |\mathrm{St}\,(w)\,|$.



In practice, we cannot check whether a simplicial map $f\colon |K| \to |L|$ satisfies the star condition...



$$f\left(|\overline{\mathrm{St}}\,(v)\,|\right) \subseteq |\mathrm{St}\,(w)\,|?$$

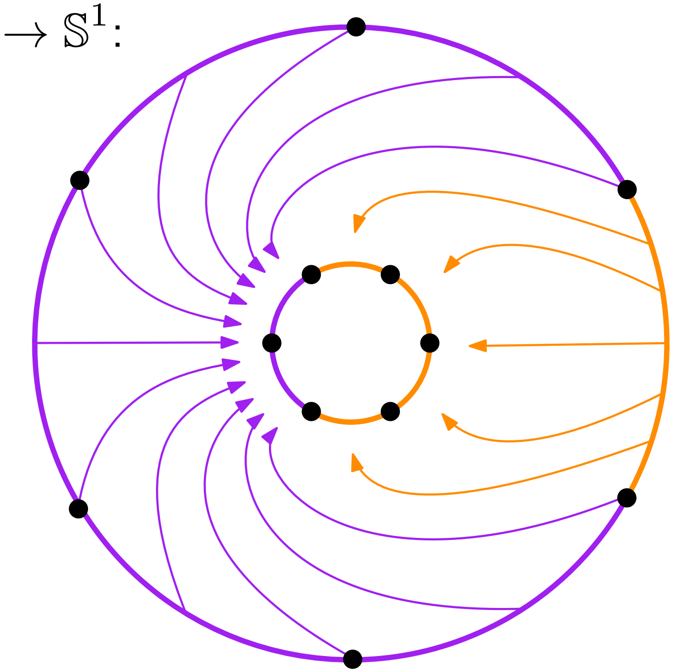
Definition: The map f satisfies the **weak star condition** if for every vertex $v \in K^{(0)}$, there exists a $w \in L^{(0)}$ such that $f\left(\overline{\mathrm{St}}\,(v)^{(0)}\right) \subseteq |\mathrm{St}\,(w)\,|$.

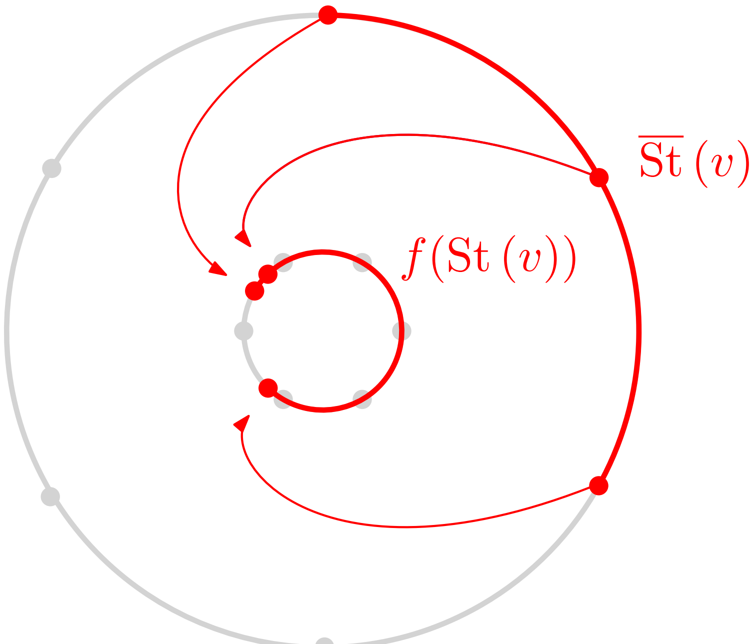
If this is the case, let $g\colon K^{(0)} \to L^{(0)}$ be any map such that $\forall v \in K^{(0)}$, $f\left((\overline{\mathrm{St}}\,(v))^{(0)}\right) \subseteq |\mathrm{St}\,(g(v))\,|$.

Such a map g is called a **weak simplicial approximation** to f. It is a simplicial map.

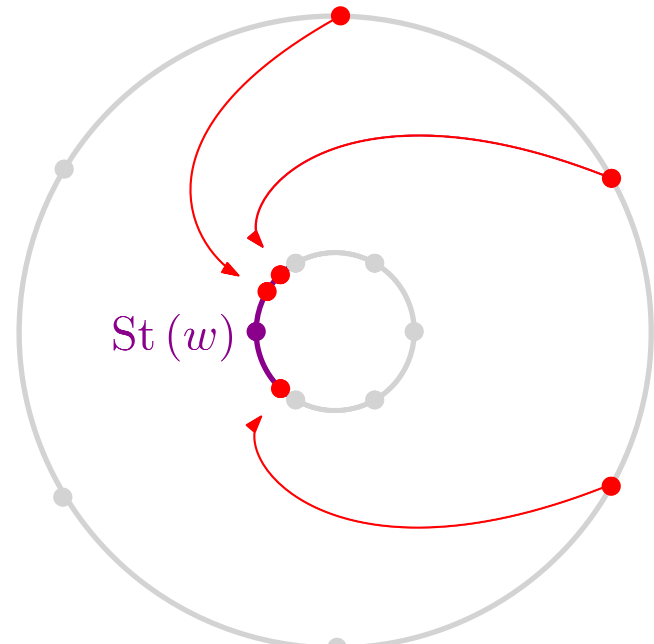
Proposition: If K is subdivised enough, then any weak simplicial approximation is a simplicial approximation.

Consider the following map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$:





f does not satisfy the star condition at v



f satisfies the star weak condition at v

One finds a constant weak simplicial approximation to f...

We have a new problem.

Input: Two simplicial complexes K, L a continuous map $f\colon |K| \to |L|$ (known through a finite number of evaluations) and a simplicial map $g\colon K \to L$.

Output: The maps f and $|g|$ are homotopic?

First idea: Verify that f actually satifies the star condition.

We endow the geometric complexes with the geodesic distance induced by the Euclidean metric.

Let λ be a Lipschitz constant for f.

- Choose $\epsilon > 0$ such that $\epsilon\lambda$ is lower than the smallest edge of L.
- For any vertex $v \in K^{(0)}$, build a subset $\mathcal{N}_\epsilon(v) \subset |\overline{\mathrm{St}}\,(v)|$ which is ϵ-dense (any point $x \in |\overline{\mathrm{St}}\,(v)|$ admits a point $y \in \mathcal{N}_\epsilon(v)$ at distance most ϵ)
- Check that there exists $w \in L^{(0)}$ such that for all $x \in \mathcal{N}_\epsilon(v)$, $f(x)$ belongs to $|\mathrm{St}\,(w)|$ and is at distance greater than $\epsilon\lambda$ from its boundary.

Lemma: Under this condition, f satisfies the star condition.

Information required for this idea: A Lipschitz constant λ for f.

We have a new problem.

Input: Two simplicial complexes K, L a continuous map $f\colon |K| \to |L|$ (known through a finite number of evaluations) and a simplicial map $g\colon K \to L$.

Output: The maps f and $|g|$ are homotopic?

Second idea: Close maps are homotopic.

Between two continuous maps $f, g\colon |K| \to |L|$, define the infinite norm

$$d_\infty(f, g) = \sup \left\{ d(f(x), g(x)),\ x \in |K| \right\}.$$

Let ϵ be such that, for all continuous maps $f, g\colon |K| \to |L|$,

$$d_\infty(f, g) < \epsilon \quad \Longrightarrow \quad f \text{ and } g \text{ are homotopic.}$$

This gives another criterion for simplicial approximation: we look for a simplicial $g\colon K \to L$ such that $d_\infty(f, g) < \epsilon$.

Information required for this idea: the quantity ϵ (linked to the injectivity radius of $|L|$).

Input: Two simplicial complexes K, L a continuous map $f\colon |K| \to |L|$ (known through a finite number of evaluations) and a simplicial map $g\colon K \to L$.

Output: The maps f and $|g|$ are homotopic?

Negative result: If L is not simply connected, deciding whether two simplicial maps $K \to L$ are homotopic in undecidable.

Third idea: Hopf theorem

Let us suppose that K is a triangulation of the d-sphere, and L is such that its top homology group is $H_d(L)$ and is isomorphic to $\mathbb{Z}$.

In this pleasant case, the Hopf theorem states that the set of homotopy classes $|K| \to |L|$ is in bijection with $\mathbb{Z}$ (through the degree).

In our case, this computation can be done trough $\mathrm{MapCone}^{\mathrm{s}}(g)$, the simplicial mapping cone of g.

Lemma: If the integral homology groups satisfy $H_{d+1}(\mathrm{MapCone}^{\mathrm{s}}(g)) \simeq \mathbb{Z}/\deg(f)\mathbb{Z}$ and $H_d(\mathrm{MapCone}^{\mathrm{s}}(g)) \simeq 0$, then the maps f and $|g|$ are homotopic.

Information required for this idea: the degree of f.
Only works for K sphere and L orientable.

Can be applied when $L = \mathbb{S}^d$, or $\mathbb{R}P^d$ with d odd.

Input: Two simplicial complexes K, L a continuous map $f\colon |K| \to |L|$ (known through a finite number of evaluations) and a simplicial map $g\colon K \to L$.

Output: The maps f and $|g|$ are homotopic?

Negative result: If L is not simply connected, deciding whether two simplicial maps $K \to L$ are homotopic in undecidable.

Fourth idea: Covering spaces

Suppose we have a (simplicial) covering space $p\colon L' \to L$ with top homology group $H_d(L') \simeq \mathbb{Z}$.

Consider a lift $f'\colon K \to L'$ of f and $g'\colon K \to L'$ of g.

$$\begin{tikzcd} & L' \arrow[d, "p"] \\ K \arrow[ur, "g'"] \arrow[r, "g"] & L \end{tikzcd}$$

Lemma: If the integral homology groups satisfy $H_{d+1}(\mathrm{MapCone}^{\mathrm{s}}(g')) \simeq \mathbb{Z}/\deg(f')\mathbb{Z}$ and $H_d(\mathrm{MapCone}^{\mathrm{s}}(g')) \simeq 0$, then the maps f and $|g|$ are homotopic.

Information required for this idea: the degree of f' and a universal cover of L.
Only works for K sphere.

Can be applied to $L = \mathbb{R}P^d$ with d even.

Cell structure for $\mathcal{G}(2, \mathbb{R}^4)$

The Grassmannian $\mathcal{G}(2, \mathbb{R}^4)$ is the set of 2-dimensional planes in $\mathbb{R}^4$. It can be given a smooth manifold structure of dimension 4. Its cohomology groups are:

$$H^0(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z} \qquad H^0(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = \mathbb{Z}_p$$
$$H^1(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z}_2 \qquad H^1(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = 0$$
$$H^2(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = (\mathbb{Z}_2)^2 \qquad H^2(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = 0$$
$$H^3(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z}_2 \qquad H^3(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = 0$$
$$H^4(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z} \qquad H^4(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = \mathbb{Z}_p$$

where p is any prime number > 2.

Consider a plane $T \in \mathcal{G}(2, \mathbb{R}^4)$. Let (u, v) be a basis, and consider the matrix

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

By mean of elementary operations, it can be reduced to a unique matrix in **orthogonal reduced echelon form** $\begin{pmatrix} v'_1 & v'_2 & v'_3 & v'_4 \\ w'_1 & w_2 & w'_3 & w'_4 \end{pmatrix}$ such that:

- $\|v'\| = \|w'\| = 1$
- v' and w' are orthogonal
- the last nonzero coordinate of v' is positive
- the last nonzero coordinate of w' is positive

Cell structure for $\mathcal{G}(2,\mathbb{R}^4)$

Consider a plane $T \in \mathcal{G}(2,\mathbb{R}^4)$ and its matrix in reduced echelon form $\begin{pmatrix} v'_1 & v'_2 & v'_3 & v'_4 \\ w'_1 & w_2 & w'_3 & w'_4 \end{pmatrix}$

Let i (resp. j) be the index of the last nonzero coordinate of v' (resp. w').

The pair (i,j) is called the *Schubert symbol* of the plane T.

There are 6 potential Schubert symbols for $\mathcal{G}(2,\mathbb{R}^4)$:

$$(1,2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (1,3) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$$

$$(1,4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix} \qquad (2,3) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

$$(2,4) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix} \qquad (1,2) \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

Cell structure for $\mathcal{G}(2, \mathbb{R}^4)$

Consider a plane $T \in \mathcal{G}(2, \mathbb{R}^4)$ and its matrix in reduced echelon form $\begin{pmatrix} v_1' & v_2' & v_3' & v_4' \\ w_1' & w_2 & w_3' & w_4' \end{pmatrix}$

Let i (resp. j) be the index of the last nonzero coordinate of v' (resp. w').

The pair (i, j) is called the *Schubert symbol* of the plane T.

There are 6 potential Schubert symbols for $\mathcal{G}(2, \mathbb{R}^4)$:

dim 0 $\quad (1,2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\qquad (1,3) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$ dim 1

dim 2 $\quad (1,4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix}$ $\qquad (2,3) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$ dim 2

dim 3 $\quad (2,4) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}$ $\qquad (1,2) \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$ dim 4

Proposition: $\mathcal{G}(2, \mathbb{R}^4)$ admits a CW-structure with 6 cells. Each cell corresponds to a pair (i, j), and contains all the planes T with Schubert symbol (i, j).

After some computations, one finds explicit expressions for characteristic maps and gluing maps.

We apply the algorithm for the Delaunay edgewise subdivision, performing edge contractions and Delaunay simplification steps.

The following table gathers the number of vertices of the complexes at each step, (in parenthesis the number before edge contractions).

Schubert symbol σ	$(1,2)$	$(1,3)$	$(2,3)$	$(1,4)$	$(2,4)$	$(3,4)$
Number of vertices	1	$3\ (4)$	$6\ (10)$	$10\ (13)$	$22\ (93)$	$825\ (3450)$

Embedding of the 4-simplex, seen as a triangulation of the 3-sphere, in the 3-cell of $\mathcal{G}(2, \mathbb{R}^4)$: