EMAp summer course

Topological Data Analysis with Persistent Homology

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Abstract. This course is intended for a 3rd year graduate student with no background on topology. The present document is a collection of notes for each lesson.

Course webpage. Various information (schedule, homework) are gathered on https://raphaeltinarrage.github.io/EMAp.html.

Numerical experiments Python notebooks containing illustrations can be found at https://github.com/raphaeltinarrage/EMAp.
Before the first tutorial (4th lesson), you should be able to run the following notebook: https://github.com/raphaeltinarrage/EMAp/blob/main/Tutorial0.ipynb.

Homework. Exercises with a vertical segment next to them are your homework. Here is the first one:

Exercise 0. Send me an email answering the following questions:

- Do you understand English well?
- Have you ever studied topology?
- Have you ever coded? In which language?
- Any remarks?

Warning I took some shortcuts in the exposition of persistent homology. Notably: we won't study basic general topology notion that are worth it (adherence, compactness, path-connectedness). We will not study singular homology, but define the homology of topological spaces via the simplicial homology of triangulations, and only with coefficients the finite fields $\mathbb{Z}/p\mathbb{Z}$. Concerning persistent homology, we will not go through the algebraic definition of persistence modules, but rather study the persistence of the simplicial filtrations.

Contents

1 General topology

1.1 Topological spaces

Topological spaces are abstractions of the concept of 'shape' or 'geometric object'.

Definition 1.1. A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
- for every infinite collection $\{O_\alpha\}_{\alpha\in A} \subset \mathcal{T}$, we have $\bigcup_{\alpha\in A} O_\alpha \in \mathcal{T}$,
- for every finite collection $\{O_i\}_{1\leq i\leq n} \subset \mathcal{T}$, we have $\bigcap_{1\leq i\leq n} O_i \in \mathcal{T}$.

The set $\mathcal{T}$ is called a *topology* on X. The elements of $\mathcal{T}$ are called the *open sets*. In other words, the previous definition says that:

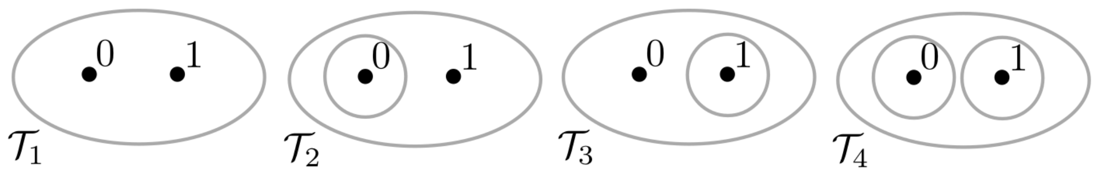
- the empty set is an open set, the set X itself is an open set,
- an infinite union of open sets is an open set,
- a finite intersection of open sets is an open set.

Note that the following is also true: an finite union of open sets is an open set.

Example 1.2. Let $X = \{0\}$ be a set with one element. There exists only one topology on X: $\mathcal{T} = \{\emptyset, \{0\}\}$.

Example 1.3. Let $X = \{0, 1\}$ be a set with two elements. There exists only four different topologies on X:

- $\mathcal{T}_1 = \{\emptyset, \{0, 1\}\}$,
- $\mathcal{T}_2 = \{\emptyset, \{0\}, \{0, 1\}\}$,
- $\mathcal{T}_3 = \{\emptyset, \{1\}, \{0, 1\}\}$,
- $\mathcal{T}_4 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.



Example 1.4. Let $X = \{0, 1, 2\}$ be a set with three elements. The set

$$\mathcal{T} = \{\emptyset\}$$

is not a topology on X because the whole set $X = \{1, 2, 3\}$ does not belong to $\mathcal{T}$. Likewise, the set

$$\mathcal{T} = \{\emptyset, \{0\}, \{1\}, \{0, 1, 2\}\}$$

is not a topology on X because the finite union $\{0\} \cup \{1\} = \{0, 1\}$ does not belong to $\mathcal{T}$.

Exercise 1. Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that X admits?
Hint: There are 29 of them.

Exercise 2. Let $\mathbb{Z}$ be the set of integers. Consider the *cofinite topology* $\mathcal{T}$ on $\mathbb{Z}$, defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or ${}^cO$ is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of O in $\mathbb{Z}$.

1. Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.
2. Exhibit an sequence of open sets $\{O_n\}_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $\bigcap\limits_{n\in\mathbb{N}} O_n$ is not an open set.

Conclusion: In general, in a given topology, an infinite intersection of open sets may not be open.
To meditate: However, if X is finite, every infinite intersection of open sets is an open set. Indeed, any topology on X must be finite, hence every infinite intersection of open sets must actually be a finite intersection.

Example 1.5. The set

$$\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{[0, a], a > 0\}$$

is not a topology on $\mathbb{R}$. Indeed, the following union of open sets is not an open set:

$$\bigcup_{a>0} [0, a] = [0, +\infty).$$

Another fondamental object of topological spaces is the following:

Definition 1.6. Let $(X, \mathcal{T})$ be a topological space. For every open set $O \in \mathcal{T}$, its complementary ${}^cO = \{x \in X, x \notin O\}$ is called a *closed set.*

We can deduce the following fact: **a subset $P \subset X$ is closed if and only if ${}^cP$ is open**. Indeed, a set P is closed if there exists an open set O such that $P = {}^cO$. Using the relation ${}^c({}^cO) = O$, we obtain ${}^cP = O$.

Proposition 1.7. *We have:*

- *the sets $\emptyset$ and X are closed sets,*
- *for every infinite collection $\{P_\alpha\}_{\alpha\in A}$ of closed set, $\bigcap\limits_{\alpha\in A} P_\alpha$ is a closed set,*
- *for every finite collection $\{P_i\}_{1\leq i\leq n}$ of closed sets, $\bigcup\limits_{1\leq i\leq n} P_i$ is a closed set.*

Proof. Proof of first point: The set $\emptyset$ is closed because ${}^c\emptyset = X$ is open. The set X is closed because ${}^cX = \emptyset$ is open.

Proof of second point: If $\{P_\alpha\}_{\alpha\in A}$ is an infinite collection of closed set, then for every $\alpha \in A$, ${}^cP_\alpha$ is open. Now, we use the relation

$${}^c\left(\bigcap_{\alpha\in A} P_\alpha\right) = \bigcup_{\alpha\in A} {}^cP_\alpha.$$

This is a union of open sets, hence it is open. Hence $\bigcap_{\alpha \in A} P_\alpha$ is closed.

Proof of third point: If $\{P_i\}_{1 \leq i \leq n}$ is a finite collection of closed set, then for every $i \in [\![1, n]\!]$, ${}^cP_i$ is open. Now, we use the relation

$$^c\left(\bigcup_{1 \leq i \leq n} P_i\right) = \bigcap_{1 \leq i \leq n} {}^cP_i.$$

This is a *finite* intersection of open sets, hence it is open. Hence $\bigcup_{1 \leq i \leq n} P_i$ is closed. □

1.2 Topology of $\mathbb{R}^n$

The study of general topological spaces is wild. In this course, we will mainly consider topological spaces that are sub-spaces of the spaces $\mathbb{R}^n$, $n \geq 0$. On $\mathbb{R}^n$, we will always consider the *Euclidean topology.*

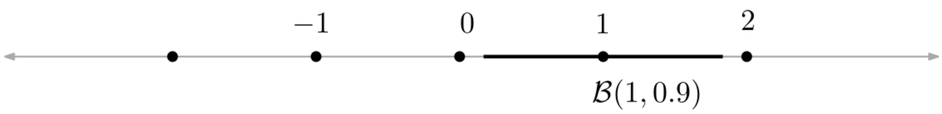
In order to define this topology, we will use open balls. Remind that the Euclidean metric on $\mathbb{R}^n$ is defined for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ as:

$$\|x\| = \sqrt{x_1^2 + ... + x_n^2}.$$

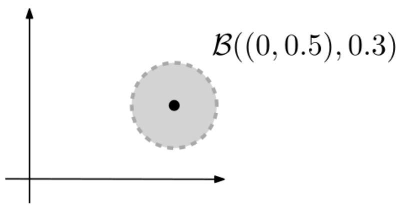
Definition 1.8. Let $x \in \mathbb{R}^n$ and $r > 0$. The *open ball* of center x and radius r, denoted $\mathcal{B}(x, r)$, is defined as:

$$\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}.$$

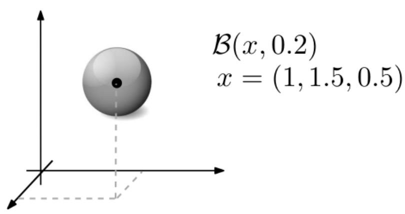
In $\mathbb{R}$:



In $\mathbb{R}^2$:

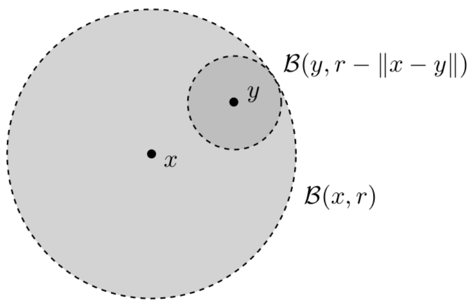


In $\mathbb{R}^3$:



Exercise 3. Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that

$$\mathcal{B}(y, \|x - y\|) \subset \mathcal{B}(x, r - \|x - y\|).$$



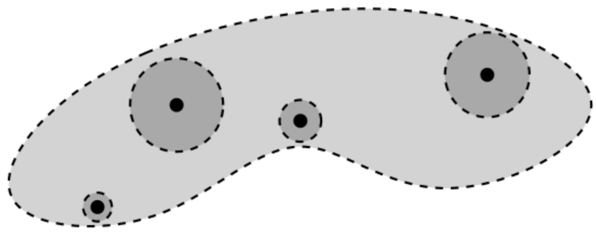
Exercise 4. Let $x, y \in \mathbb{R}^n$, and $r = \|x - y\|$. Show that

$$\mathcal{B}\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r).$$

Now we can define the Euclidean topology on $\mathbb{R}^n$.

Definition 1.9. Let $A \subset \mathbb{R}$ be a subset. Let $x \in A$. We say that A is *open around* x if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that A is *open* if for every $x \in A$, A is open around x.

We denote the set of such open sets by $\mathcal{T}_{\mathbb{R}^n}$.



Proposition 1.10. *$\mathcal{T}_{\mathbb{R}^n}$ is a topology on $\mathbb{R}^n$.*

Proof. We have to check the three axioms of a topological space.
First axiom (the empty set and the set X are open sets).
The set $\emptyset$ is clearly open according to the definition of $\mathcal{T}_{\mathbb{R}^n}$ (indeed, $\emptyset$ contains no point.) The set $\mathbb{R}^n$ also is open: for every $x \in \mathbb{R}^n$, the ball $\mathcal{B}(x, 1)$ is a subset of $\mathbb{R}^n$.

Second axiom (an infinite union of open sets is an open set).
Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}_{\mathbb{R}^n}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha \in A} O_\alpha$.
Let $x \in O$. There exists an $\alpha \in A$ such that $x \in O_\alpha$. Since O_α is open, it is open around x, i.e., there exists $r > 0$ such that $\mathcal{B}(x, r) \subset O_\alpha$.

We deduce that $\mathcal{B}(x, r) \subset O$, and that O is open around x. Since this it true for any $x \in O$, we proved that O is open.

Third axiom (a finite intersection of open sets is an open set).
Consider a finite collection $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1 \leq i \leq n} O_i$.
Let $x \in O$. For every $i \in [\![1, n]\!]$, we have $x \in O_i$. Since O_i is open, it is open around x, i.e., there exists $r_i > 0$ such that $\mathcal{B}(x, r_i) \subset O_i$. Define $r_{\min} = \min\{r_1, ... r_n\}$. For every $i \in [\![1, n]\!]$, we have $\mathcal{B}(x, r_{\min}) \subset O_i$.

We deduce that $\mathcal{B}(x, r_{\min}) \subset O$, and that O is open around x. Since this it true for any $x \in O$, we proved that O is open. $\square$

Exercise 5. Show that the open balls $\mathcal{B}(x, r)$ of $\mathbb{R}^n$ are open sets (with respect to the Euclidean topology).
Hint: You may use Exercise 3.

Exercise 6. Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?

1. $[0, 1]$,
2. $[0, 1)$,
3. $(-\infty, 1)$,
4. the singletons $\{x\}$, $x \in \mathbb{R}$,
5. the rationnals $\mathbb{Q}$.

1.3 Topology of subsets of $\mathbb{R}^n$

Definition 1.11. Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$ a subset. We define the *subspace topology on* Y as the following set:

$$\mathcal{T}_{|Y} = \{O \cap Y, O \in \mathcal{T}\}.$$

Proposition 1.12. *The set $\mathcal{T}_{|Y}$ is a topology on Y.*

Proof. We have to check the three axioms of a topological space.
First axiom (the empty set and the set X are open sets).
The set $\emptyset$ is clearly open for $\mathcal{T}_{|Y}$ because it can be written $\emptyset \cap Y$. The set Y also is open for $\mathcal{T}_{|Y}$ because it can be written $X \cap Y$, and X is open for $\mathcal{T}$.

Second axiom (an infinite union of open sets is an open set).
Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}_{|Y}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha \in A} O_\alpha$. By definition of $\mathcal{T}_{|Y}$, for every $\alpha \in A$, there exists O'_α such that $O_\alpha = O'_\alpha \cap Y$. Define $O' = \bigcup_{\alpha \in A} O'_\alpha$. It is an open set for $\mathcal{T}$. We have

$$O = \bigcup_{\alpha \in A} O_\alpha = \bigcup_{\alpha \in A} O'_\alpha \cap Y = \left(\bigcup_{\alpha \in A} O'_\alpha\right) \cap Y = O' \cap Y.$$

Hence $O \in \mathcal{T}_{|Y}$.

Third axiom (a finite intersection of open sets is an open set). Consider a finite collection $\{O_i\}_{1 \le i \le n} \subset \mathcal{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1 \le i \le n} O_i$. Just as before, for every $i \in [\![1, n]\!]$, there exists O'_i such that $O_i = O'_i \cap Y$. Define $O' = \bigcup_{1 \le i \le n} O'_i$. It is an open set for $\mathcal{T}$. We have

$$O = \bigcap_{1 \le i \le n} O_\alpha = \bigcap_{1 \le i \le n} O'_\alpha \cap Y = \left(\bigcap_{1 \le i \le n} O'_\alpha\right) \cap Y = O' \cap Y.$$

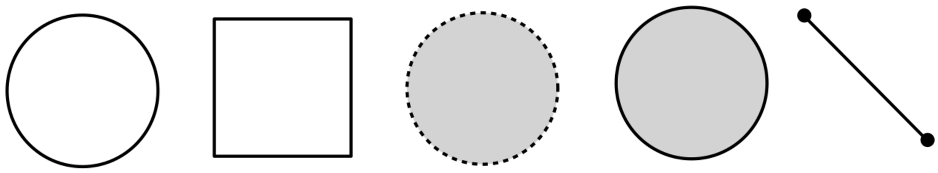
Hence $O \in \mathcal{T}_{|Y}$. □

Thanks to the subspace topology, any subset of $\mathbb{R}^n$ inherits a particular topology. This is the only topology we will consider on subsets of $\mathbb{R}^n$.

Among the subsets of $\mathbb{R}^n$ that we will consider, let us list:

- the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, \|x\| = 1\}$
- the unit cube $\mathcal{C}_{n-1} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n, \max(|x_1|, ..., |x_n|) = 1\}$
- the open balls $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}$
- the closed balls $\overline{\mathcal{B}}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| \leq r\}$
- the standard simplex

$$\Delta_{n-1} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n, x_1, ..., x_n \geq 0 \text{ and } x_1 + ... + x_n = 1\}$$



1.4 Continuous maps

The topologist's point of view allows to define the notion of continuity in great generality. In this subsection, we consider two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$.

Definition 1.13. Let $f\colon X \to Y$ be a map. We say that f is *continuous* if for every $O \in \mathcal{U}$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in $\mathcal{T}$.

In other words, a map is continuous if **the preimage of any open set is an open set**. As shown in the following example, the continuity of a map depends on the topologies that are given to X and Y.

Example 1.14. Let $X = Y = \{0, 1\}$ and $f\colon \{0, 1\} \to \{0, 1\}$ be the identity map, that is, $f(0) = 0$ and $f(1) = 1$. Let

$$\mathcal{T} = \{\emptyset, \{0, 1\}\} \qquad \text{and} \qquad \mathcal{U} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

The map f, seen as a map between the topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, is not continuous. Indeed, $\{0\}$ is an open set of $(Y, \mathcal{U})$, but $f^{-1}(\{0\}) = \{0\}$ is not an open set of $(X, \mathcal{T})$.

However, seen as a map between the topological spaces $(X, \mathcal{U})$ and $(Y, \mathcal{U})$, f is continuous. In particular, $f^{-1}(\{0\}) = \{0\}$ is an open set of $(X, \mathcal{U})$.

Remark 1.15. According to the previous Example, we should not say

$$f\colon X \to Y \textit{ is continuous},$$

without specifying the topologies on X and Y. We should say

$$f\colon (X,\mathcal{T}) \to (Y,\mathcal{U}) \textit{ is continuous.}$$

However, when it will be clear what topologies we are considering, and when there will be no risk of confusion, we will use the first sentence.

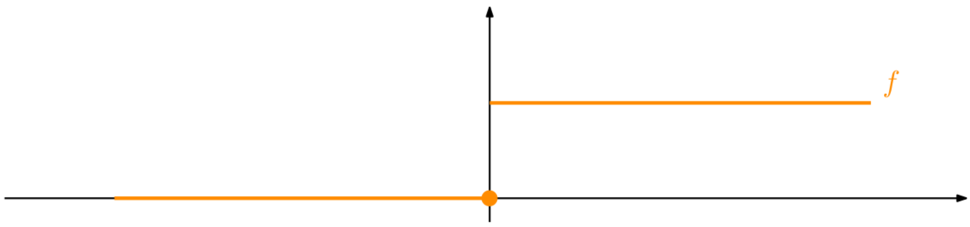
Continuity can also be stated in terms of closed sets:

Proposition 1.16. *A map is continuous if and only if the preimage of closed sets are closed sets.*

Exercise 7. Prove Proposition 1.16.
Hint: For any subset $A \subset Y$, show that $f^{-1}({}^{c}A) = {}^{c}(f^{-1}(A))$.

Example 1.17. Let $X = Y = \mathbb{R}$, endowed with the Euclidean topology. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$ for all $x \leq 0$, and $f(x) = 1$ for all $x > 0$.

The set $\{0\}$ is closed, but $f^{-1}(\{0\}) = (-\infty, 0)$ is not. Hence f is not continuous.



Proposition 1.18. *Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ and $(Z,\mathcal{V})$ be three topological spaces, and $f\colon X \to Y$, $g\colon Y \to Z$ two continuous maps. The composition $g \circ f$, defined as*

$$\begin{aligned} g \circ f\colon X &\longrightarrow Z \\ x &\longmapsto g(f(x)) \end{aligned}$$

is a continuous map.

In other words, we say that the composition of two continuous maps is a continuous map.

Proof. Let $O \in \mathcal{V}$ be an open set of Z. We have to show that $(g\circ f)^{-1}(O)$ is in $\mathcal{T}$. First, note that $(g \circ f)^{-1}(O) = f^{-1}\big(g^{-1}(O)\big)$. Since g is continuous, the set $g^{-1}(O)$ is in $\mathcal{U}$, i.e., it is an open set of Y. But since f is continuous, its preimage $f^{-1}\big(g^{-1}(O)\big)$ also is an open set (of X).

Since this is true for any open set $O \in \mathcal{V}$, we deduce that $g \circ f$ is continuous. $\square$

Link with the usual ϵ-δ calculus. We now investigate what continuity means between the Euclidean spaces $\mathbb{R}^n$. Consider a continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $\epsilon > 0$.

We have seen that the open ball $\mathcal{B}(f(x), \epsilon)$ is an open set of $\mathbb{R}^m$. By continuity of f, the preimage $f^{-1}(\mathcal{B}(f(x), \epsilon))$ is an open set.

Note that x belongs to $f^{-1}(\mathcal{B}(f(x), \epsilon))$. By definition of the Euclidean topology, we have that:

$$f^{-1}(\mathcal{B}(f(x), \epsilon)) \text{ is open around } x.$$

In other words, there exists a $\eta > 0$ such that

$$\mathcal{B}(x, \eta) \subset f^{-1}(\mathcal{B}(f(x), \epsilon)).$$

This is equivalent to

$$\forall y \in \mathcal{B}(x, \eta)\,,\ f(y) \in \mathcal{B}(f(x), \epsilon)\,.$$

We deduce that, for all $y \in \mathbb{R}^n$,

$$\|x - y\| < \eta \implies \|f(x) - f(y)\| < \epsilon.$$

We recognize **the usual definition of continuity**.

Proposition 1.19. *A map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if, for every $x \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in \mathbb{R}^n$,*

$$\|x - y\| < \eta \implies \|f(x) - f(y)\| < \epsilon.$$

Remark 1.20. As a consequence, what you already know about continuity still applies here.

Moralidade

A topologia geral contém todo o ϵ-δ cálculo, e muito mais.

The following proposition will be useful to study maps between subsets of $\mathbb{R}^n$:

Proposition 1.21. *Let f be a continuous map between $(X, \mathcal{T})$ and $(Y, \mathcal{U})$. Consider a subset $A \subset X$, and endow it with the subspace topology $\mathcal{T}_{|A}$. The induced map*

$$f_{|A}\colon (A, \mathcal{T}_{|A}) \to (Y, \mathcal{U})$$

is continuous. Moreover, for any subset $B \subset Y$ such that $f(A) \subset B$, the induced map

$$f_{|A,B}\colon (A, \mathcal{T}_{|A}) \to (B, \mathcal{U}_{|B})$$

also is continuous.

Proof. We will only prove the second statement. For every open set $O \in \mathcal{U}_{|B}$, let us show that $(f_{|A,B})^{-1}(O)$ is in $\mathcal{T}_{|A}$. By definition of $\mathcal{U}_{|B}$, there exists $O' \in \mathcal{U}$ such that $O = O' \cap B$. Now, we have

$$(f_{|A,B})^{-1}(O) = (f_{|A,B})^{-1}(O' \cap B) = (f_{|A,B})^{-1}(O') \cap (f_{|A,B})^{-1}(B).$$

Because of the assumption $f(A) \subset B$, we have $(f_{|A,B})^{-1}(B) = A$, and we deduce

$$(f_{|A,B})^{-1}(O) = (f_{|A,B})^{-1}(O') \cap A.$$

Since f is continuous, the preimage $(f_{|A,B})^{-1}(O')$ is in $\mathcal{T}$, hence the intersection $(f_{|A,B})^{-1}(O') \cap A$ is in $\mathcal{T}_{|A}$. □

Example 1.22. For any $\lambda > 0$ and $v \in \mathbb{R}^n$, we already know that the following map is continuous:

$$\begin{aligned} f\colon \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \lambda x + v \end{aligned}$$

As a consequence, the restricted map $f_{|\mathcal{B}(0,1),\mathcal{B}(v,\lambda)}\colon \mathcal{B}(0,1) \to \mathcal{B}(v,\lambda)$, seen between subspaces of $\mathbb{R}^n$ endowed with the subspace topology, is continuous.

2 Homeomorphisms

2.1 Definition

Definition 2.1. Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be two topological spaces, and $f\colon X \to Y$ a map. We say that f is a *homeomorphism* if

- f is a bijection,
- $f\colon X \to Y$ is continuous,
- $f^{-1}\colon Y \to X$ is continuous.

If there exists such a homeomorphism, we say that the two topological spaces are *homeomorphic*.

Remark 2.2. In practice, finding the inverse f^{-1} of f consists in finding a map $g\colon Y \to X$ such that

$$g \circ f = \mathrm{id} \qquad \text{and} \qquad f \circ g = \mathrm{id}.$$

In this case, g is the inverse of f.

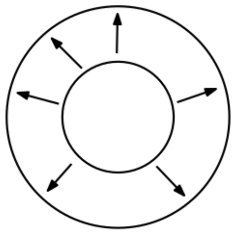
Example 2.3. Consider the following circles of $\mathbb{R}^2$:

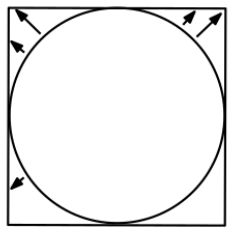
$$\begin{aligned} \mathbb{S}(0,1) &= \{x \in \mathbb{R}^2, \|x\| = 1\}, \\ \mathbb{S}(0,2) &= \{x \in \mathbb{R}^2, \|x\| = 2\} \end{aligned}$$

and the map

$$\begin{aligned} f\colon \mathbb{S}(0,1) &\longrightarrow \mathbb{S}(0,2) \\ x &\longmapsto 2x \end{aligned}$$

It is, bijective, and its inverse $f^{-1}\colon x \mapsto \frac{1}{2}x$ also is continuous. Hence f is a homeomorphism.





Example 2.4. Still in $\mathbb{R}^2$, consider a circle and a square:

$$\mathbb{S}(0,1) = \{x \in \mathbb{R}^2, \|X\| = 1\},$$
$$\mathcal{C} = \left\{(x_1, x_2) \in \mathbb{R}^2,\ \max(|x_1|, |x_2|) = 1\right\}.$$

Let $f \colon \mathbb{S}(0,1) \to \mathcal{C}$ be the map

$$f \colon (x_1, x_2) \mapsto \frac{1}{\max(|x_1|, |x_2|)}(x_1, x_2).$$

It is continuous. More over, it admits the following inverse *(check that this is true)*:

$$f^{-1} \colon x \mapsto \frac{1}{\sqrt{x_1^2 + x_2^2}}(x_1, x_2).$$

This map is continuous, hence f is a homeomorphism.

> **Exercise 8.** Show that the topological spaces $\mathbb{R}^n$ and $\mathcal{B}(0,1) \subset \mathbb{R}^n$ are homeomorphic.
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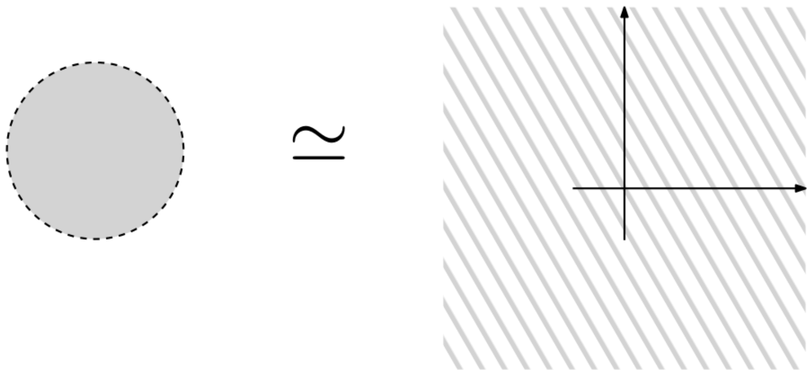
> 
>
> *Hint:* Consider the map $f \colon x \mapsto \frac{\|x\|}{(\|x\|+1)^2} x$.
> *Better hint:* Consider the map $f \colon x \mapsto \frac{1}{\|x\|+1} x$.

Exercise 9. Show that $\mathcal{B}(x, r)$ and $\mathcal{B}(y, s)$ are homeomorphic.

Exercise 10. Show that $\mathbb{S}(0,1)$, the unit circle of $\mathbb{R}^2$, is homeomorphic to the ellipse

$$\mathcal{S}(a, b) = \left\{(x_1, x_2) \in \mathbb{R}^2, \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1\right\}$$

for any $a, b > 0$.

Example 2.5. Let $\mathbb{S}(0,1)$ denote the unit circle of $\mathbb{R}^2$, and consider the map

$$\begin{aligned} f\colon [0,2\pi) &\longrightarrow \mathbb{S}(0,1) \\ \theta &\longmapsto (\cos(\theta),\sin(\theta)) \end{aligned}$$

It is continuous, and admits the following inverse:

$$\begin{aligned} g\colon \mathbb{S}(0,1) &\longrightarrow [0,2\pi) \\ (x_1,x_2) &\longmapsto \arctan\left(\frac{x_2}{x_1}\right) \end{aligned}$$

This comes from the relation $\theta = \arctan\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$ for all $\theta \in [0,2\pi)$.

The map g is **not** continuous. Indeed, $[0,\pi)$ is an open subset of $[0,2\pi)$, but $g^{-1}([0,\pi))$ is not an open subset of $\mathbb{S}(0,1)$ (it is not open around $g^{-1}(0)=(1,0)$).

We will see in Example 2.16 that there exists no homeomorphism between $[0,2\pi)$ and $\mathbb{S}(0,1)$.

Homeomorphism is an equivalence relation. Let us write $X \simeq Y$ if the two topological spaces X and Y are homeomorphic, i.e., if there exists a homeomorphism $f\colon X \to Y$. It is clear that, for any X, we have

$$X \simeq X.$$

Moreover, we have *(mental exercise)*:

$$X \simeq Y \iff Y \simeq X.$$

We also have a third property, stated in the following proposition:

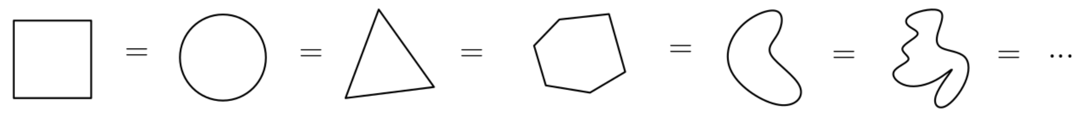
Proposition 2.6. *If three topological spaces X,Y,Z are such that X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z. In other words,*

$$X \simeq Y \text{ and } Y \simeq Z \implies X \simeq Z.$$

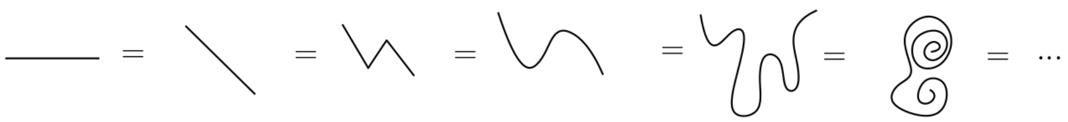
Proof. Supppose that X,Y are homeomorphic, and Y,Z too. This means that we have homeomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$. Consider the map $g\circ f\colon X \to Z$. It is continuous (by Proposition 1.18) bijective (composition of bijective maps) and its inverse $f^{-1}\circ g^{-1}\colon Z \to X$ is also continuous (by Proposition 1.18 too). Hence $g\circ f$ is a homeomorphism, and the spaces X,Z are homeomorphic. $\square$

The three previous properties are called respectively *reflexivity*, *symmetry* and *transitivity*. Hence **being homeomorphic** is what we call an **equivalence relation**. It allows to classify topological spaces in classes (called *classes of homeomorphism equivalence*):

- the class of circles:

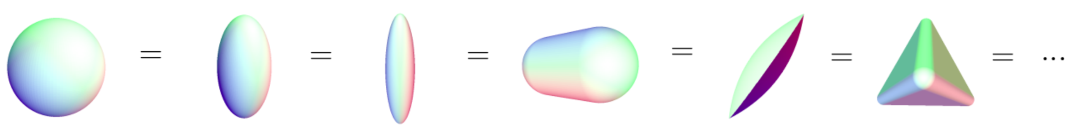


- the class of intervals:



- the class of crosses:

- the class of spheres of dimension 2:



- the class of torii, the class of Klein bottles, etc...

Moralidade
Para um·a topólogisto·a, dois espaços topológicos homeomórficos são o mesmo.

In general, it may be complicated to determine whether two topological spaces are homeomorphic. To answer this problem, we will use the notion of *invariant*. An invariant is a property, a characteristic, that is shared by all the topological space of a same class. Our first example will be connectedness.

2.2 Connected components

Definition 2.7. Let $(X, \mathcal{T})$ be a topological space. We say that X is *connected* if for every open sets $O, O' \in \mathcal{T}$ such that $O \cap O' = \emptyset$ (i.e., they are disjoint), we have

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

In other words, a connected topological space cannot be divided into two non-empty disjoint open sets.

One shows that a connected topological space cannot be divided into two non-empty disjoint **closed** sets.

Example 2.8. The subset $X = [0,1] \cup [2,3]$ of $\mathbb{R}$, endowed with the subspace topology, is not connected. Indeed, its subsets $[0,1]$ and $[2,3]$ are open disjoint non-empty sets that covers X.

We will accept the following result without proving it:

Proposition 2.9. *The balls of $\mathbb{R}^n$ are connected. More generally, any convex set is connected.*

If a space is not connected, we can consider its connected components. Let $x \in X$. The connected component of x is defined as the largest subset of X that is connected. The set of connected components of X forms a partition of X into **open** sets. Moreover, if there are only finitely many connected components, they are also **closed**.

Definition 2.10. Let $(X, \mathcal{T})$ be a topological space. Suppose that there exists a collection of n **non-empty**, **disjoint** and **connected** open sets $(O_1, ..., O_n)$ such that

$$\bigcup_{1 \leq i \leq n} O_i = X.$$

Then we say that X admits n connected components.

Remark 2.11. One shows that if there exists a collection of n **non-empty** and **disjoint** sets $(O_1, ..., O_n)$ such that

$$\bigcup_{1 \leq i \leq n} O_i = X,$$

then X admits at least n connected components.

Example 2.12. Consider the subset $X = \{0,1,2,3,4,5,6,7,8,9,10\}$ of $\mathbb{R}$. Each of its subsets $\{i\}$, $i \in X$, are open. They are all non-empty, connected and disjoint. Hence X admits ten connected components.

Lemma 2.13. *Let $f: X \to Y$ be a continuous map and O a connected component of X. Then $f(O) \subset Y$ is connected.*

Proof. Denote $O' = f(O)$. We will apply the definition of a connected topological space.

Suppose that there exists two disjoint open sets A, A' of Y such that $O' = A \cup A'$. The preimages $f^{-1}(A)$ and $f^{-1}(A')$ are disjoint open sets of X. Moreover,

$$O \subset f^{-1}(O') = f^{-1}(A \cup A') = f^{-1}(A) \cup f^{-1}(A').$$

Since O is connected, we deduce that $f^{-1}(A) = \emptyset$ or $f^{-1}(A') = \emptyset$. Therefore, $A = \emptyset$ or $A' = \emptyset$. This shows that O' is connected. $\square$

2.3 Connectedness as an invariant

Proposition 2.14. *Two homeomorphic topological spaces admit the same number of connected components.*

Proof. Let $f\colon X \to Y$ be a homeomorphism. Let n be the number of connected components of Y, and m the number of X. Let us show that $m = n$.

Suppose that Y admits n connected components. We can write $Y = \bigcup_{1\leq i\leq n} O_i$ where the O_i are disjoint non-empty connected sets. Also, we have seen that the O_i are open. For all $i \in [\![1, n]\!]$, define $O'_i = f^{-1}(O_i)$. We have:

- for all $i \in [\![1, n]\!]$ $O'_i = f^{-1}(O_i)$ is open (because f is continuous),
- $X = \bigcup_{1\leq i\leq n} O'_i$ (because f is a map)
- for all $i, j \in [\![1, n]\!]$ with $i \neq j$, $O'_i \cap O'_j = f^{-1}(O_i) \cap f^{-1}(O_j) = f^{-1}(O_i \cap O_j) = \emptyset$
- for all $i \in [\![1, n]\!]$, $O'_i = f^{-1}(O_i) \neq \emptyset$ (because f is a bijection).

Hence X can be covered by n disjoint non-empty open sets. Using Remark 2.11, we deduce that X admits at least n connected components.

Now, suppose that X admits m connected components. Using the same reasoning, one shows that Y admits at least m connected components. Hence we have $n \geq m \geq n$, that is, $n = m$. $\square$

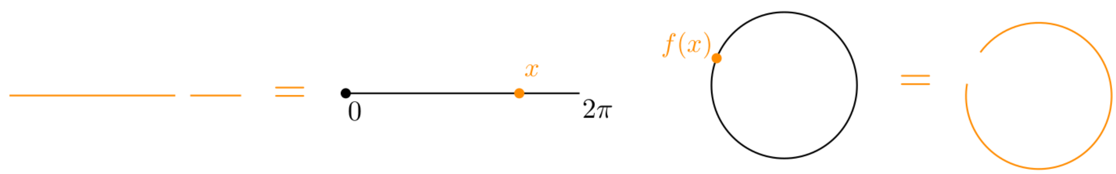
Example 2.15. The subsets $[0, 1]$ and $[0, 1] \cup [2, 3]$ of $\mathbb{R}$ are not homeomorphic. Indeed, the first one has one connected component, and the second one two.

Example 2.16. The interval $[0, 2\pi)$ and the circle $\mathbb{S}(0, 1) \subset \mathbb{R}^2$ are not homeomorphic. We will prove this by contradiction. Suppose that they are homeomorphic. By definition, this means that there exists a map $f\colon [0, 2\pi) \to \mathbb{S}(0, 1)$ which is continuous, inversible, and with continuous inverse.

Let $x \in [0, 2\pi)$ such that $x \neq 0$. Consider the subsets $[0, 2\pi) \setminus \{x\} \subset [0, 2\pi)$ and $\mathbb{S}(0, 1) \setminus \{f(x)\} \subset \mathbb{S}(0, 1)$, and the induced map

$$g\colon [0, 2\pi) \setminus \{x\} \to \mathbb{S}(0, 1) \setminus \{f(x)\}.$$

The map g is a homeomorphism. Moreover, it is clear that $[0, 2\pi) \setminus \{x\}$ has two connected components, and $\mathbb{S}(0, 1) \setminus \{f(x)\}$ only one. This contradicts Proposition 2.14.



Example 2.17. $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Just as before, we will prove this by contradiction. Suppose that there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^2$. Choose any $x \in \mathbb{R}$. The induced map

$$g\colon \mathbb{R} \setminus \{x\} \to \mathbb{R}^2 \setminus \{f(x)\}$$

is still a homeomorphism, but $\mathbb{R} \setminus \{x\}$ has two connected components, while $\mathbb{R}^2 \setminus \{f(x)\}$ has one. This is a contradiction.

The same reasoning shows that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic either.

Remark 2.18. More generally, the *invariance of domain* is a theorem that says that for every integers m, n such that $m \neq n$, the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic. We will need much more sophisticated tools to prove that (homology of spheres).

> **Exercise 11.** Show that $[0, 1)$ and $(0, 1)$ are not homeomorphic.
> *Hint:* Use the strategy of Examples 2.16 or 2.17.

Remark 2.19. The number of connected components is an example of a topological invariant: if two topological spaces are homeomorphic, they must admit the same number of connected components.

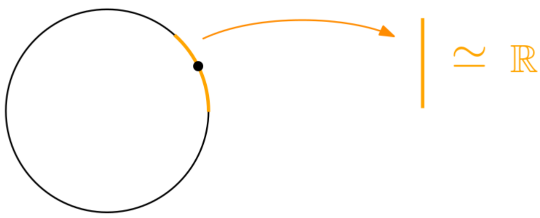
The previous examples show the general morale of a topological invariant: to prove that two spaces are not homeomorphic, prove that their invariant (here, the number of connected components) differ.

2.4 Dimension

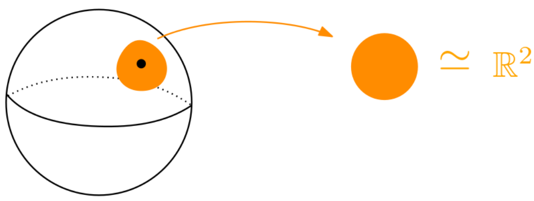
Definition 2.20. Let $(X, \mathcal{T})$ be a topological space, and $n \geq 0$. We say that it *has dimension* n if the following is true: for every $x \in X$, there exists an open set O such that $x \in O$, and a homeomorphism $O \to \mathbb{R}^n$.

In other words, a topological space of dimension n is a topological space that locally looks like the Euclidean space $\mathbb{R}^n$. For instance, one shows that

- the open intervals $(a, b) \subset \mathbb{R}$ have dimension 1,
- the circle $\mathbb{S}_1 \subset \mathbb{R}^2$ has dimension 1,



- more generally, the spheres $\mathbb{S}(v, r) \subset \mathbb{R}^n$ have dimension $n - 1$,



- the open balls $\mathcal{B}(v, r) \subset \mathbb{R}^n$ have dimension n,
- the Euclidean space $\mathbb{R}^n$ itself has dimension n.

Remark 2.21. For this definition to make sense, we have to make sure that the topological spaces $\mathbb{R}^n$, $n \geq 0$, are all not-homeomorphic. Otherwise, a topological space could have several dimensions. As we said earlier, this result, the *invariance of domain*, will be proved later.

Proposition 2.22. *Let X, Y be two homeomorphic topological spaces. If X has dimension n, then Y also has dimension n.*

Proof. Let n be the dimension of X, and consider a homeomorphism $g\colon Y \to X$.

Let $y \in Y$, and $x = g(y)$. Since x has dimension n, there exists an open set O of X, with $x \in O$, and a homeomorphism $h\colon O \to \mathbb{R}^n$.

Define $O' = g^{-1}(O)$. It is an open set of Y, with $y \in O'$. Moreover, the map $h \circ g\colon O' \to \mathbb{R}^n$ is a homeomorphism.

This being true for every $y \in Y$, we deduce that Y has dimension n. $\square$

We can read the previous proposition as follows: dimension is an invariant of homeomorphic spaces. As before, we can use it to show that two spaces are not homeomorphic.

Example 2.23. The unit circle $\mathbb{S}_1 \subset \mathbb{R}^2$ and the unit sphere $\mathbb{S}_2 \subset \mathbb{R}^3$ are not homeomorphic. Indeed, the first one has dimension 1, and the second one dimension 2.

Moralidade
Uma invariante é uma quantidade compartilhada
por todos os espaços topológicos idênticos.

3 Homotopies

3.1 Homotopy equivalence between maps

Definition 3.1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces, and $f, g\colon X \to Y$ two continuous maps. A *homotopy* between f and g is a map $F\colon X \times [0,1] \to Y$ such that:

- $F(\cdot, 0)$ is equal to f,
- $F(\cdot, 1)$ is equal to g,
- $F\colon X \times [0,1] \to Y$ is continuous.

If such a homotopy exists, we say that the maps f and g are *homotopic*.

Remark 3.2. For any $t \in [0,1]$, the notation $F(\cdot, t)$ refers to the map

$$\begin{aligned} F(\cdot, t)\colon X &\longrightarrow Y \\ x &\longmapsto F(x, t) \end{aligned}$$

Remark 3.3. Before asking for $F\colon X\times[0,1]\to Y$ to be continuous, we have to give $X\times[0,1]$ a topology. The topology we choose is the *product topology*.

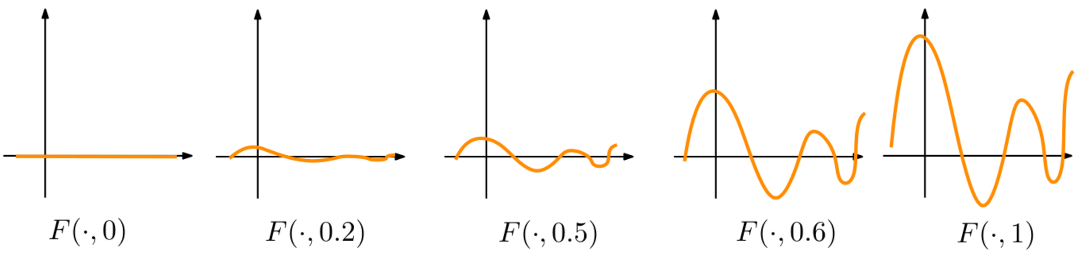
Consider the topological space $(X,\mathcal{T})$, and endow $[0,1]$ with the subspace topology of $\mathbb{R}$, denoted $\mathcal{T}_{|[0,1]}$. The product topology on $X\times[0,1]$, denoted $\mathcal{T}\otimes\mathcal{T}_{|[0,1]}$, is defined as follows: a set $O\subset X\times[0,1]$ is open if and only if it can be written as a union

$$\bigcup_{\alpha\in A} O_\alpha\times O'_\alpha$$

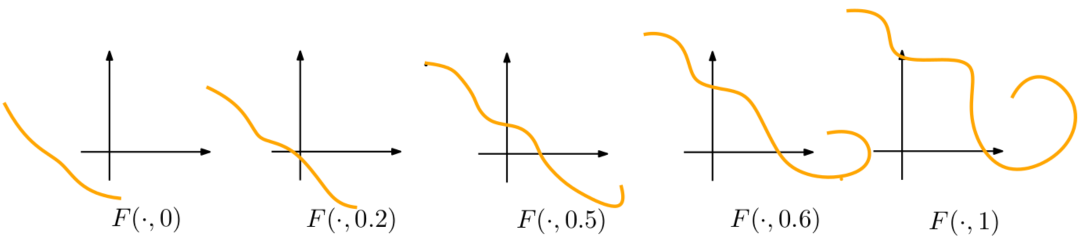
where every O_α is an open set of X and O'_α is an open set of $[0,1]$.

When $(X,\mathcal{T})$ is a subspace of $\mathbb{R}^n$ endowed with the subspace topology, we can describe the product topology in a different way. The product $X\times[0,1]$ can be seen as a subset of $\mathbb{R}^{n+1}$, and one shows that the product topology $\mathcal{T}\otimes\mathcal{T}_{|[0,1]}$ is equal to the subspace topology $\mathcal{T}_{|X\times[0,1]}$.

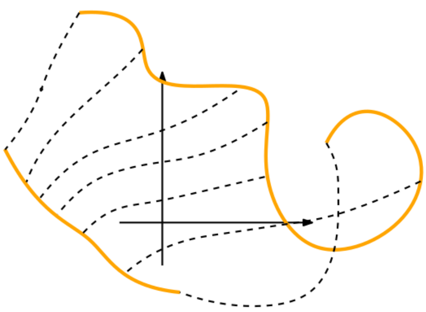
We may represent graphically a homotopy $F\colon\mathbb{R}\times[0,1]\to\mathbb{R}$ by plotting it for each value of $t\in[0,1]$:



This is an example for $F\colon[0,1]\times[0,1]\to\mathbb{R}^2$:



Sometimes we prefer to plot the deformation:



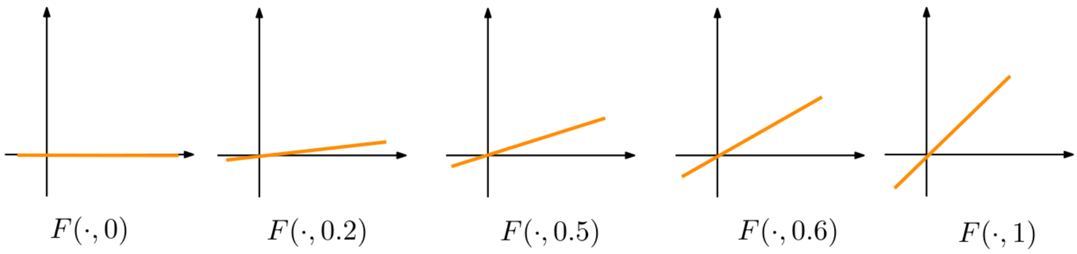
Example 3.4. Let $X = Y = [-1, 1]$ endowed with the Euclidean topology, and consider the maps $f, g \colon X \to Y$ defined as

$$f \colon x \mapsto 0$$
$$g \colon x \mapsto x$$

Let us prove that they are homotopic. Consider the map

$$\begin{aligned} F \colon X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto tx \end{aligned}$$

We see that $F(\cdot, 0) \colon x \mapsto 0$ is equal to f, and $F(\cdot, 1) \colon x \mapsto x$ is equal to g. Moreover, F is continuous. Hence, F is an homotopy between f and g. Thus these two maps are homotopic.

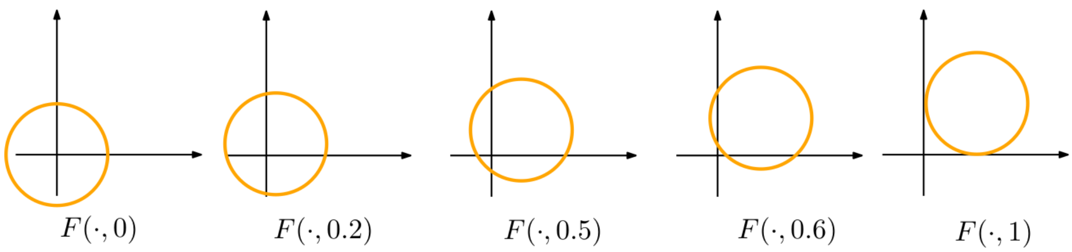


Example 3.5. The following map

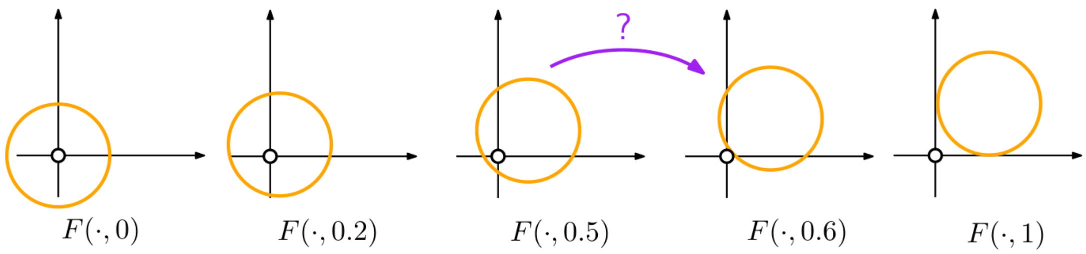
$$\begin{aligned} F \colon \mathbb{S}_1 \times [0,1] &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto (\cos(\theta) + t, \sin(\theta) + t) \end{aligned}$$

is a homotopy between the maps

$$f \colon \theta \mapsto (\cos(\theta), \sin(\theta)) \qquad \text{and} \qquad g \colon \theta \mapsto (\cos(\theta) + 1, \sin(\theta) + 1)$$



Example 3.6. Between $\mathbb{S}_1$ and $\mathbb{R}^2 \setminus \{(0,0)\}$, the plane without the origin, there is no homotopy between the maps f and g of the previous example. Indeed, the homotopy F would pass through the point $(0,0)$ at some point, which is impossible.



We have to wait for the next lessons to prove formally that such a homotopy does not exist.

From a homotopic point a view, a trivial map is a map that is homotopic to a constant map. For instance, the identity map of Example 3.4 is homotopic to the constant map $x \mapsto 0$. More generally, we have:

Proposition 3.7. *Let $f\colon X \to \mathbb{R}^n$ be a continuous map. Then f is homotopic to a constant map.*

Proof. Consider the continuous application

$$\begin{aligned} F\colon X \times [0,1] &\longrightarrow \mathbb{R}^n \\ x &\longmapsto tf(x) \end{aligned}$$

We have that $F(\cdot, 1) = f$, and $F(\cdot, 0)\colon x \mapsto 0$ is a constant map. □

Proposition 3.8. *Let $f\colon \mathbb{R}^n \to X$ be a continuous map. Then f is homotopic to a constant map.*

Exercise 12. Prove the previous proposition.

As a consequence, the theory of maps with domain or codomain $\mathbb{R}^n$ is trivial from a homotopy equivalence perspective. For instance, *knot theory*, the theory that studies maps $\mathbb{S}_1 \to \mathbb{R}^3$, does not exist for us.

However, when the domain and codomain are not Euclidean spaces, as in Example 3.6, many non-homotopic maps may exist.

Moralidade
Para um·a topólogisto·a, duas aplicações homotópicas são a mesma coisa.

Exercise 13. Let $f\colon\ \mathbb{S}_1 \to \mathbb{S}_2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map.
Hint: Let $x_0 \in \mathbb{S}_2$ be such that $x_0 \notin f(\mathbb{S}_1)$. Find a homotopy between f and the constant map $g\colon x \mapsto -x_0$.
More complicated question: Is every continuous map $f\colon\ \mathbb{S}_1 \to \mathbb{S}_2$ homotopic to a constant map?

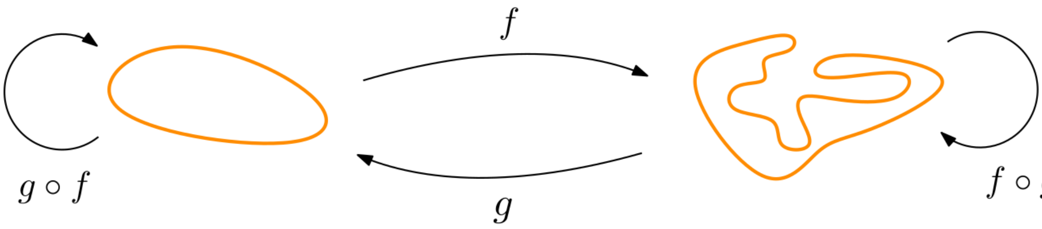
Exercise 14. Show that being homotopic is a *transitive* relation between maps: for every triplet of maps $f, g, h\colon X \to Y$, if f, g are homotopic and g, h are homotopic, then f, h are homotopic.

3.2 Homotopy equivalence between topological spaces

Definition 3.9. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces. A *homotopy equivalence* between X and Y is a pair of continuous maps $f: X \to Y$ and $g: Y \to X$ such that:

- $g \circ f: X \to X$ is homotopic to the identity map $\mathrm{id}: X \to X$,
- $f \circ g: Y \to Y$ is homotopic to the identity map $\mathrm{id}: Y \to Y$.

If such a homotopy equivalence exists, we say that X and Y are *homotopy equivalent*.



Determining whether two topological spaces are homotopy equivalent may be difficult. When one is a subset of the other, we have a handy tool:

Definition 3.10. Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $\mathcal{T}_{|Y}$. A *retraction* is a continuous map $r: X \to X$ such that $\forall x \in X, r(x) \in Y$ and $\forall y \in Y, r(y) = y$.

A *deformation retraction* is a homotopy $F: X \times [0,1] \to Y$ between the identity map $\mathrm{id}: X \to X$ and a retraction $r: X \to X$.

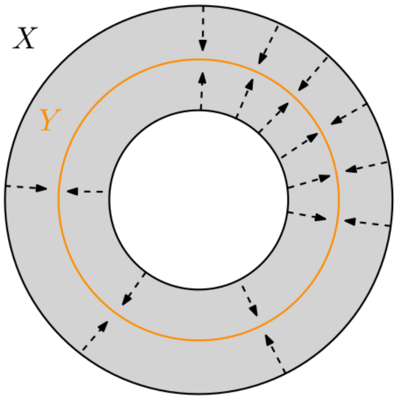
Proposition 3.11. *If a deformation retraction exists, then X and Y are homotopy equivalent.*

Proof. Let $r: X \to X$ denote the retraction, and consider the inclusion map $i: Y \to X$. Note that, since $\forall x \in X, r(x) \in Y$, we can see the retraction r as a map $r: X \to Y$. Let us prove that r, i is a homotopy equivalence.

First, let us prove that $i \circ r: X \to X$ is homotopic to the identity map $\mathrm{id}: X \to X$. This is clear because $i \circ r = r$, and r is homotopic to the identity by definition of a deformation retraction.

Second, let us prove that $r \circ i: Y \to Y$ is homotopic to the identity map $\mathrm{id}: Y \to Y$. This is obvious because $r \circ i = \mathrm{id}$ by definition of a retraction. $\square$

Example 3.12. The circle and the annulus are homotopy equivalent. Indeed, the circle can be seen as a subset of the annulus, and we have a deformation retraction:



Example 3.13. The letter O and the letter Q are homotopy equivalent. Indeed, O can be seen as a subset of Q, and Q deform retracts on it.

Example 3.14. For any $n \geq 1$, the Euclidean space $\mathbb{R}^n$ is homotopy equivalent to the point $\{0\} \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r\colon \mathbb{R}^n &\longrightarrow \{0\} \\ x &\longmapsto 0 \end{aligned}$$

It is homotopic to the identity $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ via the deformation retraction

$$\begin{aligned} F\colon \mathbb{R}^n \times [0,1] &\longrightarrow \mathbb{R}^n \\ x &\longmapsto (1-t)x \end{aligned}$$

Indeed, we have $F(\cdot, 0) = \mathrm{id}$ and $F(\cdot, 1) = r$.

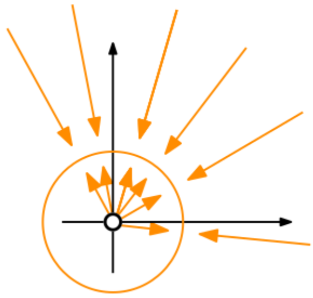
Example 3.15. For any $n \geq 1$, the Euclidean space without origin, $\mathbb{R}^n \setminus \{0\}$, is homotopy equivalent to the sphere $\mathbb{S}(0,1) \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r\colon \mathbb{R}^n \setminus \{0\} &\longrightarrow \mathbb{S}(0,1) \\ x &\longmapsto \frac{x}{\|x\|} \end{aligned}$$

It is homotopic to the identity $\mathrm{id}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ via the deformation retraction

$$\begin{aligned} F\colon (\mathbb{R}^n \setminus \{0\}) \times [0,1] &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ x &\longmapsto \left(1 - t + \frac{t}{\|x\|}\right) x \end{aligned}$$

Indeed, we have $F(\cdot, 0) = \mathrm{id}$ and $F(\cdot, 1) = r$.

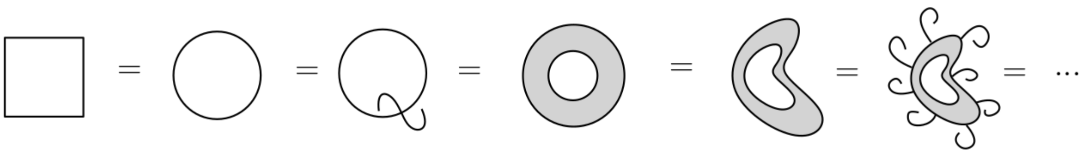


Remark 3.16. Let us denote $X \approx Y$ if the two topological spaces X and Y are homotopy equivalent. Just as for homeomorphic spaces, being homotopy equivalent is an *equivalence relation*. That is:

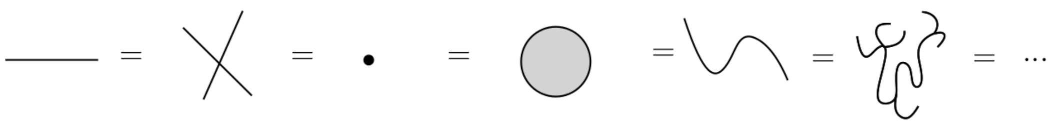
- *(Reflexivity)* $X \approx X$
- *(Symmetry)* $X \approx Y \implies Y \approx X$.
- *(Transitivity)* $X \approx Y$ and $Y \approx Z \implies X \approx Z$.

We can classify topological spaces according to this relation, and obtain *classes of homotopy equivalence*:

- the class of circles:



- the class of points:



- the class of spheres, the class of torii, the class of Klein bottles, etc...

Moralidade
Para um·a topólogisto·a, dois espaços topológicos homotópico-equivalentes são o mesmo.

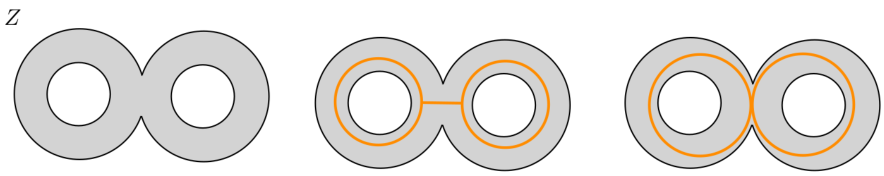
Exercise 15. Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).
Hint: You can use Exercise 14.

Remark 3.17. A method to show that two topological spaces X, Y are homotopy equivalent: find a third space Z that contains X, Y and such that there exist a deformation retraction from Z to X and from Z to Y.

If this is the case, we have $X \approx Z$ and $Y \approx Z$, and by using symmetry and transitivity, we deduce $X \approx Y$.

For instance, consider the two following subspaces of $\mathbb{R}^2$:

They are not included one in another. However, the following space contains them, and we see that it deform retracts on both X and Y.



> **Exercise 16.** Classify the letters of the alphabet into homotopy equivalence classes.

3.3 Link with homeomorphic spaces

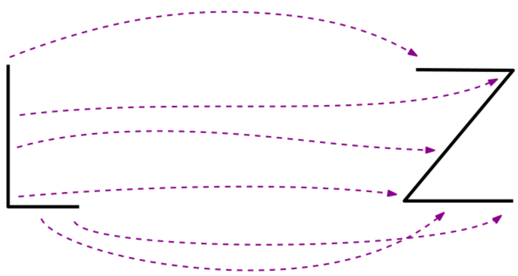
We have studied in the previous lesson another equivalence relation: the relation of homeomorphism. It turns out that it is stronger than the homotopy equivalence relation:

Proposition 3.18. *Let X, Y be two topological spaces. If they are homeomorphic, then they are homotopy equivalent. In other words:*

$$X \simeq Y \implies X \approx Y.$$

As a consequence, in order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic. However, this strategy does not always work: some spaces are homotopy equivalent but not homeomorphic. This is the case for $\mathbb{R}^n$ and $\{0\}$ for instance.

Example 3.19. The letter L and the letter Z are homeomorphic via the following homeomorphism. Hence they are homotopy equivalent.



3.4 Topological invariants

We now investigate how the invariants *connected components* and *dimension* behave with respect to the homotopy equivalence.

The following result should be compared with Proposition 2.14:

Proposition 3.20. *Two homotopy equivalent topological spaces admit the same number of connected components.*

Proof. Let X, Y be two topological spaces, and $f\colon X \to Y, g\colon Y \to X$ a homotopy equivalence. We will show that f induces a bijection between the connected components of X and Y.

Let $F\colon X \times [0,1] \to X$ be a homotopy between $g \circ f$ and $\mathrm{id}\colon X \to X$. Let $x \in X$, and O the connected component of x. The space $O \times [0,1]$ is connected. Hence its image $F(O \times [0,1]) \subset X$ is connected too (this is Lemma 2.13).

Moreover, $O = F(O \times \{1\}) \subset F(O \times [0,1])$. Hence $F(O \times [0,1])$ is a connected subset of X that contains O, and we deduce that $O = F(O \times [0,1])$. Last, notice that

$$g \circ f(O) = F(O \times \{0\}) \subset F(O \times [0,1]) = O.$$

We can now conclude from the relation $g \circ f(O) \subset O$. Suppose that X admits n connected components $O_1, ..., O_n$, and that Y admits m of them. By contradiction, suppose that $m < n$. This implies that we have two components O_i, O_j such that $f(O_i)$ and $f(O_j)$ are included in the same connected component O' of Y. Hence $g \circ f(O_i)$ and $g \circ f(O_j)$ are included in a common connected component of X. This is absurd because $g \circ f(O_i) \subset O_i$ and $g \circ f(O_j) \subset O_j$.

By exchanging the roles of X and Y in the whole reasonning, we obtain that $m > n$ also is absurd. We deduce that $m = n$. ☐

In other words, number of connected components is an invariant of homotopy equivalence. As for homeomorphic equivalence, this allows to show that two spaces are not equivalent.

Example 3.21. For any $n, m \geq 0$ such that $n \neq m$, the subspaces $\{1, ..., n\}$ and $\{1, ..., m\}$ of $\mathbb{R}$ are not homotopy equivalent. Indeed, the first one admits n connected components, and the second one m components.

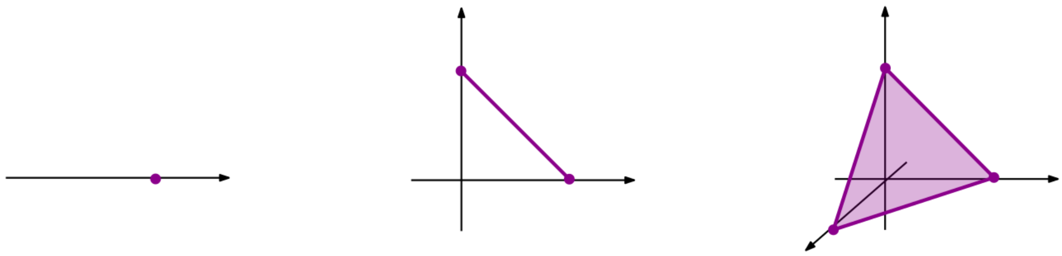
On the other hand, dimension is **not** an invariant of homotopy equivalence. Indeed, some homotopy equivalent spaces have different dimensions. This is the case, for instance, with all the Euclidean spaces $\mathbb{R}^n$, $n \geq 0$. They are all homotopy equivalent by Example 3.14, but all with different dimensions ($\mathbb{R}^n$ has dimension n).

4 Simplicial complexes

4.1 Definition

Topological spaces, such as subsets of $\mathbb{R}^n$, may be difficult to deal with on a computer. In order to describe them nicely, we may try to decompose them into simpler pieces. The pieces we shall consider are the *standard simplices*. We recall that the standard simplex of dimension n is the following subset of $\mathbb{R}^{n+1}$

$$\Delta_n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1},\ x_1, ..., x_{n+1} \geq 0 \text{ and } x_1 + ... + x_{n+1} = 1\}.$$



Δ_0 Δ_1 Δ_2

Remark 4.1. For any collection of points $a_1, ..., a_k \in \mathbb{R}^n$, we define their *convex hull* as:

$$\mathrm{conv}(\{a_1...a_k\}) = \left\{ \sum_{1 \leq i \leq k} t_i a_i, \quad t_1 + ... + t_k = 1, \quad t_1, ..., t_k \geq 0 \right\}.$$

Therefore we can say that Δ_n is the convex hull of the vectors $e_1, ..., e_{n+1} \in \mathbb{R}^{n+1}$, where

$$e_i = (0, ..., 1, 0, ..., 0) \qquad (i^{\text{th}} \text{ coordinate } 1, \text{ the other ones } 0).$$

Note that the simplex Δ_n is described by $n + 1$ vertices. Let us keep this geometric picture in mind in what follows.

Definition 4.2. Let V be a set (called the set of *vertices*). A *simplicial complex* over V is a set K of subsets of V (called the *simplices*) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$.

By convention, when talking about simplices, we write them with square brackets instead of curly brackets. For instance, the simplex $\{0, 1\}$ will be denoted $[0, 1]$

If $\sigma \in K$ is a simplex, its non-empty subsets $\tau \subset \sigma$ are called *faces* of σ, and σ is called a *coface* of τ. For instance, $[0, 1]$ is a face of $[0, 1, 2]$, and $[0, 1, 2]$ is a coface of $[0, 1]$.

Example 4.3. Let $V = \{0, 1, 2\}$ and

$$K = \{[0], [1], [2], [0, 1], [1, 2], [2, 0]\}.$$

This is a simplicial complex.

Example 4.4. Let $V = \{0, 1, 2\}$ and

$$K = \{[0], [1], [2], [0, 1], [1, 2], [0, 1, 2]\}.$$

This is not a simplicial complex. Indeed, the simplex $[0, 1, 2]$ admits a face $[2, 0]$ that is not included in V.

If σ is a simplex, its dimension is defined as $|\sigma| - 1$ (cardinality of σ minus 1). If K is a simplicial complex, its dimension is defined as the maximal dimension of its simplices.

Example 4.5. Let $V = \{0, 1, 2, 3\}$ and

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}.$$

It a simplicial complex of dimension 2.

Example 4.6. Let $V = \{0, 1, 2, 3\}$ and

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], \\ [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3], [0, 1, 2, 3]\}.$$

It a simplicial complex of dimension 3.

At the moment, a simplicial complex has no topology. It is a purely combinatorial object. However, in order to represent it, we can draw it as follows : put the points V in the plane or the space, and for each simplex of K, fill the convex hull of its vertices. For instance, the simplicial complexes of Examples 4.3 and 4.5 looks:

Remark 4.7. For reasons that will be clearer later, when drawing a simplicial complex, the simplices must not cross each other. However, it is not always possible to draw a simplicial complex in the plane (or space) this way.

As an example, the bipartite graph $K_{3,3}$ is a simplicial complex of dimension 1 (a *graph*) that cannot be drawn in the plane without crossing itself.

4.2 Topology

In this section, we will give simplicial complexes a topology. There are two ways of doing that: by embedding the simplicial complex in a Euclidean space $\mathbb{R}^n$ for n large enough, or via the gluing construction. We shall consider the first one.

Definition 4.8. Let K be a simplicial complex, with vertex $V = [\![1, ..., n]\!]$. In $\mathbb{R}^{n+1}$, consider, for every $i \in [\![0, n]\!]$, the vector $e_i = (0, ..., 1, 0, ..., 0)$ (i^{th} coordinate 1, the other ones 0). Let $|K|$ be the subset of $\mathbb{R}^{n+1}$ defined as:

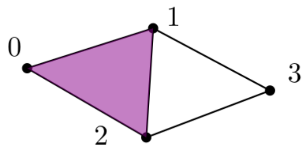
$$|K| = \bigcup_{\sigma \in K} \operatorname{conv}(\{e_j, j \in \sigma\})$$

where conv represent the convex hull (see Remark 4.1).

Endowed with the subspace topology, $(|K|, \mathcal{T}_{||K|})$ is a topological space, that we call the *topological realization of* K.

Remark 4.9. There exists another definition of topological realization, via *quotient topology*. Basically, it consists in giving each simplices a topology (namely, the subspace topology of the standard simplex), and in gluing all these simplices together.

Remark 4.10. If a simplicial complex can be drawn in the plane (or space) without crossing itself, then its topological realization simply is the subspace topology. This is the case for $K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 0], [1, 3], [2, 3], [0, 1, 2]\}$.



Definition 4.11. Let $(X, \mathcal{T})$ be a topological space. A *triangulation* of X is a simplicial complex K such that its topological realization $(|K|, \mathcal{T}_{||K|})$ is homeomorphic to $(X, \mathcal{T})$.

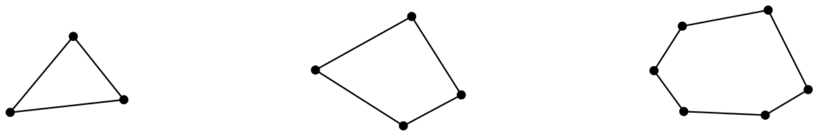
Example 4.12. The following simplicial complex, as in Example 4.3, is a triangulation of the circle:

$$K = \{[0], [1], [2], [0, 1], [1, 2], [2, 0]\}.$$

Example 4.13. The following simplicial complex, as in Example 4.5, is a triangulation of the sphere:

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}.$$

Given a topological space, it is not always possible to triangulate it. However, when it is, there exists many different triangulations. For instance, all the following simplicial complexes are triangulations of the circle.



Exercise 17. Give a triangulation of the cylinder.

4.3 Euler characteristic

Until here, we defined two invariants of topological space: number of connected components (homotopy type invariant), and dimension (homeomorphic invariant). We will now define one suited for simplicial complexes.

Definition 4.14. Let K be a simplicial complex of dimension n. Its *Euler characteristic* is the integer

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } \ i).$$

Example 4.15. The simplicial complex of Example 4.3 has Euler characteristic

$$\chi(K) = 3 - 3 = 0.$$

Exercise 18. What are the Euler characteristics of Examples 4.5 and 4.5? What is the Euler characteristic of the icosahedron?

Exercise 19. Let K be a simplicial complex (with vertex set V). A *sub-complex* of K is a set $M \subset K$ that is a simplicial complex. Suppose that there exists two sub-complexes M and N of K such that $K = M \cup N$. Show the *inclusion-exclusion principle:*

$$\chi(K) = \chi(M) + \chi(N) - \chi(M \cap N).$$

Now, let $(X, \mathcal{T})$ be a topological space, and K a triangulation of it. We would like to define the Euler characteristic of X to be equal to the Euler characteristic of K:

$$\chi(X) = \chi(K).$$

Is it well-defined? In other words, if K' is another triangulation of X, is it true that

$$\chi(K) = \chi(K')?$$

It turns out that **this is true**, but we won't be able to prove it in this summer course.

Definition 4.16. The Euler characteristic of a topological space is the Euler characteristic of any triangulation of it.

Here is a key fact: the Euler characteristic is a topological invariant.

Proposition 4.17. *If X and Y are two homotopy equivalent topological spaces, then $\chi(X) = \chi(Y)$.*

Exercise 20. What is the Euler characteristic of a sphere of dimension 1? 2? 3? *Hint:* First, find a triangulation of the sphere $\mathbb{S}_n \subset \mathbb{R}^{n+1}$. It can be triangulated with $n+2$ simplices of dimension n.

Exercise 21. Using the previous exercise, show that $\mathbb{R}^3$ and $\mathbb{R}^4$ are not homeomorphic. *Hint:* By contradiction, suppose that they are. Using Example 3.15, deduce that the unit sphere $\mathbb{S}_2 \subset \mathbb{R}^3$ and $\mathbb{S}_3 \subset \mathbb{R}^4$ are homotopy equivalent. Conclude with Proposition 4.17 and Exercise 20.

4.4 Python tutorial

Notebook available at
`https://github.com/raphaeltinarrage/EMAp/blob/main/Tutorial1.ipynb`.

In order to deal with simplicial complexes, we use the Gudhi library. We shall also use the libraries Matplotlib and Networkx (for plotting). Make sure to download the latest version!

Our code starts with

```
import gudhi
import numpy as np
import networkx as nx
```

We define a simplicial complex in GUDHI via

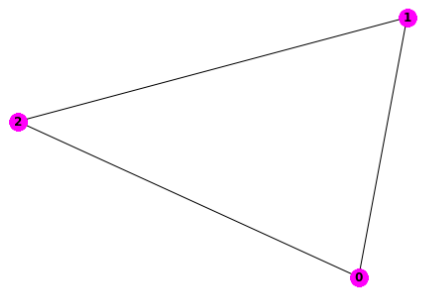
```
simpcomplex = gudhi.SimplexTree()

# We add the vertices
simpcomplex.insert([0])
simpcomplex.insert([1])
simpcomplex.insert([2])

# We add the edges
simpcomplex.insert([0,1])
simpcomplex.insert([1,2])
simpcomplex.insert([2,0])
```

The simplicial complex `simpcomplex` being created, we can use the functions

- `PrintSimplices(simpcomplex)` to print a list of its simplices:

  ```
  The simplicial complex contains the following simplices:
  Dimension 0: [0], [1], [2]
  Dimension 1: [0, 1], [0, 2], [1, 2]
  ```

- `DrawSimplicialComplex(simpcomplex)` to output a visual representation of the simplicial complex (only its vertices and edges):



- `NumberOfConnectedComponents(simpcomplex)` to give its connected components:

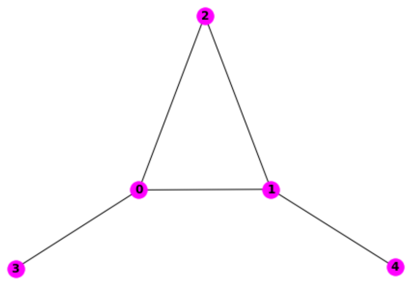
  ```
  The simplicial complex admits 1 connected component(s).
  ```

- `EulerCharacteristic(simpcomplex)` to give its Euler characteristic:

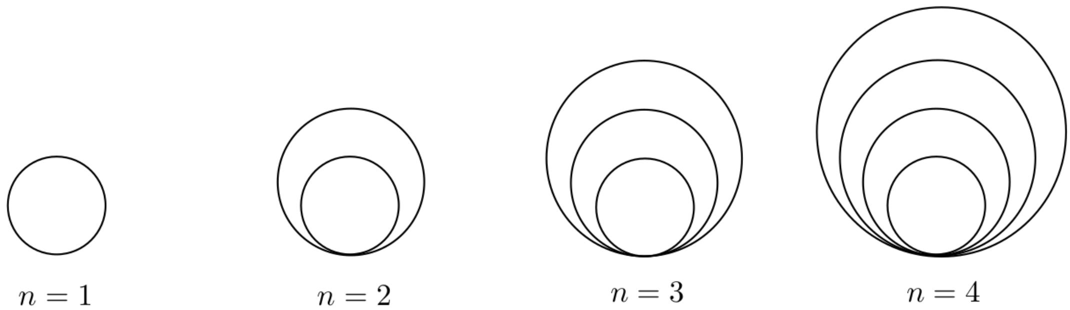
  ```
  The simplicial complex has Euler characteristic equal to 0.
  ```

Exercise 22. Build triangulations of the letters of the alphabet, and compute their Euler characteristic.

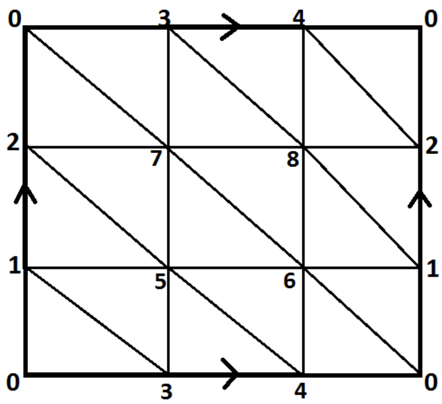
Given two letters that are homotopy equivalent, is it true thar their Euler characteristic are equal? Given two letters that are not homotopy equivalent, is it true that their Euler characteristic are different? (see Exercise 16)
Hint: For instance, the following is a triangulation of A:



Exercise 23. For every n, triangulate the bouquet of n circles (see below). Compute their Euler characteristic.



Exercise 24. Implement the following triangulation of the torus:



Compute its Euler characteristic.

> **Exercise 25.** Consider the following dataset of 30 points $x_0, ..., x_{29}$ in $\mathbb{R}^2$:
>
> ```
> {0: [0.29409772548731694, 0.6646626625013836],
>  1: [0.01625840776679577, 0.1676405753593595],
>  2: [0.15988905150272759, 0.6411323760808338],
>  3: [0.9073191075894482, -0.16417982219713312],
>  4: [-0.18661467838673884, 0.31618948583046413],
>  5: [-0.3664040542098381, 0.9098590694955988],
>  6: [-0.43753448716144905, -0.8820102274699417],
> ```

```
7: [0.4096730199915961, -0.23801426675264126],
8: [0.5903822103474676, -0.7285102954232894],
9: [0.9133851839417766, -0.6606557328320093],
10: [-0.15516122940597588, 0.7565411235103017],
11: [-0.38626186295039866, -0.3662321656058476],
12: [0.005209710070218199, 0.27655964872153116],
13: [0.670078068894711, -0.00932202688834849],
14: [-0.011268465716772091, 0.24340880308017376],
15: [-0.6441978411451603, -0.9672635759413206],
16: [-0.2841794022401025, -0.6734801188906114],
17: [-0.15473260248990717, -0.1365357396855129],
18: [0.7177096105982121, 0.9378197891592468],
19: [-0.4677068504994166, 0.1533930130294956],
20: [-0.32379909116817096, 0.9694800649768063],
21: [-0.2886940472879451, -0.039544695812395725],
22: [-0.5900701743351606, 0.8350804500575086],
23: [0.14931959728335853, 0.869106793774487],
24: [-0.14500672678238824, -0.3170082291070364],
25: [0.07324547392476122, 0.6653572287065117],
26: [-0.662990048258566, 0.1908198608241125],
27: [-0.25641262456436276, -0.9844196180941553],
28: [-0.5105685407819842, -0.4236604017060557],
29: [0.6792549581008038, -0.026215820387260003]}
```

Write a function that takes as an input a parameter $r \geq 0$, and returns the simplicial complex $\mathcal{G}(r)$ defined as follows:

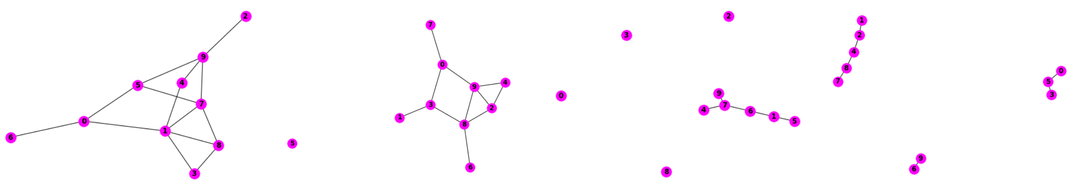
- the vertices of $\mathcal{G}(r)$ are the points $x_0, ..., x_{29}$,
- for all $i, j \in [\![0, 29]\!]$ with $i \neq j$, the edge $[i, j]$ belongs to $\mathcal{G}(r)$ if and only if $\|x_i - x_j\| \leq r$.

Compute the number of connected components of $\mathcal{G}(r)$ for several values of r. What do you observe?

Exercise 26. A *Erdős–Rényi random graph* $\mathcal{G}(n, p)$ is a simplicial complex obtained as follows:

- add n vertices $1, ..., n$,
- for every $a, b \in [\![1, n]\!]$, add the edge $[a, b]$ to the complex with probability p.

Builds a function that, given n and p, outputs a simplicial complex $\mathcal{G}(n, p)$. Observe the influence of p on the number of connected components of $\mathcal{G}(10, p)$ and $\mathcal{G}(100, p)$.



Hint: If V is a list, `itertools.combinations(V,2)` can be used to generate all the non-ordered pairs `[a,b]` in `V` (from package `itertools`).
The command `random.random()` can be used to generate a random number between 0 and 1, and `random.random()<p` is `True` with probability `p` (from package `random`).

5 Homological algebra

This subsection is devoted to defining a powerful invariant in algebraic topology, called *homology*. We will restrict to the case of *simplicial homology* over the *finite fied* $\mathbb{Z}/2\mathbb{Z}$.

5.1 Reminder on $\mathbb{Z}/2\mathbb{Z}$-vector spaces

We review some basic notions of algebra: groups and vector spaces.

Groups. We recall that a *group* $(G, +)$ is a set G endowed with an operation

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g + h \end{aligned}$$

such that:

- (associativity) $\forall a, b, c \in G,\ (a + b) + c = a + (b + c)$,
- (identity) $\exists\ 0 \in G, \forall a \in G, a + 0 = 0 + a = a$,
- (inverse) $\forall a \in G, \exists b \in G, a + b = b + a = 0$.

Moreover, we say that G is *commutative* if $\forall a, b \in G, a + b = b + a$. In this course, the only groups we consider will be commutative and finite.

A *subgroup* of $(G, +)$ is a subset $H \subset G$ such that

$$\forall a, b \in H, a + b \in H.$$

If H is a subgroup of G, the operation $+ : G \times G \to G$ restricts to an operation $+ : H \times H \to H$, making H a group on its own.

Suppose that G is commutative, and that H is a subgroup of H. We define the following equivalence relation on G: for all $a, b \in G$,

$$a \sim b \iff a - b \in H.$$

Denote by G/H the quotient set of G under this relation. For any $a \in G$, one shows that the equivalence class of a is equal to

$$a + H = \{a + h, h \in H\}.$$

Let $a_0 = 0, a_1, ..., a_n$ be a choice of representants of equivalence classes of the relation $\sim$. The quotient set can be written as

$$G/H = \big\{0 + H, a_1 + H, ..., a_n + H\big\}.$$

One defines a group structure $\oplus$ on G/H as follows: for any $i, j \in [\![0, n]\!]$,

$$(a_i + H) \oplus (a_j + H) = (a_i + a_j) + H.$$

The group $(G/H, \oplus)$ is called the *quotient group*.

Consider two groups $(G, +)$ and $(H, +)$ (for simplicity, we denote the operations with the same symbol $+$). An *morphism* between them is an application $f\colon G \to H$ such that

$$\forall a, b \in G, f(a + b) = f(a) + f(b).$$

If f is a bijection, it is called an *isomorphism*.

If $f\colon G \to H$ is a morphism, the *image* of f is defined as

$$\mathrm{Im}(f) = \{f(a), a \in G\}.$$

One shows that it is a subgroup of H. The *kernel* of f is defined as

$$\mathrm{Ker}(f) = \{a \in G, f(a) = 0\}.$$

One shows that it is a subgroup of G. The first isomorphism theorem states that the quotient group $G/\mathrm{Ker}(f)$ is isomorphic to the subgroup $\mathrm{Im}(f)$. More explicitely, an isomorphism $G/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is given by

$$a + \mathrm{Ker}(f) \longmapsto f(a).$$

The group $\mathbb{Z}/2\mathbb{Z}$. Consider the group $(\mathbb{Z}, +)$. It admits a subgroup $2\mathbb{Z} = \{2n, n \in \mathbb{Z}\}$. The equivalence relation $\sim$ admits two equivalence classes:

$$2\mathbb{Z} = \{2n, n \in \mathbb{Z}\} \qquad \text{and} \qquad 1 + 2\mathbb{Z} = \{1 + 2n, n \in \mathbb{Z}\}.$$

The quotient group can be seen as the group $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with the operations

$$\begin{aligned} 0 + 0 &= 0 \\ 0 + 1 &= 1 \\ 1 + 0 &= 1 \\ 1 + 1 &= 0 \end{aligned}$$

The group $(\mathbb{Z}/2\mathbb{Z}, +)$ is the only group with two elements. Note that it can also be given a *field* structure, via the operation

$$\begin{aligned} 0 \times 0 &= 0 \\ 0 \times 1 &= 0 \\ 1 \times 0 &= 0 \\ 1 \times 1 &= 1 \end{aligned}$$

For any $n \geq 1$, the *product group* $((\mathbb{Z}/2\mathbb{Z})^n, +)$ is the group whose underlying set is

$$(\mathbb{Z}/2\mathbb{Z})^n = \{(\epsilon_1, ..., \epsilon_n), \epsilon_1, ..., \epsilon_n \in \mathbb{Z}/2\mathbb{Z}\}$$

and whose operation is defined as

$$(\epsilon_1, ..., \epsilon_n) + (\epsilon'_1, ..., \epsilon'_n) = (\epsilon_1 + \epsilon'_1, ..., \epsilon_n + \epsilon'_n).$$

Note that the set $(\mathbb{Z}/2\mathbb{Z})^n$ has 2^n elements.

Vector spaces. Let $(\mathbb{F}, +, \times)$ be a field. We recall that a vector space over $\mathbb{F}$ is a group $(V, +)$ endowed with an operation

$$\begin{aligned} \mathbb{F} \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda \cdot v \end{aligned}$$

such that

- (compatibility of multiplication) $\forall \lambda, \mu \in \mathbb{F}, \forall v \in V, \lambda \cdot (\mu \cdot v) = (\lambda \times \mu) \cdot v$,
- (identity) $\forall v \in V, 1 \cdot v = v$ where 1 denotes the unit of $\mathbb{F}$,
- (scalar distributivity) $\forall \mu, \nu \in \mathbb{F}, \forall v \in V,\ (\lambda + \nu) \cdot v = \lambda \cdot v + \nu \cdot v$,
- (vector distributivity) $\forall \mu \in \mathbb{F}, \forall v, w \in V,\ \lambda \cdot (u + v) = \lambda \cdot v + \nu \cdot v$.

When there is no risk of confusion, we will write λv instead of $\lambda \cdot v$.

Let $\{v_1, ..., v_n\}$ be a collection of elements of V. We say that it is *free* if

$$\forall \lambda_1, ..., \lambda_n \in \mathbb{F}, \sum_{1 \leq i \leq n} \lambda_i v_i = 0 \implies \lambda_1 = ... = \lambda_n = 0.$$

We say that it is *spans* V if

$$\forall v \in V, \exists \lambda_1, ..., \lambda_n \in \mathbb{F}, \sum_{1 \leq i \leq n} \lambda_i v_i = v.$$

If the collection $\{v_1, ..., v_n\}$ is free and spans V, we say that it is a *basis*. One shows that $\{v_1, ..., v_n\}$ is a basis if and only if

$$\forall v \in V, \exists ! \lambda_1, ..., \lambda_n \in \mathbb{F}, \sum_{1 \leq i \leq n} \lambda_i v_i = v.$$

A *linear subspace* of $(V, +, \cdot)$ is a subset $W \subset V$ such that

$$\forall u, v \in W, u + v \in W \qquad \text{and} \qquad \forall v \in W, \forall \lambda \in \mathbb{F}, \lambda v \in W.$$

Just as for groups, we can define an equivalence relation $\sim$ on V, and a *quotient vector space* V/W. The quotient has dimension $\dim V/W = \dim V - \dim W$.

Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be two vector spaces. A linear map is a map $f \colon V \to W$ such that

$$\forall u, v \in V, f(u + v) = f(u) + f(v) \qquad \text{and} \qquad \forall v \in V, \forall \lambda \in \mathbb{F}, f(\lambda v) = \lambda \cdot f(v).$$

If f is a bijection, it is called an *isomorphism*, and we say that V and W are *isomorphic*. If $(V, +, \cdot)$ is a vector space of dimension n, one shows that it is isomorphic to the product vector space $\mathbb{F}^n$.

Structure of $\mathbb{Z}/2\mathbb{Z}$-vector spaces. Not all groups $(V, +)$ can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure. The following statement gives precisely when they can:

Proposition 5.1. *Le $(V, +)$ be a commutative group. It can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure if and only if $\forall v \in V, v + v = 0$.*

Proof. Suppose that $(V, +, \cdot)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space. For all $v \in V$, we have

$$0 = 0 \cdot v = (1 + 1) \cdot v = v + v,$$

which shows an application. In the other direction, if $\forall v \in V, v + v = 0$, then we can define a vector space structure on $(V, +)$ as follows: for all $v \in V$,

$$\begin{aligned} 0 \cdot v &= 0 \\ 1 \cdot v &= v \end{aligned}$$

One verifies the axioms of a vector space. $\square$

Applying the usual theory of vector spaces, we obtain the following proposition:

Proposition 5.2. *Let $(V, +, \cdot)$ be a finite $\mathbb{Z}/2\mathbb{Z}$-vector space. Then there exists $n \geq 0$ such that V has cardinal 2^n, and $(V, +, \cdot)$ is isomorphic to the vector space $(\mathbb{Z}/2\mathbb{Z})^n$.*

Exercise 27. Let V be a $\mathbb{Z}/2\mathbb{Z}$-vector space, and W a linear subspace. Using Proposition 5.2, prove that

$$\dim V/W = \dim V - \dim W.$$

Exercise 28. Let $(G, +)$ be a group, potentially non-commutative. Prove that

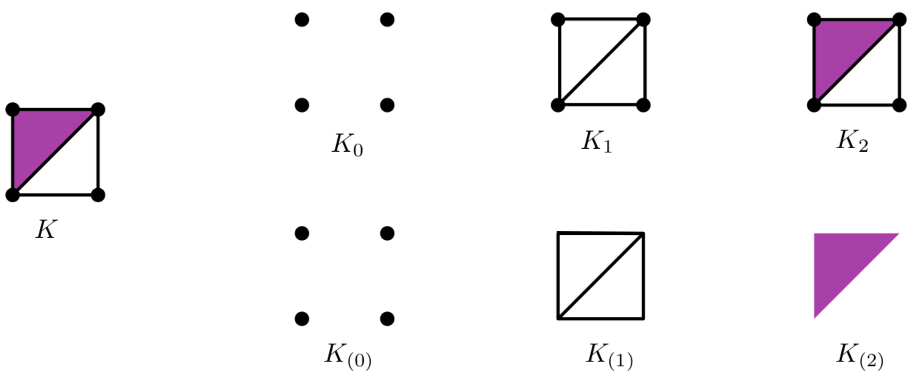
$$\forall g \in G, g + g = 0 \implies G \text{ is commutative.}$$

5.2 Chains, cycles and boundaries

Let K be a simplicial complex. For any $n \geq 0$, define the sets

$$\begin{aligned} K_n &= \{\sigma \in K, \dim(\sigma) \leq n\} \\ K_{(n)} &= \{\sigma \in K, \dim(\sigma) = n\}. \end{aligned}$$

The first set is a simplicial complex, called the *n-skeleton* of K. The second one is not a simplicial complex in general, and has no name.



Chains. Let $n \geq 0$. The *n-chains* of K is the set $C_n(K)$ whose elements are the formal sums

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \quad \text{where} \quad \forall \sigma \in K_{(n)},\ \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}.$$

We can give $C_n(K)$ a group structure via

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma + \sum_{\sigma \in K_{(n)}} \eta_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} (\epsilon_\sigma + \eta_\sigma) \cdot \sigma.$$

Moreover, $C_n(K)$ can be given a $\mathbb{Z}/2\mathbb{Z}$-vector space structure. To see this, observe that for any element of $C_n(K)$,

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma + \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} (\epsilon_\sigma + \epsilon_\sigma) \cdot \sigma = \sum_{\sigma \in K_{(n)}} 0 \cdot \sigma = 0$$

the second equality follows from $0+0 = 1+1 = 0$ in $\mathbb{Z}/2\mathbb{Z}$. We conclude with Proposition 5.1.

Example 5.3. Consider the simplicial complex

$$K = \{[0], [1], [2], [0,1], [0,2]\}.$$

The 0-chains $C_0(K)$ consists in 8 elements:

$$C_0(K) = \{0,\ [0],\ [1],\ [2],\ [0]+[1],\ [0]+[2],\ [1]+[2],\ [0]+[1]+[2]\}$$

0 $\quad$ $[0]$ $\quad$ $[1]$ $\quad$ $[2]$ $\quad$ $[0]+[1]$ $\quad$ $[0]+[2]$ $\quad$ $[1]+[2]$ $\quad$ $[0]+[1]+[2]$

As an example, in $C_0(K)$, we have

$$([0]+[1]) + ([0]+[2]) = [0]+[0]+[1]+[2] = [1]+[2].$$

Besides, the 1-chains $C_1(K)$ consists in 4 elements:

$$C_1(K) = \{0,\ [0,1],\ [0,2],\ [0,1]+[0,2]\}.$$

0 $\quad$ $[0,1]$ $\quad$ $[0,2]$ $\quad$ $[0,1]+[0,2]$

Remark 5.4. The group $C_n(K)$ can be seen a the group of maps $K_{(n)} \to \mathbb{Z}/2\mathbb{Z}$, endowed with the addition operation. For instance, the chain $[0]+[1]$ would correspond to the map $f\colon K_{(0)} \to \mathbb{Z}/2\mathbb{Z}$ defined as

$$f([0]) = 1, \qquad f([1]) = 1, \qquad f([2]) = 0,$$

and the chain $[0]+[2]$ the map $g\colon K_{(0)} \to \mathbb{Z}/2\mathbb{Z}$ defined as

$$f([0]) = 1, \qquad f([1]) = 0, \qquad f([2]) = 1.$$

Their sum is the map $f+g$ defined as

$$(f+g)([0]) = 1+1 = 0, \qquad (f+g)([1]) = 1+0 = 1, \qquad (f+g)([2]) = 0+1 = 1.$$

Remark 5.5. The group $C_n(K)$ can also be seen as the set $\mathcal{P}(K_{(n)})$ of subsets of K_n, endowed with the symmetric difference operation, defined as $A\Delta B = (A\cup B)\setminus(A\cap B)$. For instance, the chain $[0]+[1]$ would correspond to the subset $\{[0],[1]\}$, and the chain $[0]+[2]$ to $\{[0],[2]\}$. Their sum is the subset

$$\{[0],[1]\}\ \Delta\ \{[0],[2]\} = \{[1],[2]\}.$$

Boundary operator. Let $n \geq 1$, and $\sigma = [x_0, ..., x_n] \in K_{(n)}$ a simplex of dimension n. We define its *boundary* as the following element of $C_{n-1}(K)$:

$$\partial_n \sigma = \sum_{\substack{\tau\subset\sigma \\ |\tau|=|\sigma|-1}} \tau$$

where $|\tau|$ denotes the cardinal. We can extend the operator ∂_n as a linear map $\partial_n\colon C_n(K) \to C_{n-1}(K)$ as follows: for any element of $C_n(K)$,

$$\partial_n \sum_{\sigma\in K_{(n)}} \epsilon_\sigma \cdot \sigma = \sum_{\sigma\in K_{(n)}} \epsilon_\sigma \cdot \partial_n \sigma.$$

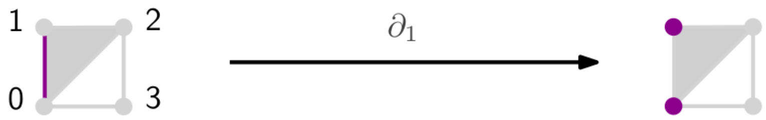
Besides, for $n=0$, we define the boundary operator ∂_0 as the zero map $C_0(K) \to \{0\}$, i.e., for all $c \in C_0(K), \partial_0(c) = 0$. In what follows, we denote $C_{-1}(K) = \{0\}$.

Example 5.6. Consider the simplicial complex

$$K = \{[0],[1],[2],[3],[0,1],[0,2],[1,2],[1,3],[2,3],[0,1,2]\}.$$

The simplex $[0,1]$ has the faces $[0]$ and $[1]$. Hence

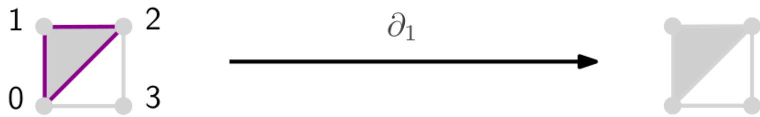
$$\partial_1[0,1] = [0]+[1].$$



Similarly, the boundary of the 1-chain $[0,1]+[1,2]+[2,0]$ is

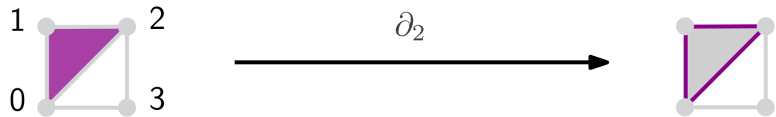
$$\begin{aligned}\partial_1\big([0,1]+[1,2]+[2,0]\big) &= \partial_1[0,1]+\partial_1[0,2]+\partial_1[2,0] \\ &= [0]+[1]+[0]+[2]+[2]+[0] \\ &= 0\end{aligned}$$

since $[0]+[0]=[1]+[1]=[2]+[2]=0$ in $C_0(K)$.



The simplex $[0,1,2]$ has the faces $[0,1]$ and $[1,2]$ and $[2,0]$. Hence

$$\partial_2[0,1,2] = [0,1] + [1,2] + [2,0].$$



Boundary and cycles. Let $n \geq 0$. We have a triplet of vector spaces

$$C_{n+1}(K) \xrightarrow{\partial n+1} C_n(K) \xrightarrow{\partial n} C_{n-1}(K).$$

The maps ∂_{n+1} and ∂_n are linear maps, and we can consider their kernel and image (see reminder in Subsection 5.1). We define:

- The *n-cycles*: $Z_n(K) = \mathrm{Ker}(\partial_n)$,
- The *n-boundaries*: $B_n(K) = \mathrm{Im}(\partial_{n+1})$.

We say that two chains $c, c' \in C_n(K)$ are *homologous* if there exists $b \in B_n(K)$ such that $c = c' + b$. In other words, two chains are homologous if they are equal up to a boundary.

Example 5.7. Consider the simplicial complex of Example 5.6. The set of cycles $Z_1(K)$ consists in the chains

$$0, \quad [0,1]+[1,2]+[0,2], \quad [0,2]+[2,3]+[0,3] \quad \text{and} \quad [0,1]+[1,2]+[2,3]+[0,3].$$

The only boundaries $B_1(K)$ is given by

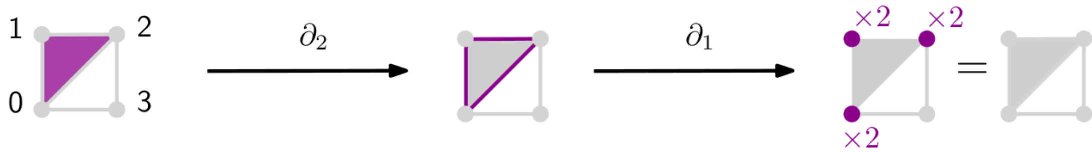
$$\partial_2(0) = 0 \quad \text{and} \quad \partial_2([0,1,2]) = [0,1]+[0,2]+[1,2].$$

We see that the chains $[0,2]+[2,3]+[0,3]$ and $[0,1]+[1,2]+[2,3]+[0,3]$ are homologous. Indeed,

$$[0,2]+[2,3]+[0,3] = [0,1]+[1,2]+[2,3]+[0,3]+[0,1]+[0,2]+[1,2].$$

Here is a key property of the boundary operator:

Lemma 5.8. *For any $n \geq 0$, for any $c \in C_n(K)$, we have $\partial_{n-1} \circ \partial_n(c) = 0$. In other words, the map $\partial_{n-1} \circ \partial_n \colon C_n(K) \to C_{n-2}(K)$ is zero.*



Proof. Suppose that $n \geq 2$, the result being trivial otherwise. Since the boundary operators are linear, it is enough to prove that $\partial_{n-1} \circ \partial_n(\sigma) = 0$ for all simplex $\sigma \in K_{(n)}$. By definition,

$$\partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \tau,$$

and

$$\partial_{n-1} \circ \partial_n(\sigma) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \partial_{n-1}(\tau) = \sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \sum_{\substack{\nu \subset \tau \\ |\nu| = |\tau| - 1}} \nu$$

We can write this last sum as

$$\sum_{\substack{\tau \subset \sigma \\ |\tau| = |\sigma| - 1}} \sum_{\substack{\nu \subset \tau \\ |\nu| = |\tau| - 1}} \nu = \sum_{\substack{\nu \subset \sigma \\ |\nu| = |\sigma| - 2}} \alpha_\nu \nu$$

where $\alpha_\nu = \{\tau \subset \sigma, |\tau| = |\sigma| - 1, \nu \subset \tau\}$. It is easy to see that for every ν such that $|\nu| = |\tau| - 2$, we have $\alpha_\nu = 2 = 0$. $\square$

Corollary 5.9. *We have $B_n(K) \subset Z_n(K)$. In other words, any boundary is a cycle.*

Proof. Let $b \in B_n(K)$ be a boundary. By definition, there exists $c \in C_{n+1}(K)$ such that $b = \partial_{n+1}(c)$. Using Lemma 5.8, we obtain

$$\partial_n(b) = \partial_n \partial_{n+1}(c) = 0,$$

hence $b \in Z_n(K)$. $\square$

5.3 Homology groups

In the previous subsection, we have defined a sequence of vector spaces, connected by linear maps

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots$$

and for every $n \geq 0$, we have defined the cycles and the boundaries $Z_n(K)$ and $B_n(K)$. According to Corollary 5.9, $B_n(K)$ is a linear subspace of $Z_n(K)$. We can consider the corresponding quotient vector space:

Definition 5.10. The n^{th} *homology group* of K is $H_n(K) = Z_n(K)/B_n(K)$.

Since $H_n(K)$ is a quotient of $\mathbb{Z}/2\mathbb{Z}$-vector spaces, it is a $\mathbb{Z}/2\mathbb{Z}$-vector space. According to Proposition 5.2, it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$, where $d = \dim H_n(K)$. We also have $|H_n(K)| = 2^d$. By applying Exercise 27, we obtain the relation

$$\dim H_n(K) = \dim B_n(K) - \dim Z_n(K).$$

Example 5.11. We consider the simplicial complex of Example 5.7. As we have seen, $Z_1(K)$ has cardinal 4, and $B_1(K)$ cardinal 2. We deduce that $\dim Z_1(K) = 2$, $\dim B_1(K) = 1$, and

$$\dim H_1(K) = 2 - 1 = 1.$$

In other words, we have an isomorphism $H_1(K) \simeq \mathbb{Z}/2\mathbb{Z}$.

Definition 5.12. Let K be a simplicial complex and $n \geq 0$. Its n^{th} *Betti number* is the integer $\beta_n(K) = \dim H_n(K)$.

Exercise 29. Compute the Betti numbers $\beta_0(K)$, $\beta_1(K)$ and $\beta_2(K)$ of the following simplicial complex:

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0]\}.$$

Exercise 30. Compute the Betti numbers $\beta_0(K)$, $\beta_1(K)$ and $\beta_2(K)$ of the following simplicial complex:

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}.$$

Moralidade
Os grupos de homologia são os espaços vetoriais dos circulos módulo os limites.

5.4 Homology groups of topological spaces

Just as we did for the Euler characteristic, we will define the homology groups of topological spaces via triangulations of it.

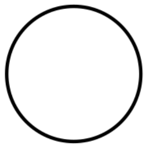
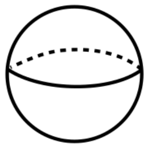
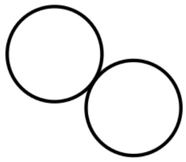
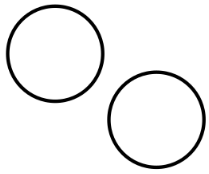
Definition 5.13. The homology groups of a topological space are the homology groups of any triangulation of it. We define their Betti numbers similarly.

For this definition to make sense, we have to make sure that the homology groups are an invariant of *homeomorphism equivalence*. We can prove an even stronger result: homology groups are an invariant of *homotopy equivalence*. We will admit this statement.

Proposition 5.14. *If X and Y are two homotopy equivalent topological spaces, then for any $n \geq 0$ we have isomorphic homology groups $H_n(X) \simeq H_n(Y)$. As a consequence, $\beta_n(X) = \beta_n(Y)$.*

Remark 5.15. Again, the previous definition suffers from the fact that all topological spaces are not triangulable. However, there exists a definition of homology that is better suited for topological spaces in many ways. It is called *singular homology*, but it is beyond the scope of this summer course.

To close this section, we give some examples of homology groups:

X				
$H_0(X)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\beta_0(X)$	1	1	1	2
$H_1(X)$	$\mathbb{Z}/2\mathbb{Z}$	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\beta_1(X)$	1	0	2	2
$H_2(X)$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0
$\beta_2(X)$	0	1	0	0

6 Incremental algorithm

In this section, we present the first algorithm of persistent homology, as in [ELZ00].

6.1 Incremental algorithm à la main

We start by presenting a version of the incremental algorithm that can be applied by hand. In Subsection 6.3 we will present a matrix version of the algorithm.

Let K be a simplicial complex with n simplices. Choose a total order of the simplices

$$\sigma^1 < \sigma^2 < ... < \sigma^n$$

such that

$$\forall \sigma, \tau \in K,\ \tau \subsetneq \sigma \implies \tau < \sigma.$$

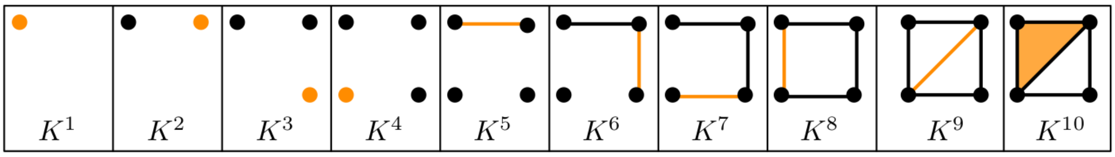
In other words, a face of a simplex is lower than the simplex itself. For every $i \leq n$, consider the simplicial complex

$$K^i = \{\sigma^1, ..., \sigma^i\}.$$

We have the relation $K^{i+1} = K^i \cup \left\{\sigma^{i+1}\right\}$. They form an inscreasing sequence of simplicial complexes

$$K^1 \subset K^2 \subset ... \subset K^n,$$

with $K^n = K$.



We will compute the homology groups of K^i incrementally. To do so, we need the following notion:

Definition 6.1. Let $i \in [\![1, n]\!]$, and $d = \dim(\sigma_i)$. The simplex σ^i is *positive* if there exists a cycle $c \in Z_d(K^i)$ that contains σ_i. Otherwise, σ^i is *negative*.

For instance:

- $\sigma^1 \in K^1$ is **positive** because it is included in the cycle $c = \sigma^1$ (indeed, $\partial_0(\sigma^1) = 0$).
- $\sigma^2 \in K^2$ is **positive** because it is included in the cycle $c = \sigma^2$ (indeed, $\partial_0(\sigma^2) = 0$).
- $\sigma^5 \in K^5$ is **negative** because it is not included in a cycle $Z_1(K^5)$. Indeed, $C_1(K^5)$ only contains 0 and σ^5, and $\partial_1(\sigma^5) = \sigma^1 + \sigma^2 \neq 0$.
- $\sigma^8 \in K^8$ is **positive** because it is included in the cycle $c = \sigma^5 + \sigma^6 + \sigma^7 + \sigma^8$ (indeed, $\partial_1(c) = 2\sigma^1 + 2\sigma^2 + 2\sigma^3 + 2\sigma^4 = 0$).

Note that, by adding σ^i in the simplicial complex, the only groups that may change are $Z_d(K^i)$ and $B_{d-1}(K^i)$. The following lemmas state precisely what happens.

Lemma 6.2. *If σ^i is positive, then $\beta_d(K^i) = \beta_d(K^{i-1}) + 1$, and for all $d' \neq d$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.*

Proof. We start by proving the following fact: if $c \in Z_d(K^i)$ is a cycle that contains σ_i, then c is not homologous (in K^i) to a cycle of $c' \in Z_d(K^{i-1})$. By contradiction: if $c = c' + b$ with $c' \in Z_d(K^{i-1})$ and $b \in B_d(K^i)$, then $c - c' = b \in B_d(K^i)$. This is absurd because we just added σ_i: it cannot appear in a boundary of K^i. As a consequence,

$$\dim Z_d(K^i) = \dim Z_d(K^{i-1}) + 1.$$

Besides, if c is a cycle of K^i that contains σ^i, then $\partial_i(\sigma^i) = \partial_i(c) + \partial_i(\sigma^i) = \partial_i(c + \sigma^i)$, and $c + \sigma^i$ is a chain of K^{i-1}. Hence

$$\dim B_{d-1}(K^i) = \dim B_{d-1}(K^{i-1}).$$

We conclude by using the relation $\beta_d(K^i) = \dim Z_d(K^i) - \dim B_d(K^i)$. $\square$

Lemma 6.3. *If σ^i is negative, then $\beta_{d-1}(K^i) = \beta_{d-1}(K^{i-1}) - 1$, and for all $d' \neq d-1$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.*

Proof. We start by proving the following fact: $\partial_d(\sigma^i)$ is not a boundary of K^{i-1}. Otherwise, we would have $\partial_d(\sigma^i) = \partial_d(c)$ with $c \in C_d(K^{i-1})$, i.e. $\partial_d(\sigma^i + c) = 0$. Hence $\sigma^i + c$ would be a cycle of K^i that contains c, contradicting the negativity of σ^i. As a consequence,

$$\dim B_{d-1}(K^i) = \dim B_{d-1}(K^{i-1}) + 1.$$

Moreover, since σ^i is negative, we have

$$\dim Z_d(K^i) = \dim Z_d(K^{i-1}).$$

We conclude by using the relation $\beta_d(K^i) = \dim Z_d(K^i) - \dim B_d(K^i)$. $\square$

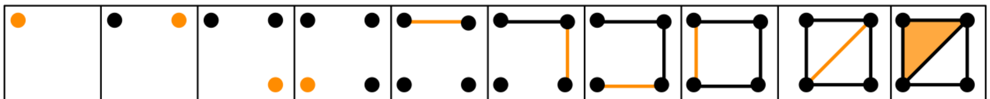
We derive the following algorithm:

Algorithm 1: Incremental algorithm for homology

Input: an increasing sequence of simplicial complexes $K^1 \subset \cdots \subset K^n = K$
Output: the Betti numbers $\beta_0(K), ...\beta_d(K)$
$\beta_0 \leftarrow 0, ..., \beta_d \leftarrow 0$;
for $i \leftarrow 1$ **to** n **do**
    $d = \dim(\sigma^i)$;
    if σ^i *is positive* **then**
        $\beta_k(K^i) \leftarrow \beta_k(K^i) + 1$;
    else if $d > 0$ **then**
        $\beta_{k-1}(K^i) \leftarrow \beta_{k-1}(K^{i-1}) - 1$;

Of course, there remains the problem of determining automatically whether the simplex is positive. We will propose a solution in Subsection 6.3.

We now apply the algorithm to our simplicial complex. The output is $\beta_0(K) = 1$ and $\beta_1(K) = 1$.



	K^1	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}
Dimension	0	0	0	0	1	1	1	1	1	2
Positivity	+	+	+	+	−	−	−	+	+	−
$\beta_0(K^i)$	1	2	3	4	3	2	1	1	1	1
$\beta_1(K^i)$	0	0	0	0	0	0	0	1	2	1

Exercise 31. Compute again the Betti numbers of the simplicial complexes of Exercises 29 and 30, using the incremental algorithm.

6.2 Applications

Number of connected components. We link the notion of connectedness with the homology groups.

Proposition 6.4. *Let X be a (triangulable) topological space. Then its 0^{th} Betti number, $\beta_0(X)$, is equal to the number of connected components of X.*

Proof. First, a definition: say that a simplicial complex L is *combinatorially connected* of for every vertex v, w of L, there exists a sequence of edges that connects v and w:

$$[v, v_1], \quad [v_1, v_2], \quad [v_2, v_3], \quad ..., \quad [v_n, w].$$

Let m be the number of connected components X, and let K be triangulation of X. We accept the following equivalent statement: there exists m **disjoint**, **non-empty** and **combinatorially connected** simplicial sub-complex $L_1, ..., L_m$ of K such that

$$K = \bigcup_{1 \leq i \leq m} L_i.$$

Now, let T be a spanning forest of K, that is, a union of spanning trees. One shows that admits m combinatorially connected components.

Consider an ordering of the simplices of K that begins with an ordering of T. We apply the incremental algorithm. First, each vertex increases β_0 by 1. Next, since T is a tree, all its edges are negative simplices (T has no cycles), and hence decrease β_0. We know that each tree of the forest contains $k-1$ edges, where k is the number of vertices of the corresponding component. At that point of the algorithm, when all T is added, β_0 is equal to m.

Now, since T is a spanning tree, each other edges of K is positive, hence β_0 does not change. Similarly, the other simplices of K do not change β_0. We deduce the result. $\square$

Homology of spheres. Let us compute the homology of spheres. For any $n \geq 1$, consider the vertex set $V = \{0, \ldots, n\}$, and the simplicial complex

$$\Delta_n = \{S \subset V, S \neq \emptyset\}.$$

We call it the *simplicial standard n-simplex.* Define its boundary as

$$\partial\Delta_n = \Delta_n \setminus V.$$

One shows that $\partial\Delta_n$ is a triangulation of the $(n-1)$-sphere $\mathbb{S}_{n-1} \subset \mathbb{R}^n$.

Δ_2 $\partial\Delta_2$ Δ_3 $\partial\Delta_3$

Exercise 32. Prove that $\partial\Delta_n$ is a triangulation of the $(n-1)$-sphere.

As a consequence, for all $i \geq 0$, we have $H_i(\mathbb{S}_n) = H_i(\partial\Delta_{n+1})$. We will use this simplicial complex to compute these homology groups.

Proposition 6.5. *The Betti numbers of $\mathbb{S}_n$ are:*

- $\beta_i(\mathbb{S}_n) = 1$ *for* $i = 0, n$,
- $\beta_i(\mathbb{S}_n) = 0$ *else.*

Proof. Consider the *simplicial standard n-simplex* Δ_n. It is homotopy equivalent to a point (its topological realization, as in Definition 4.8, deform retracts on any point of it). Hence Δ_n has the same Betti numbers as the point:

- $\beta_1(\mathbb{S}_n) = 1$,
- $\beta_i(\mathbb{S}_n) = 0$ for $i > 0$.

Now, if we run the incremental algorithm for homology on Δ_n, but stopping before adding the n-simplex V, we would obtain the Betti numbers of $\partial\Delta_n$. Also, note that the n-simplex is negative. Hence

- $\beta_n(\partial\Delta_n) = \beta_n(\Delta_n) + 1$,
- $\beta_i(\partial\Delta_n) = \beta_i(\Delta_n)$ for $i \neq n$.

We deduce the result. □

From the homology of the spheres, one deduces the theorem of Invariance of Domain.

Theorem 6.6. *For every integers m, n such that $m \neq n$, the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.*

Proof. Let m, n such that $m \neq n$. By contradiction, suppose that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic via f. Let 0 denote the origin of $\mathbb{R}^n$. By restriction, we get a homeomorphism

$$\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}.$$

We deduce the following weaker statement: $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{f(0)\}$ are homotopic equivalent. Now, using Example 3.15, we deduce that the sphere $\mathbb{S}_{n-1}$ and $\mathbb{S}_{m-1}$ are homotopic equivalent. Hence, according to Proposition 5.14, they must admit the same homology groups. This contradict Proposition 6.5. □

Euler characteristic. Finally, we prove that the Euler characteristic is an information already included in the homology groups.

Proposition 6.7. *Let X be a (triangulable) topological space. Then its Euler characteristic is equal to*

$$\chi(X) = \sum_{0 \leq i \leq n} (-1)^i \cdot \beta_i(X)$$

where n is the maximal integer such that $\beta_i(X) \neq 0$.

Proof. Let K be a triangulation of X. By definition, we have $\chi(X) = \chi(K)$ and

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Now, pick an ordering $K^1 \subset \cdots \subset K^n = K$ of K, with $K^i = K^{i-1} \cup \{\sigma^i\}$ for all $2 \leq i \leq n$. We will apply the incremental algorithm. By induction, let us show that, for all $1 \leq m \leq n$,

$$\sum_{0 \leq i \leq m} (-1)^i \cdot \beta_i(K^m) = \sum_{0 \leq i \leq m} (-1)^i \cdot (\text{number of simplices of dimension } i \text{ of } K^m). \quad (1)$$

For $m = 1$, σ^m is a 0-simplex, and the equality reads $1 = 1$. Now, suppose that the equality is true for $1 \leq m < n$, and consider the simplex σ^{m+1}. Let $d = \dim \sigma^{m+1}$. The right-hand side of Equation (1) is increased by $(-1)^d$.

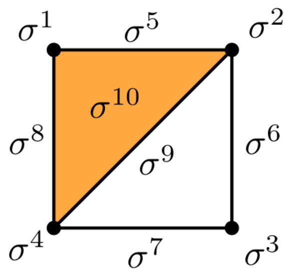
If σ^{m+1} is positive, then $\beta_d(K^{m+1}) = \beta_d(K^m) + 1$, hence the left-hand side of Equation (1) is increased by $(-1)^d$. Otherwise, it is negative, and $\beta_{d-1}(K^{m+1}) = \beta_{d-1}(K^m) - 1$, hence the left-hand side of Equation (1) is increased by $-(-1)^{d-1} = (-1)^d$. We deduce the result by induction. $\square$

6.3 Matrix algorithm

The only thing missing to apply Algorithm 1 is to determine whether a simplex is positive or negative. It turns out that this problem can be conveniently solved by using a matrix representation of the simplicial complex.

Let K be a simplicial complex, and $\sigma^1 < \sigma^2 < \cdots < \sigma^n$ and ordering of its simplices, as in Subsection 6.1. Define the *boundary matrix* of K, denoted Δ, as follows: Δ is a $n \times n$ matrix, whose (i, j)-entry (i^{th} row, j^{th} column is)

$$\begin{aligned} \Delta_{i,j} \;&=\; 1 \;\text{ if } \sigma^i \text{ is a face of } \sigma^j \text{ and } |\sigma^i| = |\sigma^j| - 1 \\ &\;\; 0 \text{ else.} \end{aligned}$$



$$\begin{array}{c c} & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\ \begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

By adding columns one to the others, we create chains. If we were able to reduce a column to zero, then we found a cycle.

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

$$\partial_1(\sigma^6) = \sigma^2 + \sigma^3$$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

$$\partial_1(\sigma^5 + \sigma^6 + \sigma^7 + \sigma^8) = 0$$

The process of reducing columns to zero is called *Gauss reduction*. For any $j \in [\![1, n]\!]$, define

$$\delta(j) = \max\{i \in [\![1, n]\!], \Delta_{i,j} \neq 0\}.$$

If $\Delta_{i,j} = 0$ for all j, then $\delta(j)$ is *undefined*. We say that the boundary matrix Δ is *reduced* if the map δ is injective on its domain of definition. The following algorithm allows to compute a reduced matrix.

Algorithm 2: Reduction of the boundary matrix

Input: a boundary matrix Δ
Output: a reduced matrix $\widetilde{\Delta}$
for $i \leftarrow 1$ **to** n **do**
  while *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column i to column j;

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \\ + & + & + & + & - & - & - & + & + & - \end{array}
\end{array}
$$

Exercise 33. Show that the algorithm stops after a finite number of steps.

Lemma 6.8. *Suppose that the boundary matrix is reduced. Let $j \in [\![1, n]\!]$. If $\delta(j)$ is defined, then the simplex σ^j is negative. Otherwise, it is positive.*

Proof. Indeed, at the end of the algorithm, $\delta(j)$ is undefined if and only if σ^i is included in a cycle of K^i, that is, if σ^i if positive. □

As a consequence, we can read on the reduced boundary matrix the positivity of the simplices. Combined with Algorithm 1, we are able to compute the Betti numbers of any simplicial complex.

Remark 6.9. Algorithm 2, also called the *standard algorithm for reduction of the boundary matrix*, is the one developped first, in the paper [ELZ00]. Then, many other algorithms have been proposed to reduce the boundary matrix. See the review [OPT+17].

Exercise 34. Apply Algorithm 2 to solve Exercise 31.

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