EMAp SUMMER COURSE

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# GENERAL AND COMBINATORIAL TOPOLOGY

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**Abstract.** This course is intended for a 3rd year graduate student with no background on topology. The present document is a collection of notes for each lesson.

**Course webpage.** Various information (schedule, homework) are gathered on `https://raphaeltinarrage.github.io/EMApTopology.html`.

**Homeworks.** Exercises with a vertical segment next to them are your homework. Here is the first one:

**Exercise 0.** Send me an email answering the following questions:

- Do you understand English well?
- Have you ever studied topology?
- Have you ever coded? In which language?
- Any remarks?

Last update: 9th January, 2023

# Contents

# Chapter I

# General topology

## 1 Topological spaces

In this section, we will introduce the basic vocabulary of topology: topological spaces, open and closed sets. We will study examples of topologies on finite sets, as well as on $\mathbb{R}^n$. In order to build topologies, we will define the notion of generated topologies. This will allow us to build the Euclidean topology, and the product topology. We will also define the subspace and quotient topologies.

In order to prepare this section, I drew inspiration from [Pau08]. We won't introduce some useful notions, such as neighborhoods, initial and final topologies, as well as basis of open sets. The reader may refer to [Mun00] for an extensive presentation.

### 1.1 Topologies

**§1.1.1 Open sets.** Topological spaces are abstractions of the concept of 'shape' or 'geometric object'. We start by defining them via open sets.

**Definition 1.1.** A *topological space* is a pair $(X, \mathscr{T})$ where $X$ is a set and $\mathscr{T}$ is a collection of subsets of $X$ such that:

1. $\emptyset \in \mathscr{T}$ and $X \in \mathscr{T}$,
2. for every (potentially infinite) collection $(O_\alpha)_{\alpha \in A} \subset \mathscr{T}$, we have $\bigcup_{\alpha \in A} O_\alpha \in \mathscr{T}$,
3. for every finite collection $(O_i)_{1 \leq i \leq n} \subset \mathscr{T}$, we have $\bigcap_{1 \leq i \leq n} O_i \in \mathscr{T}$.

The set $\mathscr{T}$ is called a *topology* on $X$. In these notes, we will use the symbol $\mathscr{P}(X)$ to denote the powerset of $X$, that is, the set of subsets of $X$. It follows that a topology on $X$ is an subset of $\mathscr{P}(X)$, i.e., $\mathscr{T} \subset \mathscr{P}(X)$.

The elements of $\mathscr{T}$ are called the *open sets*. With this vocabulary, the previous definition can be reformulated as follows;

1. the empty set is an open set, the set $X$ itself is an open set,
2. an infinite union of open sets is an open set,
3. a finite intersection of open sets is an open set.

In general, in a given topology, an infinite intersection of open sets may not be open. An example is given in Exercise 5. However, when $X$, is finite, this statement is true.

**§1.1.2 CLOSED SETS.** For every open set $O \in \mathscr{T}$, its complementary ${}^cO = \{x \in X \mid x \notin O\}$ is called a *closed set*. In other words, a subset $P \subset X$ is closed if and only if its complementary is open. As a direct consequence of Definition 1.1, one proves the following:

**Proposition 1.2.** *We have:*

1. *the sets $\emptyset$ and $X$ are closed sets,*
2. *for every (potentially infinite) collection $(P_\alpha)\alpha \in A$ of closed set, $\bigcap_{\alpha \in A} P_\alpha$ is a closed set,*
3. *for every finite collection $(P_i)_{1 \leq i \leq n}$ of closed sets, $\bigcup_{1 \leq i \leq n} P_i$ is a closed set.*

**Proof.** Point 1. The set $\emptyset$ is closed because ${}^c\emptyset = X$ is open, according to Point 1 of Definition 1.1. Same for $X$ since ${}^cX = \emptyset$ is open.

Point 2. If $(P_\alpha)_{\alpha \in A}$ is an infinite collection of closed set, then for every $\alpha \in A$, ${}^cP_\alpha$ is open. Now, we use the relation (known as De Morgan's law)

$$^c\left(\bigcap_{\alpha \in A} P_\alpha\right) = \bigcup_{\alpha \in A} {}^cP_\alpha.$$

This is a union of open sets, hence it is open by Point 2 of Definition 1.1. Hence $\bigcap_{\alpha \in A} P_\alpha$ is closed.

Point 3. Just as previously, if $(P_i)_{1 \leq i \leq n}$ is a finite collection of closed set, then each $i \in [\![1, n]\!]$, ${}^cP_i$ is open. We have the relation

$$^c\left(\bigcup_{1 \leq i \leq n} P_i\right) = \bigcap_{1 \leq i \leq n} {}^cP_i.$$

This is a *finite* intersection of open sets, hence it is open by Point 3 of Definition 1.1. Hence $\bigcup_{1 \leq i \leq n} P_i$ is closed. □

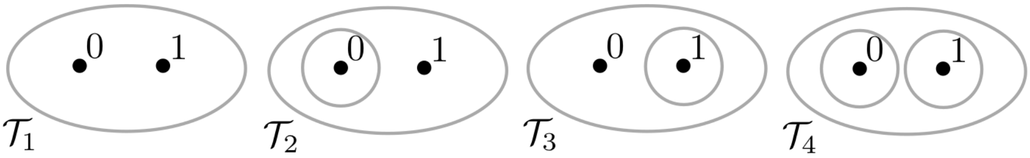
Note that the converse of Proposition 1.2 is true: if $\mathscr{T}'$ is a collection of sets satisfying 1, 2 and 3, then the collection of complementaries $\mathscr{T} = \{{}^cP \mid P \in \mathscr{T}'\}$ satisfies the axioms of Definition 1.1. Therefore, this proposition can serve as an alternative definition for topological spaces. We say that we define a topology via its closed sets.

**Example 1.3.** Let $X = \{0\}$ be a set with one element. There exists only one topology on $X$: $\mathscr{T} = \{\emptyset, \{0\}\}$.

**Example 1.4.** Let $X$ be any set. The subset $\mathscr{T} = \{\emptyset, X\}$ is a topology on $X$, called the *trivial topology*. Likewise, the power set of $X$, denoted $\mathscr{P}(X)$, is a topology on $X$, called the *discrete topology*.

**Example 1.5.** Let $X = \{0, 1\}$ be a set with two elements. There exists only four different topologies on $X$:

$$\mathscr{T}_1 = \{\emptyset, \{0,1\}\}, \quad \mathscr{T}_2 = \{\emptyset, \{0\}, \{0,1\}\}, \quad \mathscr{T}_3 = \{\emptyset, \{1\}, \{0,1\}\}, \quad \mathscr{T}_4 = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}.$$




**Example 1.6.** Let $X = \{0,1,2\}$ be a set with three elements. The set $\mathscr{T} = \{\emptyset\}$ is not a topology on $X$ because the whole set $X = \{1,2,3\}$ does not belong to $\mathscr{T}$. Likewise, the set

$$\mathscr{T} = \{\emptyset, \{0\}, \{1\}, \{0,1,2\}\}$$

is not a topology on $X$ because the finite union $\{0\} \cup \{1\} = \{0,1\}$ does not belong to $\mathscr{T}$.

**Example 1.7.** The set

$$\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{[0,a] \mid a \geq 0\}$$

is not a topology on $\mathbb{R}$. Indeed, the following union of open sets is not an open set:

$$\bigcup_{a \geq 0} [0,a] = [0,+\infty).$$

**§1.1.3 Comparison of topologies.** As illustrated in Example 1.4, any set $X$ of cardinal greater than 1 admits several different topologies. We shall compare them as follows.

**Definition 1.8.** Consider two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on $X$. If $\mathscr{T}_1 \subset \mathscr{T}_2$, we say that $\mathscr{T}_1$ is *coarser* than $\mathscr{T}_2$, and that $\mathscr{T}_2$ is *finer* than $\mathscr{T}_1$.

In other words, $\mathscr{T}_2$ is finer than $\mathscr{T}_1$ if it has 'more open sets'. The relation 'being coarser' is a partial ordering on the set of all topologies on $X$. Using this relation, we can represent the set of all topologies on $X$ as lattice, as drawn below for the case of Example 1.5. It will be called the *lattice of topologies on* $X$. In these notes, when $A$ and $B$ are sets such as $A \subset B$, the map $A \hookrightarrow B$ will denote the inclusion map.

$$\begin{tikzcd}
& \{\emptyset,\{0\},\{0,1\}\} \arrow[dr, hook] & \\
\{\emptyset,\{0,1\}\} \arrow[ur, hook] \arrow[dr, hook] & & \{\emptyset,\{0\},\{1\},\{0,1\}\} \\
& \{\emptyset,\{1\},\{0,1\}\} \arrow[ur, hook] &
\end{tikzcd}$$

Note that the relation 'being coarser' admits a lowest element (that is, an element that is coarser than any other): the trivial topology. Similarly, it admits a greatest element: the discrete topology. In the language of partially ordered sets, we say that this lattice is *bounded*.

**§1.1.4 Intersection and union of topologies.** We now wish to build new topologies, based on an arbitrary collection $\{\mathscr{T}_\alpha\}_{\alpha \in A}$ on $X$. The easiest construction is the intersection $\bigcap_{\alpha \in A} \mathscr{T}_\alpha$.

**Proposition 1.9.** *An arbitrary intersection of topologies on $X$ is a topology.*

**Proof.** It follows directly from Definition 1.1. $\square$

The intersection topology $\bigcap_{\alpha\in A}\mathscr{T}_\alpha$ has the property that is the greatest topology included in all the $\mathscr{T}_\alpha$. In other words, if $\mathscr{T}$ is any topology on $X$ such that $\mathscr{T}\subset\mathscr{T}_a$ for all $\alpha\in A$, then we must have $\mathscr{T}\subset\bigcap_{a\in A}\mathscr{T}_a$. In the language of partially ordered sets, we say that the lattice of topologies has the greatest lower bound property.

As a dual construction, one would be tempted to consider the union $\bigcup_{a\in A}\mathscr{T}_a$. However, it may not be a topology. One should instead consider the following notion.

**Definition 1.10.** Let $S\subset\mathscr{P}(X)$ be any subset. The *topology generated* by $S$ is defined as the intersection of all the topologies on $X$ that contain $S$. It is denoted $\mathscr{T}(S)$.

Using Proposition 1.9, we have that $\mathscr{T}(S)$ is a topology on $X$. Moreover, it is the smallest topology included in all the $\mathscr{T}_\alpha$. That is to say, if $\mathscr{T}$ is any topology on $X$ such that $\mathscr{T}\supset\mathscr{T}_a$ for all $\alpha\in A$, then we must have $\mathscr{T}\supset\mathscr{T}(S)$. We say that the lattice of topologies has the least upper bound property. The following proposition gives an alternative description of the generated topology.

**Proposition 1.11.** *For any $S\subset\mathscr{P}(X)$, the generated topology $\mathscr{T}(S)$ is the collection of arbitrary unions of finite intersections of element of $S$.*

**Proof.** Let $\mathscr{T}'$ denote the collection of arbitrary unions of finite intersections of element of $S$. As a direct consequence of Definition 1.1, one shows that $\mathscr{T}'$ is a topology on $X$. Moreover, since the generated topology $\mathscr{T}$ is a topology, it must contain $\mathscr{T}'$. But since $\mathscr{T}$ is the smallest topology containing $S$, we deduce that $\mathscr{T}'=\mathscr{T}$. $\square$

**Exercise 1** (Enumeration of topologies)**.** Let $X=\{0,1,2\}$ be a set with three elements. How many different topologies does $X$ admit?

*Remark:* Let $t(n)$ be the number of different topologies on a set with $n$ elements. One obtains directly the bound $2\le t(n)\le 2^{2^n}$ for $n\ge 3$. The lower bounds comes from the fact that the trivial and discrete topologies are topologies, and the upper bound from the fact that a topology on $X$ is an element of $\mathscr{P}(\mathscr{P}(X))$. A more involved bound can be found in [LA75]: $2^n\le t(n)\le 2^{n(n-1)}$.

**Exercise 2.** Let $X$ be a finite set, and $\mathscr{T}$ a topology on $X$ such that all the singletons $\{x\}$, $x\in X$, are closed. Show that $\mathscr{T}$ is the discrete topology.

**Exercise 3** (Hausdorff separability)**.** We say that a topological space $(X,\mathscr{T})$ is Hausdorff (or is a $T_2$-space) if for any $x,y\in X$ such that $x\neq y$,, there exists two open sets $U,V\in\mathscr{T}$ such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. Among the topologies on $X=\{0,1\}$ described in Example 1.5, which ones are Hausdorff?

**Exercise 4.** Show that the following set is a topology on $\mathbb{R}$:

$$\mathscr{T}=\{\emptyset,\mathbb{R}\}\cup\{(-a,a)\mid a>0\}.$$

*Hint:* Remind the least-upper-bound property of the real numbers.

**Exercise 5** (Cofinite topology)**.** Let $\mathbb{Z}$ be the set of integers. Consider the *cofinite topology* $\mathscr{T}$ on $\mathbb{Z}$, defined as follows: a subset $O\subset\mathbb{Z}$ is an open set if and only if $O=\emptyset$ or ${}^cO$ is finite.

1. Show that $\mathscr{T}$ is a topology on $\mathbb{Z}$.
2. Exhibit an sequence of open sets $\{O_n\}_{n\in\mathbb{N}} \subset \mathscr{T}$ such that $\bigcap_{n\in\mathbb{N}} O_n$ is not open.

*Remark:* If the set $X$ is finite, then the cofinite topology is the discrete topology.

**Exercise 6** (Zariski topology)**.** A subset $F \subset \mathbb{R}^n$ is a Zariski-closed set if it can be written as

$$F = \{x \in \mathbb{R}^n \ \mid \forall \alpha \in A,\ P_\alpha(x) = 0\}$$

where $(P_\alpha)_{\alpha\in A}$ is a (potentially infinite) collection of multivariate polynomials on $\mathbb{R}^n$. Show that the collection of Zariski-closed sets forms the collection of closed sets of a topology on $\mathbb{R}^n$, called Zariski topology.

*Remark:* Actually, as a consequence of Hilbert's Nullstellensatz, any Zariski-closed set can be written as the set of common roots of a *finite* family of polynomials.

**Exercise 7** (Fifth proof of the infinity of primes from [AZ99])**.** For any $a,b \in \mathbb{Z}$, define

$$N_{a,b} = \{a + bn \mid n \in \mathbb{Z}\}.$$

Call a subset $O \subset \mathbb{Z}$ closed if either $O =$ or if to every $a \in Z$ there exists a $b > 0$ such that $N_{a,b} \subset O$. Let $\mathscr{T}$ denote the collection of all open sets.

1. Show that $\mathscr{T}$ is a topology on $\mathbb{Z}$.
2. Show that any nonempty open set is infinite, and that the $N_{a,b}$ are closed sets.
3. Let $\mathbb{P}$ denotes the set of all prime numbers. Show that $\mathbb{Z} \setminus \{-1,1\} = \bigcup_{b\in\mathbb{P}} N_{0,b}$.
4. By contradiction, use 2. and 3. to deduce that $\mathbb{P}$ is infinite.

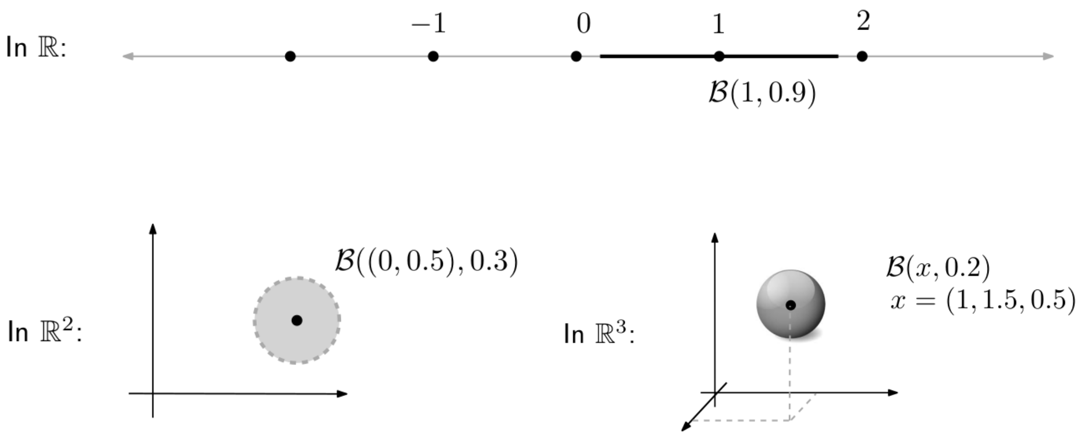
## 1.2 EUCLIDEAN TOPOLOGY

Many topological spaces encountered in practice are subsets of the Euclidean spaces $\mathbb{R}^n$. On $\mathbb{R}^n$, we will mainly consider the *Euclidean topology*. In order to define this topology, we will use open balls. Remind that the Euclidean metric on $\mathbb{R}^n$ is defined for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as:

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

For $x \in \mathbb{R}^n$ and $r > 0$, the *open ball* of center $x$ and radius $r$, denoted $\mathscr{B}(x,r)$, is defined as:

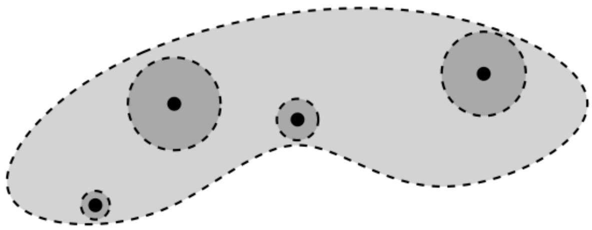
$$\mathscr{B}(x,r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}.$$



**Definition 1.12.** The *Euclidean topology* on $\mathbb{R}^n$, denoted $\mathscr{T}_{\mathbb{R}^n}$, is the topology generated by the balls $\{B(x,t) \mid x \in \mathbb{R}^n,\ t > 0\}$.

According to the discussion of §1.1.3, the Euclidean topology is the smallest topology that contains all the open balls. We now give an alternative definition of it, often more convenient to identify open sets.

**Proposition 1.13.** *A set $A$ is open (for the Euclidean topology) if and only if for every $x \in A$, there exists a $r > 0$ such that $\mathscr{B}(x,r) \subset A$.*



**Proof.** It will be convenient to use the following vocabulary: $A$ is *open around* $x$ if there exists $r > 0$ such that $\mathscr{B}(x,r) \subset A$. Note that the proposition states that $A$ is open if and only if it is open around all of its points. Let us denote by $\mathscr{U}_{\mathbb{R}^n}$ the set of all subsets $A \subset \mathbb{R}^n$ that are open around all of their points, that is,

$$\mathscr{U}_{\mathbb{R}^n} = \{A \subset \mathbb{R}^n \mid \forall x \in A,\ \exists r > 0,\ \mathscr{B}(x,r) \subset A\}.$$

In what follows, we will say that a subset $A \subset \mathbb{R}^n$ is $\mathscr{T}_{\mathbb{R}^n}$-closed (resp. $\mathscr{U}_{\mathbb{R}^n}$-closed) if it belongs to $\mathscr{T}_{\mathbb{R}^n}$ (resp. $\mathscr{U}_{\mathbb{R}^n}$). The proof consists in showing that $\mathscr{U}_{\mathbb{R}^n} \subset \mathscr{T}_{\mathbb{R}^n}$, and that $\mathscr{U}_{\mathbb{R}^n}$ is a topology that contains the open balls. Using the fact that $\mathscr{T}_{\mathbb{R}^n}$ is the smaller topology that contains the open balls, it follows that $\mathscr{T}_{\mathbb{R}^n} = \mathscr{U}_{\mathbb{R}^n}$.

First step: $\mathscr{U}_{\mathbb{R}^n} \subset \mathscr{T}_{\mathbb{R}^n}$. Let $O \in \mathscr{U}_{\mathbb{R}^n}$. For any $x \in O$, let $r_x > 0$ be such that $\mathscr{B}(x,r_x) \subset O$. We have that $O = \bigcup_{x \in O} \mathscr{B}(x,r_x)$. Moreover, by definition, this union of open balls belongs to $\mathscr{T}_{\mathbb{R}^n}$. Hence $O \in \mathscr{T}_{\mathbb{R}^n}$, and we deduce that $\mathscr{U}_{\mathbb{R}^n} \subset \mathscr{T}_{\mathbb{R}^n}$.

Second step: $\mathscr{U}_{\mathbb{R}^n}$ contains the open balls. Let $x \in \mathbb{R}^n$ and $r > 0$. Consider the ball $\mathscr{B}(x,r)$. In order to show that it is $\mathscr{U}_{\mathbb{R}^n}$-open, we must show that it is open around all of its points. Consider $y \in \mathscr{B}(x,r)$, and define $r' = r - \|x-y\|$. We will show that $\mathscr{B}(y,r') \subset \mathscr{B}(x,r)$. To prove so, let $z \in \mathscr{B}(y,r')$. We apply the triangular inequality for the Euclidean norm:

$$\begin{aligned} \|z-x\| &\leq \|z-y\| + \|y-x\| \\ &\leq r' + \|y-x\| = r. \end{aligned}$$

We deduce that $\mathscr{B}(y,r') \subset \mathscr{B}(x,r)$, hence that $\mathscr{B}(x,r)$ belongs to $\mathscr{U}_{\mathbb{R}^n}$.

Third step: $\mathscr{U}_{\mathbb{R}^n}$ is a topology. We shall verify the three axioms of Definition 1.1.

- First axiom. Since $\emptyset$ contains no point, it is open around all of its points, hence belongs to $\mathscr{U}_{\mathbb{R}^n}$. The set $\mathbb{R}^n$ also is open, since for every $x \in \mathbb{R}^n$, the ball $\mathscr{B}(x,1)$ is a subset of $\mathbb{R}^n$.
- Second axiom. Let $\{O_\alpha\}_{\alpha \in A} \subset \mathscr{T}_{\mathbb{R}^n}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha \in A} O_\alpha$. Let $x \in O$. There exists an $\alpha \in A$ such that $x \in O_\alpha$. Since $O_\alpha$ is open, it is open

around $x$, i.e., there exists $r > 0$ such that $\mathscr{B}(x,r) \subset O_\alpha$. We deduce that $\mathscr{B}(x,r) \subset O$, and that $O$ is open around $x$. Since this it true for any $x \in O$, we proved that $O$ is open.

• Third axiom. Consider a finite collection $\{O_i\}_{1\leq i\leq n} \subset \mathscr{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1\leq i\leq n} O_i$. Let $x \in O$. For every $i \in [\![1,n]\!]$, we have $x \in O_i$. Since $O_i$ is open, it is open around $x$, i.e., there exists $r_i > 0$ such that $\mathscr{B}(x,r_i) \subset O_i$. Define $r_{\min} = \min\{r_1,...r_n\}$. For every $i \in [\![1,n]\!]$, we have $\mathscr{B}(x,r_{\min}) \subset O_i$. We deduce that $\mathscr{B}(x,r_{\min}) \subset O$, and that $O$ is open around $x$. Since this it true for any $x \in O$, we have proven that $O$ is open. □

**Example 1.14.** The interval $I = (0,+\infty)$ is an open set for the Euclidean topology on $\mathbb{R}$. Indeed, for any $x \in I$, the open ball $\mathscr{B}(x,x)$ is included in $I$.

**Example 1.15.** The interval $[0,1]$ is a closed set for the Euclidean topology on $\mathbb{R}$. Indeed, its complement ${}^c[0,1] = (-\infty,0)\cup(1,+\infty)$ is open, since it is the union of two open sets.

**Example 1.16.** Let $\mathscr{C} = \{x = (x_1,...,x_n) \in \mathbb{R}^n \mid \|x\|_\infty < 1\}$ be the filled open unit cube of $\mathbb{R}^n$, where $\|x\|_\infty = \max(|x_1|,...,|x_n|)$ is the sup norm. Let $x \in \mathscr{C}$, define $r = 1-\|x\|_\infty$, and consider the open ball $\mathscr{B}(x,r)$. For any $y \in \mathscr{B}(x,r)$, we have

$$\|y\|_\infty = \max(|y_1|,...,|y_n|) < \max(|x_1|+r,...,|x_n|+r) \leq \max(|x_1|,...,|x_n|)+r \leq 1.$$

Therefore, $\|y\|_\infty < 1$, hence $\mathscr{B}(x,r) \subset \mathscr{C}$. This being true for any $x \in \mathscr{C}$, we deduce that $\mathscr{C}$ is open for the Euclidean topology.

> **Exercise 8.** Consider the real line $\mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?
>
> 1. the interval $[0,1)$,
> 2. the intervals $[x,+\infty)$, $x \in \mathbb{R}$,
> 3. the singletons $\{x\}$, $x \in \mathbb{R}$,
> 4. the rational numbers $\mathbb{Q}$.

**Exercise 9** (Sorgenfrey line)**.** Let $\mathscr{U}$ be the topology on $\mathbb{R}$ generated by the the collection

$$\{[a,b) \mid a,b \in \mathbb{R},\ a \leq b\}.$$

1. Show that $\mathscr{U}$ is finer than the Euclidean topology.
2. Show that for any $x \in \mathbb{R}$, the set $[x,+\infty)$ is open and closed.

## 1.3 Construction of topologies

### §1.3.1 Subspace topology

**Definition 1.17.** Let $(X,\mathscr{T})$ be a topological space, and $Y \subset X$ a subset. We define the *subspace topology on $Y$* as the following set:

$$\mathscr{T}_{|Y} = \{O \cap Y \mid O \in \mathscr{T}\}.$$

**Proposition 1.18.** *The set $\mathscr{T}_{|Y}$ is a topology on $Y$.*

**Proof.** We have to check the three axioms of a topological space, as in Definition 1.1.

First axiom. The set $\emptyset$ is clearly open for $\mathscr{T}_{|Y}$ because it can be written as $\emptyset \cap Y$. The set $Y$ also is open for $\mathscr{T}_{|Y}$ because it can be written $X \cap Y$, and $X$ is open for $\mathscr{T}$.

Second axiom. Let $\{O_\alpha\}_{\alpha\in A} \subset \mathscr{T}_{|Y}$ be a infinite collection of open sets, and define $O = \bigcup_{\alpha\in A} O_\alpha$. By definition of $\mathscr{T}_{|Y}$, for every $\alpha \in A$, there exists $O'_\alpha$ such that $O_\alpha = O'_\alpha \cap Y$. Define $O' = \bigcup_{\alpha\in A} O'_\alpha$. It is an open set for $\mathscr{T}$. We have

$$O = \bigcup_{\alpha\in A} O_\alpha = \bigcup_{\alpha\in A} O'_\alpha \cap Y = \left(\bigcup_{\alpha\in A} O'_\alpha\right) \cap Y = O' \cap Y.$$

Hence $O \in \mathscr{T}_{|Y}$.

Third axiom. Consider a finite collection $\{O_i\}_{1\leq i\leq n} \subset \mathscr{T}_{\mathbb{R}^n}$, and define $O = \bigcap_{1\leq i\leq n} O_i$. Just as before, for every $i \in [\![1,n]\!]$, there exists $O'_i$ such that $O_i = O'_i \cap Y$. Define $O' = \bigcup_{1\leq i\leq n} O'_i$. It is an open set for $\mathscr{T}$. We have

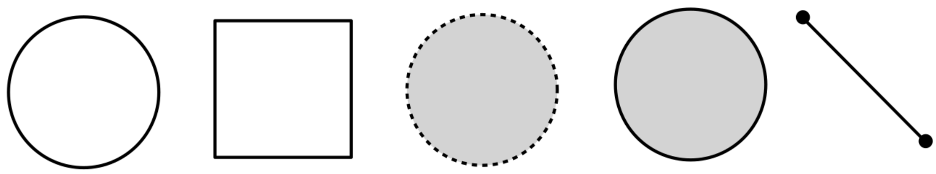
$$O = \bigcap_{1\leq i\leq n} O_\alpha = \bigcap_{1\leq i\leq n} O'_\alpha \cap Y = \left(\bigcap_{1\leq i\leq n} O'_\alpha\right) \cap Y = O' \cap Y.$$

Hence $O \in \mathscr{T}_{|Y}$. $\square$

Thanks to the subspace topology, any subset of $\mathbb{R}^n$ inherits a particular topology. Among the subsets of $\mathbb{R}^n$ that we will consider, let us list:

- the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$,
- the unit cube $\mathscr{C}_{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_\infty = 1\}$ where $\|x\|_\infty = \max(|x_1|, ..., |x_n|)$,
- the open balls $\mathscr{B}(x,r) = \{y \in \mathbb{R}^n \mid \|x-y\| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$,
- the closed balls $\overline{\mathscr{B}}(x,r) = \{y \in \mathbb{R}^n \mid \|x-y\| \leq r\}$ for $x \in \mathbb{R}^n$ and $r \geq 0$,
- the standard simplex

$$\Delta_{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1, ..., x_n \geq 0,\ x_1 + ... + x_n = 1\}.$$



**Exercise 10.** Consider the space $\mathbb{R}^n$ endowed with the Euclidean topology, and its unit sphere $\mathbb{S}^{n-1}$ endowed with the subspace topology. Define the upper hemisphere $\mathbb{S}^{n-1}_+ = \{x \in \mathbb{R}^n \mid \|x\| = 1,\ x_1 > 0\}$. Show that $\mathbb{S}^{n-1}_+$ in open in $\mathbb{S}^{n-1}$, but not in $\mathbb{R}^n$.

**Exercise 11** (Topologist's sine curve)**.** Consider the plane $\mathbb{R}^2$ endowed with the Euclidean topology. Define the set

$$X = \{(x, \sin(1/x) \mid x \in (0,\pi]\} \cup \{(0,0)\}$$

and endow it with the subspace topology. Show that the singleton $\{0\}$ is closed and not open.

**Exercise 12** (Cantor set)**.** Consider the Euclidean line $\mathbb{R}$. Let $C_0 = [0,1]$, $C_1 = [0,1/3]\cup[2/3,1]$, $C_2 = [0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]$, and in general, let $C_{n+1}$ be the union of the $2n+1$ closed intervals, each of length $(1/3)^{n+1}$, obtained by removing the open middle thirds of the $2^n$ closed intervals of $C_n$. We define

$$\mathscr{C} = \bigcap_{n\geq 0} C_n.$$

1. Show that $\mathscr{C}$ is a nonempty closed subset of $\mathbb{R}$.
2. Show that for all $x\in\mathscr{C}$, the singleton $\{x\}$ is open for the subspace topology on $\mathscr{C}$.

### §1.3.2 (Finite) product topology

**Definition 1.19.** Let $((X_\alpha, \mathscr{T}_\alpha))_{\alpha\in A}$ be a collection of topological spaces. We denote by $\Pi_{\alpha\in A}\mathscr{T}_\alpha$ the topology generated by the sets $\Pi_{\alpha\in A}O_\alpha$ where $O_\alpha\in\mathscr{T}_\alpha$ for all $\alpha\in A$. If $A$ is finite, it is called the *product topology*, and when it is infinite, it is called the *box topology*.

**Remark 1.20.** In the context where $A$ is infinite, the term *product topology* is reserved for another topology, that we will study in more details in the section about functional topology.

**Exercise 13.** Let $\mathbb{R}$ be endowed with the Euclidean topology $\mathscr{T}(\mathbb{R})$. Show that the product topology on $\mathbb{R}\times\cdots\times\mathbb{R}$ is equal to the Euclidean topology $\mathscr{T}(\mathbb{R}^n)$ on $\mathbb{R}^n$.

**Exercise 14.** Let $(X,\mathscr{T})$, $(Y,\mathscr{U})$ be two topological spaces. Show that if $A$ is a closed set of $X$ and $B$ a closed set of $Y$, then $A\times B$ is a closed set of the product topology.

**Exercise 15.** Let $(X,\mathscr{T})$ be a topological space, and consider the product topology on $X\times X$. Show that $(X,\mathscr{T})$ is Hausdorff (in the sense of Exercise 3) if and only if the diagonal $\Delta = \{(x,x)\mid x\in X\}$ is closed in $X\times X$.

**§1.3.3 Quotient topology** If $X$ is any set, we remind the reader that an *equivalence relation* on $X$ is a binary relation, denoted $\mathscr{R}$, which satisfies:

*(reflexivity)* $\forall x\in X,\ x\mathscr{R}x,$

*(symmetry)* $\forall x,y\in X,\ x\mathscr{R}y \iff y\mathscr{R}x,$

*(transitivity)* $\forall x,y,z\in X,\ (x\mathscr{R}y \text{ and } y\mathscr{R}z) \implies x\mathscr{R}z.$

For any $x\in X$, we define its equivalence class as $\mathscr{O}_x = \{y\in X\mid x\mathscr{R}y\}$. Using the fact that $\mathscr{R}$ is an equivalence relation, we deduce the following fact: $x\mathscr{R}y \iff \mathscr{O}_x = \mathscr{O}_y$. As a consequence, the set of equivalence classes form a partition of $X$. It is denoted $X/\mathscr{R}$, and is called the *quotient set*. We denote the *projection map* as

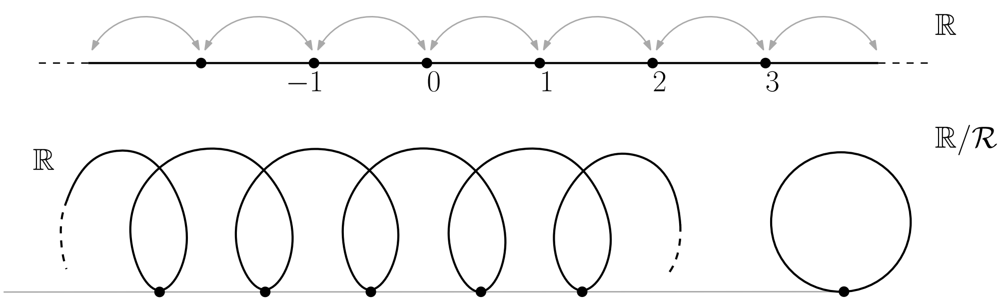
$$\begin{aligned}\pi\colon X &\longrightarrow X/\mathscr{R}\\ x&\longmapsto \mathscr{O}_x\end{aligned}$$

**Definition 1.21.** Let $(X,\mathscr{T})$ be topological space and $\mathscr{R}$ an equivalence relation on $X$. The *quotient topology* on $X/\mathscr{R}$ is defined as the topology whose open sets are the subsets $O\subset X/\mathscr{R}$ such that $\pi^{-1}(O)\in\mathscr{T}$.

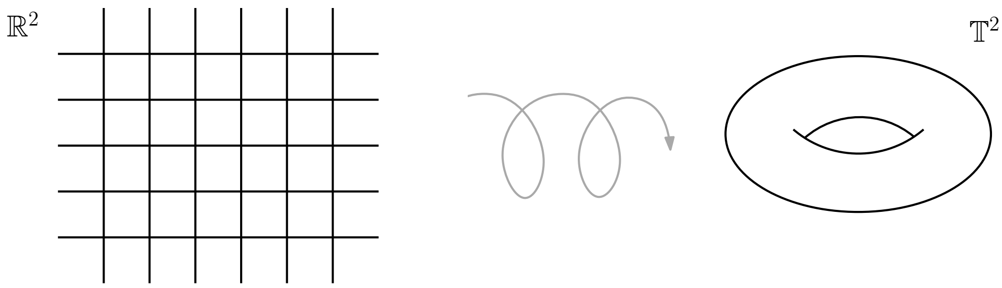
**Remark 1.22.** It is also called the *final topology* with respect to the map $\pi$.

The quotient topology gives a handy way to build new topological spaces. While quotienting the space, we 'merge', or 'identify' points that are in the same equivalence class.

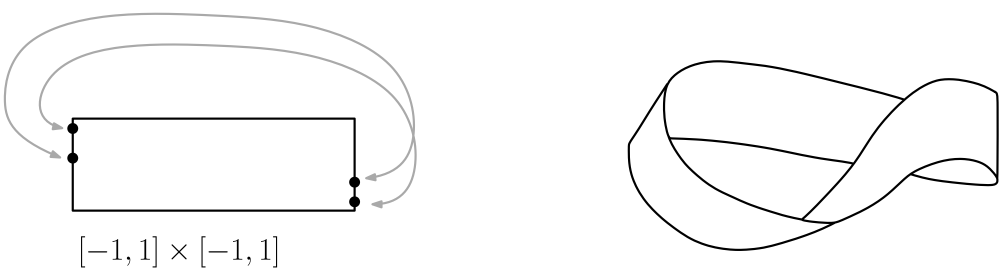
**Example 1.23** (Circle)**.** Let $\mathbb{R}$ be the real line endowed with the Euclidean topology, and consider the relation $x\mathscr{R}y \iff x-y\in\mathbb{Z}$. The equivalence classes are the sets $\mathscr{O}_x = \{x+n \mid n\in\mathbb{Z}\}$, and the quotient space $\mathbb{R}/\mathscr{R}$ can be identified with the interval $[0,1)$. While quotienting $\mathbb{R}$, we 'roll it up on itself'. The quotient topology is the one of a circle. In order to give rigorous sense to this last sentence, we will have to wait until Section 3.




**Example 1.24** (Torii)**.** More generally, the equivalence relation $x\mathscr{R}y \iff \forall i\leq n,\ x_i-y_i\in\mathbb{Z}$ on the Euclidean space $\mathbb{R}^n$ give rise to the *torus* of dimension $n$. It is denoted $\mathbb{T}^n$.



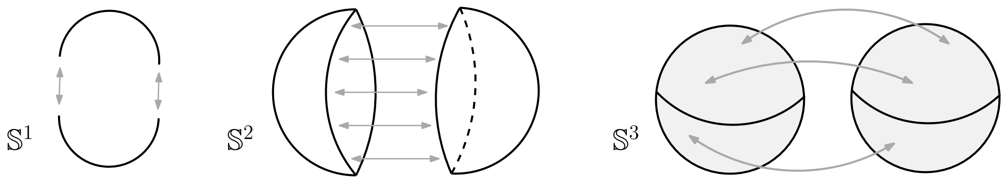

**Example 1.25** (Möbius strip)**.** Let $[-1,1]\times[-1,1]$ be the square of $\mathbb{R}^2$, endowed with the subspace topology. Consider the equivalence relation generated by $(x,y)\mathscr{R}(x',y') \iff |x|=1,\ y=-x,\ x'=-y'$. The quotient topological space is called the *Möbius strip*. The construction consists in gluing the opposite sides of a square, reversing the direction.




**Example 1.26** (Projective spaces)**.** Let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$, endowed with the subspace topology. The *antipodal relation* $x\mathscr{R}y \iff x=-y$ is an equivalence relation on $\mathbb{S}^{n-1}$. The quotient topological space is called the *real projective space* of dimension $n-1$, and is denoted $P^{n-1}\mathbb{R}$. The first projective space $P^1\mathbb{R}$ actually is a circle (make a drawing).

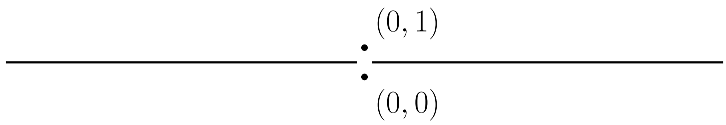
Quotient topology also allows to give a rigorous sense to the idea of 'gluing'. If $(X,\mathscr{T})$ and $(Y,\mathscr{U})$ are two topological spaces, $A \subset X$ a subset and $f\colon A \to Y$ a map, the *gluing of $A$ onto $B$ along $f$* is the quotient of the disjoint union $X \sqcup Y$ by the equivalence relation generated by $x\mathscr{R}f(x)$ for all $x \in A$. It is denoted $(X \sqcup Y)/f$.

**Example 1.27** (Spheres are gluing of disks)**.** Consider two copies $\overline{\mathscr{B}}_1, \overline{\mathscr{B}}_2$ of the unit closed ball $\overline{\mathscr{B}}(0,1)$ of $\mathbb{R}^n$. Let $\partial\overline{\mathscr{B}}_1$ denote the boundary of $\overline{\mathscr{B}}_1$, that is, the sphere. Let $f\colon \partial\overline{\mathscr{B}}_1 \to \overline{\mathscr{B}}_2$ be the inclusion map. Then the gluing $(\overline{\mathscr{B}}_1 \sqcup \overline{\mathscr{B}}_2)/f$ is the sphere $\mathbb{S}^n$.




**Exercise 16** (Double-origin interval)**.** Consider the topological space $X = [-1,1] \times \{0,1\}$, endowed with the subspace topology of $\mathbb{R}^2$. Let $\mathscr{R}$ be the relation on $X$ defined as $(t,a)\mathscr{R}(u,b) \iff (t = u \text{ and } t \neq 0)$ or $(t = u \text{ and } a = b)$.

1. Show that $\mathscr{R}$ is an equivalence relation, and describe its equivalence classes.
2. Show that the quotient topology on $X/\mathscr{R}$ is not Hausdorff (in the sense of Exercise 3).




# 2 Separation and connectedness

In this section, we will continue introducing the basic vocabulary of topological spaces. We will first define the interior, the closure and the boundary of a set. We will then introduce the notion of Hausdorff separability, and finally of connectedness.

## 2.1 Neighborhoods, interior, closure, boundary

**§2.1.1 Neighborhoods.** In what follows, $(X,\mathscr{T})$ denotes a topological space.

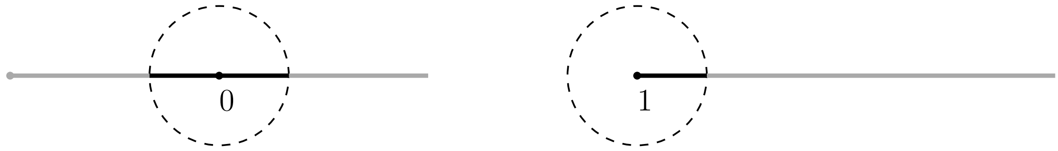
**Definition 2.1.** Let $x \in X$ be a point. We say that a subset $A \subset X$ is a *neighborhood* of $x$ if $A$ contains an open set that contains $x$, that is, if $\exists O \in \mathscr{T}$ such that $O \subset A$ and $x \in O$.

In some textbooks, the set of all neighborhoods of $x$ is denoted $\mathscr{N}(x)$, although we will not use this notation in these notes. Note that an open set is a neighborhood of all of its points. Conversely, a subset $A$ that is a neighborhood of all of its points is open. Indeed, for each point $x \in A$, we can consider an open set $O_x$ that contains $x$, and write $A = \bigcup_{x\in A} O_x$, which is open since it is an union of open sets. However, in general, a neighborhood does not have to be open.

In the case of the Euclidean topology, and as a direct consequence of Proposition 1.13, we get the following characterization:

**Proposition 2.2.** *Let $(\mathbb{R}^n, \mathscr{T}_{\mathbb{R}^n})$ be the Euclidean space, $A \subset \mathbb{R}^n$ a subset and $x \in A$ a point. The set $A$ is a neighborhood of $x$ if and only if there exists a $r > 0$ such that $\mathscr{B}(x,r) \subset A$.*

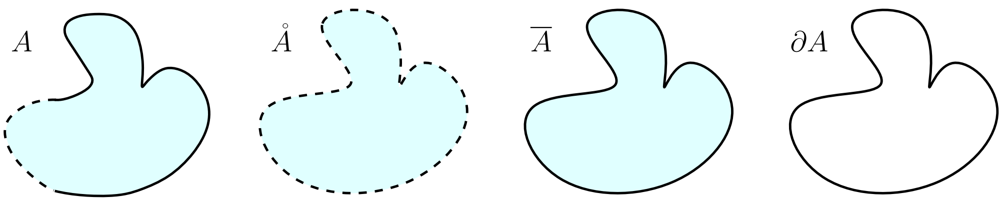
**Example 2.3.** Let $\mathbb{R}$ be the Euclidean line. The set $A = [-1,1)$ is a neighborhood of $0$, since it contains the open set $(-1,1)$. However, it is not a neighborhood of $-1$, since it does not contain any open ball of the form $(-1-r, -1+r)$.




### §2.1.2 INTERIOR, CLOSURE, BOUNDARY.

**Definition 2.4.** Let $A \subset X$ be any subset. We define

- its *interior* $\mathring{A}$, as the set of points for which $A$ is a neighborhood,
- its *closure* $\overline{A}$, as the set of points for which every neighborhood meets $A$,
- its *boundary* as $\partial A = \overline{A} \setminus \mathring{A}$.




**Lemma 2.5.** *For any $A \subset X$, we have ${}^c(\mathring{A}) = \overline{{}^cA}$ and ${}^c(\overline{A}) = \mathring{\widehat{{}^cA}}$.*

**Proof.** We shall only prove the first equality, since the second one is obtained by taking the complementary of $A$. By definition, $\overline{{}^cA}$ is the set of points for which every neighborhood meets ${}^cA$, that is, the set of points for which no neighborhood is contained in $A$. Consequently, ${}^c(\overline{{}^cA})$ is the set of points for which there exists a neighborhood contained in $A$. In other words, ${}^c(\overline{{}^cA}) = \mathring{A}$, as wanted. $\square$

**Proposition 2.6.** *Let $A \subset X$ be any subset. We have:*

- *$\mathring{A}$ is the union of open sets contained in $A$. As a consequence, it is the largest open set contained in $A$.*
- *$\overline{A}$ is the intersection of closed sets containing $A$. As a consequence, it is the smallest closed set containing $A$.*

**Proof.** The first point is a direct consequence of the definition of the interior. The second point is a consequence of the first point and Lemma 2.5. $\square$

As useful consequences of the previous proposition, we have that a set $A \subset X$ is open of and only if $\mathring{A} = A$, and $A$ is closed if and only if $\overline{A} = A$.

**Example 2.7.** Let $\mathbb{R}$ be the Euclidean line, and $A = [-1,1)$. We have $\mathring{A} = (-1,1)$, $\overline{A} = [-1,1]$ and $\partial A = \{-1,1\}$. In general, in the Euclidean space $\mathbb{R}^n$, the interior of the closed ball is the open ball, and the closure of the open ball is the closed ball. Their boundary is the sphere.

**Proposition 2.8.** *Let $A,B \subset X$. We have:*

- $\overbrace{A\cap B}^{\circ} = \mathring{A}\cap\mathring{B}$ *and* $\overline{A\cup B} = \overline{A}\cup\overline{B}$,
- $\overbrace{A\cup B}^{\circ} \supset \mathring{A}\cup\mathring{B}$ *and* $\overline{A\cap B} \subset \overline{A}\cap\overline{B}$,
- $\partial(A\cup B) \subset \partial A \cup \partial B$.

**Exercise 17.** On the Euclidean line $\mathbb{R}$, give examples of sets $A$ and $B$ for which $\overbrace{A\cup B}^{\circ} \neq \mathring{A}\cup\mathring{B}$, and for which $\overline{A\cap B} \neq \overline{A}\cap\overline{B}$.

**Exercise 18** (Kuratowski axioms)**.** Given a set $X$ and a map $c\colon \mathscr{P}(X) \to \mathscr{P}(X)$, consider the properties

| | |
|---|---|
| **(K1)** $c(\emptyset) = \emptyset$ | **(K2)** $\forall A \subset X,\ A \subset c(A)$ |
| **(K3)** $\forall A \subset X,\ c(c(A)) = c(A)$ | **(K4)** $\forall A,B \subset X,\ c(A\cup B) = c(A)\cup c(B)$ |

Such a map $c$ is called a *closure operator*.

1. Given a topological space $(X,\mathscr{T})$, show that the map $A \mapsto \overline{A}$ is a closure operator on $X$.
2. Given a set $X$ and a closure operator $c\colon \mathscr{P}(X) \to \mathscr{P}(X)$, show that the collection $\{A \subset X \mid c(A) = A\}$ forms the closed set of a topology on $X$.
3. Show that the previous constructions are inverse to one another.

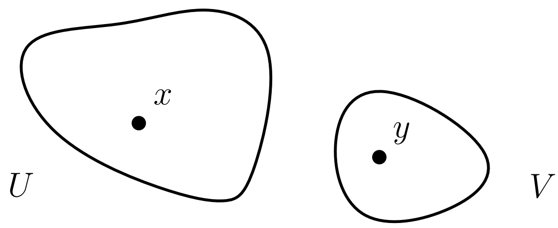
**Exercise 19** (Other formulation of Kuratowski axioms, [Per14, Exercise 5])**.** Show that the axioms **(K1)**, **(K2)**, **(K3)** and **(K4)** of Exercise 18 are equivalent to

$$\textbf{(K*)}\quad \forall A,B \subset X,\ A\cup c(A)\cup c(c(B)) = c(A\cup B)\setminus c(\emptyset).$$

## 2.2 Separation

The notion of separation captures the idea that any two points can be separated by non-intersecting open sets. Several variations of this notion exist: $T_0$-spaces, $T_1$-spaces, $T_2$-spaces, regular spaces, normal spaces, ... Here, we will only introduce one of them.

**Definition 2.9.** We say that a topological space $(X,\mathscr{T})$ is a *Hausdorff space* (or is a $T_2$-space) if for any $x,y \in X$ such that $x \neq y$, there exists neighborhoods $U,V$ of $x$ and $y$ such that $U\cap V = \emptyset$.



**Example 2.10.** The Euclidean space $(\mathbb{R}^n, \mathscr{T}_{\mathbb{R}^n})$ is Hausdorff. To prove, let $x, y \in X$ be such that $x \neq y$. Let $r = \|x-y\|$ be their distance. The balls $\mathscr{B}\left(x, \frac{r}{2}\right)$ and $\mathscr{B}\left(y, \frac{r}{2}\right)$ are neighborhoods of $x$ and $y$, and we have $\mathscr{B}\left(x, \frac{r}{2}\right) \cap \mathscr{B}\left(y, \frac{r}{2}\right) = \emptyset$.

**Proposition 2.11.** *If $(X, \mathscr{T})$ is a Hausdorff space, then all the singletons $\{x\}$, $x \in X$, are closed.*

**Proof.** Let us show that the complement ${}^c\{x\} = X \setminus \{x\}$ is open. We will show that it is a neighborhood of all of its points. Since $X$ is Hausdorff, for any $y \in X$ such that $y \neq x$, there exists an neighborhood of $y$ that does not contain $x$. Hence $X \setminus \{x\}$ is a neighborhood of $y$. □

**Exercise 20** (Separability of Zariski topology)**.** Show that the Zariski topology on $\mathbb{R}^n$ is not Hausdorff (see Exercise 6).

## 2.3 CONNECTEDNESS

**§2.3.1 CONNECTED SPACES** In a topological space, a set that is both open and closed will be called a *clopen* set.

**Definition 2.12.** Let $(X, \mathscr{T})$ be a topological space. We say that $X$ is *connected* if the only clopen sets are $\emptyset$ and $X$.

The following proposition shows that a connected topological space cannot be divided into two non-empty disjoint open sets, neither two non-empty disjoint closed sets.

**Proposition 2.13.** *The following assertions are equivalent:*

- *$(X, \mathscr{T})$ is connected,*
- *for every open sets $O, O'$ such that $O \cap O' = \emptyset$ and $X = O \cup O'$, we have $O = \emptyset$ or $O' = \emptyset$,*
- *for every closed sets $P, P'$ such that $P \cap P' = \emptyset$ and $X = P \cup P'$, we have $P = \emptyset$ or $P' = \emptyset$.*

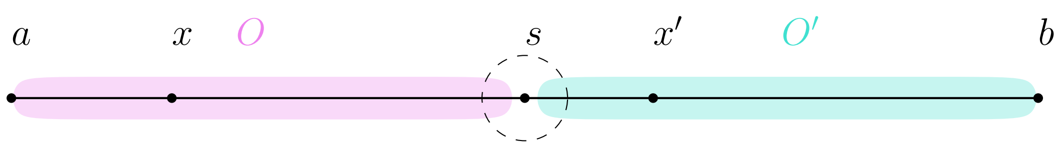
**Proof.** Let us suppose that $X$ is not connected, and let $C$ be a non-trivial clopen set. Then ${}^cC$ also is clopen, and $C \cup {}^cC$ gives the desired partition. □

If $A \subset X$ is a subset, we say that $A$ is *connected* if the topological space $(A, \mathscr{T}_{|A})$ for the subspace topology is connected (see §1.3.1).

**Example 2.14.** The subset $X = [0,1] \cup [2,3]$ of $\mathbb{R}$, endowed with the subspace topology, is not connected. Indeed, $[0,1]$ and $[2,3]$ are closed disjoint non-empty subsets that cover $X$.

**Proposition 2.15.** *Consider $\mathbb{R}$ for the Euclidean topology. For all $a, b \in \mathbb{R}$ such that $a \leq b$, the intervals $(a,b)$, $[a,b)$, $(a,b]$ and $[a,b]$ are connected.*

**Proof.** By contradiction, let us suppose that we can write $(a,b) = O \cup O'$ with $O, O'$ two non-empty disjoint open sets. Let $x \in O$ and $x' \in O'$. Without loss of generality, suppose that $x < x'$. Let $s$ be the supremum of $O \cap (a, x')$. Since $O'$ is open, we have $s < x'$.




By definition of the supremum, $O$ does not contain any open ball around $s$, hence $O$ does not contain $s$, since it is open. Similarly, $O'$ does not contain any open ball around $s$, hence $O'$ does not contain $s$. This is absurd. □

**Proposition 2.16.** *Let $(X, \mathscr{T})$ be a topological space, and $A \subset X$ a connected subset. Then its closure $\overline{A}$ is connected.*

**Proof.** Let $C$ be a clopen set of $\overline{A}$. By definition of the subspace topology, $C \cap A$ is a clopen set for $A$. Since $A$ is connected, $C \cap A$ must be $\emptyset$ of $A$. Without loss of generality, we can suppose that $C \cap A = A$ (otherwise, we replace $C$ with ${}^cC$). The relation $C \cap A = A$ is equivalent to $A \subset C$. Taking the closure, we get $\overline{A} \subset \overline{C} = C$. Moreover, $\overline{C} = C$ since $C$ is closed. Hence $\overline{A} = C$, proving the proposition. □

In the next section, we will introduce the notion of continuous function, and that of *path-connectedness*. This will be a handy tool to prove result about connectedness. In particular, we will show that the balls of $\mathbb{R}^n$ are connected, and more generally, that the convex subsets of $\mathbb{R}^n$ is connected.

> **Exercise 21.** Among the topologies on $X = \{0, 1\}$ (see Example 1.5), which ones yield connected spaces?

**§2.3.2 Connected components** If a space is not connected, we can consider its connected components.

**Definition 2.17.** Let $(X, \mathscr{T})$ be a topological space and $x \in X$. The *connected component* of $x$, denoted $\mathscr{C}(x)$, is defined as the union of all connected subsets $U \subset X$ that contain $x$.

**Proposition 2.18.** *A connected component is connected.*

**Proof.** By contradiction, suppose that $\mathscr{C}(x)$ is not connected, and let $\mathscr{C}(x) = O \cup O'$ be a partition in open sets. Without loss of generality, $x \in O$. Let $A$ be a connected subset of $X$ that contain $x$. We have a partition $A = (O \cap A) \cup (O' \cap A)$ in open sets, hence $A$ by connectedness, we must have $A \subset O$ or $A \subset O'$. Since $x \in A$, we deduce $A \subset O$. This being true for any connected subset $A$ containing $x$, we have $\mathscr{C}(x) = O$, and $O' = \emptyset$. We deduce the result. □

In other words, the connected component $\mathscr{C}(x)$ is the largest connected subspace that contains $x$. As a consequence of Proposition 2.16, every connected component is closed. In general, they may not be open, as shown in Exercise 22. However, this is true in the case of the Euclidean space, and its open subspaces.

Given two points $x, y \in X$, we have $y \in \mathscr{C}(x) \iff \mathscr{C}(x) = \mathscr{C}(y)$. Consequently, the set of connected components of $X$ forms a partition of $X$.

**Proposition 2.19.** *Let $(\mathbb{R}^n, \mathscr{T})$ be the Euclidean space. Let $O \subset \mathbb{R}^n$ be an open set, and consider the topological space $(O, \mathscr{T}_{|O})$ endowed with the subspace topology. Consider a point $x \in O$, and $\mathscr{C}(x)$ its connected component in $(O, \mathscr{T}_{|O})$. Then $\mathscr{C}(x)$ is an open set of $(\mathbb{R}^n, \mathscr{T})$, hence also of $(O, \mathscr{T}_{|O})$.*

**Proof.** Let $y \in \mathscr{C}(x)$. Since $O$ is open in $\mathbb{R}^n$, there exists a ball $\mathscr{B}(y, r)$ included in $O$. By definition of the connected component, we have $\mathscr{B}(y, r) \subset \mathscr{C}(y)$. Using that $\mathscr{C}(x) = \mathscr{C}(y)$, we deduce $\mathscr{B}(y, r) \subset \mathscr{C}(x)$, hence that $\mathscr{C}(x)$ is open in $\mathbb{R}^n$. □

**Remark 2.20.** The previous proposition is actually true for every *locally connected space*.

**Example 2.21.** Consider the subset $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ of $\mathbb{R}$. Each of its subsets $\{i\}$, $i \in X$, are open. They are all non-empty, connected and disjoint. Hence $X$ admits ten connected components.

**Exercise 22** (Connected components of $\mathbb{Q}$)**.** Let $\mathbb{Q}$ be endowed with the subspace Euclidean topology of $\mathbb{R}$.

1. Show that the connected components of $\mathbb{Q}$ are the singletons $\{x\}$, $x \in \mathbb{Q}$.
2. Show that the singletons are not open in $\mathbb{Q}$.

This shows that Proposition 2.19 is not true in general.

*Hint:* Remember that between two distinct rational numbers there exists an irrational number.

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