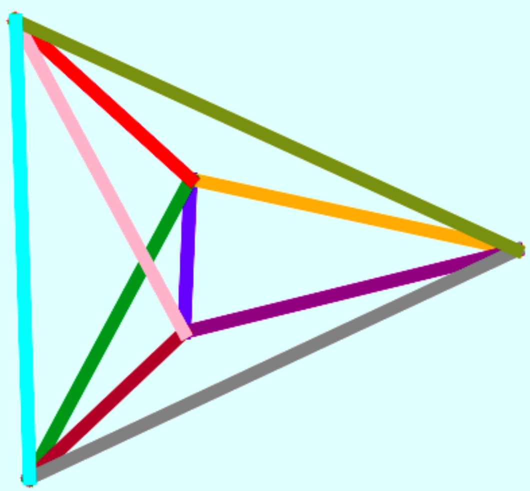
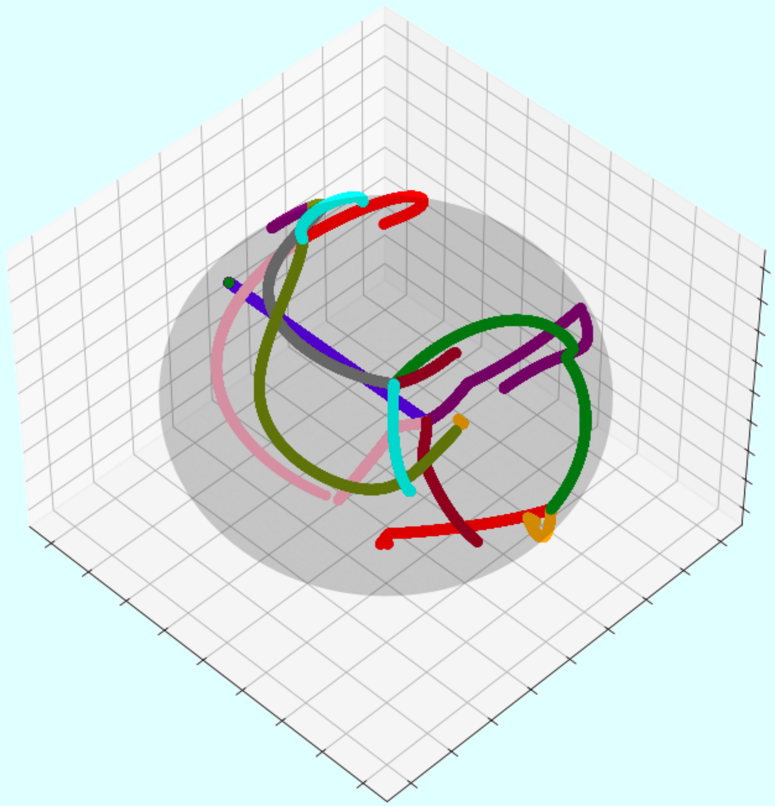
Minisymposium on Computational Topology — SoCG — 08/06/21

Simplicial approximation to CW-complexes in practice





https://raphaeltinarrage.github.io

Last update: June 8, 2021

Introduction

[T., Computing Persistent Stiefel-Whitney classes of line bundles]

Let X be a compact topological space. A **vector bundle** of dimension d on X can be described by a *classifying map* $f\colon X \to \mathcal{G}_d(\mathbb{R}^n)$ (Grassmannian of d planes in $\mathbb{R}^n$).

The map induced in cohomology, $f^*\colon \mathrm{H}^*(X) \leftarrow \mathrm{H}^*(\mathcal{G}_d(\mathbb{R}^n))$, allows to define the *Stiefel-Whitney* classes of the vector bundle

Computing these classes amouts to **triangulating** X and $\mathcal{G}_d(\mathbb{R}^n)$, and finding a simplicial approximation to f.

However, explicit triangulations of $\mathcal{G}_d(\mathbb{R}^n)$ are known only when $d = 1$ or $n - 1$ (projective spaces). Next Grassmannian: $\mathcal{G}_2(\mathbb{R}^4)$.

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[Luis Polanco, Jose A. Perea, *Coordinatizing Data With Lens Spaces and Persistent Cohomology* , 2019]

To each $\mathbb{Z}$-principal bundle over X corresponds (up to homotopy) a map $f\colon X \to \mathbb{S}^\infty/\mathbb{Z}_q$ (infinite Lens space).

This offers the Lens-PCA reduction, a non-linear dimensionality reduction algorithm.

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$\longrightarrow$ In this talk, we will construct simplicial complexes **homotopy equivalent** to a given space, based on its **CW-structure**.

I - Simplicial approximation to CW-complexes

1 - Topology CW-complexes

2 - Simplicial approximation

3 - Simplicial mapping cone

4 - Application to projective spaces

II - Simplicial approximation improved

1 - Local subdivisions

2 - Edge contractions

3 - Weak simplicial approximation

4 - Application to projective spaces, second attempt

III - Applications

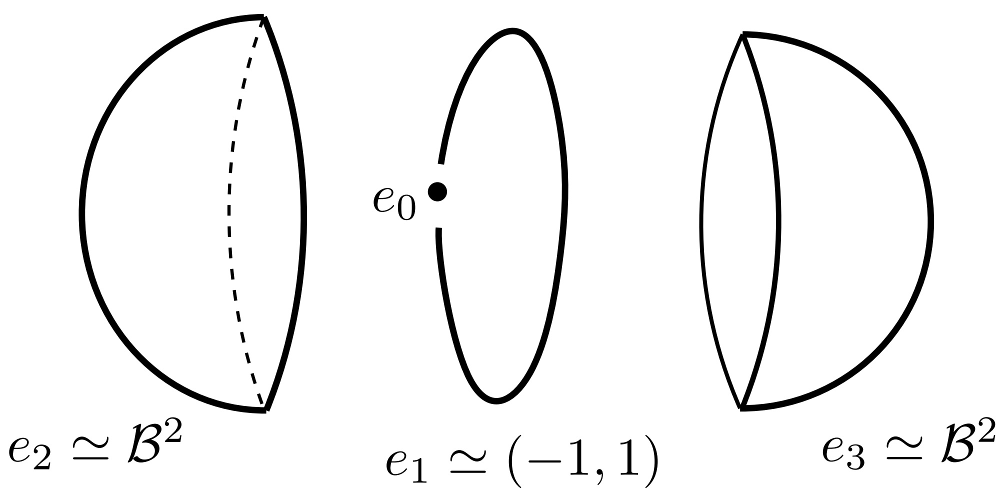
1 - Lens spaces

2 - Grassmannian $\mathcal{G}_2(\mathbb{R}^4)$

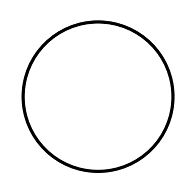
Definition: A finite **CW-complex** is a topological Hausdorff space X together with a finite partition $\{e_i\}_i$ of X (the cells) such that:

1. For each e_i, there exists an integer $n(i)$ and a homeomorphism $\mathcal{B}^{n(i)} \to e_i$, where $\mathcal{B}^{n(i)}$ is the open ball of $\mathbb{R}^{n(i)}$.
2. Moreover, this homeomorphism extends to a continuous map from the closed ball, $f_i \colon \overline{\mathcal{B}}^{n(i)} \to X$, called a **characteristic map** for e_i. We denote by $\overline{e}_i$ its image. Its restriction to the sphere, denoted $\phi_i \colon \partial\overline{\mathcal{B}}^{n(i)} \to X$, is called the **gluing map**.
3. Each point $x \in \overline{e}_i \setminus e_i$ must lie in a cell e_j of lower dimension.

Example: The sphere $\mathbb{S}^2$ admits a CW-structure with one cell of dimension 0, one cell of dimension 1, two cells of dimension 2.



$\partial(-1,1) = \{-1,1\}$ 

$\partial\mathcal{B}^2 = \mathbb{S}^1$ 

Definition of CW-complexes

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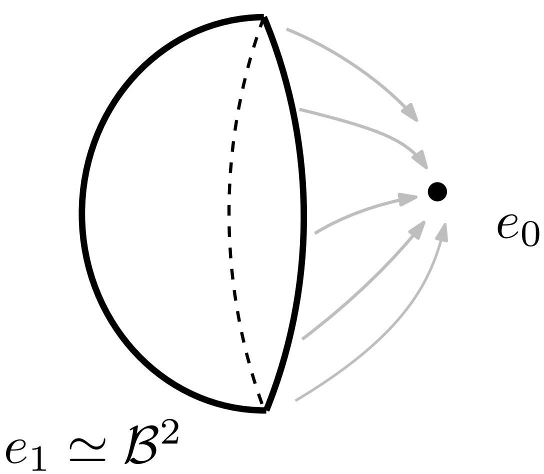

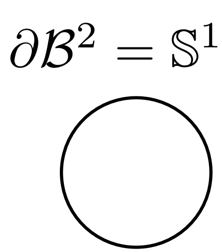

CW-complex structures

CW-complex structures provide concise descriptions of spaces:

- The **sphere** of dimension n: one 0-cell, and one n-cell
- The **projective space** of dimension n: one cell per dimension
- The **real Grassmannian** $\mathcal{G}_d(\mathbb{R}^n)$ (dimension $d(n-d)$): $p(k)$ cells of dimension k, where $p(k)$ is the number of partitions of k into at most d integers each of which is $\leq n-d$ (in total, $\binom{n}{d}$ cells) [Milnor, Stasheff, Characteristic classes]
- The **lens space** $L_p(q_1, \ldots, q_n)$ (dimension $2n-1$): one cell per dimension [Hatcher, Algebraic topology]

Triangulations are more complicated:

- The **sphere** of dimension n: at least $n+1$ vertices, $n+1$ facets, $2^n - 1$ simplices
- The **projective space** of dimension n: must have $\geq \frac{(n+1)(n+2)}{2}$ vertices [Arnoux, Marin, 1991]. We know a triangulation with $2^{n+1}-1$ vertices [Kühnel, 1987].
- The **real Grassmannian** $\mathcal{G}_d(\mathbb{R}^n)$ (dimension $d(n-d)$): a triangulation must admit $\frac{n(n+1)}{2}$ vertices and $\big(n(n-1) - 2k(n-k)\big)2^{k(n-k)+1} - 1$ simplices [Govc, Marzantowicz, Pavešić, 2020].

CW-complex structures

Cellularisation:

- Any topological manifold of dimension $d \neq 4$ is homeomorphic to a CW-complex [Kirby, Siebenmann, Quinn]
- Any topological manifold is homotopy equivalent to a CW complex [Kirby, Siebenmann, 1969]

Triangulations:

- Any PL-manifold is triangulable
- Any smooth manifold admits a PL-structure hence is triangulable [Cairns, 1935] [Whitehead, 1940]
- Any topological manifold is homotopy equivalent to a simplicial complex

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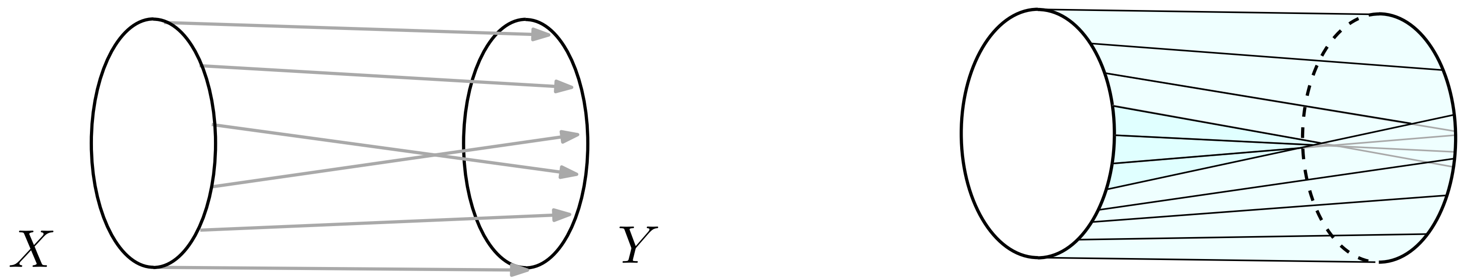
simplicial approximation to CW complexes

Mapping cones

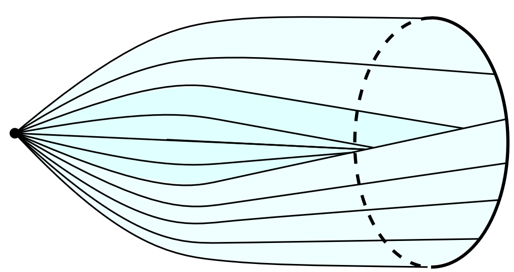
Let $f\colon X \to Y$ be a continuous map. The **mapping cylinder** is the quotient space

$$\mathrm{Cyl}(f) = X \times [0,1] \sqcup Y \;/\; (x,1) \sim f(x)$$



The **mapping cone** is obtained by identifying the upper part of the cylinder

$$\mathrm{Cone}(f) = \mathrm{Cyl}(f) \;/\; (x,0) \sim \text{point}$$



Lemma: If $f, g\colon X \to Y$ are homotopic, then so are $\mathrm{Cone}(f)$ and $\mathrm{Cone}(g)$.

Mapping cones

Let X be a CW-complex, with cells $\{e_i\}_i$, and gluing maps $\phi_i\colon \partial\mathcal{B}^{n(i)} \to X$.

One shows that X is homeomorphic to the sequence of mapping cones

$$
\begin{tikzcd}
e_0 \arrow[d, "\simeq"] & e_0 \cup e_1 \arrow[d, "\simeq"] & e_0 \cup e_1 \cup e_2 \arrow[d, "\simeq"] & e_0 \cup e_1 \cup e_2 \cup e_3 \arrow[d, "\simeq"] & \\
\{\mathrm{point}\} \arrow[r, hook] & \mathrm{Cone}(\phi_1) \arrow[r, hook] & \mathrm{Cone}(\phi_2) \arrow[r, hook] & \mathrm{Cone}(\phi_3) \arrow[r, hook] & \dots \\
& \partial\mathcal{B}^{n(1)} \arrow[ul, "\phi_1"'] & \partial\mathcal{B}^{n(2)} \arrow[ul, "\phi_2"'] & \partial\mathcal{B}^{n(3)} \arrow[ul, "\phi_3"'] &
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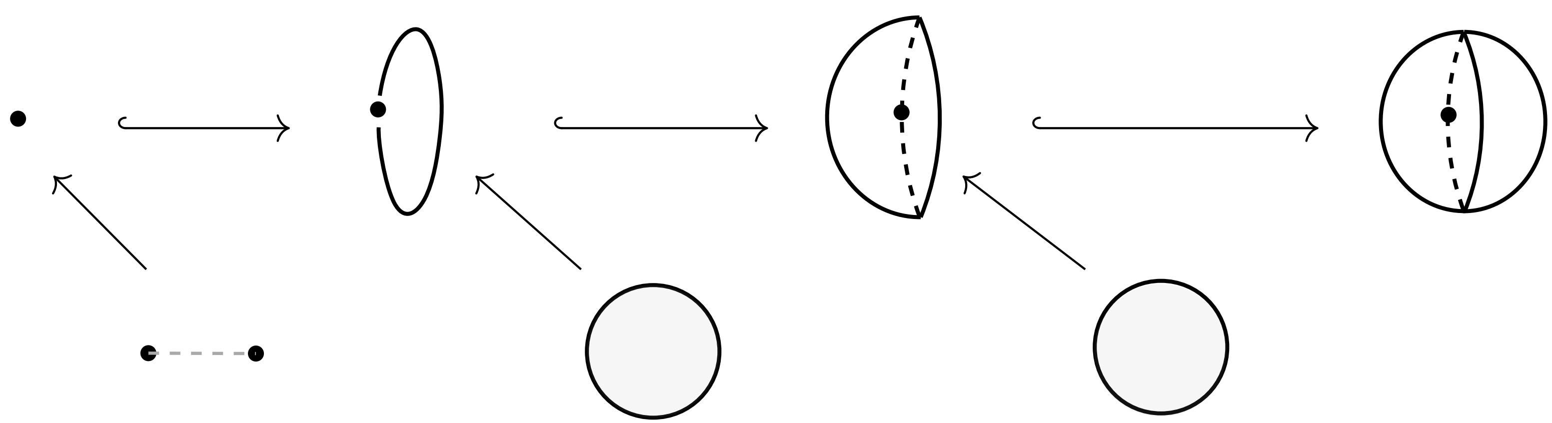

One builds X by induction: $X^i = \mathrm{Cone}(\phi_i\colon \partial\mathcal{B}^{n(i)} \to X^{i-1})$.

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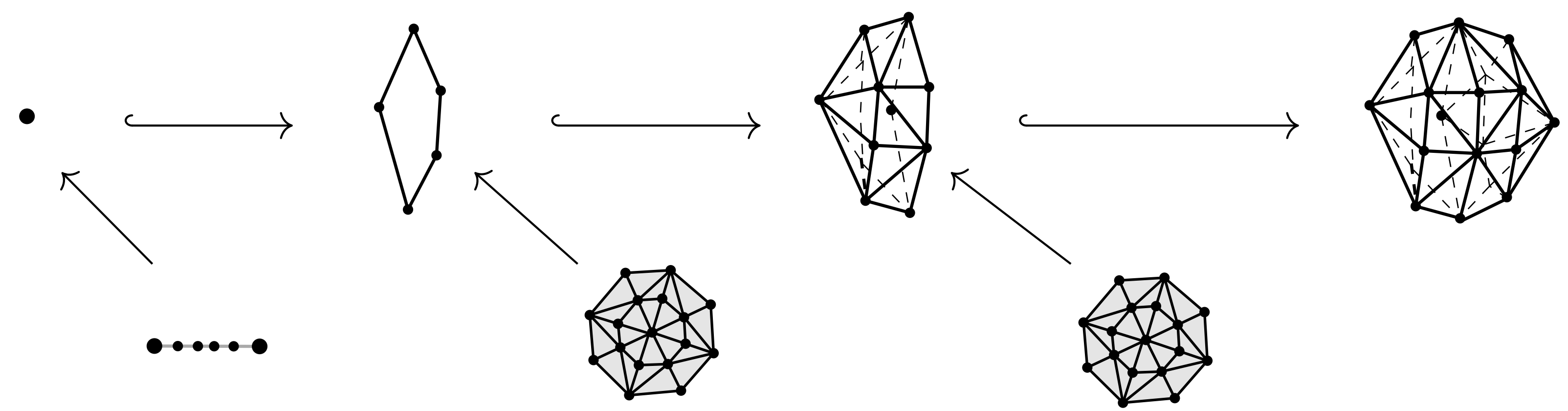
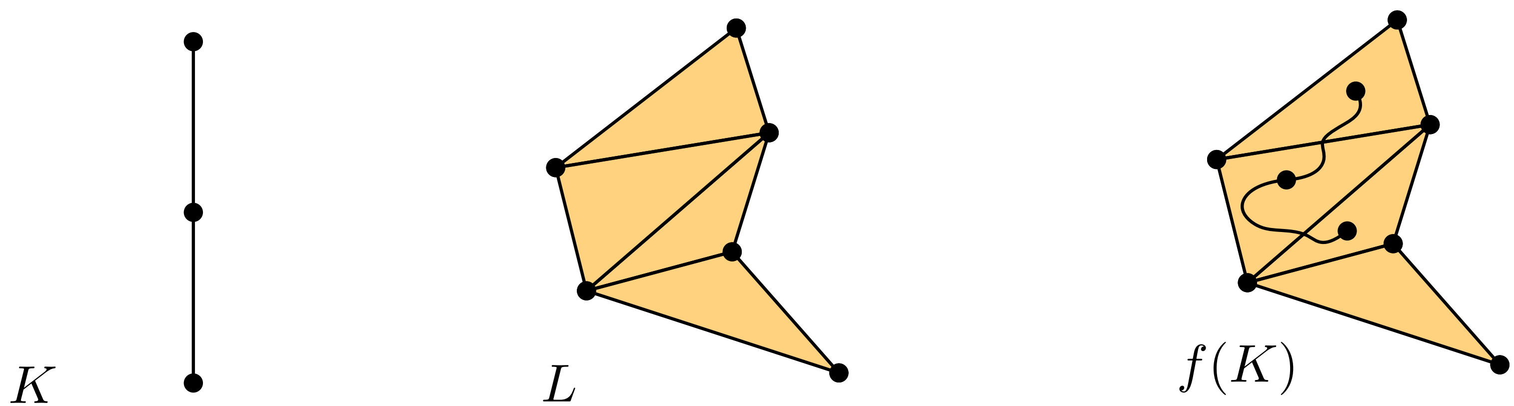


In order to triangulate X, it is enough to triangulate mapping cones!

Simplicial approximation

Consider two simplicial complexes K, L and a continuous map between their geometric realizations $f\colon\ |K| \longrightarrow |L|$.
We look for a simplicial map $g\colon K \longrightarrow L$ whose geometric realization $|g|\,:\ |K| \to |L|$ is homotopic to f.



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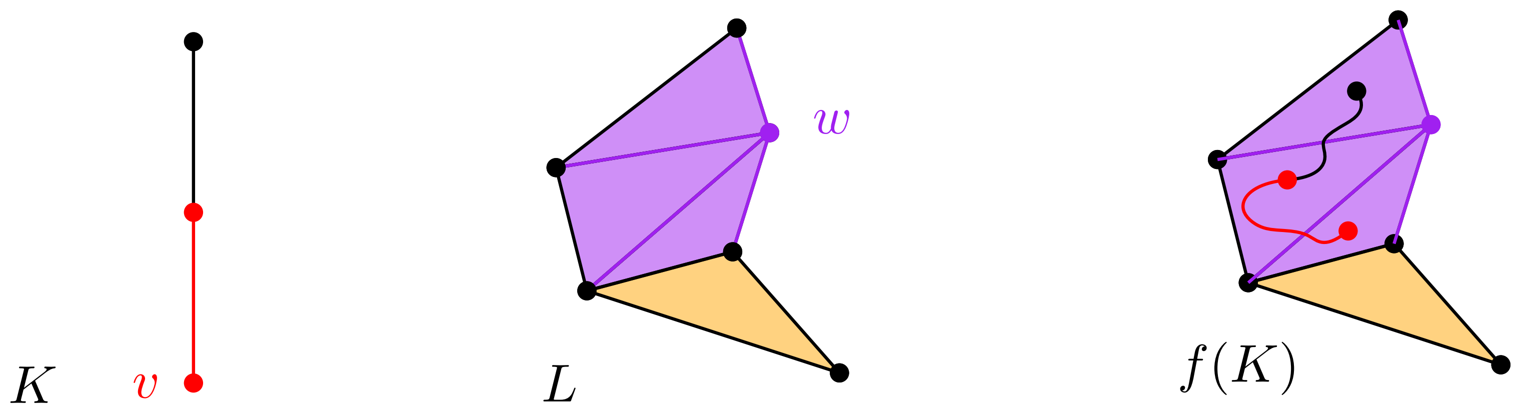
Define, for all vertex $v \in K^{(0)}$, its *open star* and its *closed star*

$$\mathrm{St}(v) = \{\sigma \in K,\ v \in \sigma\} \qquad\qquad \overline{\mathrm{St}}(v) = \{\tau \in K,\ \exists \sigma \in \mathrm{St}(v),\ \tau \subset \sigma\}$$

The map f satisfies the **star condition** for every $v \in K^{(0)}$, there exists $w \in L^{(0)}$ such that $f\left(\left|\overline{\mathrm{St}}(v)\right|\right) \subseteq |\mathrm{St}(w)|$.

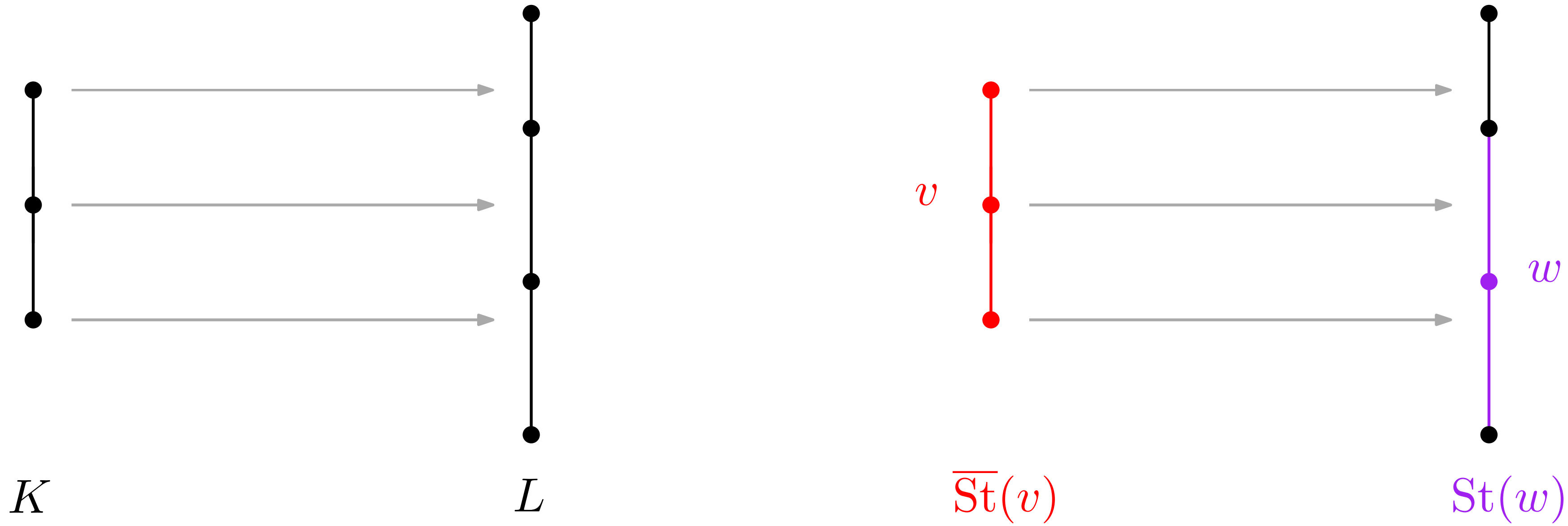
If this is the case, let $g\colon K^{(0)} \to L^{(0)}$ be such that for every $v \in K^{(0)}$, we have $f\left(\left|\overline{\mathrm{St}}(v)\right|\right) \subseteq |\mathrm{St}(g(v))|$.

Such a map g is called a **simplicial approximation** to f. It is a simplicial map.
Its geometric realization $|g|$ is homotopic to f.

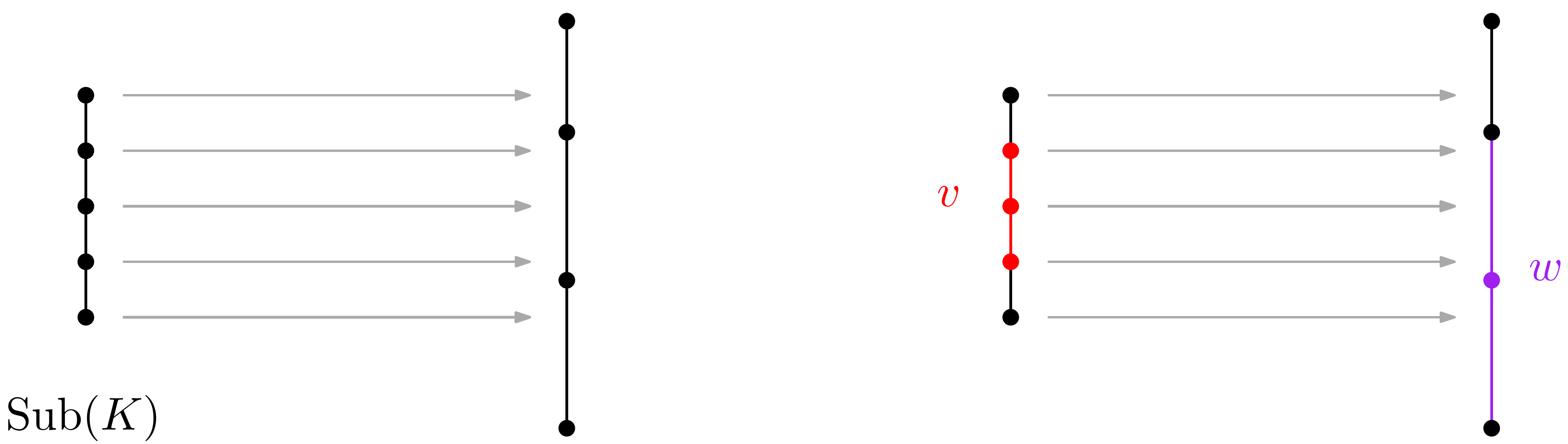


Barycentric subdivisions

What if the map f does not satisfy the star condition?

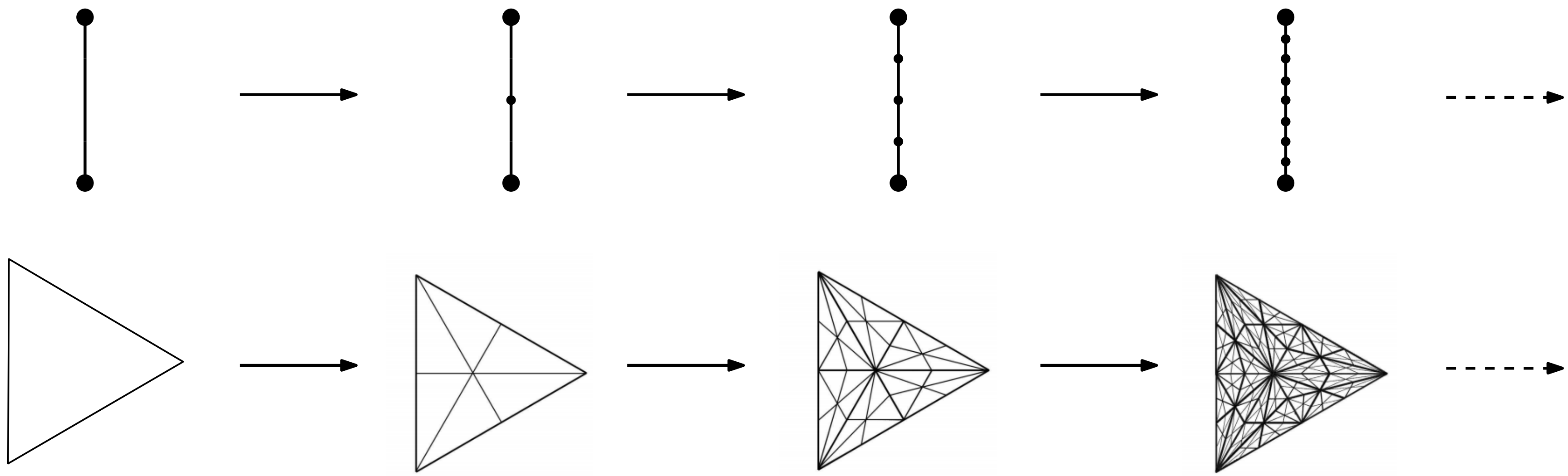


One can refine K via *barycentric subdivision*.



Barycentric subdivisions

Simplicial approximation theorem: By repeating barycentric subdivisions on K, the map $f\colon \mathrm{Sub}^k(K) \to L$ satisfies the star condition at some point.



Proof: Endow K with a metric, and by denote $\mathcal{U}$ the cover $\left\{f^{-1}(|\mathrm{St}(w)|), w \in L\right\}$ of $|K|$.
A **Lebesgue number** for $\mathcal{U}$ is a $\epsilon > 0$ such that

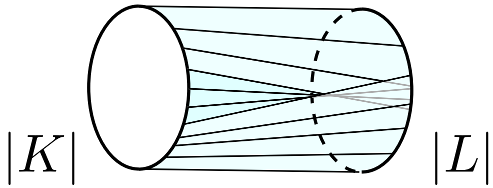
$$\forall x \in |K|,\ \exists U \in \mathcal{U},\ \mathcal{B}(x,\epsilon) \subset U \text{ (open ball of radius } \epsilon\text{)}.$$

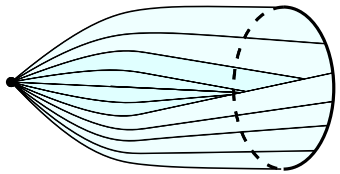
Hence, f satisfies the star condition if for every $v \in K^{(0)},\ \mathrm{Diameter}\left(\left|\overline{\mathrm{St}}(v)\right|\right) < \epsilon$.

But barycentric subdivision reduces the diameter of a d-simplex by a factor $\frac{d}{d+1}$. Hence each simplex is small enough at some point.

Triangulation of mapping cones

How to triangulate the mapping cone of $f\colon |K| \to |L|$?





$$\mathrm{Cyl}(f) = X \times [0,1] \sqcup Y \ / \ (x,1) \sim f(x)$$

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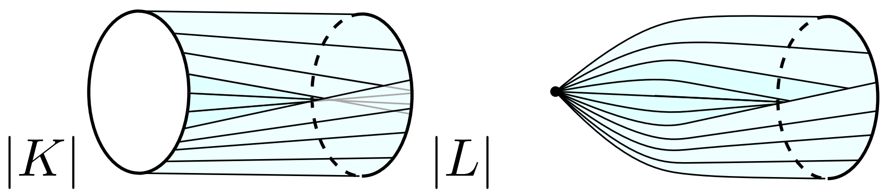

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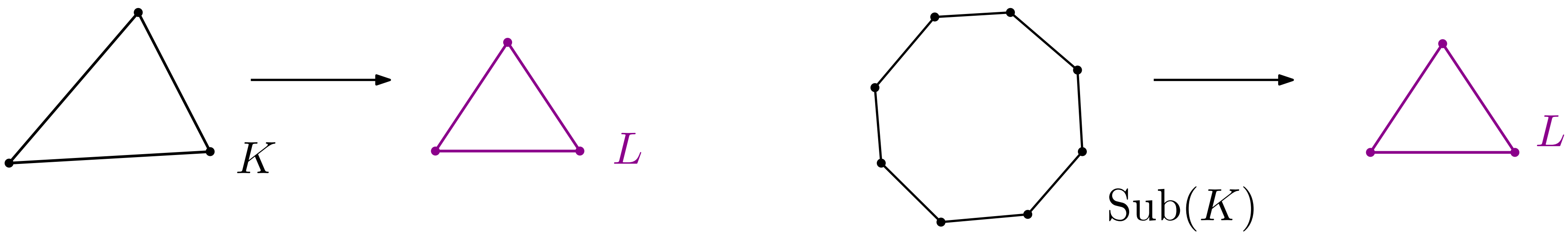
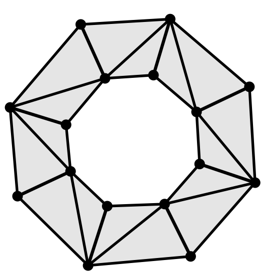
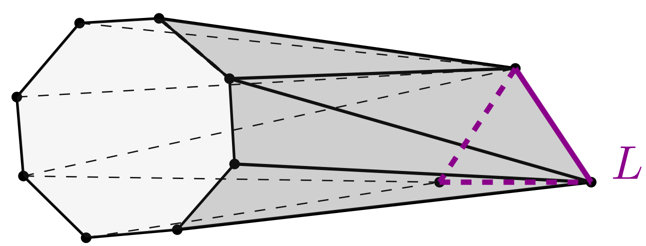
1. Find a simplicial approximation $g\colon K \to L$ to $f\colon |K| \to |L|$



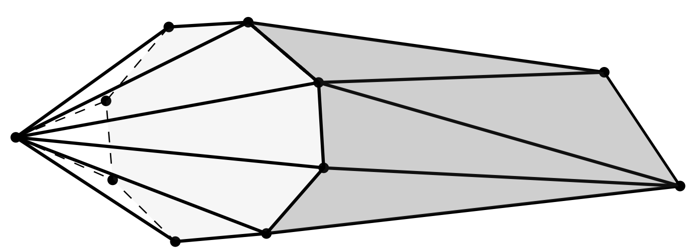
2. Triangulate $|K| \times [0,1]$ and glue L at the end





3. Cone the upper part of the cylinder

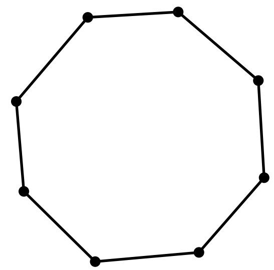
$\mathrm{Cyl}_{\mathrm{simplicial}}(g)$



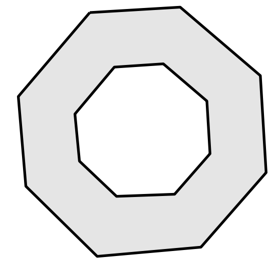
Triangulation of mapping cylinders

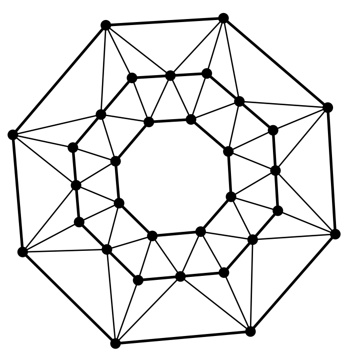
2. Let K be a simplicial complex. How to triangulate $|K| \times [0,1]$?

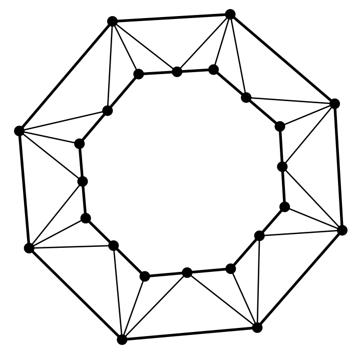
K

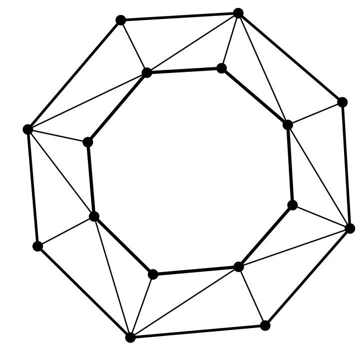


$K \times [0,1]$





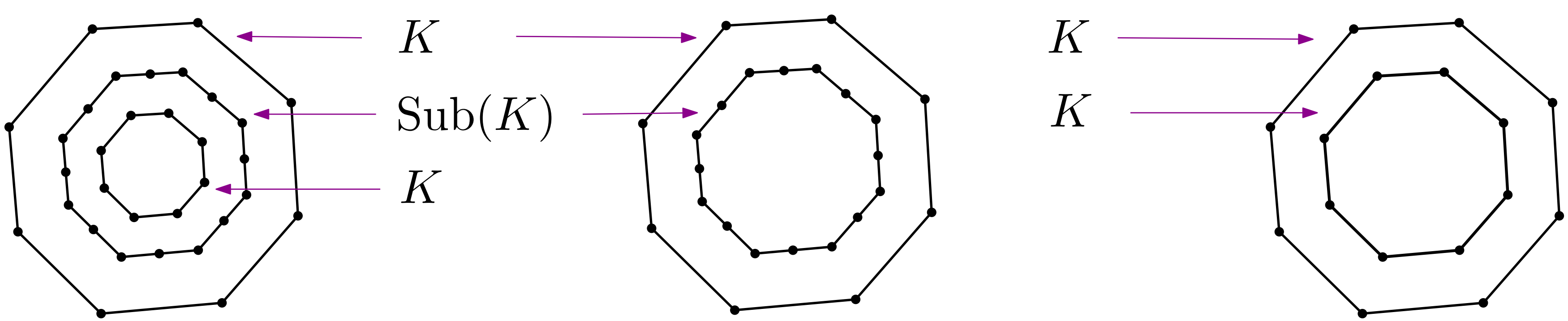




[Whitehead, *Simplicial spaces, nuclei and m-groups*, 1939]

[Cohen, *Simplicial structures and transverse cellularity*, 1967]

Simplicial set product



Triangulation of mapping cylinders

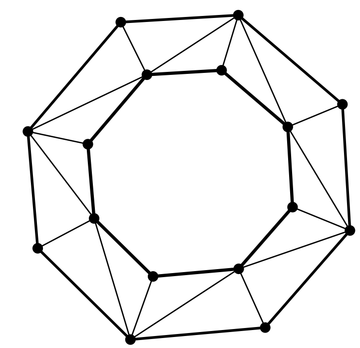
2. Let K be a simplicial complex. How to triangulate $|K| \times [0,1]$?

Suppose that the vertices of K are totally ordered.

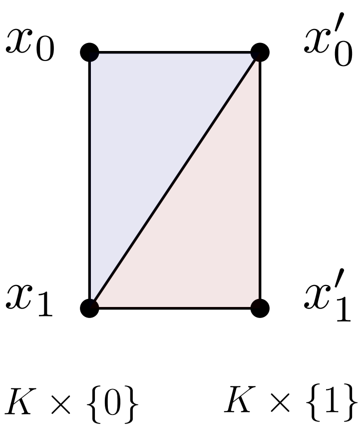
- Take two copies of K, $K \times \{0\}$ and $K \times \{1\}$.
- For every $\sigma = [x_0, \dots, x_m] \in K \times \{0\}$, with vertices ordered, consider the corresponding vertices $x_0' < \cdots < x_m'$ of $K \times \{1\}$.
- For every $0 \leq k < m$, consider the $m+1$ simplex

$$\sigma_k = [x_k, \dots, x_m, x_0', \dots x_k'] \in K \star K.$$

- The collection of all $\sigma_k \in K \star K$ for all $\sigma \in K$ form the simplicial product.



Simplicial set product



$[x_0, x_1, x_0']$

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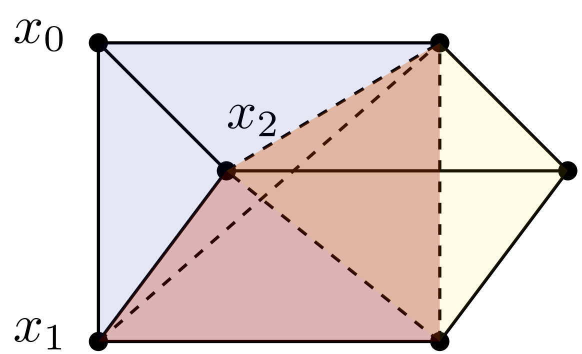

$[x_0, x_1, x_2, x_0']$

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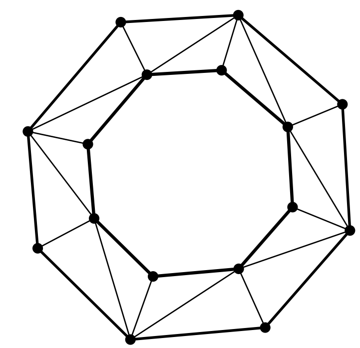
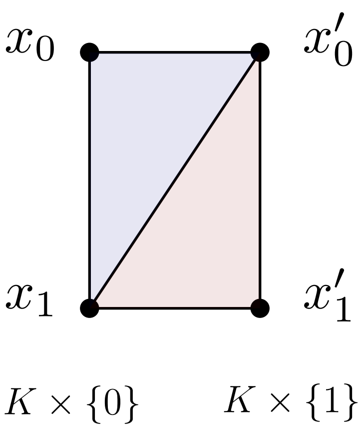
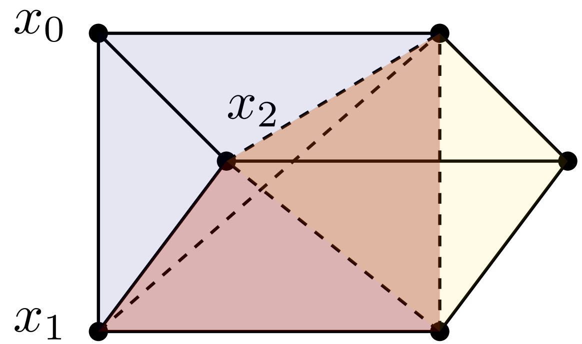


Simplicial set product



$[x_0, x_1, x'_0]$

$[x_1, x'_0, x'_1]$



$[x_0, x_1, x_2, x'_0]$

$[x_1, x_2, x'_0, x'_1]$

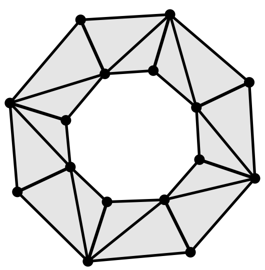
$[x_2, x'_0, x'_1, x'_2]$

$\longrightarrow$ This construction depends on a choice (of an ordering)!

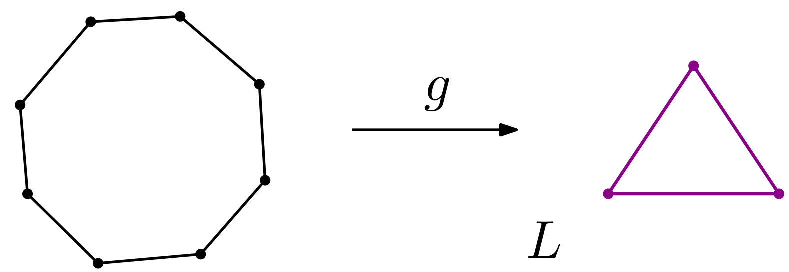
Triangulation of mapping cylinders

Once $K \times [0,1]$ is triangulated, we glue it to L via $g\colon K \to L$ as follows:

$$\mathrm{Cyl}_{\mathrm{simplicial}}(g) = \left\{ \sigma_0 \sqcup g(\sigma_1),\ \sigma \in K \times [0,1],\ \sigma = \sigma_0 \sqcup \sigma_1, \left| \begin{array}{l} \sigma_0 \in K \times \{0\} \\ \sigma_1 \in K \times \{1\} \end{array} \right. \right\}$$

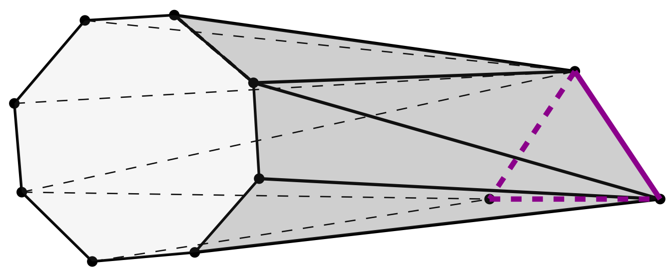


$K \times [0,1]$



K

L



$\mathrm{Cyl}_{\mathrm{simplicial}}(g)$

Triangulation of mapping cylinders

Once $K \times [0,1]$ is triangulated, we glue it to L via $g\colon K \to L$ as follows:

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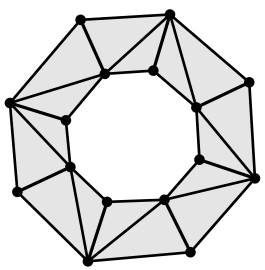
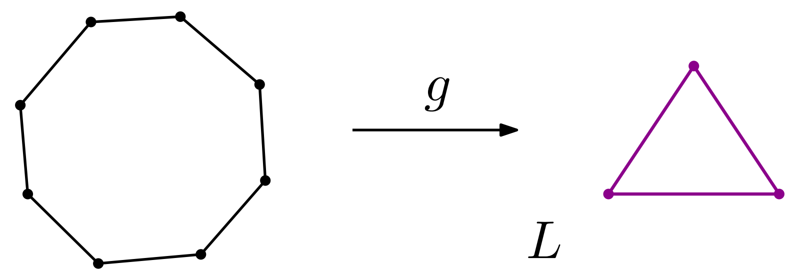
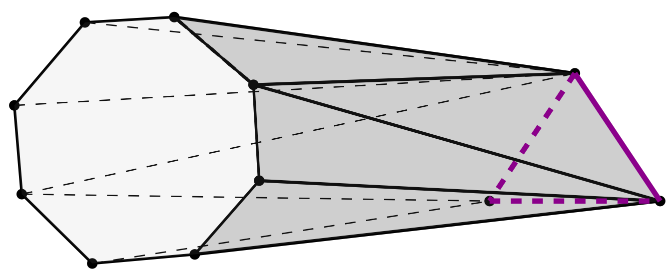


$K \times [0,1]$

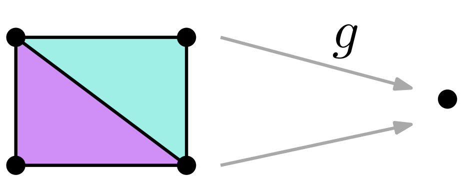


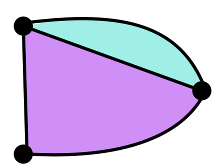
K $\qquad$ L



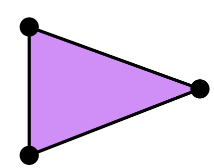
$\mathrm{Cyl}_{\mathrm{simplicial}}(g)$

The map $\mathrm{Cyl}(g) \to \mathrm{Cyl}_{\mathrm{simplicial}}(g)$ may not be a homeomorphism!





$\mathrm{Cyl}(g)$



$\mathrm{Cyl}_{\mathrm{simplicial}}(g)$

Proposition: It is a homotopy equivalence.

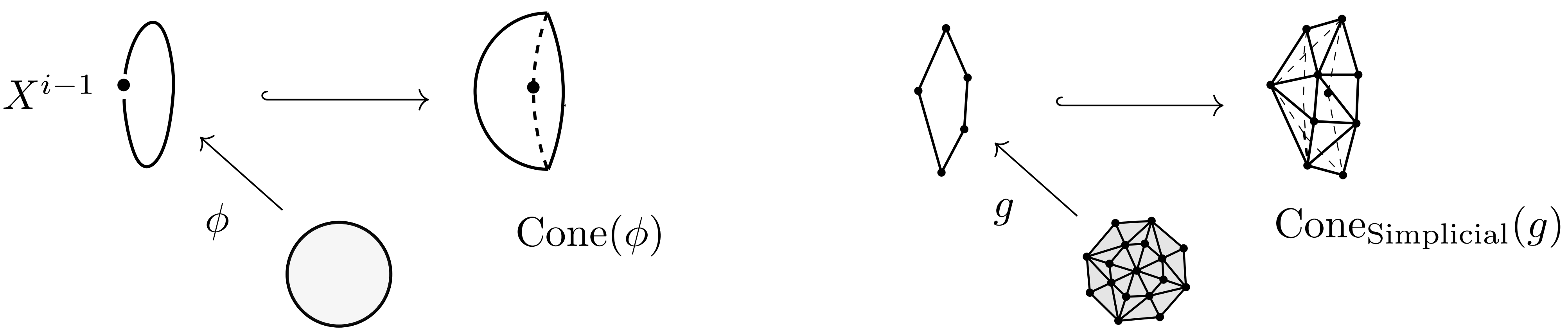
Sketch of algorithm.

Let X be a CW-complex, with cells $\{e_i\}_i$, and gluing maps $\phi_i \colon \partial\mathcal{B}^{n(i)} \to X$.

It can be built by induction: $X^i = \mathrm{Cone}(\phi_i \colon \partial\mathcal{B}^{n(i)} \to X^{i-1})$.

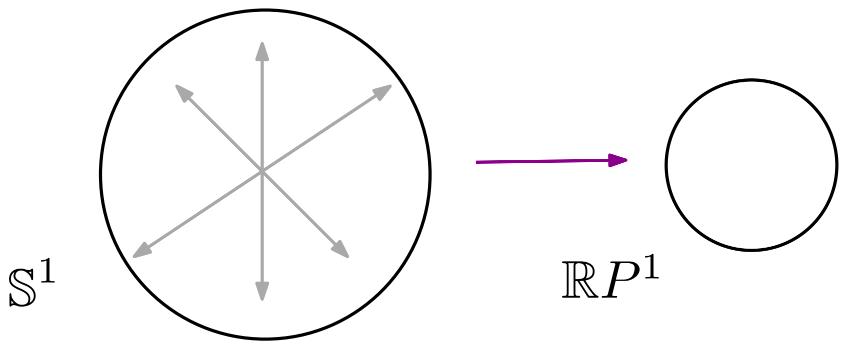
We will do the same, but with simplicial approximations g_i to ϕ_i instead:

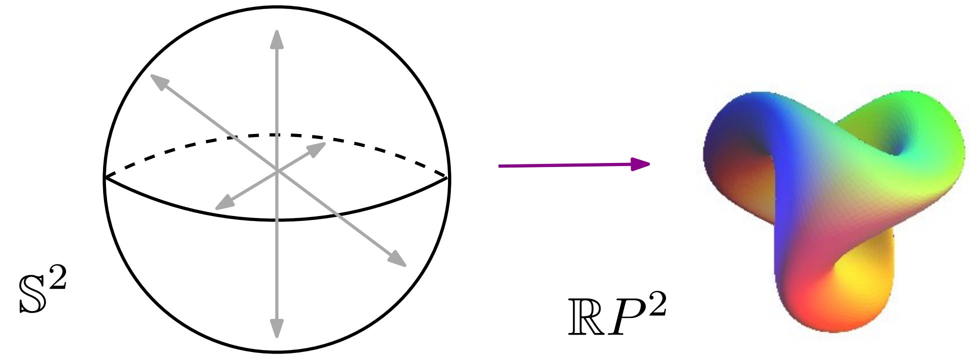
$$K^i = \mathrm{Cone}_{\mathrm{simplicial}}(g_i \colon \partial\mathcal{B}^{n(i)} \to K^{i-1}).$$



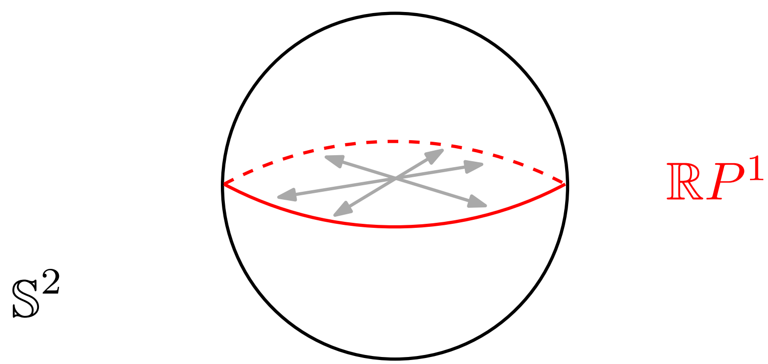
CW-structure for projective spaces

The n-dimensional projective space $\mathbb{R}P^n$ can be defined as the quotient of the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by the antipodal relation $x \sim -x$.

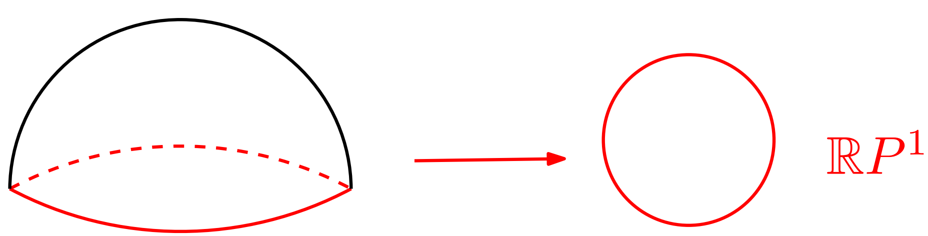




The projective space $\mathbb{R}P^n$ contains $\mathbb{R}P^{n-1}$ as its equator.



By induction, we get a CW-structure for $\mathbb{R}P^n$: take a hemisphere of $\mathbb{S}^n$, and glue its equator on $\mathbb{R}P^{n-1}$.

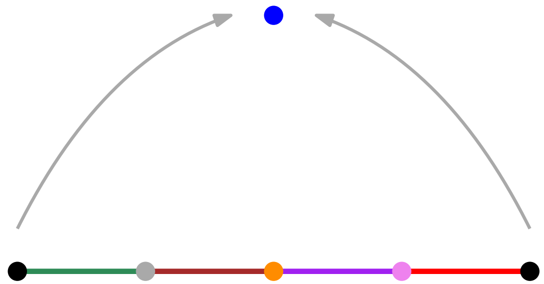


$\mathbb{R}P^0$:

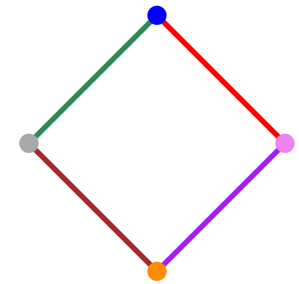
Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

1-cell:



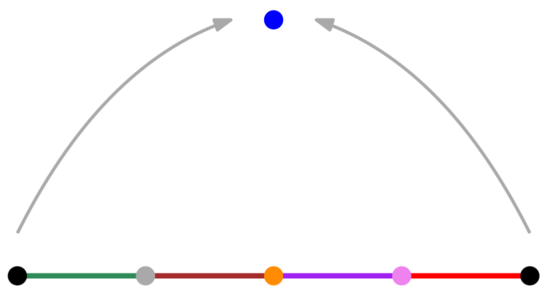
$\mathbb{R}P^1$:



Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

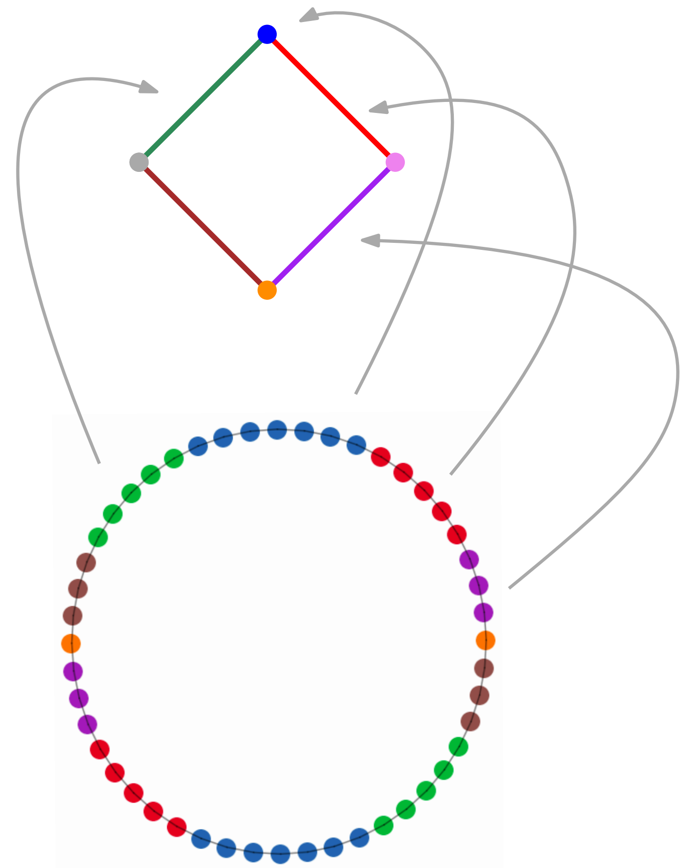
1-cell:



$\mathbb{R}P^1$:

boundary
of 2-cell:

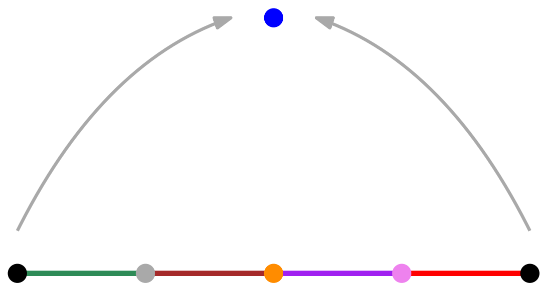
4 barycentric
subdivisions



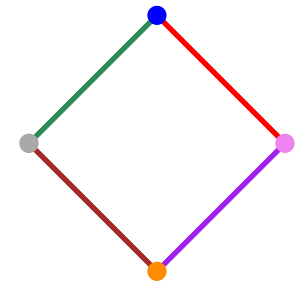
Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

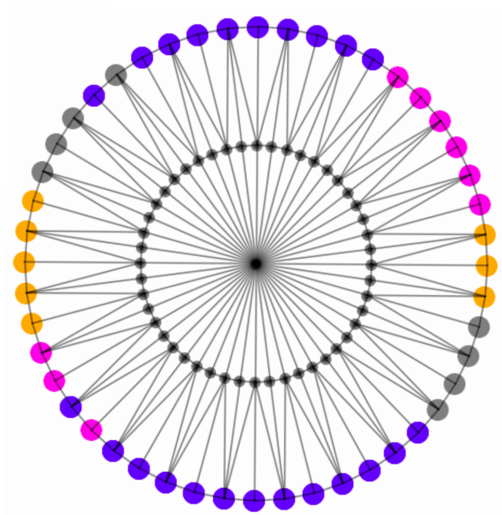
1-cell:



$\mathbb{R}P^1$:

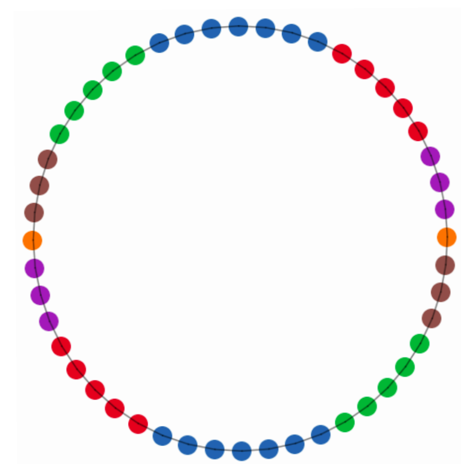


2-cell:



boundary of 2-cell:

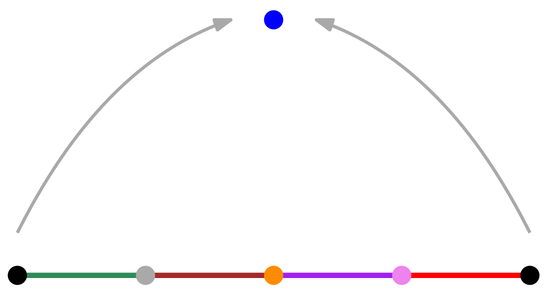
4 barycentric subdivisions



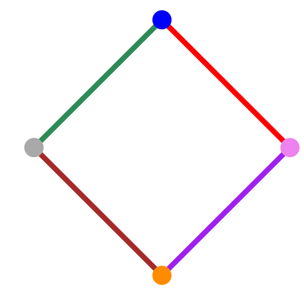
$\mathbb{R}P^2$: 53 vertices, 317 simplices.

Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

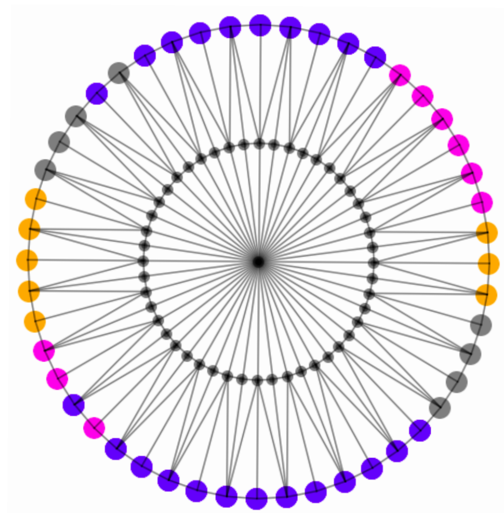


1-cell:



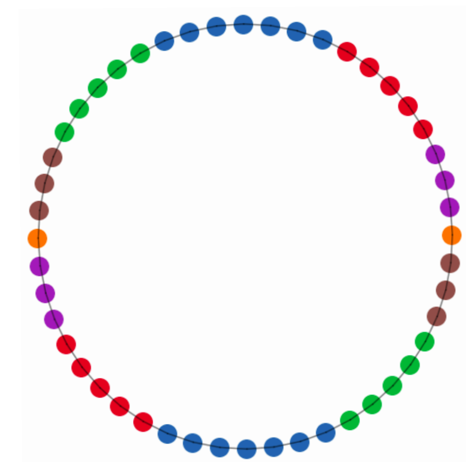
$\mathbb{R}P^1$:

2-cell:



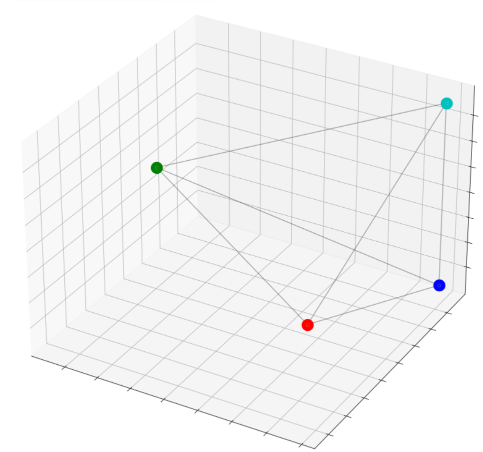
boundary of 2-cell:

4 barycentric subdivisions



$\mathbb{R}P^2$: 53 vertices, 317 simplices.

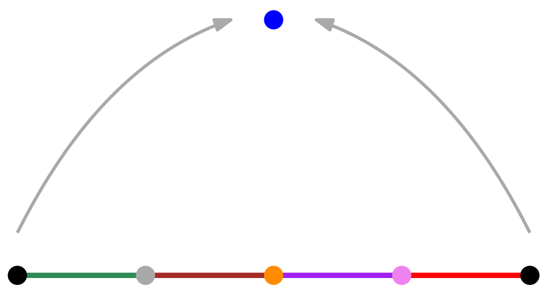
boundary of 3-cell:



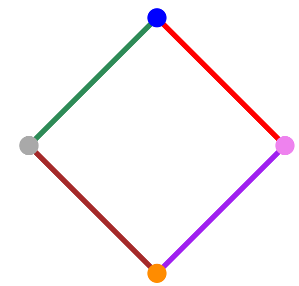
Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

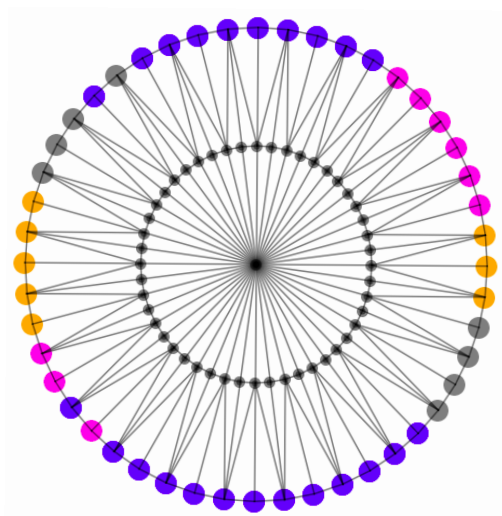
1-cell:



$\mathbb{R}P^1$:

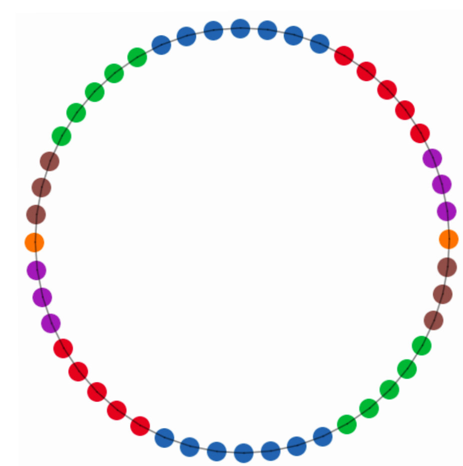


2-cell:



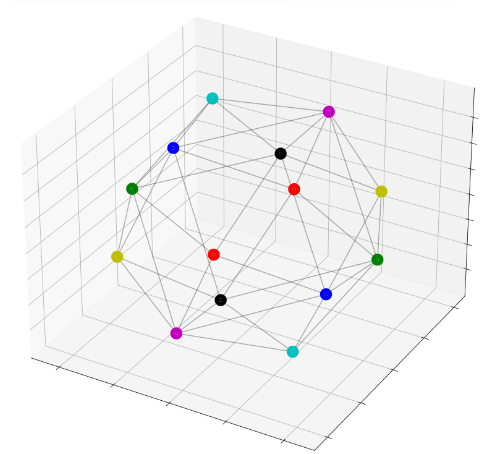
boundary of 2-cell:

4 barycentric subdivisions



$\mathbb{R}P^2$: 53 vertices, 317 simplices.

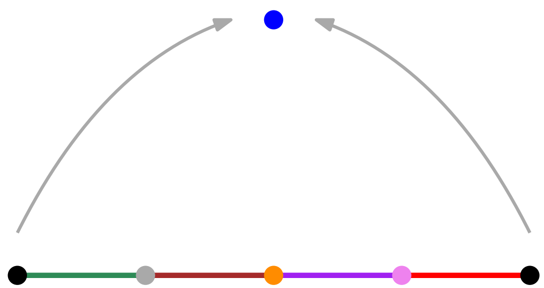
boundary of 3-cell:



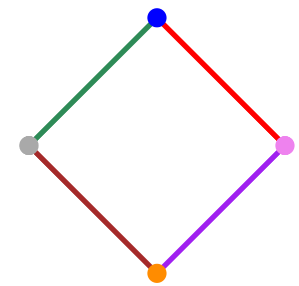
1 barycentric subdivision

Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

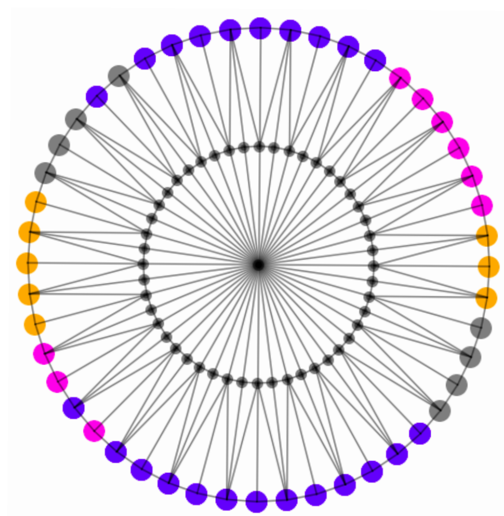


1-cell:



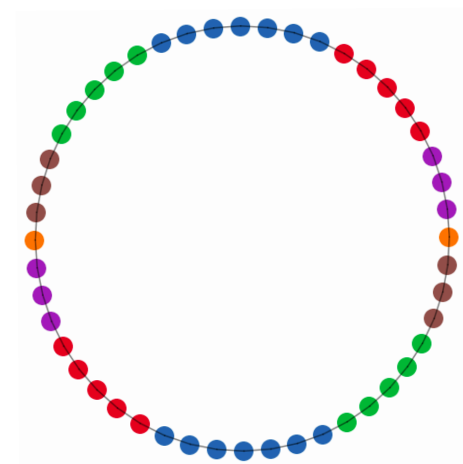
$\mathbb{R}P^1$:

2-cell:



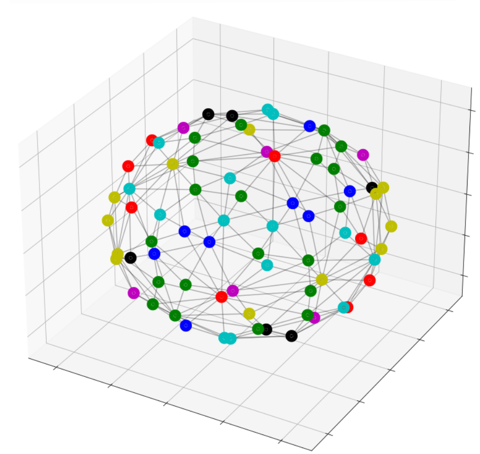
boundary
of 2-cell:

4 barycentric
subdivisions



$\mathbb{R}P^2$: 53 vertices, 317 simplices.

boundary
of 3-cell:

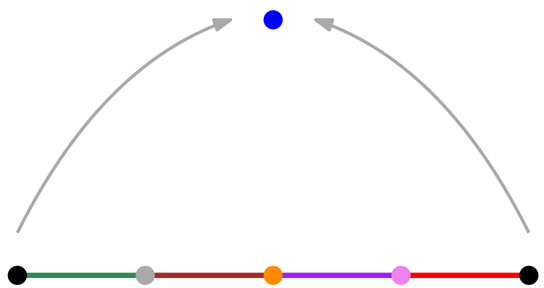


2 barycentric subdivisions

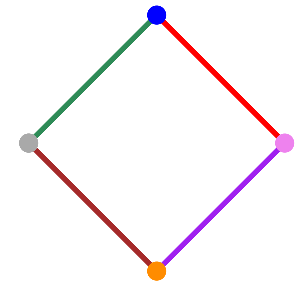
Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

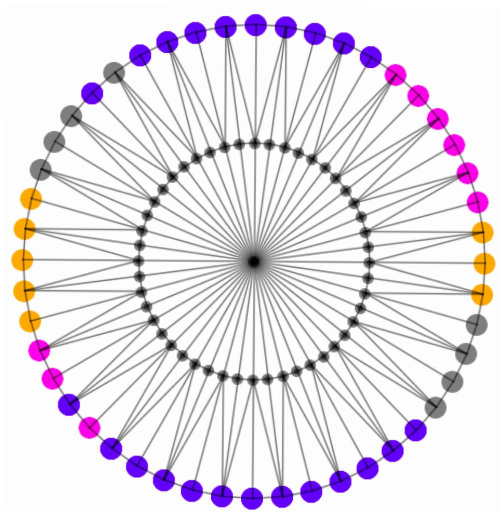
1-cell:



$\mathbb{R}P^1$:

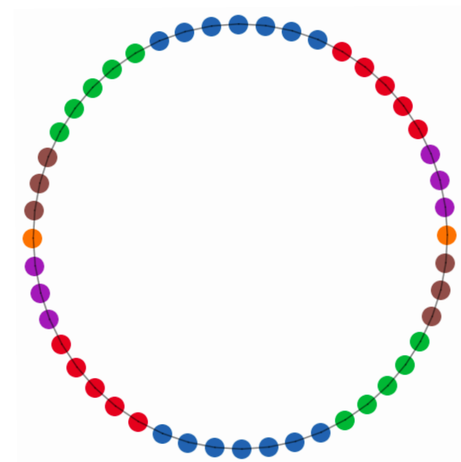


2-cell:



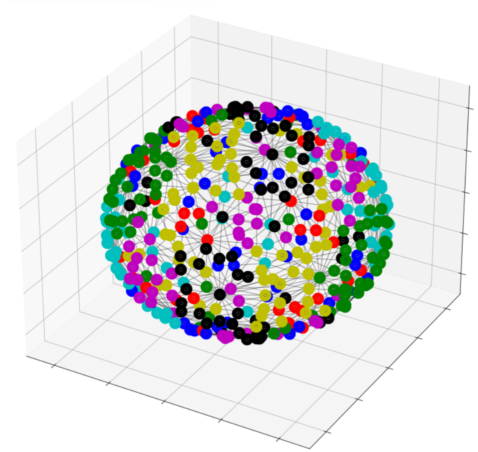
boundary of 2-cell:

4 barycentric subdivisions



$\mathbb{R}P^2$: 53 vertices, 317 simplices.

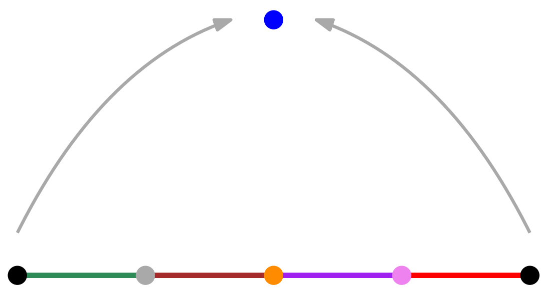
boundary of 3-cell:



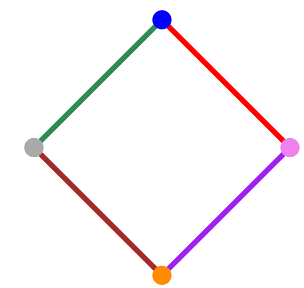
3 barycentric subdivisions

Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

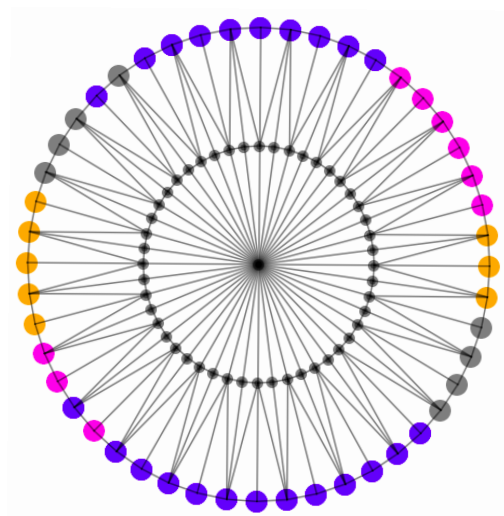


1-cell:



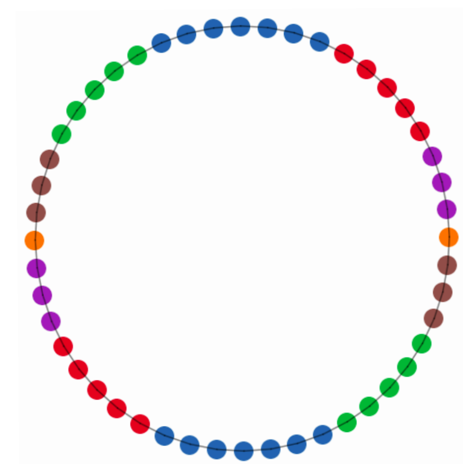
$\mathbb{R}P^1$:

2-cell:



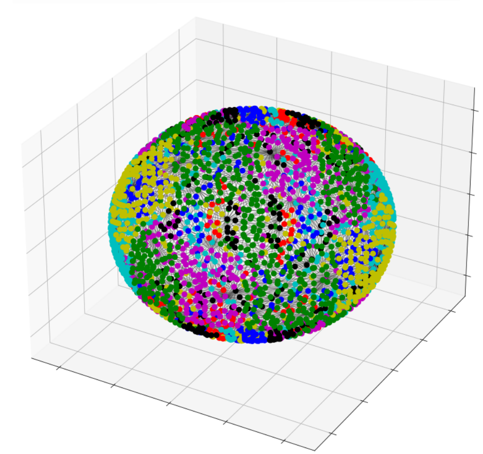
boundary of 2-cell:

4 barycentric subdivisions



$\mathbb{R}P^2$: 53 vertices, 317 simplices.

boundary of 3-cell:

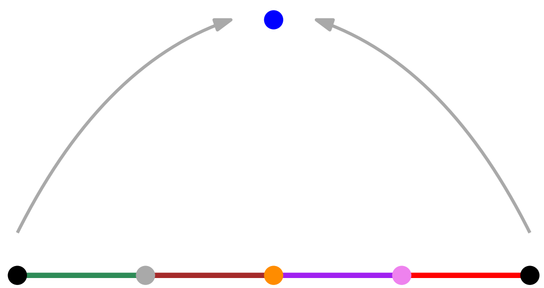


4 barycentric subdivisions

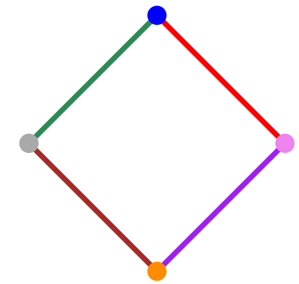
Simplicial approximation to $\mathbb{R}P^n$

$\mathbb{R}P^0$:

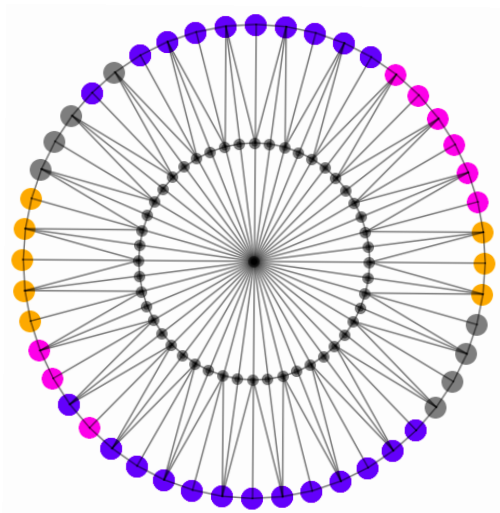
1-cell:



$\mathbb{R}P^1$:

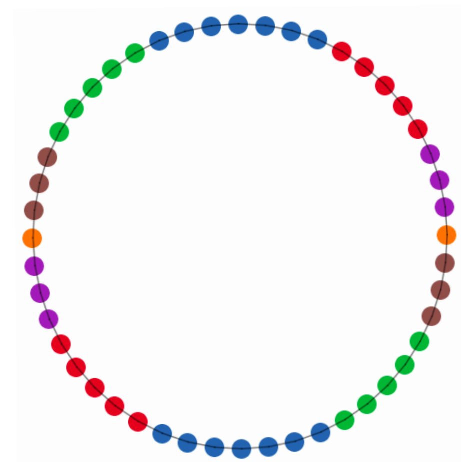


2-cell:



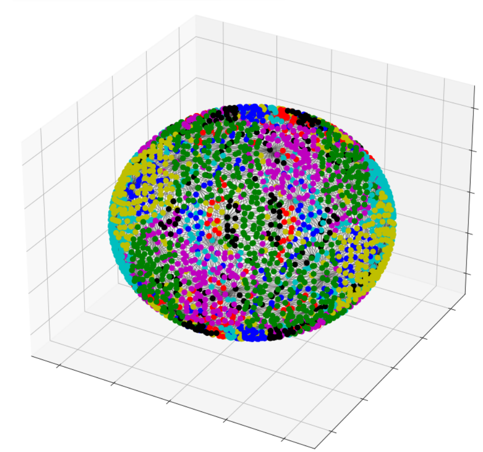
boundary of 2-cell:

4 barycentric subdivisions



$\mathbb{R}P^2$: 53 vertices, 317 simplices.

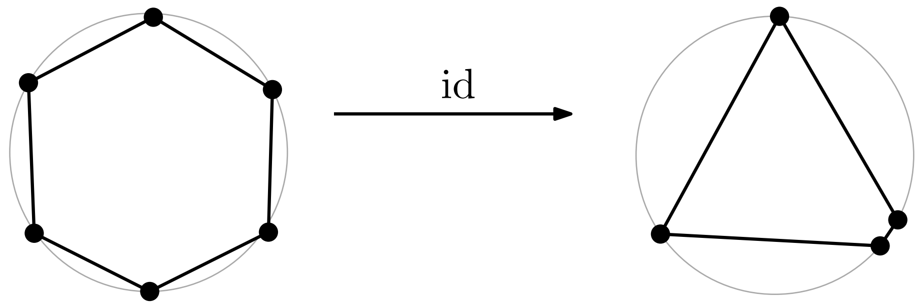
boundary of 3-cell:



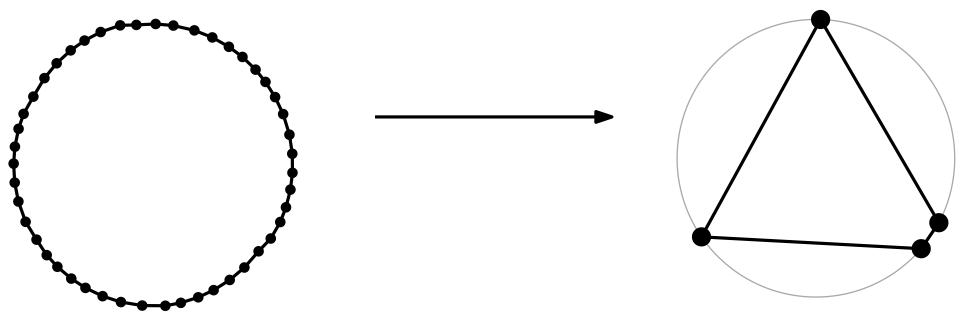
$\mathbb{R}P^3$: $560'024$ vertices, $11'521'686$ simplices (after 7 barycentric subdivisions)

Problem with barycentric subdivision

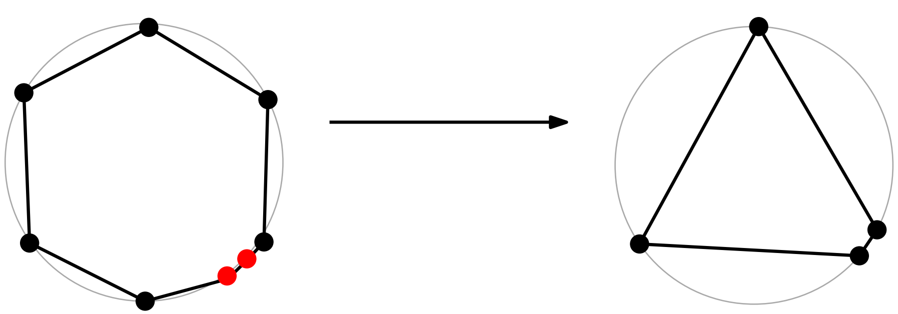
Suppose that we want to find a simplicial approximation to the identity map on $\mathbb{S}^1$, with triangulations given as:



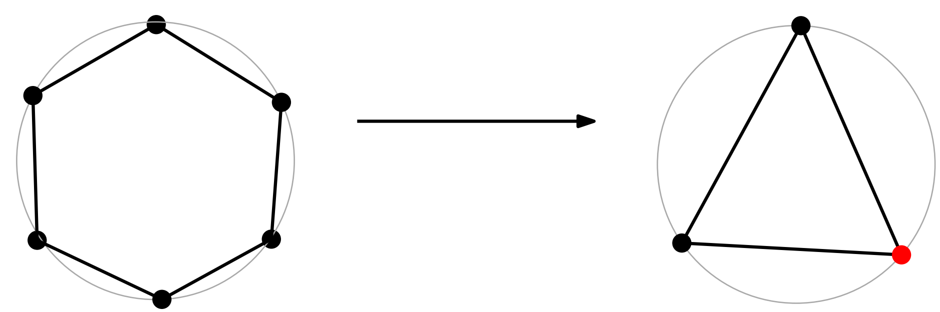
We must subdivise the domain many times to ensure that id satisfies the star condition.



We could have done simpler:



local subdivision



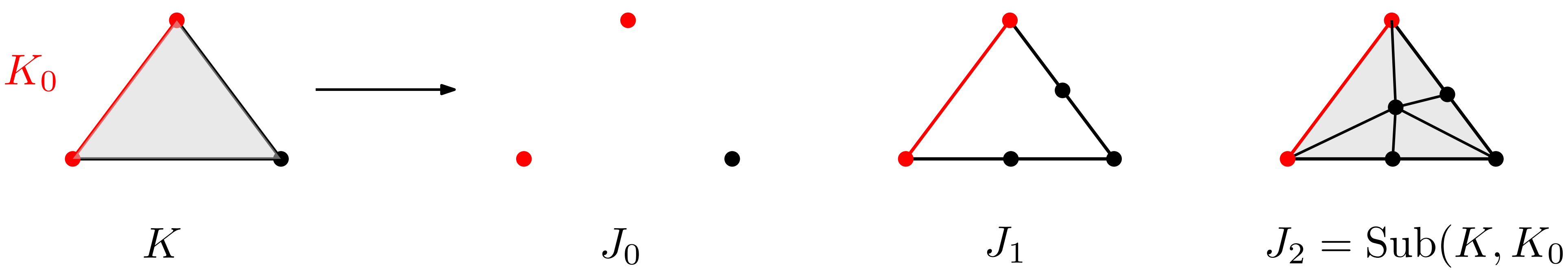
edge contraction

Generalized subdivisions

[Munkres, Elements of Algebraic Topology]

Let K be a simplicial complex, and $K_0 \subset K$ a sub-complex.
The **barycentric subdivision of K holding K_0 fixed**, denoted $\mathrm{Sub}(K, K_0)$, is defined by induction. We define simplicial complexes of increasing dimension $J_0, \dots, J_d$ as follows:

- Start with J_0 the 0-skeleton of K
- From J_{p-1}, build J_p by adding all the p-simplices of K_0
- Moreover, for any p-simplex of K not in K_0, add a point $\hat{\sigma}$, and cone it to the boundary $\partial\sigma \in J_p$

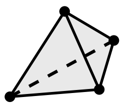


Proposition: To find a simplicial approximation to $f \colon |K| \to |L|$, it is enough to execute repeated barycentric subdivisions of K holding K_0 fixed, where K_0 is a sub-complex on which f satisfies the star condition.

Edgewise sudivisions

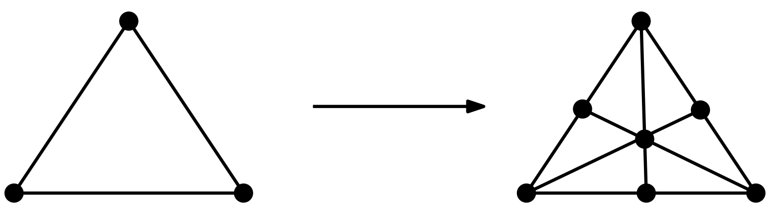
Barycentric subdivisions increase the number of simplices drastically:
a d-simplex turns into a simplicial complex with $(d+1)!$ simplices and $2^{d+1}-1$ vertices.

Example: Triangulation of the unit sphere $\mathbb{S}^2$, starting from the boundary of the 3-simplex



	K	$\mathrm{Sub}^1(K)$	$\mathrm{Sub}^2(K)$	$\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	$\mathrm{Sub}^5(K)$
vertices:	4	14	74	434	2594	15554
simplices:	14	74	434	2594	15554	93314
max diameter:	1.63	1.15	0.66	0.42	0.25	0.15

Barycentric subdivision:



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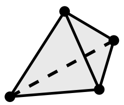

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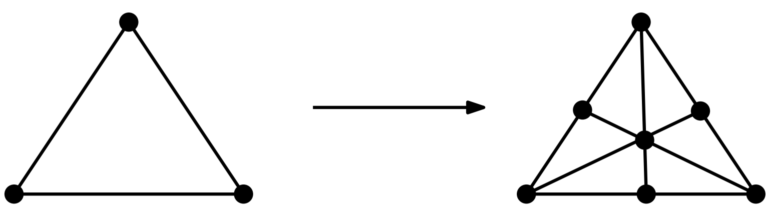
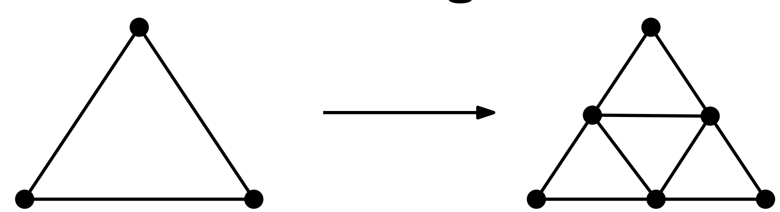


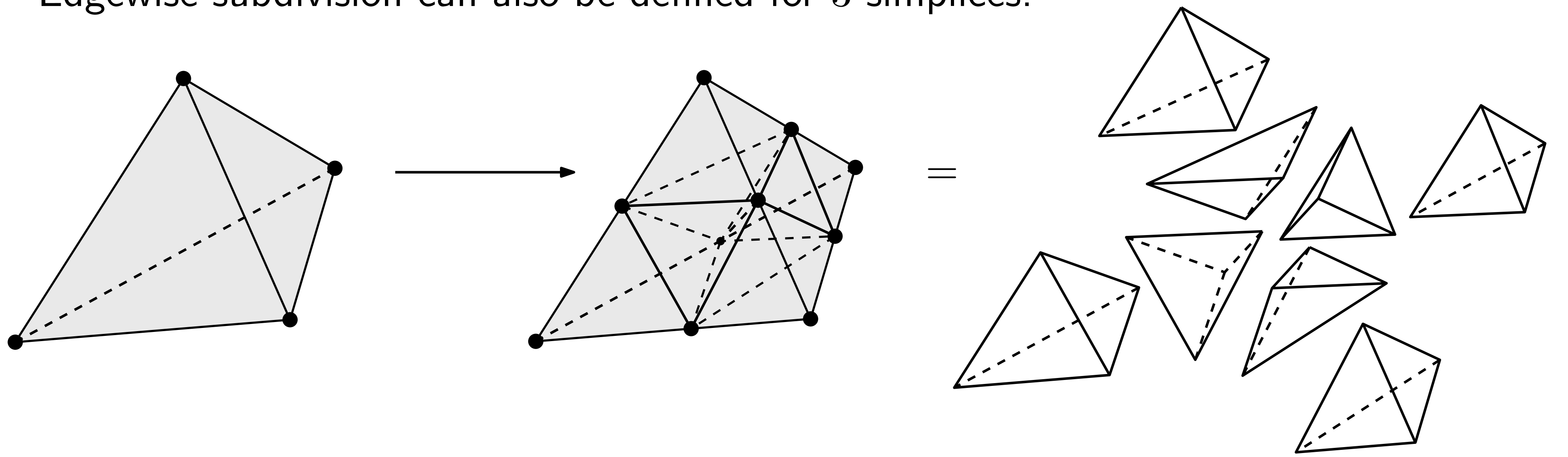
We can do better with **edgewise subdivisions**:



	K	$\mathrm{Sub}^1(K)$	$\mathrm{Sub}^2(K)$	$\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	$\mathrm{Sub}^5(K)$
vertices:	4	10	39	130	514	2050
simplices:	14	50	194	770	3074	12290
max diameter:	1.63	1.41	1	0.58	0.30	0.15

Edgewise sudivisions

Edgewise subdivision can also be defined for 3-simplices:



Example: Triangulation of the sphere $\mathbb{S}^3$, starting from the boundary of the 4-simplex.

	barycentric: $\mathrm{Sub}^3(K)$	$\mathrm{Sub}^4(K)$	edgewise: $\mathrm{Sub}^5(K)$	$\mathrm{Sub}^6(K)$
vertices:	$12'600$	$301'680$	$27'440$	$218'720$
simplices:	$301'680$	$7'238'880$	$710'240$	$5'680'320$
max diameter:	0.54	0.36	0.47	0.29

Simplifying the codomain

We look for a simplicial approximation to $f\colon |K| \to |L|$.

Endow K with a metric, and by denote $\mathcal{U}$ the cover $\left\{f^{-1}(|\mathrm{St}(w)|), w \in L\right\}$ of $|K|$.

Let $\epsilon > 0$ be a Lebesgue number for $\mathcal{U}$.

The map f satisfies the star condition if for every $v \in K^{(0)}$, $\mathrm{Diameter}\left(\left|\overline{\mathrm{St}}(v)\right|\right) < \epsilon$.

To simplify the problem, we can replace L with a homotopy equivalent complex L', given with a simplicial map $g\colon L \to L'$.

$$\begin{tikzcd} |K| \arrow[r, "f"'] \arrow[rr, bend left, "f'"] & |L| \arrow[r, "|g|"'] & |L'| \end{tikzcd}$$

Consider the new cover $\mathcal{U}' = \left\{(f')^{-1}(|\mathrm{St}(w')|), w' \in L'\right\}$.

We may hope for its Lebesgue number to be smaller.

Edge contractions

Let $[a, b]$ be an edge of L, and $c \notin L^{(0)}$ a new vertex. The *quotient map* is defined as

$$
\begin{aligned}
p\colon L^{(0)} &\longrightarrow \left(L^{(0)} \setminus \{a, b\}\right) \sqcup \{c\} \\
x &\longmapsto c \ \text{ if } \ x = a \ \text{ or } \ b \\
& \qquad x \ \text{ else.}
\end{aligned}
$$

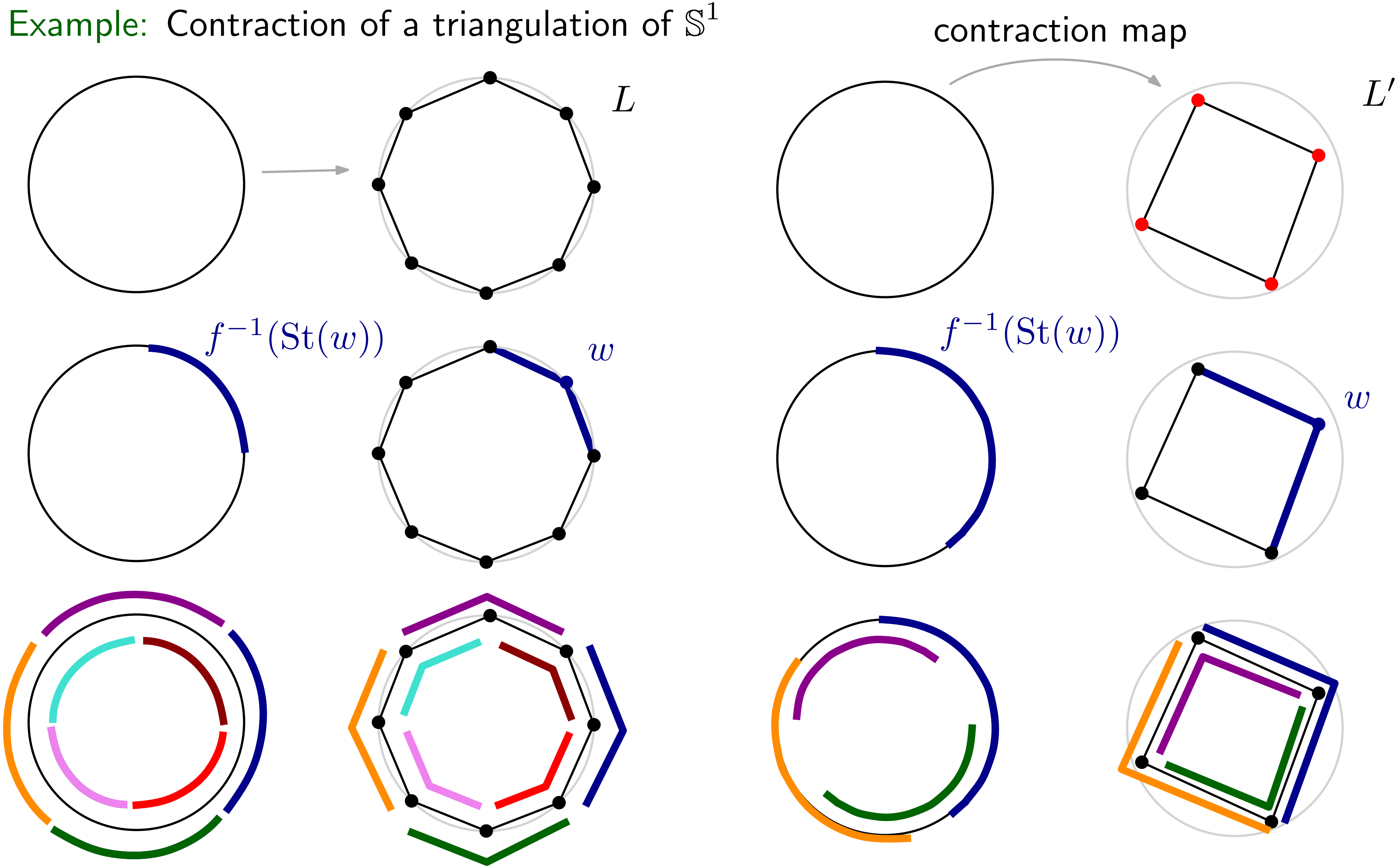
The contracted complex is defined as

$$L' = \{p(\sigma),\ \sigma \in L\}.$$

We have a surjective simplicial map $p\colon L \to L'$.

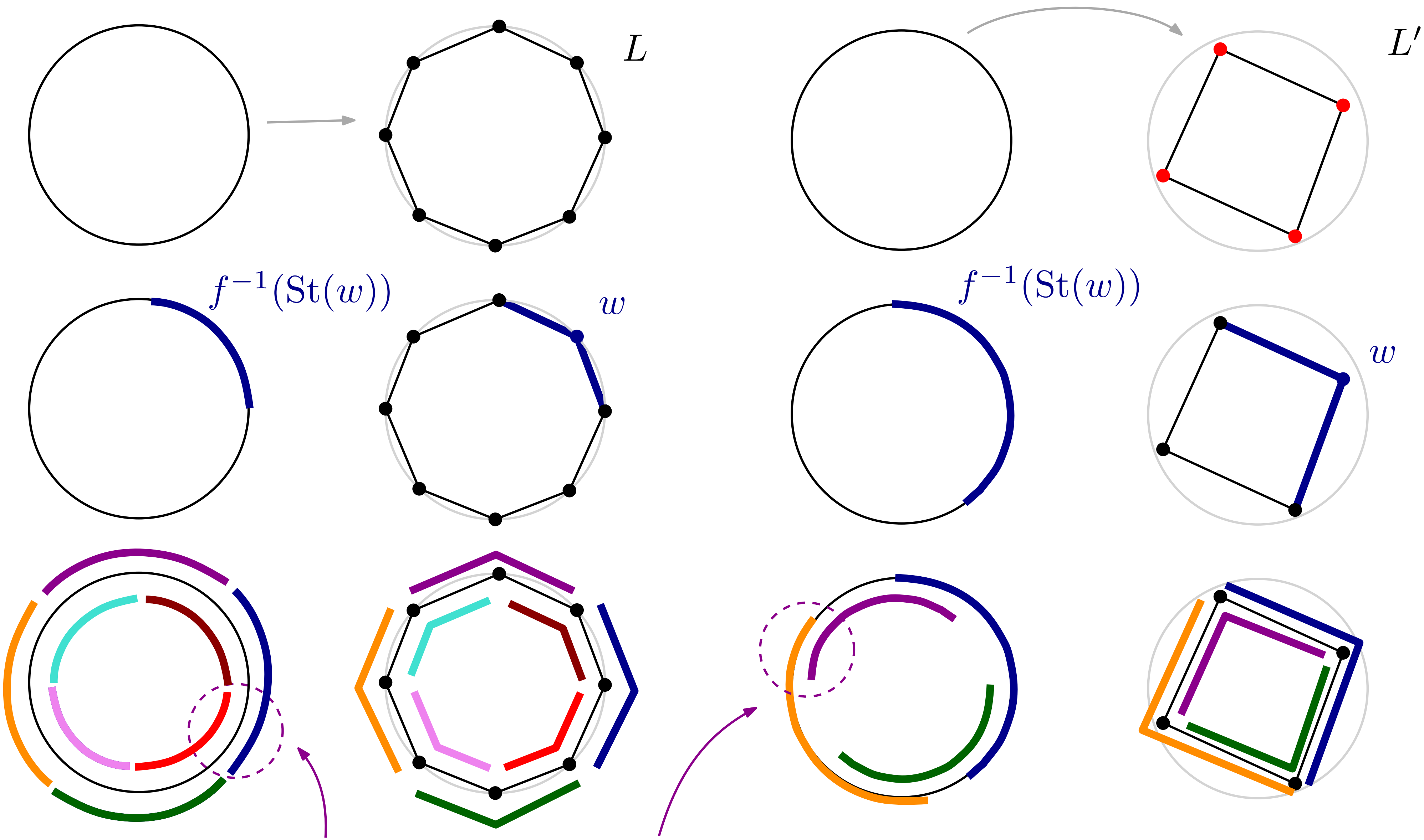
Theorem [Dey, Edelsbrunner, Guha, Nekhayev, 1998], [Attali, Lieutier, Salinas, 2012]:
The map p is a homotopy equivalence when the edge $[a, b]$ satisfies the *link condition*, that is, $\mathrm{Lk}(ab) = \mathrm{Lk}(a) \cap \mathrm{Lk}(b)$.

Example: Contraction of a triangulation of $\mathbb{S}^1$



Edge contractions

Example: Contraction of a triangulation of $\mathbb{S}^1$



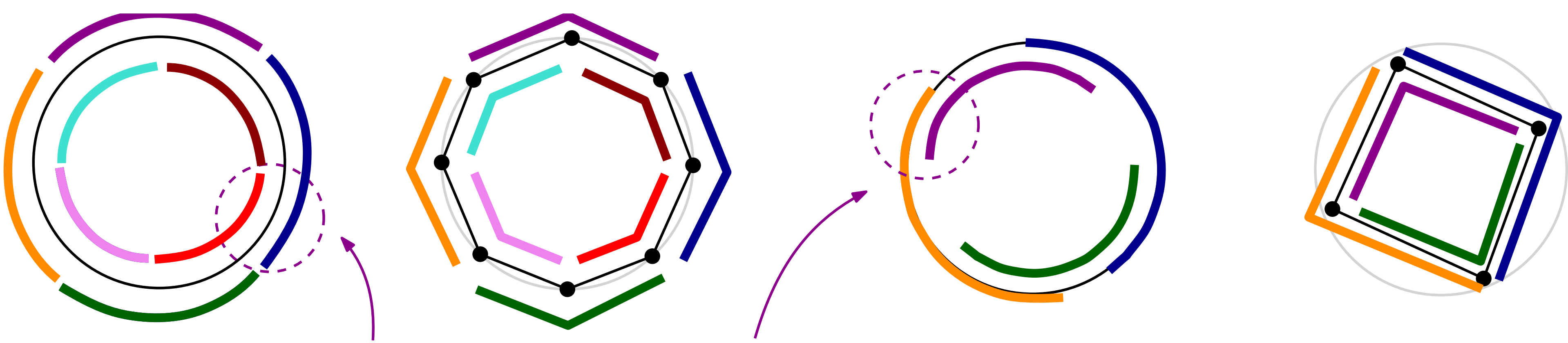
The (global) Lebesgue number remains the same

Edge contractions

We would get better results by considering a *homeomorphism* $L \to L'$ instead of the contraction map p.

→ [Dey, Edelsbrunner, Guha,Nekhayev, *Topology preserving edge contraction*, 1998]
Local unfolding for simplicial complexes of dimension 2, or combinatorial manifolds of dimension 3.

To contract large simplicial complexes, we shall use the skeleton-blockers structure [Attali, Lieutier, Salinas, *Efficient data structure for representing and simplifying simplicial complexes in high dimensions*, 2012]



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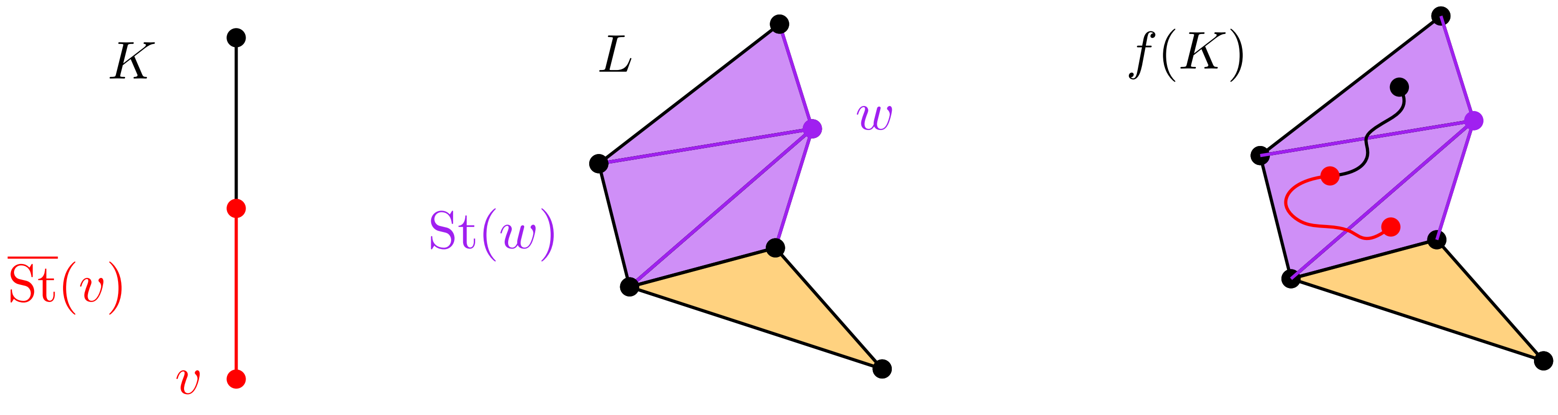
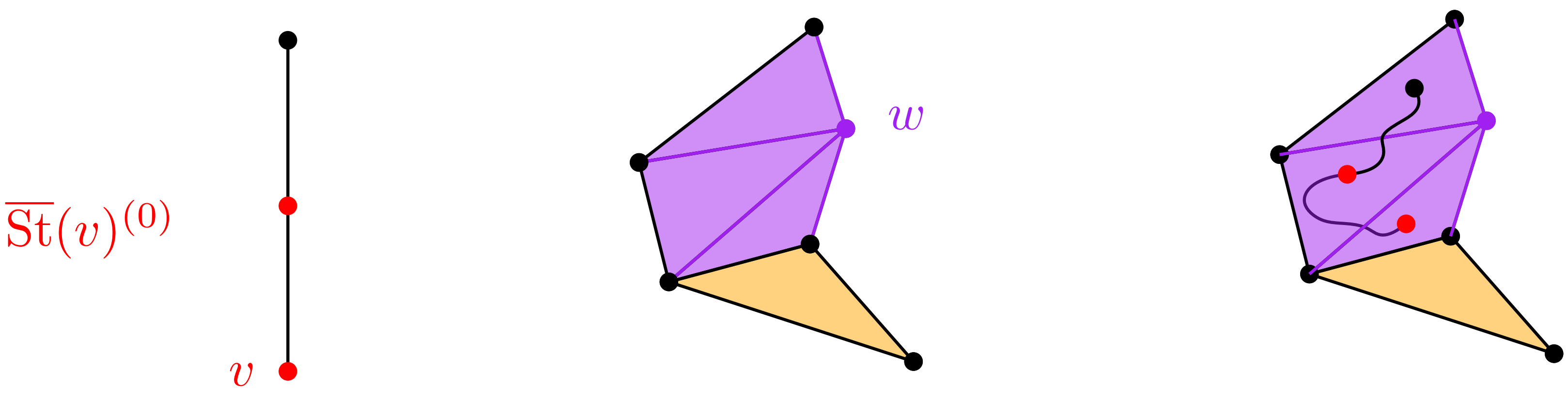
Weak star condition

In practice, we cannot check whether a simplicial map $f\colon |K| \to |L|$ satisfies the star condition...



$$f\left(\left|\overline{\mathrm{St}}(v)\right|\right) \subseteq |\mathrm{St}(w)|?$$

The map f satisfies the **weak star condition** if for every $v \in K^{(0)}$, there exists a $w \in L^{(0)}$ such that $f\left(\overline{\mathrm{St}}(v)^{(0)}\right) \subseteq |\mathrm{St}(w)|$.



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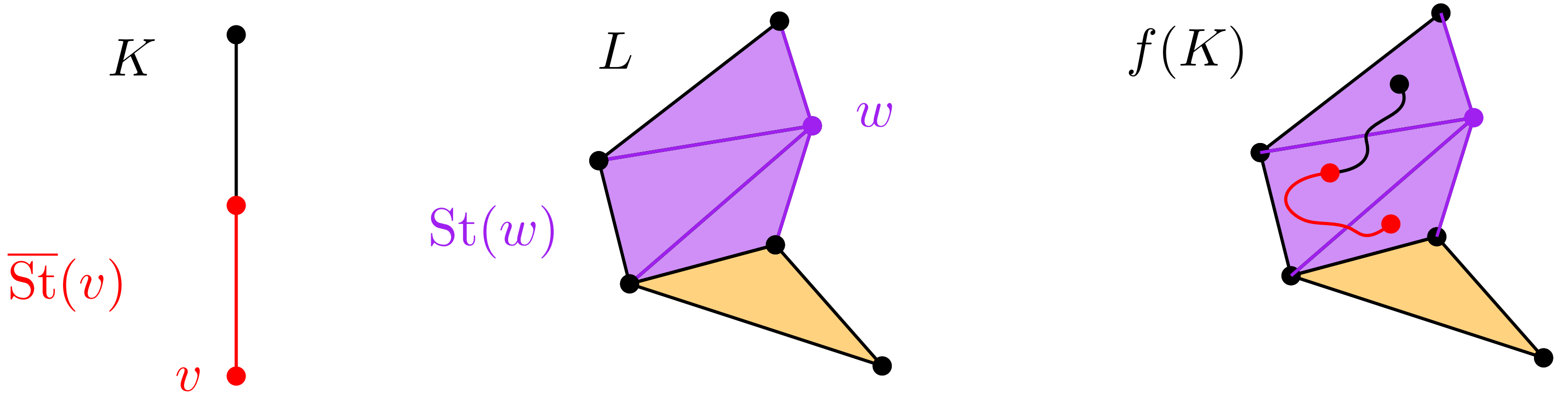

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The map f satisfies the **weak star condition** if for every $v \in K^{(0)}$, there exists a $w \in L^{(0)}$ such that $f\left(\overline{\mathrm{St}}(v)^{(0)}\right) \subseteq |\mathrm{St}(w)|$.

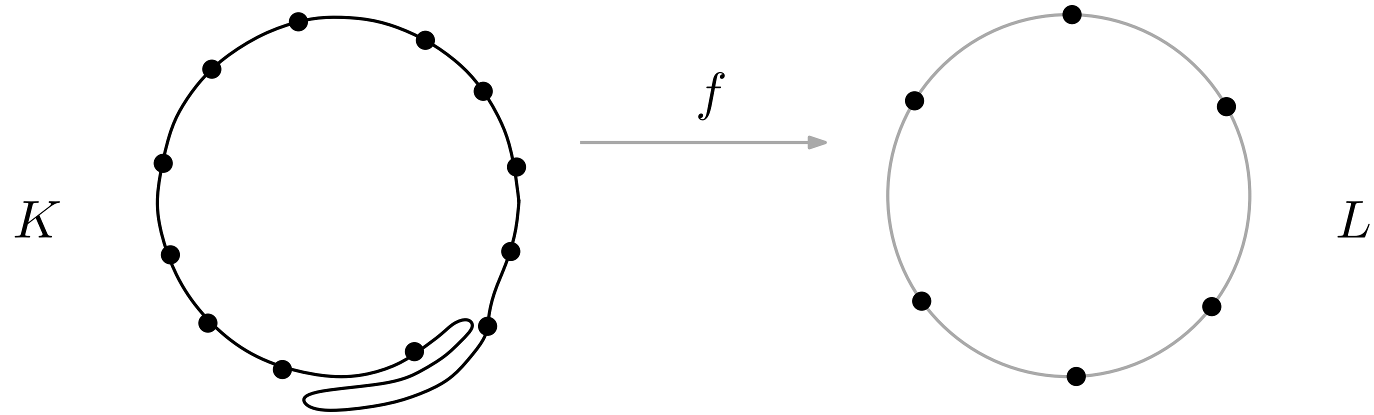
If this is the case, let $g\colon K^{(0)} \to L^{(0)}$ be any map such that for every $v \in K^{(0)}$, we have $f\left((\overline{\mathrm{St}}(v))^{(0)}\right) \subseteq |\mathrm{St}(g(v))|$.

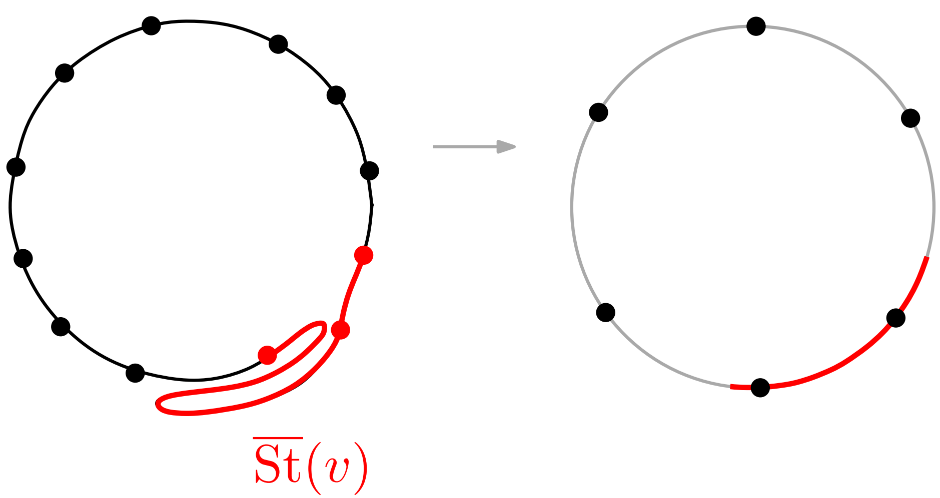
Such a map g is called a **weak simplicial approximation** to f. It is a simplicial map.

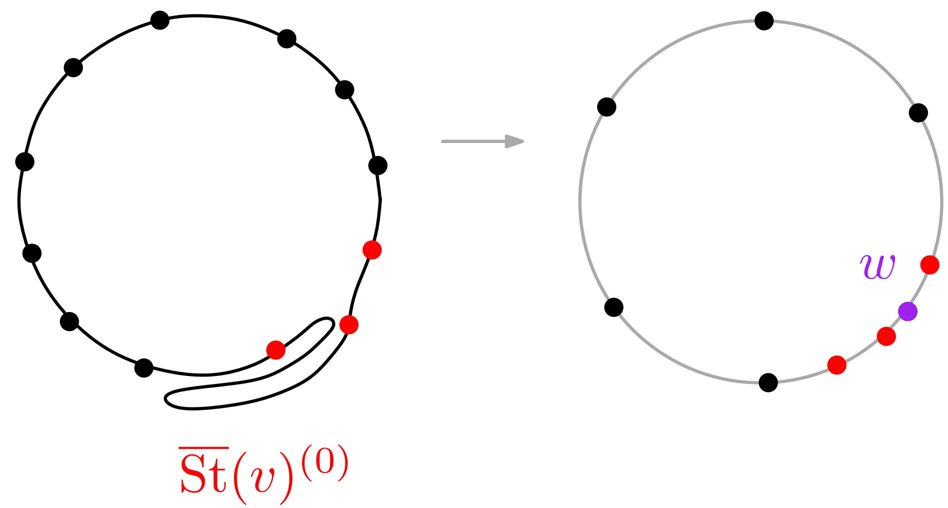
Proposition [T., 2020]: If K is subdivised enough, then any weak simplicial approximation is a simplicial approximation.

Beyond simplicial approximation

If K is not subdivided enough, weak simplicial approximations may not be simplicial approximations.







f does not satisfy the star condition

f satisfies the weak star condition

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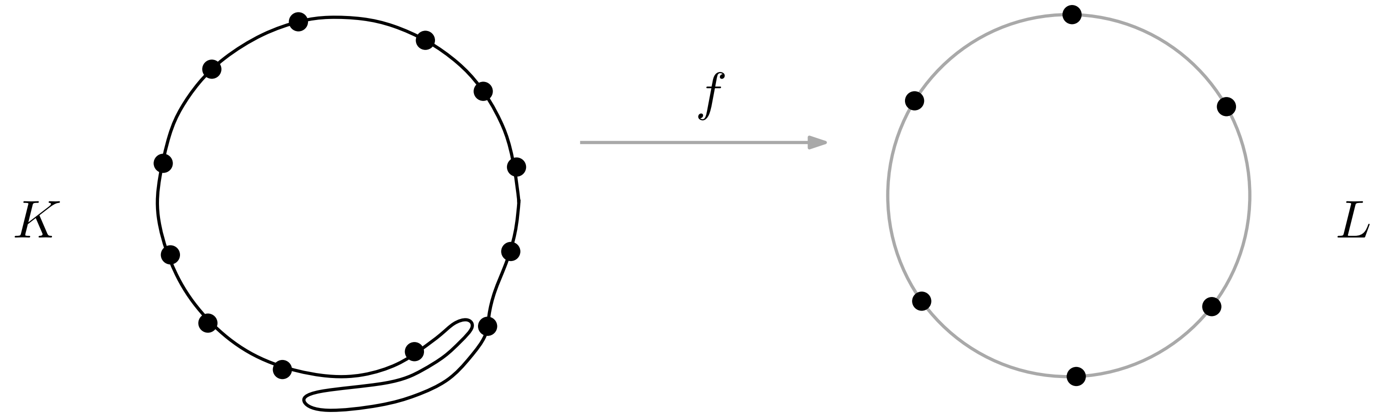


Close maps are homotopic. Suppose that $|L|$ is endowed with a distance d. Between two continuous maps $f, g\colon |K| \to |L|$, define the infinite norm

$$d_\infty(f,g) = \sup \left\{ d(f(x), g(x)),\ x \in |K| \right\}.$$

Let ϵ be such that, for all continuous maps $f, g\colon |K| \to |L|$,

$$d_\infty(f,g) < \epsilon \quad \Longrightarrow \quad f \text{ and } g \text{ are homotopic.}$$

This gives another criterion for simplicial approximation: we look for a simplicial $g\colon K \to L$ such that $d_\infty(f,g) < \epsilon$.

One could try to relate this ϵ and the *injectivity radius* of $|L|$ endowed with a Riemannian metric.

I - Simplicial approximation to CW-complexes

1 - Topology CW-complexes

2 - Simplicial approximation

3 - Simplicial mapping cone

4 - Application to projective spaces

II - Simplicial approximation improved

1 - Local subdivisions

2 - Edge contractions

3 - Weak simplicial approximation

4 - Application to projective spaces, second attempt

III - Applications

1 - Lens spaces

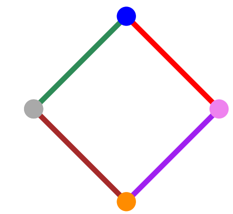
2 - Grassmannian $\mathcal{G}_2(\mathbb{R}^4)$

Approximation to $\mathbb{R}P^n$, 2^{nd} attempt

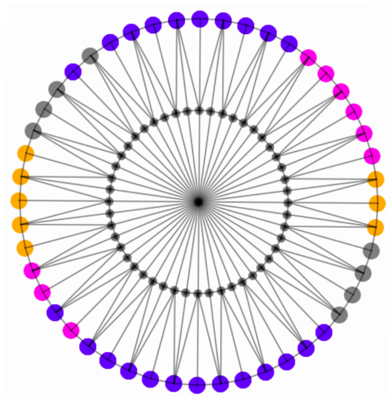
Remember our last triangulation with *global barycentric sudivisions*:

$\mathbb{R}P^0$: 1 vertex

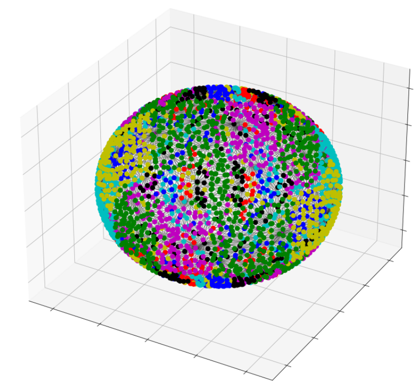
$\mathbb{R}P^1$: 4 vertices



$\mathbb{R}P^2$: 53 vertices
(4 subdivisions)



$\mathbb{R}P^3$: $560'024$ vertices
(7 subdivisions)



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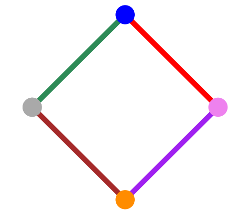

$\mathbb{R}P^2$: 53 vertices
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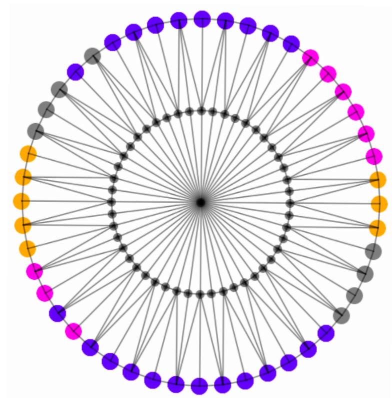

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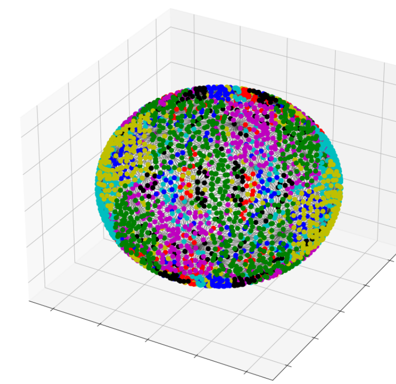
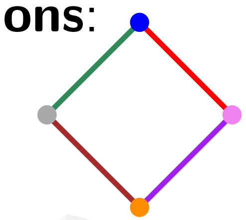


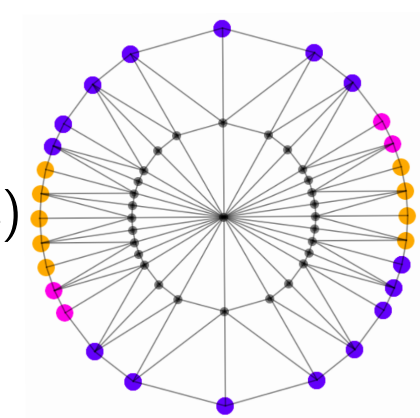
Now with **local subdivisions**, **edgewise subdivisions** and **edge contractions**:

$\mathbb{R}P^0$: 1 vertex

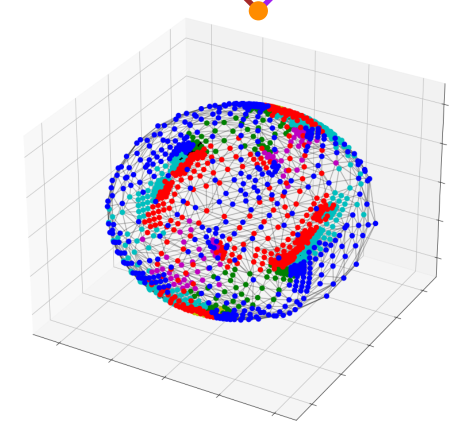
$\mathbb{R}P^1$: 4 vertices
(contract to 3 vertices)



$\mathbb{R}P^2$: 32 vertices
(contract to 7 vertices)



$\mathbb{R}P^3$: 1520 vertices
(contract to 89 vertices)



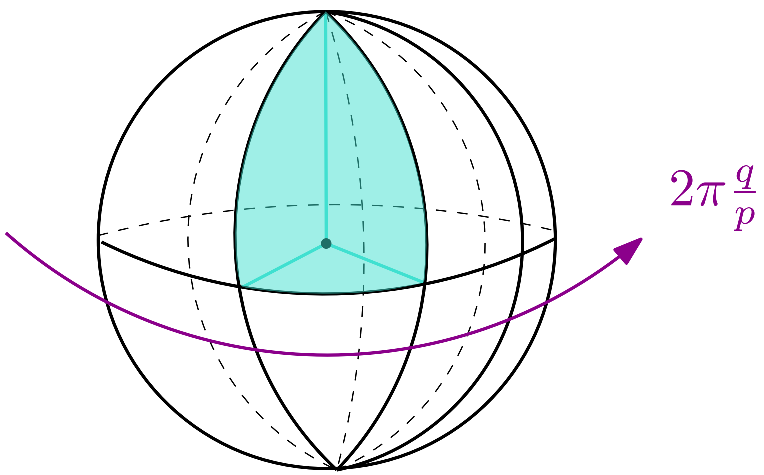
Cell structure for lens spaces

Let p, q be coprime integers, and let $\mathbb{S}^3 \subset \mathbb{C}^2$ be the unit sphere. Let $\alpha = e^{\frac{2i\pi}{p}}$ be the p^{th} root of unity.
We define an action of $\mathbb{Z}_p$ on $\mathbb{S}^3$ via $\rho \cdot (z_1, z_2) = (\alpha z_1, \alpha^q z_2)$.
It is a free action, and the orbit space is denoted $L(p, q)$.

Its homology groups are $\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}, 0, \mathbb{Z}$. It has been an important question to determine which of these spaces are homeomorphic or homotopic equivalent.

A fundamental domain for $L(p, q)$ is given by

$$\left\{\cos(\theta) \cdot (0, \alpha^t) + \sin(\theta) \cdot (\alpha^s, 0),\ t \in [0, 1]\ s \in [0, 1],\ \theta \in \left[0, \frac{\pi}{2}\right)\right\} \subset \mathbb{C}^2.$$



We deduce a CW-complex structure on L:

- L^0 is a point,
- L^1 is a circle,
- L^2 is a disk glued with a degree p map,
- L^3 is a 3-ball glued by differenciating the two hemispheres: one is glued via the identity on the 2-cell, the other one after a rotation of $2\pi\frac{q}{p}$.

Simplicial approximation to $L(p,q)$

We applied the algorithm for various values of p and q.

$(p,q) =$	$(2,3)$	$(2,5)$	$(2,11)$	$(3,5)$	$(11,2)$
number of vertices before contraction	911	929	900	$2'412$	$15'753$
number of vertices after contraction	50	39	57	212	$1'631$
number of simplices before contraction	$18'736$	$18'524$	$17'896$	$49'510$	$327'682$
number of simplices after contraction	$1'078$	820	$1'202$	$4'954$	$46'654$

Cell structure for $\mathcal{G}_2(\mathbb{R}^4)$

The Grassmannian $\mathcal{G}_2(\mathbb{R}^4)$ is the set of 2-dimensional planes in $\mathbb{R}^4$. It can be given a smooth manifold structure of dimension 4. Its cohomology groups are:

$$
\begin{aligned}
H^0(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) &= \mathbb{Z} \\
H^1(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) &= \mathbb{Z}_2 \\
H^2(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) &= (\mathbb{Z}_2)^2 \\
H^3(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) &= \mathbb{Z}_2 \\
H^4(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) &= \mathbb{Z}
\end{aligned}
\qquad
\begin{aligned}
H^0(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) &= \mathbb{Z}_p \\
H^1(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) &= 0 \\
H^2(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) &= 0 \\
H^3(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) &= 0 \\
H^4(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) &= \mathbb{Z}_p
\end{aligned}
$$

where p is any prime number > 2.

Consider a plane $T \in \mathcal{G}_2(\mathbb{R}^4)$. Let (u, v) be a basis, and consider the matrix

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

By mean of elementary operations, it can be reduced to a unique matrix in **orthogonal reduced echelon form** $\begin{pmatrix} v_1' & v_2' & v_3' & v_4' \\ w_1' & w_2 & w_3' & w_4' \end{pmatrix}$ such that:

- $\|v'\| = \|w'\| = 1$
- v' and w' are orthogonal
- the last nonzero coordinate of v' is positive
- the last nonzero coordinate of w' is positive

Cell structure for $\mathcal{G}_2(\mathbb{R}^4)$

Consider a plane $T \in \mathcal{G}_2(\mathbb{R}^4)$ and its matrix in reduced echelon form $\begin{pmatrix} v_1' & v_2' & v_3' & v_4' \\ w_1' & w_2 & w_3' & w_4' \end{pmatrix}$

Let i (resp. j) be the index of the last nonzero coordinate of v' (resp. w').

The pair (i, j) is called the *Schubert symbol* of the plane T.

There are 6 potential Schubert symbols for $\mathcal{G}_2(\mathbb{R}^4)$:

$$(1,2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (1,3) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$$

$$(1,4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix} \qquad (2,3) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

$$(2,4) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix} \qquad (1,2) \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

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There are 6 potential Schubert symbols for $\mathcal{G}_2(\mathbb{R}^4)$:

dim 0 $\quad (1,2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\qquad (1,3) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$ dim 1

dim 2 $\quad (1,4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix}$ $\qquad (2,3) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$ dim 2

dim 3 $\quad (2,4) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}$ $\qquad (1,2) \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$ dim 4

Proposition: $\mathcal{G}_2(\mathbb{R}^4)$ admits a CW-structure with 6 cells. Each cell corresponds to a pair (i, j), and contains all the planes T with Schubert symbol (i, j).

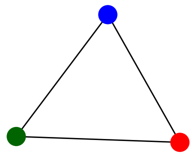
After some computations, one finds explicit expressions for characteristic maps and gluing maps.

Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$

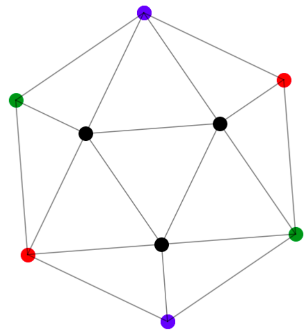
Running the algorithm as is will give a simplicial complex with millions of vertices.
Let us use handcrafted simplicial approximations for the first cells instead.

$(1,2)$: 1 vertex

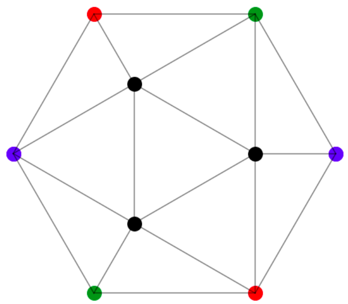
$(1,3)$: 3 vertices



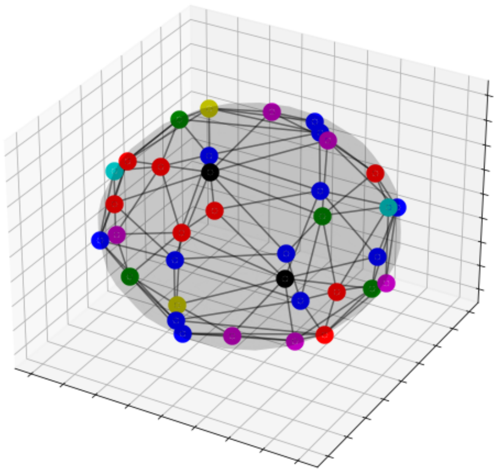
$(1,4)$: 6 vertices



$(2,3)$: 9 vertices



$(2,4)$: 46 vertices



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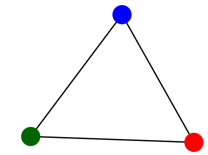
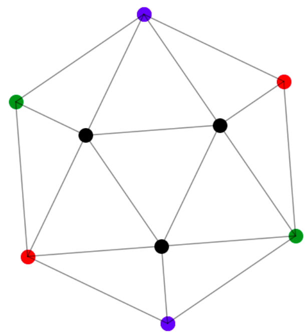
$(1,2)$: 1 vertex

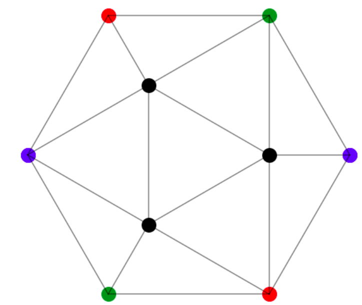
$(1,3)$: 3 vertices



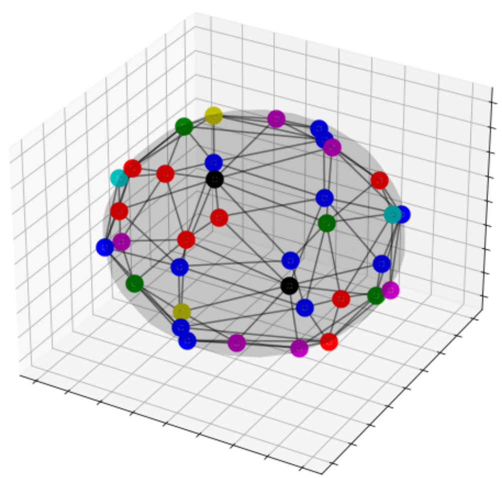
$(1,4)$: 6 vertices



$(2,3)$: 9 vertices



$(2,4)$: 46 vertices

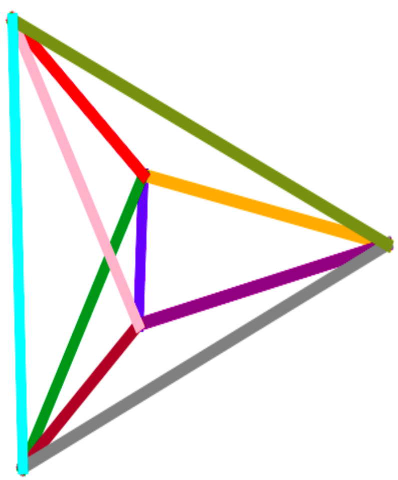
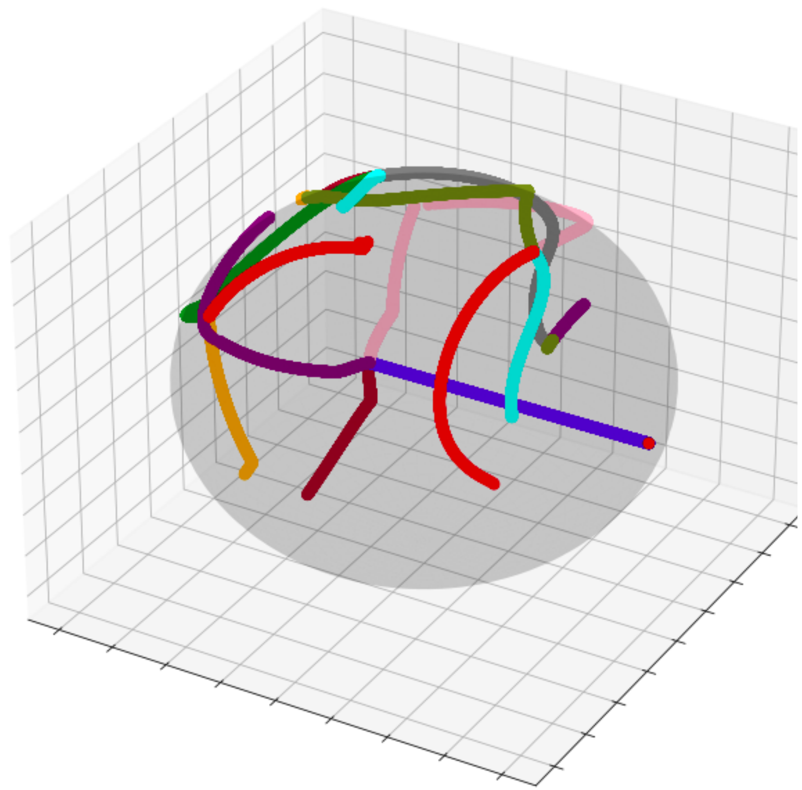
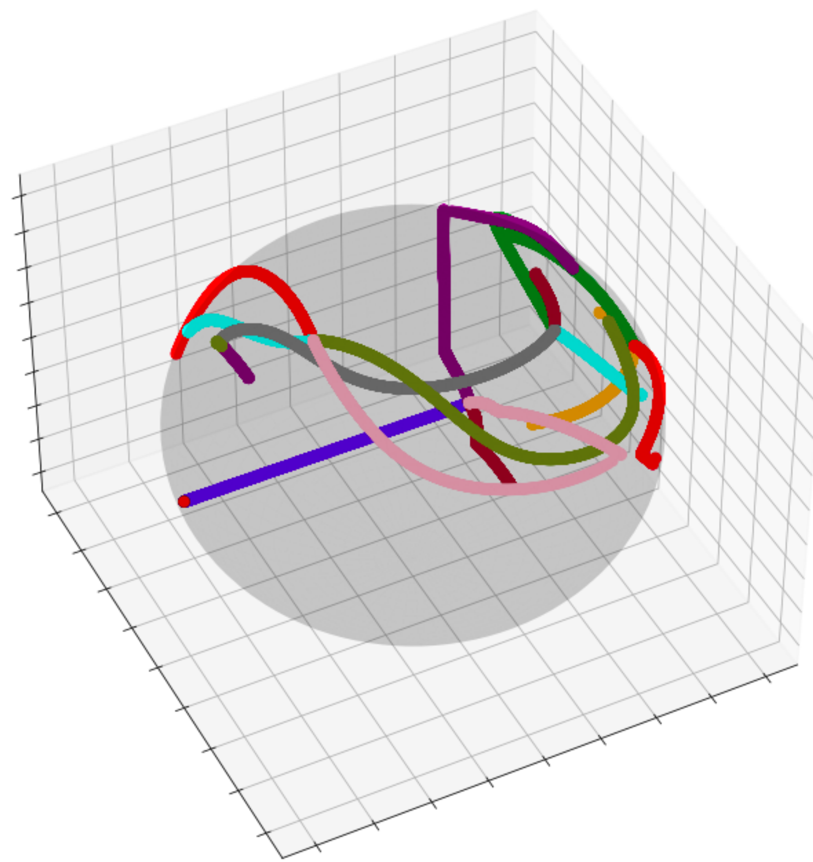


Gluing the last cell $(3,4)$: $38'951$ vertices, $3'642'258$ simplices.

After edge contractions: $7'700$ vertices, $1'061'892$ simplices (via skeleton-blockers).

Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$

Embedding of the 4-simplex, seen as a triangulation of the 3-sphere, in the 3-cell of $\mathcal{G}_2(\mathbb{R}^4)$:

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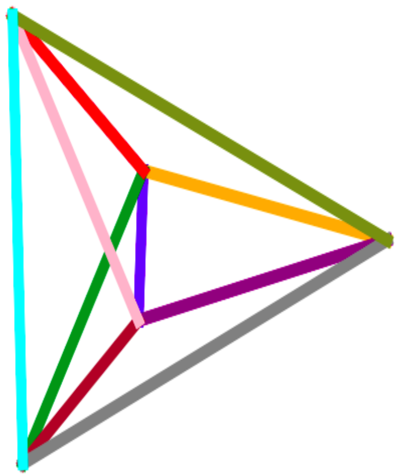
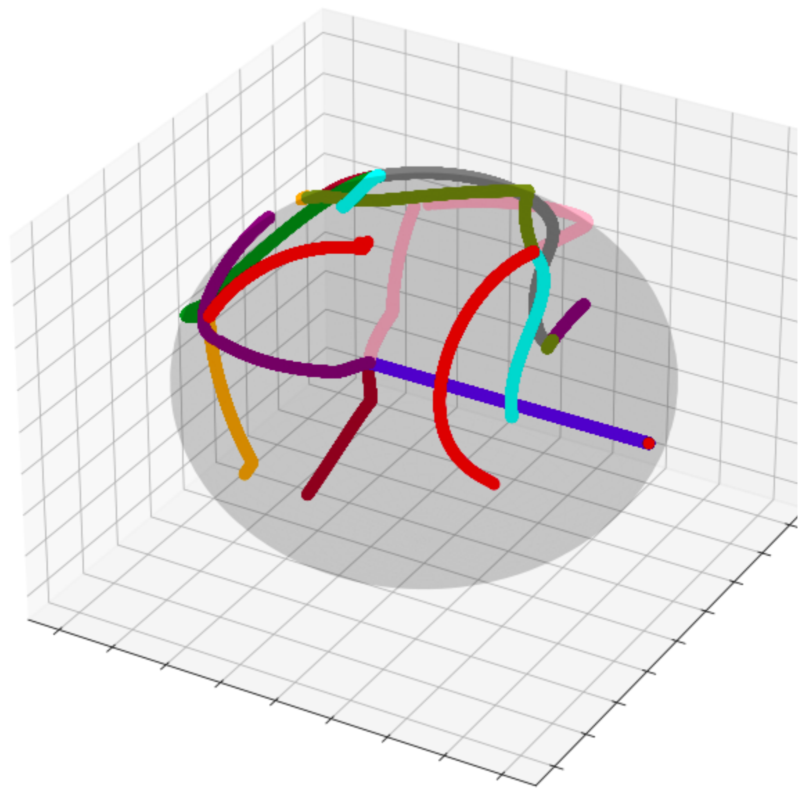
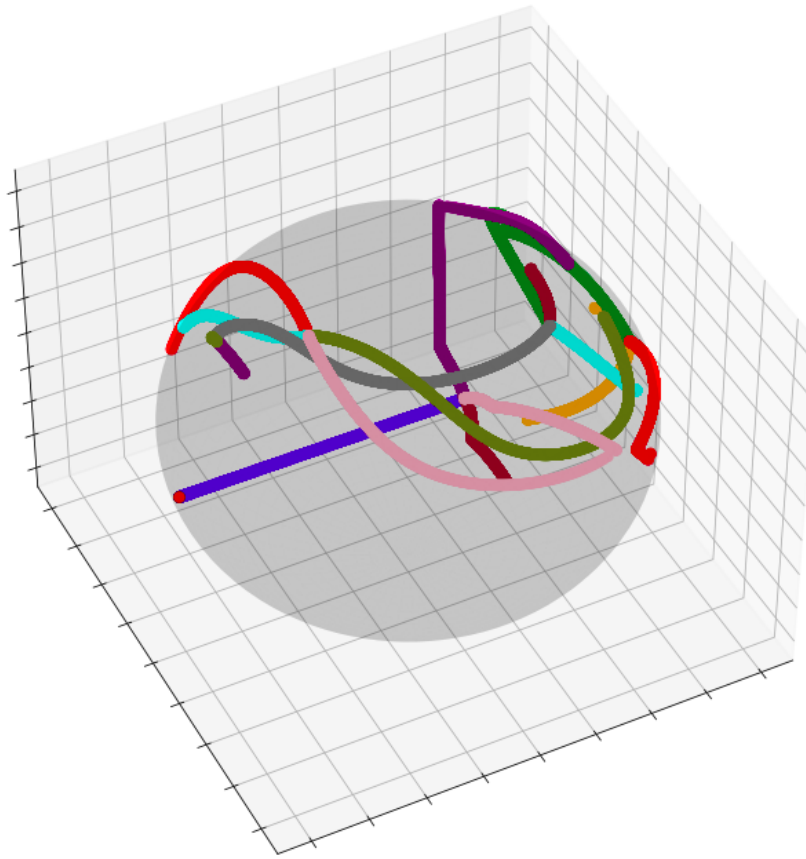
Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$

Embedding of the 4-simplex, seen as a triangulation of the 3-sphere, in the 3-cell of $\mathcal{G}_2(\mathbb{R}^4)$:







Thank you!

Gluing the last cell $(3, 4)$: $38'951$ vertices, $3'642'258$ simplices.

After edge contractions: $7'700$ vertices, $1'061'892$ simplices (via skeleton-blockers).