EMAp Summer Course

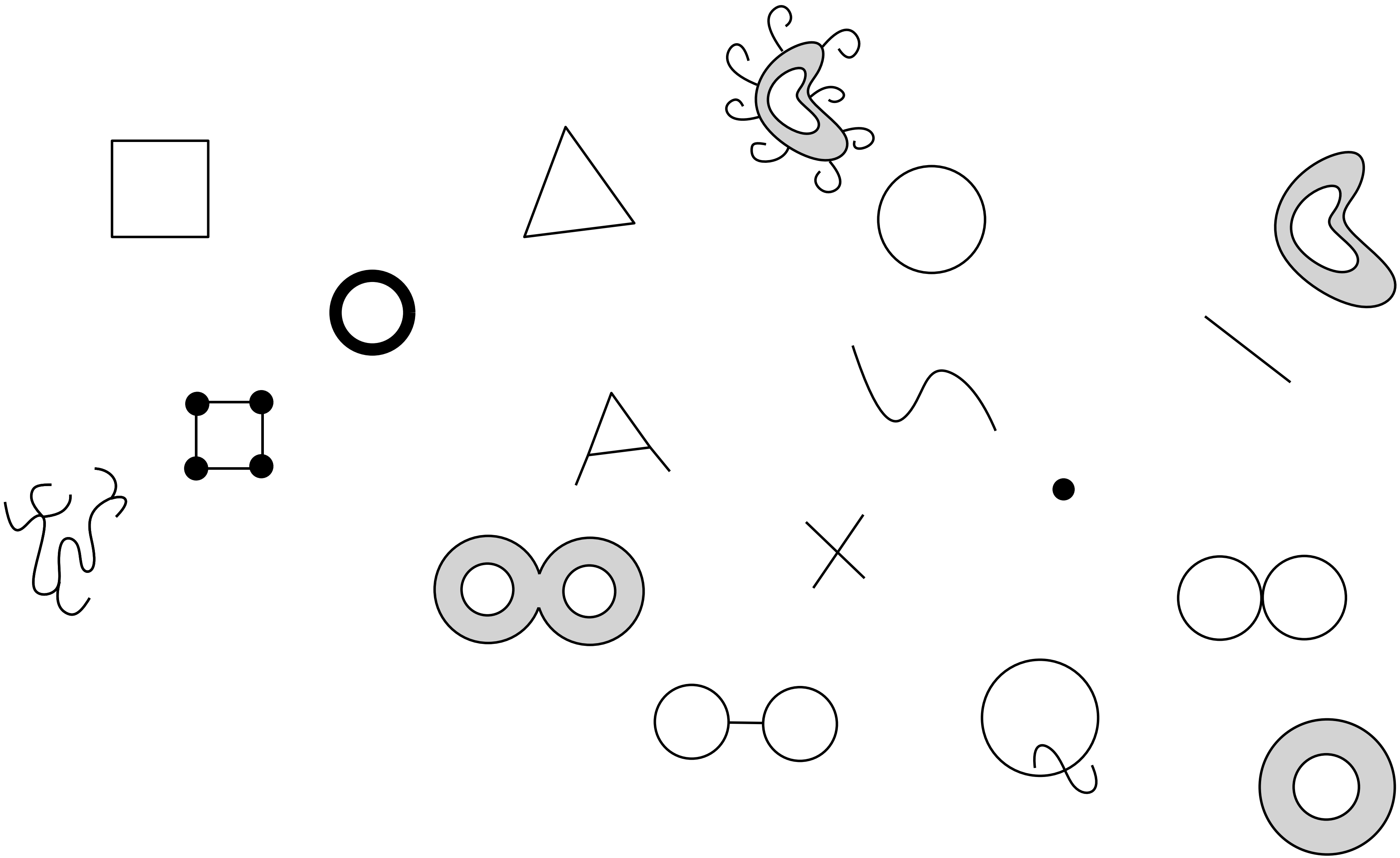
# Topological Data Analysis with Persistent Homology

https://raphaeltinarrage.github.io/EMAp.html

# Lesson 4: Simplicial complexes

Last update: January 29, 2021

# Introduction



Objective of the lesson: doing topology on a computer.

I - Combinatorial simplicial complexes

II - Topology

III - Euler characteristic

(VI - Tutorial)

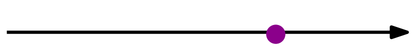
# Standard simplices



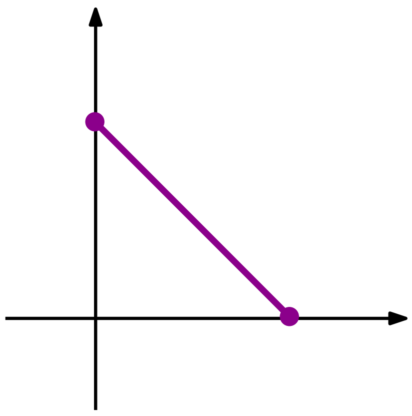
In order to describe topological spaces, we will decompose them into simpler pieces. The pieces we shall consider are the standard simplices.

The *standard simplex of dimension* $n$ is the following subset of $\mathbb{R}^{n+1}$

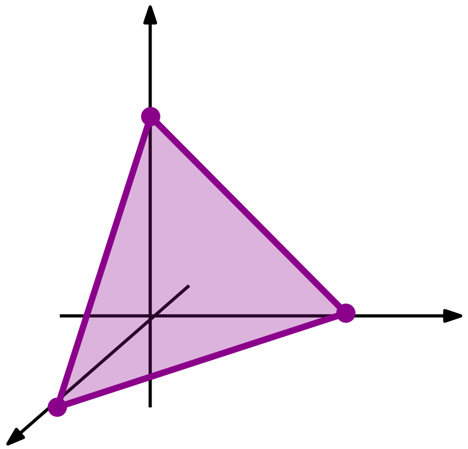
$$\Delta_n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}, x_1, ..., x_{n+1} \geq 0 \text{ and } x_1 + ... + x_{n+1} = 1\}$$



$\Delta_0$



$\Delta_1$



$\Delta_2$

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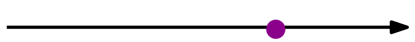

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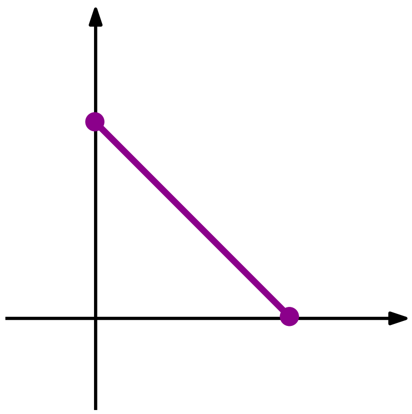

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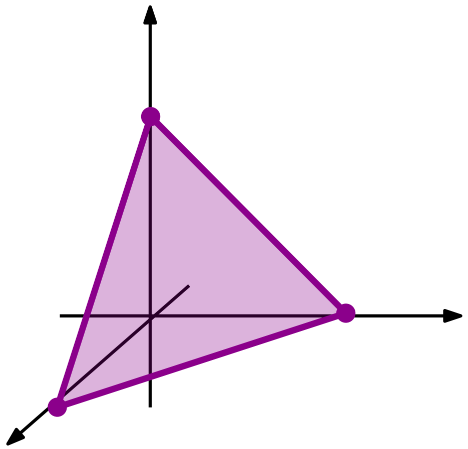

$\Delta_2$

Remark: For any collection of points $a_1, ..., a_k \in \mathbb{R}^n$, their convex hull is defined as:

$$\text{conv}(\{a_1...a_k\}) = \left\{\textstyle\sum_{1\leq i\leq k} t_i a_i, \quad t_1 + ... + t_k = 1, \quad t_1, ..., t_k \geq 0\right\}.$$

We can say that $\Delta_n$ is the convex hull of the vectors $e_1, ..., e_{n+1}$ of $\mathbb{R}^{n+1}$, where

$$e_i = (0, ..., 1, 0, ..., 0) \qquad (i^{\text{th}} \text{ coordinate } 1\text{, the other ones } 0).$$

# Simplicial complexes



First, a purely combinatorial definition (without geometry):

Definition: Let $V$ be a set (called the set of *vertices*). A *simplicial complex* over $V$ is a set $K$ of subsets of $V$ (called the *simplices*) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$.

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Example: Let $V = \{0, 1, 2\}$ and

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By convention, we write simplices with square brackets (instead of curly brackets).

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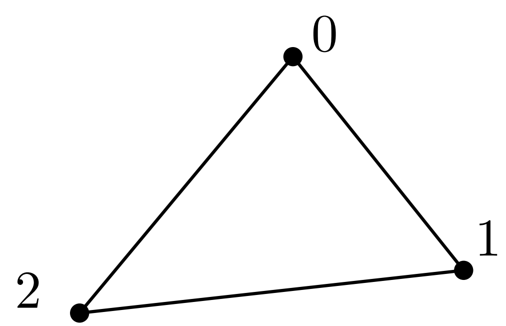

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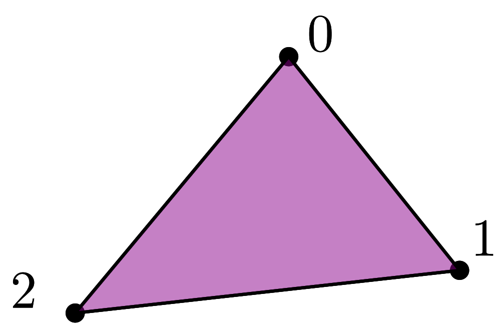

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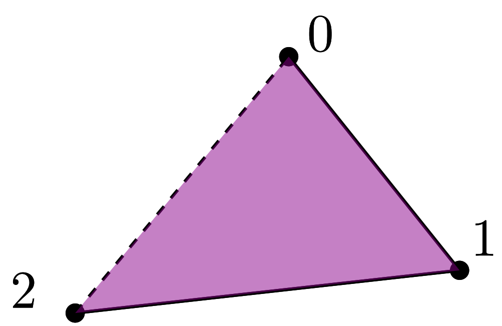
Example: Let $V = \{0, 1, 2\}$ and

$$K = \{[0], [1], [2], [0, 1], [1, 2], [0, 1, 2]\}.$$

This is not a simplicial complex.
Indeed, the simplex $[0, 1, 2]$ admits a face $[2, 0]$ that is not included in $V$.



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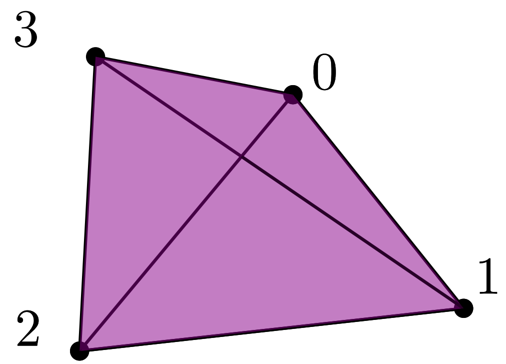
If $\sigma$ is a simplex, its dimension is defined as $|\sigma| - 1$ (cardinality of $\sigma$ minus $1$). If $K$ is a simplicial complex, its dimension is defined as the maximal dimension of its simplices.

Example: Let $V = \{0, 1, 2, 3\}$ and

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}$$

It a simplicial complex of dimension $2$.



I - Combinatorial simplicial complexes

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# Topological realization



Let us give simplicial complexes a topology.

Definition: Let $K$ be a simplicial complex, with vertex $V = \{1, ..., n\}$.
In $\mathbb{R}^n$, consider, for every $i \in [\![1, n]\!]$, the vector $e_i = (0, ..., 1, 0, ..., 0)$ ($i^{\text{th}}$ coordinate $1$, the other ones $0$).
Let $|K|$ be the subset of $\mathbb{R}^{n+1}$ defined as:

$$|K| = \bigcup_{\sigma \in K} \text{conv}\left(\{e_j, j \in \sigma\}\right)$$

where conv represent the convex hull of points.
Endowed with the subspace topology, $(|K|, \mathcal{T}_{||K|})$ is a topological space, that we call the *topological realization of* $K$.

If $a_1, ..., a_k \in \mathbb{R}^n$, the convex hull is defined as:

$$\text{conv}(\{a_1 ... a_k\}) = \left\{ \sum_{1 \leq i \leq k} t_i a_i, \;\; t_1 + ... + t_k = 1, \;\; t_1, ..., t_k \geq 0 \right\}.$$

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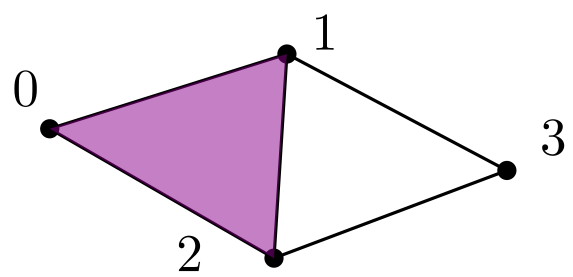
Remark: If the simplicial complex can be drawn in the plane (or space) without crossing itself, then its topological realization simply is the subspace topology.

Example: $K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 0], [1, 3], [2, 3], [0, 1, 2]\}$.



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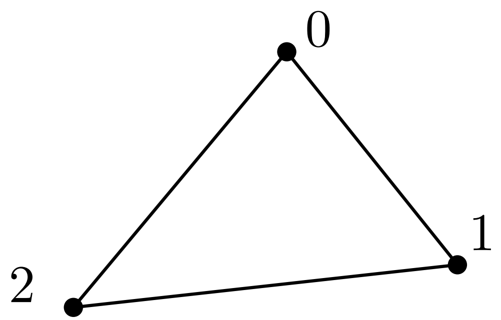

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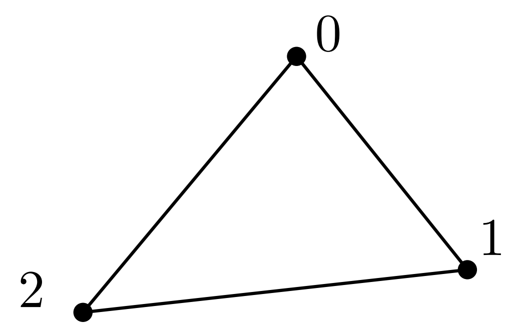
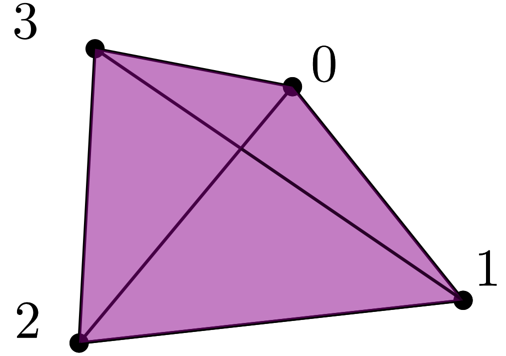



Example: The following simplicial complex is a triangulation of the sphere:

$$K = \{[0], [1], [2], [3], [0,1], [1,2], [2,3], [3,0], [0,2], [1,3], [0,1,2], [0,1,3], [0,2,3], [1,2,3]\}.$$



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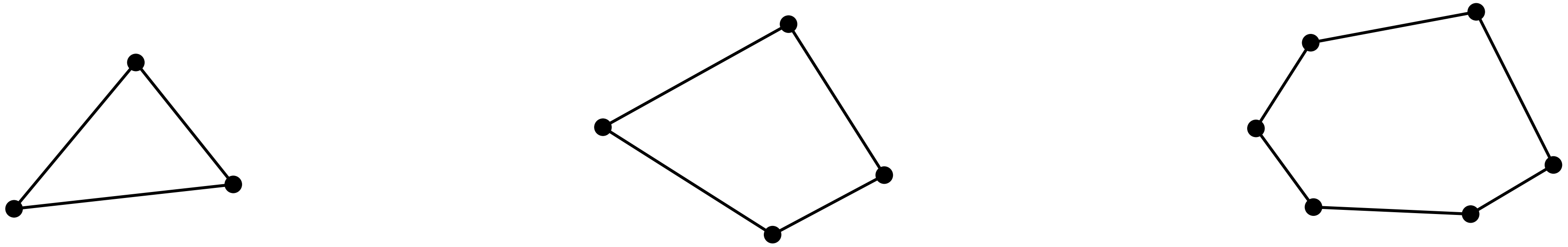
Given a topological space, it is not always possible to triangulate it.
However, when it is, there exists many different triangulations.



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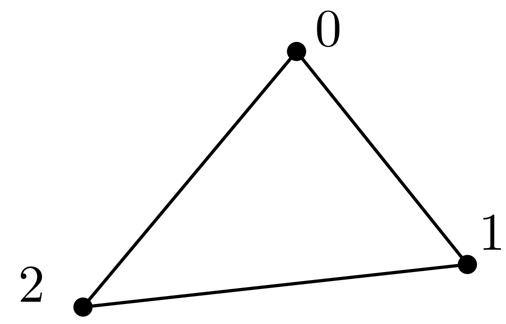
$$\chi(K) = 3 - 3 = 0$$



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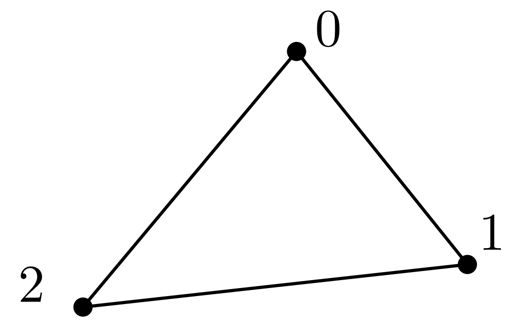


Example: The simplicial complex
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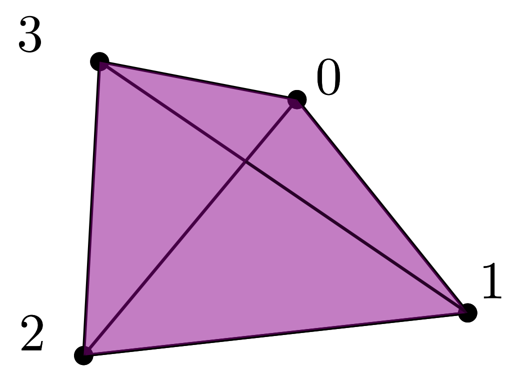
$$\chi(K) = 4 - 6 + 4 = 2$$



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Definition: Let $X$ be a topological space. Its Euler characteristic is defined as the Euler characteristic of any triangulation of it.

Two issues:

- $X$ must admit a triangulation
- we have to make sure the the triangulations of $X$ all have the same Euler characteristic. In other words, if $K$ and $K'$ are two triangulations of $X$, we must have $\chi(K) = \chi(K')$.

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  this is true!

but we won't be able to prove it in this summer course...

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Example: The circle has Euler characteristic 0 because it admits a triangulation

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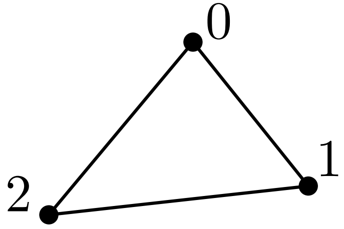


Example: The sphere has Euler characteristic 2 because it admits a triangulation

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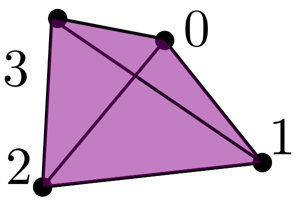

# Euler characteristic is an invariant 

Proposition: If $X$ and $Y$ are two homotopy equivalent topological spaces, then $\chi(X) = \chi(Y)$.

Therefore, the Euler characteristic is an *invariant* of homotopy equivalence classes.

We can use this information to prove that two spaces are not homotopy equivalent.

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Example: The circle has Euler characteristic 0, and the sphere Euler characteristic 2. Therefore, they are not homotopy equivalent.

Exercise (21): Show that $\mathbb{R}^3$ and $\mathbb{R}^4$ are not homotopy equivalent.

# Conclusion

We learnt how to represent topological spaces on a computer.

We defined a new topological invariant.

Homeworks for next week: Exercises 20, 23, 24

Facultative exercise: Exercises 21, 26$^{\star\star\star}$

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Obrigado!