Prova

Você tem uma hora para completar os quatro exercícios seguintes. Você pode se referir à matéria do curso.

EXERCISE 1. Let (X,d) be metric space. Prove that X is bounded[1] if and only if every countable subset of X is bounded.

EXERCISE 2. Let (X,d) be metric space.

1. Show that $\frac{d}{1+d}$ is a metric on X.
2. Show that $\frac{d}{1+d}$ and d induce the same topology.
3. If (X,d) is not bounded, show that $\frac{d}{1+d}$ and d are not equivalent.

EXERCISE 3. Let $([0,+\infty),\mathscr{T})$ be the half real line endowed with the Euclidean topology. Let $+\infty$ denote an element that is not in $[0,+\infty)$, and consider the set $[0,+\infty] = [0,+\infty) \cup \{+\infty\}$. Let $\mathscr{U}$ denote the the topology on $[0,+\infty]$ generated by $\mathscr{T}$ and the sets $(a,+\infty]$ for $a \in [0,+\infty)$. Show that $([0,+\infty],\mathscr{U})$ is compact.

EXERCISE 4. Let $(\mathbb{N},\mathscr{T})$ be the integers endowed with the discrete topology. Let $+\infty$ denote an element that is not in $\mathbb{N}$, and consider the set $\mathbb{N} \cup \{+\infty\}$. Let $\mathscr{U}$ denote the the topology on $\mathbb{N} \cup \{+\infty\}$ generated by $\mathscr{T}$ and the sets $(a,+\infty]$ for $a \in \mathbb{N}$.

1. Show that $(\mathbb{N} \cup \{+\infty\},\mathscr{U})$ is homeomorphic to the subset

   $$\{0\} \cup \bigcup_{n \geq 1} \left\{\frac{1}{n}\right\} \subset \mathbb{R}$$

   endowed with the subspace Euclidean topology.
2. Is $(\mathbb{N} \cup \{+\infty\},\mathscr{U})$ homeomorphic to $(\mathbb{N},\mathscr{T})$?

[1] A metric space (X,d) is *bounded* if there exists a $D > 0$ such that $d(x,y) < D$ for all $x,y \in X$.