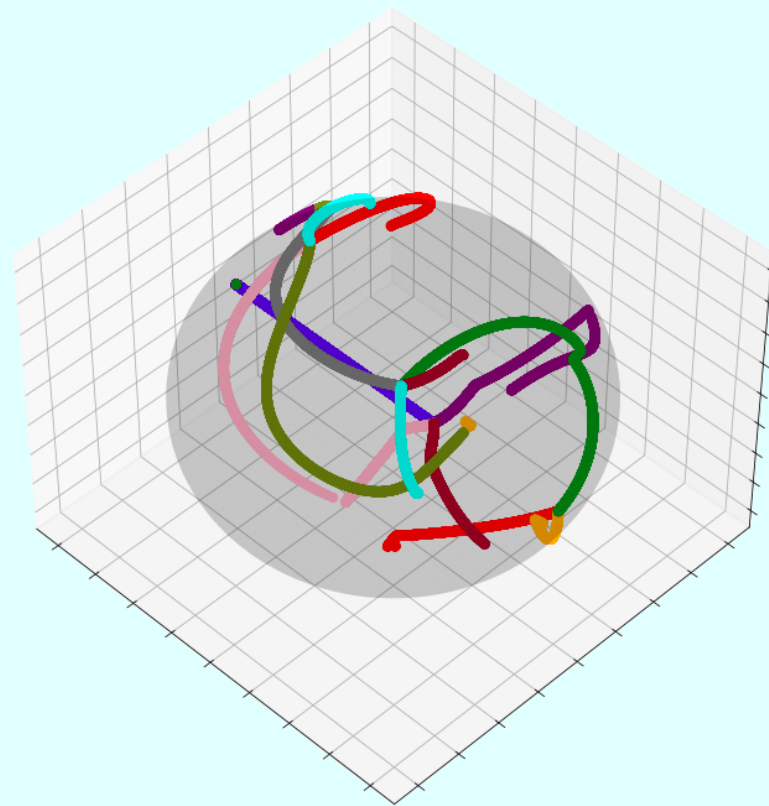
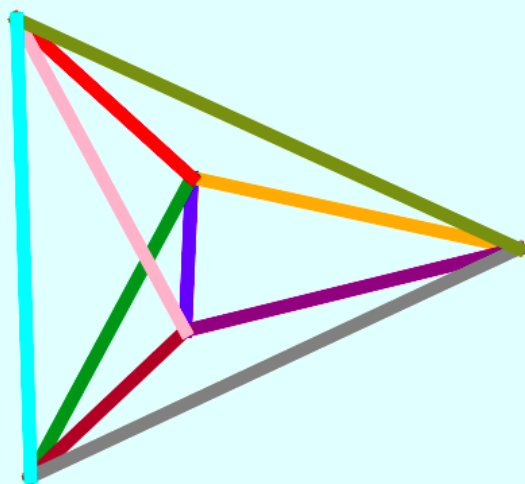


# Simplicial approximation to CW-complexes in practice



<https://raphaeltinarrage.github.io>

[T., *Computing Persistent Stiefel-Whitney classes of line bundles*, 2020]

Let  $X$  be a compact topological space. A **vector bundle** of dimension  $d$  on  $X$  can be described by a *classifying map*  $f: X \rightarrow \mathcal{G}_d(\mathbb{R}^n)$  (Grassmannian of  $d$  planes in  $\mathbb{R}^n$ ).

The map induced in cohomology,  $f^*: H^*(X) \leftarrow H^*(\mathcal{G}_d(\mathbb{R}^n))$ , allows to define the *Stiefel-Whitney* classes of the vector bundle

Computing these classes amounts to **triangulating**  $X$  and  $\mathcal{G}_d(\mathbb{R}^n)$ , and finding a simplicial approximation to  $f$ .

However, explicit triangulations of  $\mathcal{G}_d(\mathbb{R}^n)$  are known only when  $d = 1$  or  $n - 1$  (projective spaces). Next Grassmannian:  $\mathcal{G}_2(\mathbb{R}^4)$ .

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[Luis Polanco, Jose A. Perea, *Coordinatizing Data With Lens Spaces and Persistent Cohomology*, 2019]

To each  $\mathbb{Z}$ -principal bundle over  $X$  corresponds (up to homotopy) a map  $f: X \rightarrow \mathbb{S}^\infty / \mathbb{Z}/p\mathbb{Z}$  (infinite Lens space).

This offers the Lens-PCA reduction, a non-linear dimensionality reduction algorithm.

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→ In this talk, we will construct simplicial complexes **homotopy equivalent** to a given space, based on its **CW-structure**.

# I - Simplicial approximation to CW-complexes

- 1 - Topology CW-complexes
- 2 - Simplicial approximation
- 3 - Simplicial mapping cone
- 4 - Application to projective spaces

# II - Simplicial approximation improved

- 1 - Local subdivisions
- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

# III - Applications

- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

# Definition of CW-complexes

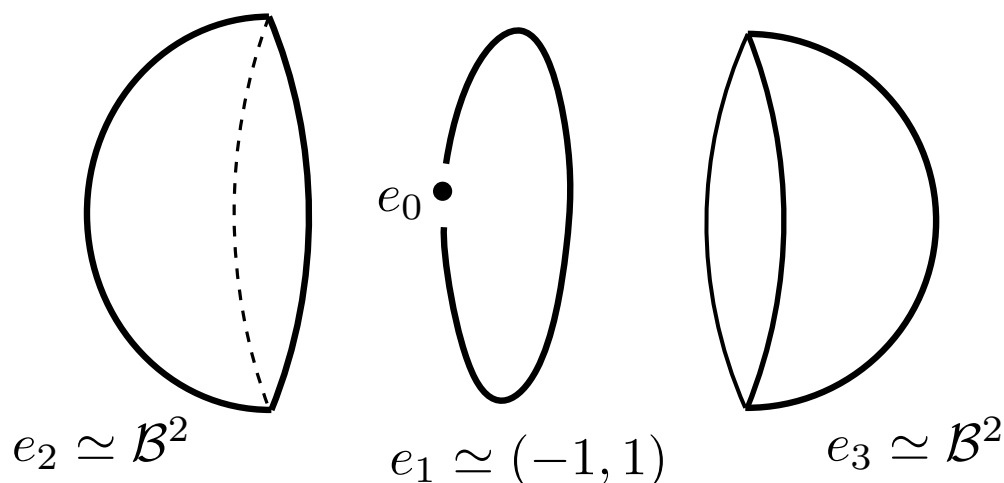
4/35 (1/2)

**Definition:** A finite **CW-complex** is a topological Hausdorff space  $X$  together with a finite partition  $\{e_i\}_i$  of  $X$  (the cells) such that:

1. For each  $e_i$ , there exists an integer  $n(i)$  and a homeomorphism  $\mathcal{B}^{n(i)} \rightarrow e_i$ , where  $\mathcal{B}^{n(i)}$  is the open ball of  $\mathbb{R}^{n(i)}$ .
2. Moreover, this homeomorphism extends to a continuous map from the closed ball,  $f_i: \overline{\mathcal{B}}^{n(i)} \rightarrow X$ , called a **characteristic map** for  $e_i$ . We denote by  $\bar{e}_i$  its image. Its restriction to the sphere, denoted  $\phi_i: \partial \overline{\mathcal{B}}^{n(i)} \rightarrow X$ , is called the **gluing map**.
3. Each point  $x \in \bar{e}_i \setminus e_i$  must lie in a cell  $e_j$  of lower dimension.

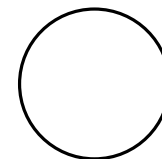
**Example:** The sphere  $\mathbb{S}^2$  admits a CW-structure with

one cell of dimension 0,
one cell of dimension 1,
two cells of dimension 2.



$$\partial(-1, 1) = \{-1, 1\} \quad \bullet \cdots \bullet$$

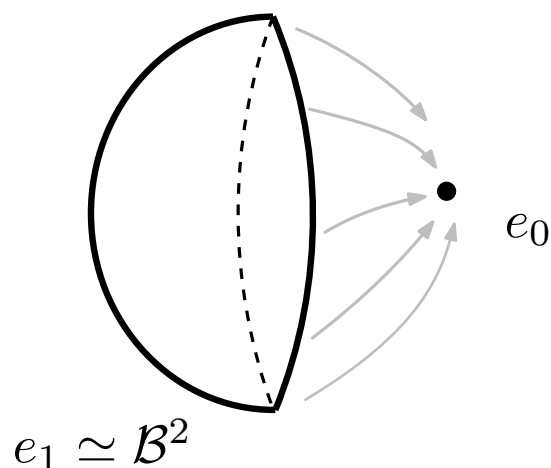
$$\partial \mathcal{B}^2 = \mathbb{S}^1$$



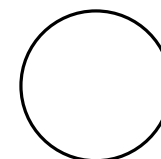
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**Example:** The sphere  $\mathbb{S}^2$  admits a CW-structure with  $\left| \begin{array}{l} \text{one cell of dimension 0,} \\ \text{one cell of dimension 2.} \end{array} \right.$



$$\partial\mathcal{B}^2 = \mathbb{S}^1$$



CW-complex structures provide concise descriptions of spaces:

- The **sphere** of dimension  $n$ : one 0-cell, and one  $n$ -cell
- The **projective space** of dimension  $n$ : one cell per dimension
- The **real Grassmannian**  $\mathcal{G}_d(\mathbb{R}^n)$  (dimension  $d(n-d)$ ):  $p(k)$  cells of dimension  $k$ , where  $p(k)$  is the number of partitions of  $k$  into at most  $d$  integers each of which is  $\leq n-d$  (in total,  $\binom{n}{d}$  cells) [Milnor, Stasheff, *Characteristic classes*]
- The **lens space**  $L_p(q_1, \dots, q_n)$  (dimension  $2n-1$ ): one cell per dimension [Hatcher, *Algebraic topology*]

Triangulations are more complicated:

- The **sphere** of dimension  $n$ : at least  $n+1$  vertices,  $n+1$  facets,  $2^n - 1$  simplices
- The **projective space** of dimension  $n$ : must have  $\geq \frac{(n+1)(n+2)}{2}$  vertices [Arnoux, Marin, 1991]. We know a triangulation with  $2^{n+1} - 1$  vertices [Kühnel, 1987].
- The **real Grassmannian**  $\mathcal{G}_d(\mathbb{R}^n)$  (dimension  $d(n-d)$ ): a triangulation must admit  $\frac{n(n+1)}{2}$  vertices and  $(n(n-1) - 2d(n-d))2^{d(n-d)+1} - 1$  simplices [Govc, Marzantowicz, Pavešić, 2020].



## Cellularisation:

- Any topological manifold of dimension  $d \neq 4$  is homeomorphic to a CW-complex [Kirby, Siebenmann] [Quinn]
- Any topological manifold is homotopy equivalent to a CW complex [Kirby, Siebenmann, 1969]

## Triangulations:

- Any PL-manifold is triangulable
- Any smooth manifold admits a PL-structure hence is triangulable [Cairns, 1935] [Whitehead, 1940]
- Any topological manifold is homotopy equivalent to a simplicial complex

## Cellularisation:

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## Triangulations:

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simplicial approximation to CW complexes

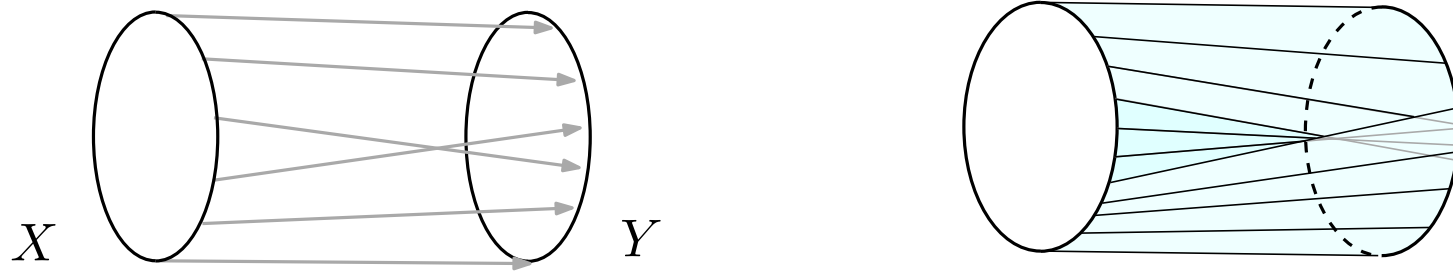


# Mapping cones

6/35 (1/3)

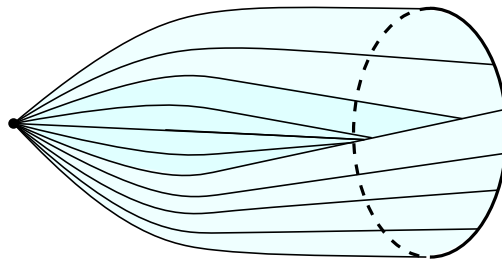
Let  $f: X \rightarrow Y$  be a continuous map. The **mapping cylinder** is the quotient space

$$\text{Cyl}(f) = X \times [0, 1] \sqcup Y / (x, 1) \sim f(x)$$



The **mapping cone** is obtained by identifying the upper part of the cylinder

$$\text{Cone}(f) = \text{Cyl}(f) / (x, 0) \sim \text{point}$$



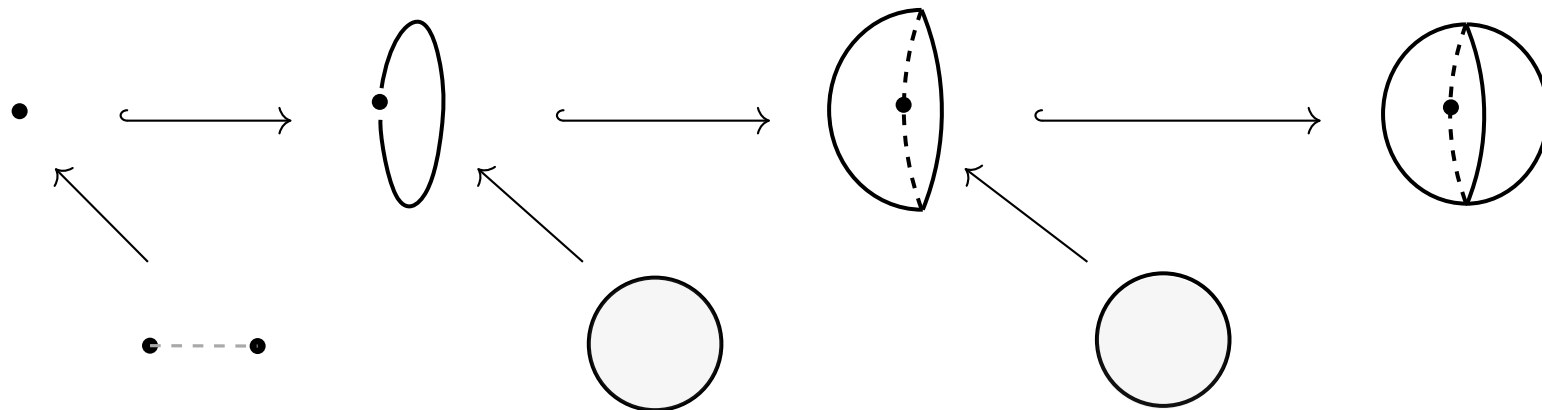
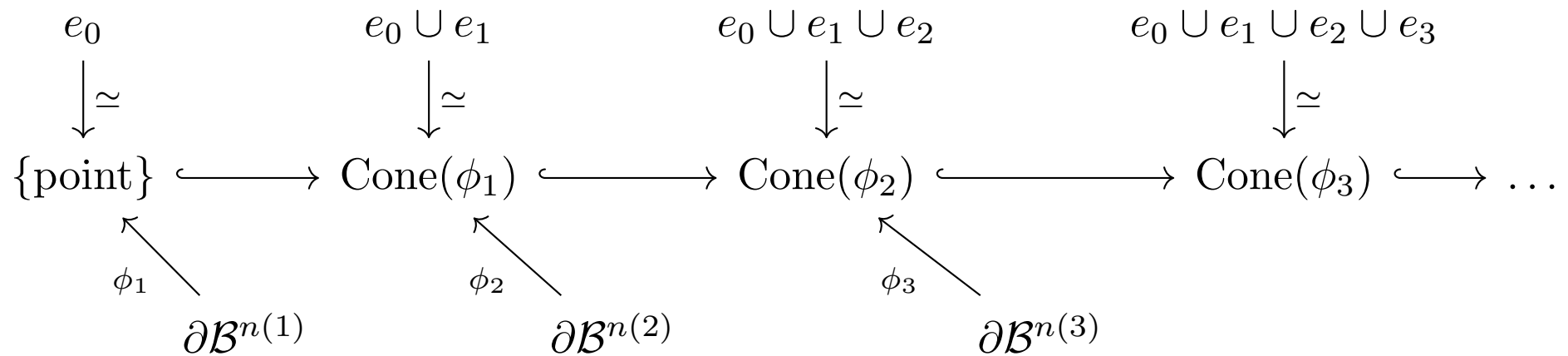
**Lemma:** If  $f, g: X \rightarrow Y$  are homotopic, then so are  $\text{Cone}(f)$  and  $\text{Cone}(g)$ .

# Mapping cones

6/35 (2/3)

Let  $X$  be a CW-complex, with cells  $\{e_i\}_i$ , and gluing maps  $\phi_i: \partial\mathcal{B}^{n(i)} \rightarrow X$ .

One shows that  $X$  is homeomorphic to the sequence of mapping cones



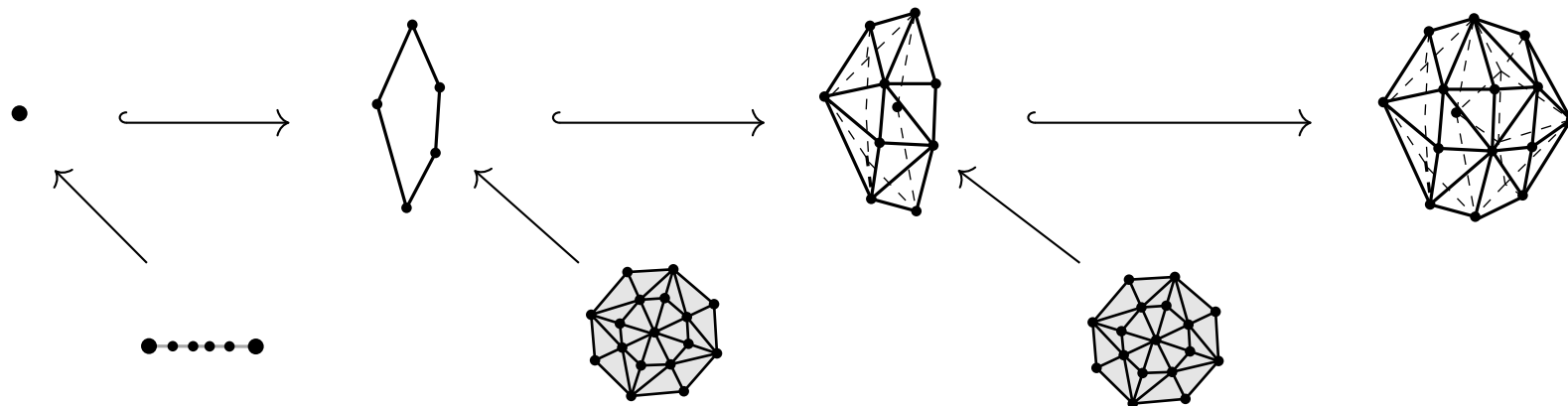
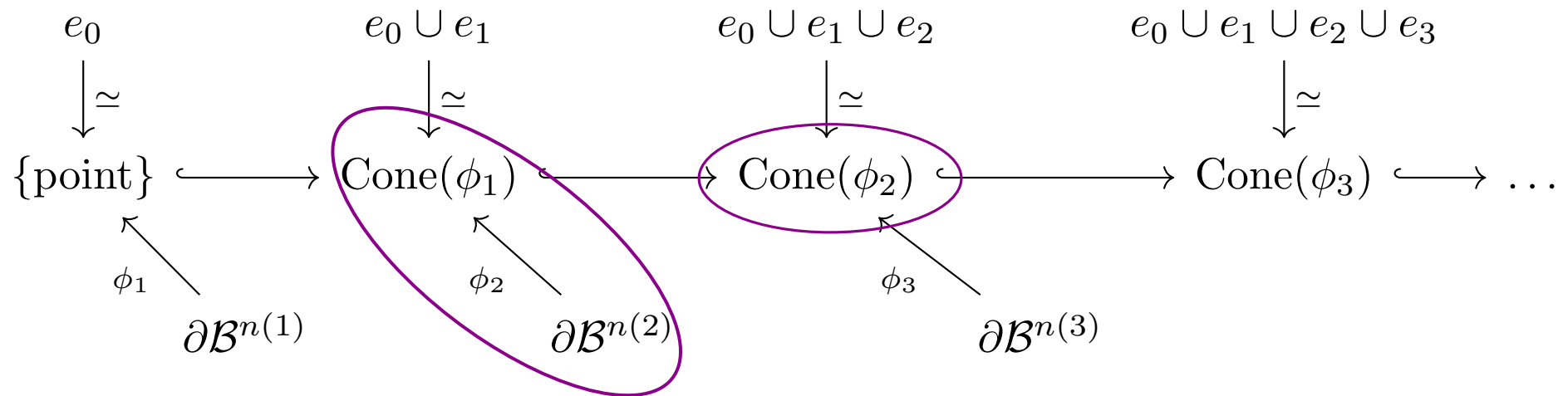
One builds  $X$  by induction:  $X^i = \text{Cone}(\phi_i: \partial\mathcal{B}^{n(i)} \rightarrow X^{i-1})$ .

# Mapping cones

6/35 (3/3)

Let  $X$  be a CW-complex, with cells  $\{e_i\}_i$ , and gluing maps  $\phi_i: \partial\mathcal{B}^{n(i)} \rightarrow X$ .

One shows that  $X$  is homeomorphic to the sequence of mapping cones



In order to triangulate  $X$ , it is enough to triangulate mapping cones!

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- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

# III - Applications

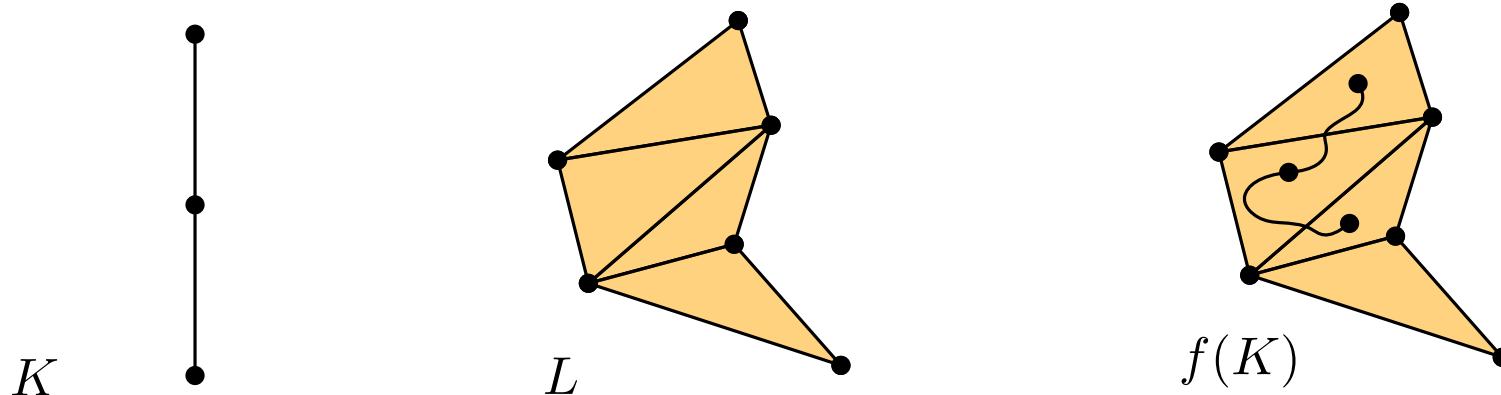
- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

# Simplicial approximation

8/35 (1/2)

Consider two simplicial complexes  $K, L$  and a continuous map between their geometric realizations  $f: |K| \longrightarrow |L|$ .

We look for a simplicial map  $g: K \longrightarrow L$  whose geometric realization  $|g|: |K| \rightarrow |L|$  is homotopic to  $f$ .



# Simplicial approximation

8/35 (2/2)

Consider two simplicial complexes  $K, L$  and a continuous map between their geometric realizations  $f: |K| \rightarrow |L|$ .

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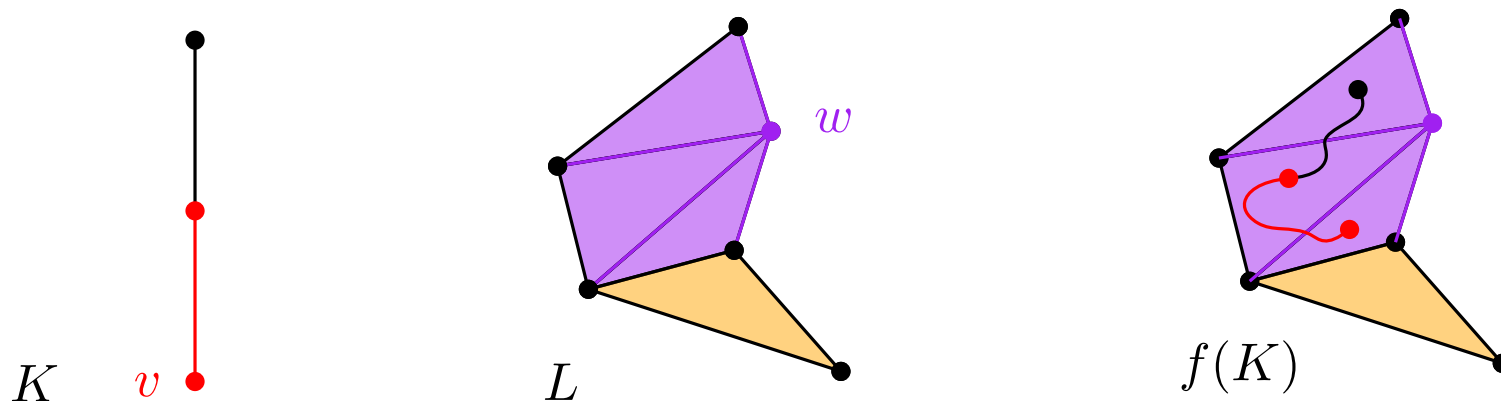
Define, for all vertex  $v \in K^{(0)}$ , its *open star* and its *closed star*

$$\text{St}(v) = \{\sigma \in K, v \in \sigma\} \quad \overline{\text{St}}(v) = \{\tau \in K, \exists \sigma \in \text{St}(v), \tau \subset \sigma\}$$

The map  $f$  satisfies the **star condition** for every  $v \in K^{(0)}$ , there exists  $w \in L^{(0)}$  such that  $f(|\overline{\text{St}}(v)|) \subseteq |\text{St}(w)|$ .

If this is the case, let  $g: K^{(0)} \rightarrow L^{(0)}$  be such that for every  $v \in K^{(0)}$ , we have  $f(|\overline{\text{St}}(v)|) \subseteq |\text{St}(g(v))|$ .

Such a map  $g$  is called a **simplicial approximation** to  $f$ . It is a simplicial map. Its geometric realization  $|g|$  is homotopic to  $f$ .

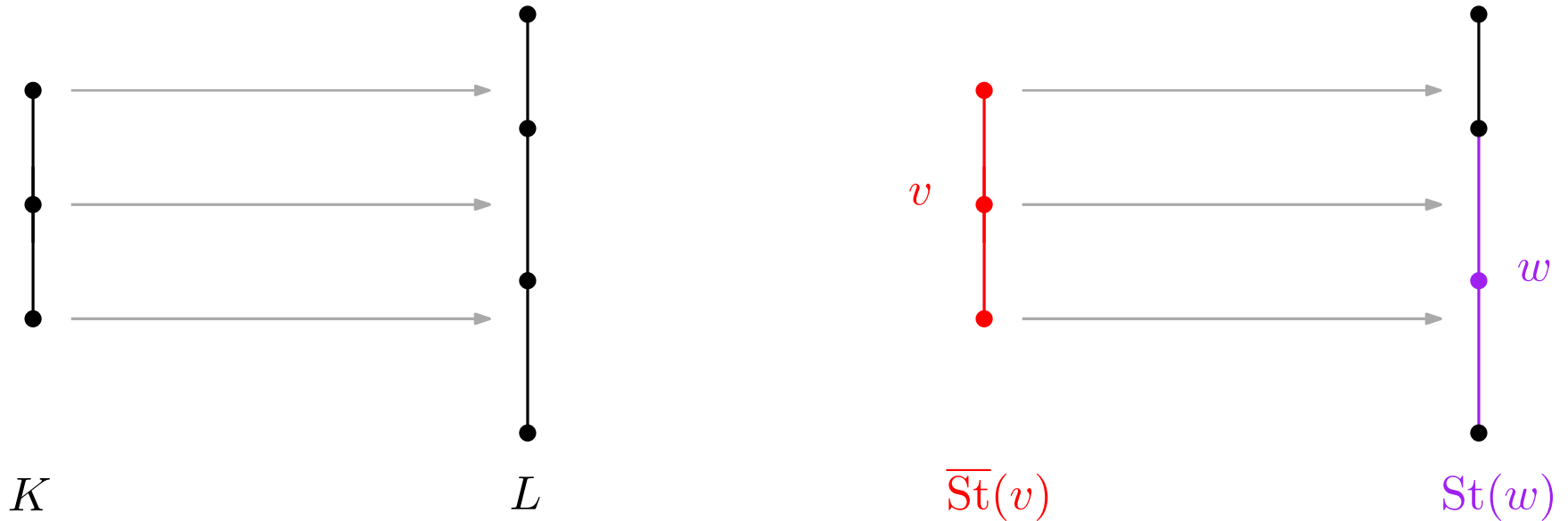




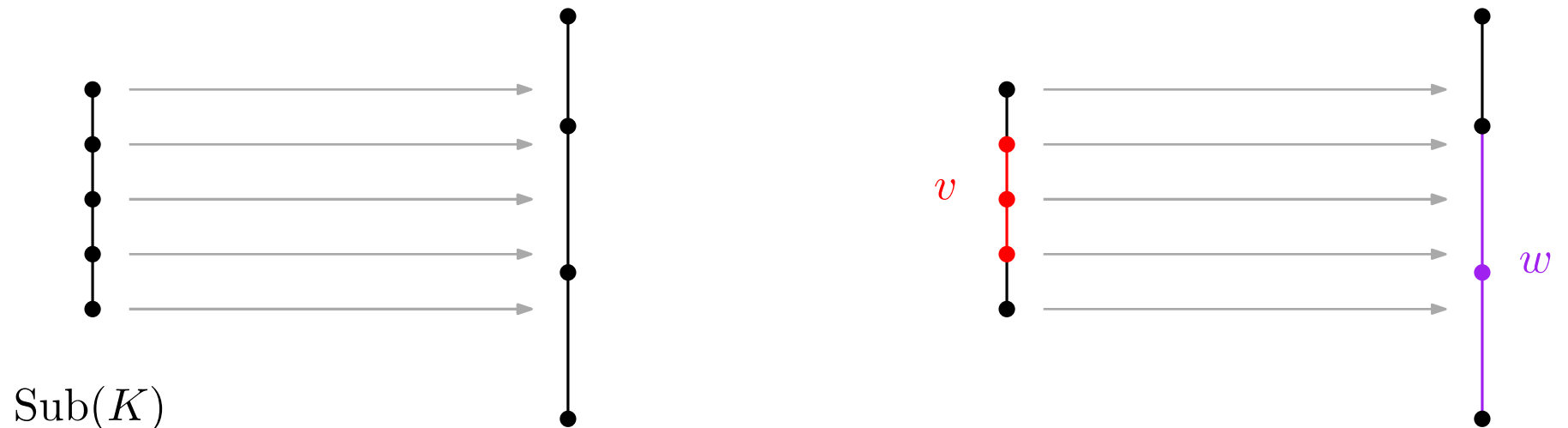
# Barycentric subdivisions

9/35 (1/2)

What if the map  $f$  does not satisfy the star condition?



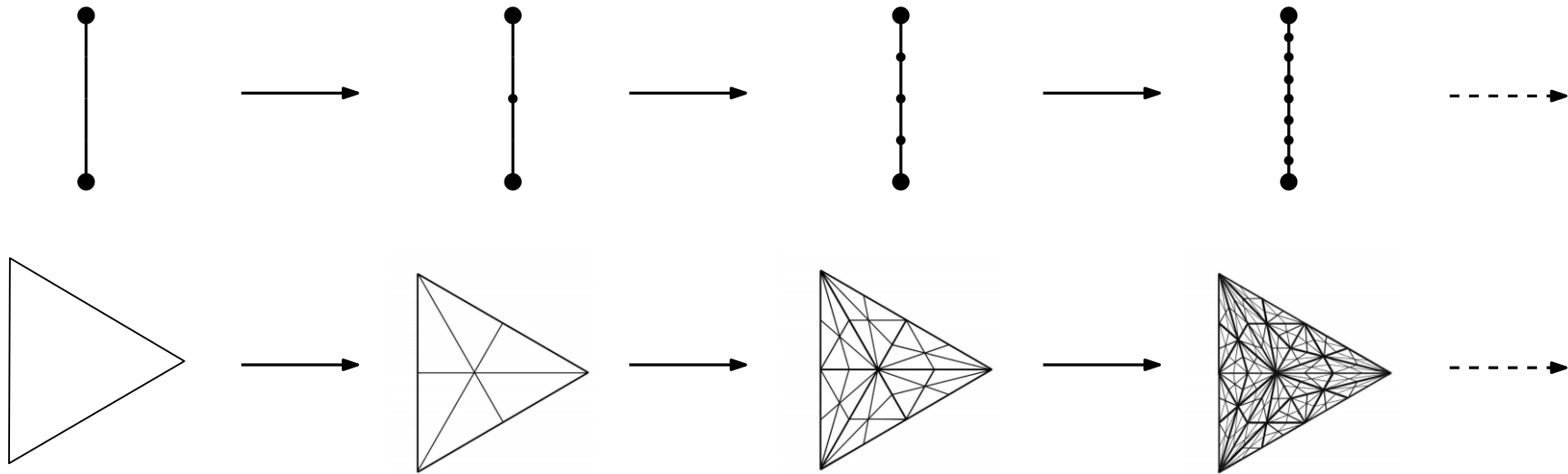
One can refine  $K$  via *barycentric subdivision*.



# Barycentric subdivisions

9/35 (2/2)

**Simplicial approximation theorem:** By repeating barycentric subdivisions on  $K$ , the map  $f: \text{Sub}^k(K) \rightarrow L$  satisfies the star condition at some point.



**Proof:** Endow  $K$  with a metric, and by denote  $\mathcal{U}$  the cover  $\{f^{-1}(|\text{St}(w)|), w \in L\}$  of  $|K|$ . A **Lebesgue number** for  $\mathcal{U}$  is a  $\epsilon > 0$  such that

$$\forall x \in |K|, \exists U \in \mathcal{U}, \mathcal{B}(x, \epsilon) \subset U \text{ (open ball of radius } \epsilon \text{)}.$$

Hence,  $f$  satisfies the star condition if for every  $v \in K^{(0)}$ ,  $\text{Diameter}(|\overline{\text{St}}(v)|) < \epsilon$ .

But barycentric subdivision reduces the diameter of a  $d$ -simplex by a factor  $\frac{d}{d+1}$ . Hence each simplex is small enough at some point.

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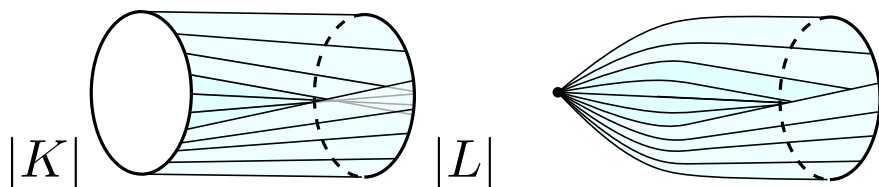
# III - Applications

- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

# Triangulation of mapping cones

11/35 (1/2)

How to triangulate the mapping cone of  $f: |K| \rightarrow |L|$ ?



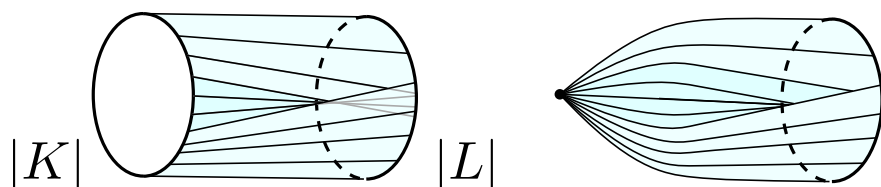
$$\text{Cyl}(f) = X \times [0, 1] \sqcup Y / (x, 1) \sim f(x)$$

$$\text{Cone}(f) = \text{Cyl}(f) / (x, 0) \sim \text{point}$$

# Triangulation of mapping cones

11/35 (2/2)

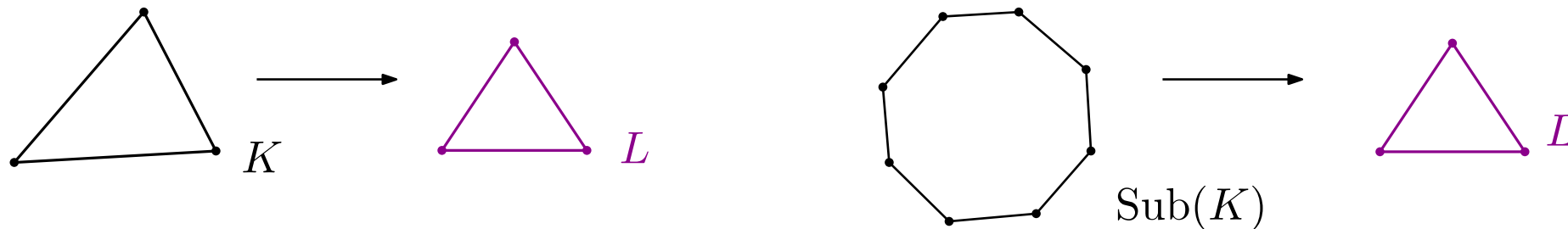
How to triangulate the mapping cone of  $f: |K| \rightarrow |L|$ ?



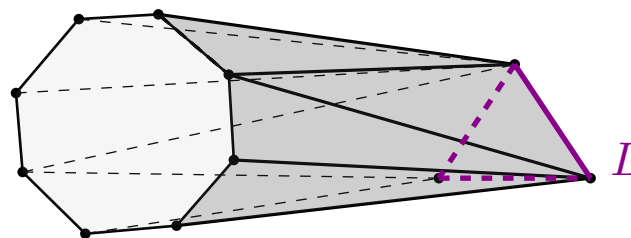
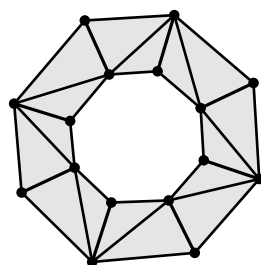
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1. Find a simplicial approximation  $g: K \rightarrow L$  to  $f: |K| \rightarrow |L|$

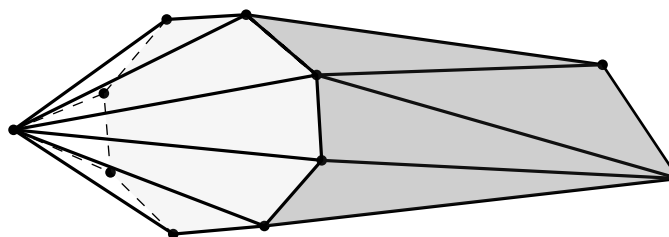


2. Triangulate  $|K| \times [0, 1]$  and glue  $L$  at the end



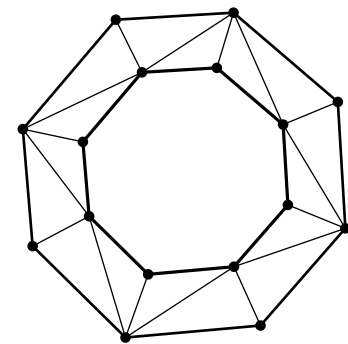
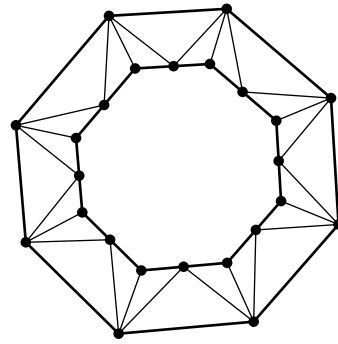
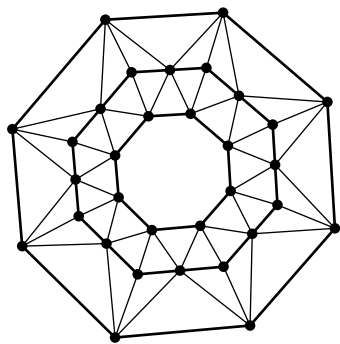
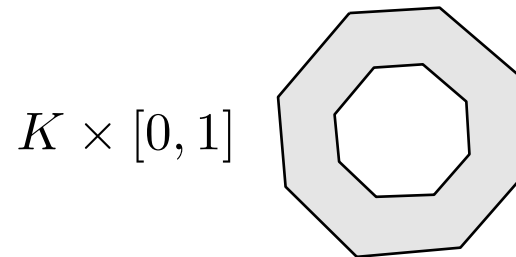
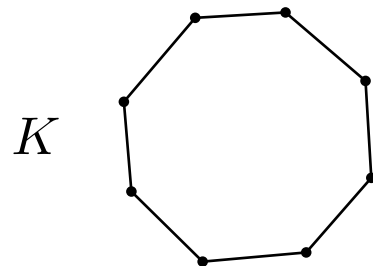
3. Cone the upper part of the cylinder

$$\text{Cyl}_{\text{simplicial}}(g)$$



# Triangulation of mapping cylinders 12/35 (1/5)

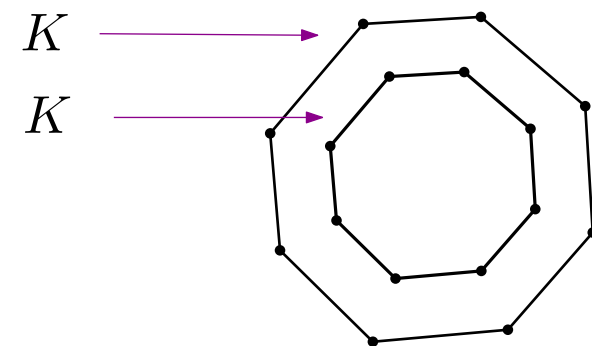
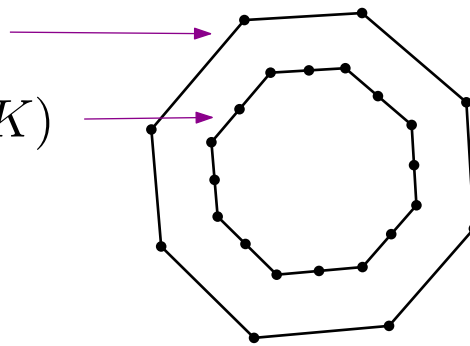
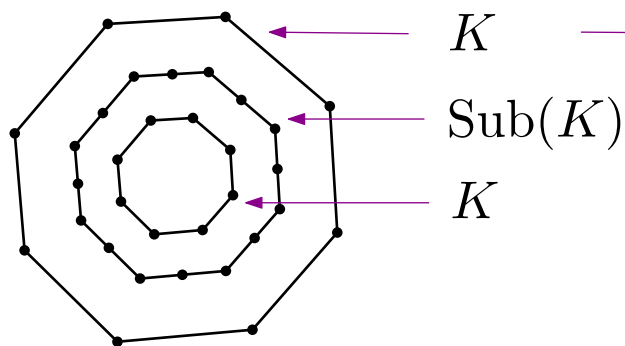
2. Let  $K$  be a simplicial complex. How to triangulate  $|K| \times [0, 1]$ ?



[Whitehead, *Simplicial spaces, nuclei and m-groups*, 1939]

[Cohen, *Simplicial structures and transverse cellularity*, 1967]

Simplicial set product



# Triangulation of mapping cylinders 12/35 (2/5)

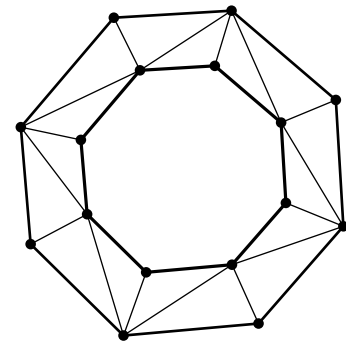
2. Let  $K$  be a simplicial complex. How to triangulate  $|K| \times [0, 1]$ ?

Suppose that the vertices of  $K$  are totally ordered.

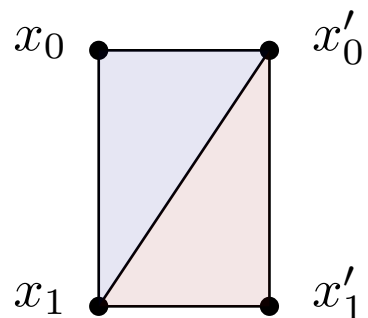
- Take two copies of  $K$ ,  $K \times \{0\}$  and  $K \times \{1\}$ .
- For every  $\sigma = [x_0, \dots, x_m] \in K \times \{0\}$ , with vertices ordered, consider the corresponding vertices  $x'_0 < \dots < x'_m$  of  $K \times \{1\}$ .
- For every  $0 \leq k < m$ , consider the  $m + 1$  simplex

$$\sigma_k = [x_k, \dots, x_m, x'_0, \dots, x'_k] \in K \star K.$$

- The collection of all  $\sigma_k \in K \star K$  for all  $\sigma \in K$  form the simplicial product.



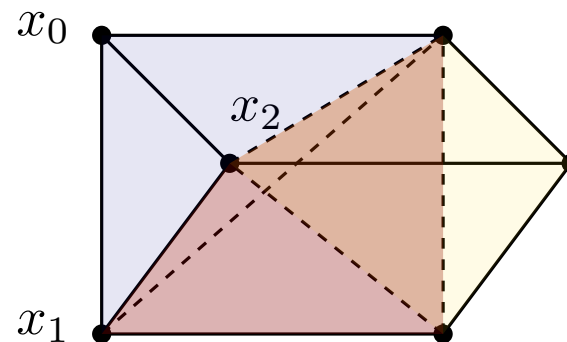
Simplicial set product



$K \times \{0\}$        $K \times \{1\}$

$$[x_0, x_1, x'_0]$$

$$[x_1, x'_0, x'_1]$$



$$[x_0, x_1, x_2, x'_0]$$

$$[x_1, x_2, x'_0, x'_1]$$

$$[x_2, x'_0, x'_1, x'_2]$$

# Triangulation of mapping cylinders 12/35 (3/5)

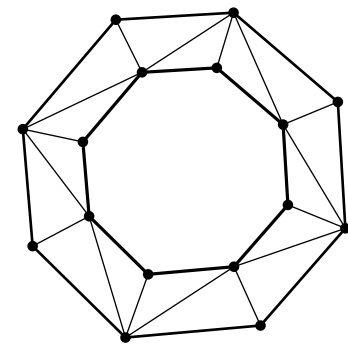
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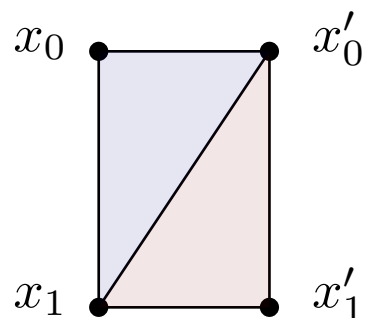
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- For every  $\sigma = [x_0, \dots, x_m] \in K \times \{0\}$ , with vertices ordered, consider the corresponding vertices  $x'_0 < \dots < x'_m$  of  $K \times \{1\}$ .
- For every  $0 \leq k < m$ , consider the  $m + 1$  simplex

$$\sigma_k = [x_k, \dots, x_m, x'_0, \dots, x'_k] \in K \star K.$$

- The collection of all  $\sigma_k \in K \star K$  for all  $\sigma \in K$  form the simplicial product.



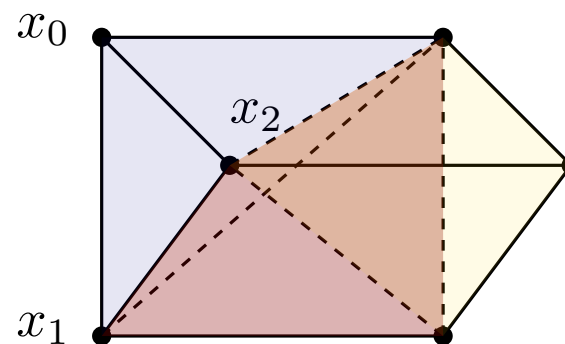
Simplicial set product



$K \times \{0\}$        $K \times \{1\}$

$$[x_0, x_1, x'_0]$$

$$[x_1, x'_0, x'_1]$$



$$[x_0, x_1, x_2, x'_0]$$

$$[x_1, x_2, x'_0, x'_1]$$

$$[x_2, x'_0, x'_1, x'_2]$$



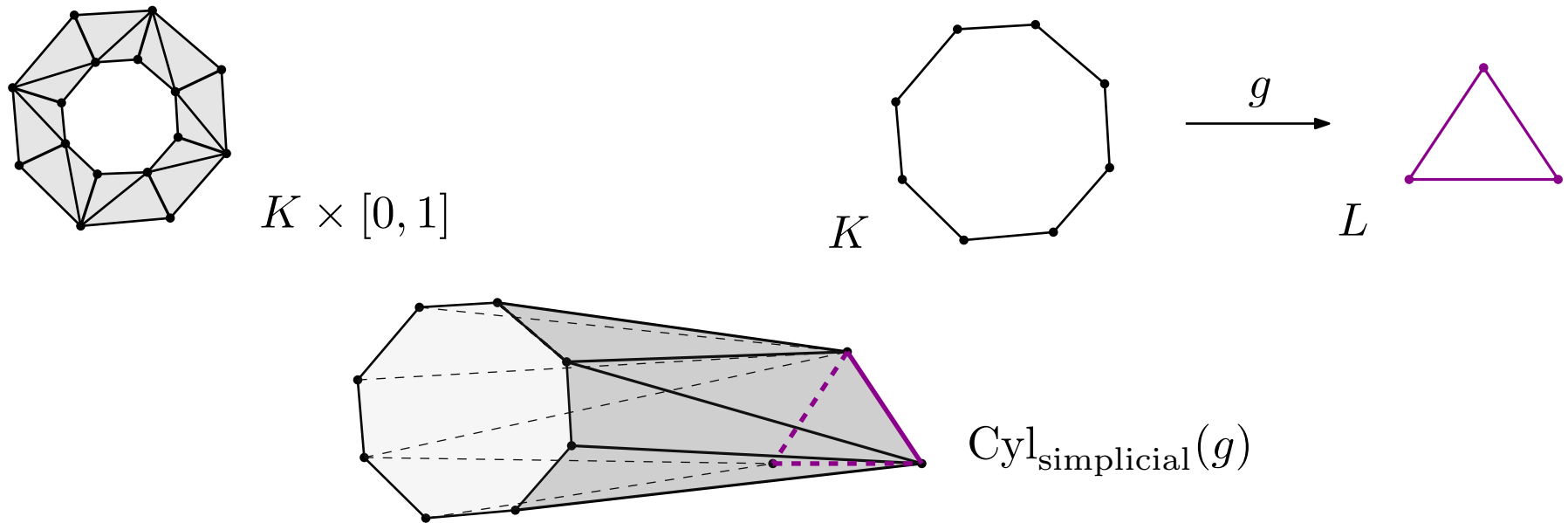
This construction depends on a choice (of an ordering)!



# Triangulation of mapping cylinders 12/35 (4/5)

Once  $K \times [0, 1]$  is triangulated, we glue it to  $L$  via  $g: K \rightarrow L$  as follows:

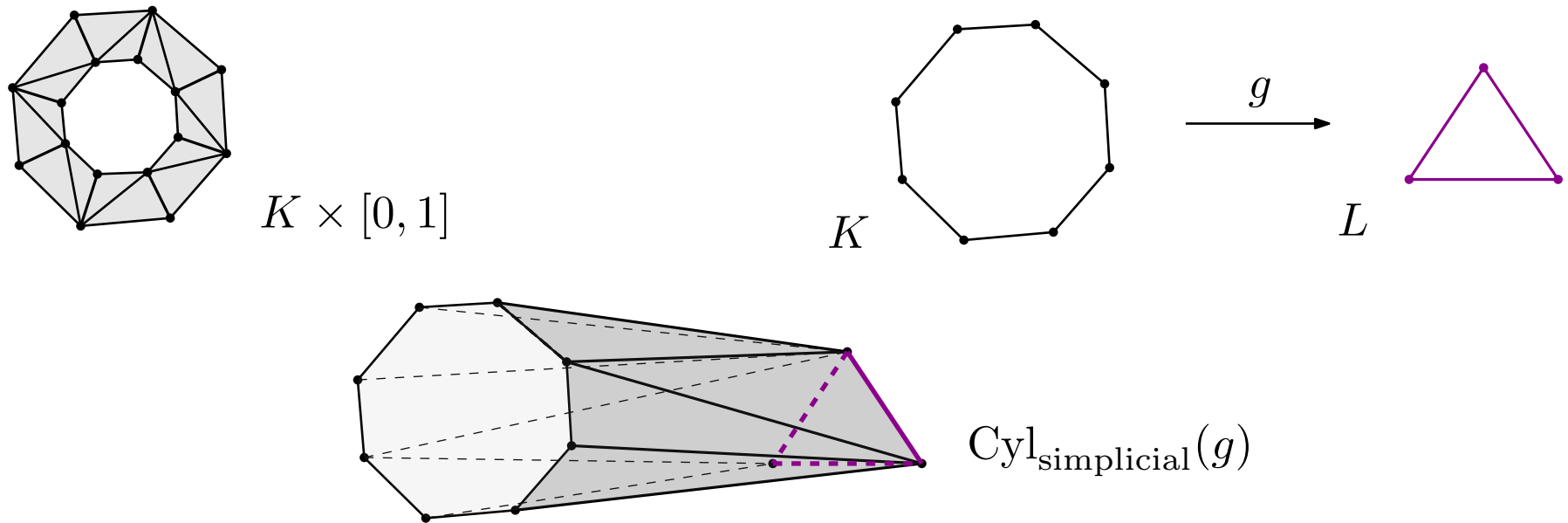
$$\text{Cyl}_{\text{simplicial}}(g) = \left\{ \sigma_0 \sqcup g(\sigma_1), \sigma \in K \times [0, 1], \sigma = \sigma_0 \sqcup \sigma_1, \begin{array}{l} \sigma_0 \in K \times \{0\} \\ \sigma_1 \in K \times \{1\} \end{array} \right\}$$



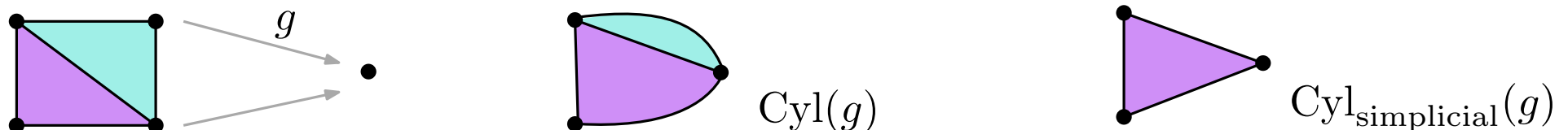
# Triangulation of mapping cylinders 12/35 (5/5)

Once  $K \times [0, 1]$  is triangulated, we glue it to  $L$  via  $g: K \rightarrow L$  as follows:

$$\text{Cyl}_{\text{simplicial}}(g) = \left\{ \sigma_0 \sqcup g(\sigma_1), \sigma \in K \times [0, 1], \sigma = \sigma_0 \sqcup \sigma_1, \begin{array}{l} \sigma_0 \in K \times \{0\} \\ \sigma_1 \in K \times \{1\} \end{array} \right\}$$



The map  $\text{Cyl}(g) \rightarrow \text{Cyl}_{\text{simplicial}}(g)$  **may not be** a homeomorphism!



**Proposition:** It is a homotopy equivalence.

# Sketch of algorithm.

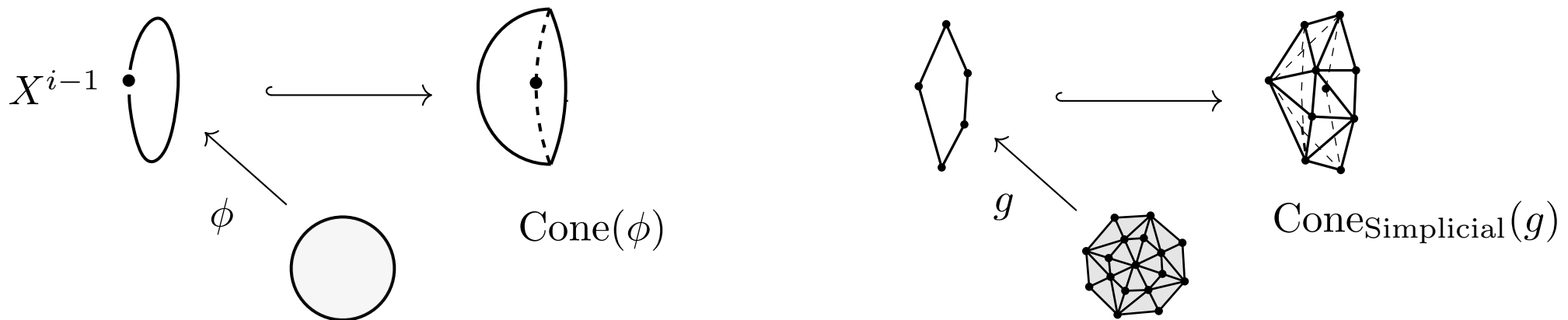
13/35

Let  $X$  be a CW-complex, with cells  $\{e_i\}_i$ , and gluing maps  $\phi_i: \partial\mathcal{B}^{n(i)} \rightarrow X$ .

It can be built by induction:  $X^i = \text{Cone}(\phi_i: \partial\mathcal{B}^{n(i)} \rightarrow X^{i-1})$ .

We will do the same, but with simplicial approximations  $g_i$  to  $\phi_i$  instead:

$$K^i = \text{Cone}_{\text{simplicial}}(g_i: \partial\mathcal{B}^{n(i)} \rightarrow K^{i-1}).$$



# I - Simplicial approximation to CW-complexes

- 1 - Topology CW-complexes
- 2 - Simplicial approximation
- 3 - Simplicial mapping cone
- 4 - Application to projective spaces

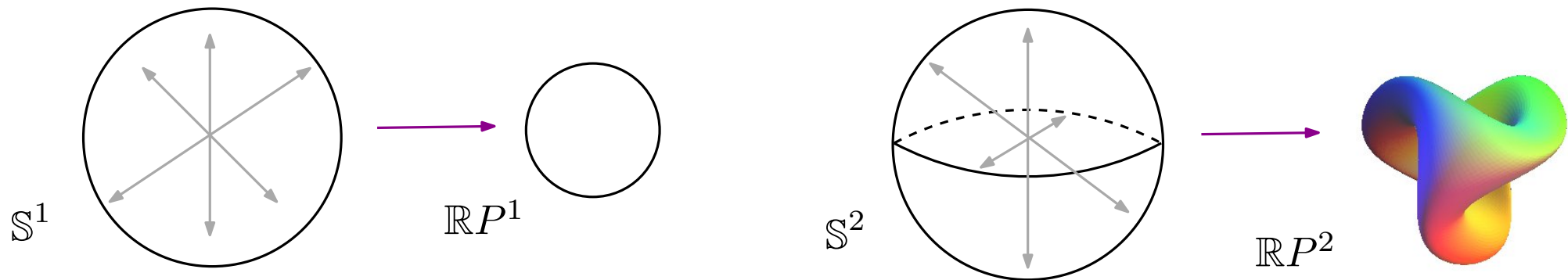
# II - Simplicial approximation improved

- 1 - Local subdivisions
- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

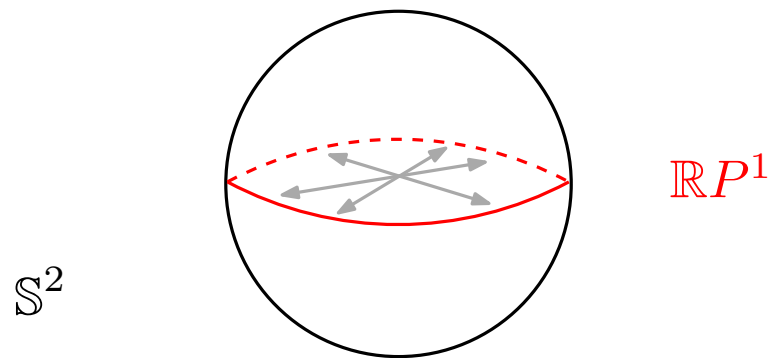
# III - Applications

- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

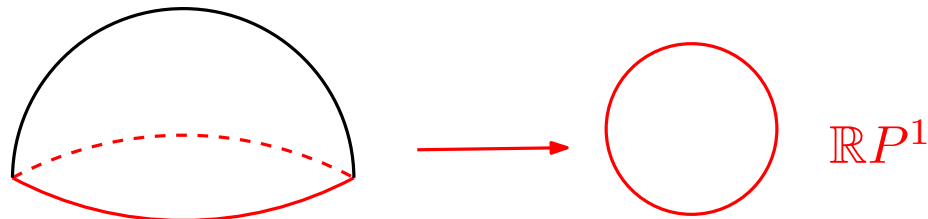
The  $n$ -dimensional projective space  $\mathbb{R}P^n$  can be defined as the quotient of the  $n$ -sphere  $S^n \subset \mathbb{R}^{n+1}$  by the antipodal relation  $x \sim -x$ .



The projective space  $\mathbb{R}P^n$  contains  $\mathbb{R}P^{n-1}$  as its equator.



By induction, we get a CW-structure for  $\mathbb{R}P^n$ : take a hemisphere of  $S^n$ , and glue its equator on  $\mathbb{R}P^{n-1}$ .



$\mathbb{R}P^0$  :

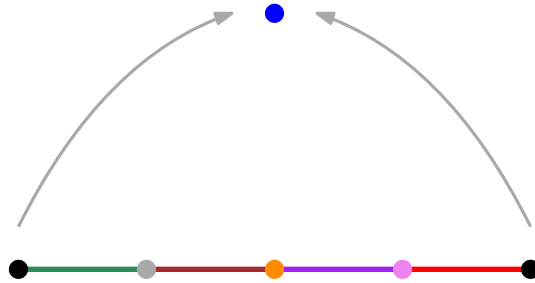


# Simplicial approximation to $\mathbb{R}P^n$

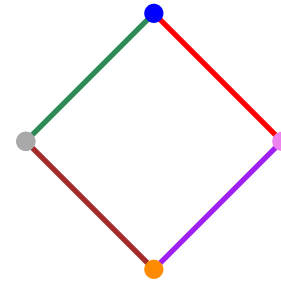
16/35 (2/10)

$\mathbb{R}P^0$  :

1-cell:



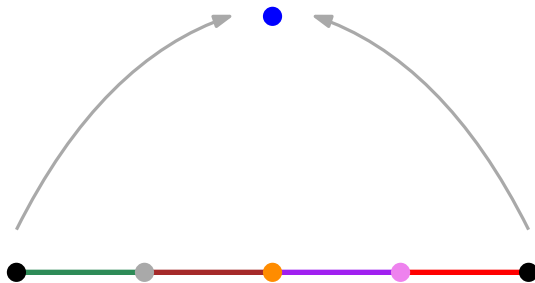
$\mathbb{R}P^1$  :



# Simplicial approximation to $\mathbb{R}P^n$

16/35 (3/10)

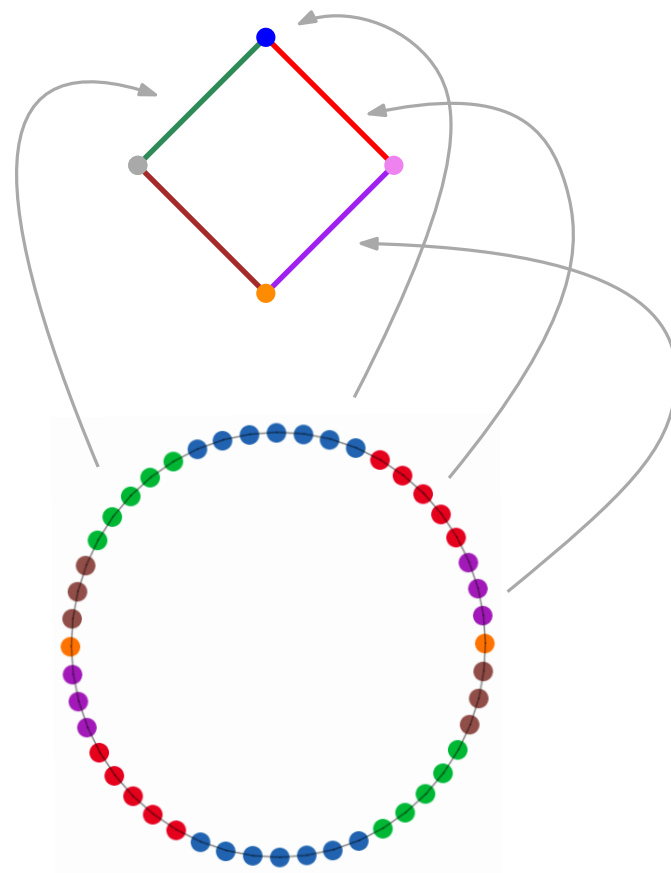
$\mathbb{R}P^0$  :



1-cell:

$\mathbb{R}P^1$  :

boundary  
of 2-cell:  
4 barycentric  
subdivisions

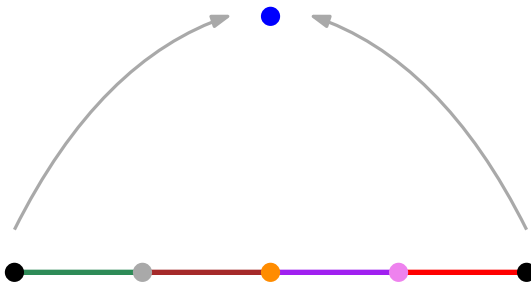




# Simplicial approximation to $\mathbb{R}P^n$

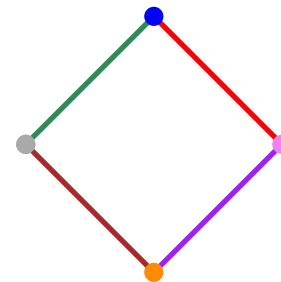
16/35 (4/10)

$\mathbb{R}P^0$  :

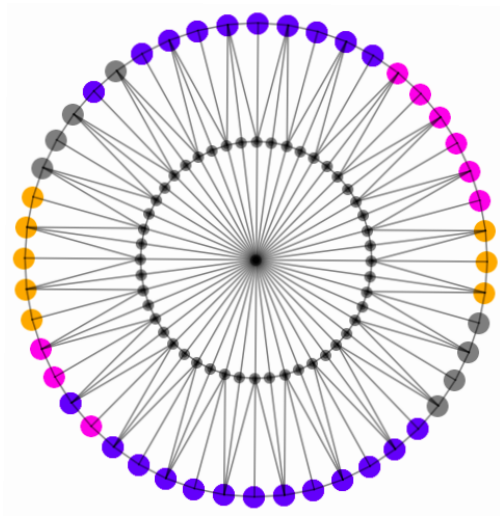


1-cell:

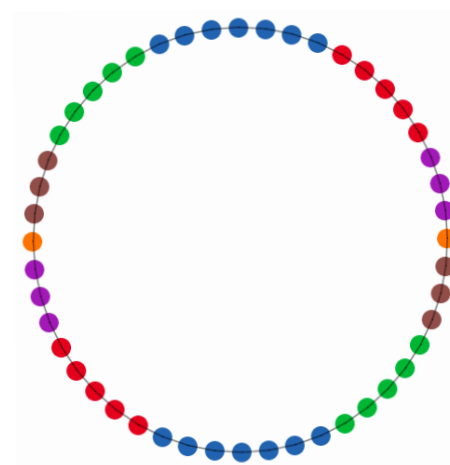
$\mathbb{R}P^1$  :



2-cell:



boundary  
of 2-cell:  
4 barycentric  
subdivisions

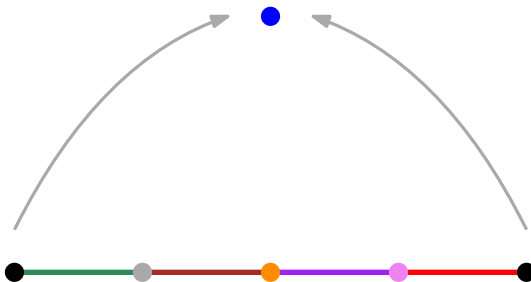


$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

# Simplicial approximation to $\mathbb{R}P^n$

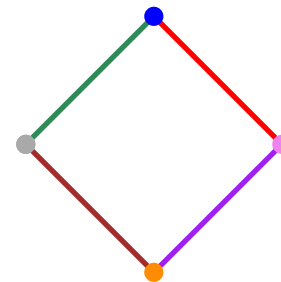
16/35 (5/10)

$\mathbb{R}P^0$  :



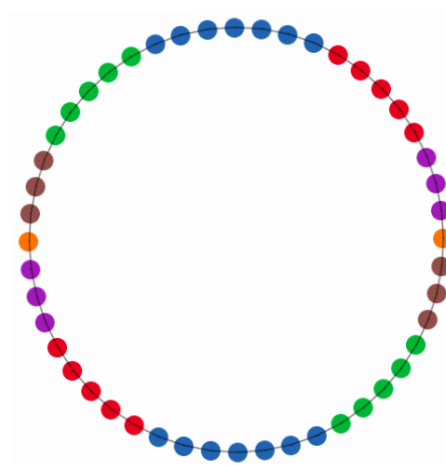
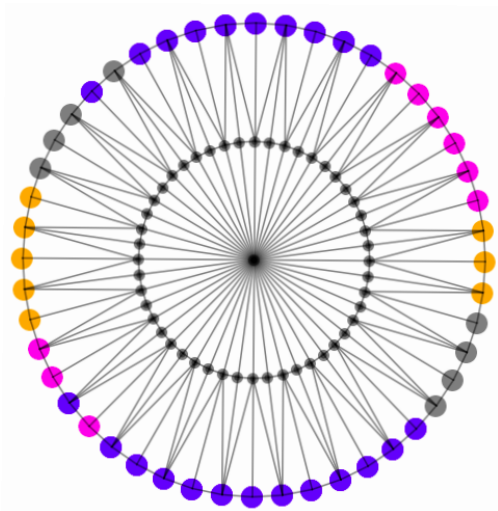
1-cell:

$\mathbb{R}P^1$  :



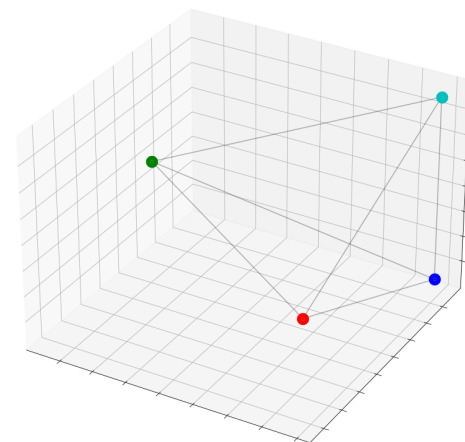
2-cell:

boundary  
of 2-cell:  
4 barycentric  
subdivisions



$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

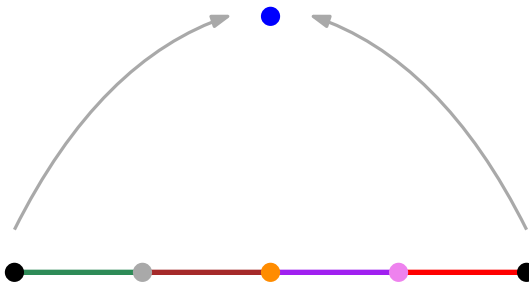
boundary  
of 3-cell:



# Simplicial approximation to $\mathbb{R}P^n$

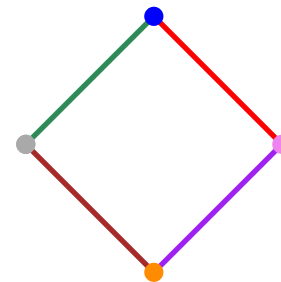
16/35 (6/10)

$\mathbb{R}P^0$  :



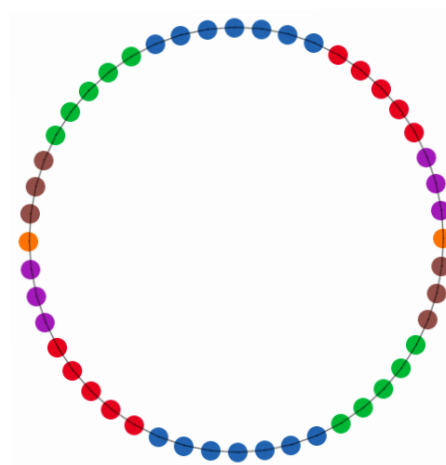
1-cell:

$\mathbb{R}P^1$  :



2-cell:

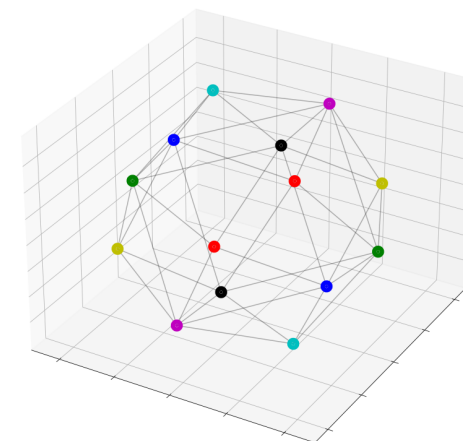
boundary  
of 2-cell:  
4 barycentric  
subdivisions



$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

boundary  
of 3-cell:

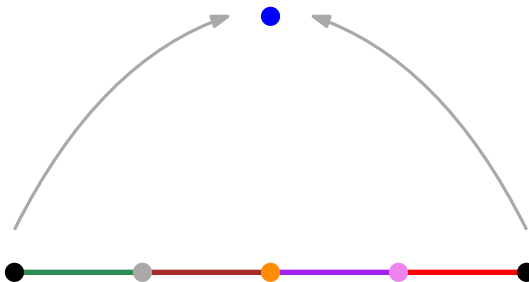
1 barycentric subdivision



# Simplicial approximation to $\mathbb{R}P^n$

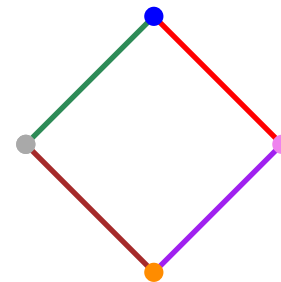
16/35 (7/10)

$\mathbb{R}P^0$  :

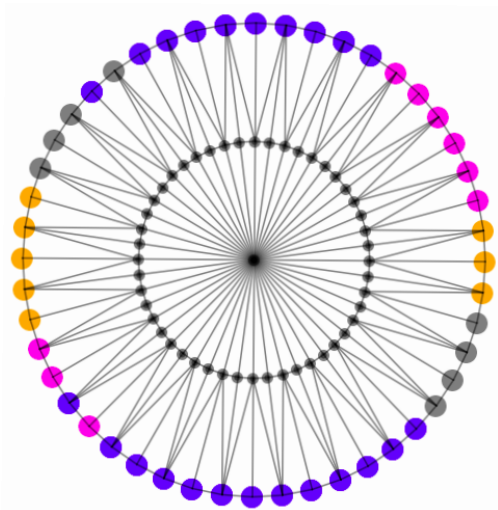


1-cell:

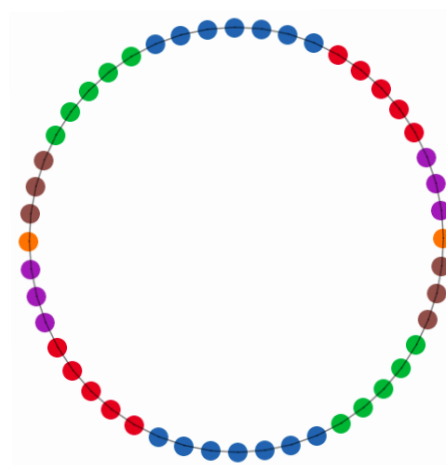
$\mathbb{R}P^1$  :



2-cell:



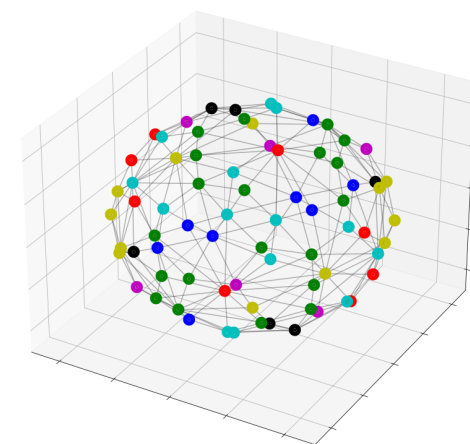
boundary  
of 2-cell:  
4 barycentric  
subdivisions



$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

boundary  
of 3-cell:

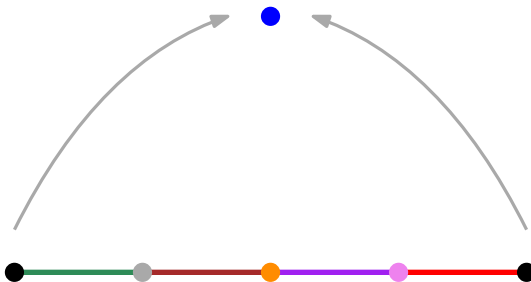
2 barycentric subdivisions



# Simplicial approximation to $\mathbb{R}P^n$

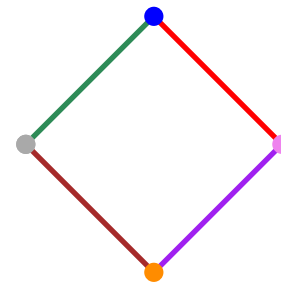
16/35 (8/10)

$\mathbb{R}P^0$  :

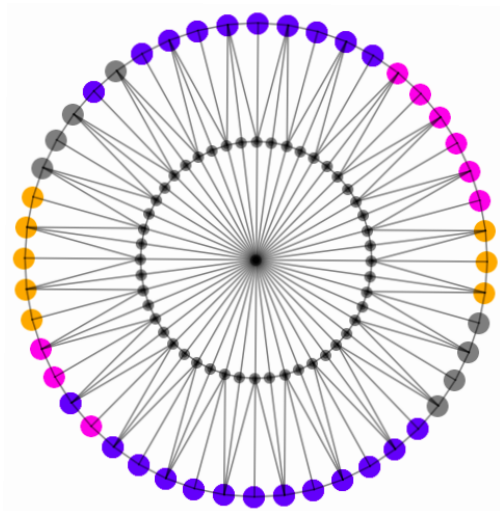


1-cell:

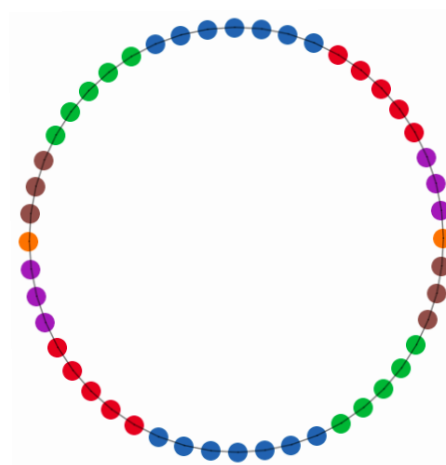
$\mathbb{R}P^1$  :



2-cell:



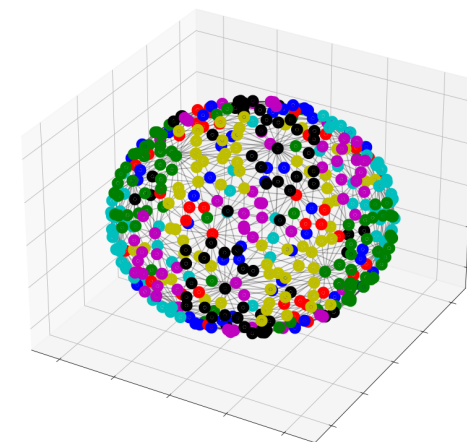
boundary  
of 2-cell:  
4 barycentric  
subdivisions



$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

boundary  
of 3-cell:

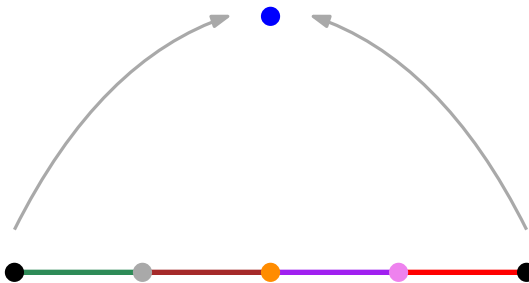
3 barycentric subdivisions



# Simplicial approximation to $\mathbb{R}P^n$

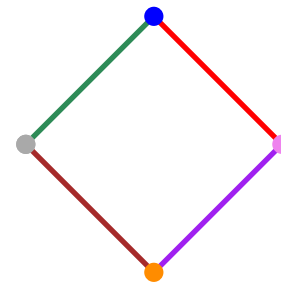
16/35 (9/10)

$\mathbb{R}P^0$  :

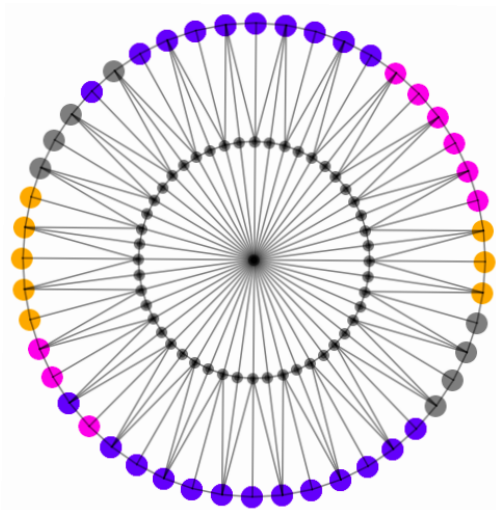


1-cell:

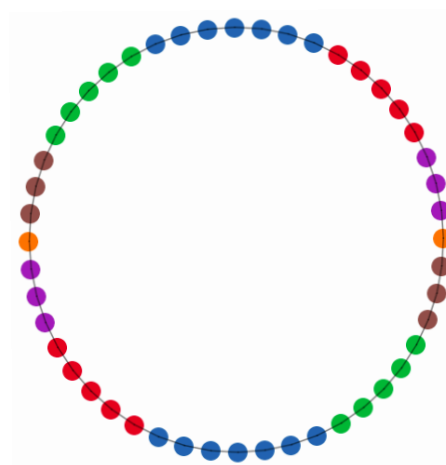
$\mathbb{R}P^1$  :



2-cell:



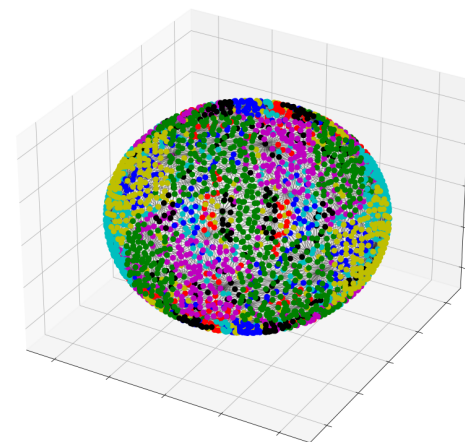
boundary  
of 2-cell:  
4 barycentric  
subdivisions



$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

boundary  
of 3-cell:

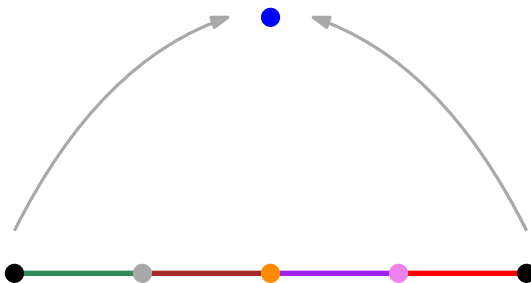
4 barycentric subdivisions



# Simplicial approximation to $\mathbb{R}P^n$

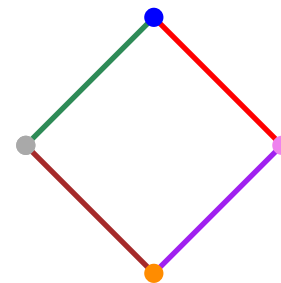
16/35 (10/10)

$\mathbb{R}P^0$  :

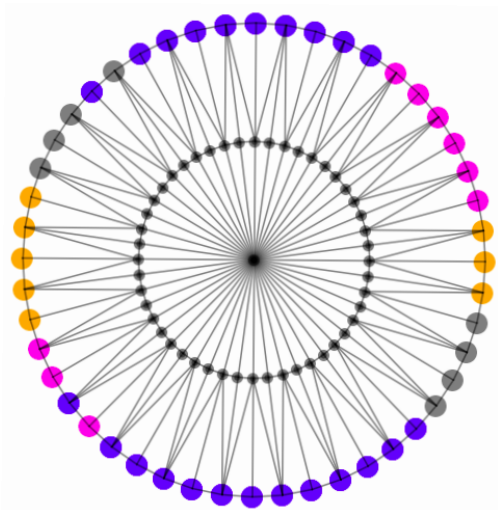


1-cell:

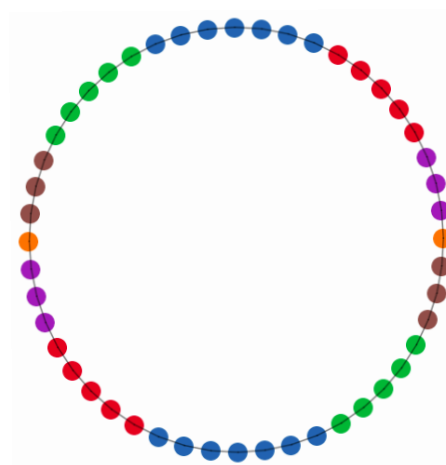
$\mathbb{R}P^1$  :



2-cell:



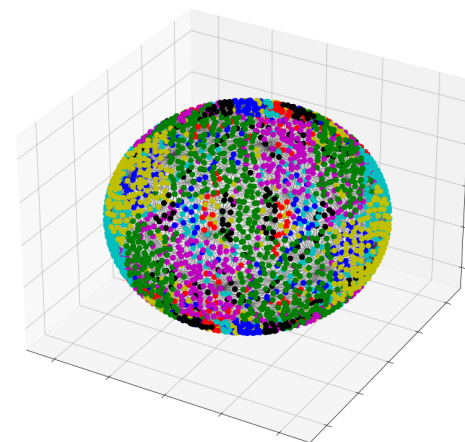
boundary  
of 2-cell:  
4 barycentric  
subdivisions



$\mathbb{R}P^2$  : 53 vertices, 317 simplices.

boundary  
of 3-cell:

$\mathbb{R}P^3$  : 560'024 vertices, 11'521'686 simplices  
(after 7 barycentric subdivisions)



# I - Simplicial approximation to CW-complexes

- 1 - Topology CW-complexes
- 2 - Simplicial approximation
- 3 - Simplicial mapping cone
- 4 - Application to projective spaces

# II - Simplicial approximation improved

- 1 - Local subdivisions
- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

# III - Applications

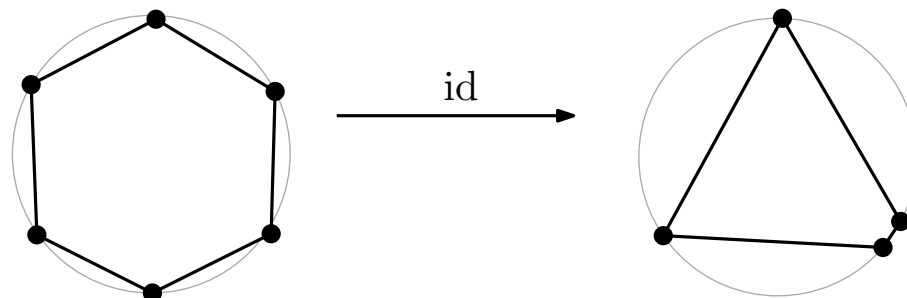
- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$



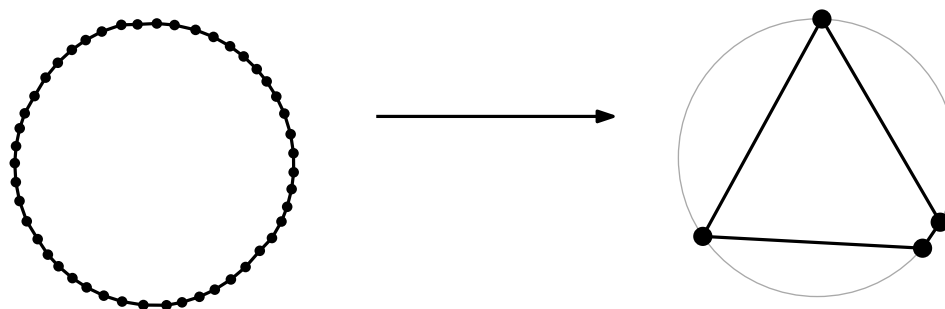
# Problem with barycentric subdivision

18/35

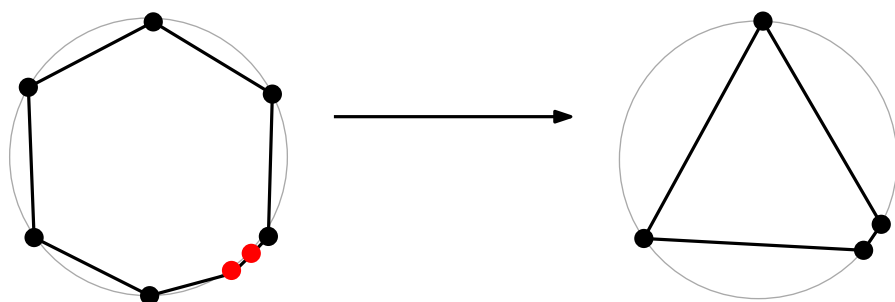
Suppose that we want to find a simplicial approximation to the identity map on  $\mathbb{S}^1$ , with triangulations given as:



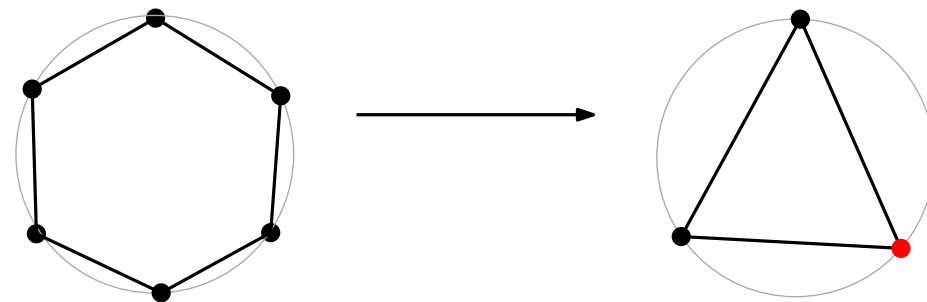
We must subdivide the domain many times to ensure that id satisfies the star condition.



We could have done simpler:



local subdivision



edge contraction

# I - Simplicial approximation to CW-complexes

- 1 - Topology CW-complexes
- 2 - Simplicial approximation
- 3 - Simplicial mapping cone
- 4 - Application to projective spaces

# II - Simplicial approximation improved

- 1 - Local subdivisions
- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

# III - Applications

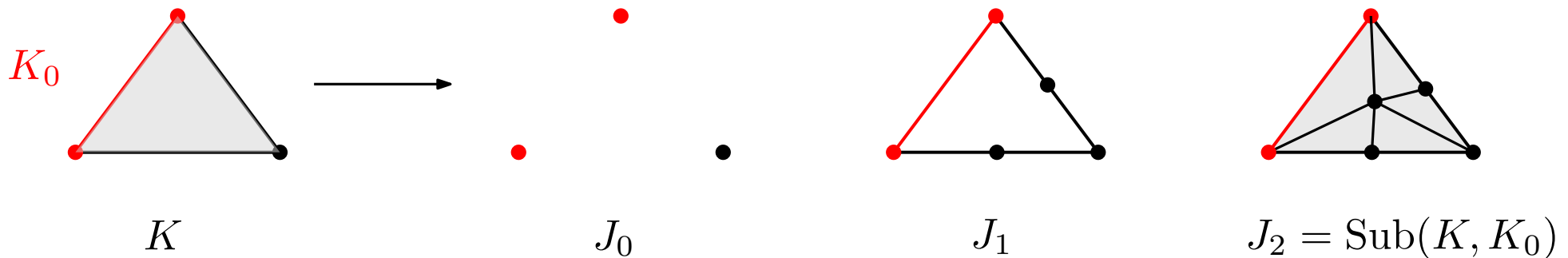
- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

[Munkres, Elements of Algebraic Topology]

Let  $K$  be a simplicial complex, and  $K_0 \subset K$  a sub-complex.

The **barycentric subdivision of  $K$  holding  $K_0$  fixed**, denoted  $\text{Sub}(K, K_0)$ , is defined by induction. We define simplicial complexes of increasing dimension  $J_0, \dots, J_d$  as follows:

- Start with  $J_0$  the 0-skeleton of  $K$
- From  $J_{p-1}$ , build  $J_p$  by adding all the  $p$ -simplices of  $K_0$
- Moreover, for any  $p$ -simplex of  $K$  not in  $K_0$ , add a point  $\hat{\sigma}$ , and cone it to the boundary  $\partial\sigma \in J_p$



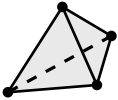
**Proposition:** To find a simplicial approximation to  $f: |K| \rightarrow |L|$ , it is enough to execute repeated barycentric subdivisions of  $K$  holding  $K_0$  fixed, where  $K_0$  is a sub-complex on which  $f$  satisfies the star condition.

# Edgewise subdivisions

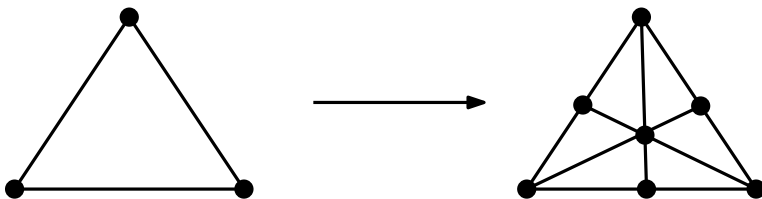
21/35 (1/3)

Barycentric subdivisions increase the number of simplices drastically:  
a  $d$ -simplex turns into a simplicial complex with  $(d+1)!$  simplices and  $2^{d+1} - 1$  vertices.

**Example:** Triangulation of the unit sphere  $\mathbb{S}^2$ , starting from the boundary of the 3-simplex

	$K$	$\text{Sub}^1(K)$	$\text{Sub}^2(K)$	$\text{Sub}^3(K)$	$\text{Sub}^4(K)$	$\text{Sub}^5(K)$
vertices:	4	14	74	434	2594	15554
simplices:	14	74	434	2594	15554	93314
max diameter:	1.63	1.15	0.66	0.42	0.25	0.15

Barycentric subdivision:

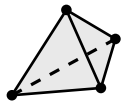


# Edgewise subdivisions

21/35 (2/3)

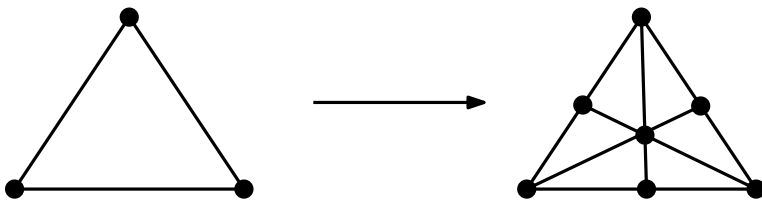
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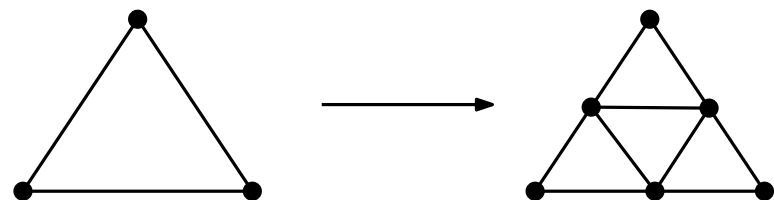


	$K$	$\text{Sub}^1(K)$	$\text{Sub}^2(K)$	$\text{Sub}^3(K)$	$\text{Sub}^4(K)$	$\text{Sub}^5(K)$
vertices:	4	14	74	434	2594	15554
simplices:	14	74	434	2594	15554	93314
max diameter:	1.63	1.15	0.66	0.42	0.25	0.15

Barycentric subdivision:



We can do better with **edgewise subdivisions**:

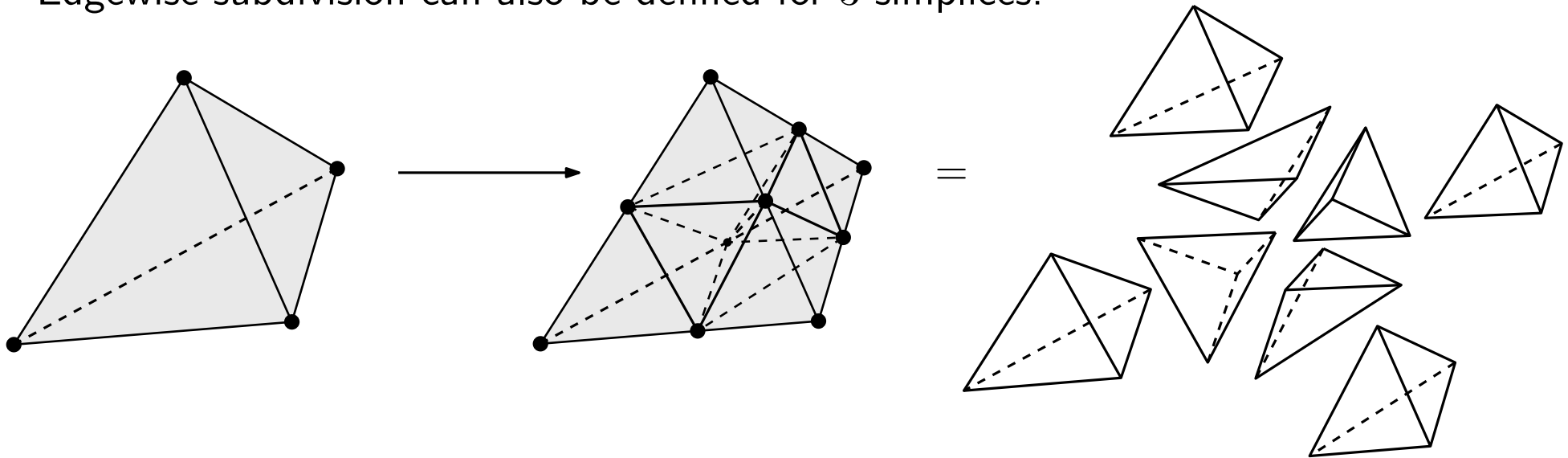


	$K$	$\text{Sub}^1(K)$	$\text{Sub}^2(K)$	$\text{Sub}^3(K)$	$\text{Sub}^4(K)$	$\text{Sub}^5(K)$
vertices:	4	10	39	130	514	2050
simplices:	14	50	194	770	3074	12290
max diameter:	1.63	1.41	1	0.58	0.30	0.15

# Edgewise subdivisions

21/35 (3/3)

Edgewise subdivision can also be defined for 3-simplices:



**Example:** Triangulation of the sphere  $\mathbb{S}^3$ , starting from the boundary of the 4-simplex.

	barycentric:	$\text{Sub}^3(K)$	$\text{Sub}^4(K)$	edgewise:	$\text{Sub}^5(K)$	$\text{Sub}^6(K)$
vertices:		12'600	301'680		27'440	218'720
simplices:		301'680	7'238'880		710'240	5'680'320
max diameter:		0.54	0.36		0.47	0.29

# I - Simplicial approximation to CW-complexes

- 1 - Topology CW-complexes
- 2 - Simplicial approximation
- 3 - Simplicial mapping cone
- 4 - Application to projective spaces

# II - Simplicial approximation improved

- 1 - Local subdivisions
- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

# III - Applications

- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

We look for a simplicial approximation to  $f: |K| \rightarrow |L|$ .

Endow  $K$  with a metric, and by denote  $\mathcal{U}$  the cover  $\{f^{-1}(|\text{St}(w)|), w \in L\}$  of  $|K|$ .

Let  $\epsilon > 0$  be a Lebesgue number for  $\mathcal{U}$ .

The map  $f$  satisfies the star condition if for every  $v \in K^{(0)}$ ,  $\text{Diameter}(|\overline{\text{St}}(v)|) < \epsilon$ .

To simplify the problem, we can replace  $L$  with a homotopy equivalent complex  $L'$ , given with a simplicial map  $g: L \rightarrow L'$ .

$$\begin{array}{ccccc} & & f' & & \\ & \nearrow & & \searrow & \\ |K| & \xrightarrow{f} & |L| & \xrightarrow{|g|} & |L'| \end{array}$$

Consider the new cover  $\mathcal{U}' = \{(f')^{-1}(|\text{St}(w')|), w' \in L'\}$ .

We may hope for its Lebesgue number to be smaller.



# Edge contractions

24/35 (1/4)

Let  $[a, b]$  be an edge of  $L$ , and  $c \notin L^{(0)}$  a new vertex. The *quotient map* is defined as

$$\begin{aligned} p: L^{(0)} &\longrightarrow \left( L^{(0)} \setminus \{a, b\} \right) \sqcup \{c\} \\ x &\longmapsto c \text{ if } x = a \text{ or } b \\ &\quad x \text{ else.} \end{aligned}$$

The contracted complex is defined as

$$L' = \{p(\sigma), \sigma \in L\}.$$

We have a surjective simplicial map  $p: L \rightarrow L'$ .

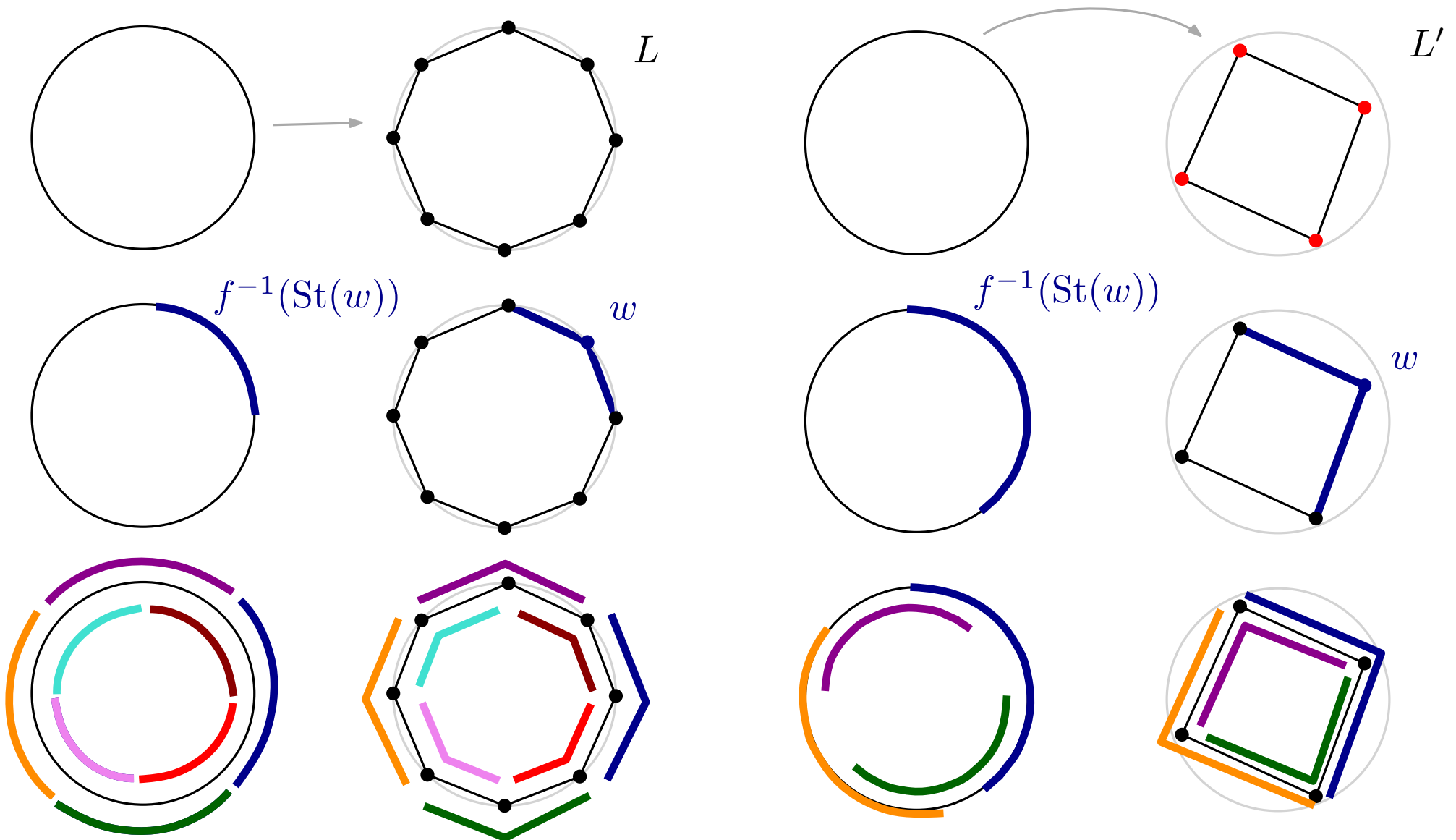


**Theorem [Dey, Edelsbrunner, Guha, Nekhayev, 1998], [Attali, Lieutier, Salinas, 2012]:**  
The map  $p$  is a homotopy equivalence when the edge  $[a, b]$  satisfies the *link condition*, that is,  $\text{Lk}(ab) = \text{Lk}(a) \cap \text{Lk}(b)$ .

# Edge contractions

24/35 (2/4)

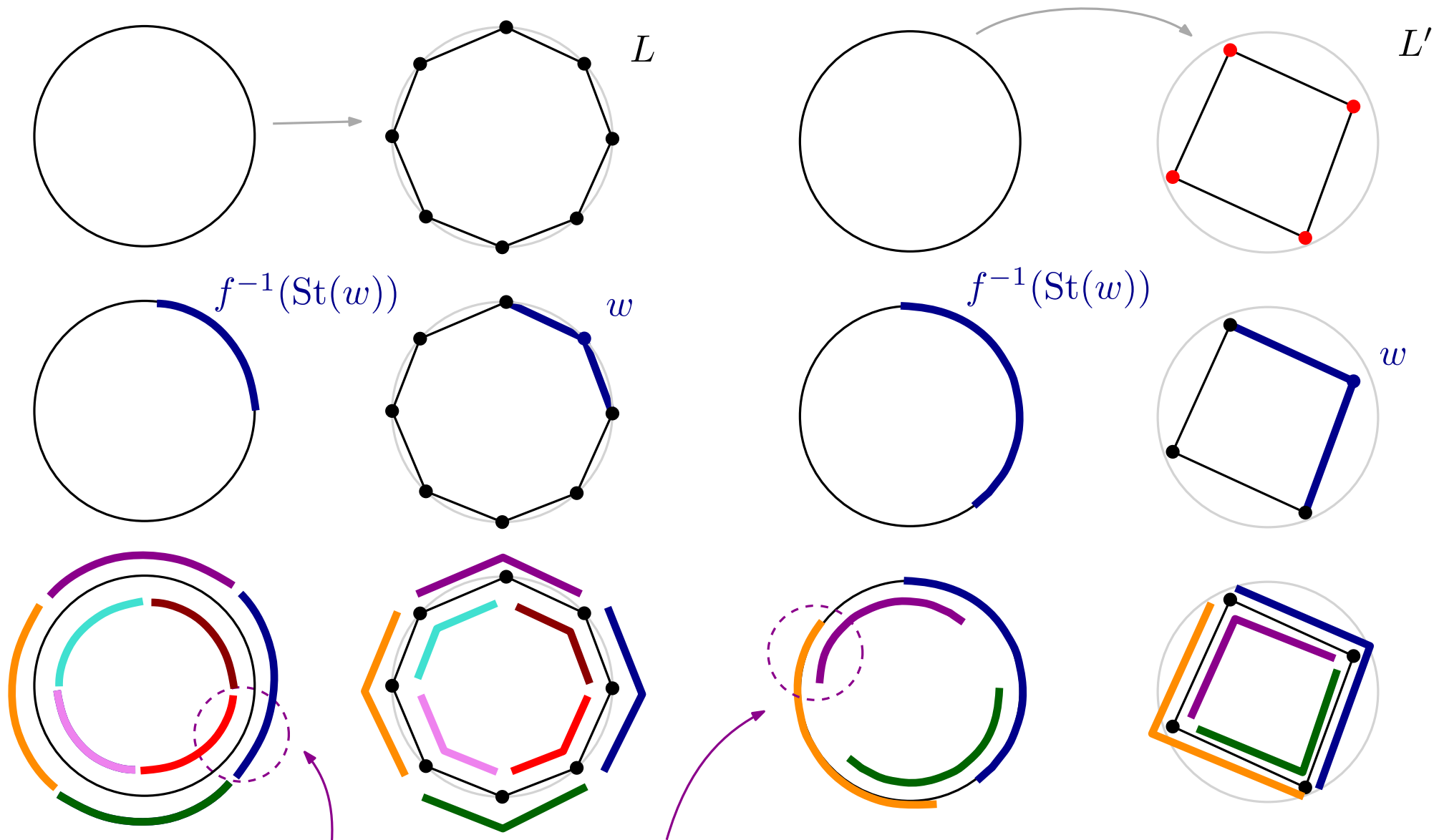
Example: Contraction of a triangulation of  $\mathbb{S}^1$



# Edge contractions

24/35 (3/4)

Example: Contraction of a triangulation of  $\mathbb{S}^1$

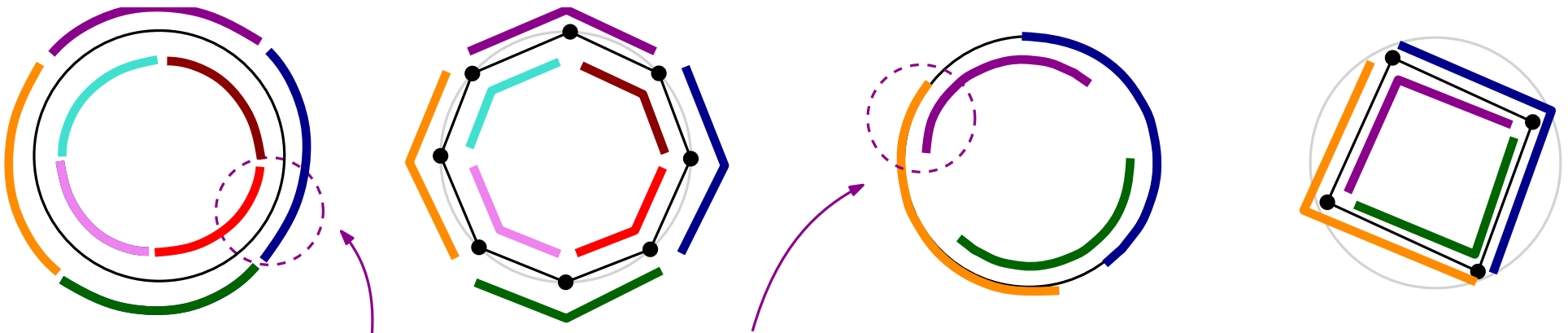


The (global) Lebesgue number remains the same

We would get better results by considering a *homeomorphism*  $L \rightarrow L'$  instead of the contraction map  $p$ .

- [Dey, Edelsbrunner, Guha, Nekhayev, *Topology preserving edge contraction*, 1998]  
Local unfolding for simplicial complexes of dimension 2, or combinatorial manifolds of dimension 3.

To contract large simplicial complexes, we shall use the skeleton-blockers structure [Attali, Lieutier, Salinas, *Efficient data structure for representing and simplifying simplicial complexes in high dimensions*, 2012]



The (global) Lebesgue number remains the same

# I - Simplicial approximation to CW-complexes

- 1 - Topology CW-complexes
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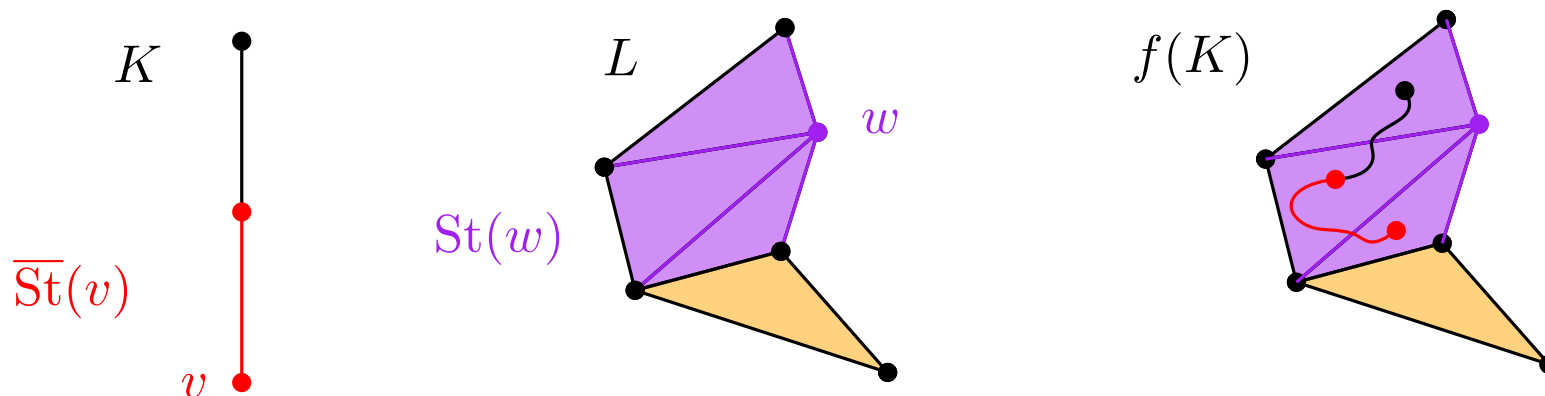
# II - Simplicial approximation improved

- 1 - Local subdivisions
- 2 - Edge contractions
- 3 - Weak simplicial approximation
- 4 - Application to projective spaces, second attempt

# III - Applications

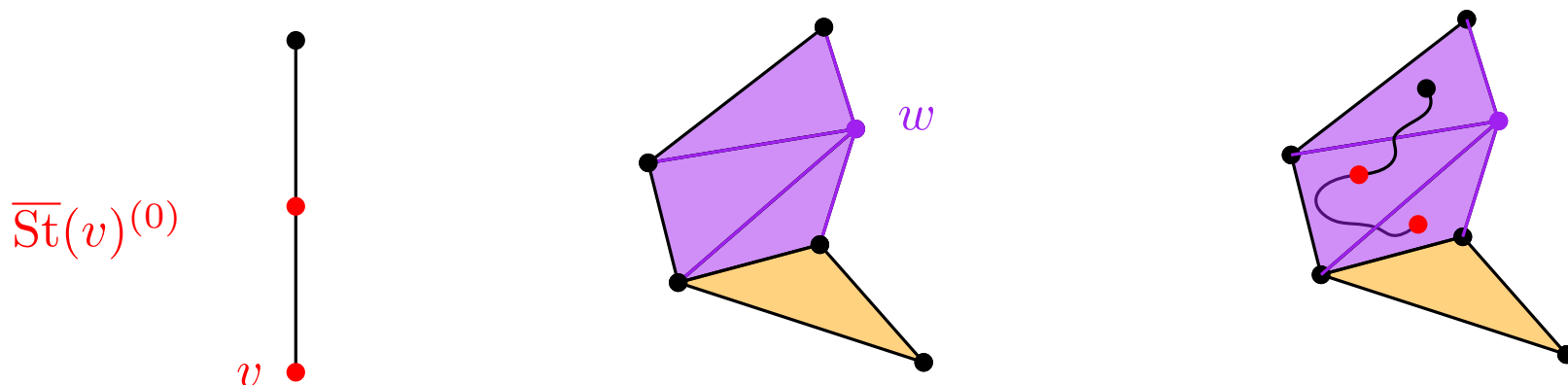
- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

In practice, we cannot check whether a simplicial map  $f: |K| \rightarrow |L|$  satisfies the star condition...

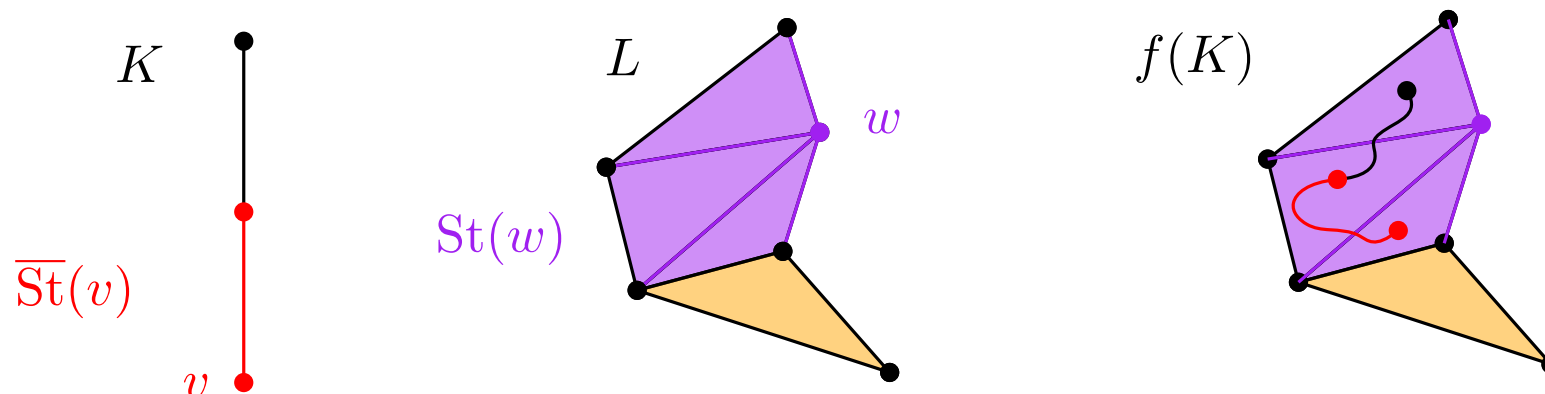


$$f(|\overline{\text{St}}(v)|) \subseteq |\text{St}(w)|?$$

The map  $f$  satisfies the **weak star condition** if for every  $v \in K^{(0)}$ , there exists a  $w \in L^{(0)}$  such that  $f(|\overline{\text{St}}(v)^{(0)}|) \subseteq |\text{St}(w)|$ .



In practice, we cannot check whether a simplicial map  $f: |K| \rightarrow |L|$  satisfies the star condition...



$$f(|\overline{\text{St}}(v)|) \subseteq |\text{St}(w)|?$$

The map  $f$  satisfies the **weak star condition** if for every  $v \in K^{(0)}$ , there exists a  $w \in L^{(0)}$  such that  $f(|\overline{\text{St}}(v)|) \subseteq |\text{St}(w)|$ .

If this is the case, let  $g: K^{(0)} \rightarrow L^{(0)}$  be any map such that for every  $v \in K^{(0)}$ , we have  $f(|\overline{\text{St}}(v)|) \subseteq |\text{St}(g(v))|$ .

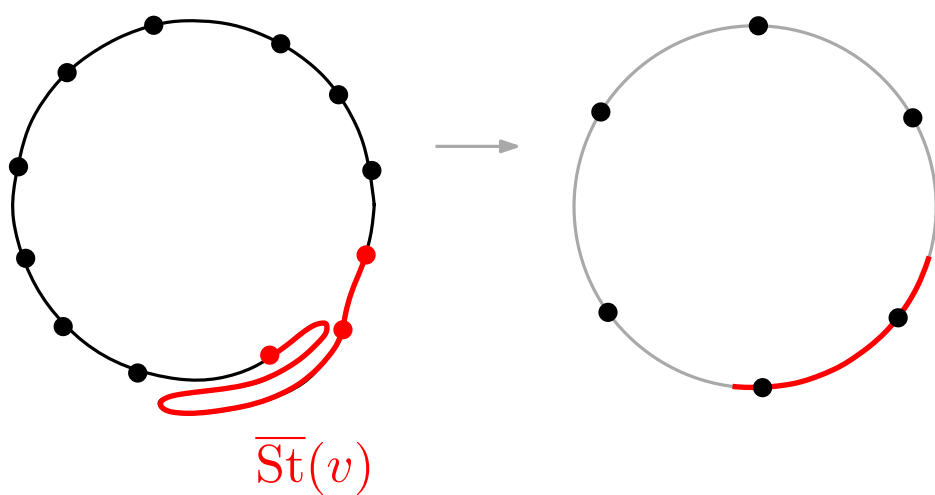
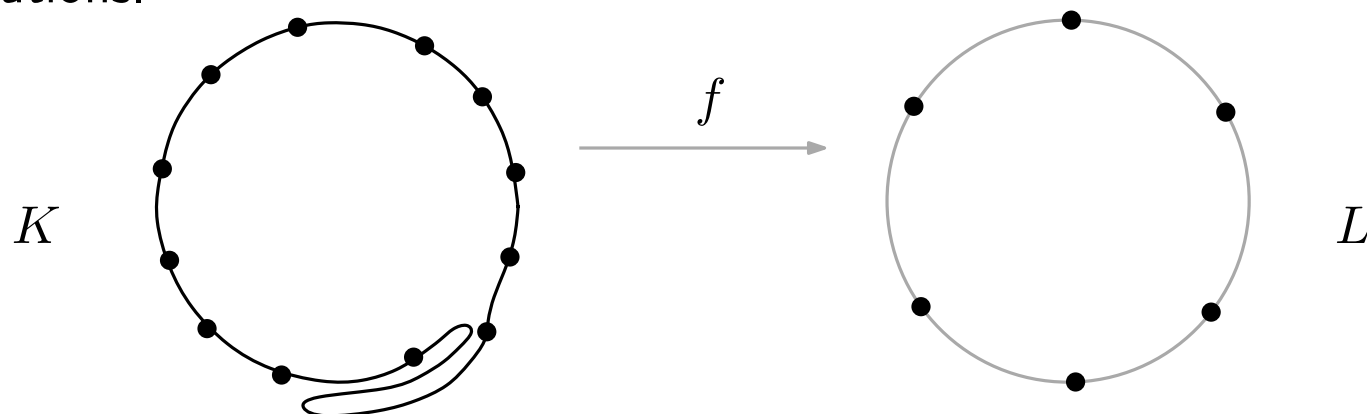
Such a map  $g$  is called a **weak simplicial approximation** to  $f$ . It is a simplicial map.

**Proposition [T., 2020]:** If  $K$  is subdivided enough, then any weak simplicial approximation is a simplicial approximation.

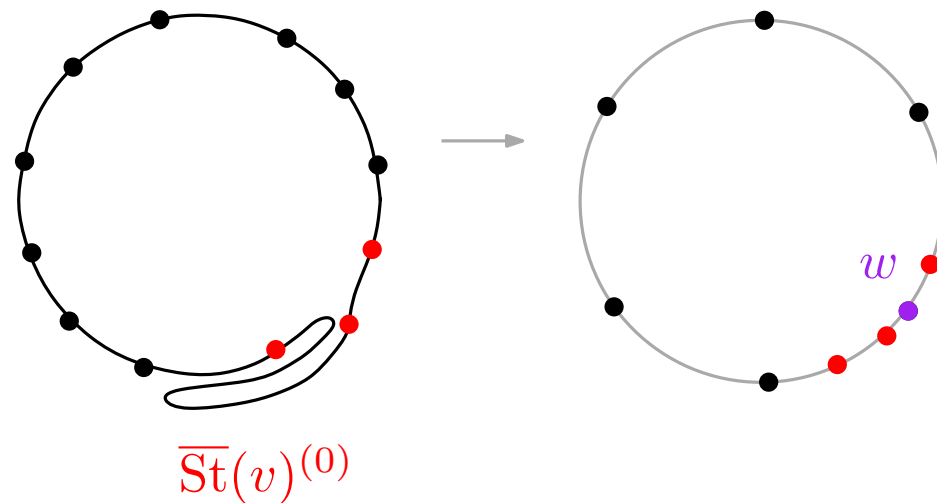
# Beyond simplicial approximation

27/35 (1/2)

If  $K$  is not subdivided enough, weak simplicial approximations may not be simplicial approximations.



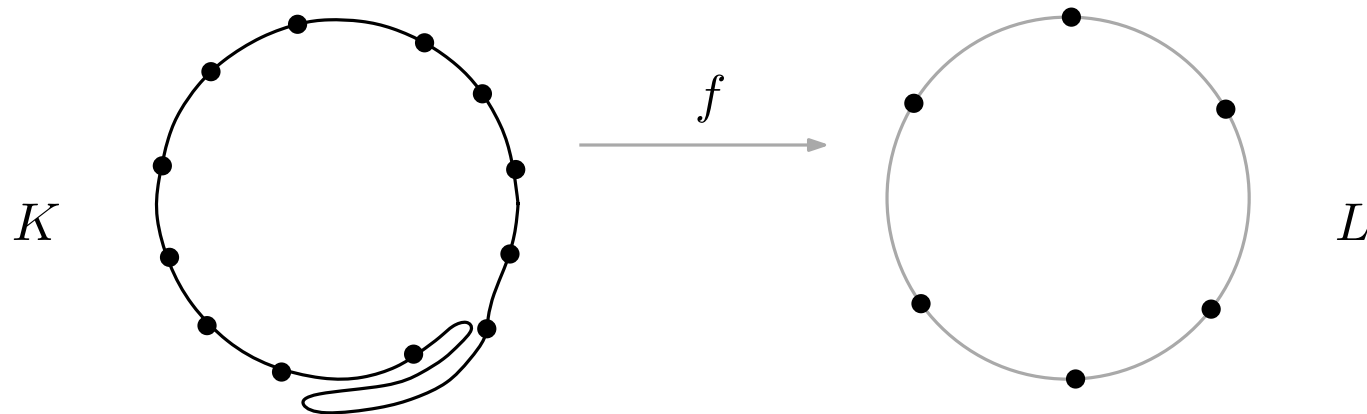
$f$  does not satisfy the star condition



$f$  satisfies the weak star condition



If  $K$  is not subdivided enough, weak simplicial approximations may not be simplicial approximations.



**Close maps are homotopic.** Suppose that  $|L|$  is endowed with a distance  $d$ . Between two continuous maps  $f, g: |K| \rightarrow |L|$ , define the infinite norm

$$d_\infty(f, g) = \sup \{ d(f(x), g(x)), x \in |K| \}.$$

Let  $\epsilon$  be such that, for all continuous maps  $f, g: |K| \rightarrow |L|$ ,

$$d_\infty(f, g) < \epsilon \implies f \text{ and } g \text{ are homotopic.}$$

This gives another criterion for simplicial approximation: we look for a simplicial  $g: K \rightarrow L$  such that  $d_\infty(f, g) < \epsilon$ .

One could try to relate this  $\epsilon$  and the *injectivity radius* of  $|L|$  endowed with a Riemannian metric.

# I - Simplicial approximation to CW-complexes

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- 1 - Lens spaces
- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

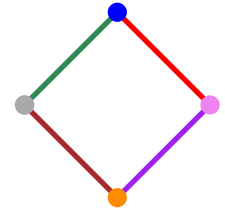
# Approximation to $\mathbb{R}P^n$ , 2<sup>nd</sup> attempt 29/35 (1/2)

Remember our last triangulation with *global barycentric subdivisions*:

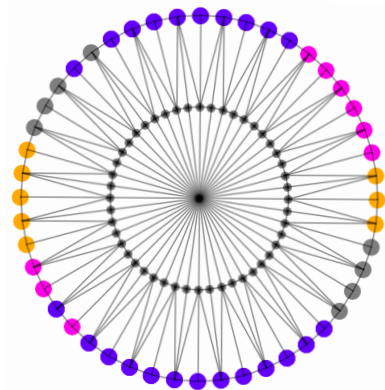
$\mathbb{R}P^0$  : 1 vertex



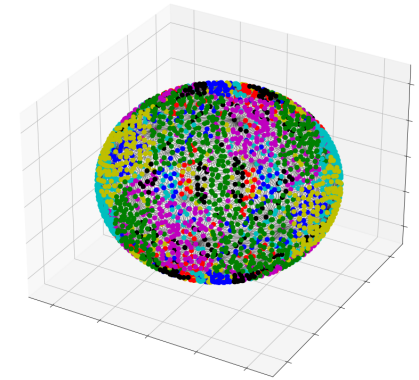
$\mathbb{R}P^1$  : 4 vertices



$\mathbb{R}P^2$  : 53 vertices  
(4 subdivisions)



$\mathbb{R}P^3$  : 560'024 vertices  
(7 subdivisions)



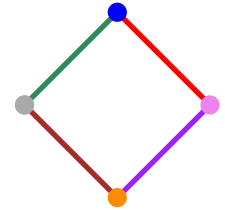
# Approximation to $\mathbb{R}P^n$ , 2<sup>nd</sup> attempt 29/35 (2/2)

Remember our last triangulation with *global barycentric subdivisions*:

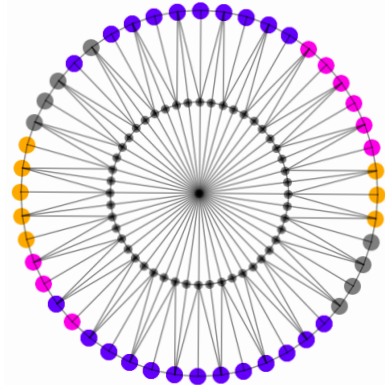
$\mathbb{R}P^0$  : 1 vertex



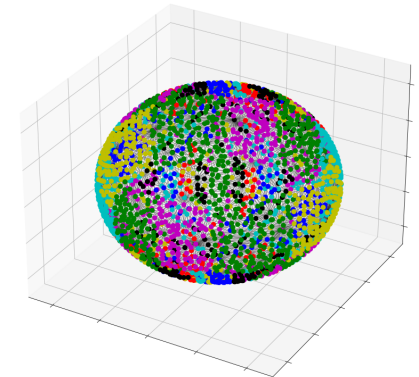
$\mathbb{R}P^1$  : 4 vertices



$\mathbb{R}P^2$  : 53 vertices  
(4 subdivisions)



$\mathbb{R}P^3$  : 560'024 vertices  
(7 subdivisions)

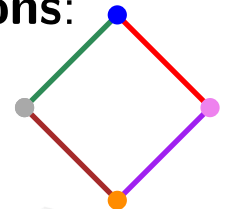


Now with **local subdivisions**, **edgewise subdivisions** and **edge contractions**:

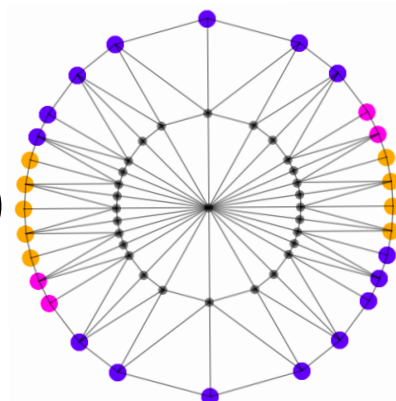
$\mathbb{R}P^0$  : 1 vertex



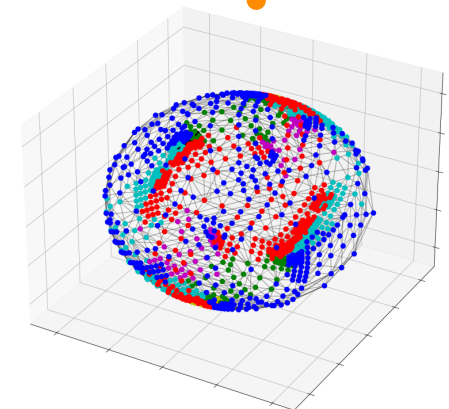
$\mathbb{R}P^1$  : 4 vertices  
(contract to 3 vertices)



$\mathbb{R}P^2$  : 32 vertices  
(contract to 7 vertices)



$\mathbb{R}P^3$  : 1520 vertices  
(contract to 89 vertices)



# I - Simplicial approximation to CW-complexes

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# III - Applications

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- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$

Let  $p, q$  be coprime integers, and let  $\mathbb{S}^3 \subset \mathbb{C}^2$  be the unit sphere. Let  $\alpha = e^{\frac{2i\pi}{p}}$  be the  $p^{\text{th}}$  root of unity.

We define an action of  $\mathbb{Z}_p$  on  $\mathbb{S}^3$  via  $\rho \cdot (z_1, z_2) = (\alpha z_1, \alpha^q z_2)$ .

It is a free action, and the orbit space is denoted  $L(p, q)$ .

Its homology groups are  $\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}, 0, \mathbb{Z}$ . It has been an important question to determine which of these spaces are homeomorphic or homotopic equivalent.

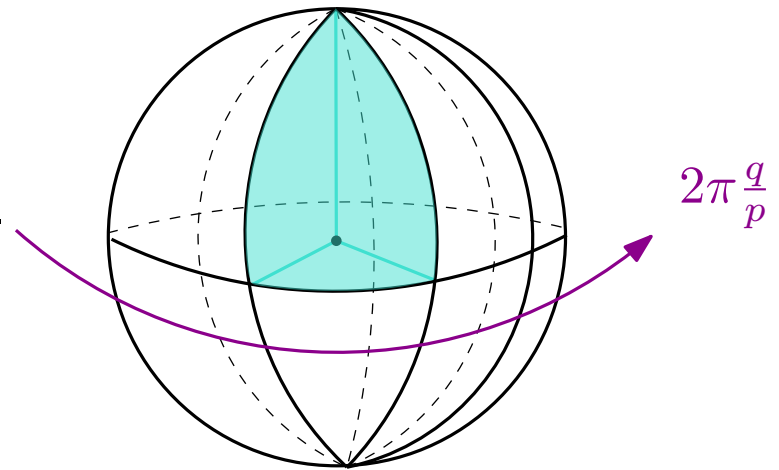
A fundamental domain for  $L(p, q)$  is given by

$$\left\{ \cos(\theta) \cdot (0, \alpha^t) + \sin(\theta) \cdot (\alpha^s, 0), \quad t \in [0, 1] \quad s \in [0, 1], \quad \theta \in \left[0, \frac{\pi}{2}\right) \right\} \subset \mathbb{C}^2.$$

$L(p, q)$  can also be seen as a quotient of the 3-ball.

We deduce a CW-complex structure:

- $L^0$  is a point,
- $L^1$  is a circle,
- $L^2$  is a disk glued with a degree  $p$  map,
- $L^3$  is a 3-ball glued by differentiating the two hemispheres: one is glued via the identity on the 2-cell, the other one after a rotation of  $2\pi \frac{q}{p}$ .



# Simplicial approximation to $L(p, q)$

32/35

We applied the algorithm for various values of  $p$  and  $q$ .

$(p, q) =$	$(2, 3)$	$(2, 5)$	$(2, 11)$	$(3, 5)$	$(11, 2)$
number of vertices before contraction	911	929	900	2'412	15'753
number of vertices after contraction	50	39	57	212	1'631
number of simplices before contraction	18'736	18'524	17'896	49'510	327'682
number of simplices after contraction	1'078	820	1'202	4'954	46'654

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- 2 - Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$



The Grassmannian  $\mathcal{G}_2(\mathbb{R}^4)$  is the set of 2-dimensional planes in  $\mathbb{R}^4$ . It can be given a smooth manifold structure of dimension 4. Its cohomology groups are:

$$H^0(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z}$$

$$H^1(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z}_2$$

$$H^2(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = (\mathbb{Z}_2)^2$$

$$H^3(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z}_2$$

$$H^4(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_2) = \mathbb{Z}$$

$$H^0(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = \mathbb{Z}_p$$

$$H^1(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = 0$$

$$H^2(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = 0$$

$$H^3(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = 0$$

$$H^4(\mathcal{G}_2(\mathbb{R}^4), \mathbb{Z}_p) = \mathbb{Z}_p$$

where  $p$  is any prime number  $> 2$ .

Consider a plane  $T \in \mathcal{G}_2(\mathbb{R}^4)$ . Let  $(u, v)$  be a basis, and consider the matrix

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

By mean of elementary operations, it can be reduced to a unique matrix in **orthogonal reduced echelon form**  $\begin{pmatrix} v'_1 & v'_2 & v'_3 & v'_4 \\ w'_1 & w'_2 & w'_3 & w'_4 \end{pmatrix}$  such that:

- $\|v'\| = \|w'\| = 1$
- $v'$  and  $w'$  are orthogonal
- the last nonzero coordinate of  $v'$  is positive
- the last nonzero coordinate of  $w'$  is positive

Consider a plane  $T \in \mathcal{G}_2(\mathbb{R}^4)$  and its matrix in reduced echelon form  $\begin{pmatrix} v'_1 & v'_2 & v'_3 & v'_4 \\ w'_1 & w_2 & w'_3 & w'_4 \end{pmatrix}$

Let  $i$  (resp.  $j$ ) be the index of the last nonzero coordinate of  $v'$  (resp.  $w'$ ).

The pair  $(i, j)$  is called the *Schubert symbol* of the plane  $T$ .

There are 6 potential Schubert symbols for  $\mathcal{G}_2(\mathbb{R}^4)$ :

$$(1, 2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(1, 3) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$$

$$(1, 4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix}$$

$$(2, 3) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

$$(2, 4) \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}$$

$$(1, 2) \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

Consider a plane  $T \in \mathcal{G}_2(\mathbb{R}^4)$  and its matrix in reduced echelon form  $\begin{pmatrix} v'_1 & v'_2 & v'_3 & v'_4 \\ w'_1 & w_2 & w'_3 & w'_4 \end{pmatrix}$

Let  $i$  (resp.  $j$ ) be the index of the last nonzero coordinate of  $v'$  (resp.  $w'$ ).

The pair  $(i, j)$  is called the *Schubert symbol* of the plane  $T$ .

There are 6 potential Schubert symbols for  $\mathcal{G}_2(\mathbb{R}^4)$ :

dim 0	(1, 2)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	(1, 3)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 0 \end{pmatrix}$	dim 1
dim 2	(1, 4)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix}$	(2, 3)	$\begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	dim 2
dim 3	(2, 4)	$\begin{pmatrix} * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}$	(1, 2)	$\begin{pmatrix} * & * & * & 0 \\ * & * & * & * \end{pmatrix}$	dim 4

**Proposition:**  $\mathcal{G}_2(\mathbb{R}^4)$  admits a CW-structure with 6 cells. Each cell corresponds to a pair  $(i, j)$ , and contains all the planes  $T$  with Schubert symbol  $(i, j)$ .

After some computations, one finds explicit expressions for characteristic maps and gluing maps.

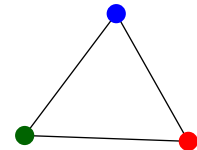
# Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$ 35/35 (1/4)

Running the algorithm as is will give a simplicial complex with **millions of vertices**.  
Let us use handcrafted simplicial approximations for the first cells instead.

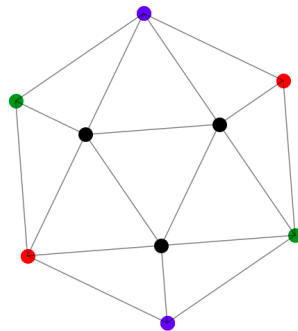
(1, 2): 1 vertex



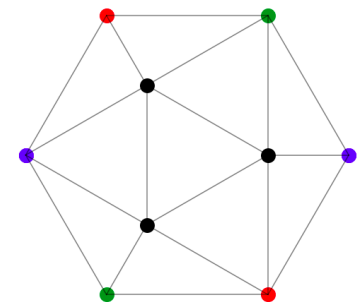
(1, 3) : 3 vertices



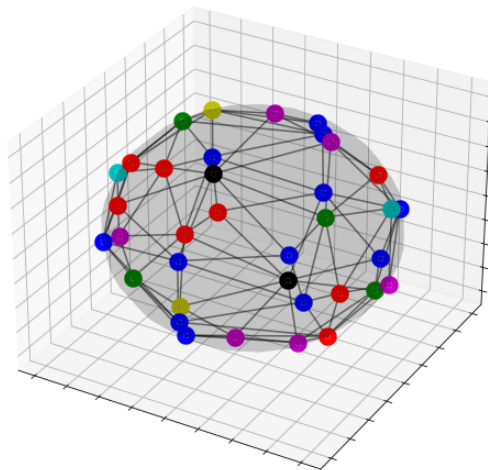
(1, 4): 6 vertices



(2, 3): 9 vertices



(2, 4): 46 vertices



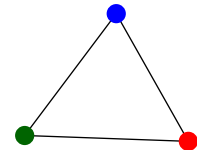
# Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$ 35/35 (2/4)

Running the algorithm as is will give a simplicial complex with **millions of vertices**.  
Let us use handcrafted simplicial approximations for the first cells instead.

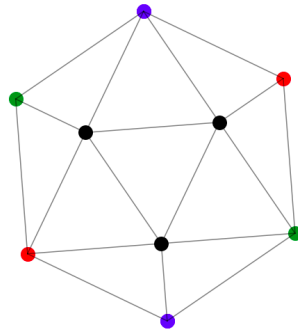
(1, 2): 1 vertex



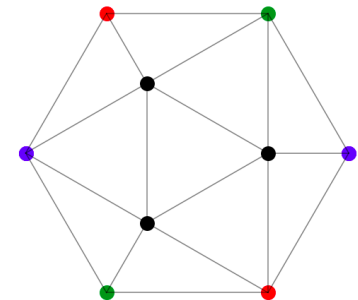
(1, 3) : 3 vertices



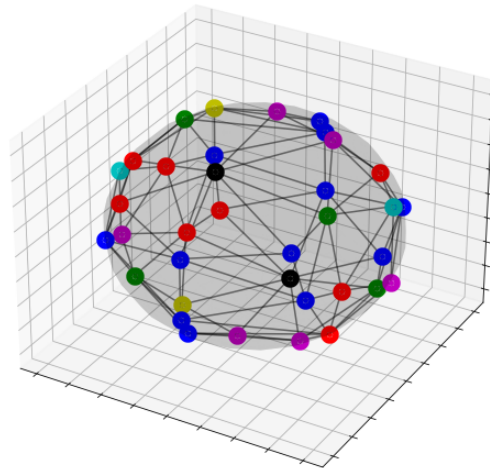
(1, 4): 6 vertices



(2, 3): 9 vertices



(2, 4): 46 vertices

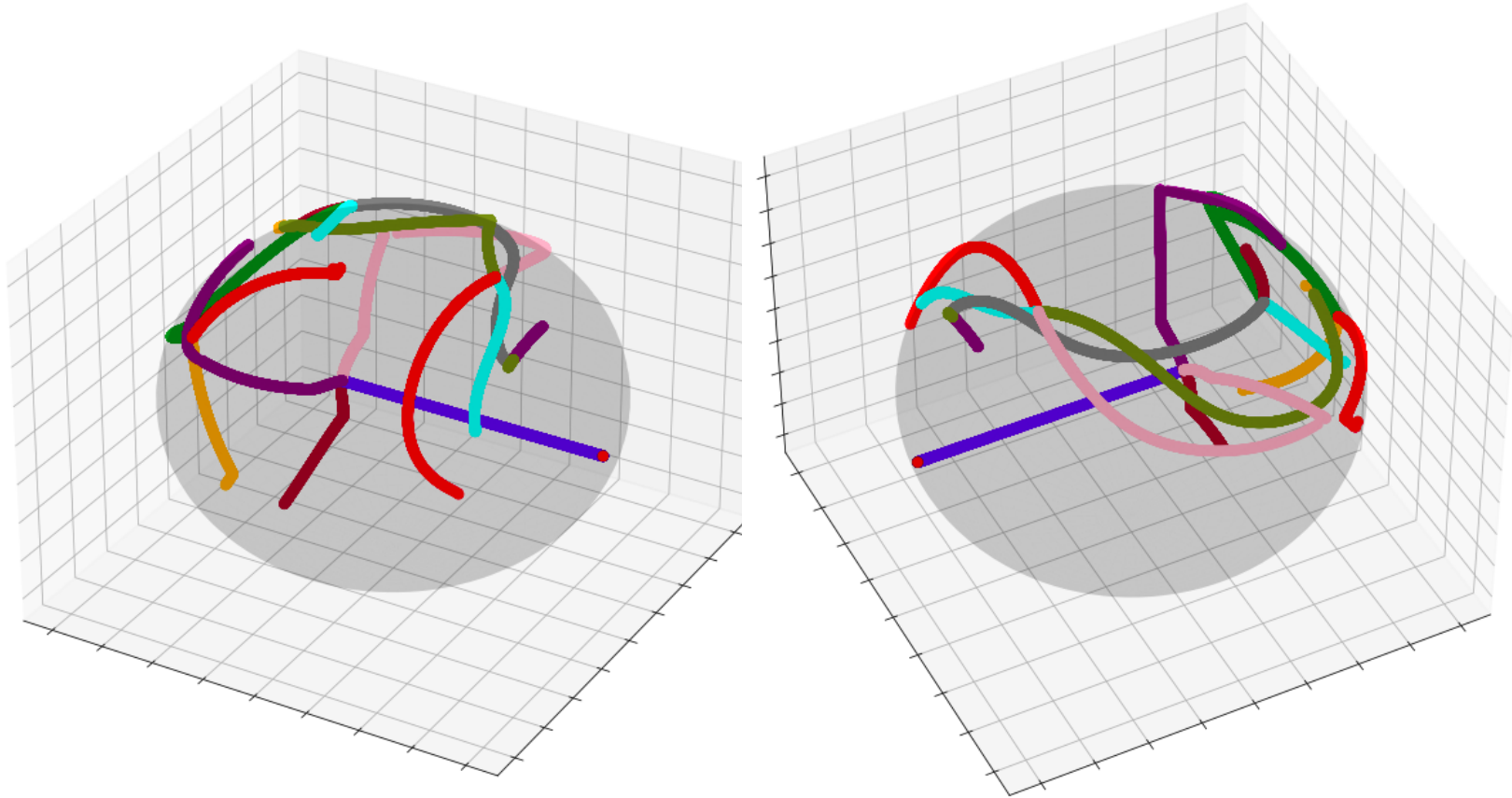
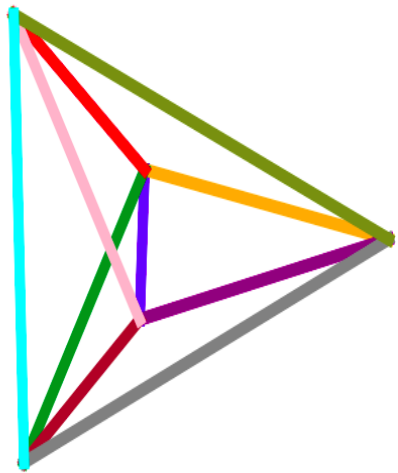


Gluing the last cell (3, 4): 38'951 vertices, 3'642'258 simplices.

After edge contractions: **7'700 vertices**, 1'061'892 simplices (via skeleton-blockers).

# Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$ 35/35 (3/4)

Embedding of the 4-simplex, seen as a triangulation of the 3-sphere, in the 3-cell of  $\mathcal{G}_2(\mathbb{R}^4)$ :

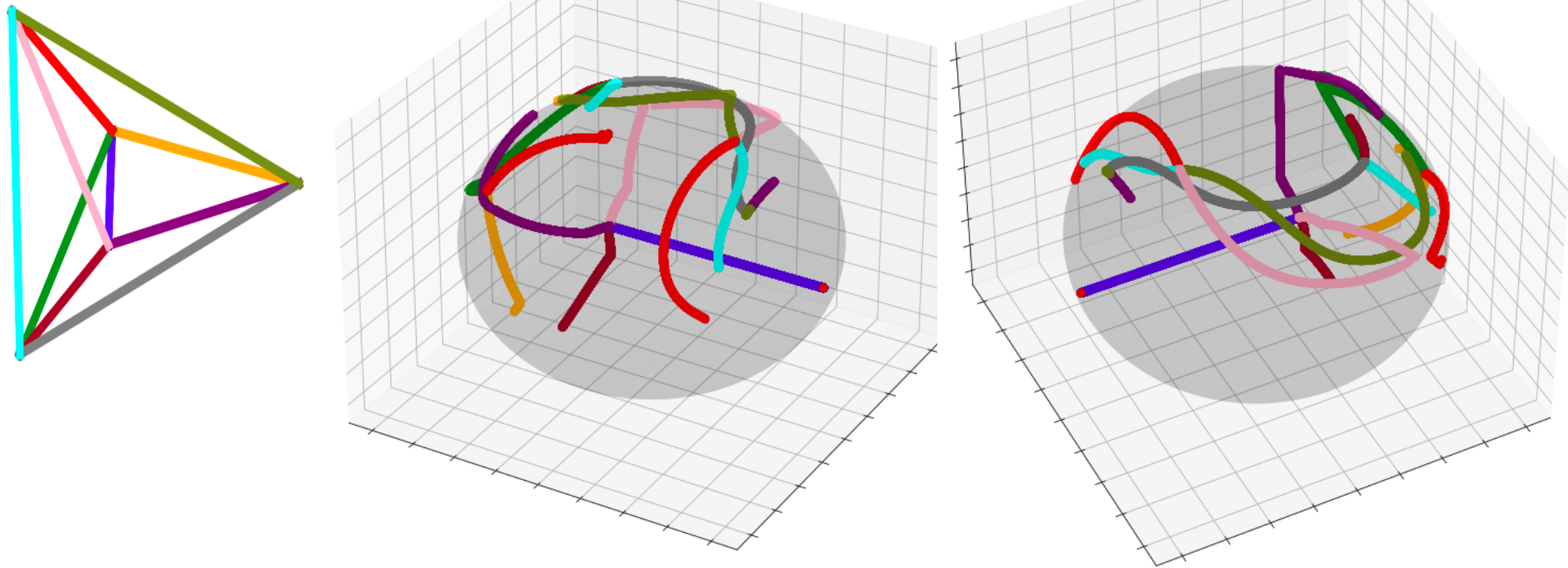


Gluing the last cell (3, 4): 38'951 vertices, 3'642'258 simplices.

After edge contractions: **7'700 vertices**, 1'061'892 simplices (via skeleton-blockers).

# Simplicial approximation to $\mathcal{G}_2(\mathbb{R}^4)$ 35/35 (4/4)

Embedding of the 4-simplex, seen as a triangulation of the 3-sphere, in the 3-cell of  $\mathcal{G}_2(\mathbb{R}^4)$ :



Gluing the last cell (3, 4): 38'951 vertices, 3'642'258 simplices.

Thank you!

After edge contractions: 7'700 vertices, 1'061'892 simplices (via skeleton-blockers).