

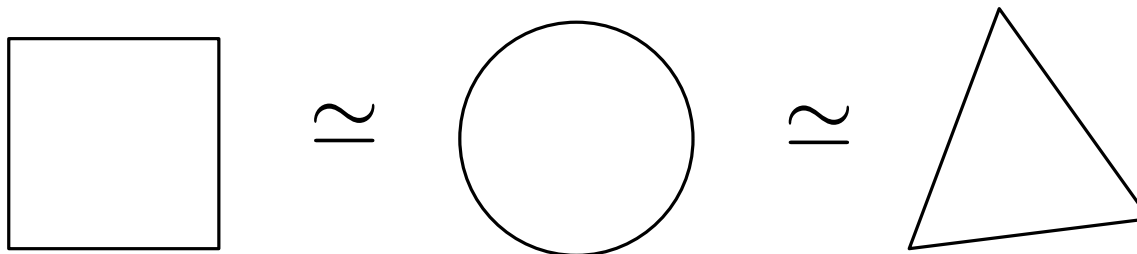
EMAp Summer Course

Topological Data Analysis with Persistent Homology

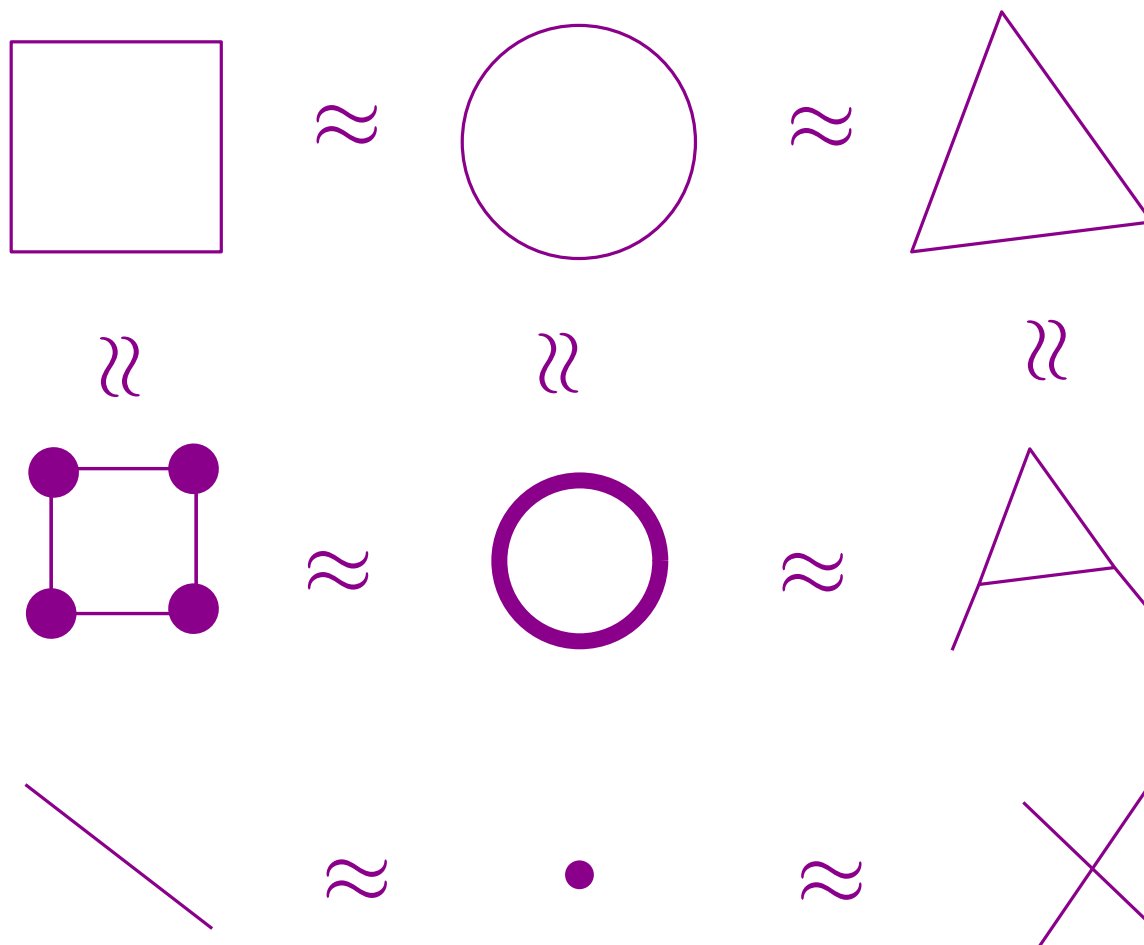
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Lesson 3: Homotopies

Homeomorphism
equivalence



Homotopy
equivalence



I - Homotopy equivalence between maps

II - Homotopy equivalence between topological spaces

III - Link with homeomorphic spaces

VI - Invariants

Definition: Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two topological spaces, and $f, g: X \rightarrow Y$ two continuous maps. A *homotopy* between f and g is a map $F: X \times [0, 1] \rightarrow Y$ such that:

- $F(\cdot, 0)$ is equal to f ,
- $F(\cdot, 1)$ is equal to g ,
- $F: X \times [0, 1] \rightarrow Y$ is continuous.

If such a homotopy exists, we say that the maps f and g are *homotopic*.

For any $t \in [0, 1]$, the notation $F(\cdot, t)$ refers to the map

$$\begin{aligned} F(\cdot, t): X &\longrightarrow Y \\ x &\longmapsto F(x, t) \end{aligned}$$

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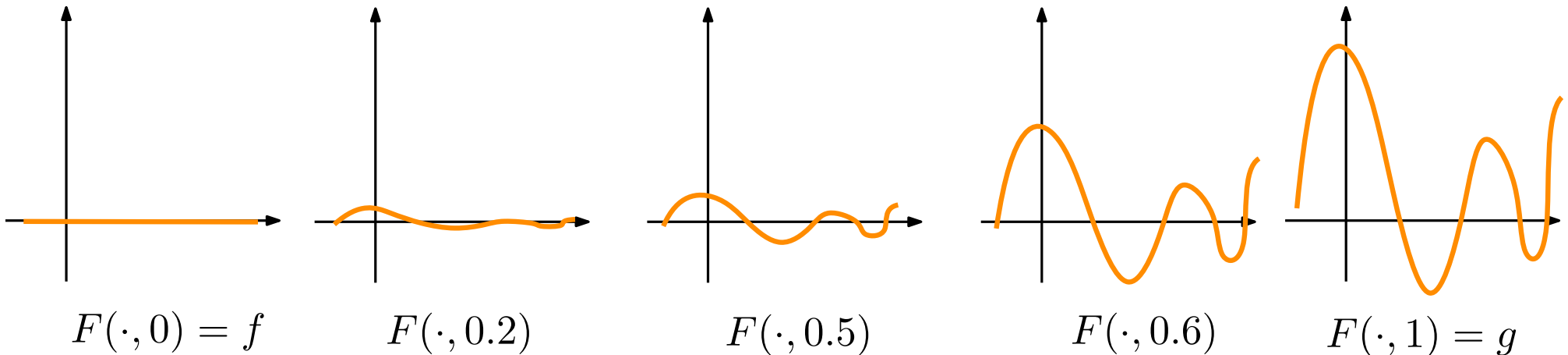
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Example: Homotopy $F: \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ between $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$.



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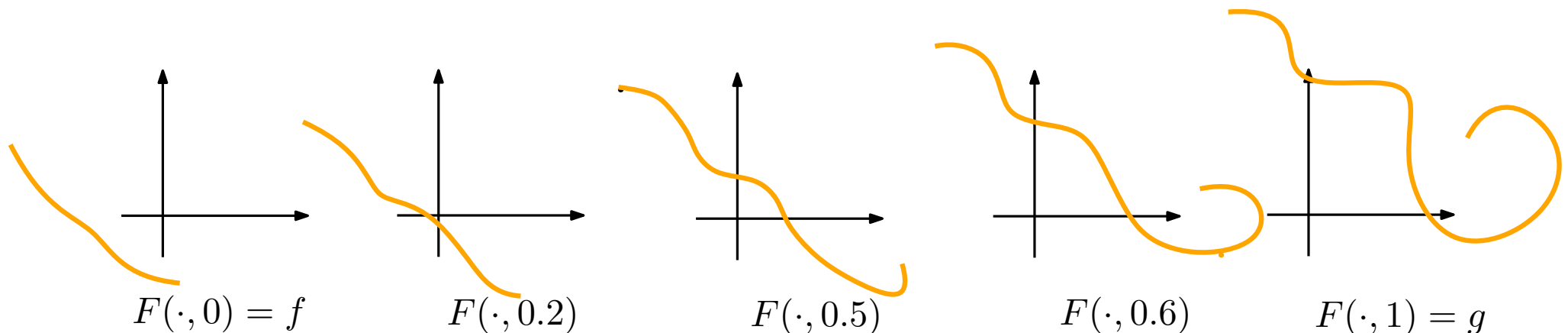
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Example: Homotopy $F: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2$ between $f: [0, 1] \rightarrow \mathbb{R}^2$ and $g: [0, 1] \rightarrow \mathbb{R}^2$.



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Example: Let $X = Y = [-1, 1]$ endowed with the Euclidean topology, and consider the maps $f, g: X \rightarrow Y$ defined as

$$f: x \mapsto 0$$

$$g: x \mapsto x$$

Let us prove that they are homotopic. Consider the map

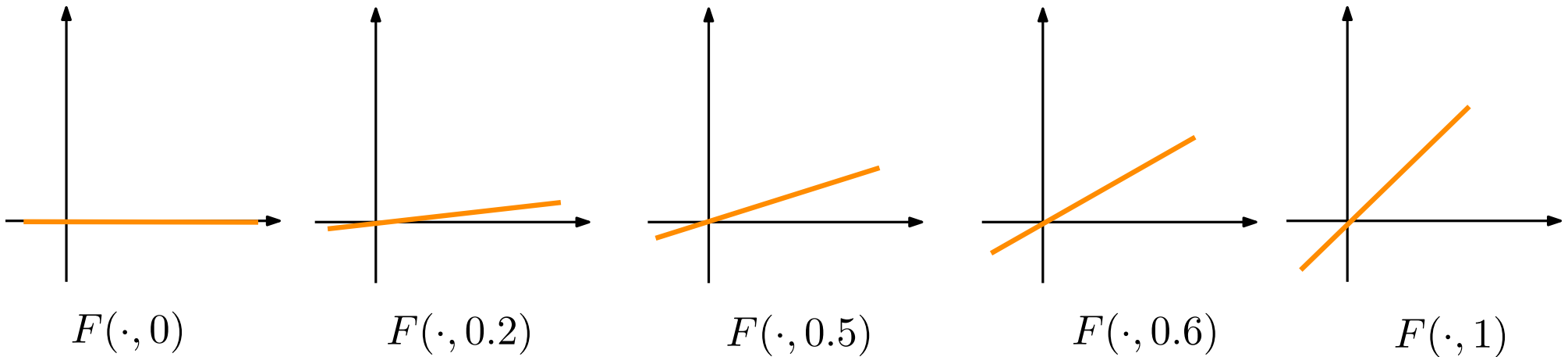
$$F: X \times [0, 1] \longrightarrow Y$$

$$(x, t) \longmapsto tx$$

We see that $F(\cdot, 0): x \mapsto 0$ is equal to f , and $F(\cdot, 1): x \mapsto x$ is equal to g . Moreover, F is continuous. Hence, F is an homotopy between f and g . Thus these two maps are homotopic.

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4/16 (5/7)



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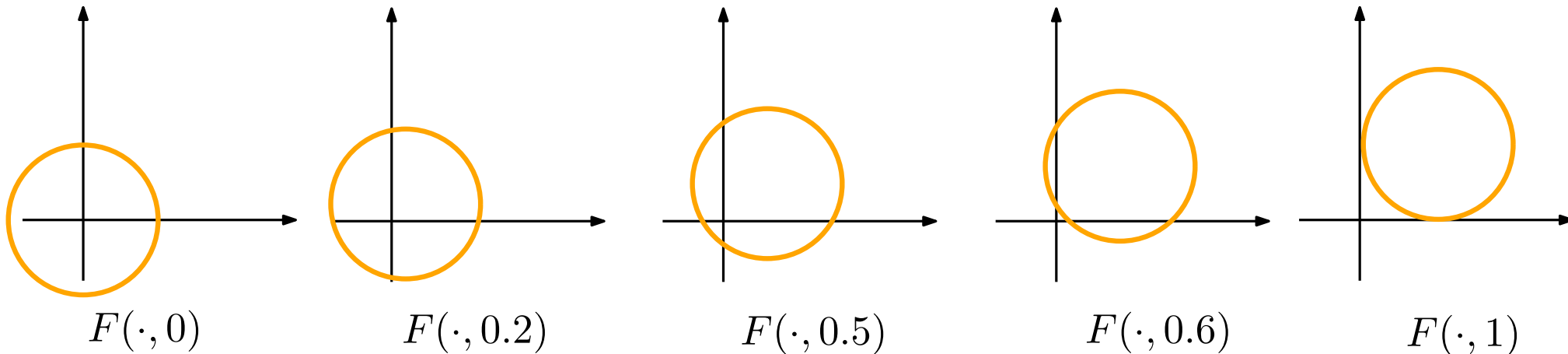
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Definition

4/16 (6/7)



Example: The map

$$F: \mathbb{S}_1 \times [0, 1] \longrightarrow \mathbb{R}^2$$
$$\theta \longmapsto (\cos(\theta) + t, \sin(\theta) + t)$$

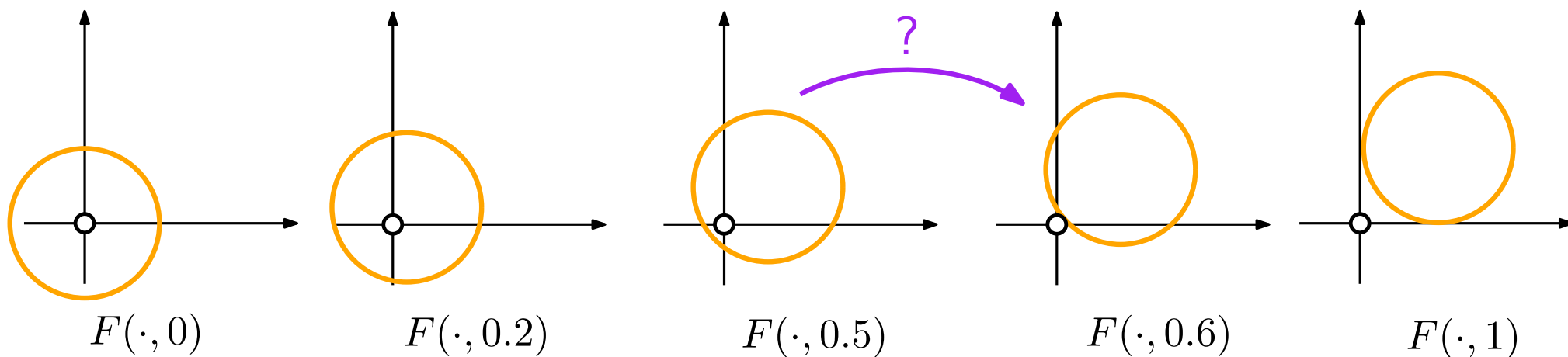
is a homotopy between the maps $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ and $g: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ defined as

$$f: \theta \mapsto (\cos(\theta), \sin(\theta)) \quad \text{and} \quad g: \theta \mapsto (\cos(\theta) + 1, \sin(\theta) + 1)$$

(**Notation:** \mathbb{S}_1 denotes the unit circle of \mathbb{R}^2 .)

Definition

4/16 (7/7)



Non-example: Between \mathbb{S}_1 and $\mathbb{R}^2 \setminus \{(0, 0)\}$, the plane without the origin, there is no homotopy between the maps

$$f: \theta \mapsto (\cos(\theta), \sin(\theta)) \quad \text{and} \quad g: \theta \mapsto (\cos(\theta) + 1, \sin(\theta) + 1)$$

The homotopy F would pass through the point $(0, 0)$ at some point, which is impossible.

(**Notation:** \mathbb{S}_1 denotes the unit circle of \mathbb{R}^2 .)

From a homotopic point a view, a *trivial map* is a map that is homotopic to a constant map.

Proposition: Let (X, \mathcal{T}) be a topological space and $f: X \rightarrow \mathbb{R}^n$ a continuous map. Then f is homotopic to a constant map.

Proof: Consider the continuous application

$$\begin{aligned} F: X \times [0, 1] &\longrightarrow \mathbb{R}^n \\ x &\longmapsto tf(x) \end{aligned}$$

We have that $F(\cdot, 1) = f$, and $F(\cdot, 0): x \mapsto 0$ is a constant map.

Hence F is a homotopy between f and the constant map $g: X \rightarrow \mathbb{R}^n$
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Exercise: Let $f: \mathbb{R}^n \rightarrow X$ be a continuous map. Then f is homotopic to a constant map.

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II - Homotopy equivalence between topological spaces

III - Link with homeomorphic spaces

VI - Invariants

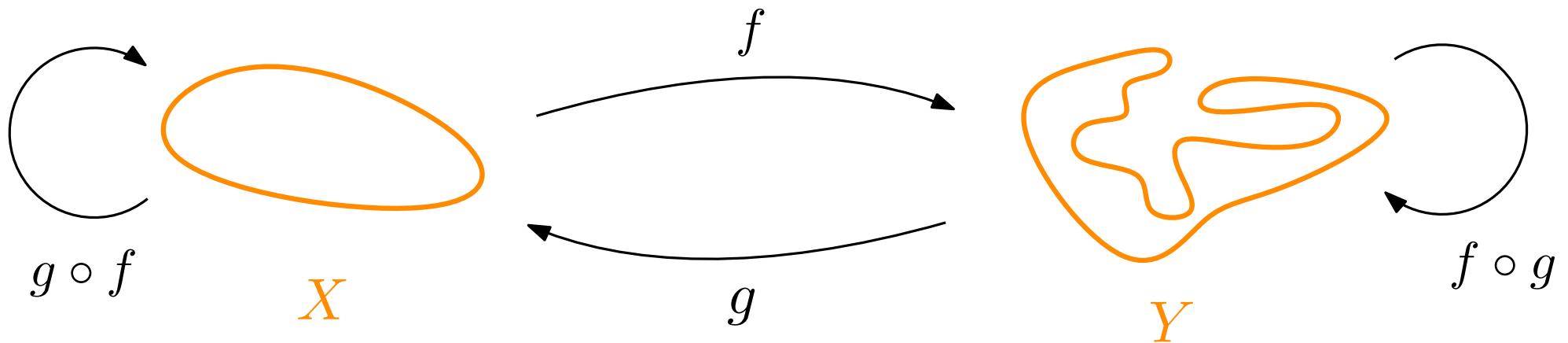
Definition

7/16 (1/2)

Defintion Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two topological spaces. A *homotopy equivalence* between X and Y is a pair of continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that:

- $g \circ f: X \rightarrow X$ is homotopic to the identity map $\text{id}: X \rightarrow X$,
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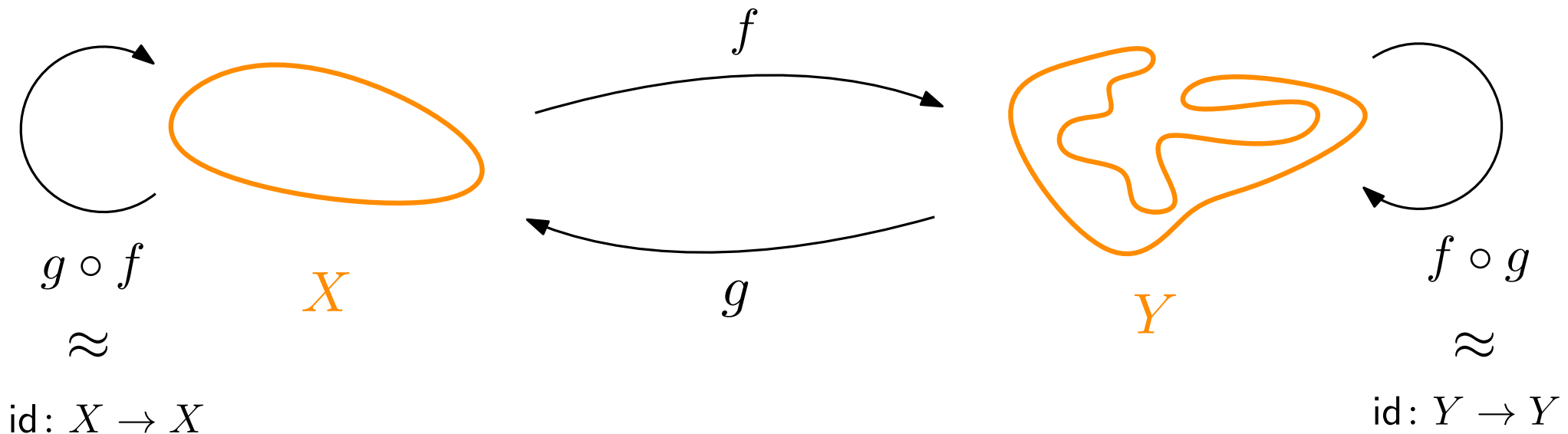
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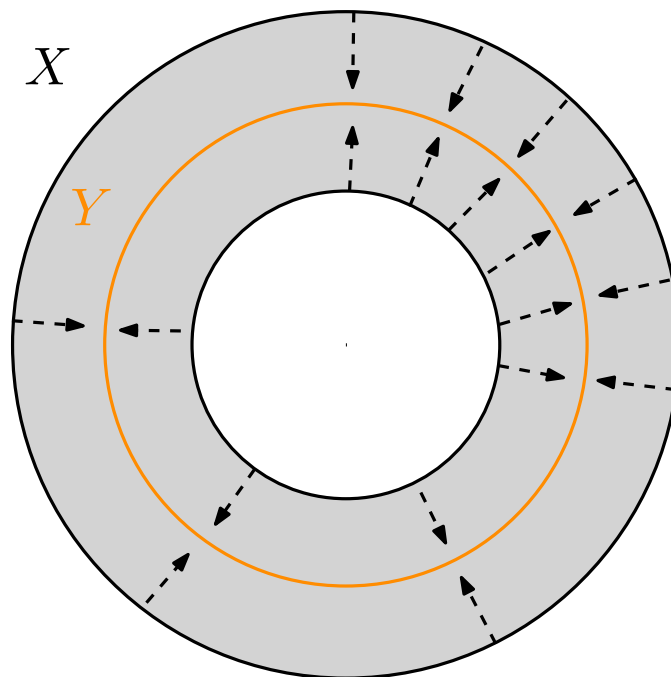


Determining whether two topological spaces are homotopy equivalent may be difficult. When one is a subset of the other, we have a handy tool:

Definition: Let (X, \mathcal{T}) be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $\mathcal{T}|_Y$.

A *retraction* is a continuous map $r: X \rightarrow Y$ such that $\forall y \in Y, r(y) = y$.

A *deformation retraction* is a homotopy $F: X \times [0, 1] \rightarrow Y$ between the identity map $\text{id}: X \rightarrow X$ and a retraction $r: X \rightarrow Y$.



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Proposition: If a deformation retraction exists, then X and Y are homotopic equivalent.

Proof: Let $r: X \rightarrow Y$ denote the retraction, and consider the inclusion map $i: Y \rightarrow X$. Let us prove that r, i is a homotopy equivalence.

First, let us prove that $i \circ r: X \rightarrow X$ is homotopic to the identity map $\text{id}: X \rightarrow X$. This is clear because $i \circ r = r$, and r is homotopic to the identity by definition of a deformation retraction.

Second, let us prove that $r \circ i: Y \rightarrow Y$ is homotopic to the identity map $\text{id}: Y \rightarrow Y$. This is obvious because $r \circ i = \text{id}$ by definition of a retraction.

Deformation retractions

8/16 (3/4)

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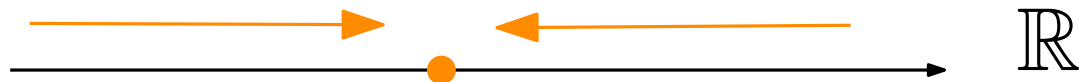
Example: For any $n \geq 1$, the Euclidean space \mathbb{R}^n is homotopy equivalent to the point $\{0\} \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r: \mathbb{R}^n &\longrightarrow \{0\} \\ x &\longmapsto 0 \end{aligned}$$

It is homotopic to the identity $\text{id}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ via the deformation retraction

$$\begin{aligned} F: \mathbb{R}^n \times [0, 1] &\longrightarrow \mathbb{R}^n \\ x &\longmapsto (1 - t)x \end{aligned}$$

Indeed, we have $F(\cdot, 0) = \text{id}$ and $F(\cdot, 1) = r$.



Deformation retractions

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Definition: Let (X, \mathcal{T}) be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $\mathcal{T}|_Y$. A *retraction* is a continuous map $r: X \rightarrow Y$ such that $\forall y \in Y, r(y) = y$. A *deformation retraction* is a homotopy $F: X \times [0, 1] \rightarrow Y$ between the identity map $\text{id}: X \rightarrow X$ and a retraction $r: X \rightarrow Y$.

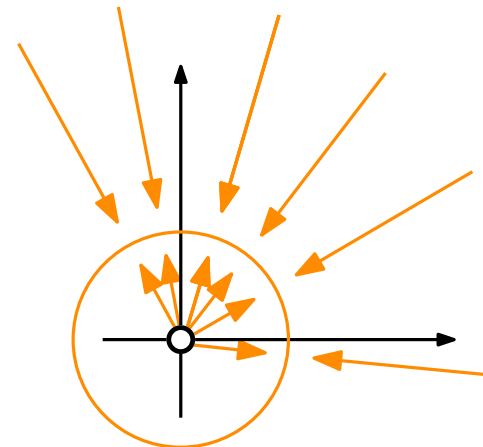
Example: For any $n \geq 1$, the Euclidean space without origin, $\mathbb{R}^n \setminus \{0\}$, is homotopy equivalent to the sphere $\mathbb{S}(0, 1) \subset \mathbb{R}^n$. To prove this, consider the retraction

$$\begin{aligned} r: \mathbb{R}^n \setminus \{0\} &\longrightarrow \mathbb{S}(0, 1) \\ x &\longmapsto \frac{x}{\|x\|} \end{aligned}$$

It is homotopic to the identity $\text{id}: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ via the deformation retraction

$$\begin{aligned} F: (\mathbb{R}^n \setminus \{0\}) \times [0, 1] &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ x &\longmapsto \left(1 - t + \frac{t}{\|x\|}\right) x \end{aligned}$$

Indeed, we have $F(\cdot, 0) = \text{id}$ and $F(\cdot, 1) = r$.



Let us denote $X \approx Y$ if the two topological spaces X and Y are homotopic equivalent.

Being homotopic equivalent is an *equivalence relation*. That is:

- (*Reflexivity*) $X \approx X$
- (*Symmetry*) $X \approx Y \implies Y \approx X$.
- (*Transitivity*) $X \approx Y$ and $Y \approx Z \implies X \approx Z$.

We can classify topological spaces according to this relation, and obtain *classes of homotopy equivalence*:

Homotopy equivalence relation

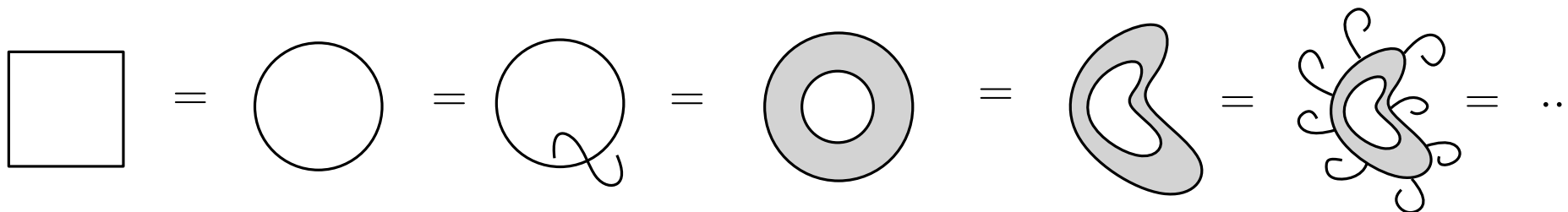
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Homotopy equivalence relation

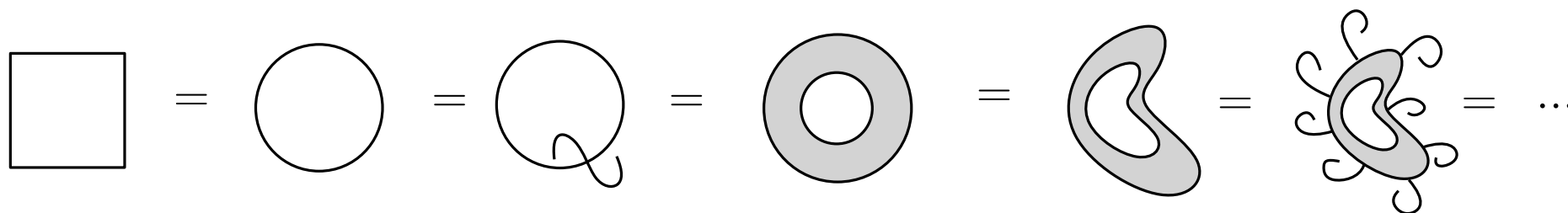
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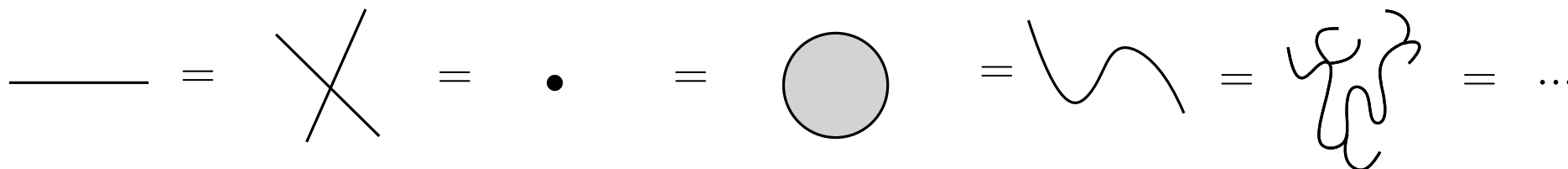
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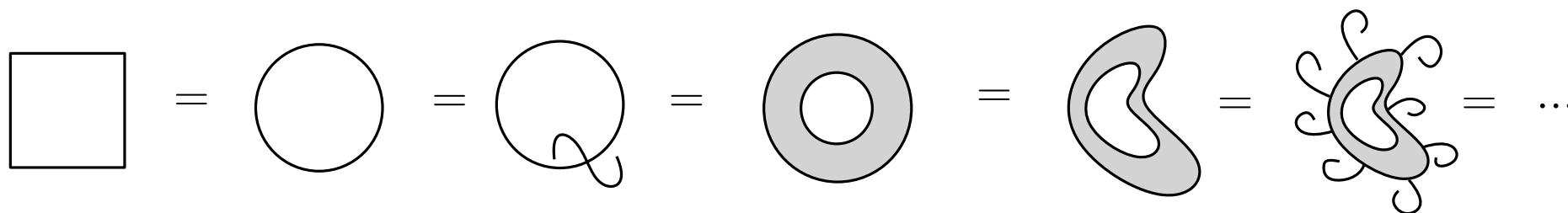
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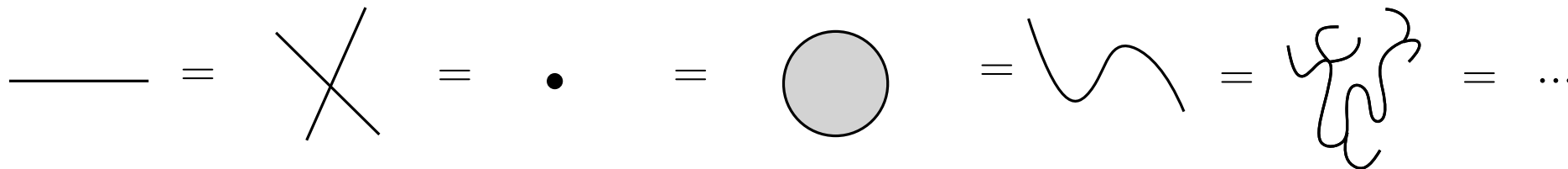
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We can classify topological spaces according to this relation, and obtain *classes of homotopy equivalence*:



the class of circles



the class of points

the class of spheres, the class of torii, the class of Klein bottles, ...

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Homeomorphic implies homotopic

11/16 (1/4)

Proposition: Let X, Y be two topological spaces. If they are homeomorphic, then they are homotopic equivalent.

In other words:

$$X \simeq Y \implies X \approx Y.$$

Consequence: In order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic.

Homeomorphic implies homotopic

11/16 (2/4)

Proposition: Let X, Y be two topological spaces. If they are homeomorphic, then they are homotopic equivalent.

In other words:

$$X \simeq Y \implies X \approx Y.$$

Consequence: In order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic.

Proof: Let $h: X \rightarrow Y$ be a homeomorphism.

We have to build a homotopy equivalence $f: X \rightarrow Y, g: Y \rightarrow X$.

Define $f = h$ and $g = h^{-1}$. Let us show $g \circ f$ is homotopic to $\text{id}: X \rightarrow X$ and $f \circ g$ is homotopic to $\text{id}: Y \rightarrow Y$.

We have $g \circ f = h^{-1} \circ h = \text{id}$. But $\text{id}: X \rightarrow X$ is homotopic to $\text{id}: X \rightarrow X$ (any map is homotopic to itself.)

Similarly, $f \circ g = h \circ h^{-1} = \text{id}$, and $\text{id}: Y \rightarrow Y$ is homotopic to $\text{id}: Y \rightarrow Y$.

Conclusion: f, g is a homotopy equivalence between X and Y , hence X and Y are homotopy equivalent.

Homeomorphic implies homotopic

11/16 (3/4)

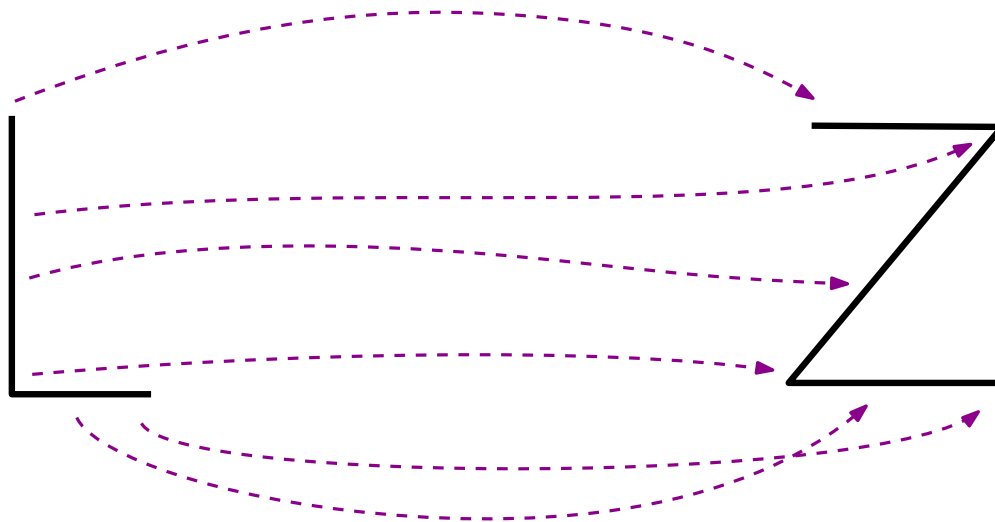
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Example: The letter L and the letter Z are homeomorphic:



Hence they are homotopy equivalent.

Homeomorphic implies homotopic

11/16 (4/4)

Proposition: Let X, Y be two topological spaces. If they are homeomorphic, then they are homotopic equivalent.

In other words:

$$X \simeq Y \implies X \approx Y.$$

Consequence: In order to prove that two spaces are homotopy equivalent, it is enough to show that they are homeomorphic.

This strategy does not always work: some spaces are homotopy equivalent but not homeomorphic!

This is the case for \mathbb{R}^n and $\{0\}$ for instance.

Consider the following letters of the alphabet, endowed with the subspace topology induced from \mathbb{R}^2 :

A B C D E F

Classify them into homotopy equivalence classes, then classify them into homeomorphism equivalence classes.

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Classify them into homotopy equivalence classes, then classify them into homeomorphism equivalence classes.

Homotopy equivalence classes:

$\rightarrow A \approx D$ $(\approx \bigcirc)$
 $\rightarrow B$ $(\approx \bigcirc \bigcirc)$
 $\rightarrow C \approx D \approx E \approx F$ $(\approx \bullet)$

Exercise

12/16 (3/3)

Consider the following letters of the alphabet, endowed with the subspace topology induced from \mathbb{R}^2 :

A B C D E F

Classify them into homotopy equivalence classes, then classify them into homeomorphism equivalence classes.

Homotopy equivalence classes:

→ A \approx D ($\approx \bigcirc$)
→ B ($\approx \bigcirc \bigcirc$)
→ C \approx D \approx E \approx F ($\approx \bullet$)

Homeomorphism equivalence classes:

→ A → C → E \simeq F
→ B → D

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Number of connected components

14/16 (1/4)

Until here, we studied two quantities associated to topological spaces: *number of connected components* and *dimension*.

Proposition: Two homotopy equivalent topological spaces admit the same number of connected components.

Proof: Let X, Y be two topological spaces, and $f: X \rightarrow Y, g: Y \rightarrow X$ a homotopy equivalence.

Let $F: X \times [0, 1] \rightarrow X$ be a homotopy between $g \circ f$ and $\text{id}: X \rightarrow X$.

Let $x \in X$, and O the connected component of x .

The space $O \times [0, 1]$ is connected. Hence its image $F(O \times [0, 1]) \subset X$ is connected too.

Moreover, $O = F(O \times \{1\}) \subset F(O \times [0, 1])$.

Hence $F(O \times [0, 1])$ is a connected subset of X that contains O , and we deduce that $O = F(O \times [0, 1])$.

Last, notice that

$$g \circ f(O) = F(O \times \{0\}) \subset F(O \times [0, 1]) = O.$$

Number of connected components

14/16 (2/4)

Until here, we studied two quantities associated to topological spaces: *number of connected components* and *dimension*.

Proposition: Two homotopy equivalent topological spaces admit the same number of connected components.

Proof: Let X, Y be two topological spaces, and $f: X \rightarrow Y, g: Y \rightarrow X$ a homotopy equivalence.

Let $F: X \times [0, 1] \rightarrow X$ be a homotopy between $g \circ f$ and $\text{id}: X \rightarrow X$.

Let $x \in X$, and O the connected component of x .

We have $g \circ f(O) \subset O$.

Suppose that X admits n connected components O_1, \dots, O_n , and that Y admits m of them.

By contradiction, suppose that $m < n$. This implies that we have two components O_i, O_j such that $f(O_i)$ and $f(O_j)$ are included in the same connected component O' of Y .

Hence $g \circ f(O_i)$ and $g \circ f(O_j)$ are included in a common connected component of X .

This is absurd because $g \circ f(O_i) \subset O_i$ and $g \circ f(O_j) \subset O_j$.

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Suppose that X admits n connected components O_1, \dots, O_n , and that Y admits m of them.

We have shown that $m \geq n$.

By exchanging the roles of X and Y in the whole reasoning, we obtain that $m \leq n$. We deduce that $m = n$.

Until here, we studied two quantities associated to topological spaces: *number of connected components* and *dimension*.

Proposition: Two homotopy equivalent topological spaces admit the same number of connected components.

In other words, *number of connected components* is an invariant of homotopy equivalence.

This allows to show that two spaces are not equivalent.

Example: For any $n, m \geq 0$ such that $n \neq m$, the subspaces $\{1, \dots, n\}$ and $\{1, \dots, m\}$ of \mathbb{R} are not homotopic equivalent.

Indeed, the first one admits n connected components, and the second one m components.

On the other hand, dimension is **not** an invariant of homotopy equivalence.

Indeed, some homotopic equivalent spaces have different dimensions.

This is the case, for instance, with all the Euclidean spaces \mathbb{R}^n , $n \geq 0$. They are all homotopic equivalent, but all with different dimensions.

Conclusion

We learnt to look at topological spaces from a homotopic-equivalence perspective.

This is a weaker notion than homeomorphism-equivalence.

Between the quantities, *number of connected components* and *dimension*, only one is invariant for the homotopic-equivalence relation.

Homework for tomorrow: Exercises 12 and 16

Facultative exercise: Exercises 13 and 14

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We learnt to look at topological spaces from a homotopic-equivalence perspective.

This is a weaker notion than homeomorphism-equivalence.

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Obrigado!