EMAp Summer Course

# Topological Data Analysis with Persistent Homology

https://raphaeltinarrage.github.io/EMAp.html

## Lesson $6$: Incremental algorithm

Last update: February 1, 2021

# Introduction



Yesterday we have defined

chain complex $\quad \dots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots$

$n$-cycles $\quad Z_n(K) = \mathsf{Ker}(\partial_n)$

$n$-boundaries $\quad B_n(K) = \mathsf{Im}(\partial_{n+1})$

$n^{\text{th}}$ homology group $\quad H_n(K) = Z_n(K)/B_n(K)$

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Today's objectives: how to compute them? what do they represent?

I - Incremental algorithm

II - Applications

III - Matrix algorithm

# Ordering the simplicial complex



Let $K$ be a simplicial complex with $n$ simplices. Choose a total order of the simplices

$$\sigma^1 < \sigma^2 < ... < \sigma^n$$

such that

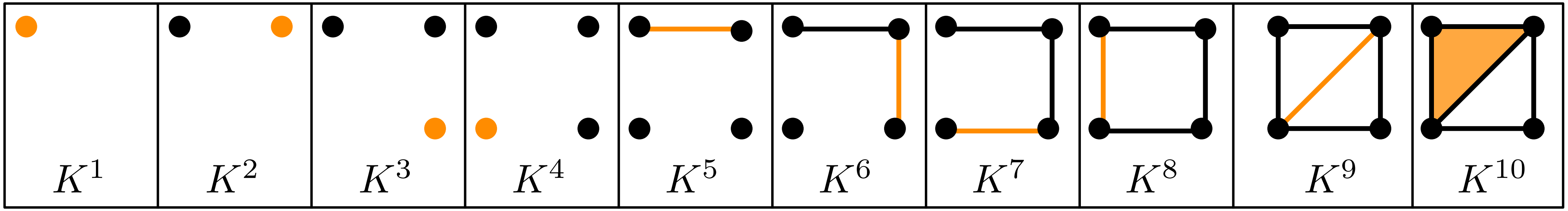
$$\forall \sigma, \tau \in K,\ \tau \subsetneq \sigma \implies \tau < \sigma.$$

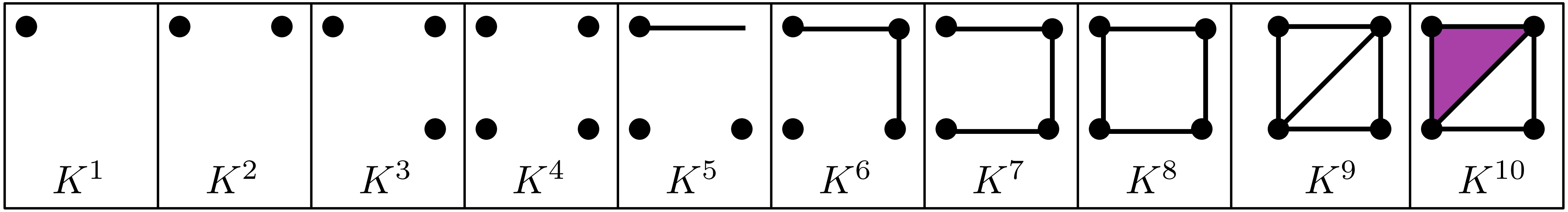
In other words, a face of a simplex is lower than the simplex itself.
For every $i \leq n$, consider the simplicial complex

$$K^i = \{\sigma^1, ..., \sigma^i\}.$$

We have $\forall i \leq n, K^{i+1} = K^i \cup \left\{\sigma^{i+1}\right\}$, and $K^n = K$. They form an inscreasing sequence of simplicial complexes

$$K^1 \ \subset \ K^2 \ \subset \ ... \ \subset \ K^n.$$



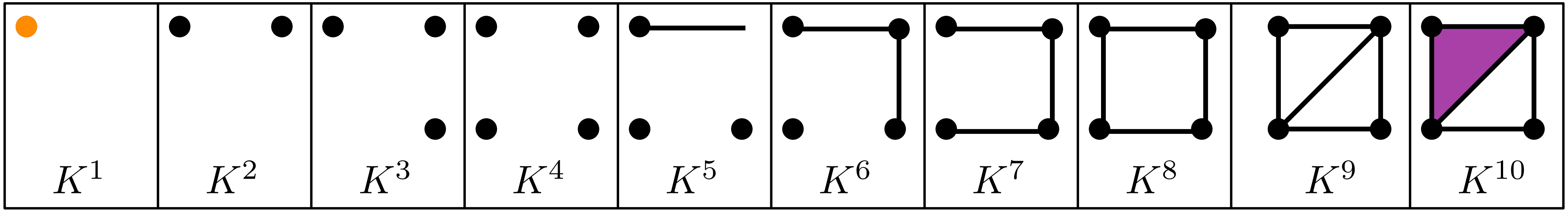



Let $k \geq 0$. We will compute the homology groups of $K^i$ incrementally:

$$H_k(K^1),\ H_k(K^2),\ H_k(K^3),\ H_k(K^4),\ H_k(K^5),\ H_k(K^6),\ H_k(K^7),\ H_k(K^8),\ H_k(K^9),\ H_k(K^{10})$$

# Positivity of simplices



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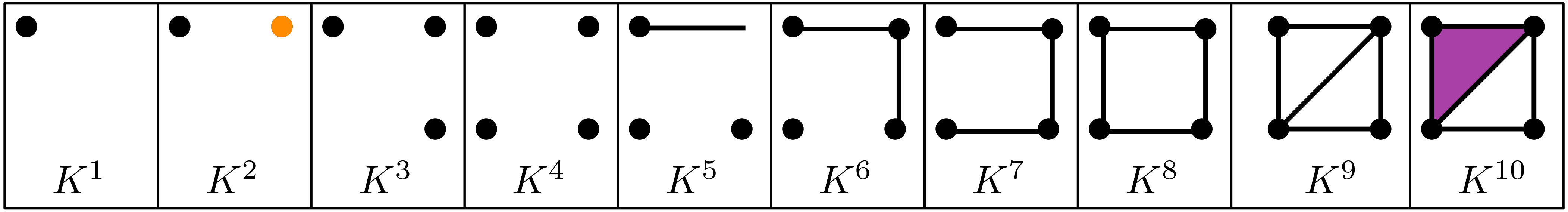
Definition: Let $i \in [\![1, n]\!]$, and $d = \dim(\sigma^i)$. Recall that $K^i = K^{i-1} \cup \{\sigma_i\}$.
The simplex $\sigma^i$ is *positive* if there exists a cycle $c \in Z_d(K^i)$ that contains $\sigma^i$.
Otherwise, $\sigma^i$ is *negative*.

Example:

- $\sigma^1 \in K^1$ is **positive** because it is included in the cycle $c = \sigma^1$ (indeed, $\partial_0(\sigma^1) = 0$).

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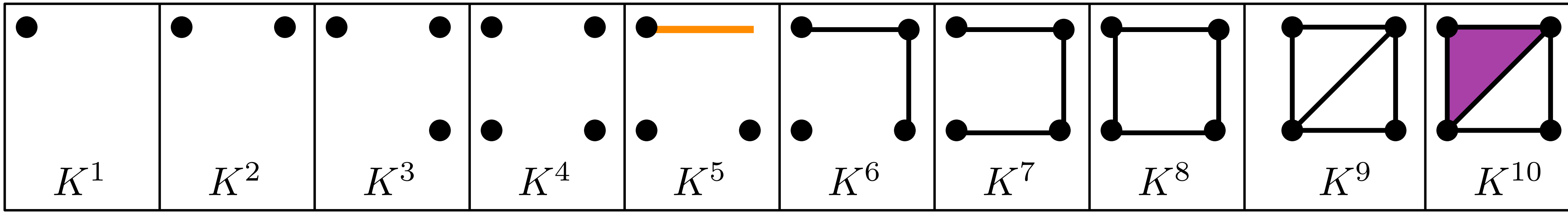


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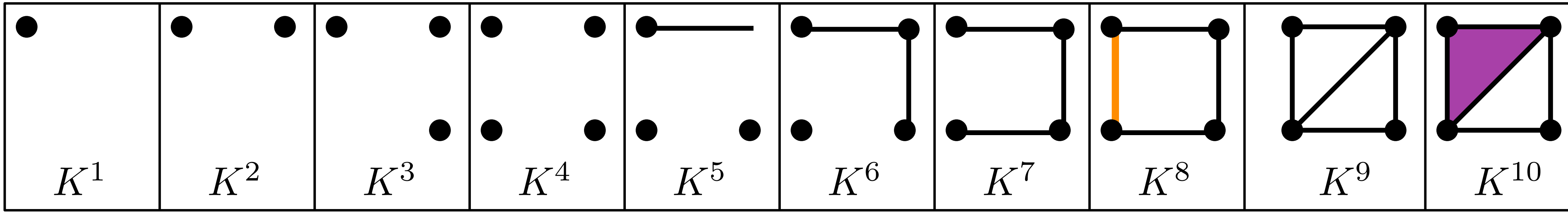


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- $\sigma^8 \in K^8$ is **positive** because it is included in the cycle $c = \sigma^5 + \sigma^6 + \sigma^7 + \sigma^8$ (indeed, $\partial_1(c) = 2\sigma^1 + 2\sigma^2 + 2\sigma^3 + 2\sigma^4 = 0$).

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Remark: By adding $\sigma^i$ in the simplicial complex, the only groups that may change are $Z_d(K^i)$ and $B_{d-1}(K^i)$.

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Lemma: If $\sigma^i$ is positive, then $\beta_d(K^i) = \beta_d(K^{i-1}) + 1$,
and for all $d' \neq d$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.

Proof: We start by proving the following fact: if $c \in Z_d(K^i)$ is a cycle that contains $\sigma_i$, then $c$ is not homologous (in $K^i$) to a cycle of $c' \in Z_d(K^{i-1})$.

By contradiction: if $c = c' + b$ with $c' \in Z_d(K^{i-1})$ and $b \in B_d(K^i)$, then $c - c' = b \in B_d(K^i)$. This is absurd because we just added $\sigma_i$: it cannot appear in a boundary of $K^i$.

As a consequence, $\dim Z_d(K^i) = \dim Z_d(K^{i-1}) + 1$.

Moreover, if $c$ is a cycle of $K^i$ that contains $\sigma^i$, then $\partial_i(\sigma^i) = \partial_i(c) + \partial_i(\sigma^i) = \partial_i(c + \sigma^i)$, and $c + \sigma^i$ is a chain of $K^{i-1}$. Hence $\dim B_{d-1}(K^i) = \dim B_{d-1}(K^{i-1})$.

We conclude by using the relation $\beta_d(K^i) = \dim Z_d(K^i) - \dim B_d(K^i)$.

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Lemma: If $\sigma^i$ is negative, then $\beta_{d-1}(K^i) = \beta_{d-1}(K^{i-1}) - 1$,
and for all $d' \neq d - 1$, $\beta_{d'}(K^i) = \beta_{d'}(K^{i-1})$.

Proof: We start by proving the following fact: $\partial_d(\sigma^i)$ is not a boundary of $K^{i-1}$.

Otherwise, we would have $\partial_d(\sigma^i) = \partial_d(c)$ with $c \in C_d(K^{i-1})$, i.e. $\partial_d(\sigma^i + c) = 0$.
Hence $\sigma^i + c$ would be a cycle of $K^i$ that contains $c$, contradicting the negativity of $\sigma^i$.

As a consequence, $\dim B_{d-1}(K^i) = \dim B_{d-1}(K^{i-1}) + 1$.

Moreover, since $\sigma^i$ is negative, we have $\dim Z_d(K^i) = \dim Z_d(K^{i-1})$.

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# Incremental algorithm



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We deduce the following algorithm:

**Input:** an increasing sequence of simplicial complexes $K^1 \subset \cdots \subset K^n = K$
**Output:** the Betti numbers $\beta_0(K), ... \beta_d(K)$
$\beta_0 \leftarrow 0, ..., \beta_d \leftarrow 0$;
**for** $i \leftarrow 1$ **to** $n$ **do**
  $d = \dim(\sigma^i)$;
  **if** $\sigma^i$ *is positive* **then**
    $\beta_k(K^i) \leftarrow \beta_k(K^i) + 1$;
  **else if** $d > 0$ **then**
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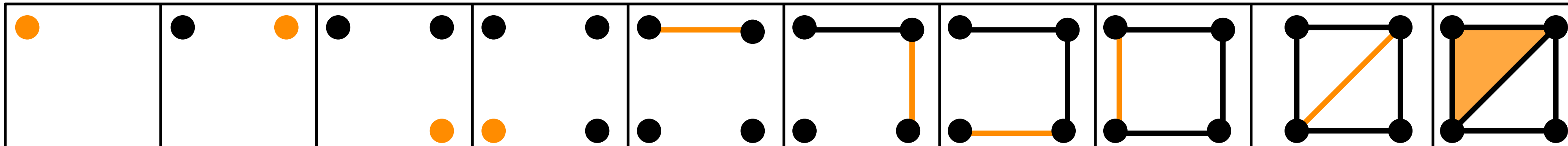
# Incremental algorithm



| | $K^1$ | $K^2$ | $K^3$ | $K^4$ | $K^5$ | $K^6$ | $K^7$ | $K^8$ | $K^9$ | $K^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Dimension | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 |
| Positivity | $+$ | $+$ | $+$ | $+$ | $-$ | $-$ | $-$ | $+$ | $+$ | $-$ |
| $\beta_0(K^i)$ | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 1 | 1 | 1 |
| $\beta_1(K^i)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 1 |

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I - Incremental algorithm

II - Applications

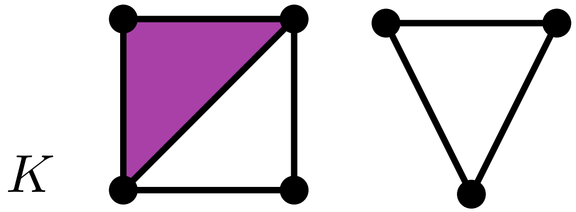
III - Matrix algorithm

# Number of connected components



Proposition: Let $X$ be a (triangulable) topological space. Then its $0^{\text{th}}$ Betti number, $\beta_0(X)$, is equal to the number of connected components of $X$.

Let $K$ be a triangulation of $X$.



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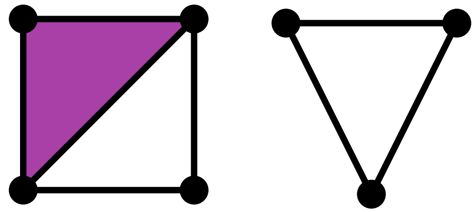

First, a definition: say that a simplicial complex $L$ is *combinatorially connected* of for every vertex $v, w$ of $L$, there exists a sequence of edges that connects $v$ and $w$:

$$[v, v_1], \quad [v_1, v_2], \quad [v_2, v_3], \quad ..., \quad [v_n, w].$$

Let $m$ be the number of connected components $X$, and let $K$ be triangulation of $X$. We accept the following equivalent statement: there exists $m$ **disjoint**, **non-empty** and **combinatorially connected** simplicial sub-complex $L_1, ..., L_m$ of $K$ such that

$$K = \bigcup_{1 \leq i \leq m} L_i.$$

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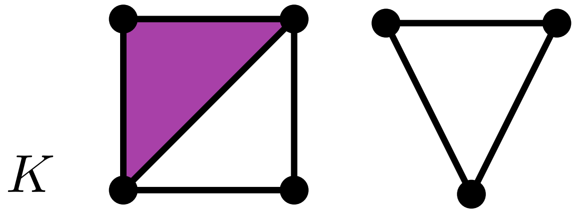
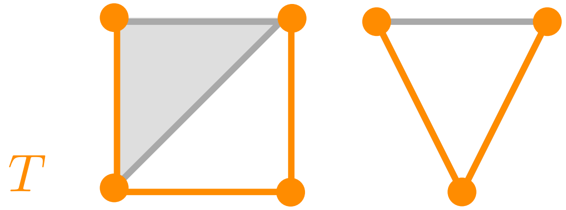



Proof: Let $T$ be a spanning forest of $K$, that is, a union of spanning trees. It admits $m$ combinatorially connected components.

Consider an ordering of the simplices of $K$ that begins with an ordering of $T$.

Apply the incremental algorithm. Each vertex increases $\beta_0$ by $1$.

Since $T$ is a tree, all its edges are negative simplices ($T$ has no cycles), hence decrease $\beta_0$. Each tree of the forest contains $k-1$ edges, where $k$ is the number of vertices of the corresponding component.

Since $T$ is a spanning tree, each other edges of $K$ is positive, hence $\beta_0$ does not change. Similarly, the other simplices of $K$ do not change $\beta_0$. We deduce the result.

# Homology of spheres



For any $n \geq 1$, consider the vertex set $V = \{0, \ldots, n\}$, and the simplicial complex

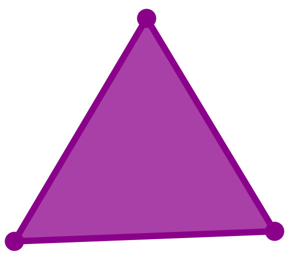
$$\Delta_n = \{S \subset V, S \neq \emptyset\}.$$

We call it the *simplicial standard $n$-simplex*. Define its boundary as

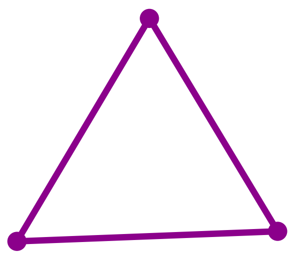
$$\partial\Delta_n = \Delta_n \setminus V.$$

The simplicial complex $\partial\Delta_n$ is a triangulation of the $(n-1)$-sphere $\mathbb{S}_{n-1} \subset \mathbb{R}^n$.

As a consequence, for all $i \geq 0$, $H_i(\mathbb{S}_n) = H_i(\partial\Delta_{n+1})$.



$\Delta_2$



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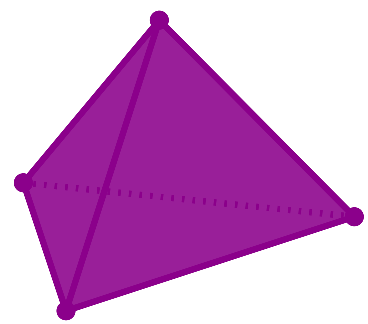
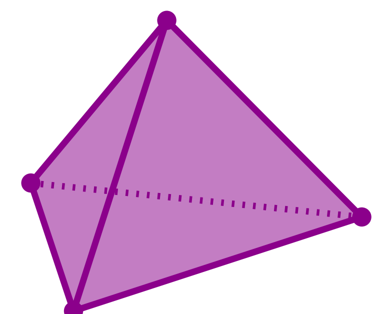


$\Delta_3$



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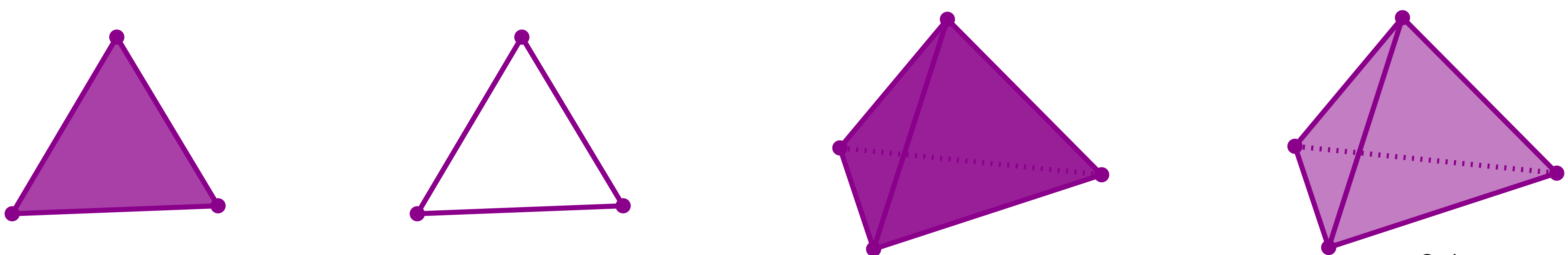


$\Delta_2$ $\partial\Delta_2$ $\Delta_3$ $\partial\Delta_3$

Proposition: The Betti numbers of $\mathbb{S}_n$ are:

- $\beta_i(\mathbb{S}_n) = 1$ for $i = 0$ or $n$,
- $\beta_i(\mathbb{S}_n) = 0$ else.

# Homology of spheres



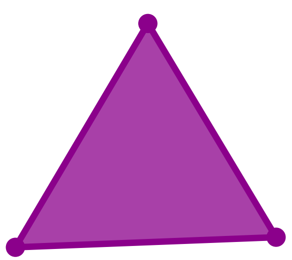
Proof: Consider the *simplicial standard $n$-simplex* $\Delta_n$. It is homotopy equivalent to a point (its topological realization deform retracts on any point of it). Hence $\Delta_n$ has the same Betti numbers as the point:

- $\beta_1(\mathbb{S}_n) = 1$,
- $\beta_i(\mathbb{S}_n) = 0$ for $i > 0$.

Now, if we run the incremental algorithm for homology on $\Delta_n$, but stopping before adding the $n$-simplex $V$, we would obtain the Betti numbers of $\partial\Delta_n$.

Note that the $n$-simplex is negative. Hence

- $\beta_n(\partial\Delta_n) = \beta_n(\Delta_n) + 1$,
- $\beta_i(\partial\Delta_n) = \beta_i(\Delta_n)$ for $i \neq n$.



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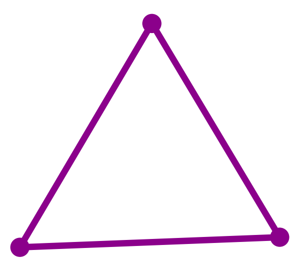

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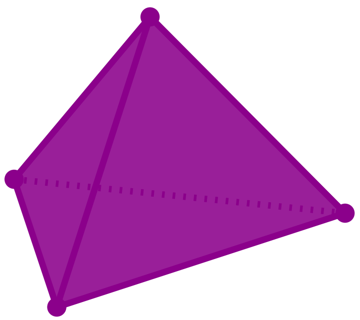

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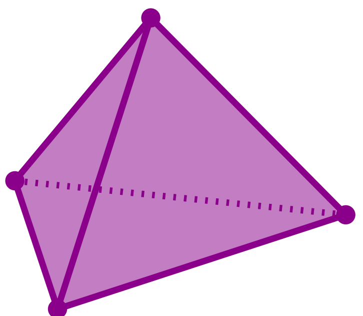

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# Invariance of domain



Theorem (Invariance of Domain): For every integers $m, n$ such that $m \neq n$, the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

Proof: Let $m, n$ such that $m \neq n$. By contradiction, suppose that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic via $f$.

Let $0$ denote the origin of $\mathbb{R}^n$. By restriction, we get a homeomorphism

$$\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}.$$

We deduce the following weaker statement: $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{f(0)\}$ are homotopic equivalent.

We deduce that the sphere $\mathbb{S}_{n-1}$ and $\mathbb{S}_{m-1}$ are homotopic equivalent.

Hence $\mathbb{S}_{n-1}$ and $\mathbb{S}_{m-1}$ must admit the same homology groups. This contradict the previous proposition.

# Euler characteristic



Reminder: the Euler characteristic of a simplicial complex $K$ is

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Proposition: Let $X$ be a (triangulable) topological space. Then its Euler characteristic is equal to

$$\chi(X) = \sum_{0 \leq i \leq n} (-1)^i \cdot \beta_i(X)$$

where $n$ is the maximal integer such that $\beta_i(X) \neq 0$.

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where $n$ is the maximal integer such that $\beta_i(X) \neq 0$.

Proof: Let $K$ be a triangulation of $X$. Pick an ordering $K^1 \subset \cdots \subset K^n = K$ of $K$, with $K^i = K^{i-1} \cup \{\sigma^i\}$ for all $2 \leq i \leq n$.

By induction, let us show that, for all $1 \leq m \leq n$,

$$\sum_{0 \leq i \leq m} (-1)^i \cdot \beta_i(K^m) = \sum_{0 \leq i \leq m} (-1)^i \cdot (\text{number of simplices of dimension } i \text{ of } K^m).$$

For $m = 1$, $\sigma^m$ is a $0$-simplex, and the equality reads $1 = 1$.

Now, suppose that the equality is true for $1 \leq m < n$, and consider the simplex $\sigma^{m+1}$. Let $d = \dim \sigma^{m+1}$. The right-hand side of the Equation is increased by $(-1)^d$.

If $\sigma^{m+1}$ is positive, then $\beta_d(K^{m+1}) = \beta_d(K^m) + 1$, hence the left-hand side of the Equation is increased by $(-1)^d$.

Otherwise, it is negative, and $\beta_{d-1}(K^{m+1}) = \beta_{d-1}(K^m) - 1$, hence the left-hand side of the Equation is increased by $-(-1)^{d-1} = (-1)^d$.

I - Incremental algorithm

II - Applications

III - Matrix algorithm

# Boundary matrix

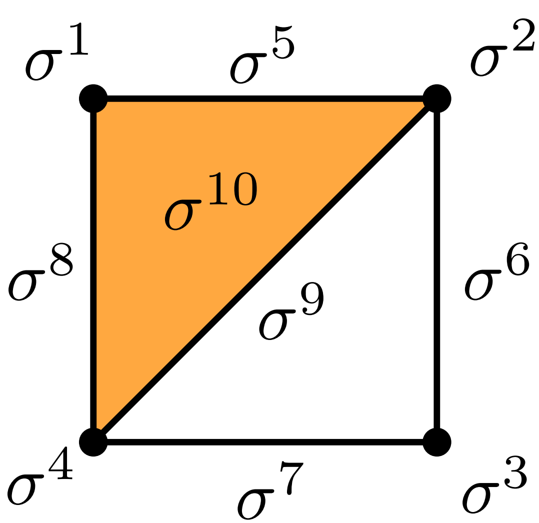


The only thing missing to apply the incremental algorithm is to determine whether a simplex is positive or negative.

Let $K$ be a simplicial complex, and $\sigma^1 < \sigma^2 < \cdots < \sigma^n$ and ordering of its simplices.

Define the *boundary matrix* of $K$, denoted $\Delta$, as follows: $\Delta$ is a $n \times n$ matrix, whose $(i,j)$-entry ($i^{\text{th}}$ row, $j^{\text{th}}$ column is)

$$\begin{aligned} \Delta_{i,j} \;&=\; 1 \;\text{ if } \sigma^i \text{ is a face of } \sigma^j \text{ and } |\sigma^i| = |\sigma^j| - 1 \\ &\phantom{=}\;\; 0 \text{ else.} \end{aligned}$$




$$\begin{array}{c c} & \begin{matrix} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{matrix} \\ \begin{matrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

# Boundary matrix

By adding columns one to the others, we create chains.
If we were able to reduce a column to zero, then we found a cycle.

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

$$\partial_1(\sigma^6) = \sigma^2 + \sigma^3$$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

$$\partial_1(\sigma^5 + \sigma^6 + \sigma^7 + \sigma^8) = 0$$

# Boundary matrix



The process of reducing columns to zero is called *Gauss reduction*.
For any $j \in [\![1, n]\!]$, define
$$\delta(j) = \max\{i \in [\![1, n]\!], \Delta_{i,j} \neq 0\}.$$

If $\Delta_{i,j} = 0$ for all $j$, then $\delta(j)$ is *undefined*.

We say that the boundary matrix $\Delta$ is *reduced* if the map $\delta$ is injective on its domain of definition.

$$\begin{array}{c@{\quad}c}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

# Boundary matrix



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**Algorithm 2:** Reduction of the boundary matrix

---

**Input:** a boundary matrix $\Delta$
**Output:** a reduced matrix $\widetilde{\Delta}$
**for** $i \leftarrow 1$ **to** $n$ **do**
  **while** *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column $i$ to column j;

---

$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \\ + & + & + & + & - & - & - & + & + & - \end{array}
\end{array}
\end{array}
$$

# Boundary matrix

**Algorithm 2:** Reduction of the boundary matrix

**Input:** a boundary matrix $\Delta$
**Output:** a reduced matrix $\widetilde{\Delta}$
**for** $i \leftarrow 1$ **to** $n$ **do**
  **while** *there exists* $i < j$ *with* $\delta(i) = \delta(j)$ **do**
    add column $i$ to column j;

Lemma: Suppose that the boundary matrix is reduced. Let $j \in [\![1, n]\!]$.
If $\delta(j)$ is defined, then the simplex $\sigma^j$ is negative.
Otherwise, it is positive.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^5+\sigma^6+\sigma^7+\sigma^8 & \sigma^9+\sigma^6+\sigma^7 & \sigma^{10} \end{array} \\
\begin{array}{c} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \\ \sigma^7 \\ \sigma^8 \\ \sigma^9 \\ \sigma^{10} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
 & \begin{array}{cccccccccc} \sigma^1 & \sigma^2 & \sigma^3 & \sigma^4 & \sigma^5 & \sigma^6 & \sigma^7 & \sigma^8 & \sigma^9 & \sigma^{10} \\ + & + & + & + & - & - & - & + & + & - \end{array}
\end{array}
$$

# Conclusion

We defined the standard algorithm for (persistent homology).

It works incrementally, by using the predicate 'test of positivity of a simplex'.

Using the algorithm, we have been able to compute the homology groups of the spheres, and as a consequence we proved the Invariance of Domain.

Homework: Exercises 31, 34

Facultative: Exercises 33

# Conclusion

We defined the standard algorithm for (persistent homology).

It works incrementally, by using the predicate ‘test of positivity of a simplex’.

Using the algorithm, we have been able to compute the homology groups of the spheres, and as a consequence we proved the Invariance of Domain.

Homework: Exercises 31, 34

Facultative: Exercises 33

Merci !