Applying Bayesian probability to Number Mind puzzles

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Overview

Number mind puzzles are a type of logic puzzle presented in Project Euler problem 185. Since the given guesses within a puzzle allow for only one possible answer, this ceases to be a probability question given enough analysis, but if the given information is reduced to occurrences and non-occurrences of a particular digit in a particular place, it's then possible to calculate the probability of that digit/place combination. The underlying assumptions are that the solution was generated randomly, and the digits of each guess were generated randomly when the puzzle was created. While the unnatural distribution of guess scores for the given puzzle indicates that guess selection was not wholly random, I believe the formula derived below applies to any method of puzzle creation in which individual guess digits are generated randomly, even if guesses are rejected based on their scores.

Derivation

To derive the expression for the probability that the solution has a particular digit in a particular place, given a distribution of guesses, we start by rearranging Bayes' theorem:

$$\begin{aligned} P(a|b) &= \frac{P(b|a)P(a)}{P(b)} \\ &= \frac{P(b|a)P(a)}{P(b|a)P(a) + P(b|\neg a)P(\neg a)} \\ &= \frac{1}{1 + \frac{P(\neg a)}{P(a)}\frac{P(b|\neg a)}{P(b|a)}} \end{aligned}$$

In this context, event a refers to digit d occurring in place i in the solution x, i.e. $x_i = d$. Assuming the solution is generated randomly at the outset, that means $\frac{P(\neg a)}{P(a)}$ can be simplified:

$$P(a|b) = \frac{1}{1 + 9\frac{P(b|\neg a)}{P(b|a)}} \qquad (1)$$

The event b refers to the specific distribution of guess scores we were given, categorized into guesses g with $g_i = d$ and those with $g_i \neq d$. For i = 0 and d = 3, in the problem's demo example, we see that digit 3 occurs in spot 0 once in a 1-correct guess, and again in a 2-correct guess. Digit 3 fails to occur in spot zero in a 0-correct guess, a 1-correct guess, and two 2-correct guesses. So the distribution of occurrences and non-occurrences for 3 in spot zero is: [0, 1, 1, 0, 0, 0], [1, 1, 2, 0, 0, 0]. Any complication introduced by guess ordering cancels out on the top and bottom of the fraction in the denominator of equation (1), meaning we can just treat $P(b|\neg a)$ as the product of a series of individual guess probabilities, and the same for $P(b|a)$. We start by splitting b into two parts, with c representing the occurrences and d the non-occurrences.

This leaves us with four probabilities to calculate, $P(c|a)$, $P(c|\neg a)$, $P(d|a)$, $P(d|\neg a)$. We'll look at the first in more depth. $P(c|a)$ refers to the probability of getting a particular set of guesses g with $g_i = d$, given that within solution x, $x_i = d$. Consider one guess in particular out of this set of guesses. Once we succeed in the $\frac{1}{10}$ chance of getting $g_i = d$, g_i contributes 1 to the score, since we know it to be correct, meaning the remaining $n-1$ digits need to contain $j-1$ correct digits. Assuming the guesses are generated randomly, that means each individual remaining digit has a $\frac{9}{10}$ chance of being wrong and a $\frac{1}{10}$ chance of being right. There are $(n-1)$ choose $(j-1)$ ways of selecting the correct digits among those that remain. This gives us the following probability:

$$P(s(g) = j, g_i = d | x_i = d) \quad = \quad \frac{9^{n-j}}{10^n}\binom{n-1}{j-1}$$

By similar means, we obtain expressions for the other three probabilities:

$$\begin{aligned}
P(s(g) = j, g_i = d | x_i \neq d) \quad &= \quad \frac{9^{n-j-1}}{10^n}\binom{n-1}{j} \\
P(s(g) = j, g_i \neq d | x_i = d) \quad &= \quad \frac{9^{n-j}}{10^n}\binom{n-1}{j} \\
P(s(g) = j, g_i \neq d | x_i \neq d) \quad &= \quad \frac{9^{n-j-1}}{10^n}\left(9\binom{n-1}{j-1} + 8\binom{n-1}{j}\right)
\end{aligned}$$

The last equation has a more complex form because there are two sub-cases, since g_i may or may not be the same as x_i. Each of these expressions will be raised to some power, depending on how many occurrences/non-occurrences there are with a particular score, and referring back to equation (1), the set of expressions with $x_i \neq d$ as a prior will be divided by the set of expressions with prior $x_i = d$. If we consider only the occurrences distribution c, and if we let e_j be the number of guesses in the distribution with score j, we get the following:

$$\frac{P(c|\neg a)}{P(c|a)} = \prod_{j=0}^{n}\left(\frac{n-j}{9j}\right)^{e_j}$$

We obtain a similar expression for the non-occurrences distribution, using f_j for the number of guesses with score j in the distribution:

$$\frac{P(d|\neg a)}{P(d|a)} = \prod_{j=0}^{n}\left(\frac{8n+j}{9(n-j)}\right)^{f_j}$$

Putting this all together, and plugging into equation (1), we get our final result:

$$\frac{1}{1 + 9\prod_{j=0}^{n}\left(\frac{n-j}{9j}\right)^{e_j}\left(\frac{8n+j}{9(n-j)}\right)^{f_j}} \tag{2}$$

When actually implementing this, some care needs to be taken with all-wrong guesses. If a digit occurs in a zero-scoring guess, the denominator will spike to infinity, and the corresponding probability will be zero. If a digit doesn't occur in a zero-scoring guess, the $(n/0)^0$ expression should be taken to be 1. In the rare event that an all-correct guess is given, similar logic applies.