6.006	Lecture 21	Dec. 1, 2011

TODAY: Dynamic Programming III (of 4)
- subproblems for strings
- parenthesization
- edit distance (& longest common subseq.)
- knapsack
- pseudopolynomial time

* 5 easy steps to dynamic programming:

① define subproblems	count # subprobs.
② guess (part of solution)	count # choices
③ relate subprob. solutions	compute time/subprob.
④ recurse + memoize	time = time/subprob.
<u>OR</u> build DP table bottom-up	· # subprobs.

- check subprobs. acyclic/topological order

⑤ solve original problem: = a subproblem
<u>OR</u> by combining subprob. solutions (⇒ extra time)

- problems from L20 (text justification, Blackjack) are on sequences (words, cards)

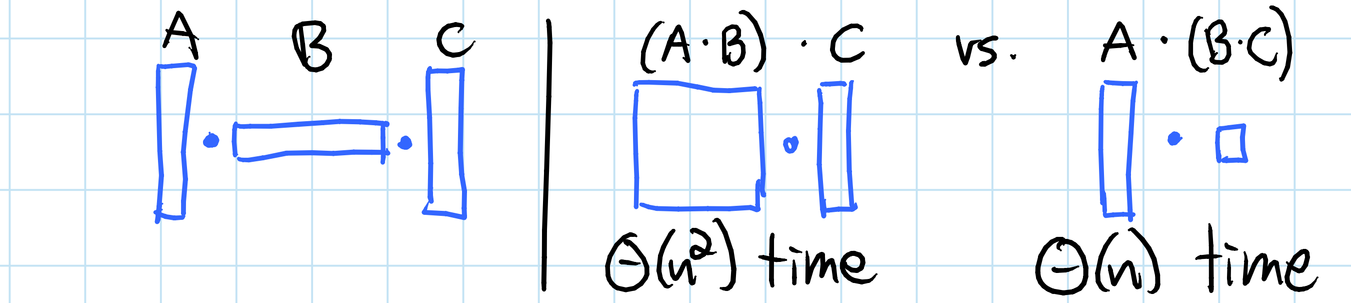
* useful subproblems for strings/sequences x:

$\boxed{\text{L20}} \nearrow$
- suffixes x[i:]
- prefixes x[:i]

$\Big\}\ \Theta(|x|)$ ⇐ cheaper ⇒ use if possible

- substrings x[i:j] $\}\ \Theta(|x|^2)$

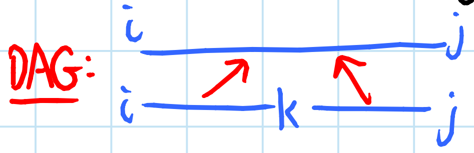
Parenthesization: optimal evaluation of an associative expression $A[\emptyset] \cdot A[1] \cdot \cdots \cdot A[n-1]$
— e.g. multiplying rectangular matrices



② guessing = outermost multiplication: $(\cdots)(\cdots)$ $\uparrow k-1$ $\uparrow k$
$\Rightarrow$ # choices $= O(n)$

① subproblems = ~~prefixes & suffixes?~~ NO
= cost of substring $A[i:j]$
$\Rightarrow$ # subproblems $= \Theta(n^2)$

③ recurrence:
— $DP[i,j] = \min \begin{pmatrix} DP[i,k] + DP[k,j] + \text{cost of } (A[i] \cdots A[k-1]) \cdot (A[k] \cdots A[j-1]) \\ \text{for } k \text{ in range}(i+1, j) \end{pmatrix}$



— $DP[i, i+1] = \emptyset$
$\Rightarrow$ cost per subproblem $= \Theta(j-i) = O(n)$

④ topological order: increasing substring size
— total time $= \Theta(n^3)$

⑤ original problem $= DP[\emptyset, n]$
(& use parent pointers to recover parens.)

NOTE: Above DP is not shortest paths in the subproblem DAG! Two dependencies ⇒ not path!

Edit distance: (used for DNA comparison, diff, CVS/SVN/…, spellchecking (typos), plagiarism detection, etc.)

given two strings x & y, what's the cheapest possible sequence of character edits to transform x into y?

$$\begin{tikzcd} & \text{edits} \arrow[dl] \arrow[d] \arrow[dr] & \\ \text{insert } c & \text{delete } c & \text{replace } c \to c' \end{tikzcd}$$

- cost of edit depends only on characters c, c'
- e.g. in DNA, C→G common mutation ⇒ low cost
- cost of sequence = sum of costs of edits

- if insert & delete cost 1, replace costs ∅, min. edit distance equivalent to finding longest common subsequence
  sequential but not necessarily contiguous

- e.g.: HIEROGLYPHOLOGY
  vs. MICHAELANGELO
  } ⇒ HELLO

Subproblems for multiple strings/sequences:
combine suffix/prefix/substring subproblems
- multiply state spaces
- still polynomial for O(1) strings

Edit distance DP:

① **subproblems**: $c(i,j) = \text{edit-distance}(x[i:], y[j:])$ for $0 \le i < |x|$, $0 \le j < |y|$

$\Rightarrow \Theta(|x| \cdot |y|)$ subproblems

② **guess** whether, to turn x into y,
- $x[i]$ deleted
- $y[j]$ inserted
- $x[i]$ replaced by $y[j]$

} 3 choices

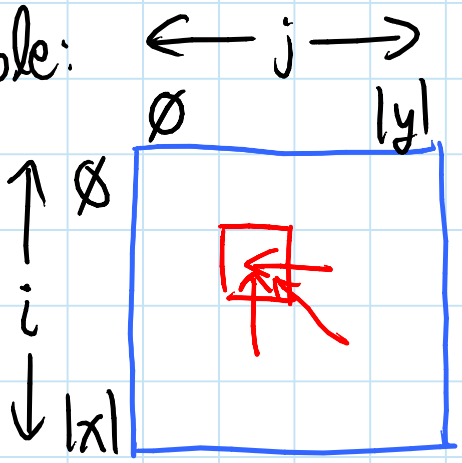
③ **recurrence**: $c(i,j) = \max\{$
$\text{cost}(\text{delete } x[i]) + c(i+1, j)$ if $i < |x|$,
$\text{cost}(\text{insert } y[j]) + c(i, j+1)$ if $j < |y|$,
$\text{cost}(\text{replace } x[i] \to y[j]) + c(i+1, j+1)$ if $i < |x|$ & $j < |y|\}$

- base case: $c(|x|, |y|) = \emptyset$

$\Rightarrow \Theta(1)$ time per subproblem

④ **topological order**: DAG in 2D table:
- bottom up OR right to left
- only need to keep last 2 rows/columns

$\Rightarrow$ linear space
- total time $= \Theta(|x| \cdot |y|)$



⑤ **original problem**: $c(\emptyset, \emptyset)$

Knapsack of size S you want to pack
- item i has integer size s_i & real value v_i
- goal: choose subset of items of max. total value subject to total size $\leq S$

First attempt:

① ~~subproblem = value for suffix i:~~ WRONG

② guessing = whether to include item i
$\Rightarrow$ #choices = 2

③ recurrence:
- $DP[i] = \max(DP[i+1], v_i + DP[i+1]$ if $\cancel{s_i \leq S}$?!$)$
- not enough information to know whether item i fits — how much space is left? GUESS!

Correct:

① subproblem = value for suffix i: given knapsack of size X
$\Rightarrow$ # subproblems = $O(nS)$ (!)

③ recurrence:
- $DP[i, X] = \max(DP[i+1, X], v_i + DP[i+1, X - s_i]$ if $s_i \leq X)$
- $DP[n, X] = \emptyset$

$\Rightarrow$ time per subproblem $= O(1)$

④ topological order: for i in $n, \dots, \emptyset$: for X in $0, \dots, S$
- total time $= O(nS)$

⑤ original problem $= DP[\emptyset, S]$
(& use parent pointers to recover subset)

AMAZING: effectively trying all possible subsets!
... but is this actually fast?

Polynomial time = polynomial in input size
- here $\Theta(n)$ if number S fits in a word
  $O(n \lg S)$ in general
- S is exponential in $\lg S$ (not polynomial)

Pseudopolynomial time = polynomial in the problem size AND the numbers in input
here: S, s_i's, v_i's
- $\Theta(nS)$ is pseudopolynomial

Remember:

polynomial	GOOD
exponential	BAD
pseudopoly.	SO SO