Problem Set #7

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MAT137 - Calculus!

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Question 1

Determine whether the following statements are true or false. If one is true, provide a proof. If one is false, provide a counterexample (proving that it is in fact a counterexample).

Question 1(a)

IF f is a positive continuous function on $[1, \infty)$ AND $\int_1^\infty (f(x))^2 dx$ converges,
THEN $\int_1^\infty f(x)dx$ converges.

Solution

This will be proven FALSE by counter example:

Hypothesis 1: f is a positive continuous function on $[1, \infty)$
Hypothesis 2: $\lim\limits_{b\to\infty} \int_1^b (f(x))^2 dx$ exists *(by definition of improper integral)*
Want to show: $\lim\limits_{b\to\infty} \int_1^b f(x)dx$ does not exist *(by definition of improper integral)*

Take $f = \frac{1}{x}$

Verify Hypothesis 1:

$\frac{1}{x}$ is positive and continuous for $x \in [1, \infty)$ *(by property of basic functions)*

Verify Hypothesis 2:

$\lim\limits_{b\to\infty} \int_1^b (f(x))^2 dx = \lim\limits_{b\to\infty} \int_1^b (\frac{1}{x})^2 dx$ *(by substitution)*

$= \lim\limits_{b\to\infty} \int_1^b \frac{1}{x^2} dx$ *(by exponent operation)*
$= \lim\limits_{b\to\infty} \int_1^b x^{-2} dx$ *(by negative exponent property)*
$= \lim\limits_{b\to\infty} [-x^{-1}]_1^b$ *(by reverse power rule)*
$= \lim\limits_{b\to\infty} [-\frac{1}{x}]_1^b$ *(by negative exponent property)*
$= \lim\limits_{b\to\infty} [-\frac{1}{b} - (-\frac{1}{1})]$ *(by Second Fundamental Theorem of Calculus)*

$= \lim_{b\to\infty} [-\frac{1}{b} + 1]$ *(by simplification)*
$= -\frac{1}{\infty} + 1$ *(by direct substitution)*
$= 0 + 1$ *(by infinity limit property)*
$= 1$ *(by addition)*

$\therefore \lim_{b\to\infty} \int_1^b (f(x))^2 dx$ exists

Verify $\lim_{b\to\infty} \int_1^b f(x)dx$ **does not exist**:

$\lim_{b\to\infty} \int_1^b f(x)dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx$ *(by substitution)*

$= \lim_{b\to\infty} \int_1^b x^{-1}dx$ *(by negative exponent property)*
$= \lim_{b\to\infty} [\ln|x|]_1^b$ *(by integral of $\frac{1}{x}$ property)*
$= \lim_{b\to\infty} [\ln(x)]_1^b$ *(by positive restriction of x)*
$= \lim_{b\to\infty} [\ln(b) - \ln(1)]$ *(by Second Fundamental Theorem of Calculus)*
$= \lim_{b\to\infty} [\ln(b) - 0]$ *(by natural log property)*
$= \lim_{b\to\infty} [\ln(b)]$ *(by subtraction)*
$= \ln(\infty)$ *(by direct substitution)*
$= \infty$ *(by natural log computation)*

$\therefore \lim_{b\to\infty} \int_1^b f(x)dx$ does not exist

Thus, there is a positive continuous function on $[1, \infty)$ where $\int_1^\infty (f(x))^2 dx$ converges, but $\int_1^\infty f(x)dx$ does not converge. The statement is FALSE. **Desmos link** $\square$

Question 1(b)

IF f is a positive continuous function on $[1, \infty)$ s.t. $\lim_{x\to\infty} f(x) = 0$ AND $\int_1^\infty f(x)dx$ converges, THEN $\int_1^\infty (f(x))^2 dx$ converges.

Solution

This will be proven TRUE directly:

Hypothesis 1: f is a positive continuous function on $[1, \infty)$
Hypothesis 2: $\forall \epsilon > 0, \exists M \in \mathbb{R}$ s.t. $x > M \Rightarrow |f(x)| < \epsilon$ *(by definition of limit)*
Hypothesis 3: $\int_1^\infty f(x)dx$ converges

Want to show: $\int_1^\infty (f(x))^2 dx$ converges

Take $\epsilon = 1$:

This means that $\exists M \in \mathbb{R}$ s.t. $x > M \Rightarrow |f(x)| < \epsilon$ *(by Hypothesis 2)*

Take M **s.t.** $x > M \Rightarrow |f(x)| < \epsilon$

Assume $x > M$:

> This means that $|f(x)| < \epsilon$ *(by Hypothesis 2)*
>
> As f is a positive function, it can be said that $f(x) < \epsilon$ *(by removing absolute)*
> Since $f(x) < 1$, it must be the case that $f(x)^2 \leq f(x)$ *(by exponent of fraction)*

Verify $\int_1^\infty (f(x))^2 dx$ **converges**:

> Know that $\lim\limits_{b \to \infty} \int_1^b (f(x))^2 dx = \int_1^M f(x)^2 dx + \int_M^\infty f(x)^2 dx$ *(by splitting integral)*
>
> Know that $\int_1^M f(x)^2 dx$ is finitely large *(by property of definite integral)*
> Know that $\int_1^M f(x)^2 dx$ is continuous *(by Hypothesis 1 and basic function property)*
> $\therefore \int_1^M f(x)^2 dx$ converges
>
> Know that $f(x)$ and $f(x)^2$ are continuous *(by Hypothesis 1)*
> Know that $f(x)$ and $f(x)^2$ are defined on $[1, \infty)$ *(by Hypothesis 1)*
> Know that $f(x)^2 \leq f(x)$ *(by previous calculation)*
> Know that $\int_M^\infty f(x) dx < \infty$ *(by definition of convergence and Hypothesis 3)*
> Know that $\int_M^\infty f(x)^2 dx < \infty$ *(by Basic Comparison Test)*
> $\therefore \int_M^\infty f(x)^2 dx$ converges
>
> Putting these together, $\lim\limits_{b \to \infty} \int_1^b (f(x))^2 dx$ converges

Thus, if f is a positive continuous function on $[1, \infty)$, $\lim\limits_{x \to \infty} f(x) = 0$, and $\int_1^\infty f(x) dx$ converges, then $\int_1^\infty (f(x))^2 dx$ converges. The statement is TRUE. $\square$

Question 1(c)

IF f is a continuous function on $[1, \infty)$ AND $\int_1^\infty f(x) dx$ converges,
THEN $\sum_{n=1}^\infty f(n)$ converges.

Solution

This will be proven FALSE by counter examples:

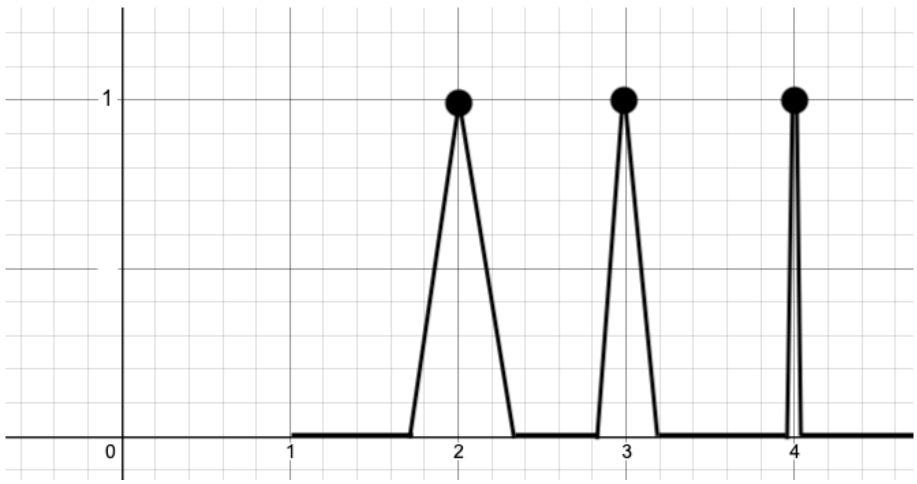
> **Hypothesis 1**: f is a continuous function on $[1, \infty)$
> **Hypothesis 2**: $\lim\limits_{b \to \infty} \int_1^b f(x) dx$ exists *(by definition of improper integral)*
> **Want to show**: $\lim\limits_{k \to \infty} S_k$ does not exist *(by definition of converging series)*

Let $n \in \mathbb{N}$ **s.t.** $n \in [2, \infty)$

Take $f =$

- $n^2x - n^3 + 1$ **for** $x \in [n - \frac{1}{n^2}, n]$
- $-n^2x + n^3 + 1$ **for** $x \in [n, n + \frac{1}{n^2}]$
- **0 elsewhere**

The resulting function will look like the following (with points at $y = 1$):



Verify Hypothesis 1:

f is continuous on $[1, \infty)$ *(by definition)*

Verify Hypothesis 2:

Calculate area of triangle:

$= \int_{n-\frac{1}{n^2}}^{n} 2(n^2x - n^3)dx$ *(by definition of function)*

$= \int_{n-\frac{1}{n^2}}^{n} (n^2x - n^3)dx$ *(by factoring constant)*
$= 2[\frac{n^2}{x^2} - n^3x + x]_{n-\frac{1}{n^2}}^{n}$ *(by limit computation)*
$= 2[\frac{1}{2n^2}]$ *(by Second Fundamental Theorem of Calculus and simplification)*
$= \frac{1}{n^2}$ *(by cancellation)*

Calculate unbounded integral:

$\lim_{b\to\infty} \int_1^b f(x)dx = \sum_{n=2}^{\infty} \frac{1}{n^2}$ *(by definition of converging series)*
Know from p-series that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges

$\therefore \lim_{b\to\infty} \int_1^b f(x)dx$ exists

Verify $\lim_{k\to\infty} S_k$ **does not exist**:

$\lim_{k\to\infty} S_k = 0+1+1+1+1+...$ *(by definition of partial sum and function)*
$= \infty$

$\therefore \lim_{k\to\infty} S_k$ does not exist

Thus, there is a continuous function on $[1,\infty)$ where $\int_1^\infty$ converges and $\sum_{n=1}^\infty f(n)$ diverges. The statement is FALSE. **Desmos link** $\square$

Question 1(d)

IF f is a continuous function on $[1,\infty)$ AND $\int_1^\infty$ diverges,
THEN $\sum_{n=1}^\infty f(n)$ diverges.

Solution

This statement will be proven FALSE by counter example:

Hypothesis 1: f is a continuous function on $[1,\infty)$
Hypothesis 2: $\lim_{b\to\infty} \int_1^b f(x)dx$ does not exist *(by definition of improper integral)*
Want to show: $\lim_{k\to\infty} S_k$ exists *(by definition of converging series)*

Take $f = \sin(\pi x)$

Verify Hypothesis 1:

$\sin(\pi x)$ is continuous for $x \in [1,\infty)$ *(by property of basic functions)*

Verify Hypothesis 2:

$\lim_{b\to\infty} \int_1^b f(x)dx = \lim_{b\to\infty} \int_1^b \sin(\pi x)dx$ *(by substitution)*

$= \lim_{b\to\infty} [-\frac{1}{\pi}\cos(\pi x)]_1^b$ *(by reverse chain rule)*
$= \lim_{b\to\infty} -\frac{1}{\pi}[\cos(\pi x)]_1^b$ *(by factoring)*
$= \lim_{b\to\infty} -\frac{1}{\pi}[\cos(\pi b) - \cos(\pi)]$ *(by Second Fundamental Theorem of Calculus)*
$= -\frac{1}{\pi}[\cos(\pi\infty) - \cos(\pi)]$ *(by direct substitution)*
$= -\frac{1}{\pi}\cos(\pi\infty) + \frac{1}{\pi}\cos(\pi)$ *(by expansion)*
$= -\frac{1}{\pi}\cos(\pi\infty) + \frac{1}{\pi}(-1)$ *(by property of cosine)*
$= -\frac{1}{\pi}\cos(\pi\infty) - \frac{1}{\pi}$ *(by multiplication)*

If x is even:
$= -\frac{1}{\pi} - \frac{1}{\pi}$ *(by property of cosine)*
$= -\frac{2}{\pi}$ *(by subtraction)*

If x is odd:
$= \frac{1}{\pi} - \frac{1}{\pi}$ *(by property of cosine)*
$= 0$ *(by subtraction)*

$\therefore \lim_{b\to\infty} \int_1^b f(x)dx$ does not exist

Verify $\lim_{k\to\infty} S_k$ **exists**:

$a_n = \sin(\pi n)$ *(by substitution)*

Know $\forall n \in \mathbb{N}, a_n = 0$ *(by property of sine)*
Know $S_k = \sum_{n=1}^{k} a_n$ *(by definition of partial sum)*
$\Rightarrow S_k = 0$ *(by property of sum of zero)*
$\Rightarrow \lim_{k\to\infty} S_k = \lim_{k\to\infty} 0$ *(by substitution)*
$\Rightarrow \lim_{k\to\infty} S_k = 0$ *(by direct substitution)*

$\therefore \lim_{k\to\infty} S_k$ exists

Thus, there is a continuous function on $[1, \infty)$ where $\int_1^\infty f(x)dx$ diverges, but $\sum_{n=1}^{\infty} f(n)$ does not diverge. The statement is FALSE. **Desmos link** $\square$

Question 2

Given a sequence (a_n), we define a sequence of averages $s_n = \frac{1}{n}\sum_{k=1}^{n} a_k$. That is to say, for a given n, s_n is the mean/average of the set $\{a_1, ..., a_n\}$. Prove the following two facts.

Question 2(a)

If $a_n \to a, a \in \mathbb{R}$, then $s_n \to a$. That is, if a_n converges to a, then the averages also converge to a. Hint, use the definition of convergence for a_n paired with squeeze theorem (you may assume squeeze theorem holds for discrete limits as well).

Solution

This will be proven directly:

Hypothesis 1: $\forall \epsilon_1 > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |a_n - a| < \epsilon_1$
Want to show: $\lim_{n\to\infty} s_n = a$ *(by definition of converging sequence)*

Let $\epsilon > 0$

Take $M = n_0$

Let $n \in \mathbb{R}$

Assume $n \geq n_0$

Take $\epsilon_1 = \epsilon$:

$\Rightarrow \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |a_n - a| < \epsilon$ *(by Hypothesis 1)*

Take this n_0:

$\Rightarrow \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow |a_n - a| < \epsilon$ *(by Hypothesis 1)*

Know $n \geq n_0$ *(by assumption)*
$\Rightarrow n \geq n_0$ *(by substitution)*
$\Rightarrow |a_n - a| < \epsilon$ *(by implication)*
$\Rightarrow a - \epsilon < a_n < a + \epsilon$ *(by absolute value)*

Verify $\lim\limits_{n\to\infty} s_n = a$:

Know $s_n = \frac{a_1+...+a_n}{n}$ *(by definition of sequence of averages)*
$= \frac{a_1+...+a_M}{n} + \frac{a_{M+1}+...+a_n}{n}$ *(by splitting sequence)*

First term:

Know $\frac{a_1+...+a_M}{n}$ is finite
$\Rightarrow \frac{a_1+...+a_M}{n} \to 0$ *(by property of infinite limits)*

Second term:

Know $\frac{a_{M+1}+...+a_n}{n} \geq \frac{(a-\epsilon)+(a-\epsilon)+(a-\epsilon)...[n-m \text{ times}]}{n}$ *(by previous calculation)*
$\Rightarrow \frac{a_{M+1}+...+a_n}{n} \geq \frac{(a-\epsilon)(n-m)}{n}$ *(by factoring)*

$\lim\limits_{n\to\infty} \frac{(a-\epsilon)(n-m)}{n} = a - \epsilon$ *(by direct substitution)*

Know $\frac{a_{M+1}+...+a_n}{n} \leq \frac{(a+\epsilon)+(a+\epsilon)+(a+\epsilon)...[n-m \text{ times}]}{n}$ *(by previous calculation)*
$\Rightarrow \frac{a_{M+1}+...+a_n}{n} \leq \frac{(a+\epsilon)(n-m)}{n}$ *(by factoring)*

$\lim\limits_{n\to\infty} \frac{(a+\epsilon)(n-m)}{n} = a + \epsilon$ *(by direct substitution)*

So, $\frac{(a-\epsilon)(n-m)}{n} \leq \frac{a_{M+1}+...+a_n}{n} < \frac{(a+\epsilon)(n-m)}{n}$ *(by concatenation)*
$\Rightarrow \lim\limits_{n\to\infty} \frac{(a-\epsilon)(n-m)}{n} \leq \lim\limits_{n\to\infty} \frac{a_{M+1}+...+a_n}{n} \leq \lim\limits_{n\to\infty} \frac{(a+\epsilon)(n-m)}{n}$ *(by limit property)*
$\Rightarrow a - \epsilon \leq \lim\limits_{n\to\infty} \frac{a_{M+1}+...+a_n}{n} \leq a + \epsilon$ *(by substitution)*
$\Rightarrow a - \epsilon \leq \lim\limits_{n\to\infty} \frac{a_{M+1}+...+a_n}{n} + 0 \leq a + \epsilon$ *(by neutral addition)*
$\Rightarrow a - \epsilon \leq \lim\limits_{n\to\infty} \frac{a_{M+1}+...+a_n}{n} + \lim\limits_{n\to\infty} \frac{a_1+...+a_M}{n} \leq a + \epsilon$ *(by substitution)*
$\Rightarrow a - \epsilon \leq \lim\limits_{n\to\infty} s_n \leq a + \epsilon$ *(by substitution)*
$\Rightarrow a \leq \lim\limits_{n\to\infty} s_n \leq a$ *(by definition of ϵ)*
$\Rightarrow \lim\limits_{n\to\infty} s_n = a$ *(by Squeeze Theorem)*

Thus, if $a_n \to a, a \in \mathbb{R}$ then $_n \to a$. $\square$

Question 2(b)

Show that if the sequence of averages converges, then we cannot conclude on whether or not a_n converges. Hint, consider a simple alternating sequence.

Solution

This will be proven by counter example:

Hypothesis 1: $\lim_{n\to\infty} s_n$ exists *(by definition of converging sequence)*
Want to show: $\forall L \in \mathbb{R}, \exists \epsilon > 0$ s.t. $\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $n \geq n_0$ and $|a_n - L| \geq \epsilon$

Take $a_n = (-1)^n$

Verify Hypothesis 1:

If n **is even**:

$s_n = \frac{1}{n}[1 + (-1) + 1 + (-1) ... + 1 + (-1)]$ *(by definition of sequence of averages)*
$s_n = \frac{1}{n}[(1 - 1) + (1 - 1) ... + (1 - 1)]$ *(by grouping)*
$s_n = \frac{1}{n}[0]$ *(by cancellation)*
$s_n = 0$ *(by multiplication)*

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 0$ *(by substitution)*
$= 0$ *(by direct substitution)*

$\therefore \lim_{n\to\infty} s_n$ exists and is 0

If n **is odd**:

$s_n = \frac{1}{n}[1 + (-1) + 1 + (-1) ... + 1 + (-1) + 1]$ *(by definition)*
$s_n = \frac{1}{n}[(1 - 1) + (1 - 1) ... + (1 - 1) + 1]$ *(by grouping)*
$s_n = \frac{1}{n}[1]$ *(by cancellation)*
$s_n = \frac{1}{n}$ *(by multiplication)*

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{n}$ *(by substitution)*
$= \frac{1}{\infty}$ *(by direct substitution)*
$= 0$ *(by property of large value)*

$\therefore \lim_{n\to\infty} s_n$ exists and is 0

Verify $\forall L \in \mathbb{R}, \exists \epsilon > 0$ **s.t.** $\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}$ **s.t.** $n \geq n_0$ **and** $|a_n - L| \geq \epsilon$:

Let $L \in \mathbb{R}$

Take $\epsilon = 1 - L$

Let $n_0 \in \mathbb{N}$

If n_0 is even:

Take $n = n_0$

Verify $n \geq n_0$:

$n_0 \geq n_0$ *(by equality)*
$\Rightarrow n \geq n_0$ *(by substitution)*

Verify $|a_n - L| \geq \epsilon$:

$a_n = 1$ *(by property of even exponent)*

Know $|1 - L| \geq 1 - L$ *(by property of absolute value)*
$\Rightarrow |a_n - L| \geq 1 - L$ *(by substitution)*
$\Rightarrow |a_n - L| \geq \epsilon$ *(by substitution)*

If n_0 is odd:

Take $n = n_0 + 1$

Verify $n \geq n_0$:

$n_0 + 1 \geq n_0$ *(by inequality)*
$\Rightarrow n \geq n_0$ *(by substitution)*

Verify $|a_n - L| \geq \epsilon$:

$a_n = 1$ *(by property of even exponent)*

Know $|1 - L| \geq 1 - L$ *(by property of absolute value)*
$\Rightarrow |a_n - L| \geq 1 - L$ *(by substitution)*
$\Rightarrow |a_n - L| \geq \epsilon$ *(by substitution)*

$\therefore \forall L \in \mathbb{R}, \exists \epsilon > 0$ s.t. $\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $n \geq n_0$ and $|a_n - L| \geq \epsilon$

Thus, it is not always that case that if the sequence of averages converges, then a_n converges. We cannot conclude whether a_n converges **Desmos link** $\square$

Question 3

Compute the volume of a tetrahedron where the edge lengths are l. A tetrahedron is a polygon where all faces are equilateral triangles with same edge length. Hint, lie the tetrahedron on one of its triangular faces and **slice it like a carrot**.

Solution

Calculate area of equilateral triangle:

Call a = area of equilateral triangle
Call l = edge length of equilateral triangle
Call h = height of equilateral triangle

Dividing triangle in half means $h = \sqrt{l^2 - (\frac{l}{2})^2}$ *(by Pythagoras Theorem)*
$= \sqrt{l^2 - \frac{l^2}{4}}$ *(by exponent operation)*
$= \sqrt{\frac{4l^2}{4} - \frac{l^2}{4}}$ *(by neutral multiplication)*
$= \sqrt{\frac{3l^2}{4}}$ *(by fraction subtraction)*
$= \frac{\sqrt{3}}{2}l$ *(by square root operation)*

Combining the two halves means $a = \frac{l \cdot h}{2}$ *(by area of triangle)*
$= \frac{l}{2} \cdot \frac{\sqrt{3}}{2}l$ *(by substitution)*
$= \frac{\sqrt{3}}{4}l^2$ *(by multiplication)*

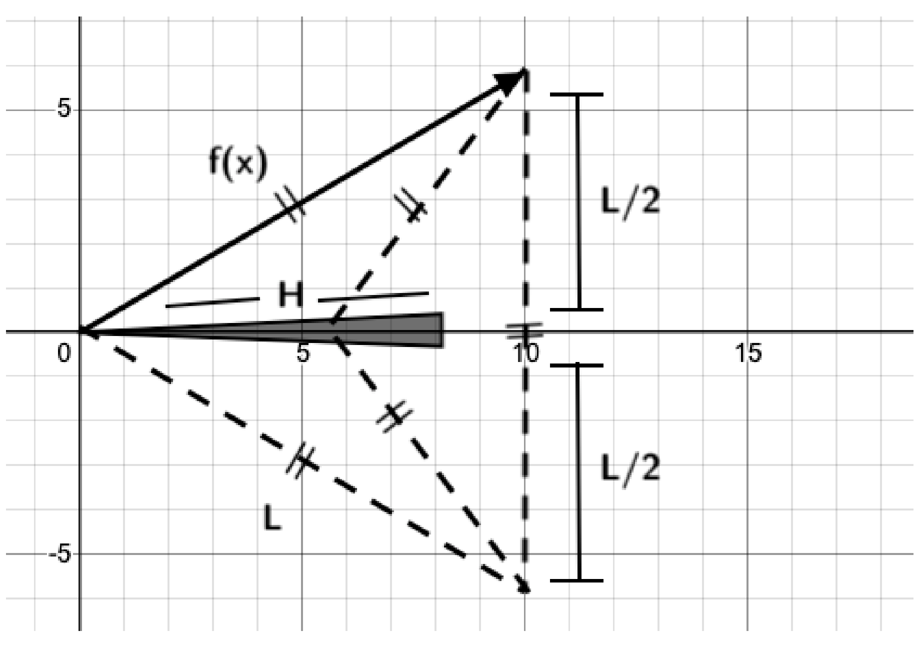
$\therefore$ The area of an equilateral triangle is $\frac{\sqrt{3}}{4}l^2$

Calculate volume of tetrahedron slice:

Call dV = volume of tetrahedron slice
Call a = edge length of tetrahedron slice
Call dx = width of tetrahedron slice
Call s = surface area of tetrahedron slice

Know that $dV = sdx$ *(by definition of volume)*
$= \frac{\sqrt{3}}{4}a^2dx$ *(by area of equilateral triangle equation)*

Call $f(x)$ = one side of the tetrahedron
Call H = the height of the tetrahedron



Know $a = 2f(x)$ *(by length of vertical slice)*
Know $f(x) = \frac{\frac{l}{2}-0}{H-0}x$ *(by slope formula)*
$= \frac{\frac{l}{2}}{H}x$ *(by subtraction)*
$= \frac{l}{2H}x$ *(by division)*

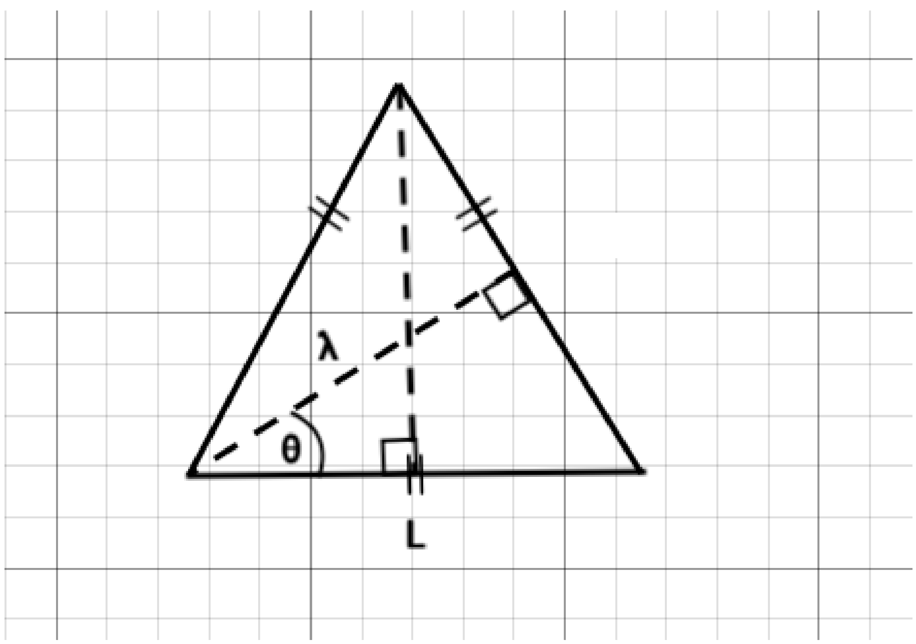
This implies that $a = 2\frac{l}{2H}x$ *(by substitution)*
$a = \frac{l}{H}x$ *(by cancellation)*

This implies that $dV = \frac{\sqrt{3}}{4}(\frac{l}{H}x)^2dx$ *(by substitution)*
$= \frac{\sqrt{3}}{4} \cdot \frac{l^2}{H^2}x^2dx$ *(by exponent operation)*
$= \frac{\sqrt{3}l^2}{4H^2}x^2dx$ *(by multiplication)*

$\therefore$ The volume of a tetrahedron slice is $= \frac{\sqrt{3}l^2}{4H^2}x^2dx$

Calculate height of tetrahedron:

Call $\lambda =$ distance between corner and centroid of tetrahedron base
Call $\theta =$ angle from bisected corner to midpoint of a tetrahedron base

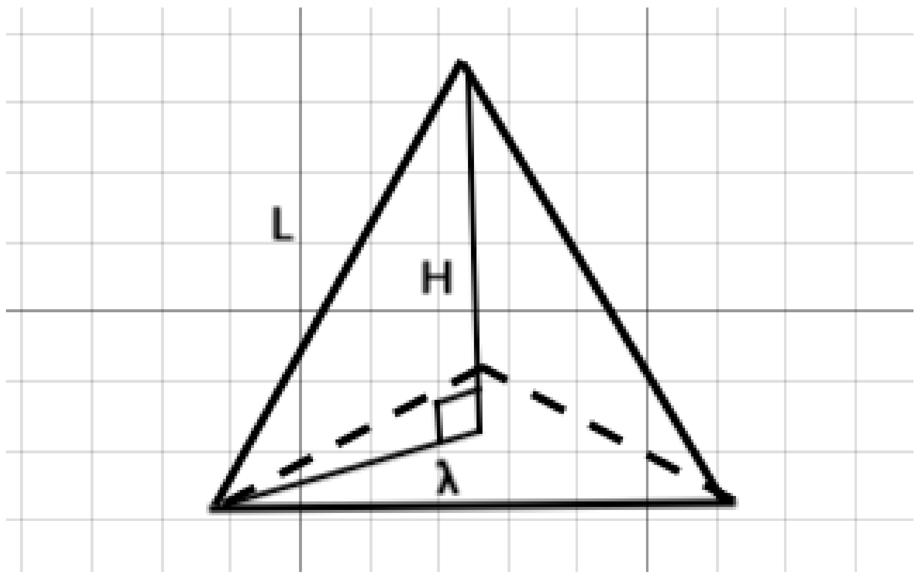


Know $\theta = \frac{60^\circ}{2}$ *(by property of equilateral triangle)*
$= 30^\circ$ *(by division)*

Know $\cos(30^\circ) = \frac{\text{adj}}{\text{hyp}}$ *(by cosine property)*
$\Rightarrow \cos(30^\circ) = \frac{\frac{l}{2}}{\lambda}$ *(by substitution)*
$\Rightarrow \frac{\sqrt{3}}{2} = \frac{\frac{l}{2}}{\lambda}$ *(by cosine property)*
$\Rightarrow \frac{\sqrt{3}}{2}\lambda = \frac{l}{2}$ *(by multiplication)*
$\Rightarrow \lambda = \frac{l}{2} \div \frac{\sqrt{3}}{2}$ *(by division)*

$\Rightarrow \lambda = \frac{l}{2} \cdot \frac{2}{\sqrt{3}}$ *(by fraction division)*
$\Rightarrow \lambda = \frac{2l}{2\sqrt{3}}$ *(by multiplication)*
$\Rightarrow \lambda = \frac{l}{\sqrt{3}}$ *(by cancellation)*

Know $H = \sqrt{l^2 - \lambda^2}$ *(by Pythagorean Theorem)*
$= \sqrt{l^2 - (\frac{l}{\sqrt{3}})^2}$ *(by substitution)*
$= \sqrt{l^2 - \frac{l^2}{3}}$ *(by exponent operation)*
$= \sqrt{\frac{3l^2}{3} - \frac{l^2}{3}}$ *(by neutral multiplication)*
$= \frac{\sqrt{2}}{\sqrt{3}}l$ *(by square root operation)*
$= \frac{\sqrt{6}}{3}l$ *(by multiplication)*



$\therefore$ The height of the tetrahedron is $\frac{\sqrt{6}}{3}l$

Calculate volume of tetrahedron:

Call V = volume of tetrahedron

Know the integration of all tetrahedron slices is $V = \int_0^H a$ *(by definition of integral)*
$= \int_0^H \frac{\sqrt{3}l^2}{4H^2}x^2dx$ *(by substitution)*
$= \frac{\sqrt{3}l^2}{4H^2} \cdot \int_0^H x^2dx$ *(by factoring out constant)*
$= \frac{\sqrt{3}l^2}{4H^2}[\frac{1}{3}x^3]_0^H$ *(by reverse power rule)*
$= \frac{\sqrt{3}l^2}{4H^2}[\frac{1}{3}H^3 - \frac{1}{3}0^3]$ *(by Second Fundamental Theorem of Calculus)*
$= \frac{\sqrt{3}l^2}{4H^2}[\frac{1}{3}H^3]$ *(by multiplication)*
$= \frac{\sqrt{3}l^2H^3}{12H^2}$ *(by multiplication)*
$= \frac{\sqrt{3}l^2H}{12}$ *(by cancellation)*

Substituting the value of H yields $V = \frac{\sqrt{3}l^2(\frac{\sqrt{6}}{3}l)}{12}$ *(by substitution)*
$= \frac{\sqrt{3}(\frac{\sqrt{6}}{3})l^3}{12}$ *(by multiplication)*
$= \frac{(\frac{\sqrt{18}}{3})l^3}{12}$ *(by multiplication)*
$= \frac{(\frac{3\sqrt{2}}{3})l^3}{12}$ *(by square root operation)*
$= \frac{\sqrt{2}l^3}{12}$ *(by cancellation)*

Thus, the volume of a tetrahedron is $\frac{\sqrt{2}l^3}{12}$. $\square$