# Problem Set #5

J. Rapson
MAT137 - Calculus!

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## Question 1

The persistent hopper once started a hop trip to travel 1 metre ahead of itself. Not being in a rush and after realizing that it was impossible for it to jump such distance at once, the persistent hopper decided to always jump half of the remaining distance. Let $S(n)$ denote the distance travelled by the persistent hopper after $n$ jumps.

### Question 1(a)

Find an expression for $S(n)$ as the sum of $n$ terms using the summation notation ($\sum$).

### Solution

Let $D$ be distance traveled *at* jump $n$, such that $S(n)$ is the sum of $D$

$D = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, ..., \frac{1}{2^n}\right\}$

$S(n) = \sum_{i=1}^{n} \left(\frac{1}{2^i}\right) = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i$

### Question 1(b)

Simplify the expression for $S(n)$ into one that easily allows you to perform the computation.
*Hint: also consider $\frac{1}{2}S(n)$ when simplifying $S(n)$.*

### Solution

$S(n) = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{2}^2 + \frac{1}{2}^3 + ... + \frac{1}{2}^{n-2} + \frac{1}{2}^{n-1} + \frac{1}{2}^n$

$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2}^2 + ... + \frac{1}{2}^{n-3} + \frac{1}{2}^{n-2} + \frac{1}{2}^{n-1}\right)$

Know that $(1-x)(1+x+x^2+x^3+...+x^n) = 1 - x^{n+1}$ (by cancellation)
Thus $1+x+x^2+x^3+...+x^n = \frac{1-x^{n+1}}{1-x}$ (by division)
Which can be written as $1 + \frac{1}{2} + \frac{1}{2}^2 + \frac{1}{2}^3 + ... + \frac{1}{2}^{n-1} = \frac{1-\frac{1}{2}^n}{1-\frac{1}{2}}$ (using $x = \frac{1}{2}$ and $n = n-1$)

Putting this back into the factored equation produces $\frac{1}{2} \cdot \frac{1-\frac{1}{2}^n}{1-\frac{1}{2}}$

$= \frac{1-\frac{1}{2}^n}{2 \cdot \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$ [**Desmos link**]

## Question 1(c)

Compute $S(137)$. Does the persistent hopper ever end its journey? Does the sequence $S(n), n \in N$ converge? If so, compute the limit.

## Solution

$S(137) = 1 - \left(\frac{1}{2}\right)^{137} = 1 - \frac{1}{2^{137}} = 1 - 5.74 \times 10^{-42}$ (very close to 1, but slightly below 1)
As there is no $n$ that would make $\left(\frac{1}{2}\right)^n = 0$, the hopper will never reach its destination.

However, it appears that the function converges to 1:

$\lim\limits_{n\to\infty} \left[1 - \left(\frac{1}{2}\right)^n\right]$

$= \lim\limits_{n\to\infty}(1) - \lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n$ (by limit law for sums)

$= \lim\limits_{n\to\infty}(1) - \lim\limits_{n\to\infty}(2)^{-n}$ (by exponent property)

$= 1 - 0$ (by direct substitution)
$= 1$

**Want to show**: $\forall \epsilon > 0, \exists N \geq 0$ s.t. $n \geq N \Rightarrow \left|\left[1 - \left(\frac{1}{2}\right)^n\right] - 1\right| < \epsilon$

Let $\epsilon > 0$
Take $N = \max\{1, -\frac{\ln(\epsilon)}{\ln(2)} + 1\}$

Assume $n \geq N$

Verify $|\left[1 - \left(\frac{1}{2}\right)^n\right] - 1| < \epsilon$:

Know $n \geq -\frac{\ln(\epsilon)}{\ln(2)} + 1$

$\Rightarrow n \geq -\frac{\ln(\epsilon)}{\ln(2)} + 1 > \frac{\ln(\epsilon)}{\ln(2)}$
$\Rightarrow n > -\frac{\ln(\epsilon)}{\ln(2)}$ (by truncation)

$\Rightarrow -\ln(2) \cdot n < \ln(\epsilon)$ (by multiplication)
$\Rightarrow \ln\frac{1}{2} \cdot n < \ln(\epsilon)$ (by log rule)
$\Rightarrow \ln\left(\left(\frac{1}{2}\right)^n\right) \leq \ln(\epsilon)$ (by log rule)
$\Rightarrow \left(\frac{1}{2}\right)^n < \epsilon$ (by log rule, both are strictly increasing)

As the positive $\left(\frac{1}{2}\right)^n < \epsilon$ (and $\left(\frac{1}{2}\right)^n$ is always positive), its negative must be too:

$\Rightarrow -\left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^n < \epsilon$ (by property of exponent)
$\Rightarrow \left|-\left(\frac{1}{2}\right)^2\right| < \epsilon$ (by absolute value)
$\Rightarrow \left|\left[1 - \left(\frac{1}{2}\right)^n\right] - 1\right| < \epsilon$ (by neutral addition)

Thus $\left|\left[1 - \left(\frac{1}{2}\right)^n\right] - 1\right| < \epsilon$ and the function converges at 1. [**Desmos link**] $\square$

## Question 1(d)

The wise hopper decided to show its fellow hoppers that jumping the first $k$-th part at each step will lead to the same conclusion as in (c). Let $S_k(n)$ be the distance travelled after $n$ jumps, each of which covering the $k$-th part of the remaining distance. Find a simplified formula for $S_k(n)$ and then justify the wise hopper's line of thought.

## Solution

Let $k \in \mathbb{N}$
Let $D_k$ be distance traveled *at* jump $n$, such that $S_k(n)$ is the sum of $D$

$k = 7$
$D_7 = \{\frac{1}{7}, \frac{6}{49}, \frac{36}{343}, \frac{216}{2401}, ..., \frac{6^{n-1}}{7^n}\}$ (as an example)

It can be shown that $S_k(n) = \sum_{i=1}^{n} \frac{(k-1)^{i-1}}{k^i}$

**Base case:**

Take $i = 1$ (the first jump)
$D_k(1) = \frac{1}{k}$ (we know that the distance traveled *at* the first jump is always $\frac{1}{k}$)

$D_k(1) = \frac{(k-1)^{i-1}}{k^i} = \frac{(k-1)^{1-1}}{k^1} = \frac{(k-1)^0}{k^1} = \frac{1}{k}$ (it is also $\frac{1}{k}$ using the proposed equation)

Thus, $D_k(i) = \frac{(k-1)^{i-1}}{k^i}$ for $i = 1$

**Induction step:**

Assume $D_k(i) = \frac{(k-1)^{i-1}}{k^i}$
Verify $D_k(i+1) = \frac{(k-1)^{i+1-1}}{k^{i+1}}$

Know that the distance traveled in a step will always be $1 - \frac{1}{k}$ times the previous step
Thus, $D_k(i+1) = \left(1 - \frac{1}{k}\right) D_k(i)$

Know that $\frac{(k-1)^{i+1-1}}{k^{i+1}} = \frac{(k-1)^i}{k^{i+1}}$ (trying to see if $D_k(i+1)$ is equal to this)

$= \left(\frac{k-1}{k}\right) \frac{(k-1)^{i-1}}{k^i}$ (by factoring)

$= \left(\frac{k-1}{k}\right) D_k(i)$ (by substituting assumption)

$= \left(\frac{k}{k} - \frac{1}{k}\right) D_k(i)$ (by breaking down fraction)

$= \left(1 - \frac{1}{k}\right) D_k(i)$ (by division)

Thus, $D_k(i+1) = \frac{(k-1)^{i+1-1}}{k^{i+1}}$

As $S_k(n)$ is the sum of $D_k(i)$, $S_k(n) = \sum_{i=1}^{n} \frac{(k-1)^{i-1}}{k^i}$

Thus, $S_k(n) = \sum_{i=1}^{n} \frac{(k-1)^{i-1}}{k^i}$ for all $i$ $\square$

This can further be simplified:

$S_k(n) = \sum_{i=1}^{n} \frac{(k-1)^{i-1}}{k^i} = \frac{1}{k} + \frac{k-1}{k^2} + \frac{(k-1)^2}{k^3} + ... + \frac{(k-1)^{n-1}}{k^n}$

$= \frac{1}{k}\left(1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + ... + \frac{(k-1)^{n-1}}{k^{n-1}}\right)$

$= \frac{1}{k}\left(1 + \frac{k-1}{k} + \left(\frac{k-1}{k}\right)^2 + ... + \left(\frac{k-1}{k}\right)^{n-1}\right)$

Know that $(1-x)(1+x+x^2+x^3+...+x^n) = 1-x^{n+1}$ (by cancellation)
Thus $1+x+x^2+x^3+...+x^n = \frac{1-x^{n+1}}{1-x}$ (by division)
Which can be written as $1 + \frac{k-1}{k} + + \left(\frac{k-1}{k}\right)^2 + ... + \left(\frac{k-1}{k}\right)^{n-1} = \frac{1-\left(\frac{k-1}{k}\right)^n}{1-\frac{k-1}{k}}$ $(x = \frac{k-1}{k},\ n = n-1)$

Putting this back into the factored equation produces $\frac{1}{k} \cdot \frac{1-\left(\frac{k-1}{k}\right)^n}{1-\frac{k-1}{k}}$
$= \frac{1-\left(\frac{k-1}{k}\right)^n}{k-k+1} = 1 - \left(\frac{k-1}{k}\right)^n$ [**Desmos link**]

This will be proven to converge at 1 directly:

> **Want to show**: $\forall \epsilon > 0, \exists N \geq 0$ s.t. $n \geq N \Rightarrow \left|\left[1 - \left(\frac{k-1}{k}\right)^n\right] - 1\right| < \epsilon$

Let $\epsilon > 0$
Take $N = \max\{1, \frac{\ln(\epsilon)}{\ln\left(\frac{k-1}{k}\right)} + 1\}$

Assume $n \geq N$

Verify $\left|\left[1 - \left(\frac{k-1}{k}\right)^n\right] - 1\right| < \epsilon$:

Know $n \geq \frac{\ln(\epsilon)}{\ln\left(\frac{k-1}{k}\right)} + 1$

$\Rightarrow n \geq \frac{\ln(\epsilon)}{\ln\left(\frac{k-1}{k}\right)} + 1 > \frac{\ln(\epsilon)}{\ln\left(\frac{k-1}{k}\right)}$
$\Rightarrow n > \frac{\ln(\epsilon)}{\ln\left(\frac{k-1}{k}\right)}$ (by truncation)

$\Rightarrow \ln\left(\frac{k-1}{k}\right) \cdot n < \ln(\epsilon)$ (by multiplication, note $\ln(\frac{k-1}{k})$ is a negative constant)
$\Rightarrow \ln\left(\left(\frac{k-1}{k}\right)^n\right) \leq \ln(\epsilon)$ (by log rule)
$\Rightarrow \left(\frac{k-1}{k}\right)^n < \epsilon$ (by log rule, both are strictly increasing)

As the positive $\left(\frac{k-1}{k}\right)^n < \epsilon$ (and $\left(\frac{k-1}{k}\right)^n$ is always positive), its negative must be too

$\Rightarrow -\left(\frac{k-1}{k}\right)^n < \left(\frac{k-1}{k}\right)^n < \epsilon$ (by property of exponent)
$\Rightarrow \left|-\left(\frac{k-1}{k}\right)^n\right| < \epsilon$ (by absolute value)
$\Rightarrow \left|\left[1 - \left(\frac{k-1}{k}\right)^n\right] - 1\right| < \epsilon$ (by neutral addition)

Thus $\left|\left[1 - \left(\frac{k-1}{k}\right)^n\right] - 1\right| < \epsilon$ and the function also converges at 1. [**Desmos link**] $\square$

# Question 2

Suppose $f$ is an integrable function on $[0,1]$ and $P$ and $Q$ are partitions of $[0,1]$. We say that $P$ is better than $Q$ if $U_P - L_P \leq U_Q - L_Q$. State whether the following statements are true or false. If true, provide a proof. If false, provide a counter example. Note: Showing that something is a counterexample requires proof.

## Question 2(a)

If $P$ is better than $Q$, then $Q \subseteq P$.

## Solution

This will be proven false by counterexample:

**Hypothesis**: $U_P - L_P \leq U_Q - L_Q$

**Want to show**: $\exists P, Q$ s.t. $Q \supset P$

Let $f$ be a function defined on interval $[0,4]$ such that $f(x) = 1$
Let $P = \{0, 2, 4\}$
Let $Q = \{0, 1, 2, 3, 4\}$

$Q \supset P$ (as $Q$ contains all the points that $P$ contains and more)

Call $\Delta x_i = x_i - x_{i-1}$
Call $m_i = \inf$ of $f$ on $[x_{i-1}, x_i]$
Call $M_i = \sup$ of $f$ on $[x_{i-1}, x_i]$

Know $m_i = M_i = 1$ (as $f$ is a constant function)

$L_P(f) = \sum_{i=1}^{N} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 = 2 + 2 = 4$
$U_P(f) = \sum_{i=1}^{N} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 = 2 + 2 = 4$

$L_Q(f) = \sum_{i=1}^{N} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + m_4 \Delta x_4 = 1 + 1 + 1 + 1 = 4$
$U_Q(f) = \sum_{i=1}^{N} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + M_4 \Delta x_4 = 1 + 1 + 1 + 1 = 4$

$U_P - L_P = 4 - 4 = 0$
$U_Q - L_Q = 4 - 4 = 0$

Know that $0 \leq 0$
$\Rightarrow U_P - L_P \leq U_Q - L_Q$

Thus, if $U_P - L_P \leq U_Q - L_Q$, $\exists Q, P$ s.t. $Q \supset P$ so the claim is false. [**Desmos link**] $\square$

## Question 2(b)

If $P$ has more points than $Q$ (not necessarily the case that $Q \subseteq P$ ), $P$ is better than $Q$.

## Solution

This will be proven false by counterexample:

**Hypothesis**: $|P| > |Q|$

**Want to show**: $\exists P, Q$ s.t. $U_P - L_P > U_Q - L_Q$

Let $f$ be a function defined on interval $[0, 4]$ such that $f(x) = \begin{cases} 2 & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$
Let $P = \{0, 1, 3, 4\}$
Let $Q = \{0, 2, 4\}$

Know that $|P| = 4$ and $|Q| = 3$, so $|P| > |Q|$

Call $\Delta x_i = x_i - x_{i-1}$
Call $m_i = \inf$ of $f$ on $[x_{i-1}, x_i]$
Call $M_i = \sup$ of $f$ on $[x_{i-1}, x_i]$

$L_P(f) = \sum_{i=1}^{N} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 = 5$
$U_P(f) = \sum_{i=1}^{N} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 = 2 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 7$

$L_Q(f) = \sum_{i=1}^{N} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 = 2 \cdot 2 + 2 \cdot 1 = 6$
$U_Q(f) = \sum_{i=1}^{N} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 = 2 \cdot 2 + 2 \cdot 1 = 6$

$U_P - L_P = 7 - 5 = 2$
$U_Q - L_Q = 6 - 6 = 0$

Know that $2 > 0$
$\Rightarrow U_P - L_P > U_Q - L_Q$

Thus, if $|P| > |Q|$, $\exists P, Q$ s.t. $U_P - L_P > U_Q - L_Q$ so the claim is false. [**Desmos link**] $\square$

## Question 2(c)

For any partitions $P$ and $Q$, $P \cup Q$ is better than $P$.

## Solution

This will be proven true directly:

**Want to show**: $U_{P \cup Q} - L_{P \cup Q} \leq U_P - L_P$

Call $\Delta x_i = x_i - x_{i-1}$
Call $m_i = \inf$ of $f$ on $[x_{i-1}, x_i]$
Call $M_i = \sup$ of $f$ on $[x_{i-1}, x_i]$

Know $P \subseteq P \cup Q$
Let $[x_{i-1}, x_i]$ be an interval in $P$

If $P \subset Q$:

There must be a point $u$ in $P \subset Q$ where $u$ splits $[x_{i-1}, x_i]$ into $[x_{i-1}, u]$ and $[u, x_i]$

Call $m_a = \inf$ of $f$ on $[x_{i-1}, u]$
Call $m_b = \inf$ of $f$ on $[u, x_i]$

Know that $m_i \leq m_a$ and $m_i \leq m_b$ (as $m_i$ is infimum for entire interval)

$m_i \Delta x_i = m_i(x_i - x_{i-1})$ (for entire interval)
$= m_i(x_i - u + u - x_{i-1})$ (by splitting into sub-intervals)
$= m_i(x_i - u) + m_i(u - x_{i-1})$ (by factoring)
$\leq m_b(x_i - u) + m_a(u - x_{i-1})$ (using definitions of sub-infimums)

This means that the lower sum is $\geq m_i \Delta x_i$ if more points are added to a partition

Call $M_a = \sup$ of $f$ on $[x_{i-1}, u]$
Call $M_b = \sup$ of $f$ on $[u, x_i]$

Know that $M_i \geq M_a$ and $M_i \geq M_b$ (as $M_i$ is supremum for entire interval)

$M_i \Delta x_i = M_i(x_i - x_{i-1})$ (for entire interval)
$= M_i(x_i - u + u - x_{i-1})$ (by splitting into sub-intervals)
$= M_i(x_i - u) + m_i(u - x_{i-1})$ (by factoring)
$\geq M_b(x_i - u) + M_a(u - x_{i-1})$ (using definitions of sub-supremums)

This means that the upper sum is $\leq M_i \Delta x_i$ if more points are added to a partition

Else $P = P \cup Q$:

$\Rightarrow L_{P \cup Q} = L_P$
$\Rightarrow U_{P \cup Q} = U_P$

From both cases, $L_{P \cup Q} \geq L_P$ and $U_{P \cup Q} \leq U_P$
$\Rightarrow -L_{P \cup Q} \leq -L_P$ (by negative multiplication)
$\Rightarrow U_{P \cup Q} - L_{P \cup Q} \leq U_P - L_P$ (by addition)

Thus, $U_{P \cup Q} - L_{P \cup Q} \leq U_P - L_P \quad \square$

# Question 3

Let $a_n$ be defined for each natural $n \geq 1$ as follows: $a_n = \frac{1}{n^3}$ if $n$ is even, and $a_n = -\frac{1}{n^2}$ if $n$ is odd. Define $A = \{a_n : n \in \mathbb{N}\}$. To simplify the notation, we will write, for instance, $\sup_{n \geq k} a_n$ an instead of the equivalent notation $\{\sup a_n : n \geq k\}$.

## Question 3(a)

Compute $\sup A$ and $\inf A$. Are they also the maximum and minimum of $A$? Justify.

## Solution

$\frac{1}{8}$ is the sup of $A$.

This will be proven true directly:

**Want to show**: $\forall\epsilon > 0, \exists a_n \in A$ s.t. $\frac{1}{8} - \epsilon < a_n \leq \frac{1}{8}$ (from definition of supremum)

Let $\epsilon > 0$

Know that $\exists n \in \mathbb{N}$ s.t. $\epsilon > -a_n + \frac{1}{8}$ (because this can be zero for $n = 2$)
Take $n$ s.t. $\epsilon > -a_n + \frac{1}{8}$

$\epsilon > -a_n + \frac{1}{8}$ (property of existential)
$\Rightarrow -\frac{1}{8} + \epsilon > -a_n$ (by subtraction)
$\Rightarrow \frac{1}{8} - \epsilon < a_n$ (by multiplication)

Further, all $a_n$ are less than or equal to $\frac{1}{8}$

Thus $\frac{1}{8} - \epsilon < a_n \leq \frac{1}{8}$ and $\frac{1}{8}$ is the supremum of $A$. [**Desmos link**] $\square$

Moreover, $a_2 = \frac{1}{8}$, so $\frac{1}{8} \in A$ meaning that the supremum of $A$ is also a maximum of $A$.

Additionally, $-1$ is the inf of $A$.

This will be proven true directly:

**Want to show**: $\forall\epsilon > 0, \exists a_n \in A$ s.t. $-1 \leq a_n < -1 + \epsilon$ (from definition of infimum)

Let $\epsilon > 0$

Know that $\exists n \in \mathbb{N}$ s.t. $\epsilon > a_n + 1$ (because this can be zero for $n = 1$)
Take $n$ s.t. $\epsilon > a_n + 1$

$\epsilon > a_n + 1$ (property of existential)
$\Rightarrow \epsilon - 1 > a_n$ (by subtraction)
$\Rightarrow a_n < -1 + \epsilon$ (by rearrangement)

Further, all $a_n$ are greater than or equal to $-1$

Thus $-1 \leq a_n < -1 + \epsilon$ and $-1$ is the infimum of $A$. [**Desmos link**] $\square$

Moreover, $a_1 = -1$, so $-1 \in A$ meaning that the infimum of $A$ is also a minumum of $A$.

## Question 3(b)

Show that $\sup_{k\geq 1}\inf_{n\geq k} a_n = \inf_{k\geq 1}\sup_{n\geq k} a_n$. Can you reach a conclusion on the convergence of the sequence $a_n, n \in N$? *Hint: draw an appropriate picture.*

## Solution

Compute $\sup_{k\geq 1}\inf_{n\geq k} a_n$:

$\inf_{n\geq k} a_n = \inf\{a_n : n \geq k\} = \inf \begin{Bmatrix} \frac{1}{k^3}, -\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}\ldots & \text{if } k \text{ even} \\ -\frac{1}{k^2}, \frac{1}{(k+1)^3}, -\frac{1}{(k+2)^2}\ldots & \text{if } k \text{ odd} \end{Bmatrix} = \begin{matrix} -\frac{1}{(k+1)^2} & \text{if } k \text{ even} \\ -\frac{1}{(k+1)^2} & \text{if } k \text{ odd} \end{matrix}$

Call $Z = \begin{cases} -\frac{1}{(k+1)^2} & \text{if } k \text{ even} \\ -\frac{1}{(k+1)^2} & \text{if } k \text{ odd} \end{cases}$

$\sup_{k\geq Z} = \sup\{Z : k \geq 1\} = \sup\{-1, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{25}, -\frac{1}{25}\ldots\} = 0$

Compute $\inf_{k\geq 1}\sup_{n\geq k} a_n$:

$\sup_{n\geq k} a_n = \sup\{a_n : n \geq k\} = \inf \begin{Bmatrix} \frac{1}{k^3}, -\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}\ldots & \text{if } k \text{ even} \\ -\frac{1}{k^2}, \frac{1}{(k+1)^3}, -\frac{1}{(k+2)^2}\ldots & \text{if } k \text{ odd} \end{Bmatrix} = \begin{matrix} \frac{1}{(k)^3} & \text{if } k \text{ even} \\ \frac{1}{(k+1)^3} & \text{if } k \text{ odd} \end{matrix}$

Call $Z = \begin{cases} -\frac{1}{(k+1)^2} & \text{if } k \text{ even} \\ -\frac{1}{(k+1)^2} & \text{if } k \text{ odd} \end{cases}$

$\inf_{k\geq Z} = \inf\{Z : k \geq 1\} = \inf\{\frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}\ldots\} = 0$

Thus, $\sup_{k\geq 1}\inf_{n\geq k} a_n = \inf_{k\geq 1}\sup_{n\geq k} a_n$. $\square$

The sequence $a_n, n \in \mathbb{N}$ will be proven to converge at 0 directly:

**Want to show**: $\forall \epsilon > 0, \exists N \geq 0$ s.t. $n \geq N \Rightarrow |a_n| < \epsilon$

Let $\epsilon > 0$
Take $N = \sqrt{\frac{1}{\epsilon}} + 1$ (this is large enough to account for cases where $n$ is both odd and even)

Assume $n \geq \sqrt{\frac{1}{\epsilon}} + 1$

Verify $|a_n| < \epsilon$:

Know $n \geq \sqrt{\frac{1}{\epsilon}} + 1$

$\Rightarrow n \geq \sqrt{\frac{1}{\epsilon}} + 1 > \sqrt{\frac{1}{\epsilon}}$
$\Rightarrow n > \sqrt{\frac{1}{\epsilon}}$ (by truncation)
$\Rightarrow n^2 > \frac{1}{\epsilon}$ (by square root property)
$\Rightarrow n^2 \cdot \epsilon > 1$ (by multiplication)
$\Rightarrow \epsilon > \frac{1}{n^2}$ (by division)
$\Rightarrow \frac{1}{n^2} < \epsilon$ (by rearrangement)

As $n^2$ is always positive, this can be written as an absolute value
$\Rightarrow |\frac{1}{n^2}| < \epsilon$ (by absolute value)

As $|\frac{1}{n^3}| \leq |\frac{1}{n^2}|$, both odd and even $n$ are less than $\epsilon$
$\Rightarrow |a_n| < \epsilon$

Thus $|a_n| < \epsilon$ and the function converges at 0. [**Desmos link**] $\square$

## Question 3(c)

Now let $B = \{f(x) : x \in [1, \infty)\}$, i.e. $B$ is the range of $f$, where $f$ is a function with domain $[1, \infty)$ is defined as follows: $f(x) = \frac{1}{x^3}$ if $x$ is rational, and $f(x) = -\frac{1}{x^2}$ if $x$ is irrational. Compute sup $B$ and inf $B$. Are they also the maximum and minimum of $B$? Justify.

## Solution

1 is the sup of $B$.

This will be proven true directly:

**Want to show**: $\forall \epsilon > 0, \exists f(x) \in A$ s.t. $1-\epsilon < f(x) \leq 1$ (from definition of supremum)

Let $\epsilon > 0$

Take $x = 1$
$\Rightarrow f(x) = \frac{1}{x^3} = 1$

Verify $1 - \epsilon < f(x) \leq 1$:

$1 > 1 - \epsilon$ (as $\epsilon > 0$)
$\Rightarrow 1 - \epsilon < 1$ (by rearrangement)

$1 \leq 1$ (by property of inequality)
$\Rightarrow f(x) \leq 1$

Thus, $1 - \epsilon < f(x) \leq 1$ and 1 is the supremum of $B$. [**Desmos link**] $\square$

Moreover, $f(1) = \frac{1}{x^3} = 1$, so $1 \in B$ meaning the supremum of $B$ is also a maximum of $B$.

Additionally, -1 is the inf of $B$.

This will be proven true directly:

**Want to show**: $\forall \epsilon > 0, \exists f(x) \in A$ s.t. $-1 \leq x < -1 + \epsilon$ (from definition of infimum)

Let $\epsilon > 0$

Know that $\exists f(x) \in B$ s.t. $\epsilon > f(x) + 1$ (this approaches zero for small, irrational $x$ values)
Take $f(x)$ s.t. $\epsilon > f(x) + 1$

Verify $-1 \leq f(x) < -1 + \epsilon$:

$\epsilon > f(x) + 1$ (by property of existential)
$\Rightarrow \epsilon - 1 > f(x)$ (by subtraction)
$\Rightarrow f(x) < -1 + \epsilon$ (by rearrangement)

$-1 \leq f(x)$ (as $x \geq 1$ so $f(x)$ cannot be less than $-1$ for any rational or irrational $x$)

Thus, $-1 \leq f(x) < -1 + \epsilon$ and $-1$ is the infimum of $B$. [**Desmos link**] $\square$

There is no $x$ s.t. $f(x) = -1$, so $-1 \notin B$ and the infimum of $B$ is *not* also a maximum.

# Question 4

Let $f$ be integrable and convex. Let $g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$.

Show that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. *Hint: Sketch the graph of the functions $f(x)$ and $g(x)$. Then, work directly from the definition of the integral.*

## Solution

It can be shown that $g(a) = f(a)$:

$g(a) = \frac{f(b)-f(a)}{b-a}(a-a) + f(a)$ (by substitution)
$= \frac{f(b)-f(a)}{b-a}(0) + f(a)$ (by subtraction)
$= 0 + f(a)$ (by multiplication)
$= f(a)$ (by addition)

It can be shown that $g(b) = f(b)$:

$g(b) = \frac{f(b)-f(a)}{b-a}(b-a) + f(a)$ (by substitution)
$= f(b) - f(a) + f(a)$ (by cancellation)
$= f(b)$ (by subtraction)

- As $g(x)$ is a constant multiplied by $x$, it is linear
- As $g(x)$ is a line that intersects curve $f(x)$ at two points, it is a secant of $f(x)$
- As $g(x)$ is continuous, it must also be integrable

Using the definition of integral:

$\int_a^b f(x)dx = \lim\limits_{||P|| \to 0} \sum_{i=1}^n S_P^*(f) = \lim\limits_{||P|| \to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$ (right Riemann sum)

$\int_a^b g(x)dx = \lim\limits_{||P|| \to 0} \sum_{i=1}^n S_P^*(g) = \lim\limits_{||P|| \to 0} \sum_{i=1}^n g(x_i^*)\Delta x_i$ (right Riemann sum)

As $f(x)$ is convex, its secant lines are all on or above the graph
Thus, $\forall x \in [a, b], g(x) \geq f(x)$

This means that:

$\lim\limits_{||P|| \to 0} \sum_{i=1}^{n} g(x_i^*) \Delta x_i \geq \lim\limits_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$ (as $g(x_i^*) \Delta x_i$ will always be $\geq f(x_i^*) \Delta x_i$)

$\Rightarrow \int_a^b g(x)dx \geq \int_a^b f(x)dx$ (by definition of integral)
$\Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$ (by rearrangement)

Thus, $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. [**Desmos link**] $\square$