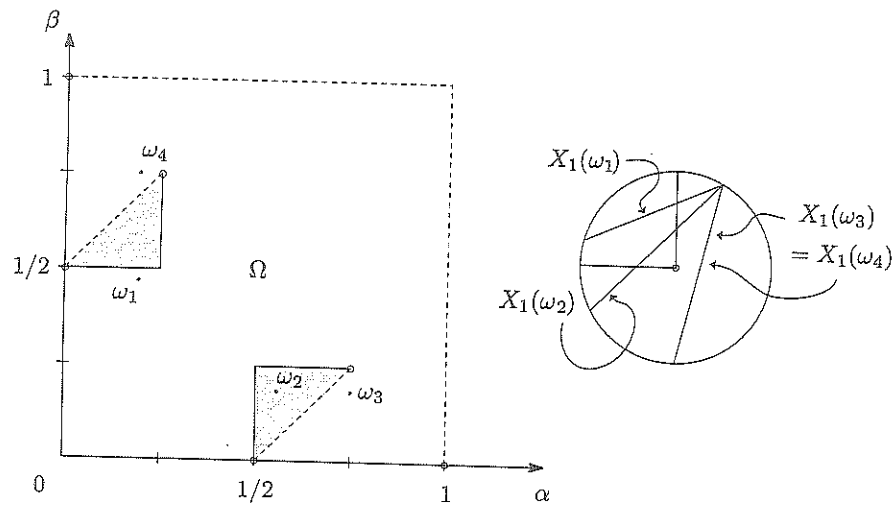

FIGURE 2.4. Random chord: four sample points

* **Problem 22.** Repeat the work done in the preceding example for each of the other three interpretations found in Example 3. That is, in each case, decide whether one-point subsets of the circle are to be considered chords, prove the necessary measurability, and compute the probability of the event that the random chord intersects the positive vertical and negative horizontal axes. Also comment on and fix an ambiguity in the definition of X_4.

CHAPTER 3
Distribution Functions

The main purpose of this chapter is to classify all probability measures on the measurable space $(\mathbb{R}, \mathcal{B})$. We will accomplish this task by establishing a one-to-one correspondence between such probability measures and a certain class of functions, known as 'distribution functions'. Many important probability measures and their corresponding distribution functions will be identified, including the binomial, normal, Poisson, gamma, and beta families of probability measures.

3.1. Basic theory

We introduce the class of functions to which the preceding paragraph refers.

Definition 1. A real-valued function F defined on $\mathbb{R}$ is called a *distribution function* for $\mathbb{R}$ if it is increasing and right-continuous and satisfies

$$\lim_{x\to-\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} F(x) = 1\,.$$

Often the phrase 'for $\mathbb{R}$' will be omitted.

Let Q denote a probability measure on the measurable space $(\mathbb{R}, \mathcal{B})$. We want to show that the function $x \rightsquigarrow Q((-\infty, x])$ is a distribution function. For this purpose we need the following useful result.

Theorem 2. [Continuity of Measure] *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(A_1, A_2, \ldots)$ be a sequence of events in $\mathcal{F}$. If $A_1 \subseteq A_2 \subseteq \ldots$, then*

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} P(A_n)\,.$$

If $A_1 \supseteq A_2 \supseteq \ldots$, then

$$P\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} P(A_n)\,.$$

PROOF. To prove the first assertion we suppose that $A_1 \subseteq A_2 \subseteq \ldots$. Let $B_1 = A_1$ and $B_m = A_m \setminus A_{m-1}$ for $m = 2, 3, \ldots$. Note that

$$A_n = \bigcup_{m=1}^{n} B_m \quad \text{and} \quad \bigcup_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} B_m .$$

Since the B_m's are disjoint, we can use the countable additivity of P to make the following calculation:

$$\begin{aligned} P\left(\bigcup_{m=1}^{\infty} A_m\right) &= P\left(\bigcup_{m=1}^{\infty} B_m\right) \\ &= \sum_{m=1}^{\infty} P(B_m) = \lim_{n\to\infty} \sum_{m=1}^{n} P(B_m) \\ &= \lim_{n\to\infty} P\left(\bigcup_{m=1}^{n} B_m\right) = \lim_{n\to\infty} P(A_n) . \end{aligned}$$

This proves the first assertion of the theorem.

Now suppose that $A_1 \supseteq A_2 \supseteq \ldots$. Then $A_1^c \subseteq A_2^c \subseteq \ldots$. Therefore, we can apply the first part of the theorem to the sequence $(A_1^c, A_2^c, \ldots)$ to obtain the following:

$$\begin{aligned} P\left(\bigcap_{n=1}^{\infty} A_n\right) &= 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \\ &= 1 - \lim_{n\to\infty} P(A_n^c) \\ &= 1 - \lim_{n\to\infty} [1 - P(A_n)] = \lim_{n\to\infty} P(A_n) . \quad \square \end{aligned}$$

Proposition 3. *Let Q be a probability measure on $(\mathbb{R}, \mathcal{B})$. Then the function $F(x) = Q((-\infty, x])$ is a distribution function.*

PROOF. Note that

$$x \le y \implies (-\infty, x] \subseteq (-\infty, y] \implies Q((-\infty, x]) \le Q((-\infty, y]) .$$

Hence, F is an increasing function. To prove right continuity, fix a real number x and let $(x_1, x_2, \ldots)$ be a decreasing sequence which converges to x. Since $(-\infty, x_1] \supseteq (-\infty, x_2] \supseteq \ldots$, Theorem 2 implies that

$$\begin{aligned} \lim_{n\to\infty} Q((-\infty, x_n]) &= Q\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right) \\ &= Q((-\infty, x]) . \end{aligned}$$

Thus, F is right-continuous. The same reasoning, with $x_n \searrow -\infty$, shows that F has the desired behavior at $-\infty$. For the behavior at ∞, let $x_n \nearrow \infty$ and use the Continuity of Measure Theorem again:

$$Q((-\infty, x_n]) \to Q\left(\bigcup_{n=1}^{\infty} (-\infty, x_n]\right) = Q(\mathbb{R}) = 1 . \quad \square$$

If a distribution Q and a distribution function F are related as in the previous theorem, then we will call F the *distribution function* of Q. If X is a random variable with distribution Q, then we will also call F the *distribution function* of X.

Problem 1. Show that the distribution function of the uniform distribution on $[0, 1]$ is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x . \end{cases}$$

What is the distribution function for the uniform distribution on $(0, 1)$?

Problem 2. Give a precise description of the probability measure Q on $(\mathbb{R}, \mathcal{B})$ that has distribution function F given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 . \end{cases}$$

Also find a probability space $(\Omega, \mathcal{F}, P)$ and a real-valued random variable X defined on $(\Omega, \mathcal{F}, P)$ that has distribution function F. The probability measure Q is sometimes called the *unit point mass* or the *delta distribution* at 0.

The preceding exercise is an example of the converse of Proposition 3: For each distribution function F there exists a unique probability Q on $(\mathbb{R}, \mathcal{B})$ such that $Q((-\infty, x]) = F(x)$. This fact is included in the next theorem, which also provides a recipe for constructing a random variable with distribution function F. From a logical point of view, this result belongs in Chapter 7, since the necessary tools for proving the existence and uniqueness of probability measures are to be found there. However, if the reader is willing to accept uniqueness in general and the existence of the uniform distribution on $(0, 1)$ in particular, then the characterization of all other distributions on $(\mathbb{R}, \mathcal{B})$ can be derived using the concepts already introduced.

Proposition 4. *Let F be a distribution function. Then there exists a unique probability measure Q on $(\mathbb{R}, \mathcal{B})$ such that $Q((-\infty, x]) = F(x)$. Moreover, a random variable X with distribution function F can be constructed as follows: Let $\Omega = (0, 1)$, let P be the uniform distribution on Ω, and define*

$$X(\omega) = \inf\{x \colon F(x) \ge \omega\} , \quad 0 < \omega < 1 . \tag{3.1}$$

PROOF. The uniqueness will follow from Theorem 3 of Chapter 7. We assume the existence of the uniform distribution P on $(0,1)$, which follows from Theorem 14 of Chapter 7. Further details about the existence of the uniform distribution are given in Example 1 of the same chapter.

By Problem 9 of Chapter 2, X is a random variable. Let Q be the distribution of X. We will complete the proof of this theorem by showing that F is the distribution function of Q, or in other words, that

$$F(y) = P(\{\omega : X(\omega) \leq y\})$$

for all $y \in \mathbb{R}$. Since X is an increasing function on $(0,1)$, the event $A = \{\omega : X(\omega) \leq y\})$ is an interval with endpoints 0 and $\sup A$. Under the uniform distribution, the probability of any sub-interval of $(0,1)$ is the length of that sub-interval, so we want to show that $F(y) = \sup A$.

The definition of X and the right continuity of F imply that $F(X(\omega)) \geq \omega$; so, if $\omega \in A$, then $F(y) \geq F(X(\omega)) \geq \omega$. Hence, $F(y)$ is an upper bound of A. On the other hand, $F(y) \in A$, because $X(F(y)) \leq y$. It follows that $F(y) = \sup A$. □

The relationship between F and X given in the preceding result is most easily understood when F is strictly increasing and continuous. In this case, X and F are inverse functions of each other and X is strictly increasing and continuous. In general X, defined by (3.1), is left-continuous. Jumps of F correspond to intervals of constancy of X, and bounded intervals of constancy of F correspond to jumps of X. Unbounded intervals of constancy of F correspond to finite limits of X at 0 and 1. It has become quite common to refer to X as the 'left-continuous inverse' of F and F as the 'right-continuous inverse' of X, even in the cases where there are jumps or intervals of constancy. Figure 3.1 shows an F that has both jumps and intervals of constancy. The corresponding X is shown below the graph of F, with its domain pictured vertically.

* **Problem 3.** Prove that X as defined in the preceding theorem is left-continuous and satisfies $X(\omega) = \sup\{x : F(x) < \omega\}$.

Problem 4. Discuss the options of making either X right-continuous or F left-continuous or both in Proposition 4.

Problem 5. Let X be an $\mathbb{R}$-valued random variable with distribution function F. Prove that for all real numbers $a < b$,

$$P(\{\omega : X(\omega) \in (a,b]\}) = F(b) - F(a).$$

Find analogous formulas involving intervals of the form (a,b), $[a,b)$, and $[a,b]$. For $x \in \mathbb{R}$, show that

$$P(\{\omega : X(\omega) = x\}) = F(x) - F(x-).$$

As a consequence, conclude that if F is continuous, then the events $\{\omega : X(\omega) = x\}$ are all null events.

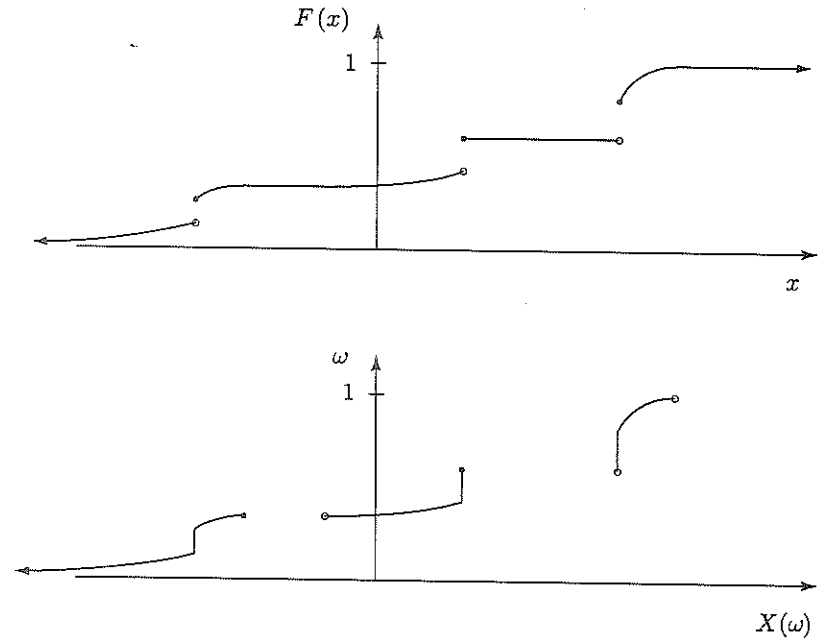

FIGURE 3.1. Distribution function and corresponding random variable

3.2. Examples of distributions

In the remainder of this chapter, we will illustrate Proposition 3 and Proposition 4 by introducing, through examples and exercises, some of the more important distributions on $(\mathbb{R}, \mathcal{B})$.

Problem 6. [Delta distributions] If X is equal to a constant a, show that the distribution function of X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x \geq a\,. \end{cases}$$

The distribution of X is called the ***delta distribution*** or ***unit point mass*** at a. Often δ_a will be used for the delta distribution at a.

Problem 7. [Bernoulli distributions] Fix $p \in [0,1]$, and let X be a random variable that equals 1 with probability p and equals 0 with probability $1-p$. Calculate the distribution function F of X. Conversely, starting with F, construct a random variable whose distribution function is F. *Hint:* See Figure 3.2.