Appendix A

Differentiation Rules

Implicit haptic rendering, as described in Chapter 5, requires the computation of several Jacobians. In this appendix I review some useful differentiation rules for the computation of Jacobians, and I derive terms necessary in the implementation of implicit integration for rigid body dynamic simulation with haptic interaction.

A.1 Vector Differentiation Rules

A.1.1 Jacobian

Given a system of equations expressed in vector form as $\mathbf{y} = \mathbf{f}(\mathbf{x})$, the Jacobian matrix can be written as:

$$J = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix}. \tag{A.1}$$

Note that, according to the above definition of the Jacobian, the derivative of each of the equations, $\frac{\partial y_i}{\partial \mathbf{x}}$, is represented as a row vector.

A.1.2 Derivative of a dot product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} \tag{A.2}$$

$$\frac{\partial (\mathbf{u} \cdot \mathbf{v})}{\partial \mathbf{w}} = \mathbf{u}^T \frac{\partial \mathbf{v}}{\partial \mathbf{w}} + \mathbf{v}^T \frac{\partial \mathbf{u}}{\partial \mathbf{w}}. \tag{A.3}$$

A.1.3 Derivative of a cross product

A cross product $\mathbf{u} \times \mathbf{v}$ can be regarded as a linear transformation on $\mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = \mathbf{u}^* \mathbf{v}, \tag{A.4}$$

where $\mathbf{u}^*$ is a matrix defined as:

$$\mathbf{u}^* = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}. \tag{A.5}$$

Note some properties of cross products:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \tag{A.6}$$

$$\mathbf{u}^* \mathbf{v} = -\mathbf{v}^* \mathbf{u}. \tag{A.7}$$

From these, one can deduce that:

$$\frac{\partial (\mathbf{u} \times \mathbf{v})}{\partial \mathbf{w}} = \mathbf{u}^* \frac{\partial \mathbf{v}}{\partial \mathbf{w}} - \mathbf{v}^* \frac{\partial \mathbf{u}}{\partial \mathbf{w}}. \tag{A.8}$$

A.1.4 Gradient

The gradient can be regarded as the computation of the derivative of a scalar function w.r.t. a vector. Using the notation of Jacobians introduced earlier, the derivative of each scalar function is represented as a row vector. Considering the gradient as a column vector leaves the following relation:

$$\nabla_{\mathbf{w}} f(\mathbf{w}) = \left(\frac{\partial (f(\mathbf{w}))}{\partial \mathbf{w}} \right)^T. \tag{A.9}$$

A.1.5 Hessian

The Hessian matrix is the Jacobian of the gradient of a scalar function. Therefore, it can be represented as:

$$\mathscr{H}_{\mathbf{w}} f(\mathbf{w}) = \frac{\partial\left(\nabla_{\mathbf{w}} f(\mathbf{w})\right)}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial^2 f}{\partial \mathbf{w}_1{}^2} & \frac{\partial^2 f}{\partial \mathbf{w}_1 \partial \mathbf{w}_2} & \cdots & \frac{\partial^2 f}{\partial \mathbf{w}_1 \partial \mathbf{w}_n} \\ \frac{\partial^2 f}{\partial \mathbf{w}_2 \partial \mathbf{w}_1} & \frac{\partial^2 f}{\partial \mathbf{w}_2{}^2} & \cdots & \frac{\partial^2 f}{\partial \mathbf{w}_2 \partial \mathbf{w}_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial \mathbf{w}_n \partial \mathbf{w}_1} & \frac{\partial^2 f}{\partial \mathbf{w}_n \partial \mathbf{w}_2} & \cdots & \frac{\partial^2 f}{\partial \mathbf{w}_n{}^2} \end{pmatrix}. \tag{A.10}$$

A.1.6 Derivative of a vector multiplied by a scalar function

$$\frac{\partial\left(f(\mathbf{w})\mathbf{u}\right)}{\partial \mathbf{w}} = f(\mathbf{w})\frac{\partial \mathbf{u}}{\partial \mathbf{w}} + \mathbf{u}\frac{\partial f(\mathbf{w})}{\partial \mathbf{w}}. \tag{A.11}$$

A.1.7 Derivative of a vector multiplied by a matrix

In this case it is more convenient to express the derivative w.r.t. each of the components of the vector separately:

$$\frac{\partial\left(M\mathbf{u}\right)}{\partial w_i} = M\frac{\partial \mathbf{u}}{\partial w_i} + \frac{\partial M}{\partial w_i}\mathbf{u}. \tag{A.12}$$

A.2 Rotations

A.2.1 Quaternions

Let us define a unit quaternion $\mathbf{q} = (\mathbf{u}, s)$, where $\mathbf{u} = (x, y, z)$ is the vector part, and s is the scalar part. The inverse of $\mathbf{q}$, $\mathbf{q}^{-1}$, is defined as:

$$\mathbf{q}^{-1} = (-\mathbf{u}, s) = (-x, -y, -z, s). \tag{A.13}$$

A.2.2 Product of Quaternions

The product of two quaternions $\mathbf{ab}$ is defined as: