SINGLE PHASE FLOW SOLVER ACROSS A SANDPACK
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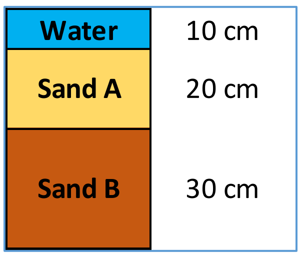
1. Abstract

This study aims to find potential and pressure profile along the sand body which water flows into it using two approach : analytical and numerical. Combination of mass conservation formula and Darcy equation are utilized to find the solution.

2. Analytical Approach :

Analytical Expression



Mass conservation equation :

$$\frac{d\rho}{dt} + \nabla(\rho v) = \rho q$$

Since the system employs incompressible fluid, and since no injection along the system:

$$\frac{dv}{dz} = 0$$

Recall Darcy's equation in 1-D to model flow along porous media :

$v = -\frac{k}{\mu}\frac{d\phi}{dz}$, thus

$$-\frac{d}{dz}\left(\frac{k}{\mu}\frac{d\phi}{dz}\right) = 0$$

$$\phi = P + \rho g z$$

At z = 0

$$\phi_{bot} = 0 \ Pa$$

at z = top sand A

$$\phi_{top} = 1000 * 9.81 * 0.6 = 5886 \, Pa$$

For sand A (sand 1) :

$$\frac{\left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{1+\frac{1}{2}} - \left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{1-\frac{1}{2}}}{Z_A} = 0$$

$$\left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{1+\frac{1}{2}} = \frac{k}{\mu}\frac{\phi_A - \phi_B}{\frac{Z_A}{2} + \frac{Z_B}{2}}$$

$$\left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{1-\frac{1}{2}} = \frac{k}{\mu}\frac{\phi_{top} - \phi_A}{\frac{Z_A}{2}}$$

Thus :

$$\frac{\frac{k_H}{\mu}\frac{\phi_A - \phi_B}{\frac{Z_A}{2} + \frac{Z_B}{2}} - \frac{k_A}{\mu}\frac{\phi_T - \phi_A}{\frac{Z_A}{2}}}{Z_A} = 0$$

For sand B (sand 2) :

$$\frac{\left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{2+\frac{1}{2}} - \left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{2-\frac{1}{2}}}{Z_B} = 0$$

$$\left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{2+\frac{1}{2}} = \frac{k}{\mu}\frac{\phi_B - \phi_{bot}}{\frac{Z_A}{2} + \frac{Z_B}{2}}$$

$$\left[\frac{k}{\mu}\frac{d\phi}{dz}\right]_{1-\frac{1}{2}} = \frac{k}{\mu}\frac{\phi_A - \phi_B}{\frac{Z_B}{2}}$$

Thus :

$$\frac{\frac{k_A}{\mu}\frac{\phi_B - \phi_{bot}}{\frac{Z_B}{2}} - \frac{k_H}{\mu}\frac{\phi_A - \phi_B}{\frac{Z_A}{2} + \frac{Z_B}{2}}}{Z_B} = 0$$

$Where\ k_H\ =\ harmonic\ average\ between\ k_A\ and\ k_B$

$$\alpha_A = \frac{k_H}{\mu\, Z_A \left(\frac{Z_A}{2} + \frac{Z_B}{2}\right)} \qquad \alpha_B = \frac{k_B}{\mu\, \frac{Z_B^2}{2}}$$

$$\beta_A = \frac{k_A}{\mu\, \frac{Z_A^2}{2}} \qquad \beta_B = \frac{k_H}{\mu\, Z_B \left(\frac{Z_A}{2} + \frac{Z_B}{2}\right)}$$

Substitute :

$$\alpha_A\,(\phi_A - \phi_B) - \beta_A\left(\phi_{top} - \phi_A\right) = 0 \qquad \alpha_B\,(\phi_B - \phi_{bot}) - \beta_B(\phi_A - \phi_B) = 0$$

$$\alpha_A\phi_A - \alpha_A\phi_{B+}\ \beta_A\phi_A = \beta_A\phi_{top} \qquad \alpha_B\phi_B + \beta_B\phi_{B-}\ \beta_B\phi_A = \alpha_B\phi_{bot}$$

$$\begin{bmatrix} \alpha_A + \beta_A & -\alpha_A \\ -\beta_B & \alpha_B + \beta_B \end{bmatrix} \begin{bmatrix} \phi_A \\ \phi_B \end{bmatrix} = \begin{bmatrix} \beta_A\phi_{top} \\ \alpha_B\phi_{bot} \end{bmatrix}$$

Plug in number and solve for $\begin{bmatrix} \phi_A \\ \phi_B \end{bmatrix} = \begin{bmatrix} 2984.8 \\ 41.8 \end{bmatrix} (Pa)$

Potential gradient A :

$$\frac{d\phi_A}{dz} = \frac{\phi_{top} - \phi_A}{Z_A - \frac{Z_A}{2}} = 29012\ {}^{Pa}/_{m}$$

Potential gradient B :

$$\frac{d\phi_B}{dz} = \frac{\phi_B - \phi_{bot}}{Z_B - \frac{Z_B}{2}} = 278\ {}^{Pa}/_{m}$$

Plug in BC to $\phi = AX + B$

Sand A (upper BC) :

$5886 = 29012 * 0.5 + B$

$B = -8619$

Sand B (lower BC) :

$0 = 278 * 0 + B$

$B = 0$

Potential equation sand A :

$\phi = 29012 * Z - 8619$

Potential equation sand B :

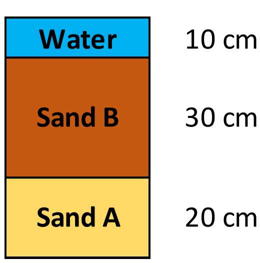
$\phi = 278 * Z$

Matlab program for analytical approach can be seen in appendix B.

Asumption :

In this study, we set a the direction of the flow as 1-D flow. Hence the main assumption is the cylinder remain standing vertically. Since any deviation of cylinder will affect true vertical height of water column, this also give difference in the amount of pressure and potential applied. Any deviation will also alter the direction of the flow from 1-D to 2-D flow.

B over A Configuration :



Several trial have been conducted to find the best configuration of sand. If sand B is placed above sand A, then will result different potential distribution. Potential will slowly and steadily decrease along the length of sand B. Unlike sand B, sand A has more rapid decrease since it needs stronger push to be able to flow same amount of water. However, both A over B and B over A configuration will drop the highest potential to zero.

Potential and pressure profile in A over B and B over A configuration is displayed in figure 1 below.

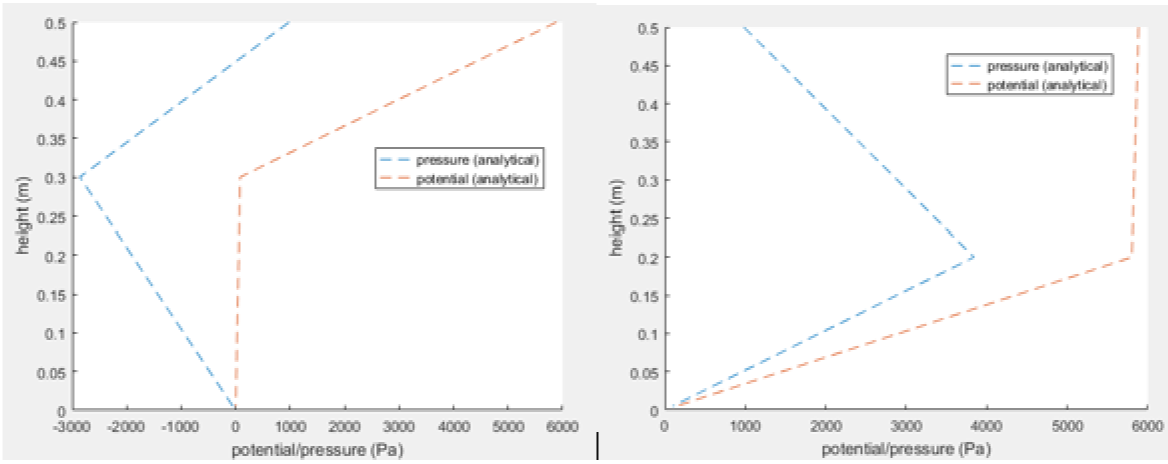


Figure 1:potential and pressure against height in A over B configuration (left) and B over A configuration (right)

3. Numerical Approach :

Analytical Expression

Solving numerical solution requires proper discretization in order to find more accurate result. A discretization consists of N number of grids is constructed along the sand body and arranged from grid 1 to grid N, from top to bottom respectively as shown in figure 3. Each center of grid has its own value of permeability which is computed using Carman-Kozeny equation. However, interface of each grid needs special treatment particularly in grid in which sand A and sand B connects in series. Permeability in grid interfaces is harmonic average value of permeability from associates grid.

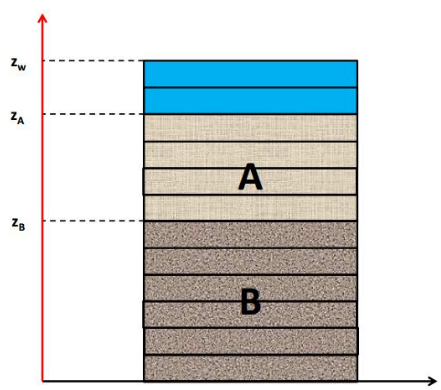


Figure 2: Discretization in System being Observed (source : HW1 AES 1310)

As well as analytical explanation, to solve the problem numerically, mass conservation equation for incompressible fluid becomes the very basic formula. Since no injection in the system, it can be written as follows :

$$\frac{dv}{dz} = 0$$

Recall Darcy's equation in 1-D and employ it for each grid:

$$-\frac{d}{dz}\left(\frac{k}{\mu}\frac{d\phi}{dz}\right) = 0$$

In discrete form

For grid i :

$$\frac{\left(\frac{k}{\mu}\frac{d\phi}{dz}\right)_{i+1/2} - \left(\frac{k}{\mu}\frac{d\phi}{dz}\right)_{i-1/2}}{\Delta z} = 0 \qquad which$$

$$\left(\frac{k}{\mu}\frac{d\phi}{dz}\right)_{i+1/2} = \frac{k^H_{i,i+1}}{\mu}\frac{\phi_i - \phi_{i+1}}{\Delta z} \qquad and \qquad \left(\frac{k}{\mu}\frac{d\phi}{dz}\right)_{i-1/2} = \frac{k^H_{i-1,\ i}}{\mu}\frac{\phi_{i-1} - \phi_i}{\Delta z}$$

Combine those equations :

$$\frac{k^H_{i,i+1}}{\mu}\ \frac{\phi_i - \phi_{i+1}}{\Delta z^2} - \frac{k^H_{i-1,\ i}}{\mu}\ \frac{\phi_{i-1} - \phi_i}{\Delta z^2} = 0$$

Define

$$\alpha_i = \frac{k^H_{i,i+1}}{\mu\Delta z^2} \ \ and\ \beta_i = \frac{k^H_{i-1,\ i}}{\mu\Delta z^2}$$

In simplified form :

$$\alpha_i(\phi_i - \phi_{i+1}) - \beta_i(\phi_{i-1} - \phi_i) = 0$$

For grid 1:

$$\frac{k^H_{1,2}}{\mu}\ \frac{\phi_1 - \phi_2}{\Delta z^2} - \frac{k^H_{1,1}}{\mu}\ \frac{\phi_{top} - \phi_1}{\Delta z^2/2} = 0$$

Define

$$\alpha_1 = \frac{k^H_{1,2}}{\mu\Delta z^2} \ \ and\ \beta_1 = \frac{k^H_{1,1}}{\Delta z^2/2}$$

In simplified form :

$$\alpha_1(\phi_1 - \phi_2) - \beta_1(\phi_{top} - \phi_1) = 0$$

$$\alpha_1\phi_1 - \alpha_1\phi_2 + \beta_1\phi_1 = \beta_1\phi_{top}$$

For grid N:

$$\frac{k^H_{N,N}}{\mu}\ \frac{\phi_N - \phi_{bot}}{\Delta z^2/2} - \frac{k^H_{N-1,N}}{\mu}\ \frac{\phi_{N-1} - \phi_N}{\Delta z^2} = 0$$

$$\alpha_N = \frac{k^H_{N,N}}{\Delta z^2/2} \ \ and\ \beta_N = \frac{k^H_{N-1,N}}{\mu\Delta z^2}$$

$$\alpha_N(\phi_N - \phi_{bot}) - \beta_N(\phi_{N-1} - \phi_N) = 0$$

$$\alpha_N\phi_N - \beta_N\phi_{N-1} + \beta_N\phi_N = \alpha_N\phi_{bot}$$

Recall BC :

$$\phi = P + \rho g z$$

At z = 0

$$\phi_{bot} = 0\ \ Pa$$

at z = top sand A

$$\phi_{top} = 1000 * 9.81 * 0.6 = 5886\ Pa$$

In matrices form :

$$\begin{bmatrix} \alpha_1 + \beta_1 & \alpha_1 & 0 \\ \\ 0 & -\beta_N & \alpha_N + \beta_N \end{bmatrix}\begin{bmatrix} \phi_1 \\ \\ \phi_N \end{bmatrix} = \begin{bmatrix} \beta_1\phi_{top} \\ \\ \alpha_N\phi_{bot} \end{bmatrix}$$

Matlab program for numerical approach can be seen in appendix C.

Plug in the number and solve for the potential and pressure of each grid. Table 1 display the potential and pressure for each gri cell. Figure 3 shows the result from numerical approach can confirm the analytical one.

Table 1: Potential and Pressure for each grid cell from numerical approach

Grid	Potential (Pa)	Pressure (Pa)
1	5161	256
2	3710	-704
3	2260	-1664
4	809	-2625
5	77	-2866
6	63	-2390
7	49	-1913
8	35	-1437
9	21	-960
10	7	-484

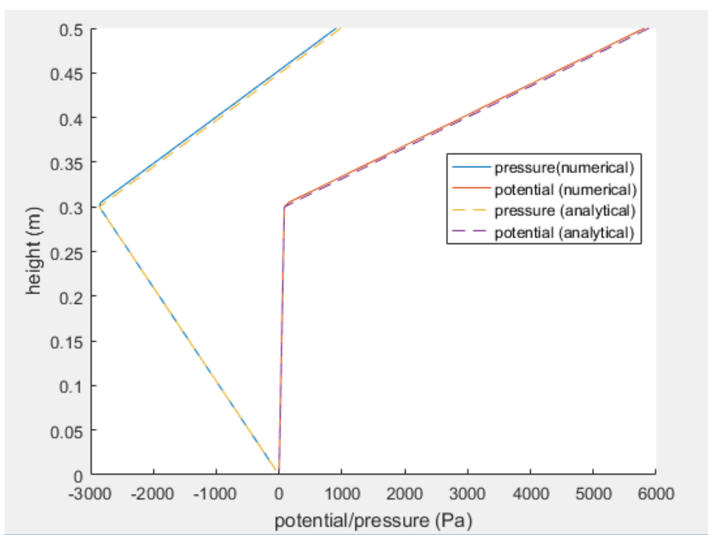


Figure 3 : overlay of potential and pressure against height in A over B configuration between analytical and numerical approach

4. Sensitivity :

Sensitivity of Water Column

Sensitivity of water column will affect mainly in potential value at top of cylinder. The higher water column, the higher potential at top cylinder. Since there is no potential loss inside the water, the potential at top sand A will always be similar to potential at top cylinder. Higher potential at top sand

A will result faster outlet velocity. Meanwhile, lower potential at top sand A will result slower velocity. Thus velocity is proportional to water column.

Sensitivity of Sand Thickness

Any difference in sand thickness would not affect outlet velocity. However, any sand thickness will surely drops the highest potential value to the least.

Sensitivity of Pressure

Since potential is summation between pressure applied and hydrostatic pressure, any pressure applied at top of water column will result higher potential at top sand and pressure along the sand body. Table 2 shows the potential and pressure when applied by 200 MPa pump pressure at top of water column.

Table 2: potential and pressure distribution against height after applied 200 MPa pump pressure

Height (m)	Potential (MPa)	Pressure (MPa)
0.5	175.4	175.4
0.45	126.1	126.1
0.4	76.8	76.8
0.35	27.5	27.5
0.3	2.6	2.6
0.25	2.1	2.1
0.2	1.7	1.7
0.15	1.2	1.2
0.1	0.7	0.7
0.05	0.2	0.2

APPENDIX A : Input Parameter for MATLAB Program

```
% single phase flow solver across a sandpack
% R B Arbarim : 4573900

%case sand A over sand B

clear all
%input parameters
opt = str2double(input('press 1 for A over B, press 2 for B over A = ','s'));

if opt==1
    Da = 2e-4; %m
    Db = 4e-4; %m
    por_a = 0.1;
    por_b =0.26;
    za = 0.2; %m
    zb = 0.3; %m

elseif opt==2
    Db = 2e-4; %m
    Da = 4e-4; %m
    por_b = 0.1;
    por_a =0.26;
    zb = 0.2; %m
    za = 0.3; %m
end

N = 100; % number of grid cells
a = zeros (N,1);
b = zeros (N,1);
Da = 2e-4; %m
Db = 4e-4; %m
rho = 1000; % [kg/m3]
g = 9.81; %m/s^2
mu = 1; %cp
zw = 0.1; %m
L = za + zb; %m
Lt = L + zw; %m

dz = L/N;
na = za/dz;
nb = zb/dz;
pressure = 0; %Pa
phi_top = pressure + rho*g*Lt; %Pa
phi_bot = 0; %Pa
Q = zeros(N,1);
A = zeros(N,N);
k = zeros(N,1);
ka = por_a^3*Da^2/(180*(1-por_a)^2); %employ 180 to constitute tortuosity
kb = por_b^3*Db^2/(180*(1-por_b)^2); %employ 180 to constitute tortuosity
```

APPENDIX B : MATLAB Program for Analytical Approach

```
%Question - Part 1 Analytical
```

```
kh = (za/2 + zb/2)/(za/2/ka + zb/2/kb);

for i = 1 : 2
    if i == 1
        alpha(i) = kh/(mu*za*(za/2 + zb/2));
        beta(i) = ka/(mu*za^2/2);
    else
        alpha(i) = kb/(mu*zb^2/2);
        beta(i) = kh/(mu*zb*(za/2 + zb/2));
    end
end

for i = 1 : 2
    X(i,i) = alpha(i) + beta(i);
    if i == 1
        X(i,i+1) = -alpha(i) ;
        Y(i) = beta(i)*phi_top;
    else
        X(i,i-1) = -beta(i);
        Y(i) = alpha(i)*phi_bot;
    end
end
potential = X\Y';
pot_A = potential(1); %potential at centre of sand A
pot_B = potential(2); %potential at centre of sand B
gradA = (phi_top - pot_A)/(za/2); %Pa/m
gradB = (pot_B - phi_bot)/(zb/2); %Pa/m
Ca = phi_top - gradA*(za+zb);
Cb = phi_bot - gradB*0;
h = (dz*(N:-1:1))';
z = [0.5:-0.1:0];

 for i = 1 : length(h)
     if h(i) >= zb
         pot_anl(i) = gradA*h(i) + Ca;
     else
         pot_anl(i) = gradB*h(i) + Cb;
     end
     P(i) = pot_anl(i) - rho*g*h(i);
 end
```

APPENDIX B : MATLAB Program for Numerical Approach

```
%Question - Part 2 Numerical
for i = 1 : N
    if i*dz <= za
        k(i) = ka;
    else
        k(i) = kb;
    end
end

for i = 1 : N
    if i == 1
    b(i) = k(i)/(mu*(dz^2)/2);
```

```
    else
    b(i) = 2*k(i).*k(i-1)/(k(i) + k(i-1))/(mu*dz^2);
    end
    if i == N
    a(i) = k(i)/(mu*(dz^2)/2);
    else
    a(i) = 2*k(i).*k(i+1)/(k(i) + k(i+1))/(mu*dz^2);
    end
end

for i = 1 : N
    if i == 1
        Q(i) = b(i)*phi_top;
    elseif i == N
        Q(i) = a(i)*phi_bot;
    end
    A(i,i) = a(i) + b(i);
    if i == 1
        A(i,i+1)= -a(i);
    elseif i == N
        A(i,i-1) = -b(i);
    else
        A(i,i+1)= -a(i);
        A(i,i-1)= -b(i);
    end
end

pot = A\Q;

for i = 1 : N
    p(i,1) = pot(i) - rho*g*h(i);
end

hold on
plot(p,h);
plot(pot,h);
plot(P,h,'--');
plot(pot_anl,h,'--');
xlabel('potential/pressure (Pa)');
ylabel('height (m)');
legend('pressure(numerical)','potential (numerical)','pressure
(analytical)','potential (analytical)');
hold off
```