Solution #1:

1. For each of the following statements, state whether it is true or false; if true give a proof; if false give a counter example: [You may assume that all functions are nonnegative, increasing and tend to infinity.]

   (a) $[f(n) = O(g(n))] \Leftrightarrow [(f(n))^2 = O((g(n))^2)]$
   Solution: This statement is true. $[f(n) = O(g(n))] \Leftrightarrow [f(n) \leq cg(n)$ for some $c, n_0 < \infty$ and $\forall n \geq n_0]$. Hence $[(f(n))^2 \leq \hat{c}(g((n))^2$ with $c^2 = \hat{c} < \infty$. Hence $\Rightarrow$ part. The reverse direction take $\hat{c} = \sqrt{c}$.

   (b) $[f(n) = O(g(n))] \Rightarrow [2^{f(n)} = O(2^{g(n)})]$
   Solution: This statement is false. Take $f(n) = 2n; g(n) = n$. Since $2^{2n} = \omega(2^n)$ the statement is not true.

   (c) $[f(n) = O(g(n))] \Leftarrow [2^{f(n)} = O(2^{g(n)})]$
   Solution: This statement is true. $[2^{f(n)} = O(2^{g(n)})] \Rightarrow [2^{f(n)} \leq c2^{g(n)}$ for $c < \infty$, and $\forall n \geq n_0] \Rightarrow [f(n) \leq g(n) + \lg c \leq 2g(n)]$ by taking logs.

2. If $\lim_{n\to\infty}[\frac{f(n)}{g(n)}]$ exists, then $[f(n) = O(g(n))] \Rightarrow \lim_{n\to\infty}[\frac{f(n)}{g(n)}] < \infty$; moreover, $[f(n) \neq o(g(n))] \Rightarrow \lim_{n\to\infty}[\frac{f(n)}{g(n)}] > 0$. Hence in this case, it is true that $f(n) = \Theta(g(n))$.

   If we assume that both these functions are nonnegative, increasing and tend to infinity as $n \to \infty$ but the $\lim_{n\to\infty}[\frac{f(n)}{g(n)}]$ does not exist, we get an interesting situation. I am viewing this as a challenge problem and will discuss solution with those who show me their answer!

3. Exercises 3.2-4: Are functions $\lceil \lg n \rceil!$ and $\lceil \lg \lg n \rceil!$ polynomially bounded?

   Solution:

   A function $f(n)$ is said to be polynomially bounded iff $f(n) = O(n^b)$ for some fixed b.

   Let $\lceil \lg n \rceil = k$. This is equivalent to $[k-1 < \lg n \leq k] \Longleftrightarrow 2^{k-1} < n \leq 2^k$. $\lceil \lg n \rceil! = k!$

   $n^b \leq (2^k)^b = 2^{bk} < k!$ since $bk < k \lg k$ for large values of k. Hence, the first function is not polynomially bounded.

   Let $\lceil \lg \lg n \rceil = k \Longleftrightarrow k - 1 < \lg \lg n \leq k \Longleftrightarrow 2^{k-1} < \lg n \leq 2^k \Longleftrightarrow 2^{2^{k-1}} < n \leq 2^{2^k}$

   $\lceil \lg \lg n \rceil! = k!$

   $n^b \leq (2^{2^k})^b = 2^{b2^k} > k!$ since $b2^k > k \lg k$ for large k. Hence, the second function is polynomially bounded.

   Alternate Solution (somewhat easier)

   $[f(n) = O(n^k)] \Leftrightarrow [\lg(f(n) = O(\lg n)]$ [verify this!]

$\lg(n!) = \Theta(n \lg n)$; $\lceil \lg n \rceil = \Theta(\lg n)$

$\lg(\lceil \lg n \rceil !) = \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil) = \Theta(\lg n \lg \lg n) = \omega(\lg n)$

Hence $\lg(\lceil \lg n \rceil !) \neq O(\lg n)$. Hence $\lceil \lg n \rceil ! \neq O(n^k)$.

$\lg(\lceil \lg \lg n \rceil !) = \Theta(\lceil \lg \lg n \rceil \lg \lceil \lg \lg n \rceil) = \Theta(\lg \lg n \lg \lg \lg n) = o((\lg \lg n)^2) = o(\lg^2(\lg n)) = o(\lg n)$

Hence $\lceil \lg \lg n \rceil ! = O(n^k)$.

4. **3-3(a)** First let's partition into some groups and then we will order within each group and merge.

   $\{2^{2^{n+1}}, 2^{2^n}, n2^n, e^n, 2^n, (\frac{3}{2})^n\}; \{(n+1)!, n!, (\lg n)!, \lg(n!)\};$

   $\{n^3, n^2 = \Theta(4^{\lg n}), n = \Theta(2^{\lg n}), (\sqrt{2})^{\lg n}\};$

   $\{n^{\lg \lg n} = \Theta((\lg n)^{\lg n}), n \lg n, \lg^2 n, \lg n, \sqrt{\lg n}, \lg \lg n\};$

   $\{2^{\lg^* n}, \lg^* n, \lg^*(\lg n), \lg(\lg^* n)\}; \{2^{\sqrt{2 \lg n}}\}; \{1, n^{\frac{1}{\lg n}}\}.$

   This is a total of 30 functions. Note: I have replaced $\ln \ln n$ by $\lg \lg n$ and $\ln n$ by $\lg n$. This can be done since these are related by Θ. Within each group it is easy to see functions have been arranged in the order required by the problem. By and large the groups have also been (almost) arranged in the order required. Some amount of merging is required to get:

   $\{2^{2^{n+1}}, 2^{2^n}\}; \{(n+1)!, n!\}; \{e^n, n2^n, 2^n, (\frac{3}{2})^n\};$

   $\{n^{\lg \lg n} = \Theta((\lg n)^{\lg n}); \{(\lg n)!\}; \{n^3, n^2 = \Theta(4^{\lg n})\};$

   $\{\lg(n!) = \Theta(n \lg n)\}, \{n = \Theta(2^{\lg n}), (\sqrt{2})^{\lg n}\}; \{2^{\sqrt{2 \lg n}}\};$

   $\{\lg^2 n, \lg n\}; \{\sqrt{\lg n}, \lg \lg n\}; \{2^{\lg^* n}, \lg^* n, \lg^*(\lg n), \lg(\lg^* n)\};$

   $\{1, n^{\frac{1}{\lg n}}\}$

5. Show that (i)$\sum_{i=1}^{i=n} i^2 = \Theta(n^3)$

   Solution: We start with the relation:

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=0}^{n} (i+1)^3 \\
&= \sum_{i=0}^{n} (i^3 + 3i^2 + 3i + 1) \\
&= \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1
\end{aligned}
$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{1}{3}[(n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)] \\
&= \frac{1}{3}(n+1)[n^2 + 2n + 1 - \frac{3n}{2} - 1] \\
&= \frac{1}{6}(n+1)[2n^2 + n] \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{2(n+1)^3 - 3(n+1)^2 + n + 1}{6} \\
&= \frac{2n^3 + 3n^2 + n}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
&= \Theta(n^3)
\end{aligned}$$

This gives you the method of finding $\sum_{i=1}^{n} i^k$ for general values of k by induction.