Solution to Assignment #2:

1. (a) (4.3-6) Show that $t(n) = O(n \lg n)$ for the relation:

$$t(n) = 2t(\left\lfloor \frac{n}{2} \right\rfloor + 17) + n$$

Solution: Let $m = n - 34$. This implies that $\lfloor \frac{n}{2} \rfloor + 17 = \lfloor \frac{m}{2} \rfloor + 34$. Hence our equation becomes:

$$\begin{aligned} t(m+34) &= 2t(\left\lfloor \frac{m}{2} \right\rfloor + 34) + (m+34) \\ s(m) &= 2s(\left\lfloor \frac{m}{2} \right\rfloor) + \Theta(m) \\ &= \Theta(m \lg m) \end{aligned}$$

if we let $S(m) = t(m+34)$. Hence,

$$t(n) = t(m+34) = s(m) = \Theta(m \lg m) = \Theta((n-34)\lg(n-34))$$

Now we show that $(n+c)\lg(n+c) = \Theta(n \lg n)$ and this will complete the proof.

$$\begin{aligned} &\lim_{n\to\infty}[\frac{(n+c)\lg(n+c) - n\lg n}{n \lg n}] \\ = &\lim_{n\to\infty}[\frac{(\lg(n+c) - \lg n)}{\lg n}] + \lim_{n\to\infty}[\frac{c\lg(n+c)}{n \lg n}] \\ = &\lim_{n\to\infty}[\frac{(\lg(1+\frac{c}{n})}{\lg n}] = 0 \end{aligned}$$

Hence

$$\lim_{n\to\infty}[\frac{(n+c)\lg(n+c)}{n \lg n}] = 1$$

and therefore, $(n+c)\lg(n+c) = \Theta(n \lg n)$.

(b) 4.3-9: Solve the relation $t(n) = 3t(\sqrt{n}) + \log n$

Solution:

Let $\lg n = m; n = 2^m$. The above equation changes to

$$t(2^m) = 3t(2^{\frac{m}{2}}) + m$$

Now let $t(2^x) = s(x)$ and the equations changes to

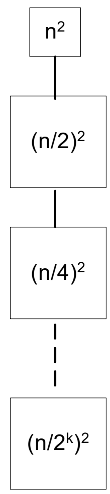
$$s(m) = 3s(\frac{m}{2}) + m$$

This equation is solved by master theorem (case 1) to yield the solution $s(m) = \Theta(m^{\lg 3})$. Hence the solution of the original equation is $t(n) = \Theta((\lg n)^{\lg 3})$.

2. **4.4-2**: Use a recursion tree to give an asymptotically tight solution to the relation:

$$t(n) = t(\frac{n}{2}) + n^2$$

Use substitution method to verify your answer.



where $k = \lg n$. Adding we get

$$\begin{aligned} t(n) &= n^2 \sum_{j=0}^{k} (\frac{1}{2})^j \\ &= \Theta(n^2). \end{aligned}$$

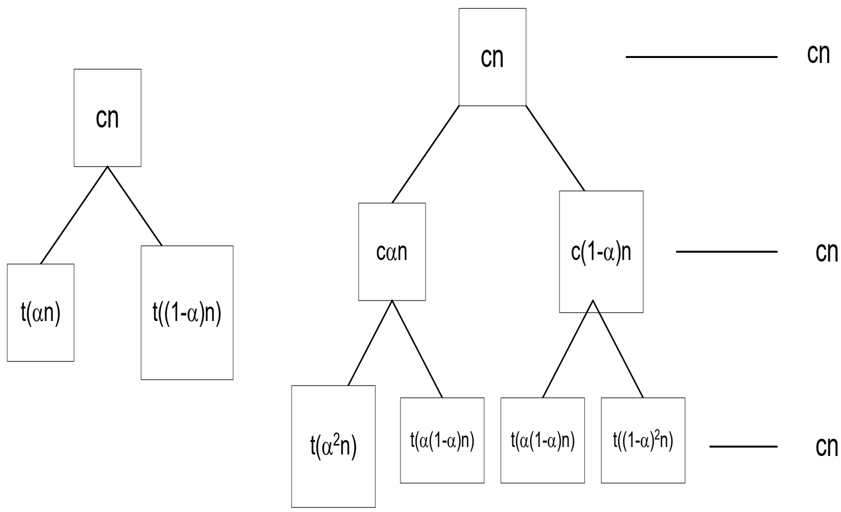
Verification: By using induction hypothesis on smaller values we get

$$\begin{aligned} t(n) &\leq n^2 + c(\frac{n}{2})^2 \\ &= n^2[1 + \frac{c}{4}] \\ &\leq cn^2 \text{if } c > \frac{4}{3} \end{aligned}$$

4.4-6: Use a recursion tree to give an asymptotically tight solution to the relation:

$$t(n) = t(\alpha n) + t((1 - \alpha)n) + cn$$

where $c > 0$, and $0 < \alpha < 1$ are given constants.



The depth of the tree is $\frac{\lg n}{-\lg \alpha} = \Theta(\lg n)$. Hence,

$$t(n) = \Theta(n \lg n)$$

Hence $t(n) = \Omega(n \lg n)$.

3. Problem 4-3: (b),(c),(f)

   b $t(n) = 3t(\frac{n}{3}) + \frac{n}{\lg n}$

   $f(n) = \frac{n}{\lg n}; n^{\log_b a} = n$. So both cases 2 and 3 do not apply. Moreover, $\lim_{n \longrightarrow \infty}[\frac{n^{1-\varepsilon}}{f(n)}] = 0$ and hence $f(n) = \omega(n^{1-\varepsilon})$. So we have $f(n) \neq O(n^{\log_b a - \varepsilon})$. So the master theorem does not apply in this case. This equation is of the form $t(n) = at(\frac{n}{b}) + n^{\log_b a}(\lg n)^k; b > 1; a \geq 1$ but $k \gneqq 0$. In this case $k = -1$. So we try iteration method after changing variables.

   Let $\log_3 n = k; n = 3^k; t(3^k) = s(k)$ and the equation becomes:

$$\begin{aligned} t(n) &= s(k) = 3s(k-1) + \frac{3^k}{ck} \\ &= \frac{3^k}{ck} + 3\frac{3^{k-1}}{c(k-1)} + ... \\ &= \frac{1}{c}3^k \sum_{i=1}^{k} \frac{1}{i} \\ &= \Theta(3^k \lg k) \\ &= \Theta(n \lg \lg n) \end{aligned}$$

   Here $\lg n = \frac{\log_3 n}{\log_2 3}; c = \frac{1}{\log_2 3}$.

c $t(n) = 4t(\frac{n}{2}) + n^2\sqrt{n}$

$f(n) = n^{2.5} = \Omega(n^2); af(\frac{n}{b}) = 4(\frac{n}{2})^{2.5} = \frac{n^{2.5}}{\sqrt{2}} \leq 0.8n^{2.5}$

Hence this is case 3 of master threorem and so the result is $t(n) = \Theta(n^{2.5})$.

f $t(n) = t(\frac{n}{2}) + t(\frac{n}{4}) + t(\frac{n}{8}) + n$

Substitution Method: Since $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} < 1$, guess $t(n) \leq cn$ and prove by induction. Using induction hypothesis

$$\begin{aligned} t(n) &\leq (\frac{1}{2} + \frac{1}{4} + \frac{1}{8})cn + n \\ &= \frac{7}{8}cn + n \\ &\leq cn \text{ for } c \geq 8 \end{aligned}$$

This completes the proof.

4. (4-4) (page 108)Fibonacci Sequence:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_i &= F_{i-1} + F_{i-2} \qquad i \geq 2 \end{aligned}$$

(a)

$$\begin{aligned} \mathcal{F}(z) &= \sum_{i=0}^{\infty} F_i z^i = z + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) z^i \\ &= z + z\sum_{i=0}^{\infty} F_i z^i + z^2 \sum_{i=0}^{\infty} F_i z^i \\ &= z + z\mathcal{F}(z) + z^2\mathcal{F}(z) \\ &= \frac{z}{1 - z - z^2} \\ &= \frac{z}{(1-\phi z)(1-\hat{\phi} z)} \end{aligned}$$

the last of these follows from the fact that $\frac{1}{\phi}$ and $\frac{1}{\hat{\phi}}$ are the roots of $[1 - z - z^2 = 0]$.

(b)

$$\frac{z}{(1-\phi z)(1-\hat{\phi} z)} = \frac{1}{\sqrt{5}}(\frac{1}{(1-\phi z)} - \frac{1}{(1-\hat{\phi} z)})$$

$\frac{1}{(1-\phi z)} = \sum_{i=0}^{\infty} \phi^i z^i$ and $\frac{1}{(1-\hat{\phi} z)} = \sum_{i=0}^{\infty} \hat{\phi}^i z^i$. Hence

(c)

$$\mathcal{F}(z) = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} (\phi^i - \hat{\phi}^i) z^i$$

(d) Hence

$$F_i = \frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i)$$

and $\left|\hat{\phi}^i\right| < 1$. Hence $F_i = [\frac{\phi^i}{\sqrt{5}}]$.

(e) Want to show that $F_{i+2} \geq \phi^i$ for $i \geq 0$

We do this by induction. Clearly true for $i = 0$. Suppose it is true for $i \leq k$ Will show for $i = k + 1$.

$F_{k+3} = F_{k+1} + F_{k+2} \geq \phi^{k-1} + \phi^k = \phi^{k+1}[\frac{1}{\phi^2} + \frac{1}{\phi}] = \phi^{k+1}[\frac{\phi+1}{\phi^2}] = \phi^{k+1}$.

$$\begin{aligned} \frac{1}{\phi^2} &= 1 - \frac{1}{\phi} \\ \phi^2 &= 1 + \phi \\ [\frac{\phi+1}{\phi^2}] &= 1 \end{aligned}$$

(Recall $\frac{1}{\phi}$ is a solution of the equation

$$1 - z - z^2 = 0$$

The result now follows.