Modelling Cavity Enhanced Raman Heterodyne

May 22, 2019

1 Setup

1.1 Set the fineness of the grids used in the calculations

The variable `patience` determines how long the calculations will take to run

- 0 - Coarse, and quick, results dubious but it runs fast (seconds)
- 1 - Results meaningful, but small number of points on graphs. (~10 min)
- 2 - Fine grids for pretty graphs. (~1 hour)

```
In [1]: patience  = 2
```

1.2 Import the required libraries

```
In [2]: import sympy as sym
        sym.init_printing()
        from sympy import I, Matrix, symbols
        from sympy.physics.quantum import TensorProduct, Dagger
        from sympy.printing import print_ccode

        import numpy as np

        import math
        from math import pi

        from scipy import integrate, signal
        import scipy.optimize
        import scipy.constants as const
        import scipy.io as sio

        import matplotlib.pyplot as plt
        from matplotlib.colors import Normalize as Norm

        import qutip

        import time

        %load_ext cython
```

1.3 Some functions to help us make Liovillian superoperators

- Here we use an approach very similar to qutip and the qotoolbox, reading the documention for those two packages might help make sense of spre, spost, etc
- This notebook uses Fortran order (column first) when flattening matricies

```
In [3]: def spre(m):
            return TensorProduct(sym.eye(m.shape[0]), m)


        def spost(m):
            return TensorProduct(m.T, sym.eye(m.shape[0]))


        def collapse(c):
            temp = Dagger(c)*c/2
            return spre(c)*spost(Dagger(c)) - spre(temp) - spost(temp)
```

1.4 Define atomic operators

For example s13 is $|1\rangle\langle 3|$

```
In [4]: s13 = Matrix([[0, 0, 1], [0, 0, 0], [0, 0, 0]])
        s23 = Matrix([[0, 0, 0], [0, 0, 1], [0, 0, 0]])
        s12 = Matrix([[0, 1, 0], [0, 0, 0], [0, 0, 0]])

        s31 = s13.T
        s32 = s23.T
        s21 = s12.T

        s11 = s12*s21
        s22 = s21*s12
        s33 = s31*s13
```

1.5 The symbolic variables we will use

- δ_2, δ_3 -- the detunings of the second and third atomic level respectively
- γ_{13}, γ_{23} -- respectively the spontaneous emission rates from the excited state down the ground state and down to the second atomic level.
- γ_{2d}, γ_{3d} -- the dephasing rates for the second atomic level and the optical excited state.
- n_b -- the number of thermal photons in the bath coupled to the microwave transition
- γ_μ -- coupling rate between the $|1\rangle \rightarrow |2\rangle$ transition and this bath.
- Ω_o -- rabi frequency for the optical pump (drives the $|2\rangle \rightarrow |3\rangle$ transition) taken to be real
- Ω_μ -- rabi frequency for the microwave input (drives the $|1\rangle \rightarrow |2\rangle$ transition)
- ρ_{xy} -- components of the atomic density matrix
- a -- cavity mode amplitude (scale is chosen like it's the operator a even though we only treat it classically)
- a_r, a_i -- real and imaginary parts of a

- g -- coupling rate of a single atom between the $|1\rangle \rightarrow |3\rangle$ transition and the output optical cavity mode.

In [5]:
```
delta2, delta3 = sym.symbols('delta_2 delta_3', real=True)

(gamma13, gamma23, gamma2d,
 gamma3d, nbath, gammamu) = sym.symbols(
    'gamma_13 gamma_23 gamma_2d gamma_3d n_b gamma_mu',
    real=True, negative=False)

omegao = sym.symbols('Omega_o', real=True, negative=False)
omegam = sym.symbols('Omega_mu', real=True, negative=False)
# omegamr, omegami = sym.symbols('Omega_mu_r Omega_mu_i',
#                                                 real = True)

(rho11, rho12, rho13, rho21, rho22,
 rho23, rho31, rho32, rho33) = sym.symbols(
    'rho_11 rho_12 rho_13 rho_21 rho_22 rho_23 rho_31 rho_32 rho_33')

a = sym.symbols('a')
ar, ai = sym.symbols('a_r a_i', real=True)

g = sym.symbols('g', real=True, negative=False)
```

2 Constructing the Liovillian analytically

In [6]:
```
# Make the Hamiltonian
H = omegam*s21 + omegao*s32 + g*a*s31
H = H + Dagger(H)
H = H + delta2*s22 + delta3*s33

# Make the Liovillian
LH = -I*spre(H)+I*spost(H)
L21 = gammamu*(nbath+1)*collapse(s12)
L12 = gammamu*(nbath)*collapse(s21)
L32 = gamma23*collapse(s23)
L31 = gamma13*collapse(s13)
L22 = gamma2d*collapse(s22)
L33 = gamma3d*collapse(s33)

L = LH + L21 + L12 + L32 + L31 + L22 + L33

# Rather than use the actual Liovillian we will use a modified Liovillian
# with the first row replaced a collections of ones and zeros that calculates
# what the trace of the density matrix would have been before it was flattened

L = L.row_insert(0, Matrix([[1, 0, 0, 0, 1, 0, 0, 0, 1]]))
```

```
        L.row_del(1)

        # The modified Liovillian is a  matrix that can be inverted
        # The condition for the steady state is now L*rho=V
        # where V = [1,0,0, 0,0,0, 0,0,0].T
```

In [7]:
```
        # The Liovillian is presented in chunks because it is
        # too wide to show together when exporting this notebook to PDF
        L[:, 0:3]
```

Out[7]:

$$\begin{bmatrix} 1 & 0 & 0 \\ -i\Omega_\mu & -i\delta_2 - \frac{\gamma_{2d}}{2} - \frac{\gamma_\mu n_b}{2} - \frac{\gamma_\mu}{2}(n_b+1) & -i\Omega_o \\ -ig & -i\Omega_o & -i\delta_3 - \frac{\gamma_{13}}{2} - \frac{\gamma_{23}}{2} - \frac{\gamma_{3d}}{2} - \frac{\gamma_\mu n_b}{2} \\ i\Omega_\mu & 0 & 0 \\ \gamma_\mu n_b & i\Omega_\mu & 0 \\ 0 & 0 & i\Omega_\mu \\ ig\bar{a} & 0 & 0 \\ 0 & ig\bar{a} & 0 \\ 0 & 0 & ig\bar{a} \end{bmatrix}$$

In [8]:
```
L[:, 3:6]
```

Out[8]:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & i\Omega_\mu & 0 \\ 0 & 0 & i\Omega_\mu \\ i\delta_2 - \frac{\gamma_{2d}}{2} - \frac{\gamma_\mu n_b}{2} - \frac{\gamma_\mu}{2}(n_b+1) & -i\Omega_\mu & -ig\bar{a} \\ -i\Omega_\mu & -\gamma_\mu(n_b+1) & -i\Omega_o \\ -ig & -i\Omega_o & i\delta_2 - i\delta_3 - \frac{\gamma_{13}}{2} - \frac{\gamma_{23}}{2} - \frac{\gamma_{2d}}{2} - \frac{\gamma_{3d}}{2} - \frac{\gamma_\mu}{2}(n_b+1) \\ i\Omega_o & 0 & 0 \\ 0 & i\Omega_o & 0 \\ 0 & 0 & i\Omega_o \end{bmatrix}$$

In [9]:
```
L[:, 6:9]
```

Out[9]:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & ig & 0 \\ 0 & 0 & ig \\ i\Omega_o & 0 & 0 \\ 0 & i\Omega_o & \gamma_{23} \\ 0 & 0 & i\Omega_o \\ i\delta_3 - \frac{\gamma_{13}}{2} - \frac{\gamma_{23}}{2} - \frac{\gamma_{3d}}{2} - \frac{\gamma_\mu n_b}{2} & -i\Omega_\mu & -ig\bar{a} \\ -i\Omega_\mu & -i\delta_2 + i\delta_3 - \frac{\gamma_{13}}{2} - \frac{\gamma_{23}}{2} - \frac{\gamma_{2d}}{2} - \frac{\gamma_{3d}}{2} - \frac{\gamma_\mu}{2}(n_b+1) & -i\Omega_o \\ -ig & -i\Omega_o & -\gamma_{13} - \gamma_{23} \end{bmatrix}$$

In [10]: H

Out[10]:

$$\begin{bmatrix} 0 & \Omega_\mu & g\bar{a} \\ \Omega_\mu & \delta_2 & \Omega_o \\ ag & \Omega_o & \delta_3 \end{bmatrix}$$

```
In [11]: rho = Matrix([[rho11, rho21, rho31],
                       [rho12, rho22, rho32],
                       [rho13, rho23, rho33]])
         rho
```

Out[11]:

$$\begin{bmatrix} \rho_{11} & \rho_{21} & \rho_{31} \\ \rho_{12} & \rho_{22} & \rho_{32} \\ \rho_{13} & \rho_{23} & \rho_{33} \end{bmatrix}$$

```
In [12]: # rho flattened out to a vector
         rho = 1*rho.T  # because we are using "fortran" style matrix flatteneing
         rho[:]
         rhoflat = 1*rho.T
         rhoflat = rhoflat[:]
         rhoflat
```

Out[12]:

$$[\rho_{11},\ \rho_{21},\ \rho_{31},\ \rho_{12},\ \rho_{22},\ \rho_{32},\ \rho_{13},\ \rho_{23},\ \rho_{33}]$$

2.1 Change of basis

We now transform the "basis" we use for our flattened density matrix so that all the elements are real. This means that we are solving a 9x9 system with real varaiables rather than a 9x9 system with complex variables and redundancy.

```
In [13]: CtoR = Matrix([[2, 0, 0, 0, 0, 0, 0, 0, 0],
                        [0, 0, 0, 0, 2, 0, 0, 0, 0],
                        [0, 0, 0, 0, 0, 0, 0, 0, 2],
                        [0, 1, 0, 1, 0, 0, 0, 0, 0],
                        [0, I, 0, -I, 0, 0, 0, 0, 0],
                        [0, 0, 1, 0, 0, 0, 1, 0, 0],
                        [0, 0, I, 0, 0, 0, -I, 0, 0],
                        [0, 0, 0, 0, 0, 1, 0, 1, 0],
                        [0, 0, 0, 0, 0, I, 0, -I, 0]
                        ])/2
```

```
In [14]: # This is our new vector to describe the atom density matrix
         t = CtoR*Matrix(rhoflat)
         t
```

Out[14]:

$$\begin{bmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{33} \\ \frac{\rho_{12}}{2} + \frac{\rho_{21}}{2} \\ -\frac{i\rho_{12}}{2} + \frac{i\rho_{21}}{2} \\ \frac{\rho_{13}}{2} + \frac{\rho_{31}}{2} \\ -\frac{i\rho_{13}}{2} + \frac{i\rho_{31}}{2} \\ \frac{\rho_{23}}{2} + \frac{\rho_{32}}{2} \\ -\frac{i\rho_{23}}{2} + \frac{i\rho_{32}}{2} \end{bmatrix}$$

In [15]:
```
# Our modified Liovillian in this new basis
Lreal = sym.simplify(CtoR*L*CtoR.inv())
Lreal = Lreal.subs(a, ar+I*ai)
```

In [16]:
```
# The Lreal matrix in chuncks because it gets truncated exporting to pdf
Lreal[:, 0:6]
```

Out[16]:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ \gamma_\mu n_b & -\gamma_\mu\left(n_b+1\right) & \gamma_{23} & 0 & 2\Omega_\mu & 0 \\ 0 & 0 & -\gamma_{13}-\gamma_{23} & 0 & 0 & 2a_i g \\ 0 & 0 & 0 & -\frac{\gamma_{2d}}{2}-\gamma_\mu n_b-\frac{\gamma_\mu}{2} & -\delta_2 & 0 \\ \Omega_\mu & -\Omega_\mu & 0 & \delta_2 & -\frac{\gamma_{2d}}{2}-\gamma_\mu n_b-\frac{\gamma_\mu}{2} & \Omega_o \\ a_i g & 0 & -a_i g & 0 & -\Omega_o & -\frac{\gamma_{13}}{2}-\frac{\gamma_{23}}{2}-\frac{\gamma_{3d}}{2}-\frac{\gamma_\mu n_b}{2} \\ a_r g & 0 & -a_r g & \Omega_o & 0 & \delta_3 \\ 0 & 0 & 0 & a_i g & a_r g & 0 \\ 0 & \Omega_o & -\Omega_o & a_r g & -a_i g & -\Omega_\mu \end{bmatrix}$$

In [17]:
```
Lreal[:, 6:9]
```

Out[17]:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\Omega_o \\ 2a_r g & 0 & 2\Omega_o \\ -\Omega_o & -a_i g & -a_r g \\ 0 & -a_r g & a_i g \\ -\delta_3 & 0 & \Omega_\mu \\ -\frac{\gamma_{13}}{2}-\frac{\gamma_{23}}{2}-\frac{\gamma_{3d}}{2}-\frac{\gamma_\mu n_b}{2} & -\Omega_\mu & 0 \\ \Omega_\mu & -\frac{\gamma_{13}}{2}-\frac{\gamma_{23}}{2}-\frac{\gamma_{2d}}{2}-\frac{\gamma_{3d}}{2}-\frac{\gamma_\mu}{2}\left(n_b+1\right) & \delta_2-\delta_3 \\ 0 & -\delta_2+\delta_3 & -\frac{\gamma_{13}}{2}-\frac{\gamma_{23}}{2}-\frac{\gamma_{2d}}{2}-\frac{\gamma_{3d}}{2}-\frac{\gamma_\mu}{2}\left(n_b+1\right) \end{bmatrix}$$

2.2 C-code from our analytic expressions

The output of this is copied an pasted into some of the functions that follow. Because it is destined for cython it's in row-major order

```
In [18]: output = 'truncated'     # truncate output
         # output = 'full'        # uncomment this line for full output

         Lflatreal = 1*Lreal.T
         Lflatreal = Lflatreal[:]

         for k in range(81):
             if k < 10 or output == 'full':
                 print("L[%d] = " % (k,), end="")
                 print_ccode(Lflatreal[k])

         if output == 'truncated':
             print('...')
```

```
L[0] = 1
L[1] = gamma_mu*n_b
L[2] = 0
L[3] = 0
L[4] = Omega_mu
L[5] = a_i*g
L[6] = a_r*g
L[7] = 0
L[8] = 0
L[9] = 1
...
```

3 Numerical code for calulating steady state density matrix

3.1 Python versions

This code uses the analytic expressions above to generate the modified Liovillian and then numerically solves the corrseponding system of equations to find the steady state density. This code is much slower than the cython versions below, but useful for confirming the correctness of the cython versions.

```
In [19]: Lfunc = sym.lambdify((a, delta2, delta3, gamma13,
                                gamma23, gamma2d, gamma3d,
                                nbath, gammamu, omegao, omegam, g), L)

         Lrealfunc = sym.lambdify((ar, ai, delta2, delta3,
                                   gamma13, gamma23, gamma2d,
                                   gamma3d, nbath, gammamu,
                                   omegao, omegam, g), Lreal)
```

```
# This function finds the steady state density matrix
# the input p is a python dictionary full of parameters.


def steady_rho_python(p):
    Lmatrix = Lfunc(p['a'], p['delta2'], p['delta3'],
                    p['gamma13'], p['gamma23'], p['gamma2d'],
                    p['gamma3d'], p['nbath'], p['gammamu'],
                    p['omegao'], p['omegam'], p['g'])
    rho = np.linalg.solve(Lmatrix.astype('complex128'),
                          np.matrix([[1, 0, 0, 0, 0, 0, 0, 0, 0]]).T)
    return np.reshape(rho, (3, 3), order='F')
```

3.2 Cython versions

Implemented in cython to (greatly) increase speed.

```
In [20]: %%cython --compile-args=-fopenmp --link-args=-fopenmp --verbose --force
         # %%cython -a # this gives "highligted" output
         # cython: boundscheck=False, wraparound=False, nonecheck=False

         cimport cython
         import numpy as np
         from cython.parallel cimport prange
         cimport scipy.linalg.cython_lapack as lapack
         import numpy as np
         from libc.math cimport exp


         # This function accepts a python dictionary of parameters (p) and returns
         # the steady state value of the the rho_13 density matrix element.

         cdef double complex steady_rho13_lots_of_args(double a_r,
                                                       double a_i,
                                                       double delta_2,
                                                       double delta_3,
                                                       double gamma_13,
                                                       double gamma_23,
                                                       double gamma_2d,
                                                       double gamma_3d,
                                                       double n_b,
                                                       double gamma_mu,
                                                       double Omega_o,
                                                       double Omega_mu,
                                                       double g) nogil:
             cdef double L[81]
             cdef double V[9]
```

```
cdef int n = 9
cdef double complex a
cdef int info
cdef int lda = 9
cdef int ldb = 9
cdef int nrhs = 1
cdef int workspace[9]
cdef int k

# Unfortunately we cant use this syntax if we dont have the GIL
#V[:] = [1,0,0, 0,0,0, 0,0,0]
#L[:] = (1, gamma_mu*n_b, 0, 0, ...

for k in range(9):
    V[k] = 0
V[0] = 1

L[0] = 1
L[1] = gamma_mu*n_b
L[2] = 0
L[3] = 0
L[4] = Omega_mu
L[5] = a_i*g
L[6] = a_r*g
L[7] = 0
L[8] = 0
L[9] = 1
L[10] = -gamma_mu*(n_b + 1)
L[11] = 0
L[12] = 0
L[13] = -Omega_mu
L[14] = 0
L[15] = 0
L[16] = 0
L[17] = Omega_o
L[18] = 1
L[19] = gamma_23
L[20] = -gamma_13 - gamma_23
L[21] = 0
L[22] = 0
L[23] = -a_i*g
L[24] = -a_r*g
L[25] = 0
L[26] = -Omega_o
L[27] = 0
L[28] = 0
L[29] = 0
L[30] = -1.0/2.0*gamma_2d - gamma_mu*n_b - 1.0/2.0*gamma_mu
```

```
L[31] = delta_2
L[32] = 0
L[33] = Omega_o
L[34] = a_i*g
L[35] = a_r*g
L[36] = 0
L[37] = 2*Omega_mu
L[38] = 0
L[39] = -delta_2
L[40] = -1.0/2.0*gamma_2d - gamma_mu*n_b - 1.0/2.0*gamma_mu
L[41] = -Omega_o
L[42] = 0
L[43] = a_r*g
L[44] = -a_i*g
L[45] = 0
L[46] = 0
L[47] = 2*a_i*g
L[48] = 0
L[49] = Omega_o
L[50] = -1.0/2.0*gamma_13 - 1.0/2.0*gamma_23 - \
    1.0/2.0*gamma_3d - 1.0/2.0*gamma_mu*n_b
L[51] = delta_3
L[52] = 0
L[53] = -Omega_mu
L[54] = 0
L[55] = 0
L[56] = 2*a_r*g
L[57] = -Omega_o
L[58] = 0
L[59] = -delta_3
L[60] = -1.0/2.0*gamma_13 - 1.0/2.0*gamma_23 - \
    1.0/2.0*gamma_3d - 1.0/2.0*gamma_mu*n_b
L[61] = Omega_mu
L[62] = 0
L[63] = 0
L[64] = 0
L[65] = 0
L[66] = -a_i*g
L[67] = -a_r*g
L[68] = 0
L[69] = -Omega_mu
L[70] = -1.0/2.0*gamma_13 - 1.0/2.0*gamma_23 - 1.0/2.0 * \
    gamma_2d - 1.0/2.0*gamma_3d - 1.0/2.0*gamma_mu*(n_b + 1)
L[71] = -delta_2 + delta_3
L[72] = 0
L[73] = -2*Omega_o
L[74] = 2*Omega_o
L[75] = -a_r*g
```

```
    L[76] = a_i*g
    L[77] = Omega_mu
    L[78] = 0
    L[79] = delta_2 - delta_3
    L[80] = -1.0/2.0*gamma_13 - 1.0/2.0*gamma_23 - 1.0/2.0 * \
        gamma_2d - 1.0/2.0*gamma_3d - 1.0/2.0*gamma_mu*(n_b + 1)
    lapack.dgesv( & n, & nrhs, L, & lda, workspace, V, & ldb, & info)
    return V[5]+1j*V[6]


# Steady state an ensemble of atoms, returns a matrix of rho13 values as
# well as the list of weights from the inhomogeneous broadening distribution.
# Convolve the two to get simulate single pass Raman heterodyne signal
# as a function of frequency for the optically thin case.

def ensemble_steady_rho13(p, delta2vals, delta3vals):
    cdef double a_r, a_i, delta_2, delta_3, gamma_13,
    cdef double gamma_23, gamma_2d, gamma_3d, n_b,
    cdef double gamma_mu, Omega_o, Omega_mu, g
    a = p['a']
    a_r = a.real
    a_i = a.imag
    delta_2 = p['delta2']
    delta_3 = p['delta3']
    gamma_13 = p['gamma13']
    gamma_23 = p['gamma23']
    gamma_2d = p['gamma2d']
    gamma_3d = p['gamma3d']
    n_b = p['nbath']
    gamma_mu = p['gammamu']
    Omega_o = p['omegao']
    Omega_mu = p['omegam']
    g = p['g']

    cdef int ii
    cdef int jj

    cdef double m2 = p['mean_delta2']
    cdef double sd2 = p['sd_delta2']
    cdef double m3 = p['mean_delta3']
    cdef double sd3 = p['sd_delta3']
    # print(sd3)

    cdef double[:] delta2valsv = delta2vals
    cdef double[:] delta3valsv = delta3vals

    cdef int ndelta2 = len(delta2vals)
    cdef int ndelta3 = len(delta3vals)
```

```
    cdef double df2 = delta2valsv[1]-delta2valsv[0]
    cdef double df3 = delta3valsv[1]-delta3valsv[0]

    # make numpy array
    coh13 = np.zeros((ndelta2, ndelta3), dtype='complex128')
    # make cython  "view" of the data
    cdef double complex[:, :] coh13v = coh13

    # make numpy array for gaussian weights
    w = np.zeros((ndelta2, ndelta3), dtype='complex128')
    # make cython  "view" of the data
    cdef double complex[:, :] wv = w

    for ii in prange(ndelta2, nogil=True):
        # for ii in range(ndelta2):
        delta_2 = delta2valsv[ii]
        for jj in range(ndelta3):
            delta_3 = delta3valsv[jj]
            coh13v[ii, jj] = steady_rho13_lots_of_args(a_r,
                                                       a_i,
                                                       delta_2,
                                                       delta_3,
                                                       gamma_13,
                                                       gamma_23,
                                                       gamma_2d,
                                                       gamma_3d,
                                                       n_b,
                                                       gamma_mu,
                                                       Omega_o,
                                                       Omega_mu,
                                                       g)
            wv[ii, jj] = 1.0/(2*3.14159265358979323846*sd3*sd2) * \
                exp(-(delta_2-m2)**2/(2*sd2*sd2)-(delta_3-m3)**2/(2*sd3*sd3))
    return (coh13, w)


# Steady state of an ensemble of atoms, returns a matrix of rho values as well
# as the list of weights from the inhomogeneous broadening distribution.
# Convolve the two to simulate single pass raman heterodyne signal
# as a function of frequency

# p is a dictionary of parameters
# delta2vals and delta3vals are lists of numbers to use for the detunings

def ensemble_steady_rho(p, delta2vals, delta3vals):
    cdef double a_r, a_i, delta_2, delta_3, gamma_13, gamma_23,
    cdef double gamma_2d, gamma_3d, n_b, gamma_mu, Omega_o, Omega_mu, g
```

```
a = p['a']
a_r = a.real
a_i = a.imag
delta_2 = p['delta2']
delta_3 = p['delta3']
gamma_13 = p['gamma13']
gamma_23 = p['gamma23']
gamma_2d = p['gamma2d']
gamma_3d = p['gamma3d']
n_b = p['nbath']
gamma_mu = p['gammamu']
Omega_o = p['omegao']
Omega_mu = p['omegam']
g = p['g']

cdef int ii
cdef int jj
cdef int kk

cdef double m2 = p['mean_delta2']
cdef double sd2 = p['sd_delta2']
cdef double m3 = p['mean_delta3']
cdef double sd3 = p['sd_delta3']
# print(sd3)

cdef double[:] delta2valsv = delta2vals
cdef double[:] delta3valsv = delta3vals

cdef int ndelta2 = len(delta2vals)
cdef int ndelta3 = len(delta3vals)

cdef double df2 = delta2valsv[1]-delta2valsv[0]
cdef double df3 = delta3valsv[1]-delta3valsv[0]

rho = np.zeros((ndelta2, ndelta3, 9),
               dtype='complex128')  # make numpy array
# make cython  "view" of the data
cdef double complex[:, :, :] rhov = rho

tmp = np.zeros((9,), dtype='complex128')  # make numpy array
# make cython  "view" for temporary array
cdef double complex[:] tmpv = tmp

# make numpy array for gaussian weights
w = np.zeros((ndelta2, ndelta3), dtype='complex128')
# make cython  "view" of the data
cdef double complex[:, :] wv = w
```

```
    for ii in prange(ndelta2, nogil=True):
        # for ii in range(ndelta2):
        delta_2 = delta2valsv[ii]
        for jj in range(ndelta3):
            delta_3 = delta3valsv[jj]
            for kk in range(9):
                rhov[ii, jj, kk] = tmpv[kk]
            wv[ii, jj] = 1.0/(2*3.14159265358979323846*sd3*sd2) * \
                exp(-(delta_2-m2)**2/(2*sd2*sd2)-(delta_3-m3)**2/(2*sd3*sd3))
    return (rho, w)


# Integrate over an ensemble using Simpson's rule to find the
# net coherence of the ensemble of atoms

# p is a dictionary of parameters
# df is the frequency resolution used when
# integrating over the inhomogeneous broadenings


cpdef ensemble_steady_rho13_integrated(p, double df):
    cdef double a_r, a_i, delta_2, delta_3, gamma_13, gamma_23
    cdef double gamma_2d, gamma_3d, n_b, gamma_mu, Omega_o, Omega_mu, g
    a = p['a']
    a_r = a.real
    a_i = a.imag
    delta_2 = p['delta2']
    delta_3 = p['delta3']
    gamma_13 = p['gamma13']
    gamma_23 = p['gamma23']
    gamma_2d = p['gamma2d']
    gamma_3d = p['gamma3d']
    n_b = p['nbath']
    gamma_mu = p['gammamu']
    Omega_o = p['omegao']
    Omega_mu = p['omegam']
    g = p['g']

    cdef int ii
    cdef int jj

    cdef double m2 = p['mean_delta2']
    cdef double sd2 = p['sd_delta2']
    cdef double m3 = p['mean_delta3']
    cdef double sd3 = p['sd_delta3']

    cdef int w2, w3
```

```
cdef int ndelta2 = 2*int(3*sd2/df)+1
cdef int ndelta3 = 2*int(3*sd3/df)+1
# print(ndelta2,ndelta3)

delta2vals = np.linspace(-3, 3, ndelta2)*sd2+m2
delta3vals = np.linspace(-3, 3, ndelta3)*sd3+m3

cdef double[:] delta2valsv = delta2vals
cdef double[:] delta3valsv = delta3vals

simpsonw2 = np.empty((ndelta2,), dtype='int32')
cdef int[:] simpsonw2v = simpsonw2
simpsonw3 = np.empty((ndelta3,), dtype='int32')
cdef int[:] simpsonw3v = simpsonw3

ii = 0
while(ii < (ndelta2-1)):
    simpsonw2v[ii] = 2
    ii += 1
    simpsonw2v[ii] = 4
    ii += 1
simpsonw2v[0] = 1
simpsonw2v[ndelta2-1] = 1

ii = 0
while(ii < (ndelta3-1)):
    simpsonw3v[ii] = 2
    ii += 1
    simpsonw3v[ii] = 4
    ii += 1
simpsonw3v[0] = 1
simpsonw3v[ndelta3-1] = 1

cdef double complex z = 0
cdef double sum_r = 0
cdef double sum_i = 0
cdef double gaussie
for ii in prange(ndelta2, nogil=True):
    # for ii in range(ndelta2):
    delta_2 = delta2valsv[ii]
    w2 = simpsonw2v[ii]
    for jj in range(ndelta3):
        delta_3 = delta3valsv[jj]
        w3 = simpsonw3v[jj]
        gaussie = 1.0/(2*3.14159265358979323846*sd3*sd2) * \
            exp(-(delta_2-m2)**2/(2*sd2*sd2)-(delta_3-m3)**2/(2*sd3*sd3))
        z = steady_rho13_lots_of_args(a_r,
                                      a_i,
```

```
                                                    delta_2,
                                                    delta_3,
                                                    gamma_13,
                                                    gamma_23,
                                                    gamma_2d,
                                                    gamma_3d,
                                                    n_b,
                                                    gamma_mu,
                                                    Omega_o,
                                                    Omega_mu,
                                                    g)*gaussie*w2*w3
                    # this is because cython doesn't know how to
                    # automatically do reductions for
                    # but does for doubles
                    sum_r += z.real
                    sum_i += z.imag
            return (sum_r+sum_i*1j)*df*df/9.0
```

```
[1/1] Cythonizing /home/jevon/.cache/ipython/cython/_cython_magic ....
```

The (classical) equations of motion of the cavity mode amplitude are given by

$$\frac{d}{dt}a = -i\delta_c a - i\sum_k g_k \sigma_{13,k} - \frac{(\kappa_i + \kappa_c)}{2}a - \sqrt{\kappa_c}a_{in}.$$

Here κ_i is the intrisic loss rate of our cavity and κ_c is the coupling induced loss.

Defining

$$S_{13}(a) = \sum_k g_k \sigma_{13,k}.$$

We write $S_{13}(a)$ to remind us (explicitly) that the atoms states are a functions of the cavity mode amplitude a.

Because we won't drive a this becomes

$$\frac{d}{dt}a = -i\delta_c a - iS_{13}(a) - \frac{(\kappa_i + \kappa_c)}{2}a.$$

So to find the steady state of the cavity we need to find the roots of the function

$$f(a) = -i\delta_c a - iS_{13}(a) - \frac{(\kappa_i + \kappa_c)}{2}a.$$

```
In [21]: # The function "f" above, parameters via the dictionary p
         def ffunc(p):
```

```
    df = p['df']
    a = p['a']
    N = p['N']
    g = p['g']
    deltac = p['deltac']
    kappa = p['kappaoi']+p['kappaoc']
    S13 = N*g*ensemble_steady_rho13_integrated(p, df)
    return -1j*deltac*a - 1j*S13 - kappa/2*a

# The same function but in a form that maps a 2d vector
# to a 2d vector rather than complex -> complex.
# This enables us to use scipy root finding routines.


def ffunc_for_root(avec, p):
    a = avec[0] + 1j*avec[1]
    # print(a)
    p['a'] = a
    f = ffunc(p)
    return [f.real, f.imag]

# Finds the steady state cavity amplitude for given parameters p


def steadya(p):
    # use value of a in p as the initial guess
    a = p['a']
    avec = [a.real, a.imag]
    df = p['df']
    result = scipy.optimize.root(ffunc_for_root, avec, args=(p))
    if not result.success:
        print(result)
        raise(AssertionError(
            'Grrr - The root finder didn\'t converge: '+result.message))
    # update p['a'] why not?
    a = result.x[0] + 1j*result.x[1]
    p['a'] = a
    return a
```

5 Modelling the experiment

5.1 Somewhere to store the results

```
In [22]: saved_output = {}
```

5.2 Anciliary functions

This calculates microwave Rabi frequency from the power incident in dBm

```
In [23]: def omegam_from_Pin(Pin, p):
             mu0 = 4*pi*1e-7
             hbar = 1.05457e-34  # in J*s

             # the volume of the Er:YSO sample
             Vsample = pi*((p['dsample']/2)**2) * p['Lsample']
             V_microwave_cavity = Vsample/p['fillfactor']

             mwP = 1e-3 * 10**(Pin/10)  # convert from dBm to Watts
             Q = 2*pi*p['freqm'] / (p['kappami'] + p['kappamc']*2)  # Q factor
             S21 = (4*p['kappamc']**2) / (p['kappami'] + p['kappamc']*2)**2

             # energy inside the microwave cavity, in J
             energy_in_cavity = math.sqrt(S21)*2 / (2*pi*p['freqm']/Q)*mwP

             # Magnetic field of the microwave
             Bmw = math.sqrt(mu0*(energy_in_cavity/V_microwave_cavity)/2)
             p['omegam'] = (p['mu12']*Bmw)/hbar*(-1)
             return p['omegam']

In [24]: # Calculates the Rabi frequency of the optical pump based
         # on the input power in dBm
         def omegao_from_Pin(Pin, p):
             epsilon0 = 8.854187817e-12
             hbar = 1.05457e-34  # in J*s

             optP = 1e-3 * 10**(Pin/10)  # incident optical power in W
             pflux = optP/(2*pi*p['freqo']*hbar)  # photon flux (photons/sec)
             n_in = pflux * p['kappaoc']*4 / \
                 (p['kappaoc']+p['kappaoi'])**2  # num intracavity photons
             Sspot = pi*p['Wcavity']**2  # cross sectional area of
             # optical mode
             V_cav = (Sspot*p['Lcavity_vac']+Sspot*p['Lsample']*p['nYSO']**3)/2
             optEfield = math.sqrt(n_in*hbar*2*pi*p['freqo']/2/epsilon0/V_cav)
             p['omegao'] = p['d23']*optEfield/hbar*(-1)
             return p['omegao']
```

5.3 Parameters for the experiment

```
In [25]: p = {}
         p['a'] = 0
         p['delta2'] = 0.
         p['delta3'] = 0.


         p['d13'] = 2e-32*math.sqrt(2/3) #in Cm
         p['d23'] = 2e-32*math.sqrt(1/3) #in Cm
```

```
# all rates, and frequencies are in s^-1
p['gamma13'] = p['d13']**2/(p['d13']**2+p['d23']**2)*1/11e-3
p['gamma23'] = p['d23']**2/(p['d13']**2+p['d23']**2)*1/11e-3
p['gamma2d'] = 1e6
p['gamma3d'] = 1e6
p['nbath'] = 20
p['gammamu'] = 1/(p['nbath']+1) * 1e3

p['g'] = 51.9  # optical coupling (s^-1)
p['N'] = 1.28e15  # number of atoms in the optical mode

p['deltac'] = 0  # detuning for optical cavity at sideband frequency
p['deltacm'] = 0  # detuning for microwave cavity at input microwave frequency

p['kappaoc'] = 2*pi*7.95e6  # intrinsic loss for optical resonator
p['kappaoi'] = 2*pi*1.7e6  # coupling loss for optical resonator
p['df'] = 0.1e6  # how small discretization step to take when integrating over the
# inhomogeneous lines

p['mean_delta2'] = 0
p['sd_delta2'] = 2*pi*25e6  # microwave inhomogeneous broadening
# 2.355is to turn FWHM into standard deviation
p['mean_delta3'] = 0
p['sd_delta3'] = 2*pi*170e6  # optical inhomogeneous broadening

p['kappami'] = 650e3*2*pi  # intrinsic loss for microwave cavity
p['kappamc'] = 70e3*2*pi  # coupling loss for optical cavity
# this is for one of the two output ports
p['Nmu'] = 2.22e16  # total number of atoms
p['gmu'] = 1.04  # coupling between atoms and microwave field


muBohr = 927.4009994e-26  # Bohr magneton in J T^-1.
# transition dipole moment for microwave cavity (J T^-1).
p['mu12'] = 4.3803*muBohr

p['Lsample'] = 12e-3  # the length of the sample, in m
p['dsample'] = 5e-3  # the diameter of the sample, in m

p['fillfactor'] = 0.8  # microwave filling factor
p['freqm'] = 5.186e9  # microwave frequency, in Hz
p['freqo'] = 195113.36e9  # optical frequency, in Hz

p['Lcavity_vac'] = 49.5e-3  # length of the vacuum part of the optical
# Fabry Perot (m)
p['Wcavity'] = 0.6e-3  # width of optical resonator beam in sample (m)
p['nYSO'] = 1.76  # refractive index of YSO
```

```
# assume 11mW (=10.4135dBm) input to optical and 0 dBm to microwave cavity
omegao_from_Pin(10.4135, p)
omegam_from_Pin(0, p)

saved_output['p'] = p
```

5.4 Rescaling

In order that the root finder doesn't get confused by very large or very small values we make the substitution

$$g \leftarrow \alpha g$$

$$N \leftarrow \frac{N}{\alpha^2}$$

This means that in order to get the true intracavity field a we need to multiply the results of our calculations by α.

Trial and error suggests that using $\alpha = \sqrt{N}/40$ (the original N) is a good choice.

```
In [26]: alpha = math.sqrt(p['N'])/40
         p['g'] = alpha * p['g']
         p['N'] = p['N']/alpha**2

         # this parameter shouldn't change
         math.sqrt(p['N'])*p['g']/1e9, " GHz"

Out[26]: (1.8568308485158254, ' GHz')
```

5.5 Efficiency of upconversion vs microwave input power

```
In [27]: # Choose the discretization of the x-axis based on
         # how long we want the calculations to take


         if patience <= 0:
             Pmu_vals_dBm = np.arange(-50, 10, 9)  # in dBm
         elif patience <= 1:
             Pmu_vals_dBm = np.arange(-50, 10, 9)  # in dBm
         elif patience <= 2:
             Pmu_vals_dBm = np.arange(-50, 10, 0.5)  # in dBm

         saved_output['Pmu_vals_dBm'] = Pmu_vals_dBm


         Pmu_vals_W = 1e-3*np.exp(np.log(10)*Pmu_vals_dBm/10)  # in W


         # We do the calulations with two different sampling intervals
```

```
        # for the inhomgeneous distribution to makes sure that the
        # integration over the inhomogneous line is accurate.

        if patience == 0:
            dffine = 5e6
            dfcoarse = 5e6
        else:
            dffine = 0.5e6
            dfcoarse = 2e6


        avals_approx = np.zeros(Pmu_vals_dBm.shape, dtype='complex128')
        avals = np.zeros(Pmu_vals_dBm.shape, dtype='complex128')

        truncate_output=True  # Used when exporting to pdf to save pages

        for (i, P) in enumerate(Pmu_vals_dBm):
            if not truncate_output or i<10:
                print("%d/%d" % (i+1, len(Pmu_vals_dBm)), end="")
            omegam_from_Pin(P, p)
            p['df'] = dfcoarse
            avals_approx[i] = steadya(p)*alpha
            p['df'] = dffine
            avals[i] = steadya(p)*alpha
            if not truncate_output or i<10:
                print(" -- completed")

        if truncate_output:
            print(" ...")
```

```
1/120 -- completed
2/120 -- completed
3/120 -- completed
4/120 -- completed
5/120 -- completed
6/120 -- completed
7/120 -- completed
8/120 -- completed
9/120 -- completed
10/120 -- completed
 ...
```

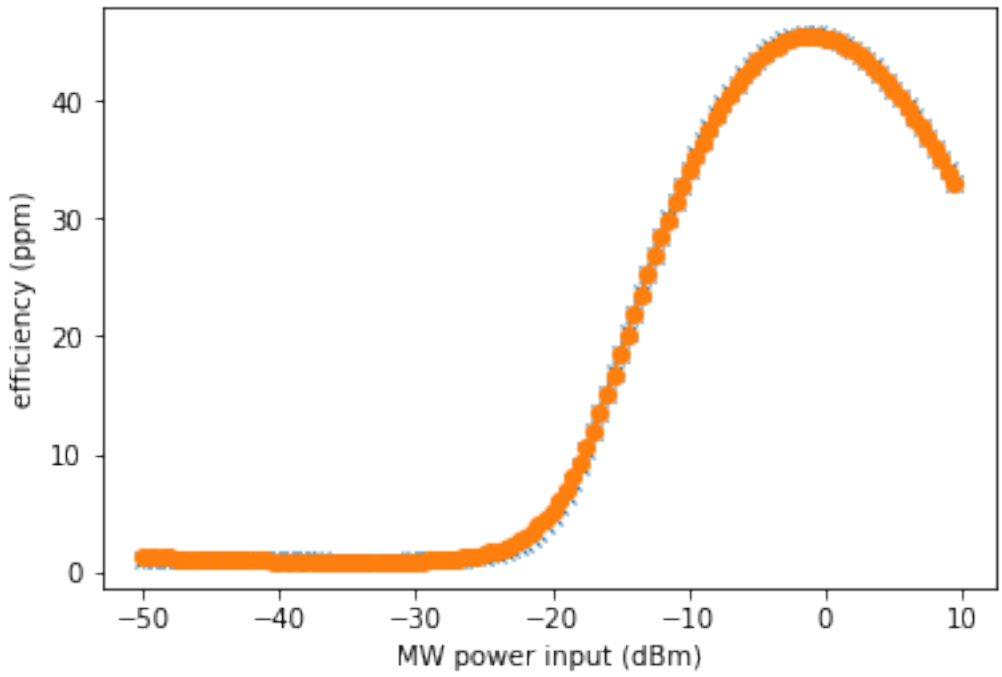
```
In [28]: Pmu_vals_dBm = Pmu_vals_dBm
         Pmu_vals_W = 1e-3*np.exp(np.log(10)*(Pmu_vals_dBm)/10)  # in W

         effic = p['kappaoc']*np.abs(avals**2)*const.h*p['freqo'] / Pmu_vals_W
```

```
effic_approx = p['kappaoc'] * \
    np.abs(avals_approx**2)*const.h*p['freqo'] / Pmu_vals_W

# quantum efficiency
effic = effic*p['freqm']/p['freqo']
effic_approx = effic_approx*p['freqm']/p['freqo']
```

5.5.1 This plot shows the difference between fine and coarser grids

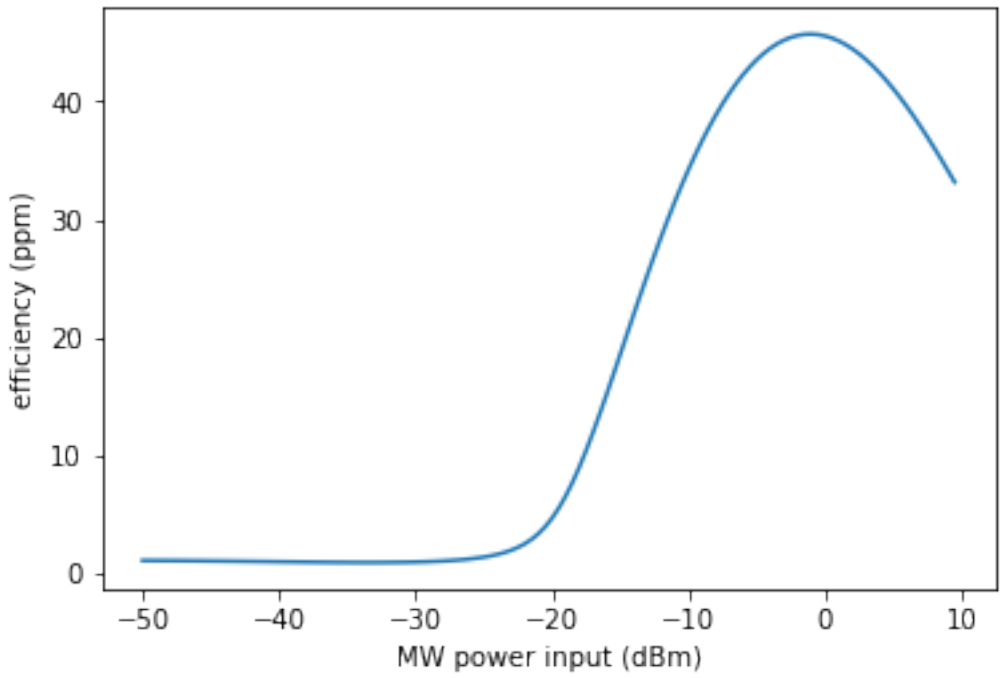
When integrating over the inhomogeneous line

```
In [29]: plt.plot(Pmu_vals_dBm, effic*1e6, 'x-',
                  Pmu_vals_dBm, effic_approx*1e6, 'o-')
         plt.ylabel('efficiency (ppm)')
         plt.xlabel('MW power input (dBm)')

Out[29]: Text(0.5,0,'MW power input (dBm)')
```



5.5.2 Upconversion efficiency versus microwave power (for paper)

```
In [30]: plt.plot(Pmu_vals_dBm, effic*1e6, '-')
         plt.ylabel('efficiency (ppm)')
         plt.xlabel('MW power input (dBm)')
         saved_output['effic'] = effic
```

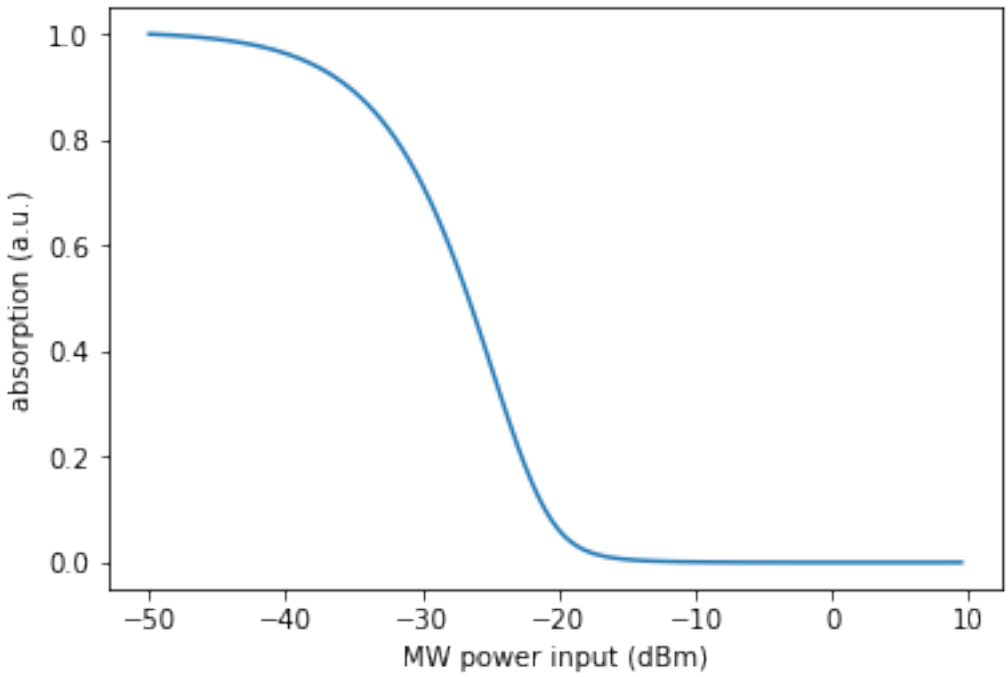


5.5.3 Absorption at the frequency of the upconverted signal as a functionof microwave power (for paper)

In [31]:
```
# fix the delta_3 = 0 (on resonance with the signal frequency)
# but integrate over the inhomogeneous broadening of the microwave transition

df = 1e+6

m2 = p['mean_delta2']
sd2 = p['sd_delta2']

ndelta2 = 2*int(3*sd2/df)+1
delta2vals = np.linspace(-3, 3, ndelta2)*sd2+m2
pop = 0*delta2vals

# absorption at signal frequency
abs_vals = 0*Pmu_vals_dBm


for ii in range(len(Pmu_vals_dBm)):
    p['a'] = avals[ii]/alpha
    omegam_from_Pin(Pmu_vals_dBm[ii], p)
    for jj in range(len(delta2vals)):
        p['delta2'] = delta2vals[jj]
```

```
            rho = steady_rho_python(p)
            pop[jj] = rho[0, 0].real-rho[2, 2].real
            #print('%d %g %g'%(jj,delta2vals[jj],pop[jj]))
        abs_vals[ii] = integrate.simps(pop)


    abs_vals = abs_vals/max(abs_vals)
```

In [32]:
```
plt.plot(Pmu_vals_dBm, abs_vals, '-')
plt.ylabel('absorption (a.u.)')
plt.xlabel('MW power input (dBm)')

saved_output['abs_vals'] = abs_vals
```



5.6 Looking at population distribution

5.6.1 Higher power microwave driving

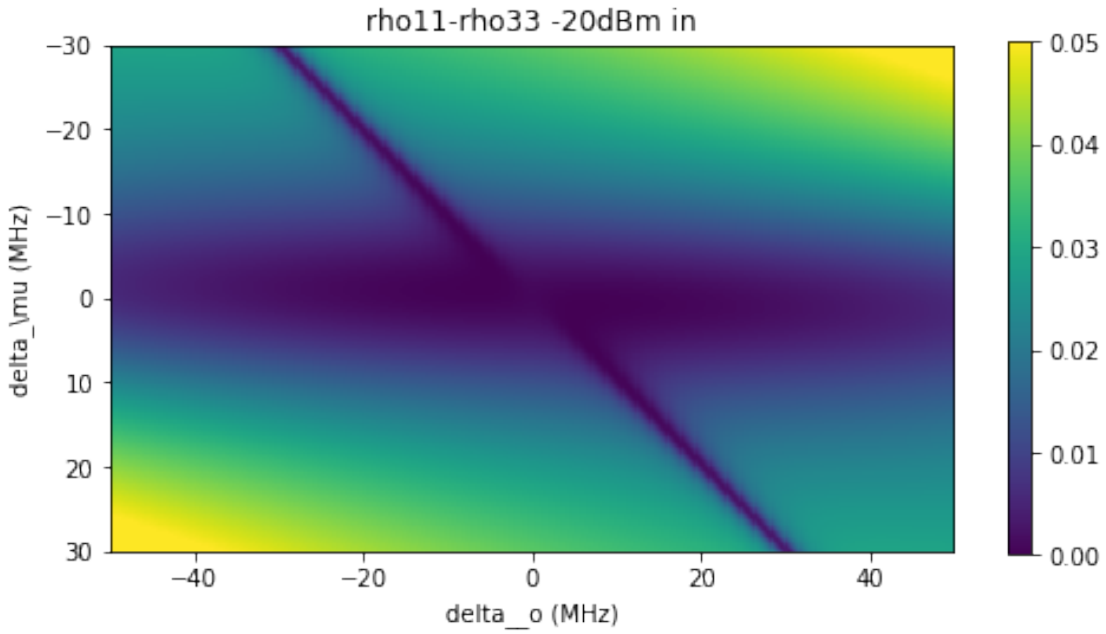
In [33]:
```
# set the microwave power to -20 dBm
omegam_from_Pin(-20, p)
saved_output['highpower'] = -20

p['a'] = 0
p['df'] = 5e6
steadya(p)
```

```
        print(p['a'])
        if patience == 0:
            p['df'] = 2e6
        else:
            p['df'] = 0.5e6

        steadya(p)
        print(p['a'])
```

```
(-1.858229419171164e-15+0.00056684108215345j)
(-5.763143725417061e-15+0.0005822635258006419j)
```

In [34]:
```
        if patience == 2:
            npts = 301
        elif patience == 1:
            npts = 101
        elif patience == 0:
            npts = 101


        delta_o_vals = np.linspace(-50e6, 50e6, npts)
        delta_mu_vals = np.linspace(-30e6, 30e6, npts)

        pop = np.zeros((len(delta_o_vals), len(delta_mu_vals), 3))

        for ii in range(len(delta_o_vals)):
            for jj in range(len(delta_mu_vals)):
                p['delta2'] = delta_mu_vals[ii]
                p['delta3'] = delta_o_vals[jj]-delta_mu_vals[ii]
                rho = steady_rho_python(p)
                for kk in range(3):
                    pop[ii, jj, kk] = rho[kk, kk].real
```

In [35]:
```
        plt.rcParams['figure.figsize'] = [8, 4]
        plt.imshow(pop[:, :, 0]-pop[:, :, 2],
                   extent=[min(delta_o_vals)/1e6,
                           max(delta_o_vals)/1e6,
                           max(delta_mu_vals)/1e6,
                           min(delta_mu_vals)/1e6],
                   vmin=0, vmax=0.05)
        plt.ylabel('delta_\mu (MHz)')
        plt.xlabel('delta__o (MHz)')
        plt.title('rho11-rho33 -20dBm in')
        plt.colorbar()

        saved_output['delta_o_vals'] = delta_o_vals
        saved_output['delta_mu_vals'] = delta_mu_vals
        saved_output['pop_diff_highpower'] = pop[:, :, 0]-pop[:, :, 2]
```



5.6.2 lower microwave power

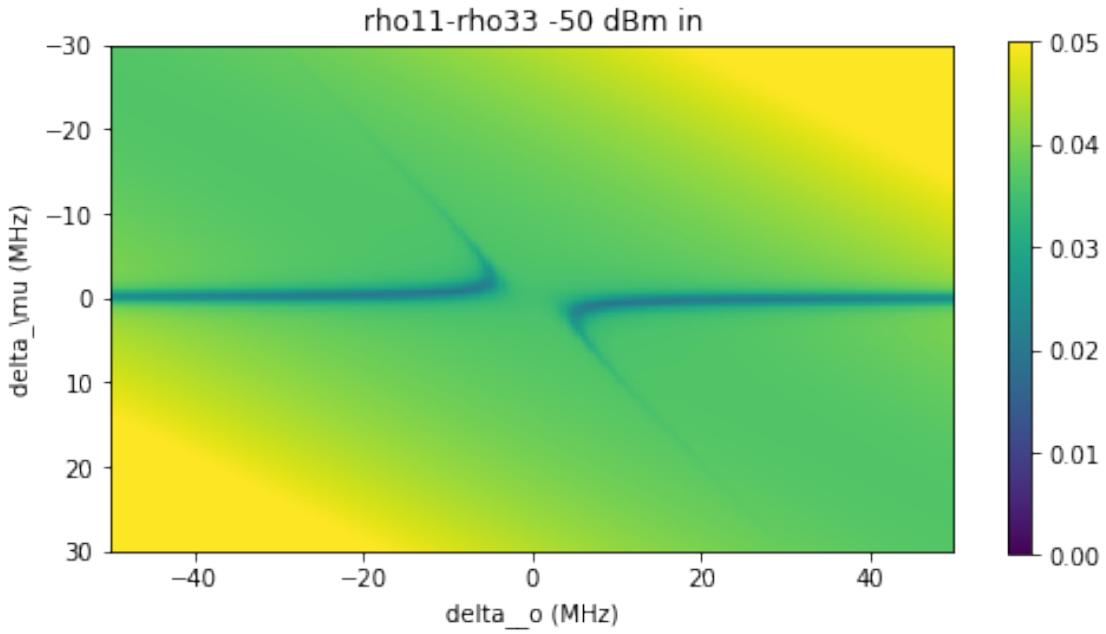
```
In [36]: # set the microwave power to -50dBm
         omegam_from_Pin(-50, p)
         saved_output['lowpower'] = -50

         p['a'] = 0
         p['df'] = 5e6
         steadya(p)
         print(p['a'])
         if patience == 0:
             p['df'] = 2e6
         else:
             p['df'] = 0.5e6

         steadya(p)
         print(p['a'])
```

```
(-2.933700541163359e-19+7.146351291391656e-06j)
(-3.7407054004981156e-19+8.82559818563289e-06j)
```

```
In [37]: if patience == 2:
             npts = 301
         elif patience == 1:
             npts = 101
```

```
        elif patience == 0:
            npts = 101


        delta_o_vals = np.linspace(-50e6, 50e6, npts)
        delta_mu_vals = np.linspace(-30e6, 30e6, npts)

        pop = np.zeros((len(delta_o_vals), len(delta_mu_vals), 3))

        for ii in range(len(delta_o_vals)):
            for jj in range(len(delta_mu_vals)):
                p['delta2'] = delta_mu_vals[ii]
                p['delta3'] = delta_o_vals[jj]-delta_mu_vals[ii]
                rho = steady_rho_python(p)
                for kk in range(3):
                    pop[ii, jj, kk] = rho[kk, kk].real
```

In [38]:
```
        plt.rcParams['figure.figsize'] = [8, 4]
        plt.imshow(pop[:, :, 0]-pop[:, :, 2],
                   extent=[min(delta_o_vals)/1e6,
                           max(delta_o_vals)/1e6,
                           max(delta_mu_vals)/1e6,
                           min(delta_mu_vals)/1e6],
                   vmin=0,vmax=0.05)


        plt.ylabel('delta_\mu (MHz)')
        plt.xlabel('delta__o (MHz)')
        plt.title('rho11-rho33 -50 dBm in')
        plt.colorbar();

        saved_output['delta_o_vals']=delta_o_vals
        saved_output['delta_mu_vals']=delta_mu_vals
        saved_output['pop_diff_lowpower']=pop[:, :, 0]-pop[:, :, 2]
```



6 Save output for later

```
In [39]: filename = time.strftime("saved_output_%Y%m%d_%H%M%S.mat", time.gmtime())
         sio.savemat(filename,saved_output)
         filename
```

```
Out[39]: 'saved_output_20190522_080448.mat'
```