Three level system interacting with fields

Stephen Chen

January 26, 2018

This note is based on the calculation for the generated optical sideband from a three level system driven by an optical pump and a microwave. The model developed here should be of broad range of applications. If you find any mistake, please correct it and leave your name down on the author list.

1 Hamiltonian and master equation

Considering a three level atom driven by an pump optical field and a microwave field, under the rotating frame we can write down the Hamiltonian

$$H = \delta_2\sigma_{22} + \delta_3\sigma_{33} + (\Omega_\mu\sigma_{21} + \Omega_o\sigma_{32} + A\sigma_{31} + \text{H.C.}) \tag{1}$$

where $\sigma_{ij} = |i\rangle\langle j|$, $\hbar\Omega_\mu = \langle 2|\,\mathbf{B}\cdot\mathbf{g}\cdot\mathbf{S}\,|1\rangle$ with $\mathbf{g}$ being the effective g factor, $\mathbf{B}$ being the magnetic field inside the microwave cavity, and $\mathbf{S}$ being the spin operator, $\hbar\Omega_o = E_o d_{23}$ with E_o being the pump electric field inside the optical cavity and d_{23} being the electric dipole moment between the ground state 2 and the excited state 3, and $\hbar A = E_s d_{13}$ with E_s being the electric field of the generated sideband signal inside the optical cavity and d_{13} being the electric dipole moment between the ground state 1 and the excited state 3. Notice that such a Hamiltonian implies

$$\omega_s = \omega_o + \omega_\mu \tag{2}$$

where ω_s is the frequency of the generated optical sideband, ω_o is the frequency of the optical pump (not necessary the frequency of the optical cavity), and ω_o is the frequency of the microwave signal (not necessary the frequency of the microwave cavity). The Hamiltonian can be written in the form of matrix, which is

$$H = \begin{bmatrix} 0 & \Omega_\mu^* & A^* \\ \Omega_\mu & \delta_2 & \Omega_o^* \\ A & \Omega_o & \delta_3 \end{bmatrix} \tag{3}$$

The density matrix of each atom is

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} \tag{4}$$

To understand the dynamics of the system, we need to solve the master equation

$$\frac{d}{dt}\rho = -i[H,\rho] + Loss \tag{5}$$

where the *Loss* includes all the decay and the dephasing terms.
The *Loss* term

$$Loss = L_{21} + L_{12} + L_{32} + L_{23} + L_{31} + L_{13} + L_{22} + L_{33} \tag{6}$$

which includes:
(1) the spin relaxation of level 2

$$\begin{aligned} L_{21} &= \frac{\gamma_\mu}{2}(n_{\text{bath}}+1)\cdot(2\sigma_{12}\rho\sigma_{21}-\sigma_{21}\sigma_{12}\rho-\rho\sigma_{21}\sigma_{12}) \\ L_{12} &= \frac{\gamma_\mu}{2}n_{\text{bath}}\cdot(2\sigma_{21}\rho\sigma_{12}-\sigma_{12}\sigma_{21}\rho-\rho\sigma_{12}\sigma_{21}), \end{aligned} \tag{7}$$

where γ_μ is the decay rate per phonon at the frequency of the applied microwave, n_{bath} is the phonon number at the operating temperature at this frequency.
(2) the spontaneous emission of level 3 to level 2

$$\begin{aligned} L_{32} &= \frac{\gamma_{32}}{2}\cdot(2\sigma_{23}\rho\sigma_{32}-\sigma_{32}\sigma_{23}\rho-\rho\sigma_{32}\sigma_{23}) \\ L_{23} &= 0, \end{aligned} \tag{8}$$

where γ_{32} is the spontaneous emission rate from level 3 to level 2. $L_{23}=0$ is due that the average photon number at normal temperature ($0\sim 300$ K) at the optical frequency is almost zero.
(3) the spontaneous emission of level 3 to level 1

$$\begin{aligned} L_{31} &= \frac{\gamma_{31}}{2}\cdot(2\sigma_{13}\rho\sigma_{31}-\sigma_{31}\sigma_{13}\rho-\rho\sigma_{31}\sigma_{13}) \\ L_{13} &= 0, \end{aligned} \tag{9}$$

where γ_{31} is the spontaneous emission rate from level 3 to level 1.
(4) the dephasing of level 2.

$$L_{22} = \frac{\gamma_{2d}}{2}\cdot(2\sigma_{22}\rho\sigma_{22}-\sigma_{22}\sigma_{22}\rho-\rho\sigma_{22}\sigma_{22}) \tag{10}$$

where γ_{2d} is the dephasing rate of level 2.
(5) the optical dephasing of level 3

$$L_{33} = \frac{\gamma_{3d}}{2}\cdot(2\sigma_{33}\rho\sigma_{33}-\sigma_{33}\sigma_{33}\rho-\rho\sigma_{33}\sigma_{33}) \tag{11}$$

where γ_{3d} is the dephasing rate of level 3.

Note that here all the bathes are considered to be bathes of harmonic oscillators, which might not be the case for spins inside crystals. The readers are refereed to *Theory of the spin bath*, Reports on Progress in Physics 669, 63(2000).

2 Steady state solutions

When Stephen was a undergraduate, over and over again, he was scared by the equations above. He didn't know how to calculate them into a form that is straightforward enough to be understood, i.e., a form that is easy enough to write a code to numerically solve the equations. Now let's do it by using matrices!
For σ_{ij}, we have, for example,

$$\sigma_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \sigma_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

Substituting them into L_{12}, then we can calculate

$$L_{12} = \frac{\gamma_\mu}{2}n_{\text{bath}}\cdot\begin{bmatrix} -2\rho_{11} & -\rho_{12} & -\rho_{13} \\ -\rho_{21} & 2\rho_{11} & 0 \\ -\rho_{31} & 0 & 0 \end{bmatrix} \tag{13}$$

By repeating this calculation and using Eq. (3), (4), (5), and (6), we can obtain nine equations for each elements of ρ (just for brevity, we use n instead of n_{bath} from now on):

$$
\begin{aligned}
\dot{\rho_{11}} =& -\gamma_\mu n\rho_{11} + i\Omega_\mu\rho_{12} + iA\rho_{13} - i\Omega_\mu^*\rho_{21} + \gamma_\mu(n+1)\rho_{22} - iA^*\rho_{31} + \gamma_{31}\rho_{33} \\
\dot{\rho_{12}} =& i\Omega_\mu^*\rho_{11} - [i\delta_\mu + \frac{\gamma_\mu(2n+1)+\gamma_{2d}}{2}]\rho_{12} + i\Omega_o\rho_{13} - i\Omega_\mu^*\rho_{22} - iA^*\rho_{32} \\
\dot{\rho_{13}} =& iA^*\rho_{11} + i\Omega_o^*\rho_{12} - [i\delta_o + \frac{\gamma_\mu n + \gamma_{31} + \gamma_{32} + \gamma_{3d}}{2}]\rho_{13} - i\Omega_\mu^*\rho_{23} - iA^*\rho_{33} \\
\dot{\rho_{21}} =& -i\Omega_\mu\rho_{11} - [i\delta_\mu + \frac{\gamma_\mu(2n+1)+\gamma_{2d}}{2}]\rho_{21} + i\Omega_\mu\rho_{22} + iA\rho_{23} - i\Omega_o^*\rho_{31} \\
\dot{\rho_{22}} =& \gamma_\mu n\rho_{11} - i\Omega_\mu\rho_{12} + i\Omega_\mu^*\rho_{21} - \gamma_\mu(n+1)\rho_{22} + i\Omega_o\rho_{23} - i\Omega_o^*\rho_{32} + \gamma_{32}\rho_{33} \\
\dot{\rho_{23}} =& -i\Omega_\mu\rho_{13} + iA^*\rho_{21} + i\Omega_o^*\rho_{22} - [i(\delta_o - \delta_\mu) + \frac{\gamma_\mu(n+1) + \gamma_{31} + \gamma_{32} + \gamma_{2d} + \gamma_{3d}}{2}]\rho_{23} - i\Omega_o^*\rho_{33} \\
\dot{\rho_{31}} =& -iA\rho_{11} - i\Omega_o\rho_{21} - [i\delta_o + \frac{\gamma_\mu n + \gamma_{31} + \gamma_{32} + \gamma_{3d}}{2}]\rho_{31} + i\Omega_\mu\rho_{32} + iA\rho_{33} \\
\dot{\rho_{32}} =& -iA\rho_{12} - i\Omega_o\rho_{22} + i\Omega_\mu^*\rho_{31} - [i(\delta_o - \delta_m u) + \frac{\gamma_\mu(n+1) + \gamma_{31} + \gamma_{32} + \gamma_{3d}}{2}]\rho_{32} + i\Omega_o\rho_{33} \\
\dot{\rho_{33}} =& -iA\rho_{13} - i\Omega_o\rho_{23} + iA^*\rho_{31} + i\Omega_o^*\rho_{32} - (\gamma_{31} + \gamma_{32})\rho_{33}
\end{aligned}
\tag{14}
$$

If we organize $\boldsymbol{\rho} = [\rho_{11}, \rho_{12}, \rho_{13}, \rho_{21}, \rho_{22}, \rho_{23}, \rho_{31}, \rho_{32}, \rho_{33}]^{\mathrm{T}}$, the above equations can be rewritten as

$$\dot{\boldsymbol{\rho}} = \mathbf{M} \cdot \boldsymbol{\rho} \tag{15}$$

where the matrix $\mathbf{M}$ is

$\mathbf{M} =$

$$
\begin{bmatrix}
-\gamma_\mu n & i\Omega_\mu & iA & -i\Omega_\mu^* & \gamma_\mu(n+1) & 0 & -iA^* & 0 & \gamma_{31} \\
i\Omega_\mu^* & \substack{i\delta_\mu + \frac{\gamma_{2d}}{2} \\ + \frac{\gamma_\mu(2n+1)}{2}} & i\Omega_o & 0 & -i\Omega_\mu^* & 0 & 0 & -iA^* & 0 \\
iA^* & i\Omega_o^* & \substack{i\delta_o + \frac{\gamma_\mu n}{2} \\ + \frac{\gamma_{31}+\gamma_{32}+\gamma_{2d}}{2}} & 0 & 0 & -i\Omega_\mu^* & 0 & 0 & -iA^* \\
-i\Omega_\mu & 0 & 0 & \substack{i\delta_\mu + \frac{\gamma_{2d}}{2} \\ + \gamma_\mu(n+1)} & i\Omega_\mu & iA & -i\Omega_o^* & 0 & 0 \\
\gamma_\mu n & -i\Omega_\mu & 0 & i\Omega_\mu^* & -\gamma_\mu(n+1) & i\Omega_o & 0 & -i\Omega_o^* & \gamma_{32} \\
0 & 0 & -i\Omega_\mu & iA^* & i\Omega^* & \substack{i(\delta_o - \delta_\mu) \\ + \frac{\gamma_\mu(n+1)}{2} \\ + \frac{\gamma_{32}+\gamma_{31}}{2} \\ + \frac{\gamma_{2d}+\gamma_{3d}}{2}} & 0 & 0 & -i\Omega_o^* \\
-iA & 0 & 0 & -i\Omega_o & 0 & 0 & \substack{i\delta_o + \frac{\gamma_\mu n}{2} + \\ \frac{\gamma_{32}+\gamma_{31}+\gamma_{3d}}{2}} & i\Omega_\mu & iA \\
0 & -iA & 0 & 0 & -i\Omega_o & 0 & i\Omega_\mu^* & \substack{i(\delta_o - \delta_\mu) \\ + \frac{\gamma_\mu(n+1)}{2} \\ + \frac{\gamma_{32}+\gamma_{31}}{2} \\ + \frac{\gamma_{2d}+\gamma_{3d}}{2}} & i\Omega_o \\
0 & 0 & -iA & 0 & 0 & -i\Omega_o & iA^* & i\Omega_o^* & -\gamma_{31} - \gamma_{32}
\end{bmatrix}
\tag{16}
$$

These nine equations are not completely independent, because

$$\dot{\rho_{11}} + \dot{\rho_{22}} + \dot{\rho_{33}} = 0 \tag{17}$$

This can be obtained by simply summing the first, the fifth and the ninth equations in Eqs. (14). Therefore if we need one more equation to find the steady solution. Noticing that

$$\rho_{11} + \rho_{22} + \rho_{33} = 1, \tag{18}$$

replacing the first row of $\mathbf{M}$ with the above equation and defining another matrix

$\mathbf{M}' =$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
i\Omega_\mu^* & \begin{matrix} i\delta_\mu+\frac{\gamma_{2d}}{2} \\ +\frac{\gamma_\mu(2n+1)}{2} \end{matrix} & i\Omega_o & 0 & -i\Omega_\mu^* & 0 & 0 & -iA^* & 0 \\
iA^* & i\Omega_o^* & \begin{matrix} i\delta_o+\frac{\gamma_\mu n}{2} \\ +\frac{\gamma_{31}+\gamma_{32}+\gamma_{2d}}{2} \end{matrix} & 0 & 0 & -i\Omega_\mu^* & 0 & 0 & -iA^* \\
-i\Omega_\mu & 0 & 0 & \begin{matrix} i\delta_\mu+\frac{\gamma_{2d}}{2} \\ +\gamma_\mu(n+1) \end{matrix} & i\Omega_\mu & iA & -i\Omega_o^* & 0 & 0 \\
\gamma_\mu n & -i\Omega_\mu & 0 & i\Omega_\mu^* & -\gamma_\mu(n+1) & i\Omega_o & 0 & -i\Omega_o^* & \gamma_{32} \\
0 & 0 & -i\Omega_\mu & iA^* & i\Omega^* & \begin{matrix} i(\delta_o-\delta_\mu) \\ +\frac{\gamma_\mu(n+1)}{2} \\ +\frac{\gamma_{32}+\gamma_{31}}{2} \\ +\frac{\gamma_{2d}+\gamma_{3d}}{2} \end{matrix} & 0 & 0 & -i\Omega_o^* \\
-iA & 0 & 0 & -i\Omega_o & 0 & 0 & \begin{matrix} i\delta_o+\frac{\gamma_\mu n}{2}+ \\ \frac{\gamma_{32}+\gamma_{31}+\gamma_{3d}}{2} \end{matrix} & i\Omega_\mu & iA \\
0 & -iA & 0 & 0 & -i\Omega_o & 0 & i\Omega_\mu^* & \begin{matrix} i(\delta_o-\delta_\mu) \\ +\frac{\gamma_\mu(n+1)}{2} \\ +\frac{\gamma_{32}+\gamma_{31}}{2} \\ +\frac{\gamma_{2d}+\gamma_{3d}}{2} \end{matrix} & i\Omega_o \\
0 & 0 & -iA & 0 & 0 & -i\Omega_o & iA^* & i\Omega_o^* & -\gamma_{31}-\gamma_{32}
\end{bmatrix} \tag{19}
$$

we have

$$
\mathbf{M}' \cdot \begin{bmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \rho_{14} \\ \rho_{21} \\ \rho_{22} \\ \rho_{23} \\ \rho_{31} \\ \rho_{32} \\ \rho_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}
$$

To get the steady solution for ρ we just need to use matlab to solve such an equation for $\boldsymbol{\rho}$

$$
\mathbf{M}' \cdot \boldsymbol{\rho} = \mathbf{T} \tag{21}
$$

which is just to do a division for matrices

$$
\boldsymbol{\rho} = \mathrm{inv}(\mathbf{M}') \cdot \mathbf{T} \tag{22}
$$

where $\mathbf{T} = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0]^{\mathrm{T}}$

3 Taking into account the optical cavity

Assuming the generated optical sideband is coupled to the experimental environment through an optical cavity, which means $A \Rightarrow g_o a$ in Eq. (1), we need to reorganize the Hamiltonian as

$$
H = \delta_c a^\dagger a + \delta_2 \sigma_{22} + \delta_3 \sigma_{33} + (\Omega_\mu \sigma_{21} + \Omega_o \sigma_{32} + g_s a \sigma_{31} + \mathrm{H.C.}) \tag{23}
$$

where $\delta_c = \omega_c - \omega_0 - \omega_\mu$ with ω_c being the original optical cavity frequency. For the evolution of a we have the following Heisenberg equation

$$
\dot{a} = -i[a, H] - \frac{\kappa_c + \kappa_i}{2} a - \sqrt{\kappa_c} A_{in} \tag{24}
$$

where κ_c is the rate of the coupling loss, κ_i is the rate of intrinsic loss, A_{in} is the input signal. Note that we have already applied the rotating frame to the above Hamiltonian, so it is advised that $A_{in} = A_{in,orignial} \exp[-i(\omega_{\text{orignial}} - \omega_o - \omega_\mu)t]$. Inserting Eq. (23) into the above equation, we have

$$\dot{a} = -i\delta_c a - ig_s \sum_k \sigma_{13,k} - \frac{\kappa_c + \kappa_i}{2} a - \sqrt{\kappa_c} A_{in} \tag{25}$$

In order to see the how the presence of the atoms affect the optical cavity mode, we can trace all the atoms

$$\langle \sigma_{13} \rangle = \text{Tr}(\rho\sigma_{13}) = \rho_{31} \tag{26}$$

If we treat a classically, from Eq. (14) we have

$$\dot{\rho_{31}} = -[i\delta_o + \frac{\gamma_\mu n + \gamma_{31} + \gamma_{32} + \gamma_{3d}}{2}]\rho_{31} - ig_s a\rho_{11} - i\Omega_o\rho_{21} + i\Omega_\mu\rho_{32} + iA\rho_{33} \tag{27}$$

so for steady solutions we have

$$\begin{aligned} &- i\delta_c a - ig_s \sum_k \rho_{31,k} - \frac{\kappa_c + \kappa_i}{2} a - \sqrt{\kappa_c} A_{in} = 0 \\ &- [i\delta_o + \frac{\gamma_\mu n + \gamma_{31} + \gamma_{32} + \gamma_{3d}}{2}]\rho_{31} - ig_s a\rho_{11} - i\Omega_o\rho_{21} + i\Omega_\mu\rho_{32} + ig_s a\rho_{33} = 0 \end{aligned} \tag{28}$$

Reorder them

$$\begin{aligned} &(i\delta_c + \frac{\kappa_c + \kappa_i}{2})a = -ig_s \sum_k \rho_{31}^k - \sqrt{\kappa_c} A_{in} \\ &(i\delta_o + \frac{\gamma_{\text{tot}}}{2})\rho_{31,k} = -ig_s a(\rho_{11,k} - \rho_{33,k}) - (i\Omega_o\rho_{21,k} - i\Omega_\mu\rho_{32,k}) \end{aligned} \tag{29}$$

where $\gamma_{\text{tot}} = \gamma_\mu n + \gamma_{31} + \gamma_{32} + \gamma_{3d}$. We can see now ρ_{31} has two parts. The fist term on the right hand side determines the absorption of the generated sideband, and the second term indicates the resultant coherence under the driving of two fields. We then obtain

$$\rho_{31,k} = \mathcal{A} + \mathcal{G} \overset{\text{def}}{=} \frac{-ig_s a(\rho_{11,k} - \rho_{33,k})}{i\delta_o + \frac{\gamma_{\text{tot}}}{2}} + \frac{-i\Omega_o\rho_{21,k} + i\Omega_\mu\rho_{32,k}}{i\delta_o + \frac{\gamma_{\text{tot}}}{2}} \tag{30}$$

From Eq. (29), if there is not input A_{in} we have

$$\begin{aligned} a &= \frac{1}{i\delta_c + \frac{\kappa_c+\kappa_i}{2} + ig_s \sum_k (\rho_{11,k} - \rho_{33,k})/(i\delta_o + \frac{\gamma_{\text{tot}}}{2})} \cdot [-ig_s \sum_k (-i\Omega_o\rho_{21,k} + i\Omega_\mu\rho_{32,k})/(i\delta_o + \frac{\gamma_{\text{tot}}}{2})] \\ &= \frac{1}{i\delta_c + \frac{\kappa_c+\kappa_i}{2} + ig_s\mathcal{A}} \cdot (-ig_s \sum_k \mathcal{G}) \end{aligned} \tag{31}$$

It can be seen that the generation term goes to the numerator while the absorption term goes to the denominator. If a calculation gives out all the elements of the steady state of ρ, which means $\mathcal{A}$ and $\mathcal{G}$ can be calculated, we can then use Eq. (31) to carry out iterations to find the exact generated sideband signal inside the cavity.

Note that taking $A_{in} = 0$ is not an proper way for considering zero inputs, although it doesn't cause any trouble for us in this case. In fact there are always quantum-noise inputs even your laser is off. A proper way to do that is to keep A_{in} until you come up with a term of the input power and then using $\langle A_{in}^\dagger A_{in} \rangle = 0$.

The final equation that you need to calculate the output is

$$A_{out} - A_{in} = \sqrt{\kappa_c} a \tag{32}$$

Then you can calculate transmissions, reflections and whatever you want. Enjoy you calculations!