Whistler parameter extraction using the *Santolik 2003* method

---

April 14, 2020

Austin Sousa
Smeade Aerospace Engineering Sciences
University of Colorado Boulder

0.1 Introduction

In studying space plasma physics, we are often interested in reducing time-domain wave measurements to a useful set of wave parameters – Poynting flux, wavenormal angle, polarization, etc. Santolík et al. [2003] provides several numerical methods to accomplish this task, in the frequency domain. Here we use the "Electromagnetic SVD technique" from Santolík et al. [2003], section 3.

The method requires at least five orthogonal measurements of the six electromagnetic wave components: three magnetic field measurements, and three electric field measurements. If any one channel is missing, it can be synthesized by a simple relation.

0.2 Santolík et al. Method summary

Santolík et al. [2003] describe several methods of calculating wave parameters. The method in section 2 uses three magnetic field measurements only; this method is similar to those of Means [1972] and Samson and Olson [1980], and cannot fully define the wave, resulting in some directional ambiguity. Section 3 uses six measurements (3 magnetic, and 3 electric) which fully defines the wave. Section 3.2 extends the model to calculate the response curves of the electric field antennas, which are difficult to define on the ground. Section 3.3 performs the same calculations, but with only five field measurements.

Here we'll deal with the full method, using all six field measurements.

The method begins with Faraday's law:

$$\mathbf{n} \times \mathbf{E} = c\mathbf{B} \tag{1}$$

where $\mathbf{n} = \mathbf{k}/\omega$ is a dimensionless vector with a magnitude of the refractive index, and the direction of the wave propagation vector. The terms $\mathbf{E}$ and $\mathbf{B}$ are expressed in the frequency domain – e.g., the (complex) amplitudes of a Fourier transform.

We then successively multiply each of the three equations in 1 by the Cartesian components of $\mathbf{E}^*$ and $\mathbf{B}^*$, resulting in 18 dependent complex equations; separating the real and imaginary parts of each equation results in 36 dependent equations in total.

We write this set of equations as a matrix operation:

$$\mathbf{A_E} \cdot \mathbf{n} = \mathbf{b} \tag{2}$$

The terms of $\mathbf{A_E}$ and $\mathbf{b}$ are formed from the *Spectral cross matrix*, Q. Q is the outer product of ζ and ζ^*.

$$\zeta = (cB_1, cB_2, cB_3, E_1, E_2, E_3) \tag{3}$$

$$Q_{ij} = \zeta_i \zeta_j^* \tag{4}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \cdots & Q_{16} \\ Q_{21} & Q_{22} & Q_{23} \cdots & Q_{26} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{61} & Q_{62} & Q_{63} \cdots & Q_{66} \end{bmatrix} \tag{5}$$

Writing the terms out explicitly, we have:

$$Q = \begin{bmatrix} c^2B_1^2 & c^2B_1B_2^* & c^2B_1B_3^* & cB_1E_1^* & cB_1E_2^* & cB_1E_3^* \\ c^2B_2B_1^* & c^2B^2 & c^2B_2B_3^* & cB_2E_1^* & cB_2E_2^* & cB_2E_3^* \\ c^2B_3B_1^* & c^2B_3B_2^* & c^2B_3^2 & cB_3E_1^* & cB_3E_2^* & cB_3E_3^* \\ cE_1B_1^* & cE_1B_2^* & cE_1B_3^* & E_1^2 & E_1E_2^* & E_1E_3^* \\ cE_2B_1^* & cE_2B_2^* & cE_2B_3^* & E_2E_1^* & E_2^2 & E_2E_3^* \\ cE_3B_1^* & cE_3B_2^* & cE_3B_3^* & E_3E_1^* & E_3E_2^* & E_3E_3^* \end{bmatrix} \tag{6}$$

Note that the diagonal elements of Q are real-valued, while the cross terms are complex. However, Q is Hermitian, or conjugate triangular:

$$Q_{ij} = Q_{ji}^* \tag{7}$$

That is, the lower triangular components are the complex conjugate of the upper triangular components. This means that, say, for a spacecraft, only the diagonal and upper-triangular terms need to be stored.

Again, the components of Q are each the result of a Fourier transform, and so we have a single $Q = Q(f)$ for each frequency bin in our analysis. We don't explicitly write out this frequency dependence, since the method here operates on a single frequency bin, and is agnostic to adjacent frequencies.

Santolík et al., equation (15) gives the elements of $\mathbf{A_E}$ and $\mathbf{b}$ in a very (very) compact form using the *Levi-Civita* symbol ϵ_{ijk}. But since we aren't concerned with brevity, we'll write all the terms out here:

$$\mathbf{A_E} \cdot \mathbf{n} = \begin{bmatrix} \Re(\mathbf{A}) \\ \Im(\mathbf{A}) \end{bmatrix} \cdot \mathbf{n} = \begin{bmatrix} \Re(b) \\ \Im(b) \end{bmatrix} \tag{8}$$

$$\begin{bmatrix}
0 & \Re(Q_{61}) & -\Re(Q_{51}) \\
-\Re(Q_{61}) & 0 & \Re(Q_{41}) \\
\Re(Q_{51}) & -\Re(Q_{41}) & 0 \\
0 & \Re(Q_{62}) & -\Re(Q_{52}) \\
-\Re(Q_{62}) & 0 & \Re(Q_{42}) \\
\Re(Q_{52}) & -\Re(Q_{42}) & 0 \\
0 & \Re(Q_{63}) & -\Re(Q_{53}) \\
-\Re(Q_{63}) & 0 & \Re(Q_{43}) \\
\Re(Q_{53}) & -\Re(Q_{43}) & 0 \\
0 & \Re(Q_{64}) & -\Re(Q_{54}) \\
-\Re(Q_{64}) & 0 & \Re(Q_{44}) \\
\Re(Q_{54}) & -\Re(Q_{44}) & 0 \\
0 & \Re(Q_{65}) & -\Re(Q_{55}) \\
-\Re(Q_{65}) & 0 & \Re(Q_{45}) \\
\Re(Q_{55}) & -\Re(Q_{45}) & 0 \\
0 & \Re(Q_{66}) & -\Re(Q_{56}) \\
-\Re(Q_{66}) & 0 & \Re(Q_{46}) \\
\Re(Q_{56}) & -\Re(Q_{46}) & 0 \\
\hdotsfor{3} \\
0 & \Im(Q_{61}) & -\Im(Q_{51}) \\
-\Im(Q_{61}) & 0 & \Im(Q_{41}) \\
\Im(Q_{51}) & -\Im(Q_{41}) & 0 \\
0 & \Im(Q_{62}) & -\Im(Q_{52}) \\
-\Im(Q_{62}) & 0 & \Im(Q_{42}) \\
\Im(Q_{52}) & -\Im(Q_{42}) & 0 \\
0 & \Im(Q_{63}) & -\Im(Q_{53}) \\
-\Im(Q_{63}) & 0 & \Im(Q_{43}) \\
\Im(Q_{53}) & -\Im(Q_{43}) & 0 \\
0 & \Im(Q_{64}) & -\Im(Q_{54}) \\
-\Im(Q_{64}) & 0 & \Im(Q_{44}) \\
\Im(Q_{54}) & -\Im(Q_{44}) & 0 \\
0 & \Im(Q_{65}) & -\Im(Q_{55}) \\
-\Im(Q_{65}) & 0 & \Im(Q_{45}) \\
\Im(Q_{55}) & -\Im(Q_{45}) & 0 \\
0 & \Im(Q_{66}) & -\Im(Q_{56}) \\
-\Im(Q_{66}) & 0 & \Im(Q_{46}) \\
\Im(Q_{56}) & -\Im(Q_{46}) & 0
\end{bmatrix} \cdot \mathbf{n} = \begin{bmatrix}
\Re(Q_{11}) \\ \Re(Q_{21}) \\ \Re(Q_{31}) \\ \Re(Q_{12}) \\ \Re(Q_{22}) \\ \Re(Q_{32}) \\ \Re(Q_{13}) \\ \Re(Q_{23}) \\ \Re(Q_{33}) \\ \Re(Q_{14}) \\ \Re(Q_{24}) \\ \Re(Q_{34}) \\ \Re(Q_{15}) \\ \Re(Q_{25}) \\ \Re(Q_{35}) \\ \Re(Q_{16}) \\ \Re(Q_{26}) \\ \Re(Q_{36}) \\ \dots \\ \Im(Q_{11}) \\ \Im(Q_{21}) \\ \Im(Q_{31}) \\ \Im(Q_{12}) \\ \Im(Q_{22}) \\ \Im(Q_{32}) \\ \Im(Q_{13}) \\ \Im(Q_{23}) \\ \Im(Q_{33}) \\ \Im(Q_{14}) \\ \Im(Q_{24}) \\ \Im(Q_{34}) \\ \Im(Q_{15}) \\ \Im(Q_{25}) \\ \Im(Q_{35}) \\ \Im(Q_{16}) \\ \Im(Q_{26}) \\ \Im(Q_{36})
\end{bmatrix} \tag{9}$$

For perfect, ideal fields, the above matrix is highly degenerate, and collapses down to the 6 independent (3 real, 3 imaginary) equations from equation 1. However, for realistic data measurements, the matrix will not be degenerate – we can exploit this fact to find a better solution in the "least-squares" sense.

An estimate of the wavenormal vector $\mathbf{n} \to \hat{\mathbf{n}}$ can be found using a singular value decomposition (SVD):

$$\mathbf{A_E} = \mathbf{U_E} \cdot \mathbf{W_E} \cdot \mathbf{V_E}^T \tag{10}$$

where $\mathbf{U_E}$, $\mathbf{W_E}$, and $\mathbf{V_E}^T$ can be computed via a numeric SVD routine (MATLAB and Scipy both have one built in).

The estimate of the wavenormal vector is given by:

$$\hat{\mathbf{n}} = \mathbf{V_E} \cdot \mathbf{W_E}^{-1} \cdot \mathbf{U_E}^T \cdot \mathbf{b} \tag{11}$$

Santolík et al. [2003] note that the amplitude of $\hat{\mathbf{n}}$ *should* contain useful information, but must be used with caution, as it does not account for the transfer function of the electric field antennas. Fortunately, we can still compute the wave direction as one normally would, by converting the 3-dimensional vector into polar coordinates:

$$\theta = \arctan(\sqrt{(}n_1^2 + n_2^2)/n_3) \tag{12}$$

$$\begin{aligned} \phi &= \arctan(n_2/n_1) && \text{for } n_1 \geq 0 && (13)\\ \phi &= \arctan(n_2/n_1) - \pi && \text{for } n_1 < 0, n_2 < 0 && (14)\\ \phi &= \arctan(n_2/n_1) + \pi && \text{for } n_1 < 0, n_2 \geq 0 && (15) \end{aligned}$$

The method includes a metric of "Planarity", which approaches 1 in the presence of an ideal plane wave. Planarity is essentially a metric of how well the SVD solution is working.

$$F_E = 1 = \sqrt{\frac{N}{D}} \tag{16}$$

where

$$N = \sum_{i=1}^{36} (\hat{\beta}_i - \hat{b}_i)^2 \tag{17}$$

$$D = \sum_{i=1}^{36} (\|\hat{\beta}_i\| + \|\hat{b}_i\|)^2 \tag{18}$$

Figure out polarization in terms of the expanded SVD

To summarize, the method works as follows:

- Assure they are in an orthogonal frame (or rotate the signals accordingly)
- For each frequency in your FFT:
  - Compute single-sided, complex-valued FFTs for each channel

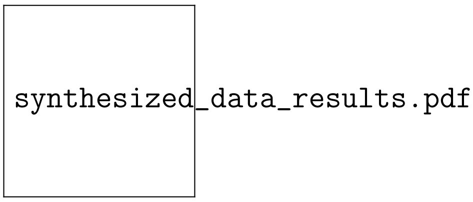

Figure 1: Results on synthetic, noiseless data

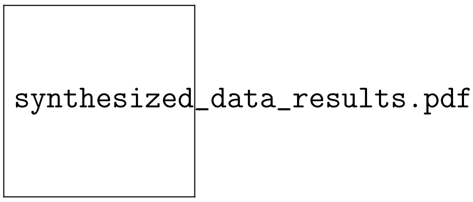

Figure 2: Results on synthetic data, with some noise added

  - Form $\mathbf{A_E}$ and $\mathbf{b}$ as described above
  - Compute the SVD, and calculate $\hat{\mathbf{n}}$ using equation 11
  - Calculate θ, ϕ, F_E, and polarization using $\hat{\mathbf{n}}$

0.3 Example outputs from Synthesized and Realistic Data

So how's it work? This section contains a few examples, as computed on data from RBSP.

0.3.1 Synthetic data

0.3.2 Real data from RBSP

0.4 varying averaged sample lengths

– peak vs average plots – errors vs window length? That was a cool viz

0.5 Synthesizing a 6th channel from 5 channels of data

– This is short, maybe we should move this part up a section

Bibliography

Joseph D. Means. Use of the three-dimensional covariance matrix in analyzing the polarization properties of plane waves. *Journal of Geophysical Research*, 77 (28):5551–5559, 1972. ISSN 0148-0227. doi: 10.1029/JA077i028p05551. URL `http://onlinelibrary.wiley.com/doi/10.1029/JA077i028p05551/abstract`.

J. C. Samson and J. V. Olson. Some comments on the descriptions of the polarization states of waves. *Geophysical Journal of the Royal Astronomical Society*, 61(1):115–129, 1980. ISSN 1365246X. doi: 10.1111/j.1365-246X.1980.tb04308.x.

O. Santolík, M. Parrot, and F. Lefeuvre. Singular value decomposition methods for wave propagation analysis. *Radio Science*, 38(1):1–13, 2003. ISSN 00486604. doi: 10.1029/2000RS002523.