EX 1 PREDICATE LOGIC

Transform the following sentences from natural language into predicate formulas.

1.3. If ~~x~~ a nonzero integer is divisible by 10, it can be decomposed in 2 factors s.t. one is divisible by 2 and the other one is divisible by 5, and x can be written as ~~f~~ sum of 2 even numbers

$(\forall x)(nonzero(x) \wedge divisible(x,10) \rightarrow (\exists y)(\exists z)(equal(product(y,z),x) \wedge divisible(y,2) \wedge divisible(z,5)) \wedge (\exists t)(\exists u)(equal(sum(t,u),x) \wedge divisible(t,2) \wedge divisible(u,2)))$

FUNCTION SYMBOLS

$product \in \mathcal{F}_2$, $product(x,y) = x * y$; the function ~~predicate~~ defined by the axioms:

→ commutativity: $(\forall x)(\forall y)(equal(product(x,y), product(y,x)))$

→ associativity
$(\forall x)(\forall y)(\forall z)($~~prod~~ $equal(product(product(x,y),z), product(x, product(y,z))))$

→ has neutral element: $(\forall x)(equal(product(x,1),x))$

$sum \in \mathcal{F}_2$, $sum(x,y) = x+y$

→ commutativity: ....

→ associativity: ....

→ has neutral element: ....

PREDICATE SYMBOLS

$nonzero \in \mathcal{P}_1$, $nonzero(x)$: "$x \neq 0$" $nonzero: D \rightarrow \{T,F\}$ $D = \mathbb{Z}$

$divisible \in \mathcal{P}_2$, $divisible(x,y)$: "$x \vdots y$" $divisible: D \times D \rightarrow \{T,F\}$

↳ reflexivity: $(\forall x) divisible(x,x)$

transitivity: $(\forall x)(\forall y)(\forall z)(divisible(x,y) \wedge divisible(y,z) \rightarrow divisible(x,z))$

not symmetric: $x \vdots y$ does not imply $y \vdots x$

$equal \in P_2$, $equal(x,y)$: "$x = y$" $equal: D \times D \to \{T; F\}$

→ reflexivity: . . .

→ symmetry: . . .

→ transitivity: . . .

Rudolph, Scrooge → constants

$H_1: (\forall x)(child(x) \to loves(x, Santa))$

$H_2: (\forall x)(\forall y)(loves(x, Santa) \wedge reindeer(y) \to loves(x, y))$

$H_3: reindeer(Rudolph) \wedge red\_nose(Rudolph)$

$H_4: (\forall x)(red\_nose(x) \to weird(x) \vee clown(x))$

$H_5: (\forall x)(reindeer(x) \to \neg clown(x))$

$H_6: (\forall x)(weird(x) \to \neg loves(Scrooge, x))$

$C: \neg child(Scrooge)$

transformation of natural language sentences into predicate formulas

THEORETICAL RESULTS

Let $U_1, U_2, \dots, U_n, V$ predicate formulas, $U_1, \dots, U_n$ hypothesis. V deducible from $U_1, \dots, U_n$ ($U_1, \dots, U_n \vdash V$) if there exists a sequence of formulas $(f_1, \dots, f_m)$ s.t $f_m = V$ and $\forall i \in \{1, \dots, m\}$ we have a), b), c)

a) $f_i \in A_{Pr}$
b) $f_i \in \{U_1, \dots, U_n\}$
c) $f_{i_1}, f_{i_2} \vdash_{mp} f_i$
d) $f_j \vdash_{univ\_gen} f_i$

$H_1 \vdash_{univ\_inst} child(Scrooge) \to loves(Scrooge, Santa)$ f_7

Scrooge used for instantiation

$H_2 \vdash_{univ\_inst} (\forall y)(loves(Scrooge, Santa) \wedge reindeer(y) \to loves(Scrooge, y))$ f_8

x=Scrooge

$f_8 \vdash_{univ\_inst} loves(Scrooge, Santa) \wedge reindeer(Rudolph) \to loves(Scrooge, Rudolph)$ f_9

y=Rudolph

$H_4 \vdash_{univ\_inst} red\_nose(Rudolph) \to weird(Rudolph) \vee clown(Rudolph)$ f_{10}

x = Rudolph

$H_5 \vdash_{univ\_inst} reindeer(Rudolph) \to \neg clown(Rudolph)$ f_{11}

x = Rudolph

$H_6 \vdash_{univ\_inst} weird(Rudolph) \to \neg loves(Scrooge, Rudolph)$ f_{12}

x = Rudolph

R = Rudolph
Sc = Scrooge
San = Santa

$f_3: reindeer(R) \wedge red\_nose(R)$

$f_7: child(Sc) \to loves(Sc, San)$

$f_9: loves(Sc, San) \wedge reindeer(R) \to loves(Sc, R)$

$f_{10}: red\_nose(R) \to weird(R) \vee clown(R)$

$f_{11}: reindeer(R) \to \neg clown(R)$

$f_{12}: weird(R) \to \neg loves(Sc, R)$

$f_3 \vdash_{simplif}$ reindeer(R) f_{13}

red_nose(R) f_{14}

SIMPLIFICATION

$U \wedge V \vdash U$

$U \wedge V \vdash V$

f_{13}: reindeer(R)

f_{14}: red_nose(R)

$f_{14}, f_{10} \vdash_{mp}$ weird(R) ∨ clown(R)

$\equiv$ ¬weird(R) → clown(R)

$\equiv$ ¬clown(R) → weird(R) f_{15}

MODUS PONENS

$U, U \to V \vdash V$

MODUS TOLLENS

$\neg V, U \to V \vdash \neg U$

$U \to V \vdash \neg V \to \neg U$

f_{15}: ¬clown(R) → weird(R)

$f_{13}, f_{11} \vdash_{mp}$ ¬clown(R) f_{16}

*write all rules you use, where it gets more complicated, explicitly state what U, V are

$f_{16}, f_{15} \vdash_{mp}$ weird(R) f_{17}

f_{17}: weird(R)

$f_{17}, f_{12} \vdash_{mp}$ ¬loves(Sc, R) f_{18}

f_{18}: ¬loves(Sc, R)

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f_9 $(\underbrace{\text{loves(Sc, San)} \wedge \text{reindeer(R)}}_{U} \to \underbrace{\text{loves(Sc, R)}}_{V}) \to$

$\to$ (¬loves(Sc, R) → ¬(loves(Sc, San) ∧ reindeer(R)))

$f_9, f_{19} \vdash_{mp}$ ¬loves(Sc, R) → ¬(loves(Sc, San) ∧ reindeer(R)) f_{20}

MODUS TOLLENS

U = loves ∧ reind

V = loves(Sc, R)

$f_{18}, f_{20} \vdash_{mp}$ ¬(loves(Sc, San) ∧ reindeer(R))

$\equiv \underbrace{\neg\text{loves(Sc, San)}}_{\neg U} \vee \underbrace{\neg\text{reindeer(R)}}_{Z}$ f_{21}

f_7: child(Sc) $\rightarrow$ loves(Sc, San)

$\equiv \neg$child(Sc) $\vee$ loves(Sc, San)

$U \quad\quad \vee V$

RESOLUTION

$U \vee V, \neg U \vee Z \vdash V \vee Z$

U = loves(Sc, San)

V = $\neg$child(Sc)

Z = $\neg$reindeer(R)

$f_7, f_{21} \vdash_{resolution}$ ~~loves~~ $\neg$child(Sc) $\vee \neg$reindeer(R)

$\equiv \neg$reindeer(R) $\vee \neg$child(Sc)

$\equiv$ reindeer(R) $\rightarrow \neg$child(Sc) f_{22}

$f_{13}, f_{22} \vdash_{mp} \neg$child(Sc)

The sequence $(f_1, \ldots, f_{22})$ is the deduction of conclusion C from hypotheses $H_1, H_2, \ldots, H_6$.

4.3. $U = (\forall x)(P(x) \wedge Q(x) \rightarrow P(sq(x)) \wedge Q(prod(x,5)))$

Interpretation: $I = \langle D, m \rangle$, $D = \mathbb{Z}$

$m(P): \mathbb{Z} \rightarrow \{T, F\}$, $m(P)(x)$: "x is even"

$m(Q): \mathbb{Z} \rightarrow \{T, F\}$, $m(Q)(x)$: "$x<0$"

$m(prod): \mathbb{Z}^2 \rightarrow \mathbb{Z}$, $m(prod)(x,y) = x * y$

$U = (\forall x)(P(x) \wedge Q(x) \rightarrow P(sq(x)) \wedge Q(prod(x,5)))$

$= (\forall x)(\text{"x is even"} \wedge \text{"}x<0\text{"} \rightarrow P(x^2) \wedge Q(5*x))$

$= (\forall x)(\text{"x is even"} \wedge \text{"}x<0\text{"} \rightarrow \text{"}x^2\text{ even"} \wedge \text{"}5x<0\text{"})$

$(\forall x)$ is semi-distributive over '$\rightarrow$' $\Rightarrow$ we have to evaluate everything in parantheses as it is

$(\forall x)(A(x) \rightarrow B(x)) \not\equiv (\forall x)A(x) \rightarrow (\forall x)B(x)$

$= (\forall x)(\text{"x is even"} \wedge \text{"}x<0\text{"} \rightarrow \text{"}x^2\text{ even"} \wedge \text{"}5x<0\text{"})$

$= T$

$$\left.\begin{array}{l} x \text{ even} \Rightarrow x = 2 \cdot k,\ k \in \mathbb{Z} \\ x<0 \\ x^2 = (2k)^2 = 4k^2 \,\vdots\, 2 \ (x^2 \text{ is even}) \\ x<0 \mid \cdot 5 \\ 5x<0 \end{array}\right\} \text{implication holds}$$

→ for this type of exercises,
use distributive laws when applicable & write them as theoretical results; and if you cannot apply distributivity, say so (something like the above example)

→ explain your evaluation, either in words or mathematical analysis