Seminar 10

1. Let $f : \mathbb{R}^2 \to R, f(x, y) = xy$. Using the definition prove that $Df(x_0, y_0)(x, y) = y_0 x + x_0 y$.

2. For $f : \mathbb{R}^2 \to R, f(x, y) = x^2 + xy$ find:

   (a) the gradient of f and the direction of steepest descent at the point $(1, 0)$.

   (b) the directional derivative at the point $(1, 0)$ in the direction of $\vec{i} + \vec{j}$.

   (c) the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 0, 1)$.

3. Find the equation of:

   (a) the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at an arbitrary point (x_0, y_0).

   (b) ★ the tangent plane to the unit sphere $x^2 + y^2 + z^2 = 1$ at an arbitrary point (x_0, y_0, z_0).

4. Let $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \|x\|^2$. Compute the directional derivative $D_v f(x)$ in two ways: directly and using the gradient.

5. For each of the following functions, compute $\dfrac{\mathrm{d}f}{\mathrm{d}t}$ directly and using the chain rule:

   (a) $f(x, y) = \ln(x^2 + y^2)$,
   $x = t,\ y = t^2$.

   (b) ★ $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$,
   $x = \cos t,\ y = \sin t,\ z = t > 0$.

6. Consider $f : \mathbb{R}^2 \to R$ and $x = g_1(u, v)$, $y = g_2(u, v)$, i.e. $f(x, y) = (f \circ g)(u, v)$. Prove that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

7. ★ A function $f : \mathbb{R}^2 \to R$ is called homogeneous of degree p if $f(tx, ty) = t^p f(x, y)$. Using the chain rule show that any such differentiable function satisfies

$$x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = pf(x, y).$$

   This is known as Euler's homogeneous function theorem.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.