EX1 PRED RES | prenex, Skolem and clausal normal forms

1.2. $U_2 = (\exists x)(\forall y)((\exists z)\neg P(z) \vee (\exists u)(R(x,u) \rightarrow (\forall z)\neg Q(u,z)))$

THEORETICAL RESULTS

A predicate formula U is in prenex normal form if it has the form:

$$\underbrace{(Q_1x_1)(Q_2x_2)\dots(Q_nx_n)}_{\text{prefix}}\;\overset{\text{matrix}}{M}$$

where $Q_{1,\dots,n}$ quantifiers, and M quantifier free.

Obtaining prenex normal form:

S1) $\rightarrow, \leftrightarrow$ replaced using $\neg, \vee, \wedge$

S2) bound vars renamed s.t. they are distinct

S3) infinitary DeMorgan's

* S4) extraction of quantifiers in front of the formula

S5) matrix into CNF

* write only ones that apply to your solution

$A \vee (\exists x)B(x) \equiv (\exists x)(A \vee B(x))$
$A \wedge (\exists x)B(x) \equiv (\exists x)(A \wedge B(x))$
$A \vee (\forall x)B(x) \equiv (\forall x)(A \vee B(x))$
$A \wedge (\forall x)B(x) \equiv (\forall x)(A \wedge B(x))$

A does not contain x as free var

$(\exists x)A(x) \vee B \equiv (\exists x)(A(x) \vee B)$
$(\exists x)A(x) \wedge B \equiv (\exists x)(A(x) \wedge B)$
$(\forall x)A(x) \vee B \equiv (\forall x)(A(x) \vee B)$
$(\forall x)A(x) \wedge B \equiv (\forall x)(A(x) \wedge B)$

B does not contain x as free var

$U_2 = (\exists x)(\forall y)((\exists z)\neg P(z) \vee (\exists u)(R(x,u) \overset{1}{\rightarrow} (\forall z)\neg Q(u,z)))$

replace 1

$\equiv (\exists x)(\forall y)((\exists z)\neg P(z) \vee (\exists u)(\neg R(x,u) \vee (\forall z)\neg Q(u,z)))$

$\equiv (\exists x)(\forall y)((\exists z)\neg P(z) \vee (\exists u)(\neg R(x,u) \vee (\forall t)\neg Q(u,t)))$ rename bound vars

$\equiv (\exists x)(\forall y)((\exists z)\neg P(z) \vee (\exists u)(\forall z)(\neg R(x,u) \vee \neg Q(u,z)))$

$$\equiv (\exists x)(\forall y)(\exists z)\neg P(z) \vee (\exists u)(\forall t)(\neg R(x,u) \vee \neg Q(u,t)))$$

$$U_1^P = (\exists x)(\forall y)(\exists z)(\exists u)(\forall t)(\neg P(z) \vee \neg R(x,u) \vee \neg Q(u,t))$$

$$U_2^P = (\exists x)(\forall y)(\exists u)(\forall t)(\exists z)(\neg P(z) \vee \neg R(x,u) \vee \neg Q(u,t))$$

SKOLEM NORMAL FORM

Let U be a first-order formula, $U^P = (Q_1 x_1)\dots(Q_n X_n)M$ be one of its conjunctive prenex NF. A formula in Skolem NF, denoted by U^S:

can summarize this in your solution

- for each exist. quantif. Q_r from prefix, we apply transformation:
  - → if on the left side of Q_r there are no universal quantif, we introduce a new constant a, and replace in M all the occurrences of x_r by a. $(Q_r x_r)$ deleted from prefix
  - → if $Q_{s_1}, \dots, Q_{s_m}$, $1 \le s_1 < \dots < s_m < r$ universal quantif. on left side of Q_r, then we introduce a new m-place fn symbol, f, and we replace in M all the occurrences of x_r by $f(x_{s_1}, \dots, x_{s_m})$. $(Q_r x_r)$ deleted

$$U_1^S = (\forall y)(\exists z)(\exists u)(\forall t)(\neg P(z) \vee \neg R(a,u) \vee \neg Q(u,t))$$
$$= (\forall y)(\exists u)(\forall t)(\neg P(f(y)) \vee \neg R(a,u) \vee \neg Q(u,t))$$
$$= (\forall y)(\forall t)(\neg P(f(y)) \vee \neg R(a,g(y)) \vee \neg Q(g(y),t))$$

$[x \leftarrow a, z \leftarrow f(y), u \leftarrow g(y)]$

a Skolem constant

f, g unary Skolem fns

$$U_2^S = (\forall y)(\exists u)(\forall t)(\exists z)(\neg P(z) \vee \neg R(a,u) \vee \neg Q(u,t))$$
$$= (\forall y)(\forall t)(\exists z)(\neg P(z) \vee \neg R(a,f(y)) \vee \neg Q(f(y),t))$$
$$= (\forall y)(\forall t)(\neg P(g(y,t)) \vee \neg R(a,f(y)) \vee \neg Q(f(y),t))$$

a Skolem constant

f unary Skolem fn

g binary Skolem fn

$U_1^C = (\neg P(f(y)) \vee \neg R(a, g(y)) \vee \neg Q(g(y), t))$. CNF with 1 clause

$U_2^C = \neg P(g(y, t)) \vee \neg R(a, f(y)) \vee \neg Q(f(y), t))$ –||–

EX2 PRED RES

Are the literals in the following pairs unifiable? If yes, find their mgu.

$x, y, z \in Var$

$a, b \in Const$

$f, g \in F_1$

$h \in F_2$

$P \in P_3$

THEORETICAL RESULTS

A. The composition of two substitutions

$$\theta_1 = [x_1 \leftarrow t_1, \dots, x_k \leftarrow t_k]$$

$$\theta_2 = [y_1 \leftarrow s_1, \dots, y_n \leftarrow s_n]$$

is defined as

$$\theta = \theta_1\theta_2 = [x_i \leftarrow \theta_2(t_i) \mid x_i \in dom(\theta_1), x_i \neq \theta_2(t_i)] \cup [y_j \leftarrow s_j \mid y_j \in dom(\theta_2) \setminus dom(\theta_1)]$$

B. A substitution θ is a unifier of terms t_1 and t_2 if $\theta(t_1) = \theta(t_2)$.

C. The most general unifier (mgu) is a unifier μ s.t. any other unifier θ can be obtained from μ by a further substitution λ,

$$\theta = \mu\lambda$$

input: $l_1 = P_1(t_{1_1}, t_{1_2}, \dots, t_{1_n})$

$l_2 = P_2(t_{2_1}, t_{2_2}, \dots t_{2_k})$

output: $mgu(l_1, l_2)$ if exists

"l_1, l_2 NOT unifiable" otherwise

begin if $(P_1 \neq P_2)$ then write

"l_1, l_2 NOT unifiable"

exit

end_if

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if (n≠k) then write
    "l1, l2 NOT unifiable"
    exit
end_if
θ:=ε
while (θ(l1) ≠ θ(l2))
    find in θ(l1), θ(l2) the terms corresponding to the
    outermost fn symbols or vars that are different &
    denote them by t1 and t2
    if (neither one of t1 and t2 is a var or one is a subterm
        of the other one) then
            write "l1, l2 NOT unifiable"
            exit
    end_if
    if (t1=var) then
        λ:=[t1←t2]
    else
        λ:=[t2←t1]
    end_if
    θ:=θλ
    if (θ not a substitution) then
        write "l1, l2 NOT unifiable"
end_while
write "l1, l2 UNIFIABLE and θ=mgu(l1,l2)"
end*
```

*can summarize algorithm
we have to first check
that we have same
predicate, then that
n=k, where n is...
k is...
(don't need to write it
in full - MAIN IDEAS)

2.2. $P(a,x,f(g(y)))$ and $P(y,f(z),f(z))$

$\theta : \varepsilon$

$\theta(l_1) = P(\underline{a},x,f(g,y))$

$\theta(l_2) = P(\underline{y},f(z),f(z))$

FIRST ITERATION

$\lambda := [y \leftarrow a]$, λ unifier of terms y and a

$\theta := \theta\lambda = [y \leftarrow a]$

$\theta(\ell_1) = P(a, \underline{x}, f(g(a)))$

$\theta(\ell_2) = P(a, \underline{f(z)}, f(z))$

SECOND ITERATION

$\lambda := [x \leftarrow f(z)]$, λ unifier of x & f(z)

$\theta := \theta\lambda = [y \leftarrow a][x \leftarrow f(z)] =$

$= [y \leftarrow a][x \leftarrow f(z)]$

$= [y \leftarrow a, x \leftarrow f(z)]$

$\theta(\ell_1) = P(a, f(z), \underline{f(g(a))})$

$\theta(\ell_2) = P(a, f(z), \underline{f(z)})$

THIRD ITERATION

$\lambda := [z \leftarrow g(a)]$ unifier of z, g(a)

$\theta := \theta\lambda = [y \leftarrow a, x \leftarrow f(z)][z \leftarrow g(a)]$

$= [y \leftarrow a, x \leftarrow f(g(a)), z \leftarrow g(a)] = mgu(\ell_1, \ell_2)$

$\theta(\ell_1) = P(a, f(g(a)), f(g(a))) = \theta(\ell_2)$ common instance

$P(x, g(f(a)), f(b))$ and $\cancel{P(y, f(z)}$ $P(f(y), z, z)$

$\theta := \varepsilon$

$\theta(\ell_1) = P(\underline{x}, g(f(a)), f(b))$

$\theta(\ell_2) = P(\underline{f(y)}, z, z)$

ITER. 1

$\lambda = [x \leftarrow f(y)]$ unifier x, f(y)

$\theta := \theta\lambda = [x \leftarrow f(y)]$

$\theta(\ell_1) = P(f(y), \underline{g(f(a))}, f(b))$

$\theta(\ell_2) = P(f(y), \underline{z}, z)$

ITER 2

$\lambda = [\cancel{z} \leftarrow g(f(a))]$

$\theta := \theta\lambda = [x \leftarrow f(y)][z \leftarrow g(f(a))]$

$= [x \leftarrow f(y), z \leftarrow g(f(a))]$

$\theta(\ell_1) = P(f(y), g(f(a)), \underline{f(b)})$

$\theta(\ell_2) = P(f(y), g(f(a)), \underline{g(f(a))})$ NOT UNIFIABLE $\Rightarrow$ ℓ_1, ℓ_2 are not unifiable

→ neither is a variable

$P(a, x, f(g(y)))$ and $P(z, h(z, u), f(b))$

$\theta := \varepsilon$

$\theta(\ell_1) = P(\underline{a}, x, f(g(y)))$

$\theta(\ell_2) = P(\underline{z}, h(z, u), f(b))$

ITER 1:

$\lambda = [z \leftarrow a]$ unifier z, a $\quad \theta := \theta\lambda = [z \leftarrow a]$

$\theta(\ell_1) = P(a, \underline{x}, f(g(y)))$

$\theta(\ell_2) = P(a, \underline{h(z, u)}, f(b))$

ITER 2

$\lambda = [x \leftarrow h(z, u)]$ unifier $x, h(z, u)$

$\theta := \theta\lambda = [z \leftarrow a][x \leftarrow h(z, u)]$

$= [z \leftarrow a, x \leftarrow h(z, u)]$

$\theta(\ell_1) = P(a, h(z, u), f(\underline{g(y)}))$

$\theta(\ell_2) = P(a, h(z, u), f(\underline{b}))$ NOT UNIFIABLE $\Rightarrow$ ℓ_1, ℓ_2 NOT UNIFIABLE*

b - const

g(y) fn

$\underbrace{}$

neither is a variable

*always write **WHY** not unifiable

EX 3 PRED RES | Prove inconsistency of the following set of clauses using lock resolution. 2 different indexings

2. $S_2 = \{P(x) \vee \neg Q(x),$
$\neg P(a) \vee R(x),$
$Q(x),$
$W(z),$
$\neg R(y) \vee \neg W(y)\}$

C: THEORETICAL RESULTS: SOUNDNESS AND COMPLETENES OF PRED. RES.

A set S of predicate (first-order) clauses is inconsistent iff $S \vdash^{Pr}_{Res} \square$.

LOCK RESOLUTION

↓ brief description of how it works (rules)
- assigning indices
- can only resolve on lowest indices
- ....

1) $C_1 = {}_{(1)}P(x) \vee {}_{(2)}\neg Q(x)$
$C_2 = {}_{(3)}\neg P(a) \vee {}_{(4)}R(x)$
$C_3 = {}_{(5)}Q(x)$
$C_4 = {}_{(6)}W(z)$
$C_5 = {}_{(8)}\neg R(y) \vee {}_{(7)}\neg W(y)$

$C_6 = Res^{lock\ Pr}_{\theta_1 = [x \leftarrow a]}(C_1, C_2) = {}_{(2)}\neg Q(a) \vee {}_{(4)}R(a)$
$C_7 = Res^{lock\ Pr}_{\theta_1 = [x \leftarrow a]}(C_2, C_3) = {}_{(4)}R(a)$
$C_8 = Res^{lock\ Pr}_{\theta_2 = [z \leftarrow y]}(C_4, C_5) = {}_{(8)}\neg R(y)$
$C_9 = Res^{lock\ Pr}_{\theta_3 = [y \leftarrow a]}(C_7, C_8) = \square \overset{(1)}{\Rightarrow}$ S inconsistent (as we obtained the empty clause

2) $C_1 = {}_{(2)}P(x) \vee {}_{(1)}\neg Q(x)$
$C_2 = {}_{(5)}\neg P(a) \vee {}_{(4)}R(x)$
$C_3 = {}_{(3)}Q(x)$
$C_4 = {}_{(8)}W(z)$
$C_5 = {}_{(6)}\neg R(y) \vee {}_{(7)}\neg W(y)$

$C_6 = Res^{lock\ Pr}(C_1, C_3) = {}_{(2)}P(x)$
$C_7 = Res^{lock\ Pr}_{\theta_1 = [y \leftarrow x]}(C_2, C_5) = {}_{(5)}\neg P(a) \vee {}_{(7)}\neg W(x)$
$C_8 = Res^{lock\ Pr}_{\theta_2 = [x \leftarrow a]}(C_6, C_7) = {}_{(7)}\neg W(a)$
$C_9 = Res^{lock\ Pr}_{\theta_3 = [z \leftarrow a]}(C_4, C_8) = \square \ldots$

It is enough to find a lock derivation of $\square$ to prove inconsistency of a set S of clauses.

THEOREM – SOUNDNESS OF LOCK RESOLUTION: $S \vdash^{lock}_{Res} \square \Rightarrow S$ inconsistent
S set of clauses with arbitrarily indexed literals. If the empty clause can be derived using lock resolution $\Rightarrow$ S is inconsistent. (1)

$\boxed{\text{Ex6 PRED RES}}$

$H_1: (\forall x)(\forall y)\, king(x) \land oldest\_son(x,y) \rightarrow king(y) = (\forall x)(\forall y)\, \neg king(x) \lor \neg oldest\_son(x,y) \lor king(y)$

$H_2: (\forall x)(\forall y)\, king(x) \land defeat(y,x) \rightarrow king(y) = (\forall x)(\forall y)\, \neg king(x) \lor \neg defeat(y,x) \lor king(y)$

$H_3: king(RIII)$

$H_4: defeat(HVII, RIII)$

$H_5: oldest\_son(HVII, HVIII)$

$C: king(HVIII)$

THEORETICAL RESULTS

1) $U_1, U_2, \dots, U_n \vdash V$ iff $\{U_1^C, U_2^C, \dots, U_n^C, (\neg V)^C\} \vdash_{Res}^{Pr}$

2) brief descr. of linear resolution

The clausal normal forms corresponding to the hypotheses and the negation of the conclusion are:

$H_1^C: \neg king(x) \lor \neg oldest\_son(x,y) \lor king(y)$ C_1

$H_2^C: \neg king(z) \lor \neg defeat(u,z) \lor king(u)$ C_2

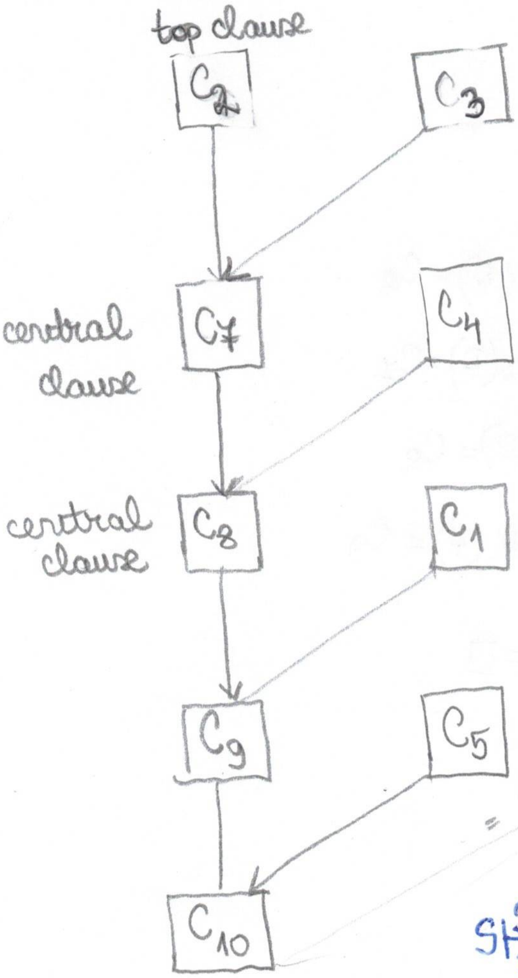
$H_3: king(RIII)$ C_3

$H_4: defeat(HVII, RIII)$ C_4

$H_5: oldest\_son(HVII, HVIII)$ C_5

$\neg C: \neg king(HVIII)$ C_6

$S = \{C_1, C_2, C_3, C_4, C_5, C_6\}$

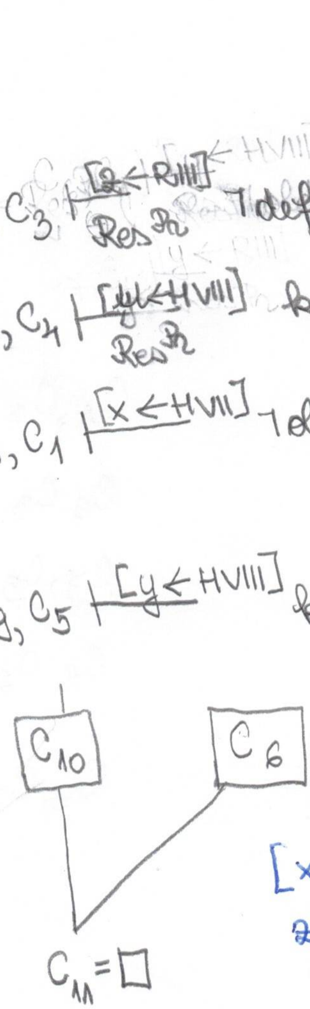


$C_2, C_3 \vdash_{Res^{Pr}}^{[z \leftarrow RIII]} \neg defeat(u, RIII) \lor king(u) = C_7$

$C_7, C_4 \vdash_{Res^{Pr}}^{[u \leftarrow HVII]} king(HVII) = C_8$

$C_8, C_1 \vdash^{[x \leftarrow HVII]} \neg oldest\_son(HVII, y) \lor king(y) = C_9$

$C_9, C_5 \vdash^{[y \leftarrow HVIII]} king(HVIII) = C_{10}$



The mgu generated during resolution:

$[x \leftarrow HVII, y \leftarrow HVIII, z \leftarrow RIII, u \leftarrow HVIII]$

$S \vdash_{Res}^{Pr} \square \Rightarrow S$ inconsistent $\Rightarrow H_1, H_2, H_3, H_4, H_5 \vdash C$ holds.