Seminar 11

1. For $f(x,y) = \dfrac{x}{y} + x\sqrt{y}$ check where $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

2. Find the second-order Taylor polynomial for the following functions at the given points:

   (a) $f(x,y) = \sin(x+2y)$ at $(0,0)$.

   (b) $f(x,y) = e^{x+y}$ at $(0,0)$ and $(1,-1)$.

   (c) $f(x,y) = \sin(x)\sin(y)$ at $(\pi/2, \pi/2)$.

   (d) $f(x,y) = e^{-(x^2+y^2)}$ at $(0,0)$.

3. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ be a diagonal $n \times n$ matrix and consider the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Dx$. Prove that $\nabla f(x) = Dx$ and $H(x) = D$. Compute the directional derivative $D_v f(x)$ in two ways.

4. ★ Compute the Hessian matrix and its eigenvalues for the following:

   (a) $f(x,y) = (y-1)e^x + (x-1)e^y$ at $(0,0)$.

   (b) $f(x,y) = \sin(x)\cos(y)$ at $(\pi/2, 0)$.

5. Find and classify the critical points for each of the following functions:

   (a) $f(x,y) = x^2 - y^2$.

   (b) $f(x,y) = x^3 - 3x + y^2$.

   (c) $f(x,y) = x^3 + y^3 - 6xy$.

   (d) $f(x,y) = x^4 + y^4 - 4(x-y)^2$.

6. Given the data points (x_i, y_i), $i = 1, \ldots, n$, find the regression line $f(x) = \alpha x + \beta$ that minimizes the least-squares error
$$\sum_{i=1}^{n} |f(x_i) - y_i|^2.$$

7. ★[Python] Let A be a 2×2 matrix and let the quadratic function $f : \mathbb{R}^2 \to R$, $f(x) = x^T Ax$.

   (a) Give two (different) matrices A such that f has a unique minimum.

   (b) Give two (different) matrices A such that f has a unique maximum.

   (c) Give two (different) matrices A such that f has a unique saddle point.

   For each matrix A plot three contour lines of f and the gradient at three different points.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.