EX PROPOSITIONAL LOGIC

1. Check the following properties using the truth table method.

1.2. associativity of '↓' connective: $\underbrace{p\downarrow(q\downarrow r)}_{U} \equiv \underbrace{(p\downarrow q)\downarrow r}_{V}$

THEORETICAL RESULTS

The formulas $U \in F_P$ and $V \in F_P$ are logically equivalent, notation: $U \equiv V$, if U and V have identical truth tables: $\forall i \in F_P \to \{T,F\}$, we have that $i(U) = i(V)$.

⇒ draw truth table → each step: p, q, r, $q\downarrow r$, $p\downarrow(q\downarrow r)$, $p\downarrow q$, $(p\downarrow q)\downarrow r$

→ check that the 2 formulas, U and V have identical truth tables

→ draw conclusion (i.e. write it out explicitly)

2. Using the truth table method decide what kind of formula is and write all models and anti-models.

2.3. $U_3 = \neg p \wedge (\neg q \vee r) \to q \wedge \neg p \vee r$

THEORETICAL RESULTS: $U(p_1, \dots, p_n)$ propositional formula

1) A model of a formula U is an interpretation....
2) An anti-model .....
3) A formula U is called consistent if ....
4) A formula U is called valid/a tautology if....
5) -ll- inconsistent if....
6) -ll- contingent if....

→ draw table for U_3 (be careful about precedence)

highest precedence

$\neg$

$\wedge$

$\vee$

$\to$

$\leftrightarrow$

i.e. don't just write "model" beside the row of table :)

→ draw conclusions (explicitly); write models as anti-models

$i_k: \{p, q, r\} \to \{T, F\}$

$i_k(p) = \dots$ $i_k(q) = \dots$ $i_k(q) = \dots$

Ex 3,4 → similar

as theoretical results ⇒ 3) formula $V \in F_P$ logical consequence of $U \in F_P$, $U \models V$ if all models ...

4) a formula is called a tautology if .....

EX 5 PROPOSITIONAL LOGIC

Transform the formulas U_j into their CNF and DNF. Use these forms to prove U_j are valid.

$$U_3 = (p \wedge q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$$

THEORETICAL RESULTS

1) A cube is a conjunction of a finite no. of literals.
2) A clause is a disjunction of a finite no of literals.
3) A formula is in disjunctive normal form (DNF) if it is written as a disjunction of cubes:

$$\vee_{i=1}^{m} (\wedge_{j=1}^{n} \ell_{ij}), \ \ell_{ij} = \text{literals}$$

4) A formula is in conjunctive normal form (CNF) if it is written as a conjunction of clauses:

$$\wedge_{i=1}^{n} (\vee_{j=1}^{m} \ell_{ij}) \ \ \ell_{ij} = \text{literals}$$

5) NORMALIZATION
- replace $U \rightarrow V$, $U \leftrightarrow V$
- apply DeMorgan's laws
- apply distributive laws

$U \rightarrow V \equiv \neg U \vee V$

$U \leftrightarrow V \equiv (\neg U \vee V) \wedge (\neg V \vee U)$

6) A formula in CNF is a tautology iff all its clauses are tautologies.

7)

$$U_3 = (p \wedge q \xrightarrow{1} r) \xrightarrow{2} (p \xrightarrow{3} (q \xrightarrow{4} r))$$

replace 2
$$\equiv \neg(p \wedge q \xrightarrow{1} r) \vee (p \xrightarrow{3} (q \xrightarrow{4} r))$$

replace 1, 3
$$\equiv \neg(\neg(p \wedge q) \vee r) \vee (\neg p \vee (q \xrightarrow{4} r))$$

replace 4
$$\equiv \neg(\neg(p \wedge q) \vee r) \vee (\neg p \vee (\neg q \vee r))$$

deMorgan
$$\equiv ((p \wedge q) \wedge \neg r) \vee (\neg p \vee \neg q \vee r)$$

$$\equiv (p \wedge q \wedge \neg r) \vee \neg p \vee \neg q \vee r$$

DNF with 4 cubes: $p \wedge q \wedge \neg r$
$\neg p$
$\neg q$
r

$\Rightarrow$ to obtain CNF, apply distributivity

$$U \wedge (V \vee Z) \equiv (U \wedge V) \vee (U \wedge Z)$$
$$U \vee (V \wedge Z) \equiv (U \vee V) \wedge (U \vee Z)$$

$$U_3 = \underset{U}{(p \wedge q \wedge \neg r)} \vee \neg p \vee \neg q \vee \neg r$$
$$\equiv (\neg p \vee \neg q \vee \neg r) \vee (p \wedge q \wedge \neg r)$$
$$\equiv (\underline{\neg p} \vee \neg q \vee r \vee \underline{p}) \wedge (\neg p \vee \underline{\neg q} \vee r \vee \underline{q}) \wedge (\neg p \vee \neg q \vee \underline{r} \vee \underline{\neg r})$$ CNF with 3 cubes

Each clause of CNF is a tautology (contains a pair of opposite literals - underlined) $\overset{\text{G}}{\Rightarrow}$ U_3 is a tautology.

6. Using the appropriate normal form, write all models:

6.3. $U_3 = (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r) \wedge q$

THEORETICAL RESULTS

- all applicable from Exercise 5)

+

7) The DNF of a propositional formula provides all ~~an~~ the models of that formula by finding all the interpretations which evaluate, one by one, the cubes as true.

$$U_3 = (p \wedge q \xrightarrow{1} r) \xrightarrow{2} (p \xrightarrow{3} r) \wedge q$$

replace 1, 3

$$\equiv (\neg(p \wedge q) \vee r) \xrightarrow{2} ((\neg p \vee r) \wedge q)$$

replace 2

$$\equiv \neg(\neg(p \wedge q) \vee r) \vee ((\neg p \vee r) \wedge q)$$

deMorgan

$$\equiv ((p \wedge q) \wedge \neg r) \vee ((\neg p \vee r) \wedge q)$$

$$\equiv (p \wedge q \wedge \neg r) \vee ((\neg p \vee r) \wedge q)$$

distributivity

$$\equiv (p \wedge q \wedge \neg r) \vee ((\neg p \wedge q) \vee (r \wedge q))$$

$$\equiv (p \wedge q \wedge \neg r) \vee (\neg p \wedge q) \vee (r \wedge q)$$

DNF with 3 cubes:

$p \wedge q \wedge \neg r$

$\neg p \wedge q$

$r \wedge q$

→ evaluate each cube as true:

(1) $p \wedge q \wedge \neg r : T \Rightarrow i_1 : \{p, q, r\} \rightarrow \{T, F\}$ model for U_3

p — T, q — T, $\neg r$ — F

$i_1(p) = T$

$i_2(q) = T$

$i_3(r) = F$

(2) $\neg p \wedge q : T \Rightarrow i_2, i_3 : \{p, q, r\} \rightarrow \{T, F\}$ models for U_3

$\neg p$ — F, q — T

$i_2(p) = F$	$i_3(p) = F$
$i_2(q) = T$	$i_3(q) = T$
$i_2(r) = T$	$i_3(r) = F$

③ $\underset{T}{r} \wedge \underset{T}{q} : T \Rightarrow i_4, i_5 : \{p, q, r\} \to \{T, F\}$ models for U_3

$$i_4(p) = T \qquad i_5(p) = F$$
$$i_4(q) = T \qquad i_5(q) = T$$
$$i_4(r) = T \qquad i_5(r) = T$$

We have obtained models i_1, i_2, i_3, i_4, i_5. However, $i_2 = i_5 \Rightarrow U_3$ has 4 distinct models i_1, i_2, i_3, i_4:

$$i_1(U_3) = i_2(U_3) = i_3(U_3) = i_4(U_3) = T$$

Ex.9 Using the definition of deduction, prove the following deductions:

2. $p \to r,\ p \vee r_3 \to q,\ r \vdash q$

$f_1: p \to r$

$f_2: p \vee r \to q$

$f_3: r$

$f_3 \vdash_{addition} p \vee r$ (where $U = r$, $V = p$, $\cancel{p}\ r \vee p \equiv p \vee r$) i.e. $r \vdash r \vee p$

ADDITION

$U \vdash U \vee V$ (inference rule)

($\vdash U \to U \vee V$ (theorem))

$f_4: p \vee r$

$f_2: p \vee r \to q$

$\cancel{f_4}: p \vee r$

$f_2, f_4 \vdash_{mp} q$

MODUS PONENS

$U, U \to V \vdash V$ (inference rule)

($\vdash U \wedge (U \to V) \to V$ (theorem))

*write all inference rules you use

$f_5: q$

The sequence $(\cancel{f}1, \cancel{f}2, f_3, f_4, f_5)$ is the deduction of q from the hypotheses

$p \to r$

$p \vee r \to q$

r

THEORETICAL RESULTS

DEFINITION OF DEDUCTION:

Let $U_1, \dots, U_m$ be propositional formulas (= hypotheses) and V a formula called conclusion. V is deducible from $U_1, \dots, U_m$ and we denote by $U_1, U_2, \dots, U_m \vdash V$ if there exists a sequence $(f_1, \dots, f_m)$ of formulas such that $f_m = V$ and $\forall i \in \{1, \dots, m\}$ we have a), b) or c)

a) $f_i \in A_p$

b) $f_i \in \{U_1, \dots, U_m\}$

c) $f_{i_1}, f_{i_2} \vdash_{mp} f_i$

Ex 10 Prove the following theorems using the theorem of deduction & reverse.

2. $\vdash (p \to (\neg r \to q)) \to (r \vee \neg p \vee q)$

STEP 1

if $\vdash (p \to (\neg r \to q)) \to (r \vee \neg p \vee q)$

then $(p \to (\neg r \to q)) \vdash r \vee \neg p \vee q$ ← apply reverse of theorem of deduction

if $(p \to (\neg r \to q)) \vdash \neg(r \vee \neg p) \to q$

then $p \to (\neg r \to q), \neg(r \vee \neg p) \vdash q$

$\cancel{f_1}$

STEP 2: Prove deduction obtained in STEP 1

$f_1: p \to (\neg r \to q)$

$f_2: \neg(r \vee \neg p) \equiv \neg r \wedge p$

SIMPLIFICATION

$U \wedge V \vdash U$ (inference rule)

($\vdash U \wedge V \to U$ (theorem))

$f_2 \vdash_{simplification} \neg r$

(i.e. $U = \neg r$

$V = p$)

$f_2 \vdash_{simplification} p$

$f_3: \neg r$

$f_4: p$

MODUS PONENS (mp)

$U, U \to V \vdash V$ (inf. rule)

($\vdash U \wedge (U \to V) \to V$ (theorem))

$f_1, f_4 \vdash_{mp} \neg r \to q$

$f_5: \neg r \to q$

$f_3, f_5 \vdash_{mp} q$ $\quad f_6: q$

The sequence $(f_1, f_2, f_3, f_4, f_5, f_6)$ is the deduction of q from premises

We can deduce 2 theorems depending on the order in which we apply theorem of deduction (see Ex 11).

Ex 11 Using the theorem of deduction and reverse, prove that:

$\vdash (p \to q) \to ((\neg r \vee p) \to (r \to q))$

STEP 1: Apply reverse of theorem of deduction ~~is/applies~~ to obtain starting point deduction.

if $\vdash (p \to q) \to ((\neg r \vee p) \to (r \to q))$

then if $p \to q \vdash (\neg r \vee p) \to (r \to q)$

then if $p \to q, \neg r \vee p \vdash r \to q$

then if $p \to q, \neg r \vee p, r \vdash q$

STEP 2: Prove deduction obtained in Step 1

$f_1: p \to q$

$f_2: \neg r \vee p \equiv r \to p$

$f_3: r$

~~$f_4: q$~~

MODUS PONENS

$U, U \to V \vdash_{mp} V$

$f_3, f_2 \vdash_{mp} p$

$f_4: p$

$f_4, f_1 \vdash_{mp} q$

$f_5: q$

The sequence $(f_1, f_2, f_3, f_4, f_5)$ is the deduction of q from the premises: $p \to q, \neg r \vee p, r$.

STEP 3:

To the deduction

$p \to q, \neg r \vee p, r \vdash q$ we apply 3 times the theorem of deduction.

There are $3! = 6$ possibilities to move the premises to the right-hand side of the meta-symbol $\vdash$) and we prove 6 theorems: $T_1, T_2, T_3, T_4, T_5, T_6$

Possibility 1: move in order: $p \to q, \neg r \vee p, r$

if $p \to q, \neg r \vee p, r \vdash q$

then $p \to q, \neg r \vee p \vdash r \to q$

then $p \to q \vdash (\neg r \vee p) \to (r \to q)$

then $\cancel{p} \vdash (p \to q) \to ((\neg r \vee p) \to (r \to q))$

~~then $\vdash p \to (q$~~

Possibility 2: move in order: $r, p \to q, \neg r \vee p$

if $r, p \to q, \neg r \vee p \vdash q$

then $p \to q, \neg r \vee p \vdash \cancel{q}\ r \to q$

then $\neg r \vee p \vdash (p \to q) \to (r \to q)$

then $\vdash (\neg r \vee p) \to ((p \to q) \to (r \to q))$

Possibility 3: move in order $r, \neg r \vee p, p \to q$

if $r, \neg r \vee p, p \to q \vdash q$

then $\neg r \vee p, p \to q \vdash (r \to q)$

then $p \to q \vdash (\neg r \vee p) \to (r \to q)$

then $\vdash (p \to q) \to ((\neg r \vee p) \to (r \to q))$

We need to also write ~~possibilities for~~ theorems obtained with possib. 4, 5, 6

Ex 12

H_1: It is not sunny this afternoon and it is colder than yesterday.
$\underbrace{\text{It is not sunny}}_{sun}$ $\qquad$ $\underbrace{\text{it is colder than yesterday}}_{col}$

H_2: We will go swimming only if it is sunny.
$\underbrace{\text{We will go swimming}}_{swim}$

H_3: If we do not go swimming, then we will take canoe trip
$\underbrace{\text{we will take canoe trip}}_{can}$

H_4: If we take a canoe trip, then we will be home by sunset.
$\underbrace{\text{we will be home by sunset}}_{home}$

C: We will be home by sunset.

sun: It is not sunny. $\qquad$ sun: It is sunny

H_1: sun $\wedge$ cold
H_2: swim $\to \neg$sun
H_3: $\neg$swim $\to$ can
H_4: can $\to$ home

C: home

V1

H_1: $\neg$sun $\wedge$ cold
H_2: swim $\to$ sun
H_3: $\neg$swim $\to$ can
H_4: can $\to$ home

C: home

V2

sun $\wedge$ col, swim $\to \neg$sun, $\neg$swim $\to$ can, can $\to$ home $\vdash$ home V1

$\neg$sun $\wedge$ col, swim $\to$ sun, $\neg$swim $\to$ can, can $\to$ home $\vdash$ home V2

V2

H_5: (swim $\to$ sun) $\to$ ($\neg$sun $\to \neg$swim) $\quad$ obtained from A_3: $(U \to V) \to (\neg V \to \neg U)$
replacing U by swim
and V by sun

$H_2, H_5 \vdash_{mp} \neg sun \to \neg swim$

$U =$ ~~sun~~ swim $\to$ sun

$V = \neg sun \to \neg swim$

MODUS PONENS
$U, U \to V \vdash V$

H_6: $\neg$sun $\to \neg$swim

$H_1 \vdash_{simplification} \neg sun$

SIMPLIFICATION
$U \wedge V \vdash U$

H_7: $\neg$sun

$H_7, H_6 \vdash_{mp} \neg snow$

$\downarrow$

$U = \neg sun$

$V = \neg snow$

$H_8 : \neg snow$

$H_8, H_3 \vdash_{mp} \cancel{\neg}\, car$

$\downarrow$

$U = \neg snow$

$V = car$

$H_9 : car$

$H_9, H_4 \vdash_{mp} home$ $\qquad H_{10} : home$

$\downarrow$

$U = car$

$V = home$

$(H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10})$ deduction of our conclusion from premises