Seminar 12

1. Let A be a symmetric $n \times n$ matrix and the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Ax$. Prove that $\nabla f(x) = Ax$ and $H(x) = A$. *Hint: use the Taylor expansion.*

2. Let A be an $m \times n$ matrix, b a vector in R^m and the least squares minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2.$$

   Prove that the solution x^* of this problem satisfies (the so-called normal equations)

$$A^T Ax^* = A^T b.$$

3. Constrained optimization: for $f, g \in \mathbb{R}^n \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, the problem

$$\text{minimize/maximize } f(x) \text{ subject to } g(x) = c$$

   is solved by looking for the critical points of the Lagrange function (λ – Lagrange multiplier)

$$L(x, \lambda) = f(x) - \lambda\big(g(x) - c\big).$$

   Find the extrema of the following functions subject to the constraints:

   (a) $x^2 + y^2$ subject to $x - y + 1 = 0$.
   (b) $(x + y)^2$ subject to $x^2 + y^2 = 1$.
   (c) $x^2 - y^2$ subject to $x^2 + y^2 = 1$.
   (d) $x + 2y + 3z$ subject to $x^2 + y^2 + z^2 = 1$.
   (e) $2x^2 + y^2 + 3z^2$ subject to $x^2 + y^2 + z^2 = 1$.
   (f) $x^3 + y^3 + z^3$ subject to $x^2 + y^2 + z^2 = 1$.

4. ★[Python] Consider the quadratic function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{1}{2}(x^2 + by^2)$ with $b > 0$ and the gradient descent algorithm for finding its minimum

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - s_k \nabla f(x_k, y_k),$$

   where the step size $s_k > 0$ is chosen (exact line search) to minimize the function

$$\varphi(s) = f(x_{k+1}, y_{k+1}) = f\big((x_k, y_k) - s\nabla f(x_k, y_k)\big), \quad \varphi'(s_k) = 0.$$

   For $b = 1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}$ plot some gradient descent iterations and the relevant contour lines of f. What do you notice as b gets smaller?

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.