AA
lab 4

1. $T(n) = a\,T(n/b) + f(n)$

a) $T(n) = 16\,T(n/4) + n^2$

$a = 16$
$b = 4$
$f(n) = n^2$

CAZ I: dacă $\exists\, \varepsilon > 0$ a.î. $f(n) = O(n^{\log_b a - \varepsilon})$ atunci $T(n) = \Theta(n^{\log_b a})$

$n^2 = O(n^{2-\varepsilon}), \varepsilon > 0 \Rightarrow 2 - \varepsilon < 2 \Rightarrow n^2 \notin O(n^{2-\varepsilon}) \Rightarrow$

$\Rightarrow$ CAZ I nu e bun.

CAZ II: dacă $f(n) = \Theta(n^{\log_b a})$ atunci $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$

$n^2 = \Theta(n^{\log_4 16}) = \Theta(n^2) \Rightarrow \boxed{T(n) = \Theta(n^2 \lg n)}$

b) $T(n) = 4\,T(n/2) + \Theta(n)$, $\Theta(n) = k n$

$a = 4$
$b = 2$
$f(n) = \Theta(n)$

CAZ II: dacă $f(n) = \Theta(n^{\log_b a})$ atunci $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$

$\left.\begin{array}{l} f(n) = \Theta(n), \\ f(n) = \Theta(n^{\log_2 4}) = \Theta(n^2) \end{array}\right\} \Rightarrow \Theta(n) = \Theta(n^2)$ F $\Rightarrow$
$\Rightarrow$ CAZ II nu este bun.

CAZ I: dacă $\exists\, \varepsilon > 0$ a.î. $f(n) = O(n^{\log_b a - \varepsilon})$ atunci $T(n) = \Theta(n^{\log_b a})$

$$\left.\begin{array}{l} a=4 \\ b=2 \end{array}\right\} \Rightarrow n^{\log_2 4} = n^2$$

$$f(n) = \Theta(n) = \Theta(n^{2-1}) = \Theta(n^{\log_b a - 1}) \Rightarrow$$

$$\Rightarrow f(n) = O(n^{\log_b a - 1}) \Rightarrow \varepsilon = 1 > 0 \overset{CAZ\ I}{\Rightarrow} \boxed{T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)}$$

$\cancel{= \Theta(n^2)}$

c) $T(n) = 2T(n/2) + \Theta(n^3)$, $\Theta(n^3) = k\, n^3$

CAZ II: Dacă $f(n) = \Theta(n^{\log_b a})$, atunci $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$

$$\left.\begin{array}{l} a=b=2 \Rightarrow f(n) = \Theta(n^{\log_2 2}) = \Theta(n) \\ f(n) = \Theta(n^3) \end{array}\right\} \Rightarrow \Theta(n) = \Theta(n^3) \ ↯ \Rightarrow$$

$\Rightarrow$ CAZ II nu e bun.

CAZ I: Dacă $\exists\, \varepsilon > 0$ a.î. $f(n) = O(n^{\log_b a - \varepsilon})$ atunci $T(n) = \Theta(n^{\log_b a})$

$f(n) = \Theta(n^3)$

$a = b = 2 \Rightarrow f(n) = O(n^{1-\varepsilon})$ cu $\varepsilon > 0$

$\varepsilon > 0 \Rightarrow 1 - \varepsilon < 1 < 3 \Rightarrow 3 > 1 - \varepsilon \Rightarrow k\, n^3 \notin O(n^{1-\varepsilon}), \forall \varepsilon > 0$

$\Rightarrow$ CAZ I nu este bun.

CAZ III: Dacă $\exists \varepsilon > 0$ a.î. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ și $\exists c \in (0,1)$, $n_0 \in \mathbb{N}^*$ a.î. $a \cdot f(n/b) \leq c f(n), \forall n \geq n_0$ atunci $T(n) = \Theta(f(n))$.

a) $\exists \varepsilon > 0$ a.î. $f(n) = \Omega(n^{\log_b a + \varepsilon})$

$\exists c' \in \mathbb{R}_+^*, \exists n_0' \in \mathbb{N}^*$ a.î. $c' n^{\log_b a + \varepsilon} \leq f(n), \forall n \geq n_0'$

$c' n^{\log_2 2 + \varepsilon} \leq k n^3$

$c' n^{1+\varepsilon} \leq k n^3 \Rightarrow$ pt. $\varepsilon = 2 > 0 \Rightarrow c' n^3 \leq k n^3 \Rightarrow$

$\Rightarrow \varepsilon = 2$ pt. $\forall n \geq n_0$ cu $n_0 = 1$ și $c' \leq k$

b) $\exists c \in (0,1), n_0 \in \mathbb{N}^*$ a.î. $af(n/b) \leq c f(n), \forall n \geq n_0$

$$\left.\begin{array}{l} 2 f(n/2) \leq c f(n) \\ f(n) = \Theta(n^3) \end{array}\right\} \Rightarrow 2 \cancel{k} \frac{n^3}{8} \leq c n^3 \cancel{k} \cancel{\Rightarrow \frac{k}{4} \leq ck \Rightarrow}$$

$\cancel{\Rightarrow \frac{1}{4} \leq c}$ $\quad 0 \leq kcn^3 - \frac{1}{4} n^3 k \Rightarrow 0 \leq n^3 (c - \frac{1}{4}) k$

1. $k n^3 \geq 0 \Rightarrow c - \frac{1}{4} \geq 0 \Rightarrow c \geq \frac{1}{4} \Rightarrow \exists c \in (\frac{1}{4}, 1)$ a.î. $0 \leq (c - \frac{1}{4}) k n^3$

2. $k n^3 \leq 0 \Rightarrow c - \frac{1}{4} \leq 0 \Rightarrow c \leq \frac{1}{4} \Rightarrow \exists c \in (0; \frac{1}{4})$ a.î. $0 \leq (c - \frac{1}{4}) k n^3$

$1. \wedge 2. \Rightarrow$ b) corect.

$$a) \wedge b) \overset{CAZ\ III}{\Rightarrow} \boxed{T(n) = \Theta(\Theta(n^3)) = \Theta(k n^3) = \Theta(n^3)}$$

d) $T(n) = 2T(n/2) + \Theta(n \cdot \lg n)$

$a = b = 2$

$f(n) = \Theta(n \cdot \lg n)$

CAZ I: Dacă $\exists\, \varepsilon > 0$ a.î. $f(n) = O(n^{\log_b a - \varepsilon})$ atunci $T(n) = \Theta(n^{\log_b a})$

$f(n) = \Theta(n \cdot \lg n) = O(n^{1-\varepsilon}) \Leftrightarrow k n \lg n = O(n^{1-\varepsilon})$ F $\Rightarrow$

$\Rightarrow$ CAZ I nu e bun

CAZ II: Dacă $f(n) = \Theta(n^{\log_b a})$ atunci $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$

$f(n) = k \cdot n \cdot \lg n = \Theta(n)$ F $\Rightarrow$ CAZ II nu e bun.

CAZ III: Dacă $\exists\, \varepsilon > 0$ a.î. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ și $\exists\, c \in (0,1)$, $n_0 \in \mathbb{N}^*$ a.î. $a f(n/b) \le c f(n)$, $\forall n \ge n_0$ atunci $T(n) = \Theta(f(n))$

~~$\exists\, c' \in \mathbb{R}^*_+$, $\exists\, n_0' \in \mathbb{N}^*$ a.î. $c' \cdot n \lg n \le n^{1+\varepsilon}$~~

$\exists\, c' \in \mathbb{R}^*_+$, $\exists\, n_0' \in \mathbb{N}^*$ a.î. $c' n^{1+\varepsilon} \le n \lg n$, $\forall n \ge n_0'$

$c' n \cdot n^{\varepsilon} \le n \lg n \quad | :n$

$c' n^{\varepsilon} \le \lg n \Rightarrow c' \le \frac{\lg n}{n^{\varepsilon}} \xrightarrow{n \to \infty} 0$ F $(c' \in \mathbb{R}^*_+) \Rightarrow$

$\Rightarrow$ CAZ III nu e bun

I, II, III $\Rightarrow$ Nu se poate aplica ~~Metoda~~ Teorema Master.

2. a) $T(n) = 2T(n/2) + \Theta(n \lg n)$

$$T(n) = \begin{cases} K_1, & n=1 \\ T(n/2)+T(n/2)+\Theta(n\lg n), & n>1 \end{cases} \Rightarrow$$

$\Rightarrow T(n) = 2T(n/2)+\Theta(n\lg n)$

$T(1)=\Theta(1) = K_1$

$$T(n) = 2T(\cancel{n/2}) + \Theta(n\lg n) \qquad | \cdot 2^0$$

$$T(\cancel{n/2}) = 2T(n/2^2) + \Theta(n/2 \lg \tfrac{n}{2}) \qquad |\cdot 2^1$$

$\vdots$

$$T(\cancel{n/2^k}) = 2T(\overbrace{n/2^{k+1}}^{=1}) + \Theta(n/2^k \lg(n/2^k)) \qquad | \cdot 2^k$$

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$$T(n) = 2^{k+1}\cdot T(n/2^{k+1}) + \sum_{i=0}^{k} 2^i \Theta(n/2^i \lg(n/2^i))$$

$n/2^{k+1} = 1 \Rightarrow n = 2^{k+1} \Rightarrow k+1 = \lg n$

$k = (\lg n) - 1$

$$T(n) = 2^{k+1} T(1) + \sum_{i=0}^{k} \Theta(\cancel{2^i}\cdot \frac{n}{\cancel{2^i}} \lg(n/2^i)) =$$

$$= 2^{\lg n} T(1) + \sum_{i=0}^{n} \Theta(n(\lg n - \lg 2^i)) = n + \sum_{i=0}^{k} \Theta(n\lg n) - \Theta(n\cdot i)$$

$$= n + \sum_{i=0}^{k} \Theta(n\lg n) - \sum_{i=0}^{k} \Theta(n\cdot i)$$

$$= n \cdot \Theta(1) + \Theta(n \lg n) \frac{k(k+1)}{2} - \Theta(n) \cdot \frac{k(k+1)}{2} =$$

$$= \Theta\left(n + n \lg n \frac{(n-1)n}{2} - n \frac{(n-1)n}{2} \right) =$$

$$= \Theta\left(n + \frac{n^2(n-1)}{2} (\lg n - 1) \right)$$

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b) $T(n) = 2T(n-1) + \Theta(1)$

$$T(n) = 2\cancel{T(n-1)} + \Theta(1) \quad | \cdot 2^0$$

$$\cancel{T(n-1)} = 2T(n-2) + \Theta(1) \quad | \cdot 2^1$$

$$\vdots$$

$$\cancel{T(n-k)} = 2T(\overbrace{n-k-1}^{=1}) + \Theta(1) \quad | \cdot 2^k$$

$+$

$$T(n) = 2^{k+1} T(n-k-1) + 2^k \Theta(1)$$

$$n = k+2 \Rightarrow T(n) = 2^{n-1} T(1) + 2^{n-2} \Theta(1)$$

$$T(n) = \Theta(2^{n-1} + 2^{n-2}) = \Theta(2^{n-1}) = \frac{1}{2} \Theta(2^n) = \Theta(2^n)$$

$$T(n) = \Theta(2^n)$$