UNIVERSITATEA BABEŞ-BOLYAI CLUJ-NAPOCA
FACULTATEA DE MATEMATICĂ ŞI INFORMATICĂ
SPECIALIZAREA Matematică

LUCRARE DE LICENŢĂ

Metode de Regularizare pentru Probleme Inverse de Ecuații cu Derivate Parțiale

Conducător științific
Lect. dr. Nechita Mihai

Absolvent
Răhăian Rareș

2025

BABEŞ-BOLYAI UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
SPECIALIZATION Mathematics

DIPLOMA THESIS

Regularisation Methods for Inverse Problems related to Partial Differential Equations

Supervisor
Lect. dr. Nechita Mihai

Author
Răhăian Rareș

2025

Abstract

Inverse problems related to partial differential equations (PDEs) arise in many fields, including medical imaging, geophysics, meteorology, and fluid mechanics. When only partial data are available, e.g. if measurements can only be done on a part of the boundary or inside a subdomain, the numerical reconstruction of the solution becomes an ill-posed problem. Such problems must be regularised to obtain stable, physically meaningful solutions.

This thesis first presents a concise abstract framework that unifies several regularisation techniques widely used in the inverse problems literature with an emphasis on linear inverse problems. This is done in Chapters 1, 2 and 3. Chapter 4 then presents the general abstract framework for inverse problems related to PDEs. These inverse problems usually arise from the weak formulation of the PDE.

In Chapter 5 the presented regularisation methods are then applied to solve the Cauchy and the Unique Continuation problem for elliptic operators. Both are recast as PDE-constrained optimisation problems and solved using the method of Lagrange multipliers, by introducing the test function in the weak formulation of the PDE as a multiplier. The resulting Lagrangians are then regularised using Tikhonov regularisation, following ideas presented in [33]. Two methods for the Cauchy problem are introduced, although one is very similar to a method developed in [9], but with a different approach. Well-posedness results are proved on the continuous level for one of the methods for both problems. The regularised optimality systems are then discretised using the finite element method. Discretisations and numerical experiments for the Laplace and Helmholtz operators are presented in Chapter 6.

Contents

Introduction

In this thesis we focus on two ill-posed inverse problems for elliptic partial differential equations—the Cauchy problem, in which Dirichlet and Neumann data are available only on a part of the boundary, and the unique continuation problem, where the solution must be reconstructed from measurements in a subdomain. Both tasks are ill-posed in the sense of Hadamard[22]: the solution doesn't depend continuously on the data. To tame this instability we use Tikhonov regularisation, drawing on ideas presented in [33]. The work is motivated by applications ranging from non-destructive testing to medical imaging, where only partial or interior data can be accessed experimentally. The goal of this thesis is to introduce or present some techniques of solving the considered problems, prove well-posedness results for these methods and illustrate the behavior of the methods through various numerical illustrations for two elliptic operators.

Chapter 1 gathers the functional-analytic tools, namely fundamental results regarding the space of bounded operators between normed vector spaces and compact operators, that underpin our later analysis. These results are presented with an eye toward operator equations of the form $\mathcal{A}u = f$ in Banach/Hilbert spaces.

Chapter 2 introduces linear inverse problems in an abstract setting. We revisit the notions of well-posedness, ill-posedness and generalised solutions, recall the generalised Moore–Penrose inverse, and discuss the Picard criterion and singular-value decay for compact operators. The chapter focuses on the case when the forward operator $\mathcal{A}$ is compact and has infinite dimensional range.

Chapter 3 emphasizes the definitions of regularisation methods and operators and surveys classical regularisation strategies such as spectral regularisation methods, including Tikhonov regularisation. Variational regularisation is also briefly drawing on the variational formulation of Tikhonov regularisation, which will be applied throughout the thesis.

Chapter 4 specialises in the abstract framework of inverse problems arising from PDEs, where the abstract linear operator could be modeled using the bilinear form in the weak formulation of the PDE. Starting from weak formulations, we present general well-posedness results in order to introduce the abstract context for the two considered inverse problems. Galerkin approximation is also presented as a discretisation technique for such problems and discrete well-posedness results are mentioned.

Chapter 5 introduces the two considered problems and presents their weak formulation, in order to reformulate both model problems as PDE-constrained optimisation tasks. Introducing the test function of the weak formulation as a Lagrange multiplier, the optimisation problems are equivalent to the minimisation of some Lagrangians, whose saddle-point systems are stabilised via Tikhonov regularisation. These optimality systems are then gathered in a bilinear form, and the final formulation matches the abstract one presented in Chapter 5. Two methods for solving the Cauchy problem are introduced using this technique arriving at similar results as presented in [9], by taking a different approach however. Well-posedness results are proved for the first method for both the Cauchy and the Unique Continuation Problem using results presented in Chapter 5 and [32]. These proofs and methods were presented together with some numerical illustrations in the ”Sesiunea de Comunicări Științifice ale Studenților - Matematică, 2025”, where I received the second prize for my presentation.

Chapter 6 turns to computation. We present the discretisations of our problems using the finite element method. Numerical results are presented and discussed for various examples for Laplace's, Poisson's or Helmholtz's equations.

Declaration of Generative AI in the writing process

During the preparation of this work, generative artificial intelligence was used, namely ChatGPT(4o, o3, o4-mini), to assist in specific technical tasks. ChatGPT was employed for generating blocks of code for the numerical implementations and LaTeX formatting code to enhance the document structure and clarity. ChatGPT was also used in the process of installing all needed libraries and dependencies in order to be able to develop the code.

The integration of these tools was limited to technical support tasks, enabling me to focus on the mathematical concepts, problem-solving, and analytical content that form the core of this thesis. After utilizing these tools for their specific purposes, all content was reviewed and edited as needed and I take full responsibility for the accuracy and integrity of the thesis.

Chapter 1

Basics of Functional Analysis

1.1 Bounded Operators between Normed Vector Spaces

In the following, let U and V be real normed vector spaces and let $\mathcal{A} : U \to V$ be a linear operator mapping these 2 spaces. Let us denote by:

- Dom($\mathcal{A}$), the *domain* of $\mathcal{A}$;
- Ker($\mathcal{A}$) $= \{u \in U : \mathcal{A}u = 0\}$, the *kernel* of $\mathcal{A}$;
- Im($\mathcal{A}$) $= \{\mathcal{A}u \in V : u \in U\}$, the *range* of $\mathcal{A}$;
- Gr($\mathcal{A}$) $= \{(u, \mathcal{A}u) \in U \times V : u \in \text{Dom}(\mathcal{A})\}$, the graph of $\mathcal{A}$.

We say that $\mathcal{A}$ is *continuous* or *bounded* if

$$\exists c > 0, \text{ such that } \|\mathcal{A}u\| \leq c\|u\|, \forall u \in U.$$

The space of bounded linear operators from U to V will be denoted by $\mathcal{LC}(U, V)$, and if V is complete it becomes a Banach space when endowed with the norm given by:

$$\|\mathcal{A}\| = \inf\{c > 0 : \|\mathcal{A}u\| \leq c\|u\|, \forall u \in U\}.$$

If $V = \mathbb{R}$, $\mathcal{LC}(U, V)$ will be the space of bounded linear functionals defined on U, namely the dual space of U and will be denoted by U'. We will present some fundamental results that will be used throughout this thesis. However, the proofs will only be referenced, since we will only use these results in order to prove some other results.

Theorem 1.1. *(Algebraic Hahn-Banach Theorem) Let U be a vector space and X a subspace of U. Let $p : U \to \mathbb{R}$ be sublinear and $l_0 : X \to \mathbb{R}$ linear such that $l_0(x) \leq p(x), \forall x \in X$. Then, there exists $l : U \to \mathbb{R}$ linear such that $l|_X = l_0$ and $l(x) \leq p(x), \forall x \in U$.*

Proof. See [35], Prop.III.1.2. □

This theorem actually implies the existence of a bounded linear functional defined on a normed vector space: let U be a normed vector space and let the sublinear functional in Theorem 1.1 be $p(x) = \|x\|$. Choose the subspace $X = \{0_U\}$ and $l_0 = 0$. We thus have the existence of a linear functional $l : U \to \mathbb{R}$ such that $l(x) \leq \|x\|, \forall x \in U$, and since $l(-x) = -l(x)$, we have $|l(x)| \leq \|x\|, \forall x \in U$ and thus $l \in U'$.

Theorem 1.2. *(Open Mapping Theorem) Let $\mathcal{A} \in \mathcal{LC}(U, V)$, where U, V are real Banach spaces. If $\mathcal{A}$ is surjective, then $\mathcal{A}$ is open.*

Proof. See [35], Th.IV.3.3. □

Corollary 1.2.1. *(Boundedness of the Inverse Operator) Let $\mathcal{A} \in \mathcal{LC}(U, V)$, where U, V are real Banach spaces. If $\mathcal{A}$ is bijective, then $\mathcal{A}^{-1}$ is bounded.*

Proof. See [10], Corollary 2.7. □

Theorem 1.3. *(Closed Graph Theorem) Let $\mathcal{A} \in \mathcal{L}(U, V)$, where U, V are real Banach spaces. Assume that Gr($\mathcal{A}$) is closed in $U \times V$. Then $\mathcal{A}$ is bounded, i.e. $\mathcal{A} \in \mathcal{LC}(U, V)$.*

Proof. See [10], Th.2.10. □

Theorem 1.4. *(BLT Theorem) Suppose that X is a dense subspace of a normed space U and let V be a complete normed space. Let $\mathcal{A}_0 : X \to V$ be a bounded linear operator. Then there exists a unique linear bounded operator $\mathcal{A} : U \to V$ such that $\mathcal{A}|_X = \mathcal{A}_0$ and $\|\mathcal{A}\| = \|\mathcal{A}_0\|$.*

Proof. See [30], Th.1.9.1. □

Now let U and V be normed vector spaces, not necessarily complete.

Definition 1.1. *Let $(\mathcal{A}_j)_{j\in J} \subset \mathcal{LC}(U, V)$. This family of bounded linear operators is called:*

- ***uniformly bounded**, if the set $\{\mathcal{A}_j : j \in J\}$ is bounded in $\mathcal{LC}(U, V)$;*
- ***pointwise bounded**, if for all $u \in U$ the set $\{\mathcal{A}_j u : j \in J\}$ is bounded in V.*

Theorem 1.5. *(Uniform Boundedness Principle) Let U be a Banach space and V be a normed vector space. Let $(\mathcal{A}_j)_{j\in\mathcal{J}}$ be a family of pointwise bounded operators in $\mathcal{LC}(U, V)$. Then $(\mathcal{A}_j)_{j\in\mathcal{J}}$ is uniformly bounded.*

Proof. See [10], Th.2.2. □

Corollary 1.5.1. *(Banach-Steinhaus) Let $(\mathcal{A})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{LC}(U, V)$, where U is a real Banach space and V is a normed vector space. Then the following conditions are equivalent:*

- *There exists* $\mathcal{A} \in \mathcal{LC}(U, V)$ *such that* $\mathcal{A}u = \lim_{n\to\infty} \mathcal{A}_n u, \forall u \in U$;
- *There is a dense subset* M *of* U *such that* $\lim_{n\to\infty} \mathcal{A}_n u$ *exists* $\forall u \in M$ *and* $(\mathcal{A}_n)_{n\in\mathbb{N}}$ *is uniformly bounded.*

Proof. See [36], Chapter 3.3, Prop.2. □

We will now introduce the concept of a reflexive Banach space. To this end, let us follow [20], Appendix C and define the isometry into the double dual:

Proposition 1.1. *Let* U *be a Banach space. The bounded linear map* $J_U : U \to U''$ *such that*

$$\langle J_U u, \phi\rangle_{U'',U'} := \langle \phi, u\rangle_{U',U}, \forall (u, \phi) \in U \times U',$$

is called the canonical isometry in the double dual and is an isometry.

Proof. See [20], App.C, Prop.C.17. □

Definition 1.2. *A Banach space* U *is said to be* ***reflexive*** *if* J_U *is an isomorphism.*

Let us note that J_U is always injective, since it is an isometry and we can identify U with the subspace $J_U(U) \subset U''$.

Let U and V be Banach spaces and consider $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset U \to V$ a linear operator that is densely defined. In analogy to the transpose of a matrix, one can define the *adjoint operator* of $\mathcal{A}$. We will follow the definition by Brezis, see [10].

Definition 1.3. *Let us introduce a linear operator* $\mathcal{A}^* : Dom(\mathcal{A}^*) \subset V' \to U'$ *as follows. The domain of* $\mathcal{A}^*$ *is:*

$$Dom(\mathcal{A}^*) = \{v \in V' : \exists c \geq 0 \text{ such that } |\langle v, \mathcal{A}u\rangle| \leq c\|u\|, \forall u \in Dom(\mathcal{A}).\}$$

We have that $Dom(\mathcal{A}^*)$ *is a linear subspace of* V'. *Given* $v \in Dom(\mathcal{A}^*)$, *let us consider the map* $g : Dom(\mathcal{A}) \to \mathbb{R}, g(u) = \langle v, \mathcal{A}u\rangle, \forall u \in Dom(\mathcal{A})$. *We have that* $|g(u)| \leq c\|u\|, \forall u \in Dom(\mathcal{A})$ *and thus, by Theorem 1.1 there exists a unique* $f \in U'$ *such that* $|f(u)| \leq c\|u\|, \forall u \in U$. *Set* $\mathcal{A}^* v = f$. *The operator* $\mathcal{A}^*$ *is called the* ***adjoint*** *of* $\mathcal{A}$. *The fundamental relation between* $\mathcal{A}$ *and* $\mathcal{A}^*$ *is given by:*

$$\langle v, \mathcal{A}u\rangle_{V',V} = \langle \mathcal{A}^* v, u\rangle_{U',U}$$

Definition 1.4. *Let* U *be a reflexive Banach space and* $\mathcal{A} \in \mathcal{LC}(U, U')$, *so that* $\mathcal{A}^* \in \mathcal{LC}(U'', U')$. $\mathcal{A}$ *is said to be* ***self-adjoint*** *if* $\mathcal{A} = \mathcal{A}^* \circ J_U$, *where* J_U *is the canonical isometry in the double dual. In this case, the following holds true:*

$$\langle \mathcal{A}u, w\rangle_{U',U} = \langle \mathcal{A}w, u\rangle_{U',U}, \forall u, w \in U.$$

Theorem 1.6. *(Orthogonal Decomposition of Hilbert Spaces) Let U be a Hilbert space and $X \subset U$ be a nonempty closed subspace. Then, $U = X \oplus X^\perp$.*

Proof. See [35], Th.V.3.4. □

Thus, any $u \in U$ can be written uniquely as $u = x + x^\perp$, where $x \in X, x^\perp \in X^\perp$. We can thus define the *orthogonal projection* onto X by $\mathcal{P}_X u = x, \forall u \in U$.

Lemma 1.1. *([31], Section 5.16) Let U be a Hilbert space and $X \subset U$ be a non-empty closed subspace. The orthogonal projection onto X satisfies the following:*

- *$\mathcal{P}_X$ is self-adjoint;*
- *$\|\mathcal{P}_X\| = 1$;*
- *$\mathcal{I} - \mathcal{P}_X = \mathcal{P}_{X^\perp}$;*
- *$\|u - \mathcal{P}_U x\| \leq \|u - v\|, \forall v \in U$;*
- *$x = \mathcal{P}_X u \iff x \in X$ and $u - x \in X^\perp$.*

Proof. See [31], Section 5.16. □

Following [6], for $\mathcal{A} \in \mathcal{LC}(U,V)$ we have that: $\text{Im}(\mathcal{A})^\perp = \text{Ker}(\mathcal{A}^*)$, $\text{Im}(\mathcal{A}^*)^\perp = \text{Ker}(\mathcal{A})$ and thus, using the fact that for a closed subspace X we have $(X^\perp)^\perp = X$, we obtain: $\overline{\text{Im}(\mathcal{A})} = \text{Ker}(\mathcal{A}^*)^\perp$ and $\overline{\text{Im}(\mathcal{A}^*)} = \text{Ker}(\mathcal{A})^\perp$. By the orthogonal decomposition of Hilbert spaces, we have that:

$$U = \text{Ker}(\mathcal{A}) \oplus \overline{\text{Im}(\mathcal{A}^*)}, V = \text{Ker}(\mathcal{A}^*) \oplus \overline{\text{Im}(\mathcal{A})}.$$

Theorem 1.7. *(Riesz-Frechet Representation Theorem) Let U be a Hilbert space. For every $\phi \in U'$, there exists a unique $u_\phi \in U$, called the Riesz representation of ϕ such that*

$$\langle \phi, u \rangle_{U',U} = \langle u_\phi, u \rangle_U, \forall u \in U.$$

Thus, the operator $J_U^R : U \to U'$ defined by

$$\langle J_U(w), u \rangle_{U',U} = \langle w, u \rangle_U, \forall w, u \in U,$$

is a linear isometric isomorphism.

Proof. See [10], Th.5.5. □

Corollary 1.7.1. *Hilbert spaces are reflexive.*

Proof. The proof follows immediately by Theorem 1.7 since one can identify $U'' = (U')' = U$ using the defined isomorphism. □

1.2 Compact Operators

We will introduce compact operators between Banach spaces and the singular-value decomposition of a compact operator between two Hilbert spaces.

Definition 1.5. *Let $\mathcal{A} \in \mathcal{LC}(U, V)$, where U and V are Banach spaces. $\mathcal{A}$ is said to be compact, if for any bounded set $B \subset U$, $\overline{\mathcal{A}(B)}$ is compact in V.*

An equivalent definition for compact operators would the following: $\mathcal{A} \in \mathcal{LC}(U, V)$ is said to be compact if the image of any bounded sequence $(u_n)_{n\in\mathbb{N}} \subset U$ has a convergent subsequence $(\mathcal{A}u_{n_j})_{j\in\mathbb{N}} \subset V$.

This type of operators is very common in the solution of inverse problems, since most inverse problems involve the inversion of a compact operator. We will see that if the forward operator is compact, the inverse problem will be ill-posed.

Theorem 1.8. *(Singular value decomposition of compact operators) Let U and V be Hilbert spaces. Let $\mathcal{A} \in \mathcal{LC}(U, V)$ be a compact operator. There exist orthonormal bases $(u_n)_{n\in\mathbb{N}^*}$ for U and $(v_n)_{n\in\mathbb{N}^*}$ for V and a sequence of positive real numbers $\sigma_1 \geq \sigma_2 \geq ... \geq 0$ with $\lim_{n\to\infty} \sigma_n = 0$, such that:*

$$\mathcal{A}u = \sum_{n=1}^{\infty} \sigma_n \langle u, u_n \rangle v_n, \forall u \in U.$$

Proof. See [35], Prop.VI.3.6. □

Chapter 2

Linear Inverse Problems

2.1 Well-posed and ill-posed problems

We cast a linear inverse problem in abstract form as

$$\mathcal{A}u = f, \tag{2.1}$$

where $\mathcal{A} : U \to V$ is a linear operator between two Banach/Hilbert spaces.

Definition 2.1. *(Well-posed problem) Problem (2.1) is called well-posed in the sense of Hadamard if it satisfies the following three conditions:*

1. *For any $f \in V$ there exists a solution $u \in U$ such that (2.1) holds.*
2. *The solution $u \in U$ is unique.*
3. *The solution depends continuously on the data, i.e. there exists $c > 0$ such that for any $f \in V$,*
$$\|u\| \leq c\|f\|.$$

In terms of the operator $\mathcal{A}$, problem (2.1) is said to be well-posed if $\mathcal{A}$ is bijective and its inverse operator $\mathcal{A}^{-1}$ is bounded. One can observe that the three conditions are not independent: if $\mathcal{A}$ is bounded, in our context where U and V are both Banach spaces, the bijectivity implied by the first two conditions, by Corollary 1.2.1, also forces the inverse operator $\mathcal{A}^{-1}$ to be continuous.

If any of these conditions is violated, problem (2.1) is said to be *ill-posed*. Our focus will be on problems whose direct problem is considered to be well-posed, whilst the inverse problem is harder to solve and is ill-posed. The first two conditions for well-posedness can be assured by considering a generalised solution, as will be presented in the following section. However, ensuring that the third condition isn't violated requires more advanced techniques such as *regularisation*, which will be the main focus of Chapter 3. A naive approach to solving inverse problems would be to solve problem (2.1) by inverting $\mathcal{A}$. However, since most data in real world scenarios are contaminated with errors, these errors will be significantly amplified by $\mathcal{A}^{-1}$, when $\mathcal{A}^{-1}$ is not bounded.

2.2 Generalised Solutions

In order to cope with the conditions of existence and uniqueness for the problem (2.1), one might consider a generalised least squares solution of minimal norm. In the following, we will consider the two spaces U and V involved in the inverse problem (2.1) to be Hilbert Spaces. Most of the results and proofs in this chapter will follow the lecture notes [27] and [6].

Definition 2.2. *An element $u \in U$ is called:*

- *a **least-squares solution (LSS)** of (2.1) if*

$$\|\mathcal{A}u - f\| = \inf\{\|\mathcal{A}v - f\| : v \in U\};$$

- *a **minimal-norm solution** of (2.1), denoted by $u^\dagger$, if*

$$\|u^\dagger\| \leq \|v\|, \text{ for all least squares solutions } v.$$

Theorem 2.1. *Consider problem (2.1). The following are equivalent:*

1. *$u \in U$ satisfies $\mathcal{A}u = \mathcal{P}_{\overline{Im(\mathcal{A})}} f$*
2. *u is a LSS for (2.1)*
3. *u solves the normal equation $\mathcal{A}^*\mathcal{A}u = \mathcal{A}^* f$*

Proof. $1 \Rightarrow 2$: Let $u \in U$ such that $\mathcal{A}u = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f$ and $v \in U$. Using the properties of the orthogonal projection(see Lemma 1.1), we have that:

$$\|\mathcal{A}u - f\|^2 = \|(\mathcal{I} - \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}})f\|^2 \leq \inf_{g \in \overline{\mathrm{Im}(\mathcal{A})}} \|g - f\|^2 \leq \inf_{g \in \mathrm{Im}(\mathcal{A})} \|g - f\|^2 = \inf_{v \in U} \|\mathcal{A}v - f\|^2,$$

which is equivalent to u being a LSS for (2.1).
$2 \Rightarrow 3$: Let u be a LSS for (2.1) and $v \in U$. We define the function $F : \mathbb{R} \to \mathbb{R}$, $F(\lambda) = \|\mathcal{A}(u + \lambda v)\|^2$. We have that $F(\lambda) = \lambda^2 \|\mathcal{A}v\|^2 - 2\lambda\langle \mathcal{A}v, f - \mathcal{A}u\rangle + \|f - \mathcal{A}u\|^2$. Hence, $F'(0) = -2\langle \mathcal{A}v, f - \mathcal{A}u\rangle \in \mathbb{R}$, which shows that F is differentiable at $\lambda = 0$. Since u is a LSS, we have that $F(0) \leq F(\lambda), \forall \lambda \in \mathbb{R}$. By Fermat's Theorem we thus have that $F'(0) = 0 \iff \langle \mathcal{A}v, f - \mathcal{A}u\rangle = 0 \iff \langle v, \mathcal{A}^*(f - \mathcal{A}u)\rangle = 0$, which will hold for any

$v \in U$. It follows that $\mathcal{A}^*\mathcal{A}u = \mathcal{A}^*f$.
$3 \Rightarrow 1$: $\mathcal{A}^*\mathcal{A}u = \mathcal{A}^*f \Rightarrow \mathcal{A}^*(f - \mathcal{A}u) = 0 \Rightarrow 0 = \langle v, \mathcal{A}^*(f - \mathcal{A}u)\rangle = \langle \mathcal{A}v, \mathcal{A}u - f\rangle, \forall v \in U \Rightarrow \mathcal{A}u - f \in \mathrm{Im}(\mathcal{A})^\perp$. We have that $\mathrm{Im}(\mathcal{A})^\perp = \overline{\mathrm{Im}(\mathcal{A})}^\perp$, $\mathcal{A}u \in \mathrm{Im}(\mathcal{A}) \subset \overline{\mathrm{Im}(\mathcal{A})}$, and by Lemma 1.1 we have that $\mathcal{A}u = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}f$. □

However, since $\mathrm{Im}(\mathcal{A})$ is not necessarily closed, in fact it is closed for compact operators only if the range is finite-dimensional, a LSS may not exist. In case a LSS exists, then the minimal-norm solution will be unique. In the following, we will present and prove some conditions that ensure the existence of a LSS.

Lemma 2.1. *There exists a LSS solution of the problem (2.1) if and only if $f \in$ Im$(\mathcal{A}) \oplus$ Im$(\mathcal{A})^\perp$.*

Proof. Let u be a LSS for (2.1). Then, by Theorem 2.1, $f - \mathcal{A}u \in \mathrm{Im}(\mathcal{A})^\perp$. Thus, $f = \mathcal{A}u + (f - \mathcal{A}u) \in \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp$. Now, let $f \in \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp \Rightarrow \exists u \in U, g \in \mathrm{Im}(\mathcal{A})^\perp = \overline{\mathrm{Im}(\mathcal{A})}^\perp$, such that $f = \mathcal{A}u + g \Rightarrow \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}f = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}\mathcal{A}u + \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}g = \mathcal{A}u + 0 = \mathcal{A}u$, which again by Theorem 2.1 is equaivalent to u being a LSS. □

It is now obvious by the orthogonal decomposition of Hilbert Spaces that a LSS always exists when $\mathrm{Im}(\mathcal{A})$ is closed. In a finite dimensonal setting, there will always exist a LSS. Since we have proven a necessary and suffcient condition for the existence of a LSS, we now turn our attention to the existence of a minimal-norm solution:

Theorem 2.2. *Consider problem (2.1) with $f \in$ Im$(\mathcal{A}) \oplus$ Im$(\mathcal{A})^\perp$. Then there exists a minimal-norm solution $u^\dagger$ and all LSSs are given by $\{u^\dagger\} +$ Ker$(\mathcal{A})$.*

Proof. By Lemma 2.1 there exists a LSS for problem (2.1) and thus the minimal-norm solution $u^\dagger$ is unique (it is the orthogonal projection of 0_U on an affine subspace given by all $u \in U$ s.t. $\|\mathcal{A}u - f\| = \min\{\|\mathcal{A}v - f\| : v \in U\}$). Let $v \in U$ be an arbitrary LSS: $\mathcal{A}(v - u^\dagger) = \mathcal{A}v - \mathcal{A}u^\dagger = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}f - \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}f = 0 \Rightarrow v - u^\dagger \in \mathrm{Ker}(\mathcal{A})$. □

Corollary 2.2.1. *The minimal-norm solution $u^\dagger$ is the unique solution of the normal equation $\mathcal{A}^*\mathcal{A}u = \mathcal{A}^*f$ in Ker$(\mathcal{A})^\perp$.*

Proof. By Theorem 2.1 we have that u is a LSS if and only if it satisfies the normal equation $\mathcal{A}^*\mathcal{A}u = \mathcal{A}^*f$. Any LSS u is also a solution of $\mathcal{A}u = \mathcal{P}_{\overline{Im(\mathcal{A})}}f$ and can be written uniquely as $u = u_0 + u_p, u_0 \in \mathrm{Ker}(\mathcal{A}), u_p \in \mathrm{Ker}(\mathcal{A})^\perp$. Let us now prove that $\mathcal{A}$ is injective on $\mathrm{Ker}(\mathcal{A})^\perp$: let $x, y \in \mathrm{Ker}(\mathcal{A})^\perp$ such that $\mathcal{A}x = \mathcal{A}y \Rightarrow \mathcal{A}(x - y) = 0 \Rightarrow x - y \in \mathrm{Ker}(\mathcal{A})$. Thus, $\langle x - y, x - y\rangle = \langle x, x - y\rangle - \langle y, x - y\rangle = 0$, since $x, y \in \mathrm{Ker}(\mathcal{A})^\perp \Rightarrow \|x - y\| = 0 \Rightarrow x = y$. It follows that $\mathcal{A}$ is injective on $\mathrm{Ker}(\mathcal{A})^\perp$. By the injectivity, the element u_p must be independent of the LSS u, otherwise by the fact that all LSSs are solutions of $\mathcal{A}u = \mathcal{P}_{\overline{Im(\mathcal{A})}}f$ we would contradict the injectivity of $\mathcal{A}$ on $\mathrm{Im}(\mathcal{A})^\perp$. Further, we have that:

$$\|u\|^2 = \|u_0\|^2 + 2\langle u_0, u_p\rangle + \|u_p\|^2 = \|u_0\|^2 + \|u_p\|^2 \geq \|u_p\|^2,$$

which implies that $u_p = u^\dagger \in \mathrm{Ker}(\mathcal{A})^\perp$ is the unique minimum-norm solution. $\square$

2.3 Moore-Penrose Inverse and Compact Operators

We have seen so far that if a LSS exists for a given right-hand side of the inverse problem (2.1), the existence of the minimal-norm solution is assured. The minimal-norm solution can be viewed as the unique solution of the normal equation in $\mathrm{Ker}(\mathcal{A})^\perp$. In theory, we can compute this minimal-norm solution using the *Moore-Penrose Generalised Inverse*, which we will define in this section.

Definition 2.3. *Let $\mathcal{A} \in \mathcal{LC}(U, V)$. We consider the restriction*

$$\tilde{\mathcal{A}} := \mathcal{A}|_{Ker(\mathcal{A})^\perp} : Ker(\mathcal{A})^\perp \to Im(\mathcal{A})$$

of $\mathcal{A}$ to $Ker(\mathcal{A})^\perp$. The ***Moore-Penrose Inverse*** *$A^\dagger$ is defined as the unique linear extension of $\tilde{\mathcal{A}}^{-1}$ to*

$$Dom(\mathcal{A}^\dagger) = Im(\mathcal{A}) \oplus Im(\mathcal{A})^\perp$$

with

$$Ker(\mathcal{A}^\dagger) = Im(\mathcal{A})^\perp.$$

Let us now analyse the fact that the Moore-Penrose inverse is correctly defined and unique: the injectivity of $\tilde{\mathcal{A}}$ was proven in the proof of Corollary 2.2.1. For the surjectivity, let $f \in \mathrm{Im}(\mathcal{A}) \Rightarrow \exists u \in U : \mathcal{A}u = f$. u can be written as $u = u_0 + u_p, u_0 \in \mathrm{Ker}(\mathcal{A}), u_p \in \mathrm{Ker}(\mathcal{A})^\perp$, thus $\mathcal{A}u = \mathcal{A}u_p = f \Rightarrow \exists u_p \in \mathrm{Ker}(\mathcal{A})^\perp : \mathcal{A}u_p = f$. It follows that $\tilde{\mathcal{A}}$ is bijective. Now let $f \in \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp \Rightarrow \exists! f_0 \in \mathrm{Im}(\mathcal{A}), f_p \in \mathrm{Im}(\mathcal{A})^\perp : f = f_0 + f_p$. We have that $\mathcal{A}^\dagger f = \mathcal{A}^\dagger f_0 + \mathcal{A}^\dagger f_p = \mathcal{A}^\dagger f_0 = \tilde{\mathcal{A}}^{-1} f_0 = \tilde{\mathcal{A}}^{-1}\mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f$ and thus $\mathcal{A}^\dagger$ is well defined on $\mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp$.

Proposition 2.1. *The four "Moore-Penrose equations" hold:*

$$\mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{A}; \tag{2.2}$$

$$\mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger = \mathcal{A}^\dagger \tag{2.3}$$

$$\mathcal{A}^\dagger\mathcal{A} = \mathcal{I} - \mathcal{P}_{Ker(\mathcal{A})} \tag{2.4}$$

$$\mathcal{A}\mathcal{A}^\dagger = \mathcal{P}_{\overline{Im(\mathcal{A})}}|_{Dom(\mathcal{A}^\dagger)} \tag{2.5}$$

Proof. See [18], Prop. 2.3. □

We can observe that $\overline{\mathrm{Dom}(\mathcal{A}^\dagger)} = \overline{\mathrm{Im}(\mathcal{A})} \oplus \mathrm{Im}(\mathcal{A})^\perp = V$, so $\mathcal{A}^\dagger$ is densely defined. If $\mathrm{Im}(\mathcal{A})$ is closed, we have that $\mathrm{Dom}(\mathcal{A}^\dagger) = V$ and the reverse, which is usually not the case for ill-posed problems since most compact operators don't have closed ranges. In fact, the closedness of the range of $\mathcal{A}$ is equivalent to the boundedness of the Moore-Penrose inverse as can be seen in the next theorem.

Theorem 2.3. *([18], Prop. 2.4) $Gr(\mathcal{A}^\dagger)$ is closed. Furthermore, $\mathcal{A}^\dagger$ is bounded if and only if $Im(\mathcal{A})$ is closed.*

Proof. For the first assertion of the theorem, the reader can check the proof of Proposition 2.4 in [18]. For the second assertion of the theorem, let us first assume that $\mathrm{Im}(\mathcal{A})$ is closed, so $\mathrm{Dom}(\mathcal{A}^\dagger) = V$. By Theorem 1.3 $\mathcal{A}^\dagger$ will be bounded.
Now, suppose $\mathcal{A}^\dagger$ is bounded. It will then have a unique continuous extension to V by Theorem 1.4, which we will denote by $\overline{\mathcal{A}^\dagger}$. From (2.5) and using the continuity of $\mathcal{A}$ we have that $\mathcal{A}\overline{\mathcal{A}^\dagger} = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}$. So, for $f \in \overline{\mathrm{Im}(\mathcal{A})}$ we have that $f = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f = \mathcal{A}\overline{\mathcal{A}^\dagger} f \in \mathrm{Im}(\mathcal{A})$. It follows that $\overline{\mathrm{Im}(\mathcal{A})} \subset \mathrm{Im}(\mathcal{A})$ which implies the closedness of the range of $\mathcal{A}$. □

The main theoretical result of this section is the following result which links the minimal-norm solution to the Moore-Penrose generalised inverse:

Theorem 2.4. *Let $f \in Dom(\mathcal{A}^\dagger)$. Then the minimal-norm solution $u^\dagger$ of the inverse problem (2.1) is given by*

$$u^\dagger = \mathcal{A}^\dagger f.$$

Proof. Since $f \in \mathrm{Dom}(\mathcal{A}^\dagger) = \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp$, Theorem 2.2 assures us that the minimal-norm solution $u^\dagger$ exists and is unique. We have that $u^\dagger \in \mathrm{Ker}(\mathcal{A})^\perp$ and $u^\dagger = (\mathcal{I} - \mathcal{P}_{\mathrm{Ker}(\mathcal{A})})u^\dagger$, which by (2.3 - 2.5), and Theorem 2.1 becomes $u^\dagger = \mathcal{A}^\dagger \mathcal{A} u^\dagger = \mathcal{A}^\dagger \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f = \mathcal{A}^\dagger \mathcal{A}\mathcal{A}^\dagger f = \mathcal{A}^\dagger f$. □

We can now observe that by Corollary 2.2.1 and Theorem 2.4 the minimal-norm solution $u^\dagger$ of problem (2.1) is actually a minimal-norm solution for the associated normal equation and thus:

$$u^\dagger = \mathcal{A}^\dagger f = (\mathcal{A}^*\mathcal{A})^\dagger \mathcal{A}^* f.$$

This fact will be used when constructing regularisation methods further on.

2.4 Compact Operators and the Picard Criterion

As was already discussed in the previous chapter compact operators are a class of operators that come up very often in inverse problems. Since the range of a compact operator will only be closed if it is finite-dimensional, the Moore-Penrose generalised inverse will be unbounded in most cases, which will be a source of ill-posedness for the inverse problem.

Proposition 2.2. *Let $\mathcal{A} \in \mathcal{K}(U, V)$ with $\dim Im(\mathcal{A}) = \infty$. Then the Moore-Penrose Inverse $\mathcal{A}^\dagger$ is unbounded.*

Proof. Since $\dim \mathrm{Im}(\mathcal{A}) = \infty \Rightarrow \dim X = \dim \mathrm{Ker}(\mathcal{A})^\perp = \infty$, which means we can construct a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{Ker}(\mathcal{A})^\perp$ such that $\|u_n\| = 1, \forall n \in \mathbb{N}$ and $\langle u_n, u_m\rangle = 0, n \neq m$. The sequence $(u_n)_{n\in\mathbb{N}}$ is obviously bounded and since $\mathcal{A}$ is compact, the sequence $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ will have a convergent, and in particular Cauchy subsequence, which we will denote by $(\mathcal{A}u_{n_k})_{k\in\mathbb{N}}$. Thus, $\forall \epsilon > 0, \exists m, k \in \mathbb{N} : \|\mathcal{A}u_{n_m} - \mathcal{A}u_{n_k}\| < \epsilon$. We have by (2.4) that $\|\mathcal{A}^\dagger\mathcal{A}u_{n_m} - \mathcal{A}^\dagger\mathcal{A}u_{n_k}\|^2 = \|u_{n_m} - u_{n_k}\|^2 = \|u_{n_m}\|^2 - 2\langle u_{n_m}, u_{n_k}\rangle + \|u_{n_k}\|^2 = 2$, which proves that $\mathcal{A}^\dagger$ is discontinuous. $\square$

By considering the singular value decomposition for compact operators in the case when $\mathcal{A} \in \mathcal{K}(U, V)$, we will be able to describe when we have $f \in \mathrm{Dom}(\mathcal{A}^\dagger)$.

Definition 2.4. *Let $\mathcal{A} \in \mathcal{K}(U, V)$ with the singular system $\{\sigma_j, u_j, v_j\}_{j\in\mathbb{N}}$. We say that the data f satisfy the **Picard Criterion** if:*

$$\sum_{j=1}^{\infty} \frac{|\langle f, v_j\rangle|^2}{\sigma_j^2} < \infty.$$

By the SVD we know that $\sigma_j \to 0$ as $j \to \infty$. This tells us that the Picard criterion is a condition on the decay speed of the Fourier coefficients $\langle f, v_j\rangle$, which is equivalent to the smoothness of f.

Theorem 2.5. *([18], Thm. 2.8) Let $\mathcal{A} \in \mathcal{K}(U, V)$ with the singular system $\{\sigma_j, u_j, v_j\}_{j\in\mathbb{N}}$. We then have:*

1. *$f \in Dom(\mathcal{A}^\dagger) \iff \displaystyle\sum_{j=1}^{\infty} \frac{|\langle f, v_j\rangle|^2}{\sigma_j^2} < \infty$.*

2. *If $f \in Dom(\mathcal{A}^\dagger)$, $\mathcal{A}^\dagger f = \displaystyle\sum_{j=1}^{\infty} \frac{\langle f, v_j\rangle}{\sigma_j} u_j$.*

Proof. 1. Let $f \in \mathrm{Dom}(\mathcal{A}^\dagger) \Rightarrow \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f \in \mathrm{Im}(\mathcal{A})$. We know by the SVD that $(v_j)_{j\in\mathbb{N}}$ are an orthonormal basis for $\overline{\mathrm{Im}(\mathcal{A})}$, which means we can represent the orthogonal projector as: $\mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} = \displaystyle\sum_{i=1}^{\infty} \langle \cdot, v_i\rangle v_i$. By $\mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f \in \mathrm{Im}(\mathcal{A})$ we have that there exists $u \in \mathrm{Ker}(\mathcal{A})^\perp$ such that $\mathcal{A}u = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}} f$. We know that $(u_j)_{j\in\mathbb{N}}$ are an orthonormal basis for $\overline{\mathrm{Im}(\mathcal{A}^*)} = \mathrm{Ker}(\mathcal{A})^\perp$, which means we can represent u as $u = \displaystyle\sum_{i=1}^{\infty} \langle u, u_i\rangle u_i$.

We then have: $\sum_{i=1}^{\infty}\langle f, v_i\rangle v_i = \mathcal{A}u = \sum_{i=1}^{\infty}\langle u, u_i\rangle \mathcal{A}u_i = \sum_{i=1}^{\infty}\sigma_i\langle u, u_i\rangle v_i \Rightarrow \langle f, v_j\rangle = \sigma_j\langle u, u_j\rangle, \forall j \in \mathbb{N}$. By the characterization of an orthonormal basis in a Hilbert spaces we know that the sequence of Fourier coefficients $(\langle u, u_j\rangle)_{j\in\mathbb{N}} \in \ell^2$, which implies that $(\frac{\langle f, v_j\rangle}{\sigma_j})_{j\in\mathbb{N}} \in \ell^2$, which is equivalent to the Picard criterion being satisfied. For the reverse implication, assume that the Picard criterion is satisfied and let us define $u = \sum_{i=1}^{\infty}\frac{\langle f, v_i\rangle}{\sigma_i}u_i \in U$, which will then be well-defined. We then have $\mathcal{A}u = \sum_{i=1}^{\infty}\frac{\langle f, v_i\rangle}{\sigma_i}\mathcal{A}u_i = \sum_{i=1}^{\infty}\langle f, v_i\rangle v_i = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}f \in \mathrm{Im}(\mathcal{A})$ and thus $f \in \mathrm{Dom}(\mathcal{A}^\dagger)$.

2. Let $u = \sum_{i=1}^{\infty}\frac{\langle f, v_i\rangle}{\sigma_i}u_i \in U$. Since $(u_j)_{j\in\mathbb{N}}$ are an orthonormal basis for $\overline{\mathrm{Im}(\mathcal{A}^*)} = \mathrm{Ker}(\mathcal{A})^\perp$ it follows that $u \in \mathrm{Ker}(\mathcal{A})^\perp$. We have seen that $\mathcal{A}u = \mathcal{P}_{\overline{\mathrm{Im}(\mathcal{A})}}f$, which shows that u is a LSS. Since $u \in \mathrm{Ker}(\mathcal{A})^\perp$, it follows that $u = u^\dagger$. But, $u^\dagger = \mathcal{A}^\dagger f \Rightarrow \mathcal{A}^\dagger f = \sum_{j=1}^{\infty}\frac{\langle f, v_j\rangle}{\sigma_j}u_j$. □

We can now conclude that a minimal-norm solution will only exist if the Fourier coefficients of the data f will decay "sufficiently" fast with respect to the singular values. From this representation of the Moore-Penrose inverse we might as well see how the errors in the data are amplified by the inversion: Let us assume we are given noisy data $f^\delta = f + \delta v_j$, for some $j \in \mathbb{N}, \delta > 0$. We have that:

$$\|\mathcal{A}^\dagger f^\delta - \mathcal{A}^\dagger f\| = \delta\|\mathcal{A}^\dagger v_j\| = \frac{\delta}{\sigma_j} \to \infty,\ j \to \infty.$$

It can be observed that errors in high frequencies(large j and small σ_j) are amplified stronger then those in low frequencies(small j). We are interested in how strong high frequency errors are amplified: this depends on the decay speed of the singular values of our operator $\mathcal{A}$. By following [11] and [17] let us divide ill-posed inverse problems into the following two classes of "ill-posedness", depending on the decay of the singular values to 0:

- *mildly ill-posed*, when there exist $\gamma, C > 0$ such that $\sigma_j \geq Cj^{-\gamma}$ for all $j \in \mathbb{N}$ (the decay is at most polynomial).
- *severely ill-posed*, when for all $\gamma, C > 0$ such that $\sigma_j \leq Cj^{-\gamma}$ for sufficiently large j (the decay is faster than any polynomial rate). If $\sigma_j \leq Ce^{-j^\gamma}$, for all $j \in \mathbb{N}$, then the problem is called *exponentially ill-posed* (the decay is at least exponential).

Chapter 3

Regularisation Methods for Linear Inverse Problems

In [18], the authors describe regularisation as ”the approximation of an ill-posed problem by a family of neighbouring well-posed problems”. The idea consists of constructing a so called regularisation operator and method, in order to be able to approximate the minimal-norm solution of (2.1), which is given via $u^\dagger = \mathcal{A}^\dagger f$, when we don’t know the exact data, but only a noisy measurement f^δ that satisfies:

$$\|f - f^\delta\| \leq \delta, \tag{3.1}$$

where $\delta > 0$ is the noise level.

When dealing with ill-posed problems we can be sure that $\mathcal{A}^\dagger y^\delta$ will approximate $u^\dagger$ very badly, since the Moore-Penrose inverse is unbounded or in fact because y^δ may not even be in $\mathrm{Dom}(\mathcal{A}^\dagger)$. We will look for an approximation u_α^δ of $u^\dagger$ that will depend continuously on the noisy data and will converge to $u^\dagger$ when $\delta \to 0$. This can be done by approximating $\mathcal{A}^\dagger$ by a family of operators $\mathcal{R}_\alpha$ (called regularisation operator) and by taking $u_\alpha^\delta = \mathcal{R}_\alpha y^\delta$.

Definition 3.1. *Consider problem (2.1). A family $(\mathcal{R}_\alpha)_{\alpha \in I}$ of bounded operators is called **regularisation** or **regularisation operator** for $\mathcal{A}^\dagger$, if for all $f \in \mathrm{Dom}(\mathcal{A}^\dagger)$ there exists a parameter choice rule $\alpha : \mathbb{R}_+ \times V \to I$ such that:*

$$\limsup_{\delta \to 0} \left\{ \|\mathcal{R}_{\alpha(\delta, f^\delta)} f^\delta - \mathcal{A}^\dagger f\| \;\middle|\; f^\delta \in V, \|f - f^\delta\| \leq \delta \right\} = 0 \tag{3.2}$$

and

$$\limsup_{\delta \to 0} \left\{ \alpha(\delta, f^\delta) \;\middle|\; f^\delta \in V, \|f - f^\delta\| \leq \delta \right\} = 0. \tag{3.3}$$

*We will call $(\mathcal{R}_\alpha, \alpha)$ a convergent **regularisation method** of (2.1) if (3.2) and (3.3) hold. If* *each $\mathcal{R}_\alpha$ is a linear operator, we will say that $(\mathcal{R}_\alpha)_{\alpha\in I}$ is a linear regularisation for $\mathcal{A}^\dagger$.*

There are 3 types of parameter choice rules in practice, which are characterised by the dependence on the noisy data f^δ:

- *a-priori choice if $\alpha = \alpha(\delta)$ (the parameter choice depends only on the noise level);*
- *a-posteriori choice if $\alpha = \alpha(\delta, f^\delta)$;*
- *heuristic choice if $\alpha = \alpha(f^\delta)$ (the paremeter choice is independent of the noise level).*

In practice, heuristic parameter choice rules are often in use although it was proven by Bakushinskii in [3] that a convergent regularisation method with such parameter choice rule can't work for ill-posed problems:

Theorem 3.1. *(Bakushinskii veto, [3]) Let $\mathcal{A} \in \mathcal{LC}(U, V)$ and $(\mathcal{R}_\alpha)_\alpha$ be a regularisation for $\mathcal{A}^\dagger$. Let $\alpha = \alpha(f^\delta)$ such that $(\mathcal{R}_\alpha, \alpha)$ is a convergent regularisation. Then $\mathcal{A}^\dagger$ can be extended to a bounded operator from V to U.*

Proof. For $\alpha = \alpha(f^\delta)$, by (3.2) we have that $\mathcal{R}_{\alpha(f)}f = \mathcal{A}^\dagger f, \forall f \in \mathrm{Dom}(\mathcal{A}^\dagger)$. Thus, for any sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathcal{A}^\dagger)$ convergent to some $f \in \mathrm{Dom}(\mathcal{A}^\dagger)$ we have that: $\mathcal{A}^\dagger f_n = \mathcal{R}_{\alpha(f_n)} f_n \to \mathcal{R}_{\alpha(f)} f = \mathcal{A}^\dagger f$, which is equivalent to $\mathcal{A}^\dagger$ being continuous on $\mathrm{Dom}(\mathcal{A}^\dagger)$. Since $\mathrm{Dom}(\mathcal{A}^\dagger)$ is dense in V, by Theorem 1.4 we have that there exists a unique continuous extension of $\mathcal{A}^\dagger$ to V. □

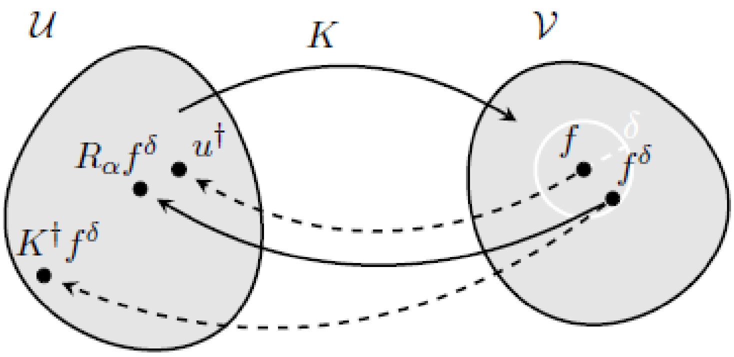


Figure 3.1: ([6], Fig.3.1) A visualization of reconstruction from noisy data using the Regularisation Operator vs. the Moore-Penrose inverse

Proposition 3.1. *([18], Prop. 3.4) Let $\mathcal{R}_\alpha : V \to U$ be a family of bounded operators, $\alpha \in \mathbb{R}_+$. Then, the family $\{\mathcal{R}_\alpha\}$ is a regularisation of $\mathcal{A}^\dagger$ if*

$$\mathcal{R}_\alpha \to \mathcal{A}^\dagger \text{ pointwise on } Dom(\mathcal{A}^\dagger) \text{ as } \alpha \to 0.$$

In this case, there exists an a-priori parameter choice rule such that $(\mathcal{R}_\alpha, \alpha)$ is a convergent regularisation method.

Proof. See [18], Prop. 3.4. $\square$

By (3.3), if the parameter choice rule is a continuous function, we have that the convergent regularisation method implies the pointwise convergence of the regularisation operators to $\mathcal{A}^\dagger$. This leads to questions regarding the boundedness of the regularisation operators:

Theorem 3.2. *Let $\mathcal{A} \in \mathcal{LC}(U, V)$ and consider a linear regularisation $(\mathcal{R}_\alpha)$ for $\mathcal{A}^\dagger$. If $\mathcal{A}^\dagger$ is not bounded, the family $(\mathcal{R}_\alpha)$ cannot be uniformly bounded. There exists $f \in V$ such that: $\|\mathcal{R}_\alpha f\| \to \infty, \alpha \to 0$.*

Proof. Let us assume that $(\mathcal{R}_\alpha)$ is uniformly bounded. We have that

$$\mathcal{R}_\alpha \to \mathcal{A}^\dagger \text{ pointwise on } \mathrm{Dom}(\mathcal{A}^\dagger) \text{ as } \alpha \to 0.$$

Since $\mathrm{Dom}(\mathcal{A}^\dagger)$ is dense in V we have by Corollary 1.5.1 that $\mathcal{A}^\dagger$ is bounded, which contradicts our assumption.
Let us now assume that there exists no element $f \in V$ such that: $\|\mathcal{R}_\alpha f\| \to \infty, \alpha \to 0$. It follows that the family $(\mathcal{R}_\alpha)$ is pointwise bounded. Since U is a Banach space, by Theorem 1.5 we would conclude that $(\mathcal{R}_\alpha)$ is uniformly bounded, which contradicts the first part of this proof. $\square$

It can be shown ([18], Prop. 3.6) that under the additional assumption that $\mathcal{A}\mathcal{R}_\alpha$ is a bounded operator, $\mathcal{R}_\alpha f$ diverges for all $f \notin \mathrm{Dom}(\mathcal{A}^\dagger)$.

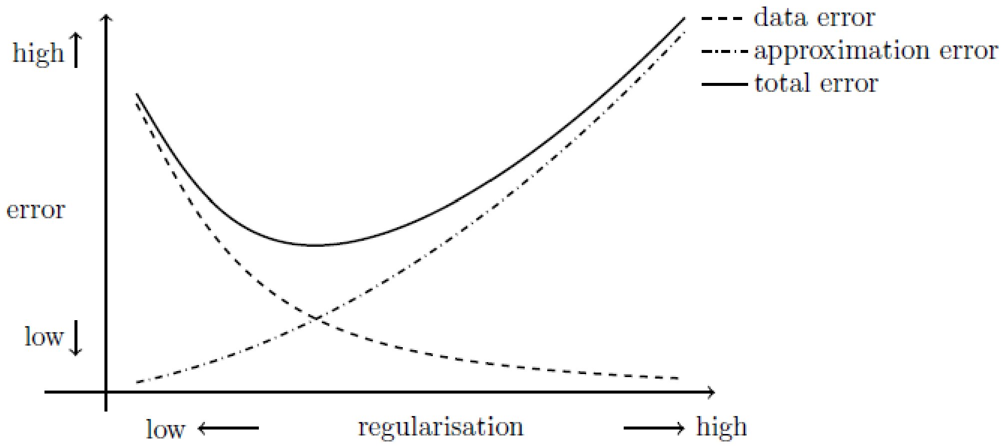


Figure 3.2: ([6], Fig.3.2) The total error between the regularised and the minimal-norm solution depending on the choice of the regularisation parameter α

We could split the error of approximating the noise-free minimal-norm solution in the following manner:

$$\|\mathcal{R}_\alpha f^\delta - u^\dagger\| \leq \|\mathcal{R}_\alpha f^\delta - \mathcal{R}_\alpha f\| + \|\mathcal{R}_\alpha f - u^\dagger\|$$

$$\leq \delta\|\mathcal{R}_\alpha\| + \|\mathcal{R}_\alpha f - \mathcal{A}^\dagger f\|.$$

The first term in the sum will be the *data error* and the second one the *regularisation error*. By Theorem 3.2 we can conclude that the data error term will not stay bounded when $\delta \to 0$, whilst the second term vanishes to 0 for a convergent regularisation method as $\alpha \to 0$. Thus, in order to construct a good regularisation method, one needs to balance between these two error terms and to choose the parameter α dependent on the noise level δ.

We will now focus on a-priori parameter choice rules, since for every regularisation, there is an a-priori choice such that the regularisation method converges, as was already seen in (3.1) and because these will be the ones used for the regularisation methods that will be presented in the following sections.

Theorem 3.3. *Let $\{\mathcal{R}_\alpha\}_{\alpha>0}$ be a linear regularisation and $\alpha : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ an a-priori parameter choice rule. $(\mathcal{R}_\alpha, \alpha)$ is a convergent regularisation method if and only if the following relations are satisfied:*

1. $\lim\limits_{\delta\to 0} \alpha(\delta) = 0;$
2. $\lim\limits_{\delta\to 0} \delta\|\mathcal{R}_{\alpha(\delta)}\| = 0$

Proof. See [6], Theorem 3.5. □

3.1 Spectral Methods for Regularisation

The idea behind spectral regularisation starts from the spectral representation of the Moore-Penrose inverse presented in Theorem 2.5. The issue there was with the decay of the singular values to 0. Thus, spectral regularisation draws on the idea of modifying these singular values in the following manner:

Theorem 3.4. *([6], Theorem 3.8) Let $\mathcal{A} \in \mathcal{K}(U, V)$ and $g_\alpha : \mathbb{R}_+ \to \mathbb{R}$ be a piecewise continuous function such that:*

1. $\exists C_\alpha > 0$ *such that* $g_\alpha(\sigma) \leq C_\alpha, \forall\sigma \in \mathbb{R}_+;$
2. $\lim\limits_{\alpha\to 0} g_\alpha(\sigma) = \dfrac{1}{\sigma};$
3. $\exists\gamma > 0$ *such that* $\sup\limits_{\alpha,\sigma} \sigma g_\alpha(\sigma) < \gamma;$

and define $\mathcal{R}_\alpha(\cdot) = \sum\limits_{i=1}^{\infty} g_\alpha(\sigma_i)\langle\cdot, v_i\rangle u_i$. *We have that* $\mathcal{R}_\alpha f = \mathcal{A}^\dagger f$ *as* $\alpha \to 0$, *for all* $f \in Dom(\mathcal{A}^\dagger)$.

Proof. We have that:

$$\mathcal{R}_\alpha f - \mathcal{A}^\dagger f = \sum_{i=1}^{\infty}(g_\alpha(\sigma_i) - \frac{1}{\sigma_i})\langle f, v_i\rangle u_i = \sum_{i=1}^{\infty}(\sigma_i g_\alpha(\sigma_i) - 1)\langle u^\dagger, u_i\rangle u_i.$$

From 3. we can use the following upper bound:

$$|(\sigma_i g_\alpha(\sigma_i) - 1)\langle u^\dagger, u_i\rangle u_i| \leq (1+\gamma)\|u^\dagger\|,$$

which means that each element in the sum will be bounded. We have that

$$\|\mathcal{R}_\alpha f - \mathcal{A}^\dagger f\|^2 \leq (1+\gamma)^2\|u^\dagger\|^2.$$

We can now apply the Reverse Fatou Lemma, and estimate:

$$\limsup_{\alpha\to 0}\|\mathcal{R}_\alpha f - \mathcal{A}^\dagger f\|^2 \leq \limsup_{\alpha\to 0}\sum_{i=1}^{\infty}|\sigma_i g_\alpha(\sigma_i) - 1|^2|\langle u^\dagger, u_i\rangle u_i|^2$$

$$\leq \sum_{i=1}^{\infty}|\lim_{\alpha\to 0}\sigma_i g_\alpha(\sigma_i) - 1|^2|\langle u^\dagger, u_i\rangle u_i|^2.$$

Using 2. we have that $\|\mathcal{R}_\alpha f - \mathcal{A}^\dagger f\| \to 0$ as $\alpha \to 0$ for all $f \in \mathrm{Dom}(\mathcal{A}^\dagger)$. □

Proposition 3.2. *Consider the same assumptions hold as the ones in Theorem 3.4. Let α be an a-priori parameter choice rule. Then $(\mathcal{R}_\alpha, \alpha)$ will be a convergent method if*

$$\lim_{\delta\to 0}\delta C_\alpha(\delta) = 0.$$

Proof. Let $f \in \mathrm{Dom}(\mathcal{A}^\dagger)$. We have that:

$$\|\mathcal{R}_\alpha f\| = \sum_{i=1}^{\infty}(g_\alpha(\sigma_i))^2|\langle f, v_i\rangle|^2 \leq C_\alpha^2\sum_{i=1}^{\infty}|\langle f, v_i\rangle|^2 = C_\alpha^2\|f\|^2,$$

which implies that $\|\mathcal{R}_\alpha\| \leq C_\alpha$. Thus, by $\lim_{\delta\to 0}\delta C_\alpha(\delta) = 0$ and Theorem 3.3 we have that $(\mathcal{R}_\alpha, \alpha)$ is a convergent method. □

We will follow [11] and give a few examples of spectral regularisations:

3.1.1 Truncated Singular Value Decomposition

In this method, we will ignore all singular values below a certain threshold, which will actually be given by the regularisation parameter α. Thus, let us define:

$$g_\alpha(\sigma) = \begin{cases} \frac{1}{\sigma} & \text{if } \sigma \geq \alpha \\ 0 & \text{if } \sigma < \alpha \end{cases}$$

We can choose $C_\alpha = \frac{1}{\alpha}$ and the method will converge for $\frac{\delta}{\alpha} \to 0$. The regularised solution is given by:

$$u_\alpha = \mathcal{R}_\alpha f = \sum_{\sigma_j \geq \alpha} \frac{1}{\sigma_j} \langle f, v_j \rangle u_j.$$

Since 0 is the only accumulation point for the singular values, the sum will always be finite. However, for very small α the number of singular values and vectors that have to be computed will be very high.

3.1.2 Lavrentiev Regularisation

In this method, we shift all singular values by the regularisation parameter α. Let us define:

$$g_\alpha(\sigma) = \frac{1}{\sigma + \alpha}$$

We can again choose $C_\alpha = \frac{1}{\alpha}$ and the method will converge for $\frac{\delta}{\alpha} \to 0$. The regularised solution is given by:

$$u_\alpha = \sum_{j=1}^{\infty} \frac{1}{\sigma_j + \alpha} \langle f, v_j \rangle u_j.$$

In this case, all singular values and vectors come up in the sum and have to be computed which may be very impractical. However, let us see that if $\mathcal{A}$ is a positive semidefinite operator($\lambda_j = \sigma_j$, $u_j = v_j$), we have that:

$$(\mathcal{A} + \alpha \mathcal{I}) u_\alpha = \sum_{j=1}^{\infty} \frac{1}{\sigma_j + \alpha} \langle u_\alpha, u_j \rangle u_j = \sum_{j=1}^{\infty} \langle f, u_j \rangle u_j = f.$$

We can thus recover the regularised solution without knowledge of the singular system by solving:

$$(\mathcal{A} + \alpha \mathcal{I}) u_\alpha = f.$$

3.1.3 Tikhonov Regularisation

For the Tikhonov regularisation method, the idea is to consider the following function:

$$g_\alpha(\sigma) = \frac{\sigma}{\sigma^2 + \alpha},$$

where α is the regularisation parameter. We can choose $C_\alpha = \dfrac{1}{2\sqrt{\alpha}}$ and thus, the method will converge if $\dfrac{\delta}{\sqrt{\alpha}} \to 0$. The regularised solution in this case is given by:

$$u_\alpha = \sum_{j=1}^{\infty} \frac{\sigma_j}{\sigma_j^2 + \alpha} \langle f, v_j \rangle u_j.$$

We can actually compute the regularised solution without any knowledge of the singular system, since x_α is in this case the solution of the well-posed linear system:

$$(\mathcal{A}^*\mathcal{A} + \alpha\mathcal{I})u_\alpha = \mathcal{A}^* f.$$

It can be seen that Tikhonov regularisation is actually just Lavrentiev regularisation applied to the Gaussian normal equation.

3.2 Variational Regularisation Methods

Let us see that the Tikhonov Regularisation can be seen not only as a Spectral Regularisation Method but also as the minimiser of a functional:

Theorem 3.5. *Let $f \in V$. The Tikhonov-regularised solution $u_\alpha = \mathcal{R}_\alpha f$ is the unique solution of the global minimisation problem:*

$$\min_{u \in U} \frac{1}{2}\|\mathcal{A}u - f\|^2 + \frac{\alpha}{2}\|u\|^2.$$

Proof. Let us denote $f_\alpha(\cdot) = \dfrac{1}{2}\|\mathcal{A}u - f\|^2 + \dfrac{\alpha}{2}\|u\|^2$. Assume that u_α is a Tikhonov-regularised solution. We have that:

$$\begin{aligned} f_\alpha(u) - f_\alpha(u_\alpha) &= \frac{1}{2}\|\mathcal{A}u - f\|^2 + \frac{\alpha}{2}\|u\|^2 - \frac{1}{2}\|\mathcal{A}u_\alpha - f\|^2 + \frac{\alpha}{2}\|u_\alpha\|^2 \\ &= \frac{1}{2}\|\mathcal{A}u\|^2 - \langle \mathcal{A}u, f\rangle + \frac{\alpha}{2}\|u\|^2 - \frac{1}{2}\|\mathcal{A}u_\alpha\|^2 + \langle \mathcal{A}u_\alpha, f\rangle - \\ &- \frac{\alpha}{2}\|u_\alpha\|^2 + \langle (\mathcal{A}^*\mathcal{A} + \alpha\mathcal{I})u_\alpha - \mathcal{A}^* f, u_\alpha - u\rangle \\ &= \frac{1}{2}\|\mathcal{A}u - \mathcal{A}u_\alpha\|^2 + \frac{\alpha}{2}\|u = u_\alpha\|^2 \\ &\geq 0, \end{aligned}$$

which implies that u_α is a global minimiser for $f-\alpha$. Now assume that u' is a global minimiser for f_α. We have that $f_\alpha(u') \leq f_\alpha(u)$, for all $u \in U$. Consider $u = u' + \lambda v$, with arbitrary $\lambda > 0$ and fixed $v \in V$. We have that:

$$\begin{aligned} 0 &\leq f_\alpha(u) - f_\alpha(u') \\ &= \frac{\lambda^2}{2}\|\mathcal{A}v\|^2 + \frac{\lambda^2\alpha}{2}\|v\|^2 + \lambda\langle(\mathcal{A}^*\mathcal{A} + \alpha\mathcal{I})u' - \mathcal{A}^*f, v\rangle. \end{aligned}$$

Let us divide by λ and take the limit as $\lambda \to 0$ in order to obtain $\langle(\mathcal{A}^*\mathcal{A} + \alpha\mathcal{I})u' - \mathcal{A}^*f, v\rangle = 0$. Since v was fixed arbitrarily in V we have that $u' = u_\alpha$. $\square$

This result now paves the way to a very broad class of regularisation methods called *variational regularisation* or *Tikhonov-type regularisation* that are based on the following idea:

$$u_\alpha^\delta \in \arg\min_{u\in C}\{\mathcal{J}_\alpha(u) = \phi(u, f^\delta) + \alpha\psi(u)\},$$

where $C \subset U$ is a convex and closed subset of U, ϕ is some error metric (fidelity term) and ψ is a functional (regularisation term). The choice of the fidelity and regularisation terms is done with respect to the application and will produce a nearby well-posed optimization problem. For classical Tikhonov-regularisation we have that $\phi(u, f^\delta) = \frac{1}{2}\|\mathcal{A}u - f^\delta\|^2$ and $\psi(u) = \frac{1}{2}\|u\|^2$.

The most common type of variational regularisation is of the form:

$$\mathcal{R}_\alpha f := u_\alpha := \arg\min\{\mathcal{J}_\alpha(u) = \frac{1}{2}\|\mathcal{A}u - f\|^2 + \alpha\psi(u)\}. \tag{3.4}$$

This is also the regularisation method form we will use in the presented methods for the two considered problems in Chapter 5. For a more comprehensive analysis of the existence and uniqueness of the solution of the variational problem (3.4), one can inspect [25], Chapter 3. The variational regularisation methods can also be extented to inverse problems where the forward operator $\mathcal{A}$ is nonlinear (e.g. parameter identification problems for partial differential equations). Variational regularisation methods are also presented in [5], alongside state of the art regularisation methods that are widely used in practice nowadays.

Chapter 4

Weak Formulations and Galerkin Approximations

4.1 Weak Formulations

Let us now shift our attention to inverse PDE-related problems, where our operator $\mathcal{A}$ in the model problem (2.1) will come from the weak formulation of our PDE-problem, which is mainly of the form:

$$a(u, w) = f(w), \forall w \in W, \tag{4.1}$$

where $u \in U$ (called solution/trial space and its elements are called trial functions), W (called test space and its elements are called test functions) is a reflexive Banach Space (e.g. Hilbert Spaces, L^p spaces for $1 < p < \infty$), a is a continuous bilinear form on $U \times W$ and $f \in W'$. We can then define the operator $\mathcal{A} : U \to W'$ as:

$$\langle \mathcal{A}u, w \rangle_{W' \times W} := a(u, w), \tag{4.2}$$

where the scalar product denotes the outer scalar product on $W' \times W$. By defining $\mathcal{A}$ as in (4.2), problem (4.1) is equivalent to problem (2.1) with $V = W'$.

Let us introduce a weak formulation that can be modeled by the abstract problem (4.1) for a homogeneous Dirichlet problem for the Poisson equation:

$$-\Delta u = f \text{ in } \Omega, \tag{4.3}$$

$$u = 0 \text{ on } \partial\Omega. \tag{4.4}$$

We search for the solution $u : \Omega \to \mathbb{R}$ with some appropriate degree of smoothness that satisfies this problem. For the weak formulation, let us consider a test function

$\phi \in C_0^\infty(\Omega)$ and multiply the PDE by ϕ in order to then integrate over Ω to obtain:

$$-\int_\Omega (\Delta u)\phi \, dx = \int_\Omega f\phi \, dx.$$

We now apply Green's formula for u and ϕ to get:

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f\phi \, dx, \forall \phi \in C_0^\infty(\Omega).$$

Let $f \in L^2(\Omega)$. A natural solution space that also encodes the boundary conditions is $H_0^1(\Omega)$, which is a Hilbert Space when endowed with the scalar product given by $\langle u, v\rangle_{H_0^1} := \int_\Omega \nabla u \cdot \nabla v \, dx$. We can obtain the following weak formulation by extending the test space to its closure in $H^1(\Omega)$:

$$\begin{cases} \text{Find } u \in U = H_0^1(\Omega) \text{ such that} \\ \langle u, w\rangle_{H_0^1} = \langle f, w\rangle_{L^2}, \forall w \in H_0^1(\Omega). \end{cases}$$

4.1.1 Well-Posedness

Let us now return to our abstract problem (4.1). Since a is bounded, we have that:

$$\|a\|_{U\times W} = \sup_{u\in U} \sup_{w\in W} \frac{|a(u,w)|}{\|u\|_U \|w\|_W} < \infty.$$

Since f is bounded we have that:

$$\|f\|_{W'} = \sup_{w\in W} \frac{|f(w)|}{\|w\|_W} < \infty.$$

Introducing the operator $\mathcal{A}$ as in (4.2) we have that $\|\mathcal{A}\|_{\mathcal{LC}(U,W')} = \|a\|_{U\times W}$ and the existence and uniqueness of the solution to (4.1) is equivalent to $\mathcal{A}$ being bijective, i.e. $\mathcal{A}$ is injective, $\mathrm{Im}(\mathcal{A})$ is closed and $\mathcal{A}^*$ is injective, since the last two are equivalent to $\mathcal{A}$ being surjective (this is due to the fact that the closure of $\mathrm{Im}(\mathcal{A})$ is $\mathrm{Ker}(\mathcal{A}^*)^\perp$).

The first sufficient result for well-posedness is the Lax-Milgram Lemma, which is only applicable is the solution and test spaces are identical.

Lemma 4.1. *(Lax-Milgram Lemma) Let U be a Hilbert space, $a : U \times U \to \mathbb{R}$ be a bounded bilinear form and $f \in U'$. If a is coercive, i.e there exists $\alpha > 0$ such that for any $u \in U$, $a(u,u) \geq \alpha\|u\|_U^2$, then problem (4.1) is well-posed with the a-priori estimate:*

$$\|u\|_U \leq \frac{1}{\alpha}\|f\|_{U'}.$$

Proof. Define $\mathcal{A} : U \to U'$ as in (2.1). We will first show that $\mathcal{A}$ is injective, $\mathrm{Im}(\mathcal{A})$ is closed and $\mathcal{A}^*$ is injective, which is equivalent to the bijection of $\mathcal{A}$. By the coercivity of a, we have that:

$$\alpha\|u\|_U \leq \frac{a(u,u)}{\|u\|_U} \leq \sup_{w\in U}\frac{a(u,w)}{\|w\|_U} = \sup_{w\in U}\frac{|a(u,w)|}{\|w\|_U} = \|\mathcal{A}u\|_{U'},$$

which implies that $\mathcal{A}$ is injective. Let us now prove that Im($\mathcal{A}$) is closed: assume that $\mathcal{A}u_n \to y \in U$. We have that $(\mathcal{A}u_n)$ is Cauchy, but by the above inequality we have that: $\|\mathcal{A}u_n - \mathcal{A}u_m\|_{U'} = \|\mathcal{A}(u_n - u_m)\|_{U'} \geq \alpha\|u_n - u_m\|_U$, and thus (u_n) is Cauchy. However, since U is complete, $u_n \to u \in U$ and since $\mathcal{A}$ is continuous: $\mathcal{A}u_n \to \mathcal{A}u = y$ and thus $y \in$ Im($\mathcal{A}$), which implies the closedness of Im($\mathcal{A}$). $\mathcal{A}^* : U'' = U \to U'$ is defined by $\langle \mathcal{A}^*u, w\rangle_{U',U} = \langle \mathcal{A}w, u\rangle_{U',U}, \forall u, w \in U$. Let $u \in U$ such that $\mathcal{A}^*u = 0$. Then $0 = \langle \mathcal{A}^*u, u\rangle_{U',U} = a(u,u)$ which implies $\|u\|_U = 0$ and thus $\mathcal{A}^*$ is injective. We have that $\mathcal{A}$ is bijective, which is equivalent to the existence and uniqueness conditions for problem (4.1). For the continuity condition, we have that:

$$\alpha\|u\|_U \leq \frac{a(u,u)}{\|u\|_U} = \frac{|a(u,u)|}{\|u\|_U} = \frac{|f(u)|}{\|u\|_U} \leq \|f\|_{U'} \Rightarrow \|u\|_U \leq \frac{1}{\alpha}\|f\|_{U'}.$$

This completes the proof of the well-posedness of problem (4.1). □

Proposition 4.1. *(Variational Formulation) Let U be a Hilbert space, $a : U \times U \to \mathbb{R}$ be a bounded bilinear form and $f \in U'$. Assume that a is coercive and symmetric, i.e $a(u,w) = a(w,u), \forall u, w \in U$.. Then u is the unique solution of (4.1) if and only if*

$$u = \arg\min_{v\in U}\{\mathcal{J}(v) = \frac{1}{2}a(v,v) - f(v)\}.$$

Proof. Let $u, w \in U$ and $t \in \mathbb{R}$ arbitrarily chosen. We have that:

$$\mathcal{J}(u + tw) = \mathcal{J}(u) + t(a(u,w) - f(w)) + \frac{1}{2}t^2 a(w,w).$$

Let us assume u solves (4.1). Since $a(w,w) \geq 0$, we have that $\mathcal{J}(u+tw) \leq \mathcal{J}(u)$, and thus u minimizes $\mathcal{J}$ over U. Let us now assume that u minimizes $\mathcal{J}$ over U. $\mathcal{J}(u + tw)$ is a quadratic polynomial in t that reaches its minimum value for $t = 0$. We have that the derivative of this polynomial evaluated at $t = 0$ is equal to 0 and thus $a(u,w) = f(w)$. □

We will now present a necessary and sufficient role for the well-posedness of problem (4.1) in the general context where U and W can be different:

Theorem 4.1. *(Banach-Nečas-Babuška(BNB)) Let U be a Banach space and W be a reflexive Banach space. Let $a : U \times W \to \mathbb{R}$ be a continuous bilinear form and $f \in W'$. Then problem (4.1) is well-posed if and only if the following conditions hold:*

(BNB1) There exists $\alpha > 0$ *such that*

$$\inf_{u\in U}\sup_{w\in W}\frac{|a(u,w)|}{\|u\|_U\|w\|_W}\geq\alpha.$$

(BNB2) Let $w\in W$*. If* $a(u,w)=0$ *for all* $u\in U$*, then* $w=0$*.*

We have the following a-priori estimate: $\|u\|_U\leq\frac{1}{\alpha}\|f\|_{W'}$*.*

Proof. Let us define $\mathcal{A}$ as in (4.2). We will prove that the two conditions in the theorem are equivalent to the injectivity of $\mathcal{A}$, the fact that $\mathrm{Im}(\mathcal{A})$ is closed and the injectivity of $\mathcal{A}^*$. Let $u\in U$. We have by (BNB1) that: $\|\mathcal{A}u\|_{W'}=\sup_{w\in W}\frac{|\langle\mathcal{A}u,w\rangle_{W',W}|}{\|w\|_W}=\sup_{w\in W}\frac{|a(u,w)|}{\|w\|_W}$. Dividing by $\|u\|_U$ and taking the infimum over $u\in U$ we get that $\mathcal{A}$ is injective and $\mathrm{Im}(\mathcal{A})$ is closed (by the same argument as in the proof of Lemma 4.1). Let us now prove that the condition (BNB2) together with the reflexiveness of W is equivalent to $\mathcal{A}^*$ being injective. Let $J_W:W\to W''$ be the canonical isometry. Since W is reflexive we have that J_W is an isomorphism. We have that:

$$\langle\mathcal{A}^*(J_Ww),u\rangle_{U',U}=\langle J_Ww,\mathcal{A}u\rangle_{W'',W'}=\langle\mathcal{A}u,w\rangle_{W',W}=a(u,w),\forall(u,w)\in U\times W.$$

If $a(u,w)=0,\forall u\in U$ we have that $(\mathcal{A}^*\circ J_W)w=0$ and using (BNB2) we have that $\mathcal{A}^*\circ J_W$ is injective, and thus $\mathcal{A}^*$ is injective. By now we have proved the existence and uniqueness of the solution to problem (4.1). For the continuity, let us prove the a-priori estimate, which will follow from:

$$\alpha\|u\|_U\leq\sup_{w\in W}\frac{|a(u,w)|}{\|w\|_W}=\sup_{w\in W}\frac{|f(w)|}{\|w\|_W}=\|f\|_{W'}.$$

□

Let us now see the fact that we can actually obtain a two-sided bound on the solution of problem (4.1): We have that $\|f\|_{W'}=\|\mathcal{A}u\|_{W'}\leq\|\mathcal{A}\|_{\mathcal{LC}(U,W')}\|u\|_U=\|a\|_{U\times W}\|u\|_U$. We thus have that:

$$\frac{1}{\|a\|_{U\times W}}\|f\|_{W'}\leq\|u\|_U\leq\frac{1}{\alpha}\|f\|_{W'}.$$

4.2 Galerkin Approximation

Let us consider problem (4.1). We will make the following notations:

$$\alpha:=\inf_{u\in U}\sup_{w\in W}\frac{|a(u,w)|}{\|u\|_U\|w\|_W}\leq\sup_{u\in U}\sup_{w\in W}\frac{|a(u,w)|}{\|u\|_U\|w\|_W}:=\|a\|_{U\times W},$$

and we will assume that the problem is well-posed, i.e. $0 < \alpha$ and $\|a\|_{U\times W} < \infty$.

The Galerkin method replaces the infinite-dimensional spaces U and W by two finite-dimensional spaces U_h and W_h. In our context, these finite dimensional spaces will be constructed using finite elements and a mesh $\mathcal{T}_h$. The discrete formulation of our problem will then look like this:

$$\begin{cases} \text{Find } u_h \in U_h \text{ such that} \\ a_h(u_h, w_h) = f_h(w_h), \quad \forall w_h \in W_h, \end{cases} \tag{4.5}$$

where a_h will be a continuous bilinear form on $U_h \times W_h$ and f_h a continuous linear functional on W_h. The discrete problem (4.5) will be called a *standard Galerkin approximation* when $U_h = W_h$ and *Petrov-Galerkin* otherwise. The approximation is said to be *conforming* if $V_h \subset V$ and $W_h \subset W$.

4.2.1 Discrete Well-Posedness

We will consider the two spaces U_h and W_h to be equipped with the norms $\|\cdot\|_{U_h}$ and $\|\cdot\|_{V_h}$. Let us now state the two well-posedness results from the last subsection for the discrete problem (4.5):

Lemma 4.2. *(Discrete Lax-Milgram) Consider problem (4.5) with a standard Galerkin approximation. Assume that a_h is coercive on U_h, i.e there exists $\alpha_h > 0$ such that for all $u_h \in U_h$ we have that $a_h(u_h, u_h) \geq \alpha_h \|u_h\|_{U_h}^2$. Then (4.5) is well-posed with the a-priori estimate $\|u_h\|_{U_h} \leq \dfrac{1}{\alpha_h}\|f_h\|_{U_h'}$.*

Proof. See [[20], Lemma 26.3] □

Theorem 4.2. *(Discrete BNB) Let U_h, W_h be finite-dimensional spaces, a_h be a continuous bilinear form on $U_h \times W_h$ and let $f_h \in W_h'$. Then problem (4.5) is well-posed if and only if the following conditions are satisfied:*

- *There exists $\alpha_h > 0$ such that*

$$\inf_{u_h \in U_h} \sup_{w_h \in W_h} \frac{|a_h(u_h, w_h)|}{\|u_h\|_{U_h}\|w_h\|_{W_h}} \geq \alpha_h.$$

- $\dim U_h = \dim(W_h)$.

We have the a-priori estimate $\|u_h\|_{U_h} \leq \dfrac{1}{\alpha_h}\|f_h\|_{W_h'}$.

Proof. See [[20], Theorem 26.6] □

When dealing with conforming approximations, we equip U_h and W_h with the norms of U and W and consider $a_h := a\big|_{U_h \times W_h}$. The following result by Fortin is a criterion for the satisfaction of the discrete inf-sup condition using its continuous counterpart on $U \times W$.

Lemma 4.3. *Let U, W be Hilbert spaces and a be a continuous bilinear form on $U \times W$. Let $U_h \subset U$ and $W_h \subset W$ be equipped with the norms of U and W, respectively. Let us consider the following statements:*

1. *There exists a map $\Pi_h : W \to W_h$, called Fortin operator such that:*
   (a) $a(u_h, \Pi_h w - w) = 0, \forall (u_h, w) \in U_h \times W;$
   (b) *There is $\gamma_{\Pi_h} > 0$ such that $\gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W, \forall w \in W$.*
2. *The discrete inf-sup condition holds true.*

Then (1) $\Rightarrow$ (2) with $\alpha_h \geq \gamma_{\Pi_h} \alpha$. Conversely, (2) $\Rightarrow$ (1) with $\gamma_{\Pi_h} \geq \dfrac{\alpha_h}{\|a\|_{U \times W}}$ and Π_h can be constructed to be linear and idempotent.

Proof. See [[20], Lemma 26.9] □

Chapter 5

The Cauchy Problem and Unique Continuation

5.1 The Cauchy Problem for Elliptic Operators

The elliptic Cauchy problem is a well known PDE-related ill-posed inverse problem. Let us consider a bounded domain $\Omega \subset \mathbb{R}^n, n \in \{2, 3\}$ with Lipschitz boundary $\partial\Omega$, which can be partitioned in the following manner: $\partial\Omega = \overline{\Gamma} \cup \overline{\tilde{\Gamma}}$, where $\Gamma, \tilde{\Gamma}$ are disjoint open sets of non-zero measure in the topology induced on $\partial\Omega$. The idea of the Cauchy problem is that Dirichlet and Neumann data are only given on Γ, which will be referred to as the *accessible* part of the boundary, while $\tilde{\Gamma}$ will be the *inaccessible* part of the boundary, where no information about u is known. We will consider an elliptic operator $\mathcal{L}$ (Laplace or Helmholtz) and formulate the following problem: find $u \in H^1(\Omega)$ such that

$$\begin{cases} \mathcal{L}u = -f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ \partial_\nu u = g_1 & \text{on } \Gamma, \end{cases} \tag{5.1}$$

where $f \in L^2(\Omega), g_0 \in H^{\frac{1}{2}}(\Gamma), g_1 \in H^{-\frac{1}{2}}(\Gamma)$ and ν is the outward unit normal to $\partial\Omega$.

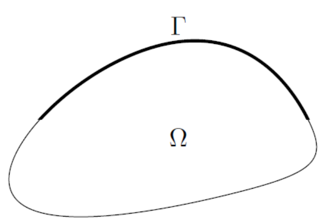


Figure 5.1: ([33], Fig.1.1) Sketch of the domain Ω in the Cauchy problem with accessible boundary part Γ

The Cauchy problem for elliptic operators was considered by Hadamard in [22] where he has shown that it is ill-posed because it doesn't satisfy the third condition of well-posedness. The example refers to the case when the elliptic operator $\mathcal{L}$ is Laplace's operator.

Example 5.1. *(Hadamard's Example) Consider* $\Omega = (0,\pi) \times (0,1)$, *the source term* $f = 0$ *and the Dirichlet-Neumann data given on* $\Gamma = (0,\pi) \times \{0\}$ *by* $g_0 = 0$ *and* $g_1 = -\frac{1}{n}\sin(nx)$. *We formulate the problem:*

$$\begin{cases} -\Delta u = f & \textit{in } \Omega, \\ u = g_0 & \textit{on } \Gamma, \\ \partial_\nu u = g_1 & \textit{on } \Gamma, \end{cases}$$

The exact solution of this problem is $u(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$. *We have that*

$$\|g_1\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|g_1\|_{L^\infty(\Gamma)} \to 0, n \to \infty.$$

However, we have that:

$$\|u\|_{H^1(\Omega)} \to \infty \textit{ and } u(x,y) \to \infty \textit{ a.e. as } n \to \infty,$$

and thus the solution doesn't depend continuously on data.

It can be shown that the Cauchy problem is actually an example of a severely ill-posed problem. The author of [4] proved for a smooth domain in $\mathbb{R}^2$ that the problem is equivalent to the inversion of a compact operator with infinite dimensional range and whose singular values decay faster than any polynomial rate.

The Cauchy problem is however a conditionally stable problem, i.e. if we assume an a priori bound on our solution, we can obtain continuous dependence on data. This is very thoroughly studied in [1] for a wide class of elliptic differential operators. For results regarding both the uniqueness and the stability of the solution of the Cauchy problem for second-order differential operators, one can see [24], Chapter 3.

5.1.1 Weak Formulation for the Cauchy Problem

Let us follow [1] and [13] and consider the elliptic operator $\mathcal{L}$ to be of the form: $\mathcal{L}u = \nabla \cdot (A\nabla u) + \mu u$, where $\mu \in \mathbb{R}$ and $A \in \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix, called the diffusivity matrix. The Cauchy problem will be formulated in the following manner:

$$\begin{cases} \nabla \cdot (A\nabla u) + \mu u = -f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma, \\ (A\nabla u) \cdot \nu = g_1 & \text{on } \Gamma, \end{cases} \tag{5.2}$$

For the Poisson and Helmholtz equations we have that $A = I_n$, where I_n is the $n \times n$ identity matrix and $\mu = 0$, respectively $\mu = k^2, k > 0$.

Let us now derive a weak formulation for problem (5.2). We will define the following spaces both equipped with the $H^1(\Omega)$ norm:

$$U := \{u \in H^1(\Omega) : u|_\Gamma = g_0\}$$

and, for $\tilde{\Gamma} = \partial\Omega \setminus \Gamma$,

$$W := \{w \in H^1(\Omega) : w|_{\tilde{\Gamma}} = 0\}.$$

We will multiply the PDE in (5.2) by an element $\phi \in W$ and integrate over Ω using Green's Formula to get:

$$\int_\Omega (A\nabla u \cdot \nabla \phi - \mu u \phi)\, dx = \int_{\partial\Omega} \phi (A\nabla u \cdot \nu)\, d\sigma + \int_\Omega f\phi\, dx$$

and since $\phi \in W$ and $(A\nabla u) \cdot \nu = g_1$ on Γ we get:

$$\int_\Omega (A\nabla u \cdot \nabla \phi - \mu u \phi)\, dx = \int_\Gamma g_1 \phi\, d\sigma + \int_\Omega f\phi\, dx.$$

We can thus introduce the bilinear form:

$$a(u, w) := \int_\Omega (A\nabla u \cdot \nabla w - \mu u w)\, dx, \forall (u, w) \in U \times W, \tag{5.3}$$

and the linear functional:

$$l(w) = \int_\Gamma g_1 w\, d\sigma + \int_\Omega f w\, dx, \forall w \in W, \tag{5.4}$$

and our weak formulation will read as:

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u, w) = l(w), \forall w \in W. \end{cases} \tag{5.5}$$

For the Laplacian and the Helmholtz operator, the bilinear form will have the form:

$$a(u, w) := \int_\Omega (\nabla u \cdot \nabla w - k^2 u w)\, dx, \forall (u, w) \in U \times W, k \geq 0$$

and problem (5.2) will be equivalent to (5.1).

5.1.2 Regularised Optimisation Formulation for the Cauchy Problem

In the following we will present three different regularisation methods for solving the Cauchy problem that all start from a constrained optimisation problem.

We will first cast the Cauchy problem (5.2) as an optimisation problem that has a PDE-constraint:

$$\min_{u\in H^1(\Omega)} \frac{1}{2}\|u-g_0\|^2_{L^2(\Gamma)} \qquad \text{subject to } \mathcal{L}u = -f \text{ in } \Omega.$$

By considering the proposed weak formulation, we will follow [26], [12], [33] and introduce our test function as a Lagrange multiplier and consider the following Lagrangian functional:

$$\tilde{L}(u,\lambda) = \frac{1}{2}\|u-g_0\|^2_{L^2(\Gamma)} + a(u,\lambda) - l(\lambda).$$

However, trying to find the saddle-point of $\tilde{L}$, denoted by $(\tilde{u},\tilde{\lambda}) \in H^1(\Omega)\times W$ will lead to an ill-posed problem again, which might lead to an ill-conditioned system of equations when discretised. The saddle point is the solution to:

$$\begin{cases} 0 = \partial_u \tilde{L}v = \langle \tilde{u}-g_0, v\rangle_{L^2(\Gamma)} + a(v,\tilde{\lambda}), \\ 0 = \partial_\lambda \tilde{L}w = a(\tilde{u},w) - l(w), \end{cases} \qquad \forall (v,w)\in H^1(\Omega)\times W$$

In order to overcome the ill-posedness of solving for $(\tilde{u},\tilde{\lambda}) \in H^1(\Omega)\times W$ we can add a regularising term for the objective function of the initial PDE-constrained minimization problem on the continuum level by following a Tikhonov-type regularisation method: let $\alpha_T > 0$ be a regularisation parameter and let us penalise the norm of the gradient in order to force for a smoother solution. This is an almost identical approach to the one presented in [9], namely the relaxed mixed formulation of Tikhonov regularisation. However, in this method, the dual variable is introduced directly considering the PDE residual, whilst our method introduces the dual variable as a Lagrange multiplier. Thus, let us consider the optimisation problem:

$$\min_{u\in H^1(\Omega)} \frac{1}{2}\|u-g_0\|^2_{L^2(\Gamma)} + \frac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)} \qquad \text{subject to } \mathcal{L}u = -f \text{ in } \Omega,$$

and by using the weak form of the PDE-constraint we will introduce the following Lagrangian functional, by also regularising the Lagrange multiplier, which will then help prove an inf-sup condition on our problem:

$$L_T(u,\lambda) = \frac{1}{2}\|u-g_0\|^2_{L^2(\Gamma)} + \frac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{\alpha_T}{2}\|\nabla \lambda\|^2_{L^2(\Omega)} + a(u,\lambda) - l(\lambda).$$

Our functional will contain three terms: the data term, the regularisation and the PDE-constraint. Its saddle point $(u_{\alpha_T}, \lambda_{\alpha_T}) \in H^1(\Omega) \times W$ is the solution to:

$$\begin{cases} 0 = \langle u_{\alpha_T} - g_0, v\rangle_{L^2(\Gamma)} + \alpha_T \langle \nabla u_{\alpha_T}, \nabla v\rangle_{L^2(\Omega)} + a(v, \lambda_{\alpha_t}), \\ 0 = a(u_{\alpha_T}, w) - \alpha_T \langle \nabla \lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} - l(w), \end{cases}$$

$\forall (v, w) \in H^1(\Omega) \times W$.
This is equivalent to finding $(u_{\alpha_T}, \lambda_{\alpha_T}) \in H^1(\Omega) \times W$ such that:

$$\begin{cases} \langle u_{\alpha_T}, v\rangle_{L^2(\Gamma)} + \alpha_T \langle \nabla u_{\alpha_T}, \nabla v\rangle_{L^2(\Omega)} + a(v, \lambda_{\alpha_T}) = \langle g_0, v\rangle_{L^2(\Gamma)}, \\ \qquad a(u_{\alpha_T}, w) - \alpha_T \langle \nabla \lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} = l(w), \end{cases}$$

$\forall (v, w) \in H^1(\Omega) \times W$.
This system is also almost identical to the one in [9], namely equations (3.5), when considering Laplace's operator. If there exists a solution $u \in H^1(\Omega)$ to the Cauchy problem (5.2), it is the limit of the regularised solutions u_{α_T} in the H^1-norm, while the multiplier λ_{α_T} will tend to 0 in the H^1-norm, when $\alpha_T \to 0$, as can be seen in [9].
Let us now collect the left-hand sides of the optimality conditions in a bilinear form:

$$A_T[(u, \lambda), (v, w)] = \langle u, v\rangle_{L^2(\Gamma)} + \alpha_T \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - \alpha_T \langle \nabla \lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} + a(v, \lambda) + a(u, w),$$

and thus the optimality system can be rewritten as:

$$A_T[(u_{\alpha_T}, \lambda_{\alpha_T}), (v, w)] = \langle g_0, v\rangle_{L^2(\Gamma)} + l(w), \forall (v, w) \in H^1(\Omega) \times W.$$

Let us note that:

$$A_T[(u, \lambda), (u, -\lambda)] = \|u\|^2_{L^2(\Gamma)} + \alpha_T \|\nabla u\|^2_{L^2(\Omega)} + \alpha_T \|\nabla \lambda\|^2_{L^2(\Omega)}.$$

We will define the following functional on the product space $H^1(\Omega) \times W$:

$$\|(u, \lambda)\|^2_T = \|u\|^2_{L^2(\Gamma)} + \alpha_T \|\nabla u\|^2_{L^2(\Omega)} + \alpha_T \|\nabla \lambda\|^2_{L^2(\Omega)}, (u, \lambda) \in H^1(\Omega) \times W.$$

This is evidently a seminorm, and in the following we will prove the positive definiteness. To this end, we will present a Poincare-Friedrich's type inequality:

Theorem 5.1. *([32], Ch.1, Th.1.9) Let Ω be a bounded domain with Lipschitz boundary. Let $\Gamma \subset \partial\Omega$ with $meas(\Gamma) > 0$ and $u \in H^1(\Omega)$. There exists $C > 0$ such that:*

$$\|u\|_{H^1(\Omega)} \leq C(\|u\|^2_{L^2(\Gamma)} + \|\nabla u\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Proof. See [32], Ch.1, Th.1.9. □

Let us now assume that $\|(u,\lambda)\|_T^2 = 0$. It follows that $\|\nabla\lambda\|_{L^2(\Omega)} = 0$ and thus, since $\lambda \in W$, it follows that also $\|\lambda\|_{L^2(\Gamma')} = 0$. By the above equality we have that $\|\lambda\|_{H^1(\Omega)} = 0$, which implies that $\|\lambda\|_{L^2(\Omega)} = 0$ and $\lambda = 0$ a.e. in Ω. Following a similar argument we can show that $\|u\|_{H^1(\Omega)} = 0$ and $u = 0$ a.e. in Ω. By this argument, we have proven that our seminorm is actually a norm. In this norm, we will prove an inf-sup condition for the bilinear form A:

$$A_T[(u,\lambda),(u,-\lambda)] = \|(u,\lambda)\|_T^2 \Rightarrow \exists c > 0 \text{ s.t. } c\|(u,\lambda)\|_T^2 \le A_T[(u,\lambda),(u,-\lambda)].$$

We have that:

$$\begin{aligned} c\|(u,\lambda)\|_T &\le \frac{A_T[(u,\lambda),(u,-\lambda)]}{\|(u,\lambda)\|_T} \\ &= \frac{A_T[(u,\lambda),(u,-\lambda)]}{\|(u,-\lambda)\|_T} \\ &\le \sup_{(v,w)\in H^1\times W} \frac{A_T[(u,\lambda),(v,w)]}{\|(v,w)\|_T}, \end{aligned}$$

which proves the first condition of Theorem 4.1. For the second condition consider a fixed $(v,w) \in H^1 \times W$. If we have that $A_T[(u,\lambda),(v,w)] = 0, \forall (u,\lambda) \in H^1 \times W$ we can let $(u,\lambda) = (v,-w)$ and thus $A_T[(u,\lambda),(v,w)] = \|(v,w)\|_T^2 = 0$, which implies that $(v,w) = 0_{H^1\times W}$.

By Theorem 4.1 we have that our problem

$$A_T[(u,\lambda),(v,w)] = \langle g_0, v\rangle_{L^2(\Gamma)} + l(w), \forall (v,w) \in H^1(\Omega) \times W,$$

is well-posed.

Another way of using Tikhonov regularisation to solve the considered Cauchy problem is to drop the weak formulation and consider another unconstrained minimisation problem. In this formulation, we will try to minimise the L^2-norm of the PDE-residual over the H^1-space of functions that already satisfy the boundary conditions (both Dirichlet and Neumann) and also add a regularising term (which will be the H^1-norm). Let us define the following spaces:

$$D_{\phi_0,\phi_1}(\mathcal{L},\Omega,\Gamma) = \{u \in \text{Dom}(\mathcal{L}) : u = \phi_0, \partial_\nu u = \phi_1 \text{ on } \Gamma\},$$

where $\mathcal{L}u = \nabla \cdot (A\nabla u) + \mu u$. The minimisation problem will then have the form:

$$\min_{u \in D_{g_0,g_1}(\mathcal{L},\Omega,\Gamma)} \frac{1}{2}\|\mathcal{L}u + f\|_{L^2(\Omega)}^2 + \frac{\alpha_{qr}}{2}\|u\|_{H^1(\Omega)}^2,$$

where $\alpha_{qr} > 0$ is the regularisation parameter. The optimality condition for the solution $u_{qr} \in D_{g_0,g_1}(\mathcal{L}, \Omega, \Gamma)$ is:

$$\langle \mathcal{L}u_{qr}, \mathcal{L}v\rangle_{L^2(\Omega)} + \alpha_{qr}\langle u_{qr}, v\rangle_{H^1(\Omega)} = \langle -f, \mathcal{L}v\rangle_{L^2(\Omega)}, \forall v \in D_{0,0}(\mathcal{L}, \Omega, \Gamma).$$

This is a well-posed fourth-order problem obtained by a Tikhonov-type regularisation method named *quasi-reversibility*, which was presented in [8] for the Cauchy problem for Laplace's equation.

Let us now reformulate the PDE-constrained optimisation problem by not trying to force the Dirichlet data in the optimisation problem. We will again construct a Lagrangian and enforce the Dirichlet boundary condition weakly by using a similar method as the one presented in [21], which is a modification of the classic Nitsche method (see [7] for a general description). However, we will only apply this method after constructing the optimality system and discretising the system using the finite element method. The optimisation problem will have the form:

$$\min_{u \in H^1(\Omega)} \frac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)} \quad \text{subject to } \mathcal{L}u = -f \text{ in } \Omega.$$

For this purpose we will introduce the following Lagrangian functional by also introducing a regulariser for the multiplier:

$$L^*_T(u, \lambda) = \frac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{\alpha_T}{2}\|\nabla \lambda\|^2_{L^2(\Omega)} + a(u, \lambda) - l(\lambda).$$

The saddle point $(u^*_{\alpha_T}, \lambda^*_{\alpha_T}) \in H^1(\Omega) \times W$ will satisfy the following optimality conditions:

$$\begin{cases} \alpha_T\langle \nabla u^*_{\alpha_T}, \nabla v\rangle_{L^2(\Omega)} + a(v, \lambda^*_{\alpha_T}) = 0, \\ a(u^*_{\alpha_T}, w) - \alpha_T\langle \nabla \lambda^*_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} = l(w), \end{cases}$$

$\forall (v, w) \in H^1(\Omega) \times W$.

This is very similar to system (3.4) in [9], namely the system for the Cauchy problem for Laplace's equation when using the mixed formulation for Tikhonov regularisation. However, in [9], the Dirichlet boundary condition is enforced strongly (the solution space will be restricted to the space U from our weak formulation), whilst we take an approach that enforces the condition weakly on a discrete level.

Let us again collect the left-hand sides of the optimality conditions in a bilinear form:

$$A^*_T[(u, \lambda), (v, w)] = \alpha_T\langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - \alpha_T\langle \nabla \lambda, \nabla w\rangle_{L^2(\Omega)} + a(u, w) + a(v, \lambda),$$

and thus the optimality system will be equivalent to the equation:

$$A_T^*[(u_{\alpha T}^*, \lambda_{\alpha T}^*), (v, w)] = l(w), \forall (v, w) \in H^1(\Omega) \times W.$$

As was already said, this time we will modify the variational problem by weakly enforcing the Dirichlet condition $u = g_0$ on Γ on a discrete level. However, this will only be presented in the next chapter, namely in the section Discretisation.

5.2 Unique Continuation for Elliptic Operators

The unique continuation problem is another known example of a PDE-related ill-posed inverse problem. Let us consider again a bounded domain with continuous/Lipschitz boundary $\Omega \subset \mathbb{R}^n, n \in \{2, 3\}$ and an open subset $\omega \subset \Omega$. The idea of the unique continuation problem is that we only have information about the solution in the open subset ω and we have no boundary information. We will consider an elliptic operator $\mathcal{L}$ and formulate the following problem: find $u \in H^1(\Omega)$ such that

$$\begin{cases} \mathcal{L}u = -f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega, \end{cases} \tag{5.6}$$

where $f \in H^{-1}(\Omega) \supset L^2(\Omega), u_\omega \in H^1(\omega)$.

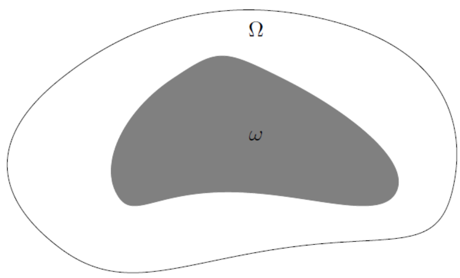


Figure 5.2: ([33], Fig.1.3) Sketch of domain Ω and subdomain ω (in gray) for the Unique Continuation Problem

It is obvious that for a solution to exist, the data need to be compatible, i.e. $\mathcal{L}u_\omega = f$ in ω. If there exists a solution $u \in H^1(\Omega)$, it is unique since we could consider the difference v of two solutions which will satisfy:

$$\begin{cases} \mathcal{L}v = 0 & \text{in } \Omega, \\ v = 0 & \text{in } \omega. \end{cases}$$

By the weak unique continuation principle for elliptic operators (see [29], Theorem 3.8) we have that $v = 0$ in Ω and thus, the solution is unique. There is an equivalence between the weak unique continuation property for an elliptic operator $\mathcal{L}$ and the uniqueness of the Cauchy problem. A general proof can be found in [34]. However, we will follow [33] and offer a quick explanation for one implication for Laplace's operator: if $v|_\omega = 0$ and $\Delta v = 0$ in Ω, we can take a ball $B \subset \omega$ and we can obtain $v = 0$ in $\Omega \setminus B$ as the unique solution of the Cauchy problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \setminus B, \\ v, \partial_\nu v = 0 & \text{on } \partial B. \end{cases}$$

Since the equivalence, it is easy to argue that unique continuation is also severely ill-posed just like the Cauchy problem. However, we'd like to use Hadamard's example, as done in [12], to prove the ill-posedness of unique continuation problem: Let us consider the continuous linear operator $\mathcal{A} : H^1(\Omega) \to H^{-1}(\Omega) \times H^1(\omega), \mathcal{A}u = (-\Delta u, u|_\omega)$. The Moore-Penrose generalised inverse of this operator, $\mathcal{A}^\dagger$ is discontinuous, since it's range is not closed. If we assume the contrary, there should exist $c > 0$ such that for any $u \in H^1(\Omega)$ we have that:

$$\|u\|_{H^1(\Omega)} \leq c(\|\Delta u\|_{H^{-1}(\Omega)} + \|u\|_{H^1(\omega)}),$$

where we considered the norm on the cartesian product of the two spaces to be the sum of the norms of the two spaces. However, considering now Hadamard's example for ω close to the origin, we have that the constant c must depend on n since $\|u\|_{H^1(\Omega)} \to \infty$ as $n \to \infty$.

5.2.1 Weak Formulation for Unique Continuation

When we consider a weak solution to problem (5.6), not just in the sense of $\mathcal{A}^\dagger$ acting on $(f, u|_\omega)$, the problem is conditionally stable, as can be seen in [33], Chapter 2 for different elliptic operators. This is the reason for which we will turn our attention to a weak formulation for problem (5.6).

We will again consider the elliptic operator $\mathcal{L}$ to be of the form $\mathcal{L}u = \nabla \cdot (A\nabla u) + \mu u$, where $\mu \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix. We will formulate the unique continuation problem as follows: find $u \in H^1(\Omega)$ such that

$$\begin{cases} \nabla \cdot (A\nabla u) + \mu u = -f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega, \end{cases} \tag{5.7}$$

where $f \in H^{-1}(\Omega) \supset L^2(\Omega), u_\omega \in H^1(\omega)$. Let us now follow [15] and derive a weak formulation for problem (5.7). We will define the following spaces:

$$U := \{u \in H^1(\Omega) : u|_\omega = u_\omega\}$$

and

$$W := H_0^1(\Omega).$$

Let us consider U with the $H^1(\Omega)$ norm. We will now multiply the PDE in (5.7) by an element $\phi \in W$ and integrate over Ω using Green's formula and the fact that $\phi \in W = H_0^1(\Omega)$ to get:

$$\int_\Omega (A\nabla u \cdot \nabla\phi - \mu u\phi)\,dx = \int_\Omega f\phi\,dx.$$

Now, we will define the bilinear form:

$$a(u, w) := \int_\Omega (A\nabla u \cdot \nabla w - \mu uw)\,dx, \forall (u, w) \in U \times W, \tag{5.8}$$

and the linear functional:

$$l(w) := \int_\Omega fw\,dx, \forall w \in W, \tag{5.9}$$

and our weak formulation will read as:

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u, w) = l(w), \forall w \in W. \end{cases} \tag{5.10}$$

5.2.2 Regularised Optimisation Formulation for Unique Continuation

We will again cast this problem as a PDE-constrained optimisation problem

$$\min_{u \in H^1(\Omega)} \frac{1}{2}\|u - u_\omega\|^2_{L^2(\omega)} \text{ subject to } \mathcal{L}u = -f \quad \text{in } \Omega.$$

Using the weak formulation we can introduce the Lagrangian functional which will encode the PDE-constraint and also have a regularising term:

$$L_T(u, \lambda) = \frac{1}{2}\|u - u_\omega\|^2_{L^2(\omega)} + \frac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{\alpha_T}{2}\|\nabla\lambda\|^2_{L^2(\Omega)} + a(u, \lambda) - l(\lambda).$$

The saddle point is the pair $(\tilde{u}, \tilde{\lambda}) \in H^1(\Omega) \times W$ that satisfies:

$$\begin{cases} \langle u_{\alpha_T}, v\rangle_{L^2(\omega)} + \alpha_T \langle \nabla u_{\alpha_T}, \nabla v\rangle_{L^2(\Omega)} + a(v, \lambda_{\alpha_T}) = \langle u_\omega, v\rangle_{L^2(\omega)}, \\ a(u_{\alpha_T}, w) - \alpha_T \langle \nabla \lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} = l(w), \end{cases}$$

$\forall (v, w) \in H^1(\Omega) \times W$.
If there exists a solution $u \in H^1(\Omega)$, when $\alpha_T \to 0$, u_{α_T} will converge to u in H^1-norm and λ_{α_T} will converge to 0 in H^1-norm. This can again be seen in [9].
Let us now consider the bilinear form:

$$A_T[(u, \lambda), (v, w)] = \langle u, v\rangle_{L^2(\omega)} + \alpha_T \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - \alpha_T \langle \nabla \lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} + a(v, \lambda) + a(u, w),$$

and thus the optimality system will be equivalent to the equation:

$$A_T[(u_{\alpha_T}, \lambda_{\alpha_T}), (v, w)] = \langle u_\omega, v\rangle_{L^2(\omega)} + l(w), \forall (v, w) \in H^1(\Omega) \times W.$$

Let us note that:

$$A_T[(u, \lambda), (u, -\lambda)] = \|u\|^2_{L^2(\omega)} + \alpha_T \|\nabla u\|^2_{L^2(\Omega)} + \alpha_T \|\nabla \lambda\|^2_{L^2(\Omega)}$$

We will define the following functional on $H^1 \times W$:

$$\|(u, \lambda)\|^2_T = \|u\|^2_{L^2(\omega)} + \alpha_T \|\nabla u\|^2_{L^2(\Omega)} + \alpha_T \|\nabla \lambda\|^2_{L^2(\Omega)}$$

This is evidently a seminorm, and in fact a norm as we will see: Let $\|(u, \lambda)\|_T = 0$. We have that $\|\nabla \lambda\|^2_{L^2(\Omega)} = 0$ and since $\lambda \in W := H^1_0(\Omega)$, we have that $\lambda = 0$ in W. We will now use the following result:

Theorem 5.2. *([32], Ch.2, Th.7.4) Let Ω be a bounded domain with continuous boundary and ω an open nonempty subset of Ω. Let $u \in H^1(\Omega)$. There exists $C > 0$ such that the following inequality holds:*

$$\|u\|_{H^1(\Omega)} \leq C(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\omega)})^{\frac{1}{2}}.$$

Proof. See [32], Ch.2, Th.7.4. □

Since $\|(u, \lambda)\|_T = 0$ we have that $\|u\|^2_{L^2(\omega)} = \|\nabla u\|^2_{L^2(\Omega)} = 0$ and using the above theorem, we conclude that $\|u\|_{H^1(\Omega)} = 0$. This means that $\|\cdot\|_T$ is a norm on $H^1 \times W$. Now, following the exact steps as for the Cauchy problem, an inf-sup condition can be proved in this norm and also the second condition in Theorem 4.1 in the same manner as was already done. Thus, by Theorem 4.1, the problem

$$A_T[(u, \lambda), (v, w)] = \langle u_\omega, v\rangle_{L^2(\omega)} + l(w), \forall (v, w) \in H^1(\Omega) \times W$$

is well posed.

Let us now again consider an unconstrained minimisation problem related to our unique continuation problem. To this end, let us define the space:

$$D_g(\mathcal{L}, \Omega, \omega) = \{u \in \mathrm{Dom}(\mathcal{L}) : u = g \text{ in } \omega\},$$

and the minimisation problem:

$$\min_{u \in D_{u_\omega}(\mathcal{L}, \Omega, \omega)} \frac{1}{2}\|\mathcal{L}u + f\|^2_{L^2(\Omega)} + \frac{\alpha_{qr}}{2}\|u\|^2_{H^1(\Omega)},$$

where $\alpha_{qr} > 0$ is the regularisation parameter. The optimality condition for the solution $u_{qr} \in D_{u_\omega}(\mathcal{L}, \Omega, \omega)$ is:

$$\langle \mathcal{L}u_{qr}, \mathcal{L}v \rangle_{L^2(\Omega)} + \alpha_{qr}\langle u_{qr}, v \rangle_{H^1(\Omega)} = \langle -f, \mathcal{L}v \rangle_{L^2(\Omega)}, \forall v \in D_0(\mathcal{L}, \Omega, \omega),$$

which is again a well-posed fourth order problem obtained by quasi-reversibility.

Chapter 6

Discretisation and Implementation

6.1 Discretisation

In the following we will present the finite element discretisations of the optimisation problems presented in Chapter 5, for both considered problems. Throughout this section, we will abuse notation and consider our discrete solution and test spaces to be denoted by U_h and W_h throughout the whole section, although their definition will vary. First, we will give a short overview of the abstract Finite Element Method, following [28] and [20].

6.1.1 Abstract Finite Element Method

Consider the same abstract weak formulation (4.1) as in Chapter 4. By following a Galerkin approximation as was already mentioned in Chapter 4, we introduce finite-dimensional spaces U_h and W_h such that $dim(U_h) = dim(W_h) = m$ and assume that the discrete problem (4.5) is well-posed by Theorem 4.2. In our context, U_h and W_h will be mainly subspaces of H^1 consisting of continuous piecewise linear polynomials on a uniform mesh $\mathcal{T}_h$ of Ω with mesh size h.

Let us now consider $\{\phi_i\}_{i\in\{1:m\}}$ be a basis for U_h and $\{\psi_i\}_{i\in\{1:m\}}$ be a basis for W_h. Let us define the following isomorphisms, which actually reconstruct functions from coordinates in the given basis:

$$R_\phi : \mathbb{R}^m \to U_h, R_\phi(U) = \sum_{i\in\{1:m\}} U_i\phi_i, \forall U = (U_i)_{i\in\{1:m\}},$$

$$R_\psi : \mathbb{R}^m \to W_h, R_\psi(W) = \sum_{i\in\{1:m\}} W_i\psi_i, \forall W = (W_i)_{i\in\{1:m\}}.$$

We will define the *stiffness matrix* of our problem to be:

$$A \in \mathbb{R}^{m\times m} : A_{ij} = a_h(\phi_j, \psi_i), \forall i, j \in \{1 : m\}.$$

We will also define the *load vector* of our problem as:

$$B \in \mathbb{R}^m : B_i = f_h(\psi_i), \forall i \in \{1 : m\}.$$

Using these notations, we have that:

$$u_h \text{ solves (4.5)} \iff AU = B,$$

where $u_h = R_\phi(U)$. In order to find our finite element solution we thus have to compute the coordinate vector U by solving the above linear system and reconstruct the solution in the given basis $\{\phi_i\}_{i\in\{1:m\}}$.

6.1.2 Discretisation of proposed methods for the Cauchy Problem

Consider a quasi-uniform family $\{\mathcal{T}_h\}$ of geometrically conformal triangulations K of Ω (see [19], Section 8.2). We define $h_K = diam(K), h = \max_{K\in\mathcal{T}_h} h_K$. For our first proposed method, we have the following variational problem:

$$A_T[(u,\lambda),(v,w)] = \langle g_0, v\rangle_{L^2(\Gamma)} + l(w), \forall (v,w) \in H^1(\Omega)\times W,$$

where our bilinear form is:

$$A_T[(u,\lambda),(v,w)] = \langle u,v\rangle_{L^2(\Gamma)} + \alpha_T\langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - \alpha_T\langle \nabla\lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} + a(v,\lambda) + a(u,w),$$

and $W := \{w \in H^1(\Omega) : w|_{\tilde{\Gamma}} = 0\}$. We will define the following finite-dimensional spaces:

$$U_h := \left\{u \in C(\overline{\Omega}) : u|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h\right\},$$
$$W_h := U_h \cap W.$$

Our discretised problem will read: find $(u_h, \lambda_h) \in U_h \times W_h$ such that

$$A_{T_h}[(u_h,\lambda_h),(v_h,w_h)] = \langle g_0, v_h\rangle_{L^2(\Gamma)} + l_h(w_h), \forall (v_h,w_h) \in U_h \times W_h,$$

For the third proposed method, we have the following variational problem:

$$A_T^*[(u,\lambda),(v,w)] = l(w), \forall (v,w) \in H^1(\Omega)\times W,$$

where our bilinear form is:

$$A_T^*[(u,\lambda),(v,w)] = \alpha_T\langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - \alpha_T\langle \nabla\lambda, \nabla w\rangle_{L^2(\Omega)} + a(u,w) + a(v,\lambda).$$

We will define the same finite dimensional spaces as before:

$$U_h := \Big\{u \in C(\overline{\Omega}) : u|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h\Big\},$$

$$W_h := U_h \cap W.$$

As was already mentioned, we will however change our problem on the discrete level in order to weakly enforce boundary conditions following a similar approach as the one introduced in [21]. To this end, consider the following modified discrete problem: find $(u_h, \lambda_h) \in U_h \times W_h$ such that

$$A^*_{\mathcal{T}_h}[(u_h, \lambda_h), (v_h, w_h)] = l_h(w) + \langle g_0, \nabla v_h \cdot \nu\rangle_{L^2(\Gamma)} + \gamma \sum_{K\in\mathcal{T}_h} \langle h_K^{-1} g_0, v_h\rangle_{\Gamma\cap\partial K},$$

$\forall (v_h, w_h) \in U_h \times W_h$, where our bilinear form will be:

$$\begin{aligned} A^*_{\mathcal{T}_h}\big[(u_h, \lambda_h), (v_h, w_h)\big] \ &= \alpha_T \langle \nabla u_h, \nabla v_h\rangle_{L^2(\Omega)} \ - \ \alpha_T \langle \nabla \lambda_h, \nabla w_h\rangle_{L^2(\Omega)} \ + \ a_h(u_h, w_h) \ + \\ &+ \ a_h(v_h, \lambda_h) \ + \ \langle u_h, \nabla v_h \cdot \nu\rangle_{L^2(\Gamma)} \ + \ \gamma \sum_{K\in\mathcal{T}_h} \langle h_K^{-1} u_h, v_h\rangle_{\Gamma\cap\partial K}\,. \end{aligned}$$

6.1.3 Discretisation of proposed method for Unique Continuation

For the first proposed method we have the following variational problem:

$$A_T[(u, \lambda), (v, w)] = \langle u_\omega, v\rangle_{L^2(\omega)} + l(w), \forall (v, w) \in H^1(\Omega) \times W,$$

where our bilinear form is:

$$A_T[(u, \lambda), (v, w)] = \langle u, v\rangle_{L^2(\omega)} + \alpha_T \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - \alpha_T \langle \nabla \lambda_{\alpha_T}, \nabla w\rangle_{L^2(\Omega)} + a(v, \lambda) + a(u, w),$$

and $W := H^1_0(\Omega)$. We will define the following finite dimensional spaces:

$$U_h := \Big\{u \in C(\overline{\Omega}) : u|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h\Big\},$$

$$W_h := U_h \cap W = U_h \cap H^1_0(\Omega).$$

Our discretised problem will be: find $(u_h, \lambda_h) \in U_h \times W_h$ such that

$$A_{\mathcal{T}_h}[(u_h, \lambda_h), (v_h, w_h)] = \langle u_\omega, v_h\rangle_{L^2(\omega)} + l_h(w_h), \forall (v_h, w_h) \in U_h \times W_h.$$

6.2 Implementation and Numerical Results

In this section we will present numerical results for different examples for the proposed methods in Chapter 4, using the discretised problems we have formulated in the last section. All implementations were made in Python using FEniCS[2] for the finite element discretisation. All plots were made using Matplotlib[23]. For each particular example, the Tikhonov regularisation parameter was heuristically tuned by carrying out the proposed methods on a 128×128 mesh for $\alpha_T \in (10^{-1}, 10^{-7})$. The "best choice" for α_T will be mentioned for each considered numerical example. Solution plots, error distribution plots and convergence plots (evolution of error with respect to meshsize) will also be presented. Whenever there will be considered an example that was already presented in the literature, it will be in order to be able to compare this method to other methods and the corresponding articles will be cited. The whole code can be found under https://github.com/raresicar/Regularisation-Methods-for-Inverse-Problems-related-to-PDEs.

6.2.1 Numerical Results for the proposed methods for the Cauchy Problem

The first two examples, namely Example 6.1 and Example 6.2 are solved using the first proposed method for the Cauchy problem, where the discrete bilinear form is A_{T_h}. The last two examples, namely Example 6.3 and Example 6.4 are solved using the third proposed method(the penalty method) for the Cauchy problem, where the discrete bilinear form is $A^*_{T_h}$.

Example 6.1 (Example 1: Polynomial Function). *Let us consider the domain* $\Omega = (0,1) \times (0,1)$ *and the accessible part of the boundary* $\Gamma = \{(x,0) : |x - 0.5| < 0.1\}$. *Following [16], we investigate the following Cauchy problem:*

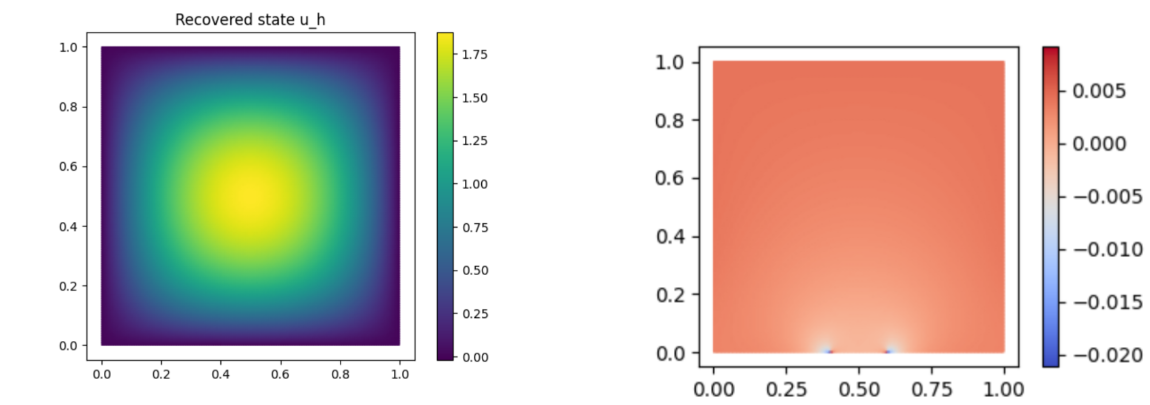
$$\begin{cases} -\Delta u = 60(x - x^2 + y - y^2) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ \partial_\nu u = -30(x - x^2) & \text{on } \Gamma, \end{cases}$$

The exact solution to this problem is given by

$$u(x,y) = 30xy(1-x)(1-y).$$

We implemented the method on a 128×128 mesh and tested various values of the regularisation parameter α_T in the interval $(10^{-6}, 10^{-1})$. The results indicate that $\alpha_T = 10^{-3}$ offers a good heuristic choice.

The relative errors obtained using the method are:



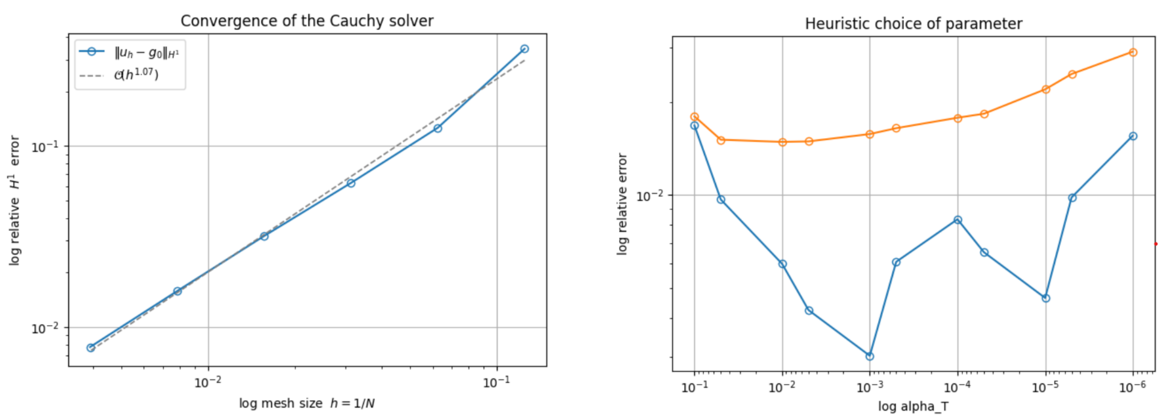
(a) Recovered solution u_h.

(b) Error distribution $u_h - u$.

Figure 6.1: Solution and error plots for Example 1.

- Relative L^2 norm error: `0.003017`
- Relative H^1 norm error: `0.015767`
- Relative H^1_0 seminorm error: `0.016142`

It is interesting to note that the highest errors are distributed at the ends of the accessible boundary Γ, namely in the neighborhood of the points of coordinates $(0.4, 0)$ and $(0.6, 0)$. It is however clear that, even with poor boundary data, the method can recover the solution with good approximation. However, this is also due to the fact that this is a rather simple example, since it is a symmetric polynomial of degree 2 in both variables with respect to $(0.5, 0.5)$ and with a homogeneous Dirichlet boundary condition.



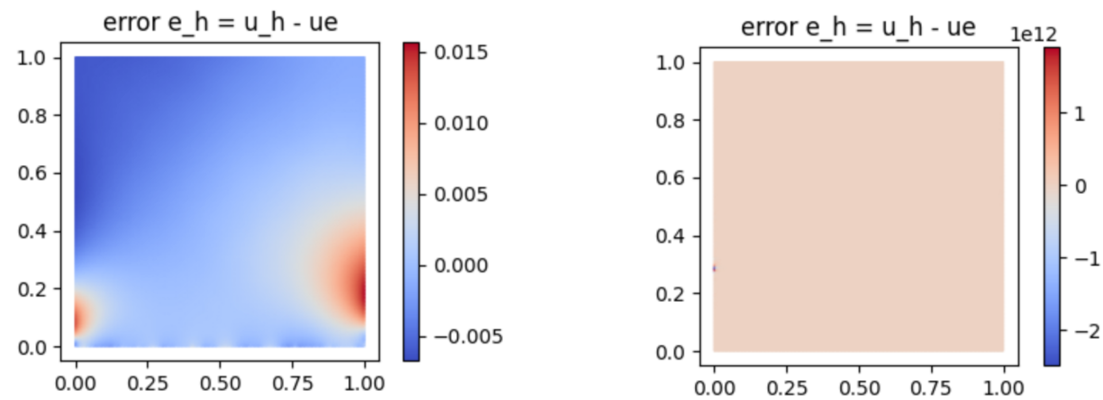
(a) $H^1(\Omega)$ relative error vs. mesh size $h = 1/N$.

(b) Tuning α_T: L^2 (blue) and H^1 (yellow) errors.

Figure 6.2: Convergence and parameter tuning plots for Example 1.

We will now add noise to the given data, namely uniformly sampled noise of level 0.05. The accessible boundary part will now be the whole bottom edge, namely

$\Gamma = (0, 1) \times \{0\}$. The solution and error distribution plots after solving the "noisy" Cauchy problem for Example 6.1 are presented in Fig. 6.3. Also, an error distribution plot is added in order to show that the "naive" proposed Lagrangian doesn't turn the problem into a well posed one. The relative error in all norms is of the order 10^6 in this case, highlighting the highly unstable numerical properties of the discretised optimality system.



(a) Error plot with noise level 0.05 for data on the bottom edge.

(b) Error plot for the naive method with $\alpha_T = 0$(no regularisation).

Figure 6.3: Comparison of error distributions for noisy data (left) and unregularised method (right).

We now present a challenging example, illustrating a situation where the proposed method exhibits poor performance.

Example 6.2 (Example 2: Hadamard Example). *Let $\Omega = (0, \pi) \times (0, 1)$ and let the accessible boundary be $\Gamma = (0, \pi) \times \{0\}$. Let $n \in \mathbb{N}^*$ and consider the following Cauchy problem:*

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ \partial_\nu u = -\dfrac{1}{n}\sin(nx) & \text{on } \Gamma. \end{cases}$$

The exact solution is given by

$$u(x, y) = \frac{1}{n^2}\sin(nx)\sinh(ny).$$

Fig. 6.5 shows the recovered numerical solution and the corresponding error distribution for the case $n = 2$. The method ran on a 256×128 mesh with $\alpha_T = 10^{-6}$, which was heuristically chosen by running the proposed method on a 128×64 mesh with different regularisation parameters.

The relative errors obtained in this case are significantly larger:

- Relative L^2 norm error: `0.1324`

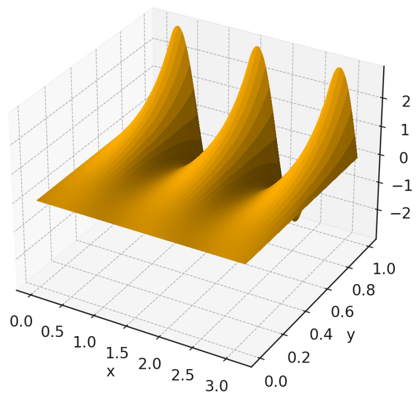
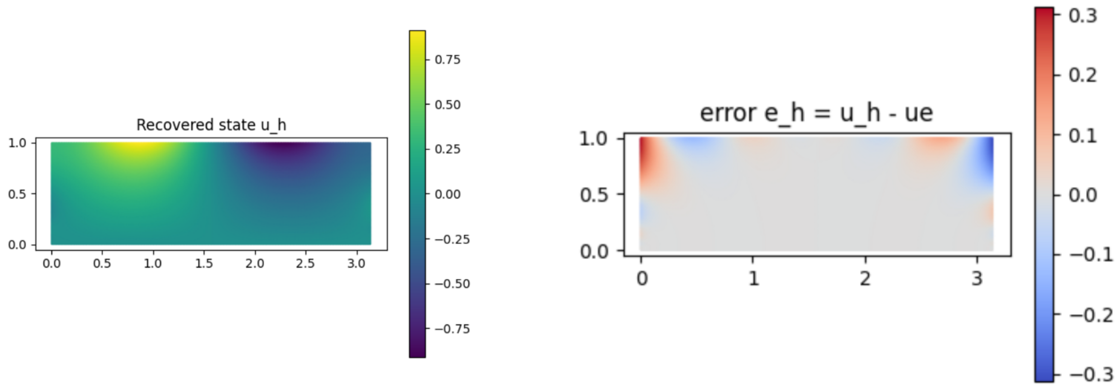


Figure 6.4: Visualization of Hadamard's example for $n = 5$.



(a) Recovered solution u_h.

(b) Error distribution $u_h - u$.

Figure 6.5: Numerical results for the Hadamard-type example with $n = 2$.

- Relative H^1 norm error: `0.3255`
- Relative H^1_0 seminorm error: `0.3397`

This is due to the fact that the accessible part of the boundary was the one where the oscillations of the function have the smallest amplitude, as can be seen in Fig. 6.4. The method seems to not be able to recover the exponential growth in the amplitude of the oscillations, which are due to the $\sinh(ny)$ term. Numerical simulations were performed for larger values of n, but the obtained relative errors were even higher.

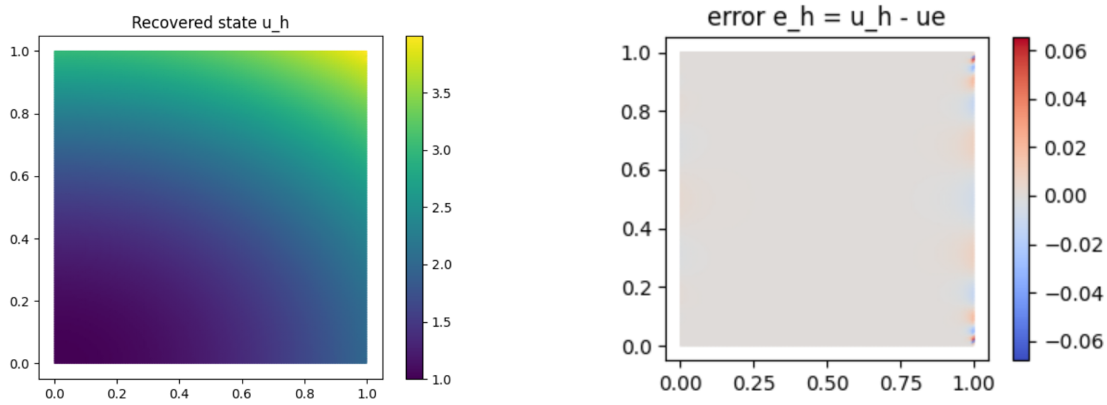
Example 6.3 (Example 3: Polynomial Function). *Let us consider the domain* $\Omega = (0, 1) \times (0, 1)$ *and the accessible part of the boundary* $\Gamma = (0, 1) \times \{0, 1\}$*. Let us formulate* *the following Cauchy problem:*

$$\begin{cases} -\Delta u = -6 & \textit{in } \Omega, \\ u = 1 + x^2 & \textit{on } (0,1) \times \{0\}, \\ u = 3 + x^2 & \textit{on } (0,1) \times \{1\}, \\ \partial_\nu u = 0 & \textit{on } (0,1) \times \{0\}, \\ \partial_\nu u = 4 & \textit{on } (0,1) \times \{1\}. \end{cases}$$

The exact solution to this problem is given by

$$u(x, y) = 1 + x^2 + 2y^2.$$

The regularisation parameter was again tuned on a 128×128 mesh and the "best choice" was chosen accordingly to the lowest L^2 relative error, namely $\alpha_T = 10^{-5}$. Also, the penalty parameter γ was chosen heuristically based on multiple simulations to be $\gamma = 10$. This seems to be a value that fits the data in a good manner in most cases and that doesn't bias the method to choose data fitting in the detriment of satisfying the PDE or regularising the solution.



(a) Recovered solution u_h.

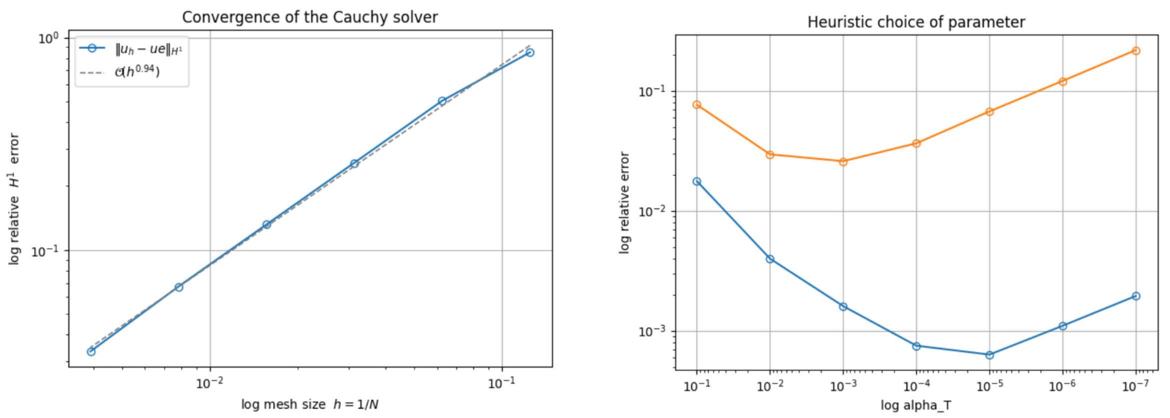
(b) Error distribution $u_h - u$.

Figure 6.6: Solution and error plots for Example 3.

The relative errors obtained using the method are:

- Relative L^2 norm error: `0.000632`
- Relative H^1 norm error: `0.067258`
- Relative H_0^1 seminorm error: `0.086828`

What is interesting to note is how small the L^2 relative error is in comparison to the H^1 or H_0^1 relative errors. Also, the error distribution plot seems to show that the solution was recovered almost perfectly in Ω, with small errors on the left and right edges of the boundary(on $\tilde{\Gamma}$).



(a) $H^1(\Omega)$ relative error vs. mesh size $h = 1/N$.

(b) Tuning α_T: L^2 (blue) and H^1 (yellow) errors.

Figure 6.7: Convergence and parameter tuning plots for Example 3.

We will now add a noise level of 0.01 to our data. The regularisation parameter was set to $\alpha_T = 10^{-2}$. We can thus notice, that since the data terms are noisy, more regularisation needs to be done in the detriment of data fitting in order to obtain a better approximation of the exact solution u. Fig. 6.8 shows the error distribution in the noisy case. The error distribution seems to now also show higher errors in the interior of Ω, not only on the inaccessible part of the boundary.

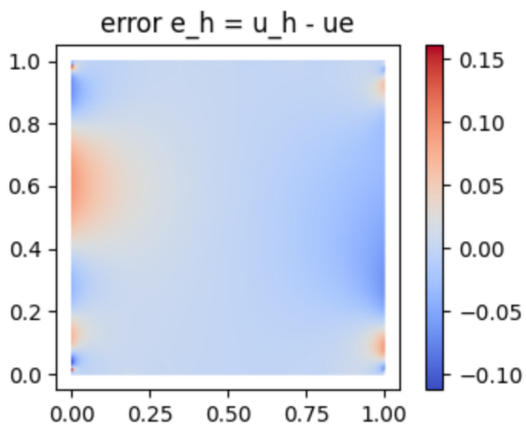
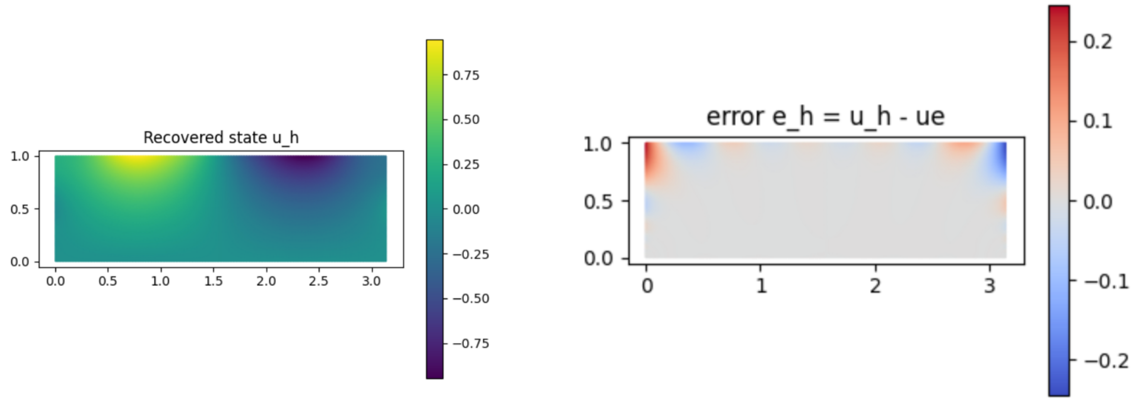


Figure 6.8: Error distribution for noisy data with noise level 0.01

Example 6.4 (Example 2: revisited)**.** *Let us consider Example 6.2 once again, this time using the penalty method for weakly enforcing the Dirichlet boundary condition.*

The regularisation parameter was chosen $\alpha_T = 10^{-7}$ in this case and the penalty parameter was $\gamma = 10$. The method ran on a 256×128 mesh.

The relative errors obtained in this case are:



(a) Recovered solution u_h. (b) Error distribution $u_h - u$.

Figure 6.9: Numerical results for the Hadamard-type example with $n = 2$ for the penalty method

- Relative L^2 norm error: `0.080951`
- Relative H^1 norm error: `0.257996`
- Relative H_0^1 seminorm error: `0.270117`

It seems that this method recovers the Hadamard-type solution somewhat better. However, it still has an issue with recovering the exponential amplitude growth of the oscillations along the y-axis. The error is distributed very similarly as in the first method's case.

6.2.2 Numerical Results for the proposed method for Unique Continuation

All considered examples are solved using the first proposed method for the unique continuation problem, where the discrete bilinear form is A_{T_h}.

Example 6.5 (Example 1: Polynomial Function). *Let us follow [14] and consider the domain* $\Omega = (0,1) \times (0,1)$*, the subdomain* $\omega = (0.25, 0.75) \times (0, 0.5)$ *and a subdomain* $B = (0.125, 0.875) \times (0, 0.95)$ *where the errors will be computed. Following [16], we investigate the following unique continuation problem:*

$$\begin{cases} -\Delta u = 60(x - x^2 + y - y^2) & \text{in } \Omega, \\ \quad\; u = 30xy(1-x)(1-y) & \text{in } \omega. \end{cases}$$

The exact solution to this problem is given by

$$u(x,y) = 30xy(1-x)(1-y).$$

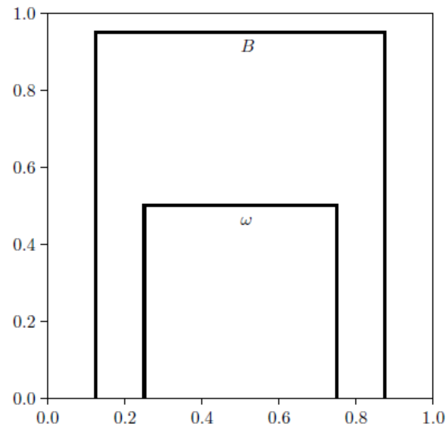
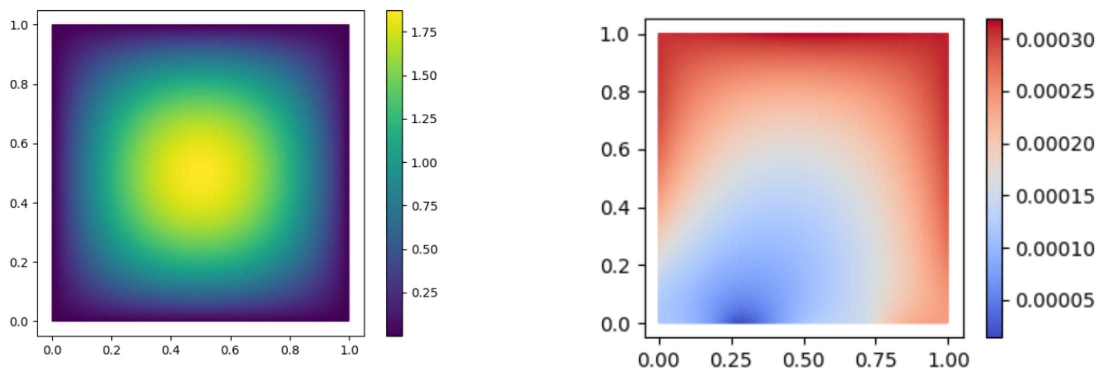


Figure 6.10: ([14], Fig.3.b) Domains for the proposed problem.

The method ran on a 128×128 mesh with the regularisation parameter $\alpha_T = 10^{-3}$. The relative errors in the three norms were computed in the subdomain B.



(a) Recovered solution u_h.

(b) Error distribution $u_h - u$.

Figure 6.11: Numerical results for the Unique Continuation problem.

The relative errors for this example are remarkably low, indicating a highly accurate reconstruction:

- Relative L^2 norm error in B: `8.83e-05`
- Relative H^1 norm error in B: `0.0114`
- Relative H^1_0 seminorm error in B: `0.01183`

Example 6.6 (Example 2: Gaussian Bump)**.** *Let Ω, ω and B be defined as in 6.6. We investigate the following unique continuation problem, also proposed in [14] (Section 4, Example 1):*

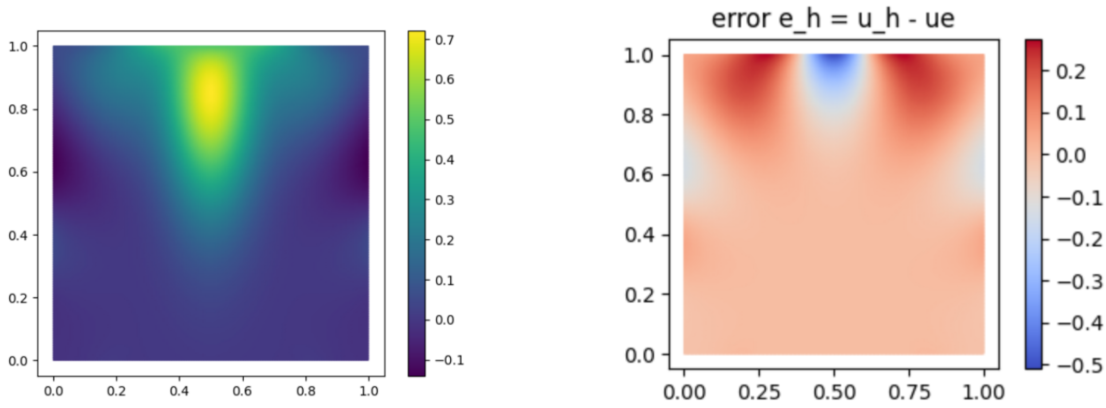
$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega, \end{cases}$$

where $k = 10$, $\sigma_x = 0.01$, $\sigma_y = 0.1$, $u_\omega = u|_\omega$, *and* $f = -\Delta u - k^2 u$, *with*

$$u(x, y) = \exp\left(-\frac{(x - 0.5)^2}{2\sigma_x} - \frac{(y - 1)^2}{2\sigma_y}\right)$$

as the exact solution.

This problem is a harder one, because this function has a bump at $(0.5, 1)$ which won't be recovered optimally by our method, since the data are given in the subdomain $\omega = (0.25, 0.75) \times (0, 0.5)$, which is far from $(0.5, 1)$. The method ran on a 128×128 mesh with $\alpha_T = 10^{-6}$.



(a) Recovered solution u_h. (b) Error distribution $u_h - u$.

Figure 6.12: Numerical results for the Unique Continuation problem for the Helmholtz equation.

It can be seen that the method can recover the bump partially, but the obtained relative errors are very high:

- Relative L^2 norm error in B: `0.286945`
- Relative H^1 norm error in B: `0.427334`
- Relative H^1_0 seminorm error in B: `0.429421`

Bibliography

[1] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella. The stability for the cauchy problem for elliptic equations. *Inverse Problems*, 25(12):123004, 2009.

[2] M. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. Rognes, and G. Wells. The fenics project version 1.5. *Archives of Numerical Software*, 3(100):9–23, 2015.

[3] A.B. Bakushinskii. Remarks on choosing a regularization parameter using the quasi-optimality and ratio criterion. *USSR Computational Mathematics and Mathematical Physics*, 24(4):181–182, 1984.

[4] F. B. Belgacem. Why is the cauchy problem severely ill-posed? *Inverse Problems*, 23(2):823–836, 2007.

[5] M. Benning and M. Burger. Modern regularization methods for inverse problems. *Acta Numerica*, 27:1–111, 2018.

[6] M. Benning and M. J. Ehrhardt. Inverse problems in imaging. Lecture Notes, Michaelmas Term, University of Cambridge, 2016. https://mehrhardt.github.io/data/201611_lecture_notes_invprob.pdf.

[7] J. Benzaken, J. A. Evans, and R. Tamstorf. Constructing nitsche's method for variational problems. *Archives of Computational Methods in Engineering*, 31(4):1867–1896, 2024.

[8] L. Bourgeois. Convergence rates for the quasi-reversibility method to solve the cauchy problem for laplace's equation. *Inverse Problems*, 22:413 – 430, 2006.

[9] L. Bourgeois and A. Recoquillay. A mixed formulation of the tikhonov regularization and its application to inverse pde problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 52:123–145, 2018.

[10] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, New York, 2010.

[11] M. Burger. Inverse problems. Lecture Notes, Winter Semester, University of Münster, 2007. https://www.uni-muenster.de/AMM/num/Vorlesungen/IP_WS07/skript.pdf.

[12] E. Burman, P. Hansbo, and M. G. Larson. Solving ill-posed control problems by stabilized finite element methods: an alternative to tikhonov regularization. *Inverse Problems*, 34(3):035004, 2018.

[13] E. Burman, M. G. Larson, and L. Oksanen. Primal-dual mixed finite element methods for the elliptic cauchy problem. *SIAM Journal on Numerical Analysis*, 56(6):3480–3509, 2018.

[14] E. Burman, M. Nechita, and L. Oksanen. Unique continuation for the helmholtz equation using stabilized finite element methods. *Journal de Mathématiques Pures et Appliquées*, 129:1–22, 2019.

[15] E. Burman, M. Nechita, and L. Oksanen. Optimal approximation of unique continuation. *Foundations of Computational Mathematics*, 25:1025–1045, 2024.

[16] E. Burman and L. Oksanen. Weakly consistent regularisation methods for ill-posed problems. In D. Di Pietro, A. Ern, and L. Formaggia, editors, *Numerical Methods for PDEs*, volume 15 of *SEMA SIMAI Springer Series*. Springer, Cham, 2018.

[17] Clason C.. Regularization of inverse problems. Lecture Notes, University of Duisburg-Essen, 2021. https://arxiv.org/abs/2001.00617.

[18] H.W. Engl, M. Hanke, and G. Neubauer. *Regularization of Inverse Problems*. Mathematics and Its Applications. Springer, Dordrecht, 1996.

[19] A. Ern and J.L. Guermond. *Finite Elements I: Approximation and Interpolation*. Texts in Applied Mathematics. Springer, Cham, 2021.

[20] A. Ern and J.L. Guermond. *Finite Elements II: Galerkin Approximation, Elliptic and Mixed PDEs*. Texts in Applied Mathematics. Springer, Cham, 2021.

[21] J. Freund and R. Stenberg. On weakly imposed boundary conditions for second order problems. *Proceedings of the Ninth International Conference on Finite Elements in Fluids*, pages 327–336, 1995.

[22] J. Hadamard. *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, 1923.

[23] J. D. Hunter. Matplotlib: A 2d graphics environment. *Computing in Science & Engineering*, 9(3):90–95, 2007.

[24] V. Isakov. *Inverse Problems for Partial Differential Equations*. Applied Mathematical Sciences. Springer, Cham, 3rd edition, 2017.

[25] K. Ito and B. Jin. *Inverse Problems: Tikhonov Theory And Algorithms*. Series On Applied Mathematics. World Scientific Publishing Company, Singapore, 2014.

[26] I. Kazufumi and K. Karl. Augmented lagrangian methods for nonsmooth, convex optimization in hilbert spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 41(5):591–616, 2000.

[27] L. F. Lang and M. J. Ehrhardt. Inverse problems. Lecture Notes, Lent Term, University of Cambridge, 2018. https://mehrhardt.github.io/data/201803_lecture_notes_invprob.pdf.

[28] M.G. Larson and F. Bengzon. *The Finite Element Method: Theory, Implementation, and Applications*. Texts in Computational Science and Engineering. Springer Berlin, Heidelberg, 2013.

[29] N. Lerner. *Carleman Inequalities: An Introduction and More*. Grundlehren der mathematischen Wissenschaften. Springer, Cham, 1st edition, 2019.

[30] R.E. Megginson. *An Introduction to Banach Space Theory*. Graduate Texts in Mathematics. Springer, New York, 1998.

[31] Arch W. Naylor and George R. Sell. *Linear Operator Theory in Engineering and Science*. Applied Mathematical Sciences. Springer Science & Business Media, New York, 2nd edition, 2000.

[32] J. Necas, Š. Necasová, C.G. Simader, G. Tronel, and A. Kufner. *Direct Methods in the Theory of Elliptic Equations*. Springer Monographs in Mathematics. Springer Berlin, Heidelberg, 2011.

[33] M. Nechita. *Unique continuation problems and stabilised finite element methods*. PhD thesis, University College London, London, 2020. https://discovery.ucl.ac.uk/id/eprint/10113065/.

[34] L. Nirenberg. Uniqueness in cauchy problems for differential equations with constant leading coefficients. *Communications on Pure and Applied Mathematics*, 10:89–105, 1957.

[35] D. Werner. *Funktionalanalysis*. Springer-Lehrbuch. Springer Berlin, Heidelberg, 2018.

[36] E. Zeidler. *Applied Functional Analysis: Main Principles and Their Applications*. Applied Mathematical Sciences. Springer, New York, 2012.