61A Lecture 5

Friday, January 30

Environments for Higher-Order Functions

Environments Enable Higher-Order Functions

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Environment diagrams describe how higher-order functions work!

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(Demo)

Names can be Bound to Functional Arguments

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1  def apply_twice(f, x):
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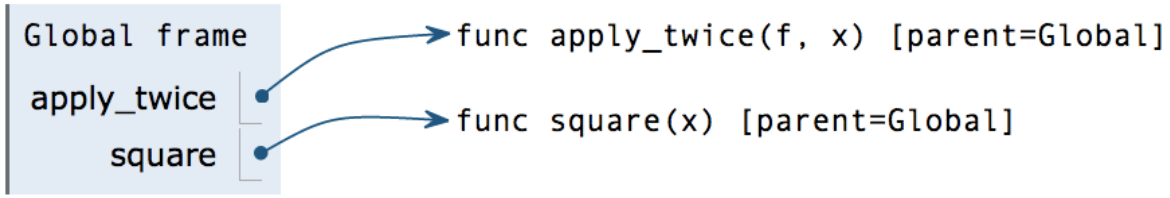

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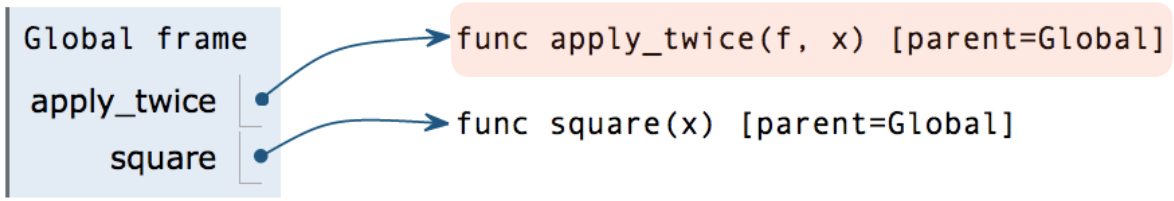

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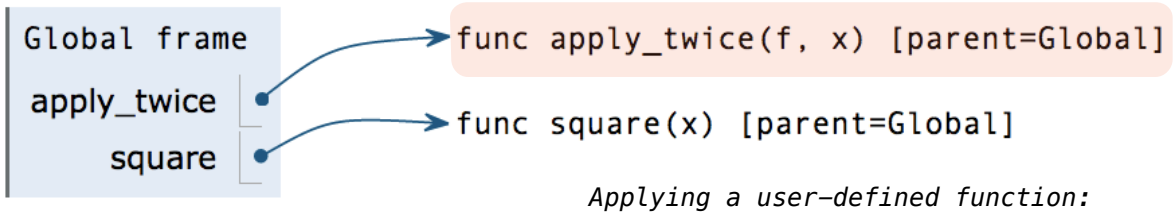

Applying a user-defined function:

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- Bind formal parameters (f & x) to arguments
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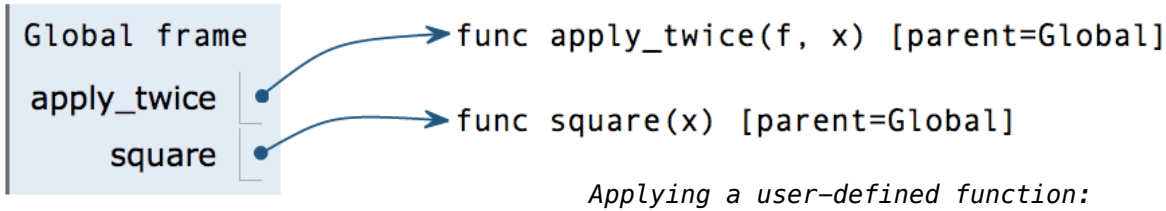

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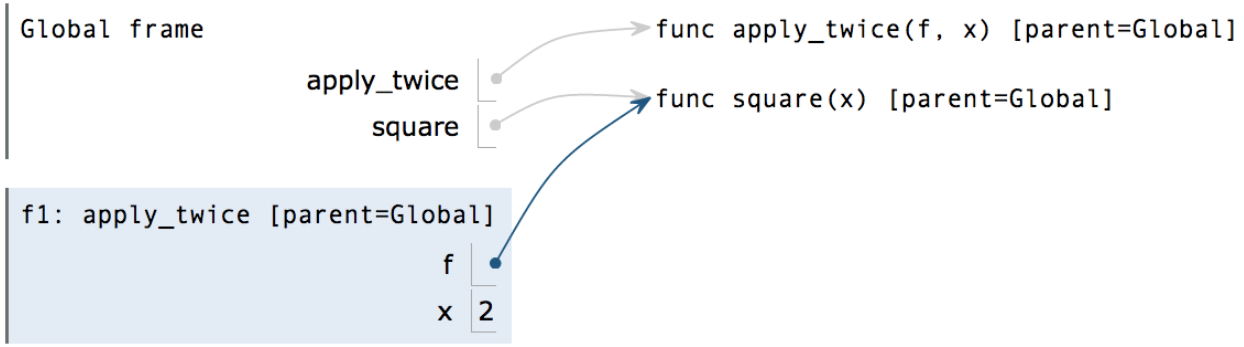

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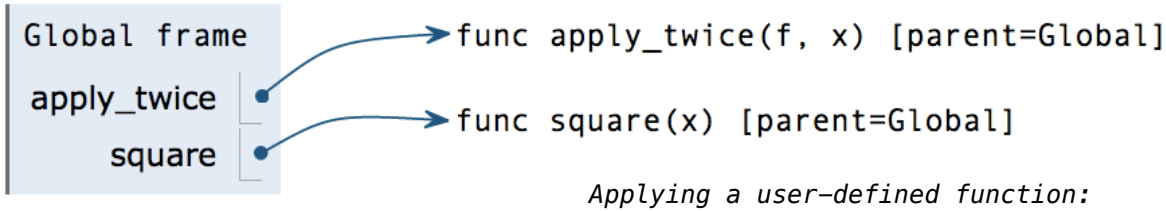

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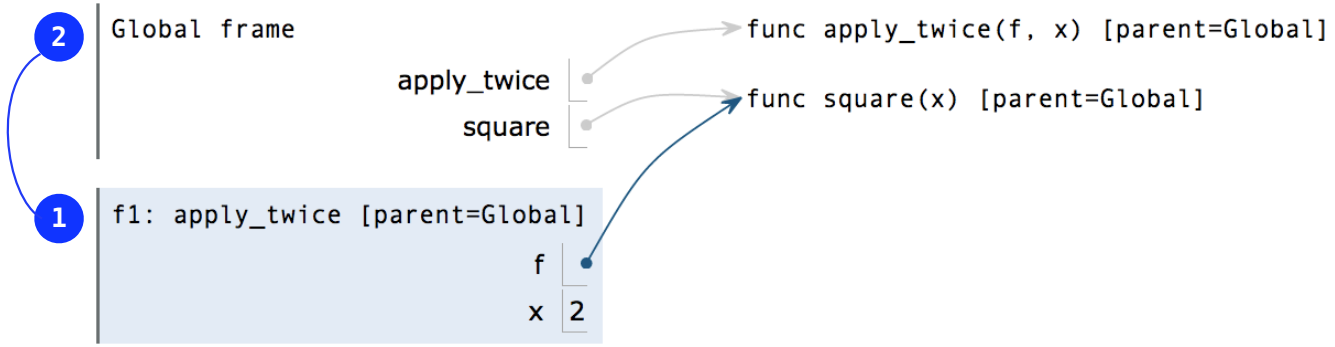

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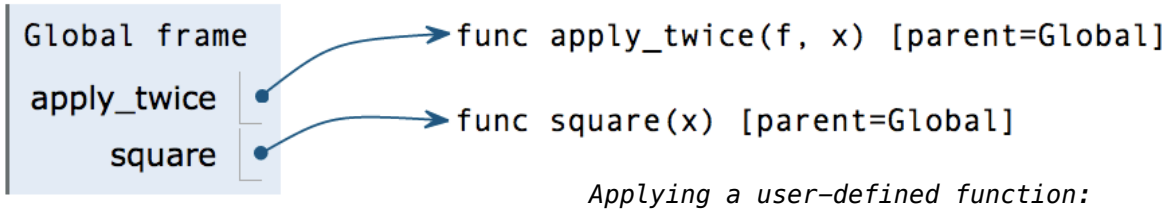

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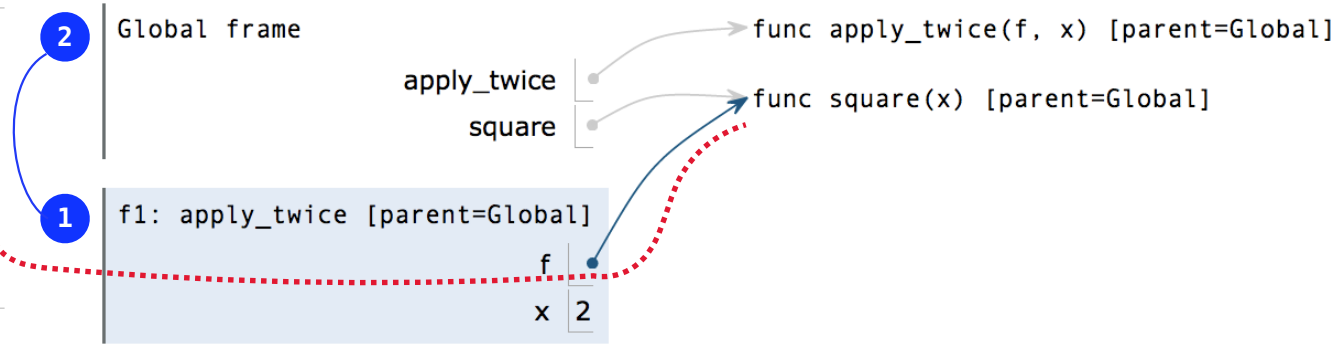

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Environments for Nested Definitions

(Demo)

Environment Diagrams for Nested Def Statements

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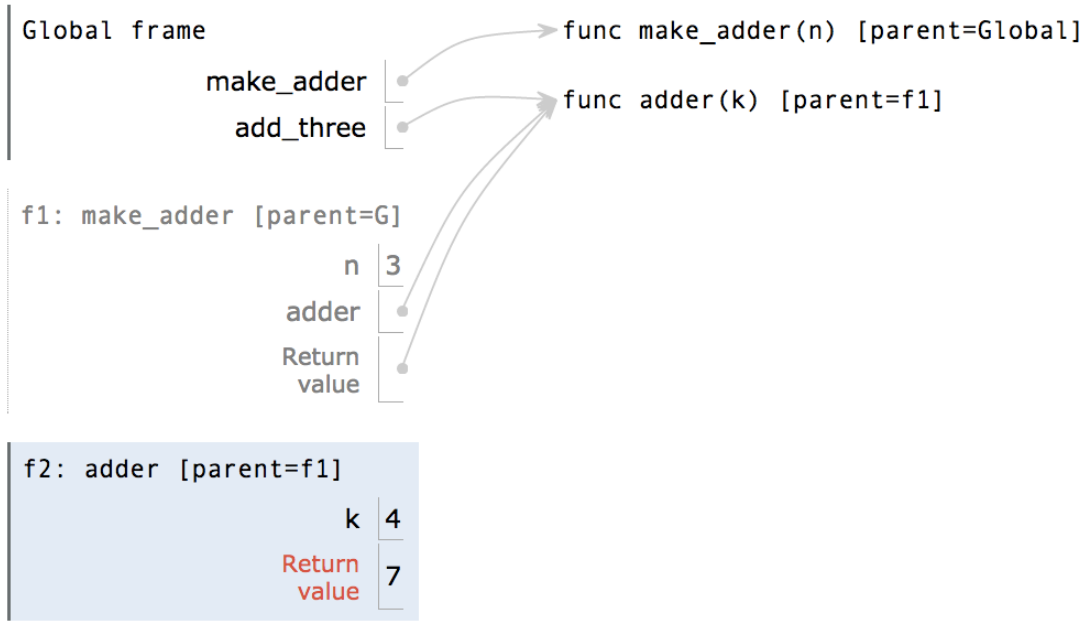

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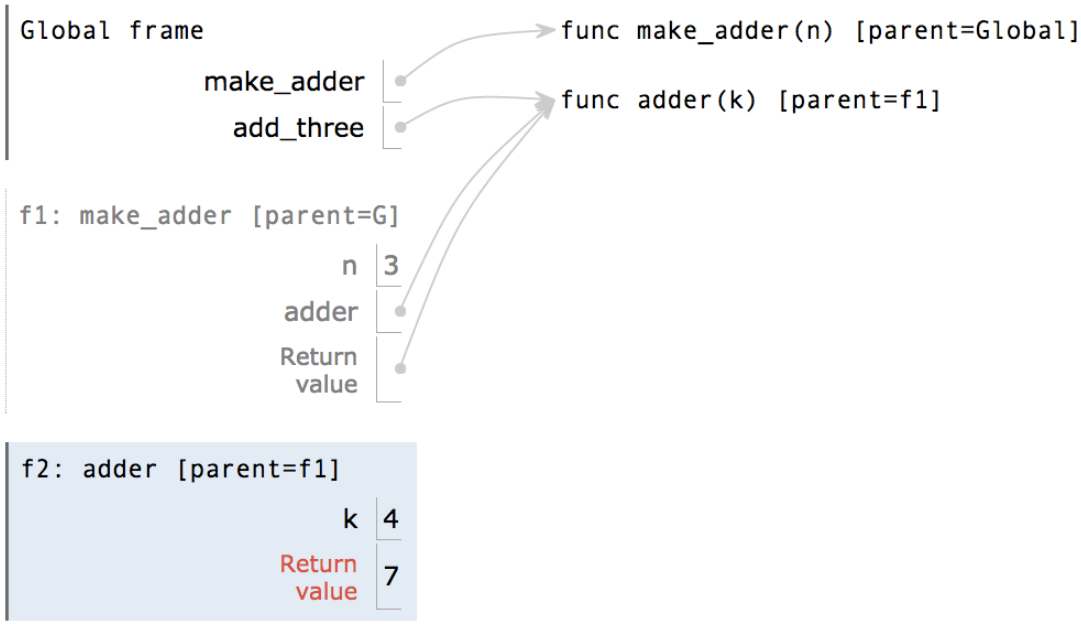

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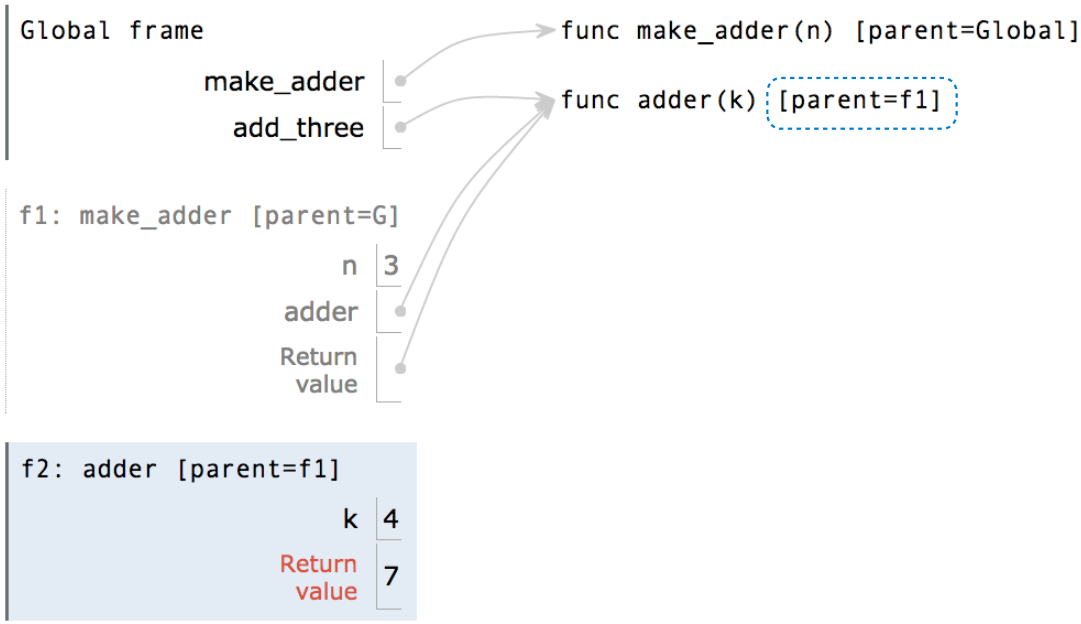

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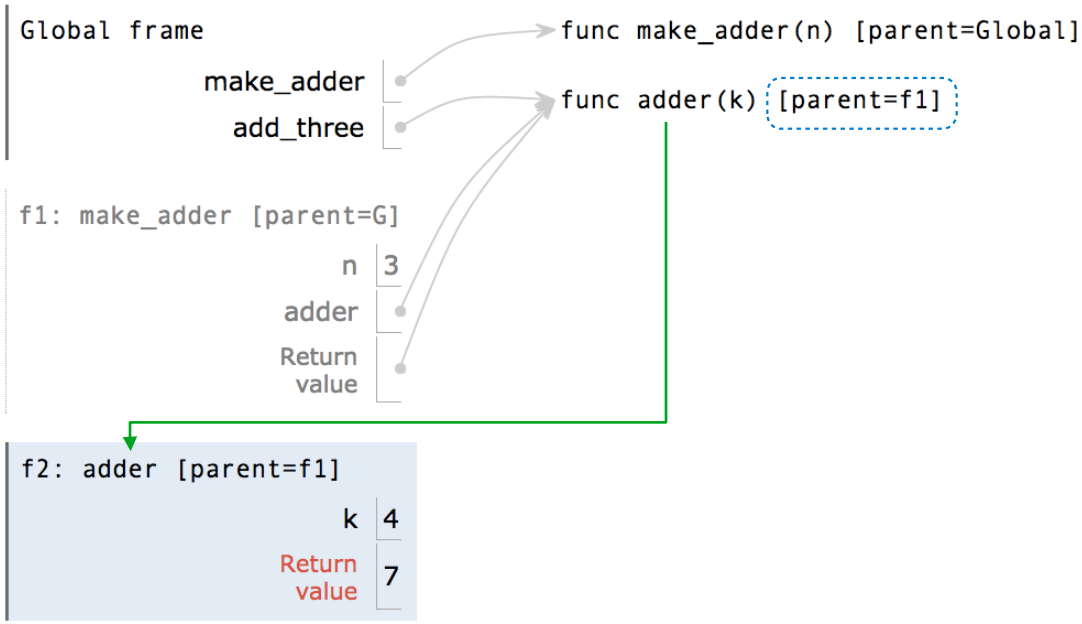

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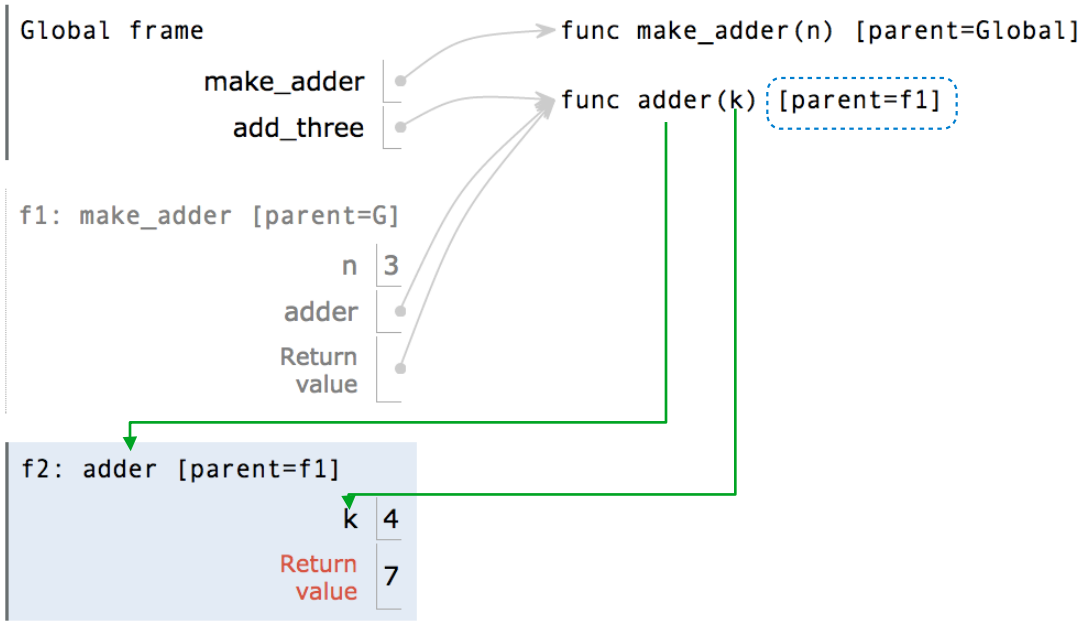

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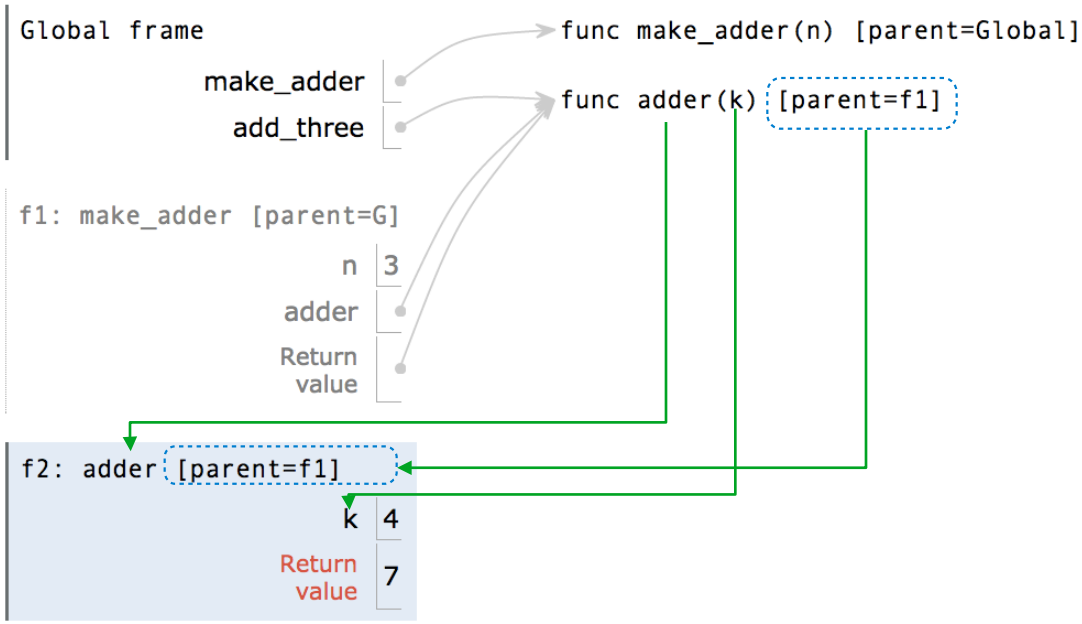

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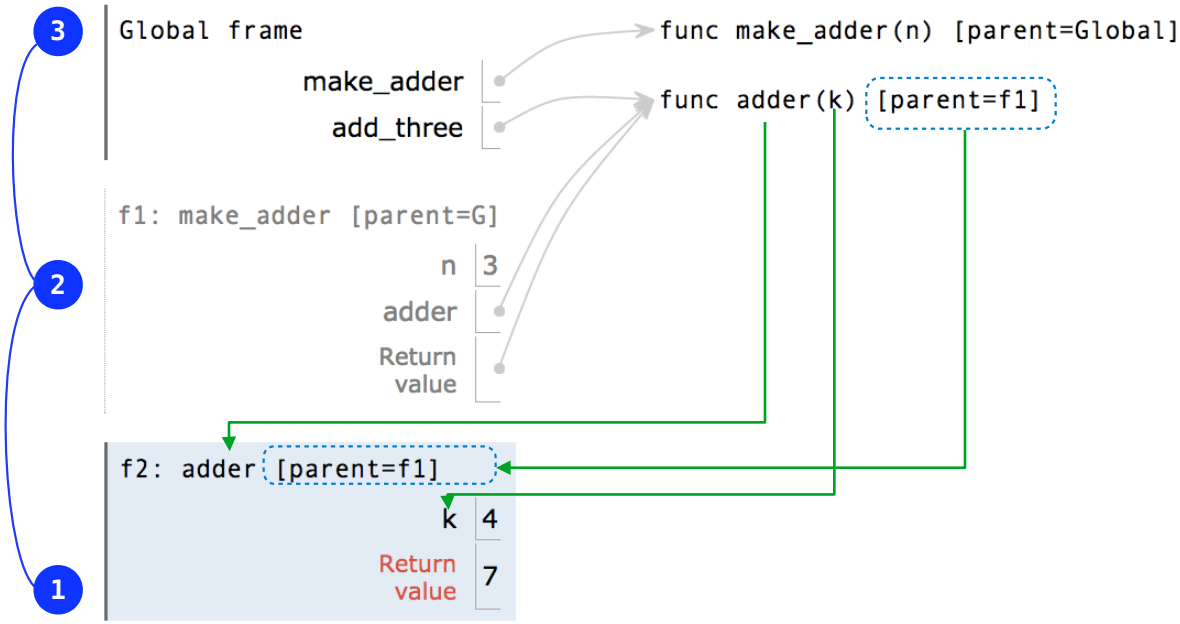

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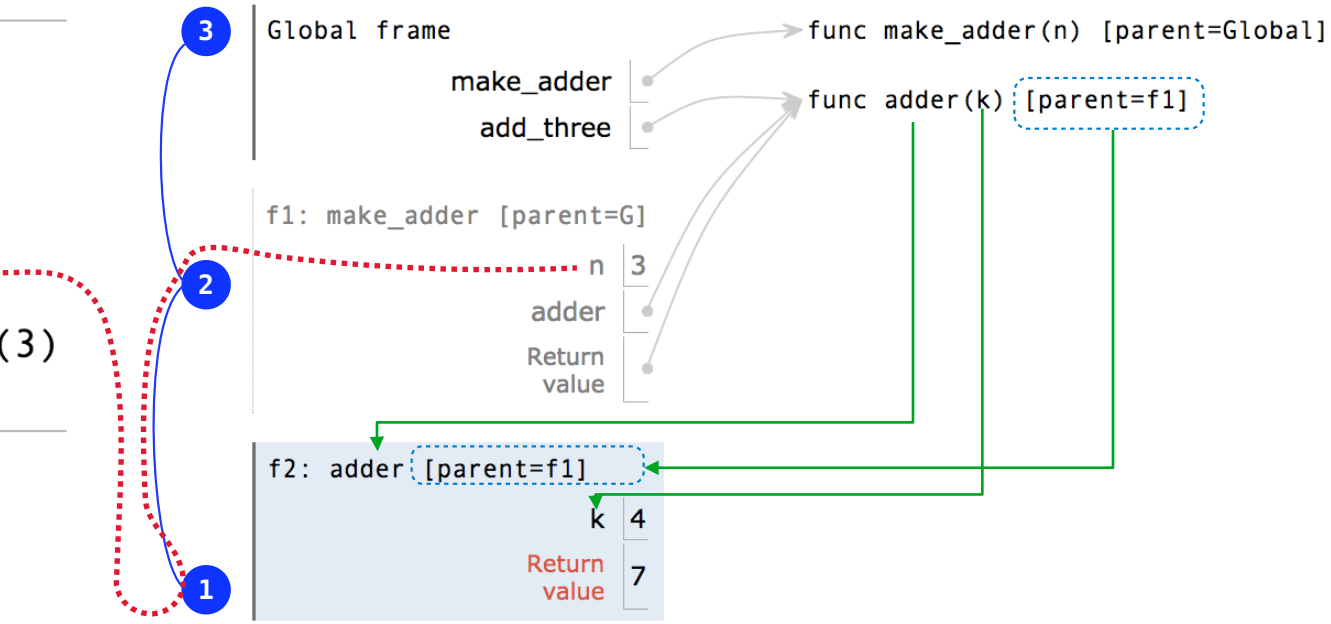

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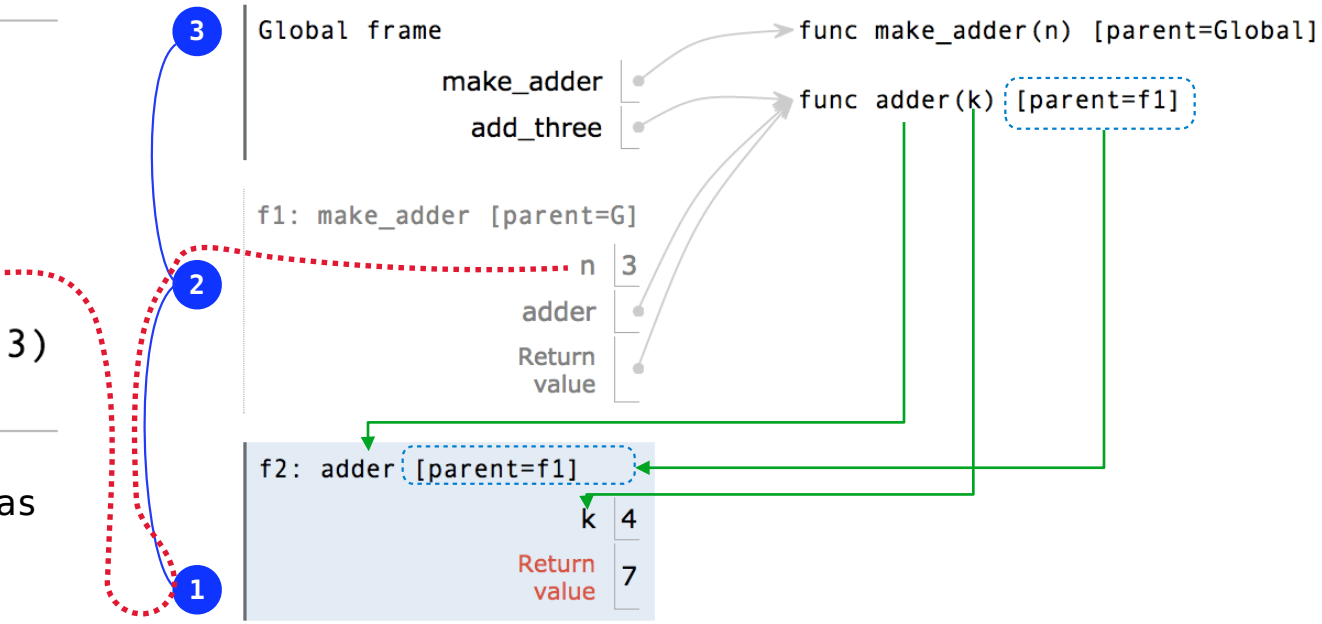

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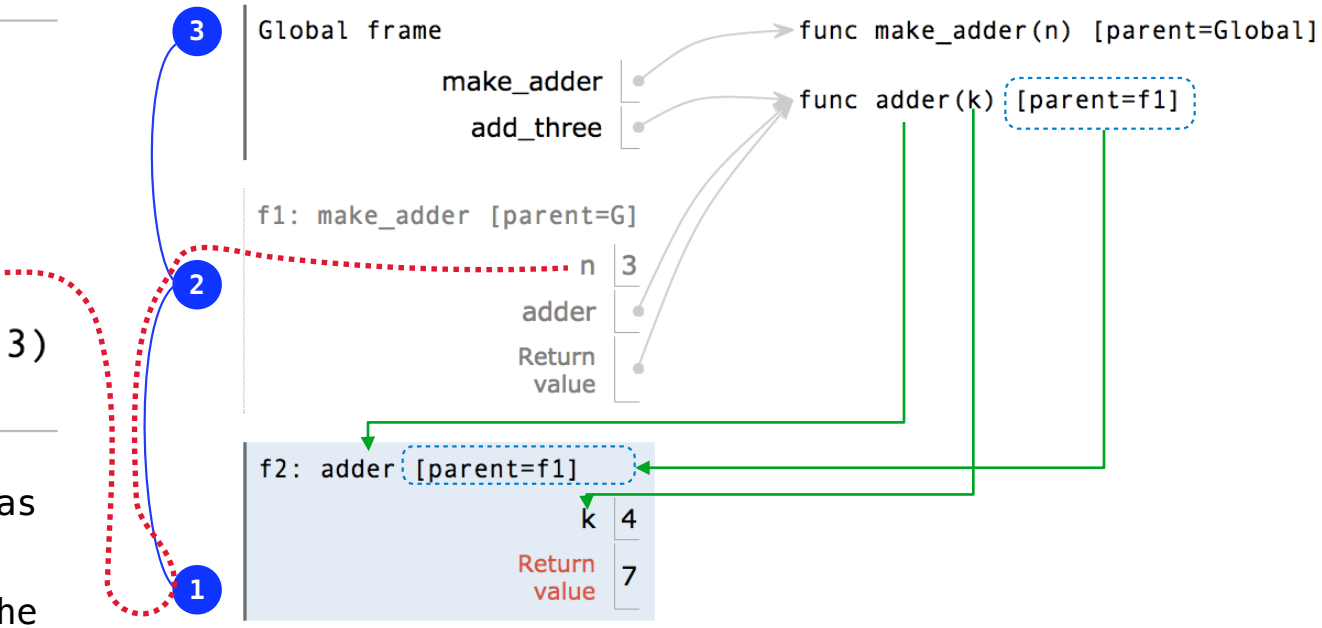

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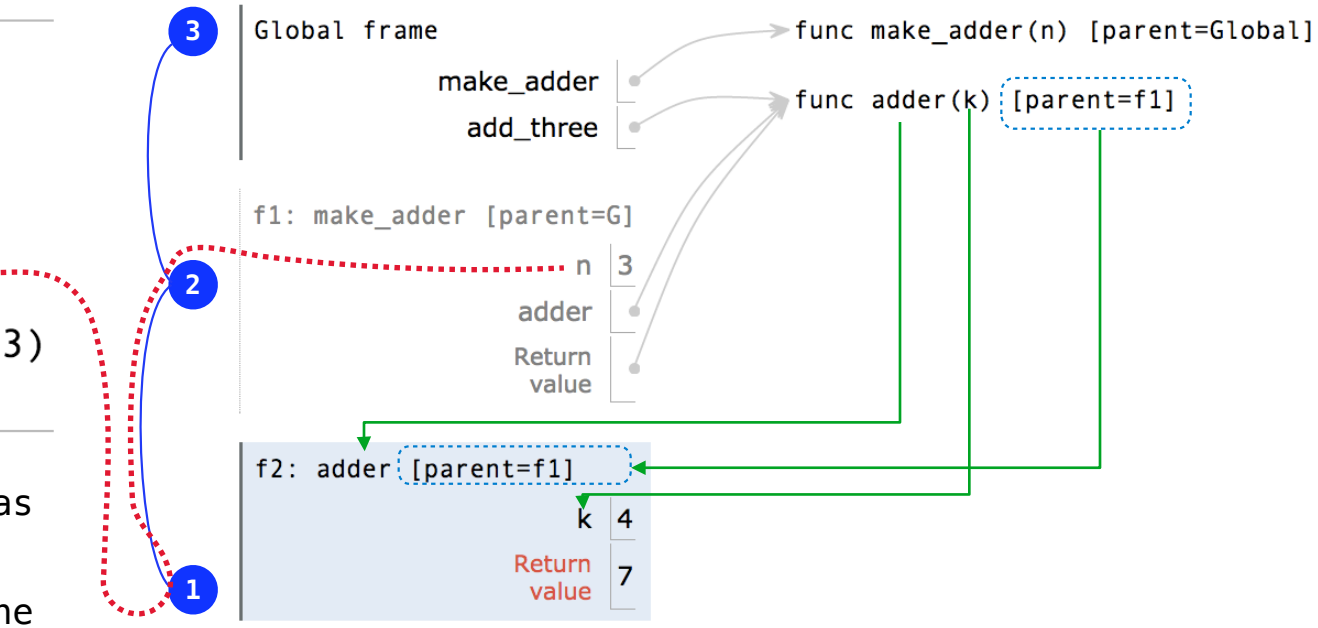

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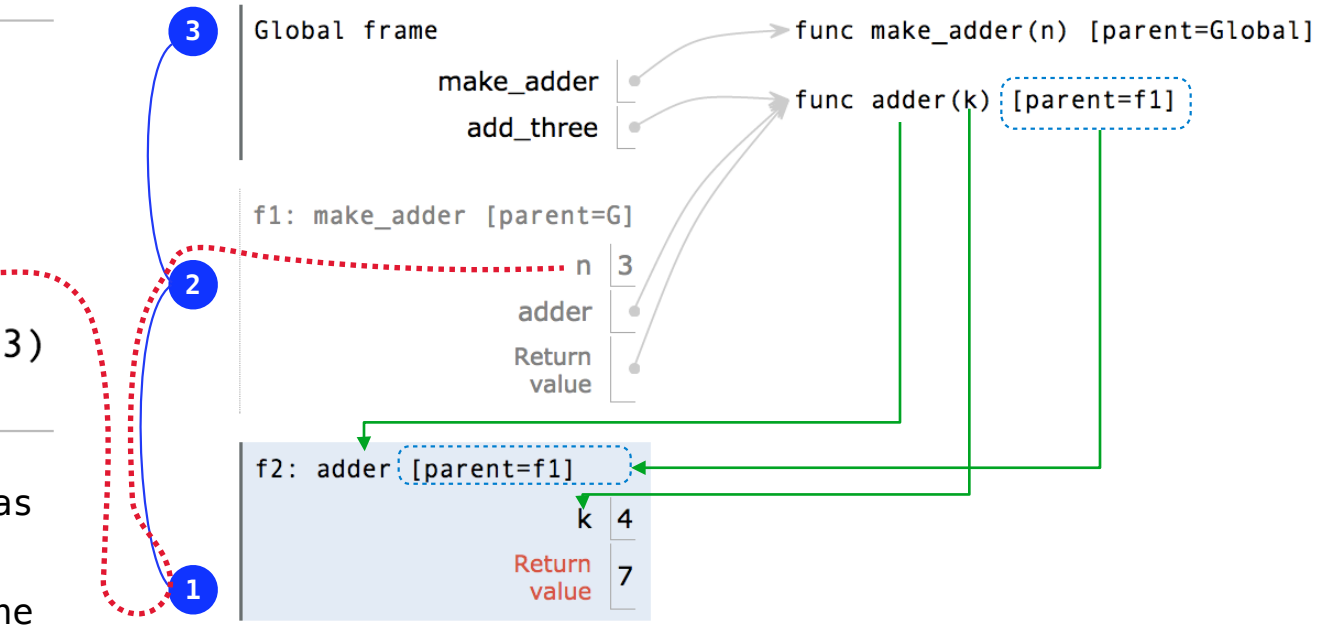

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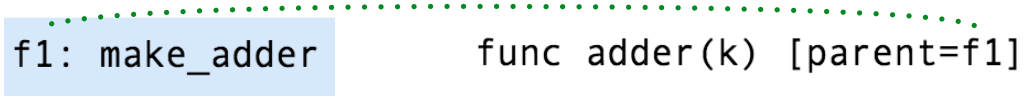

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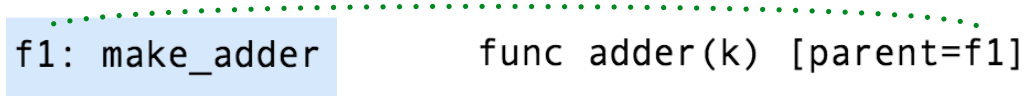

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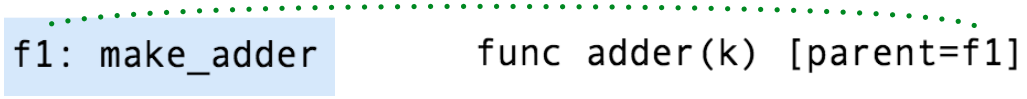

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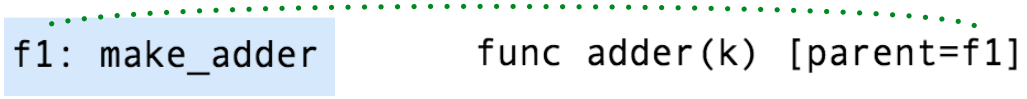

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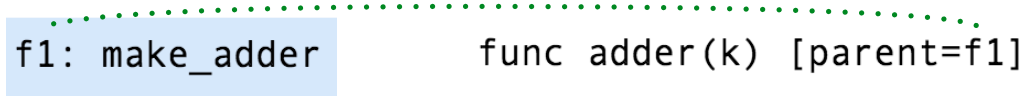

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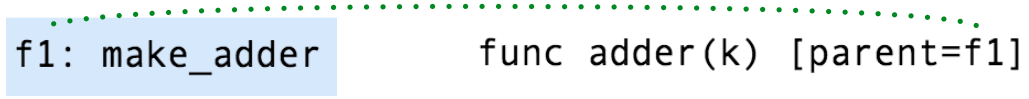

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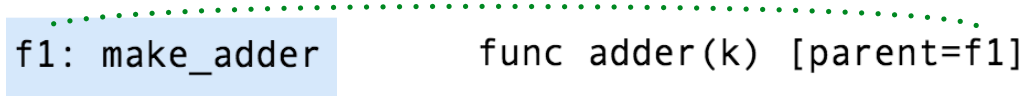

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3. Bind the `<formal parameters>` to the arguments in the local frame.
4. Execute the body of the function in the environment that starts with the local frame.

Local Names

(Demo)

Local Names are not Visible to Other (Non-Nested) Functions

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2      return g(x)
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4  def g(a):
5      return a + y
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7  result = f(1, 2)
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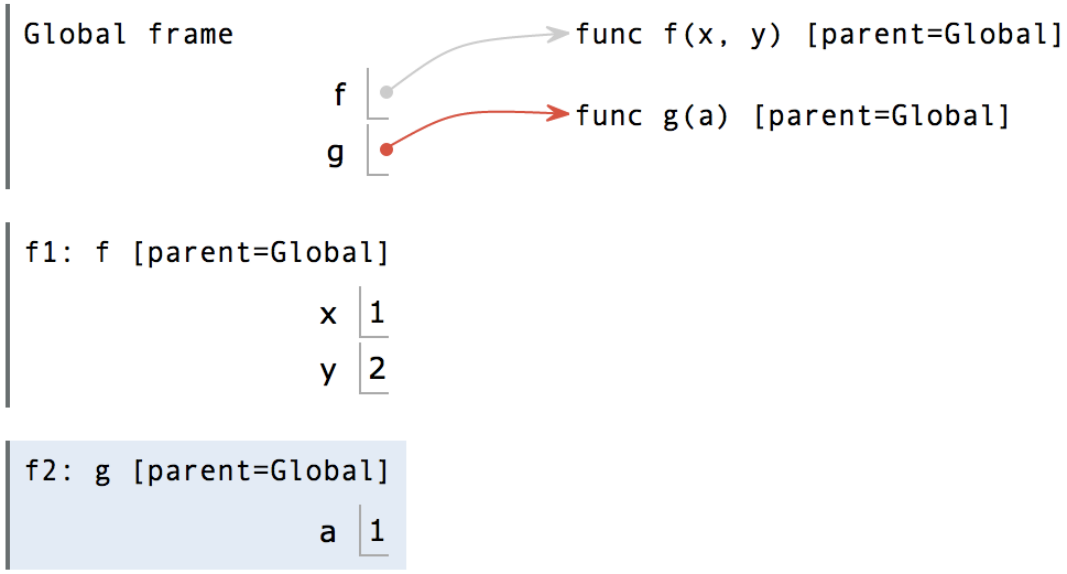

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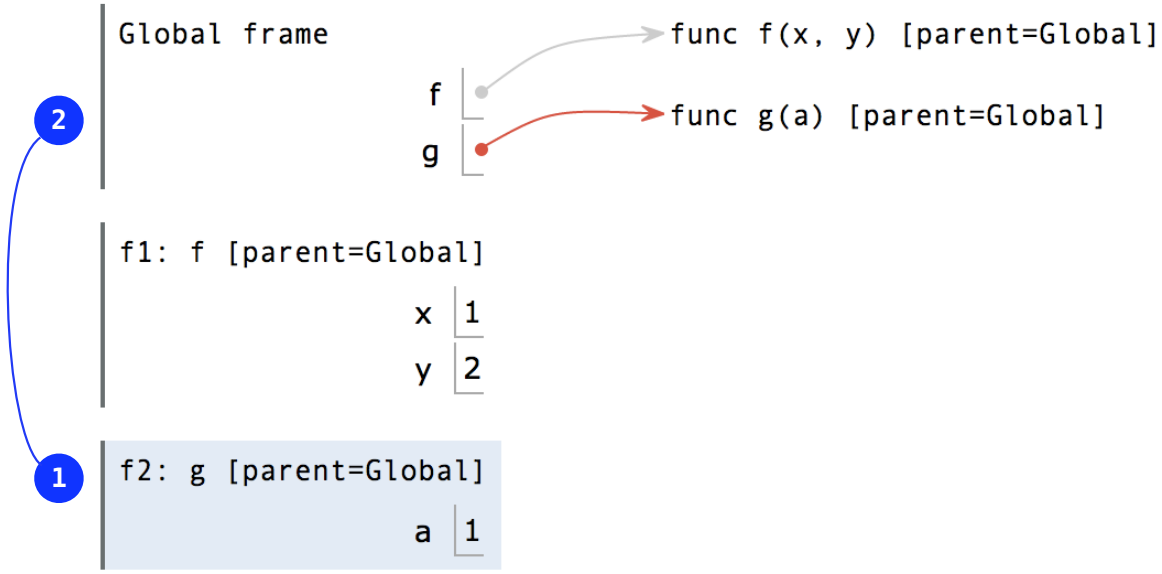

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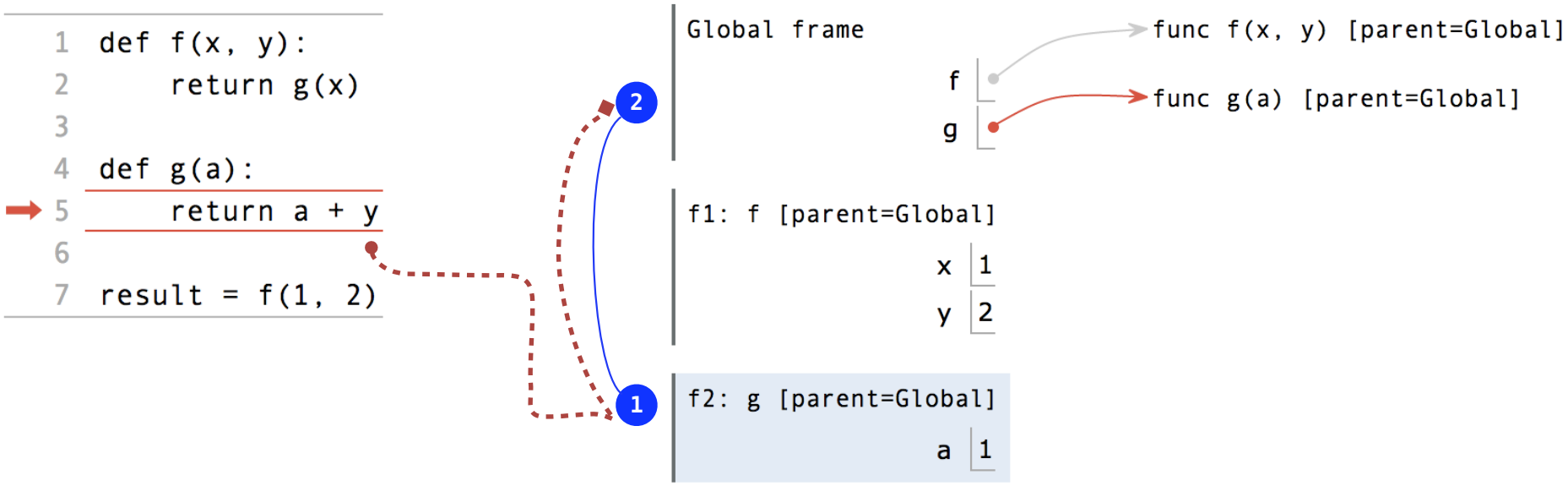

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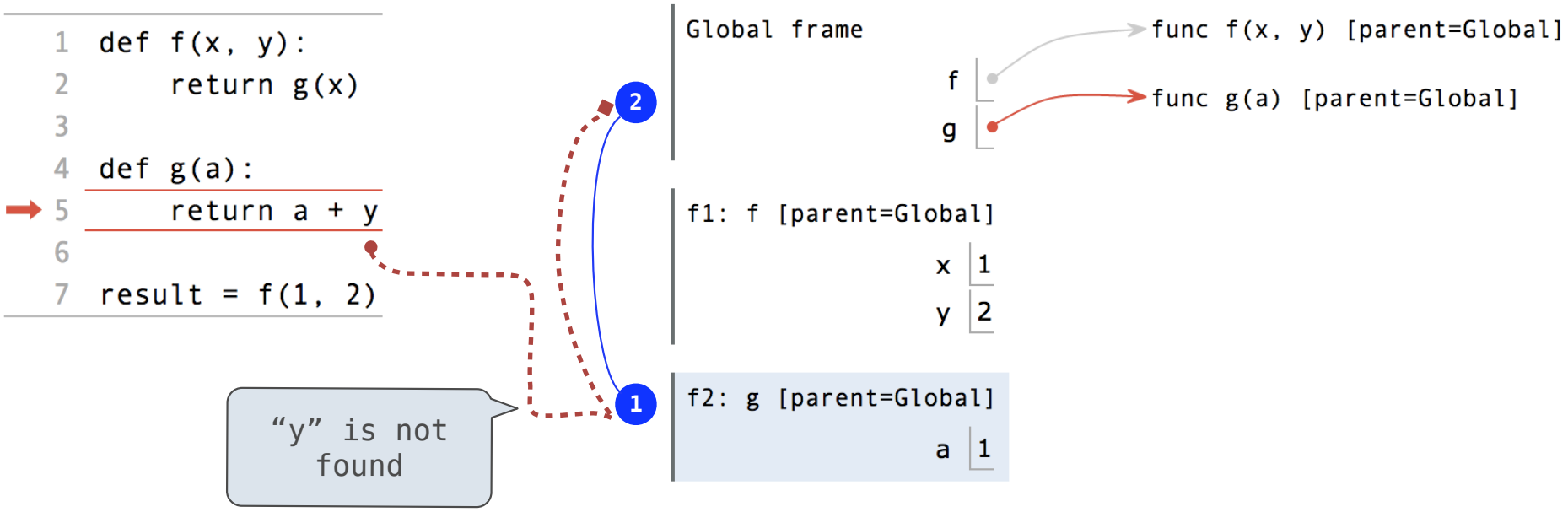

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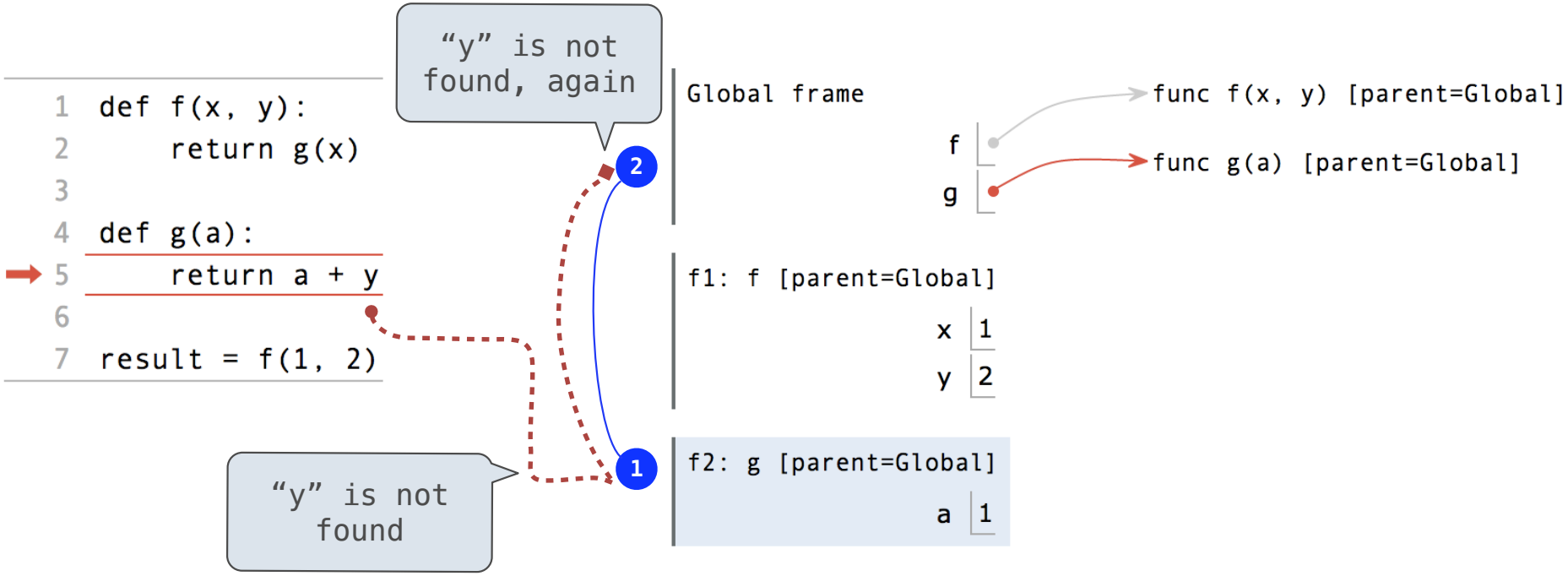

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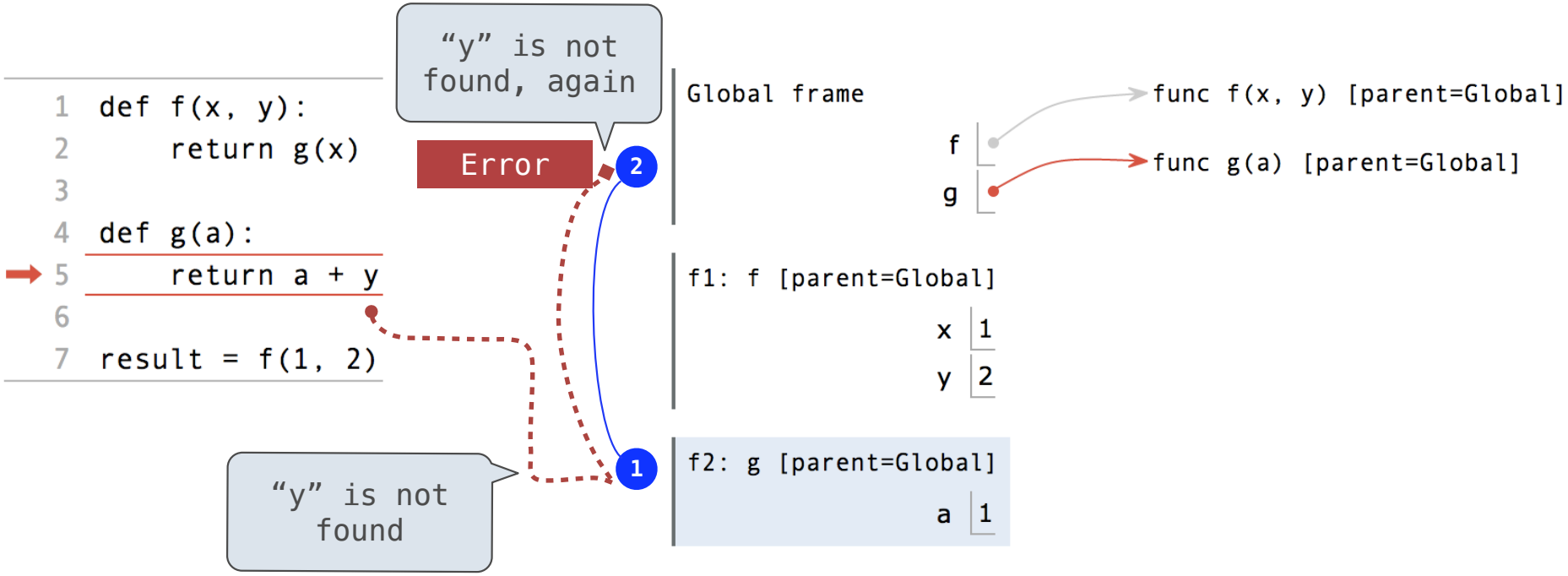

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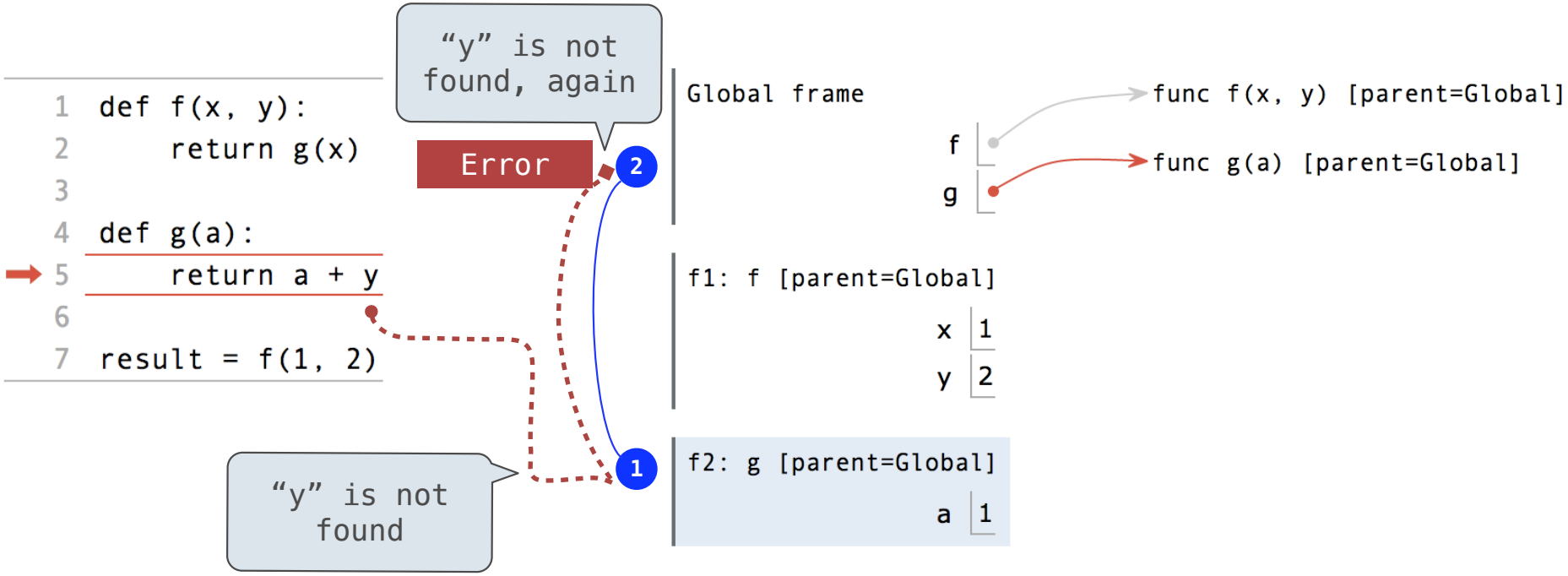

- An environment is a sequence of frames.

Interactive Diagram

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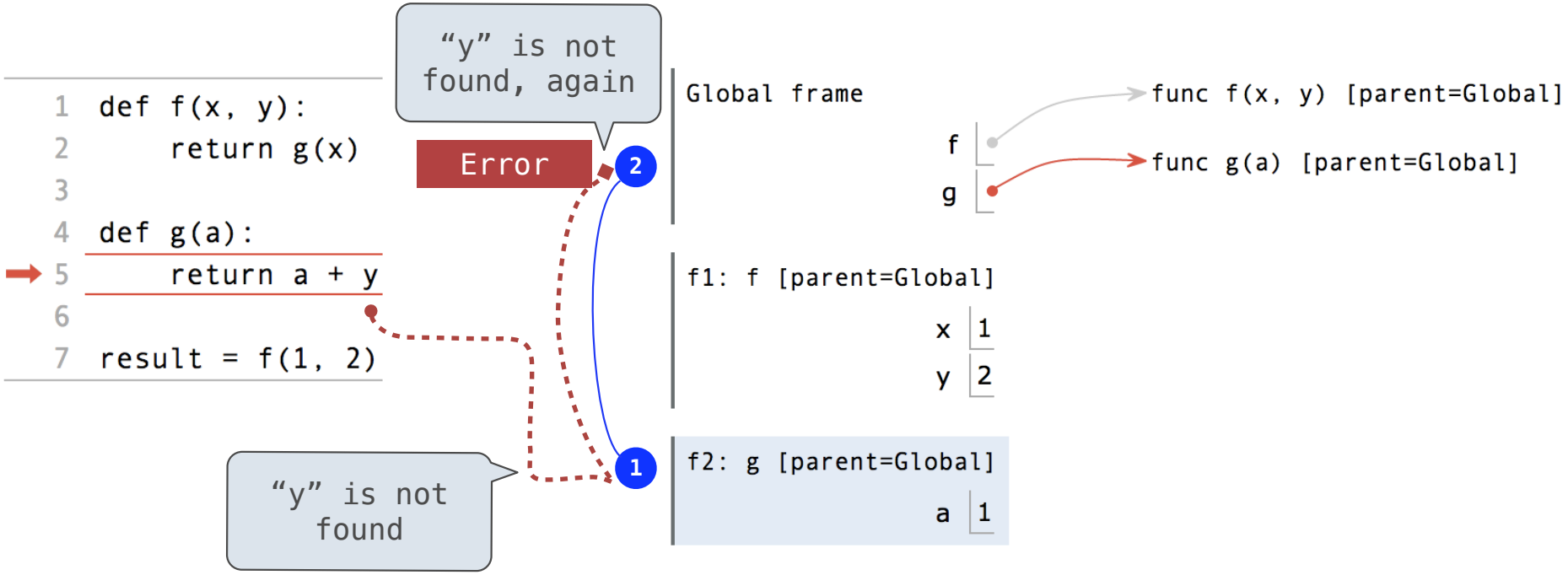

- An environment is a sequence of frames.
- The environment created by calling a top-level function (no def within def) consists of one local frame, followed by the global frame.

Interactive Diagram

Function Composition

(Demo)

The Environment Diagram for Function Composition

```
1  def square(x):
2      return x * x
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4  def make_adder(n):
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10     def h(x):
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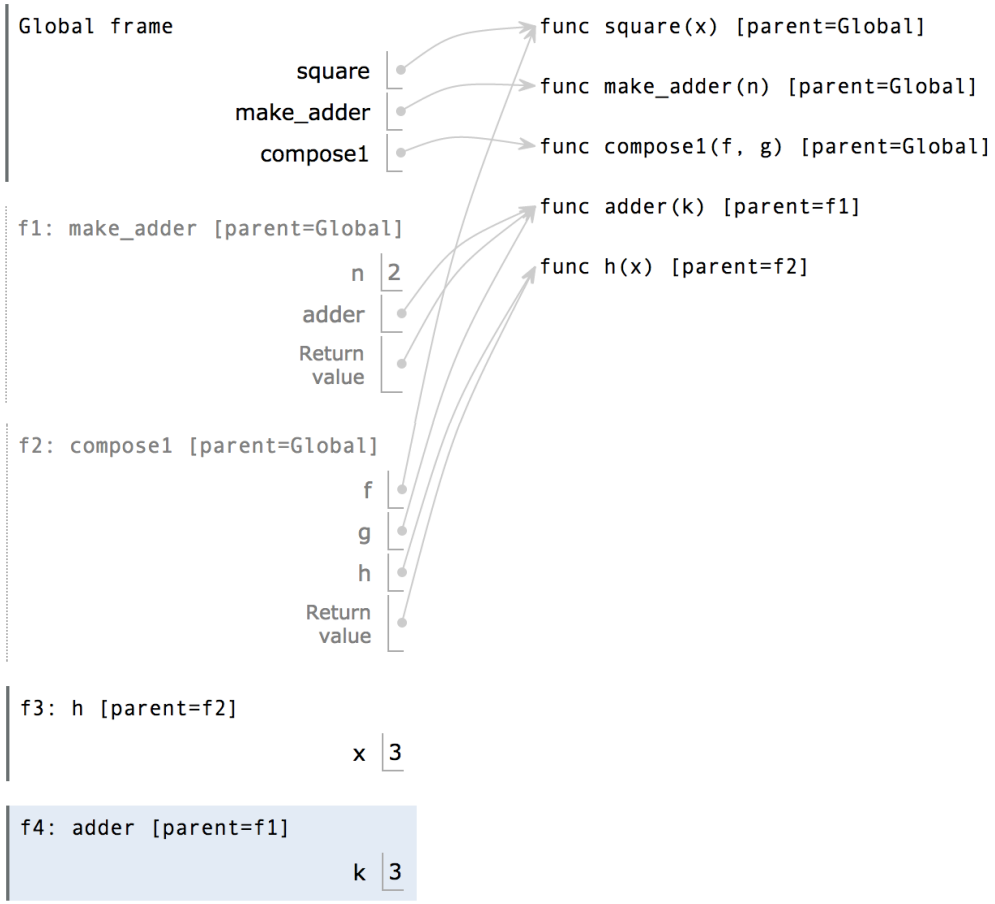

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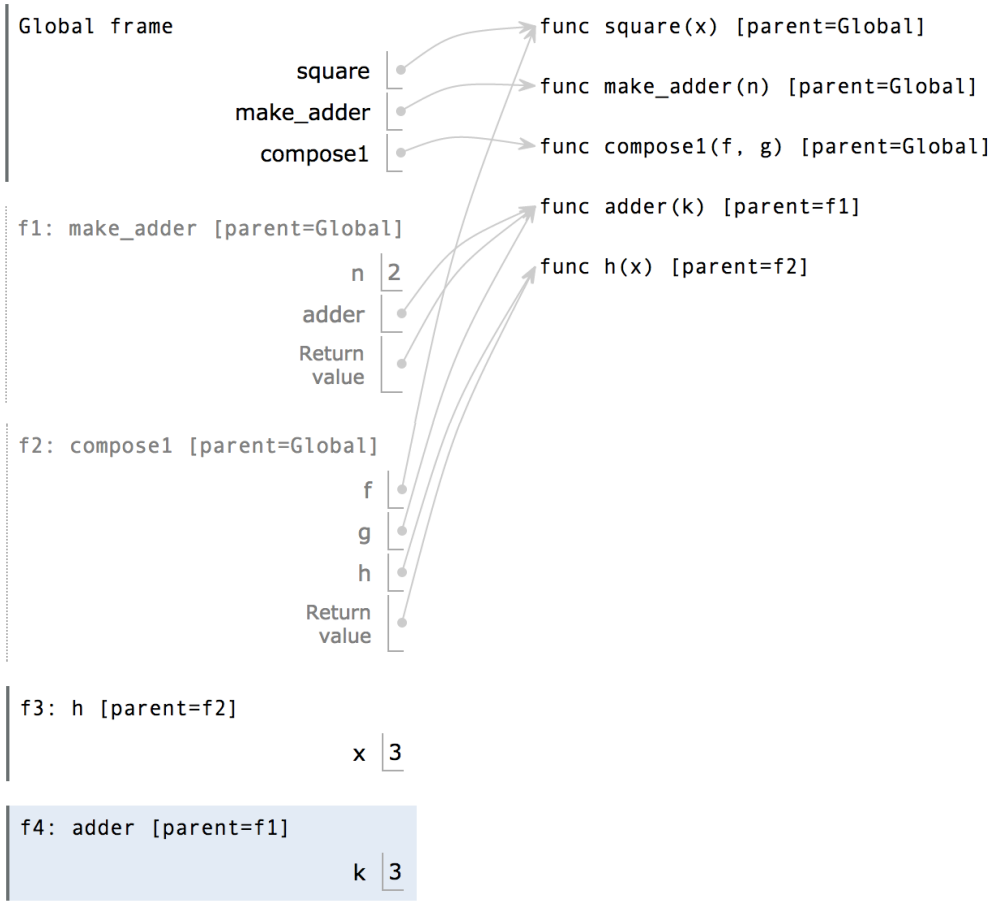

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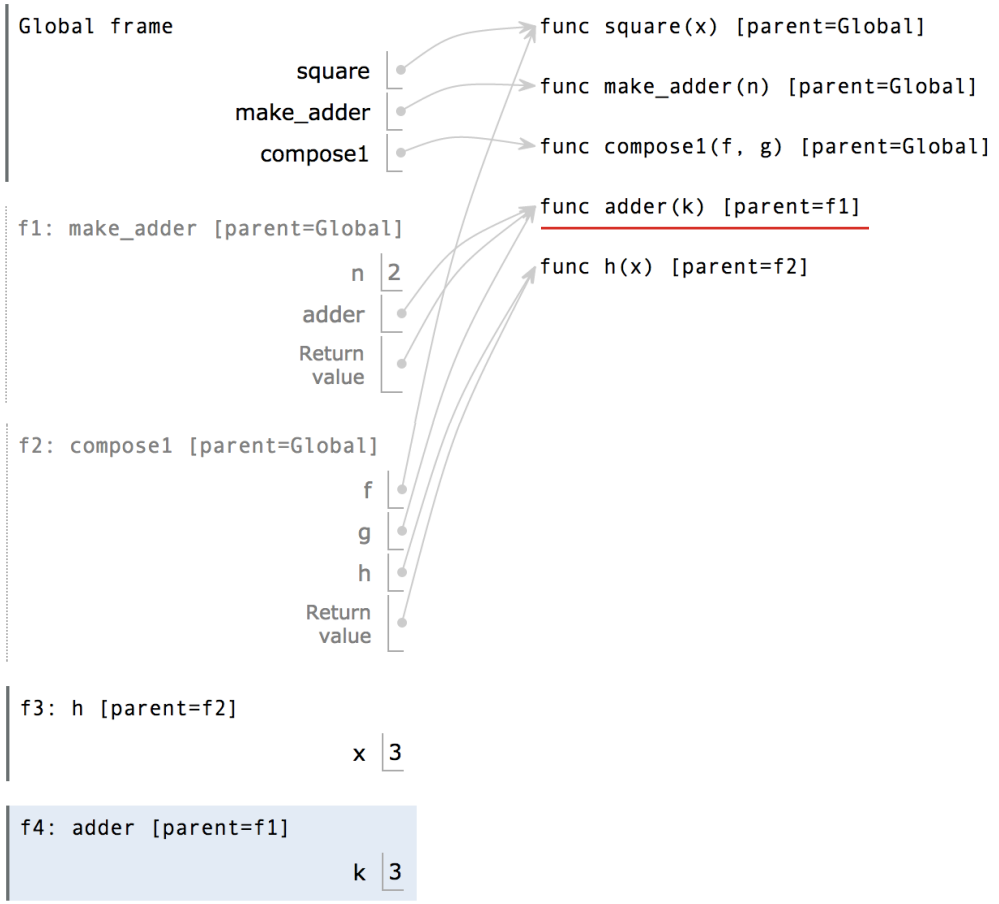

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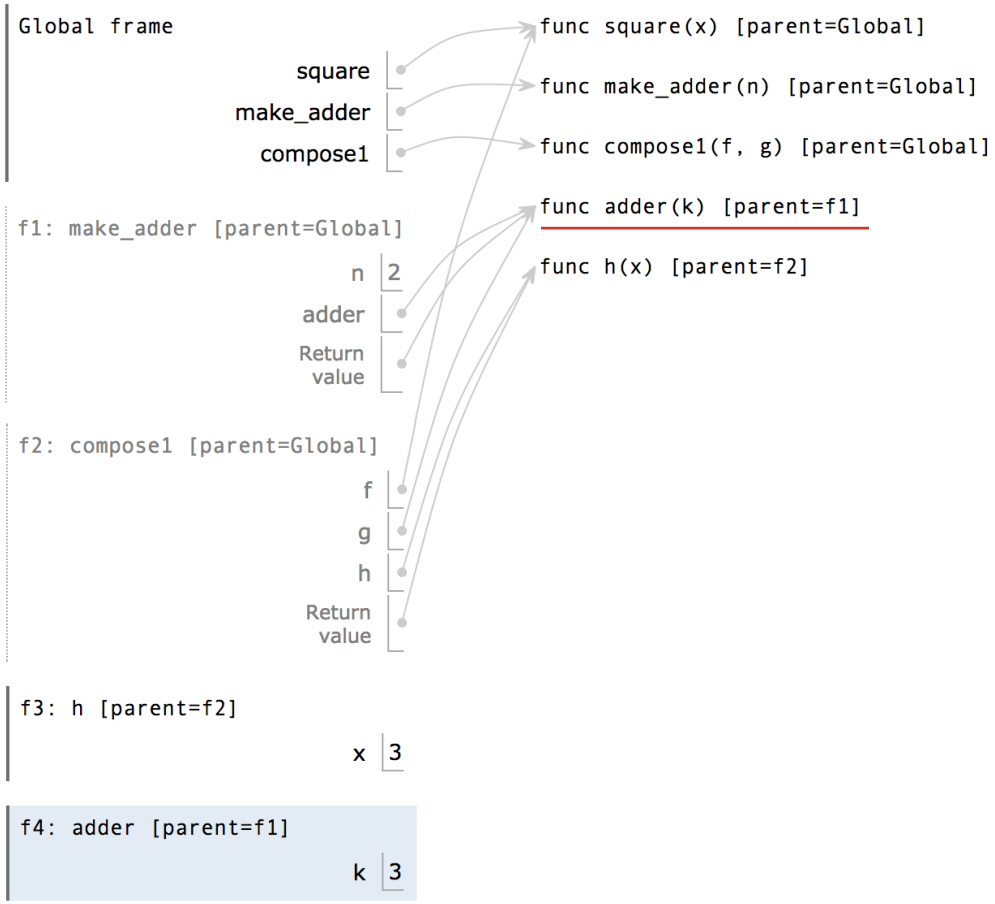

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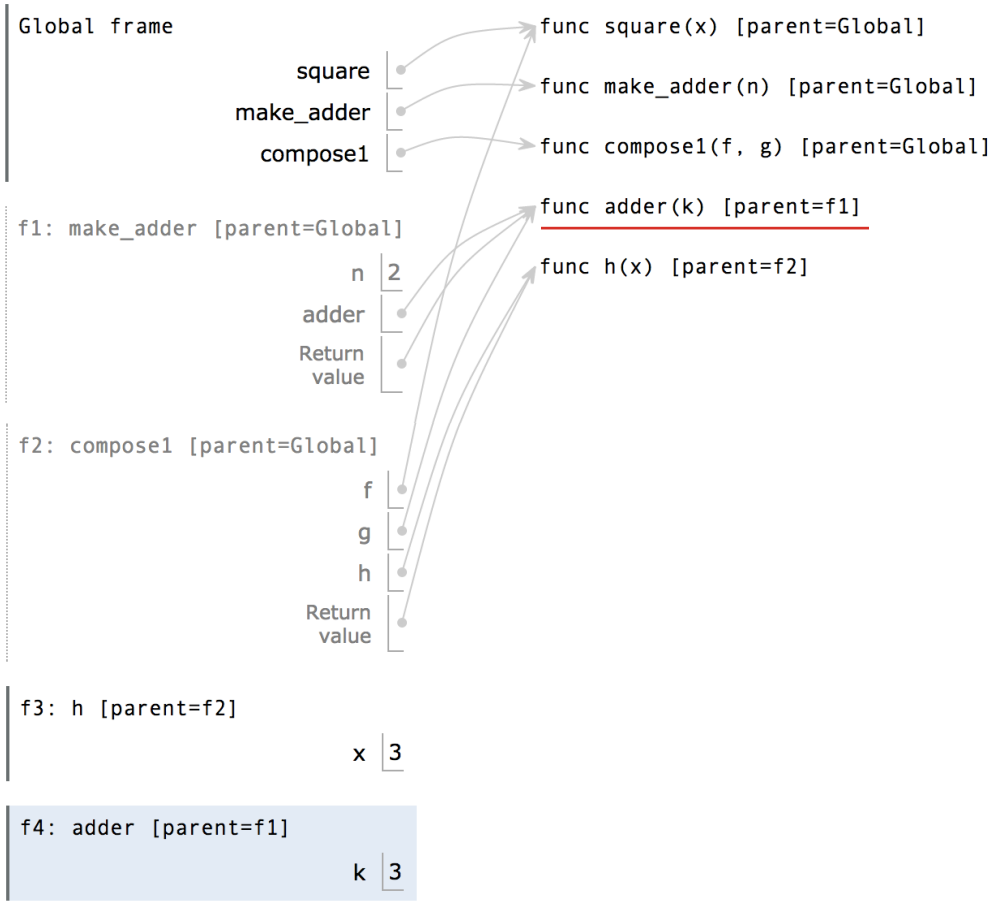

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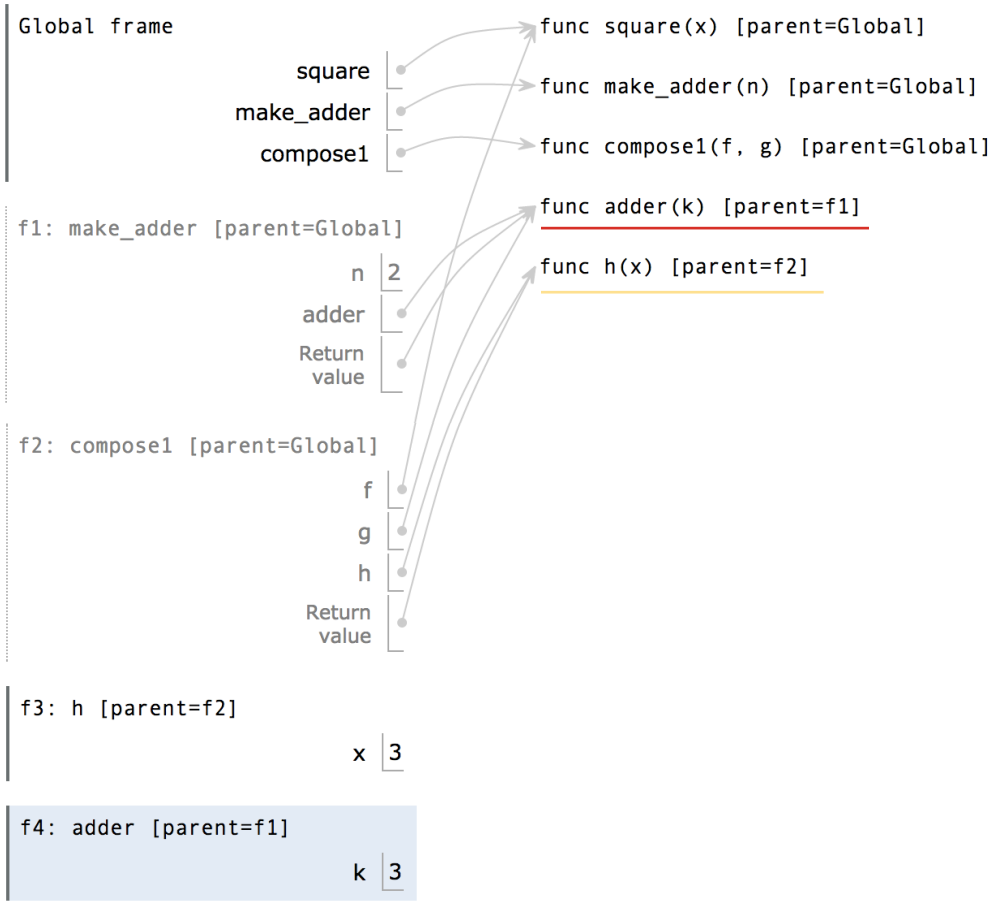

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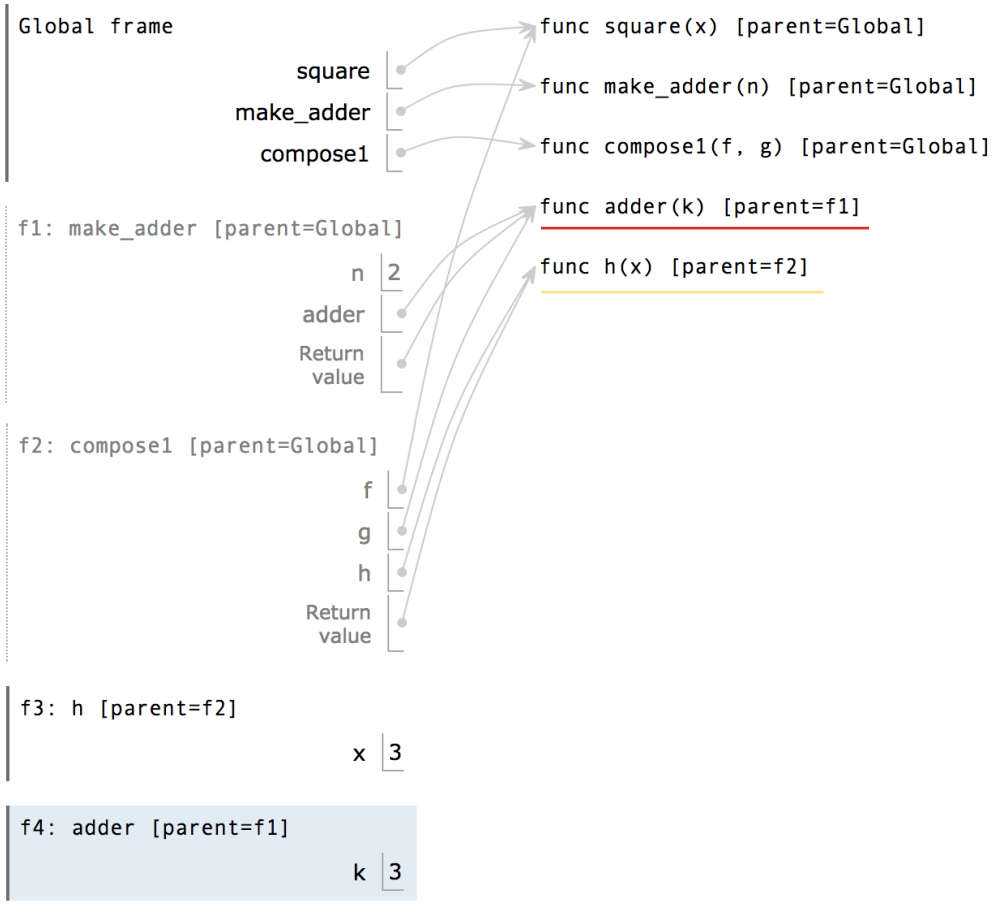

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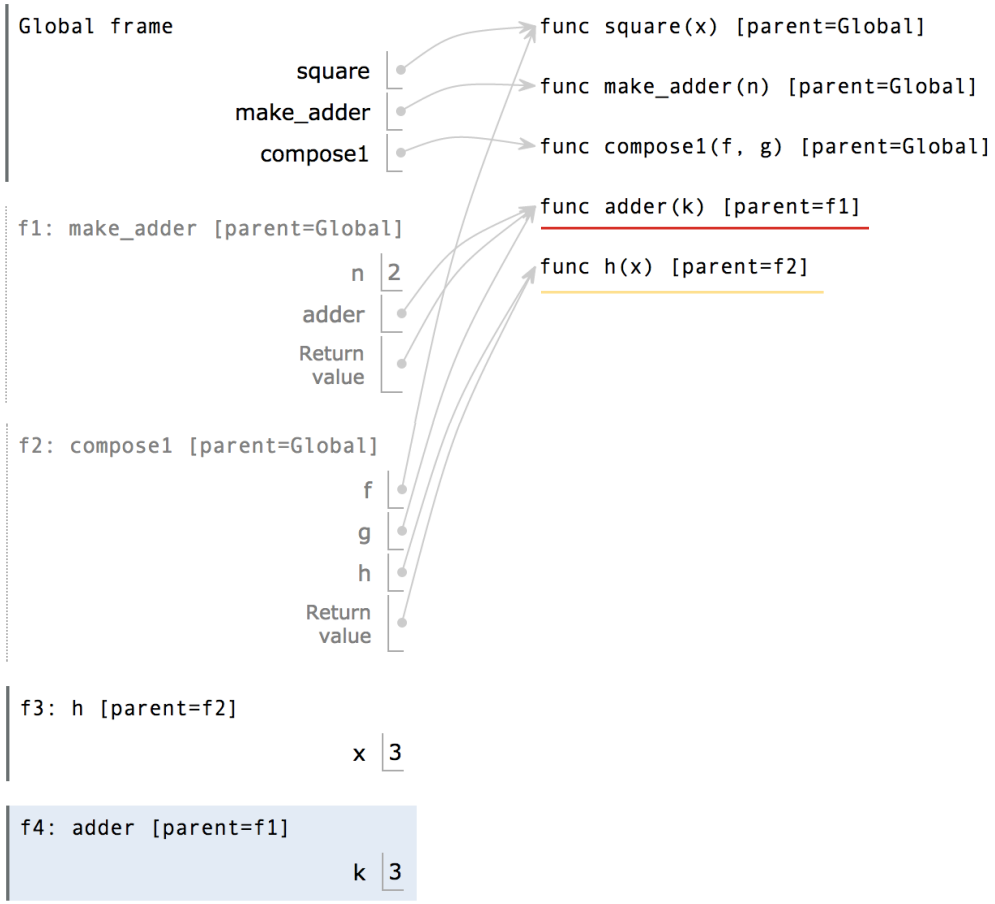

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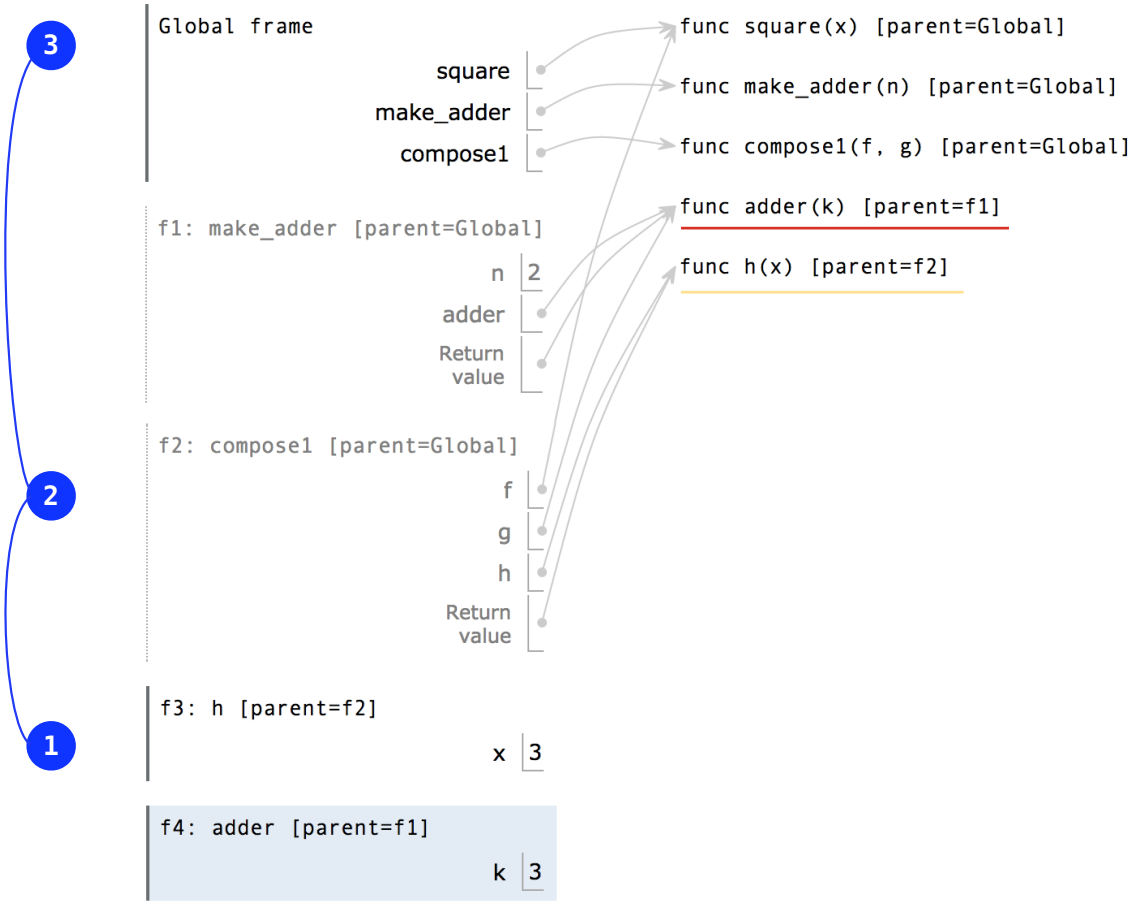

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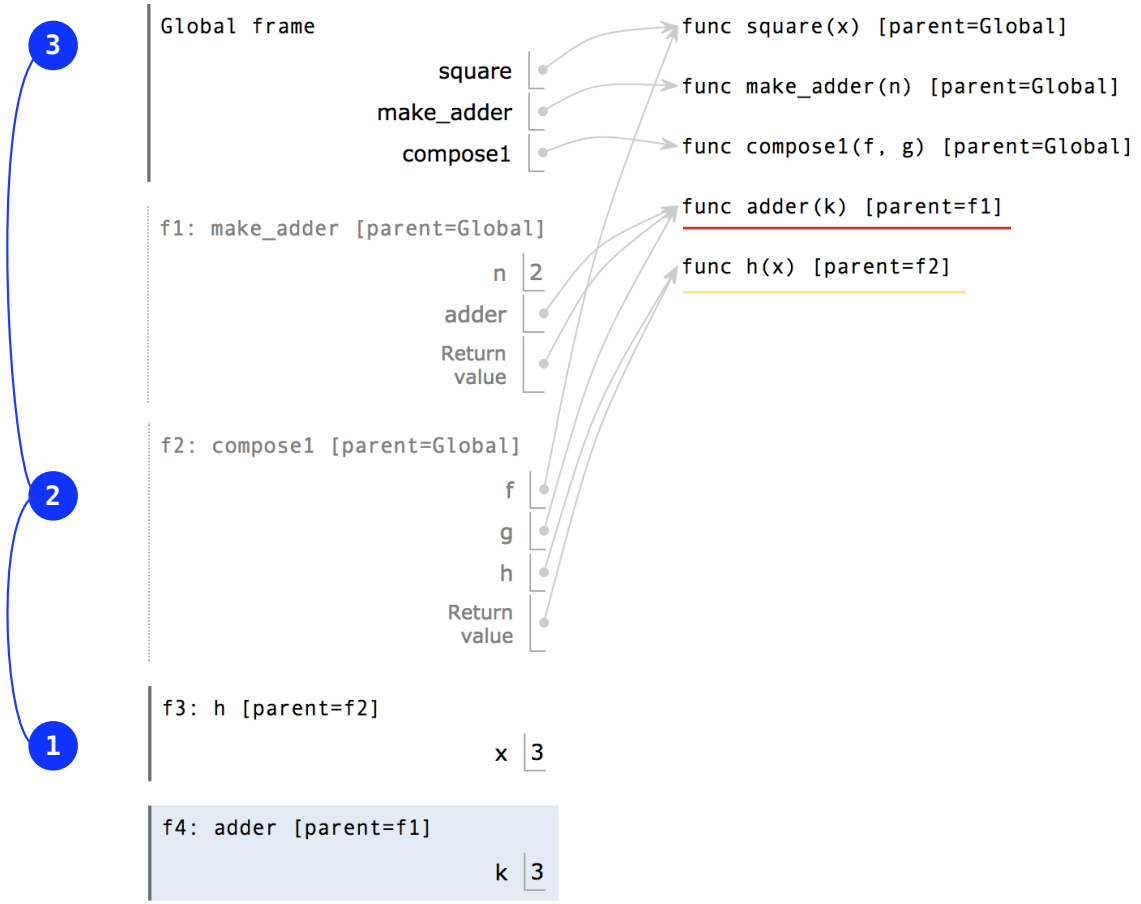

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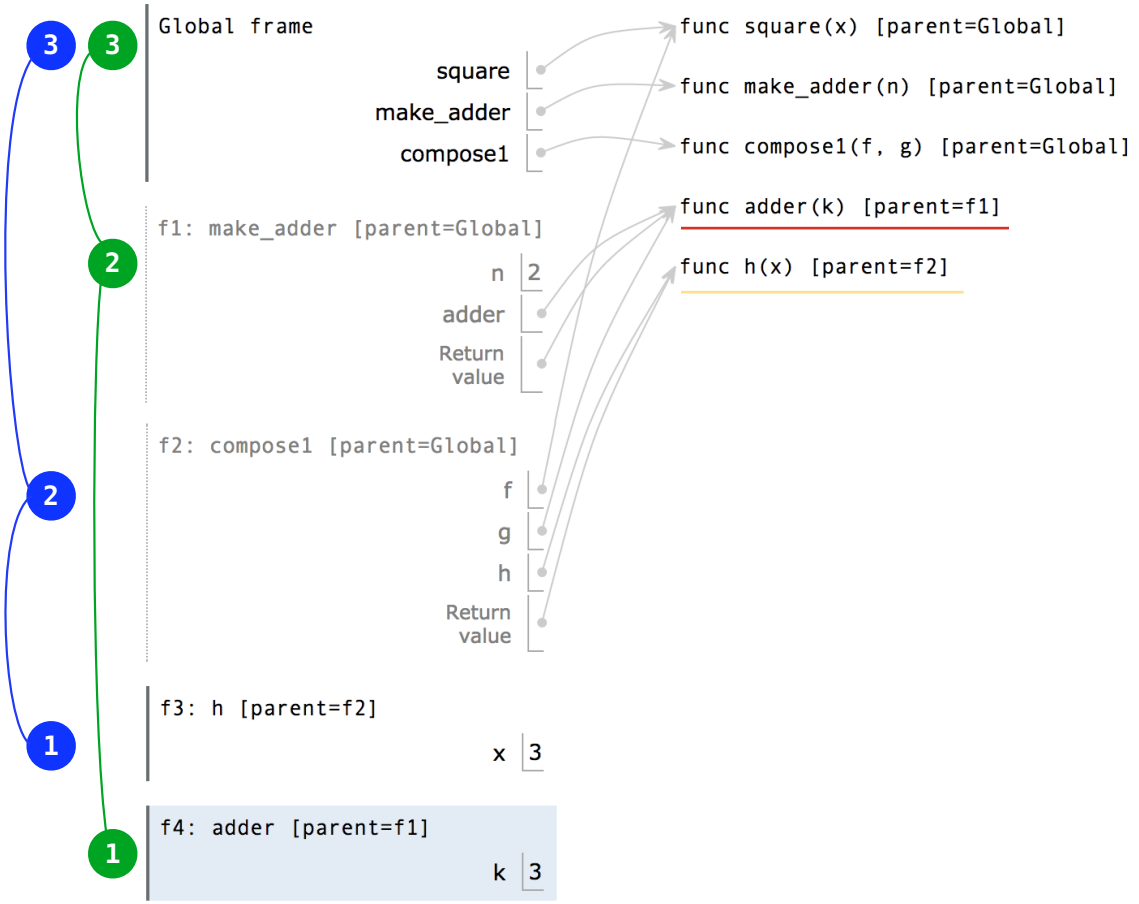

Interactive Diagram

Lambda Expressions

(Demo)

Lambda Expressions

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>>> x = 10
```

Lambda Expressions

```
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```
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An expression: this one evaluates to a number

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A function

with formal parameter x

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Must be a single expression

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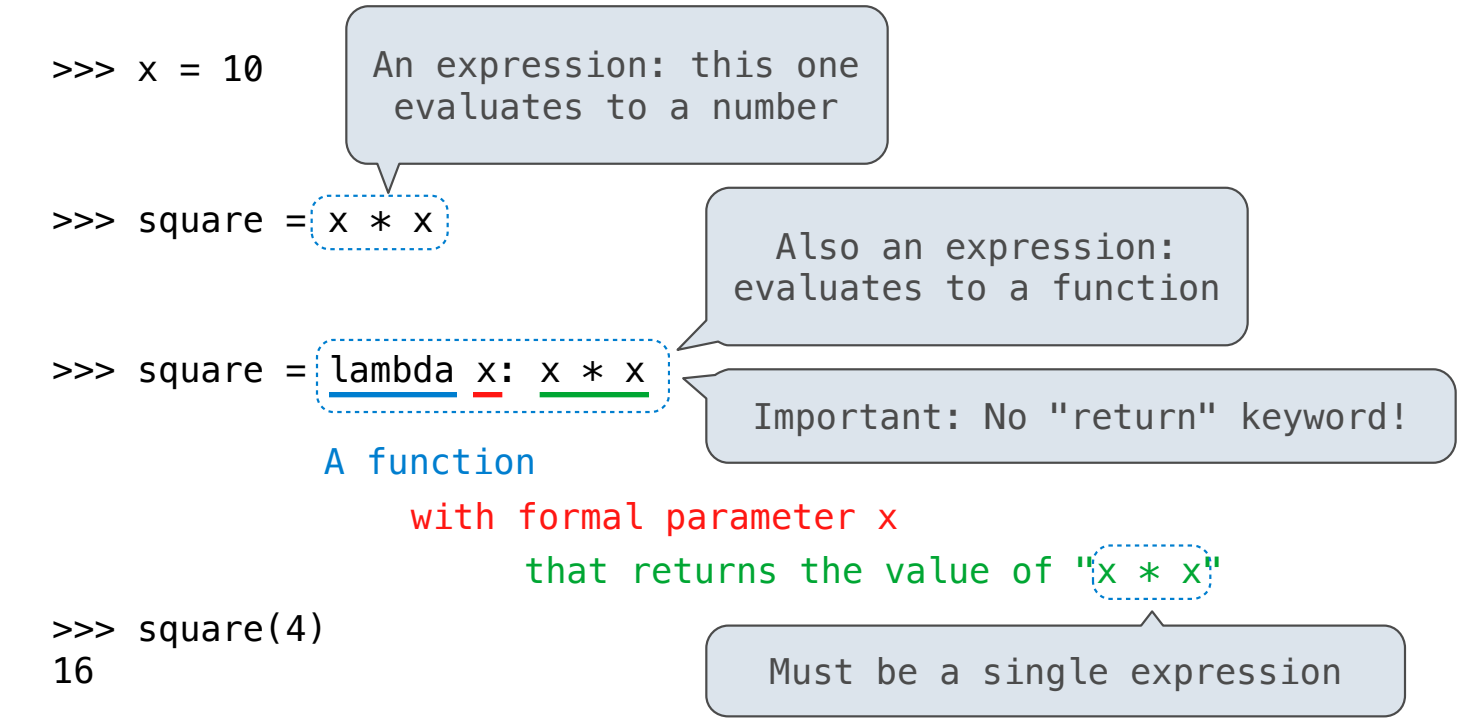
Must be a single expression

Lambda expressions are not common in Python, but important in general

Lambda Expressions



Lambda expressions are not common in Python, but important in general

Lambda expressions in Python cannot contain statements at all!

Lambda Expressions Versus Def Statements

Lambda Expressions Versus Def Statements

VS

Lambda Expressions Versus Def Statements

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square = lambda x: x * x
```

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Lambda Expressions Versus Def Statements

```
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Lambda Expressions Versus Def Statements

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- Both create a function with the same domain, range, and behavior.

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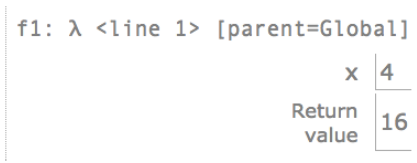

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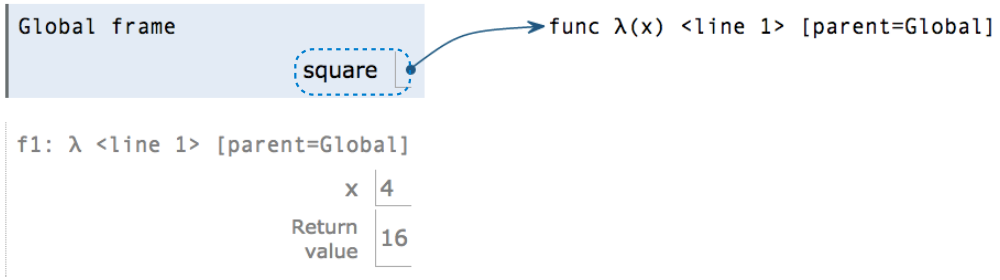

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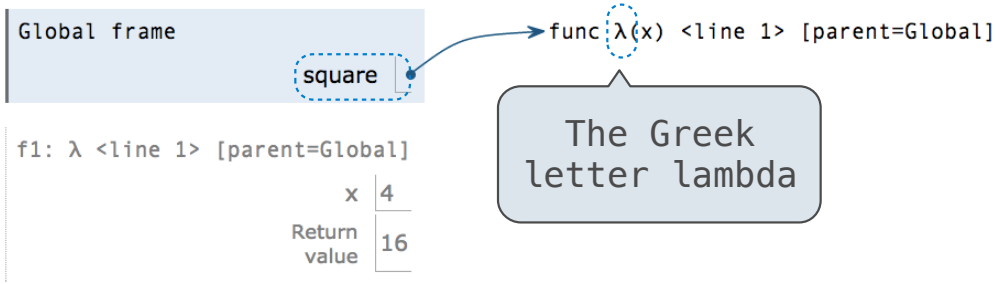

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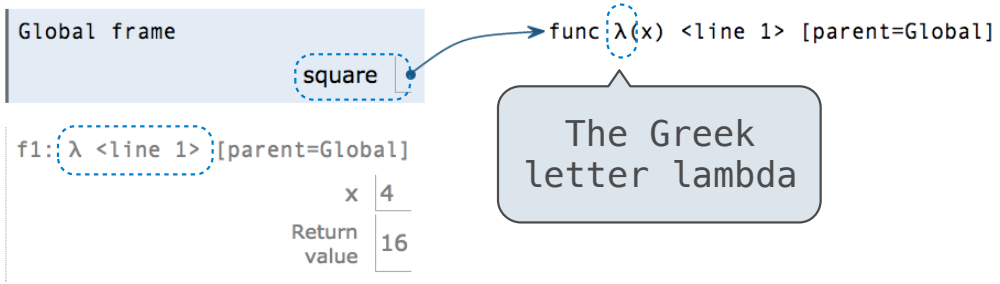

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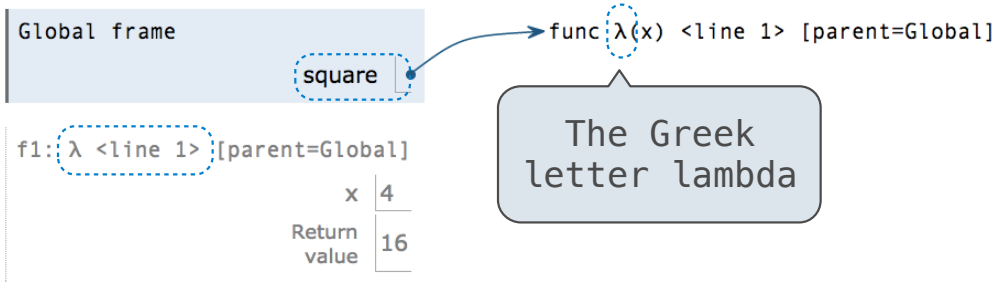

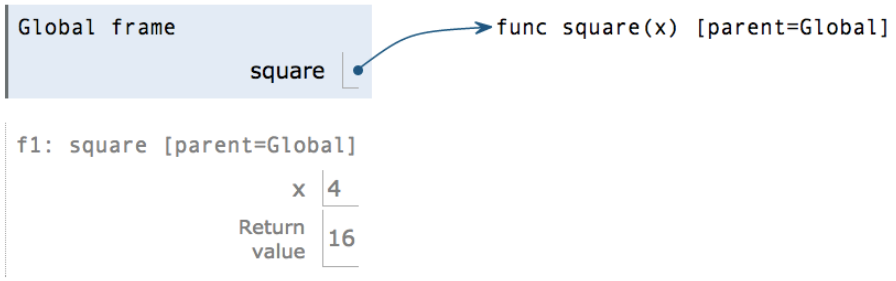

Currying

Function Currying

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def make_adder(n):
    return lambda k: n + k
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>>> make_adder(2)(3)
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>>> add(2, 3)
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There's a general relationship between these functions

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(Demo)

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There's a general relationship between these functions

(Demo)

Curry: Transform a multi-argument function into a single-argument, higher-order function