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September 11, 2017
System Analysis with Laplace Transforms

> *restart*

Helpful functions

> $with(inttrans):$

> $u := t \to \mathrm{Heaviside}(t):$

> $PAR := (Za, Zb) \to simplify\left(\frac{Za \cdot Zb}{Za + Zb}\right):$

> $SCS := X \to sort(collect(simplify(expand(numer(X))/expand(denom(X))), s), s):$

> $IL := (X, s, t) \to simplify(convert(invlaplace(convert(X, parfrac, s), s, t), expsincos)):$

> $ILTS := (X, s, t) \to simplify(convert(invlaplace(X, s, t), expsincos)):$

# Transfer Function

> $G := (M, fv, K) \to \frac{fv \cdot s + K}{M \cdot s^2 + fv \cdot s + K}:$

*Case one*

> $M := 1$

$$M := 1 \quad \textbf{(1)}$$

> $fv := 2$

$$fv := 2 \quad \textbf{(2)}$$

> $K := 5$

$$K := 5 \quad \textbf{(3)}$$

> $x1a := \sin(2 \cdot t)$

$$x1a := \sin(2\,t) \quad \textbf{(4)}$$

> $X1a := SCS(laplace(x1a, t, s))$

$$X1a := \frac{2}{s^2 + 4} \quad \textbf{(5)}$$

> $X2a := G(M, fv, K) \cdot X1a$

$$X2a := \frac{2\,(2\,s + 5)}{(s^2 + 2\,s + 5)\,(s^2 + 4)} \quad \textbf{(6)}$$

> $x2a := IL(X2a, s, t)$

$$x2a := \frac{(16\,e^{-t} - 16)\cos(2\,t)}{17} - \frac{13\sin(2\,t)\,e^{-t}}{17} + \frac{21\sin(2\,t)}{17} \quad \textbf{(7)}$$

*Case two*

> $M := 1$

$$M := 1 \quad \textbf{(8)}$$

> $fv := 6$

$$fv := 6 \quad \textbf{(9)}$$

> $K := 5$

$$K := 5 \tag{10}$$

> $x1b := \sin(2 \cdot t)$

$$x1b := \sin(2\,t) \tag{11}$$

> $X1b := SCS(laplace(x1b, t, s))$

$$X1b := \frac{2}{s^2 + 4} \tag{12}$$

> $X2b := G(M, fv, K) \cdot X1b$

$$X2b := \frac{2\,(6\,s + 5)}{\left(s^2 + 6\,s + 5\right)\,\left(s^2 + 4\right)} \tag{13}$$

> $x2b := IL(X2b, s, t)$

$$x2b := -\frac{e^{-t}}{10} - \frac{48\cos(2\,t)}{145} + \frac{149\sin(2\,t)}{145} + \frac{25\,e^{-5\,t}}{58} \tag{14}$$

## Case three

> $M := 1$

$$M := 1 \tag{15}$$

> $fv := 2$

$$fv := 2 \tag{16}$$

> $K := 5$

$$K := 5 \tag{17}$$

> $x1c := \sin(10 \cdot t)$

$$x1c := \sin(10\,t) \tag{18}$$

> $X1c := SCS(laplace(x1c, t, s))$

$$X1c := \frac{10}{s^2 + 100} \tag{19}$$

> $X2c := G(M, fv, K) \cdot X1c$

$$X2c := \frac{10\,(2\,s + 5)}{\left(s^2 + 2\,s + 5\right)\,\left(s^2 + 100\right)} \tag{20}$$

> $x2c := IL(X2c, s, t)$

$$x2c := \frac{(80\cos(2\,t) + 55\sin(2\,t))\,e^{-t}}{377} - \frac{80\cos(10\,t)}{377} - \frac{3\sin(10\,t)}{377} \tag{21}$$

## Case four

> $M := 1$

$$M := 1 \tag{22}$$

> $fv := 6$

$$fv := 6 \tag{23}$$

> $K := 5$

$$K := 5 \tag{24}$$

> $x1d := \sin(10 \cdot t)$

$$x1d := \sin(10\,t) \tag{25}$$

> $X1d := SCS(laplace(x1d, t, s))$

$$X1d := \frac{10}{s^2 + 100} \tag{26}$$

> $X2d := G(M, fv, K) \cdot X1d$

$$X2d := \frac{10\,(6\,s + 5)}{(s^2 + 6\,s + 5)\,(s^2 + 100)} \tag{27}$$

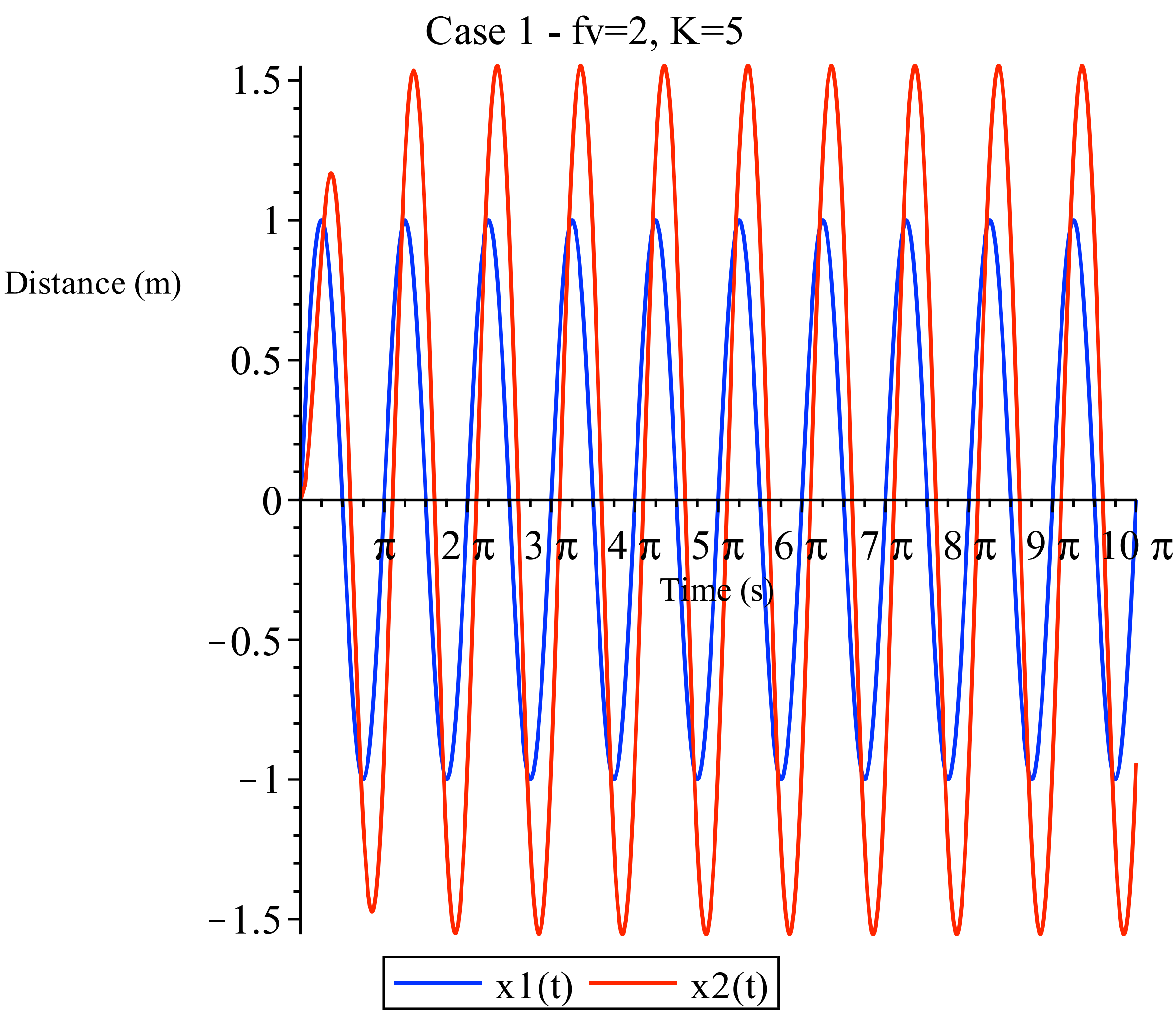
> $x2d := IL(X2d, s, t)$

$$x2d := -\frac{5\,e^{-t}}{202} + \frac{e^{-5\,t}}{2} - \frac{48\cos(10\,t)}{101} + \frac{25\sin(10\,t)}{101} \tag{28}$$

# Plots

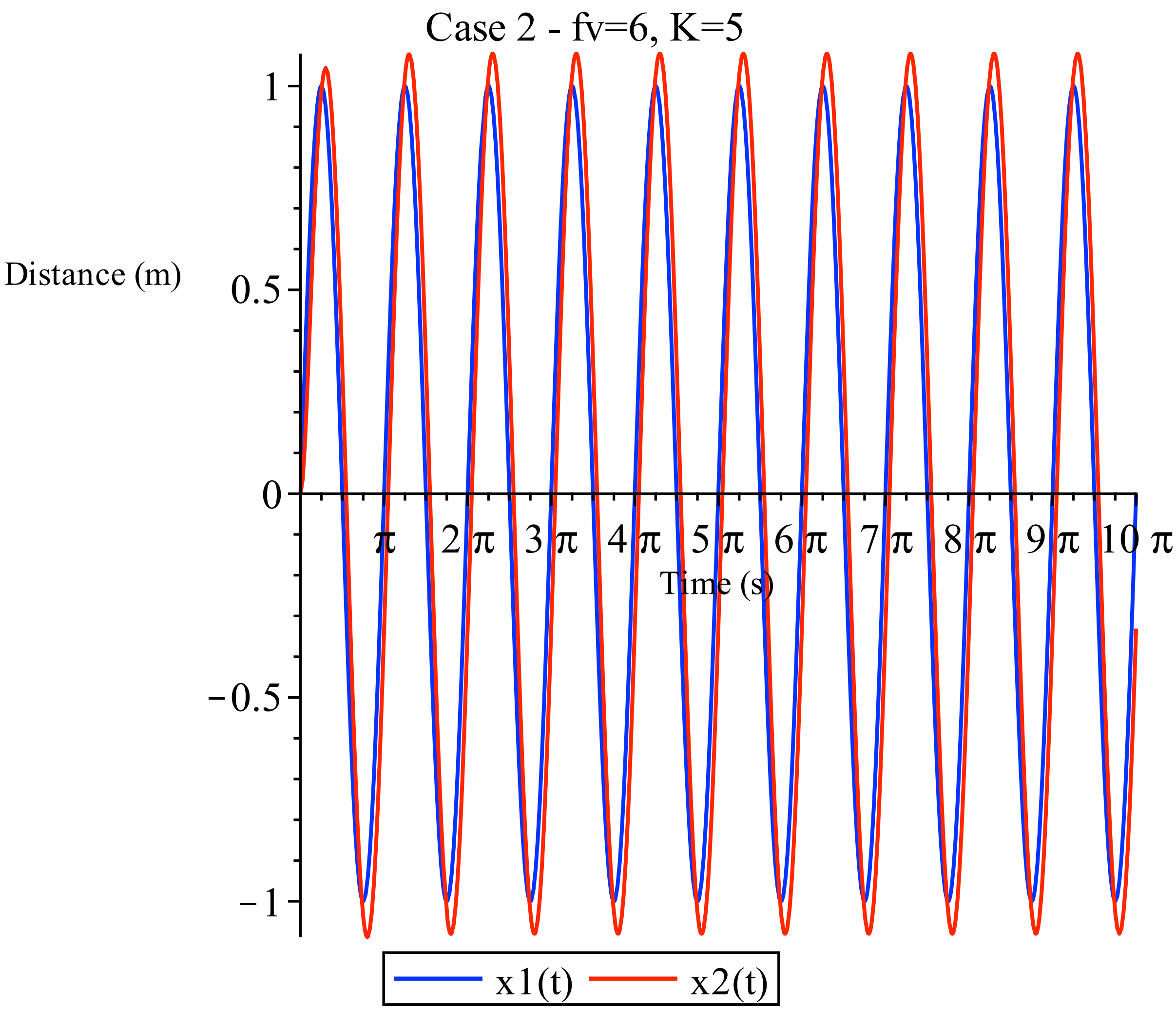
*Case 1*

> plot( [x1a, x2a], t = 0 .. 10·Pi, color = ["Blue", "Red"], title = "Case 1 - fv=2, K=5", legend = ["x1(t)", "x2(t)"], labels = ["Time (s)" ,"Distance (m)"]);



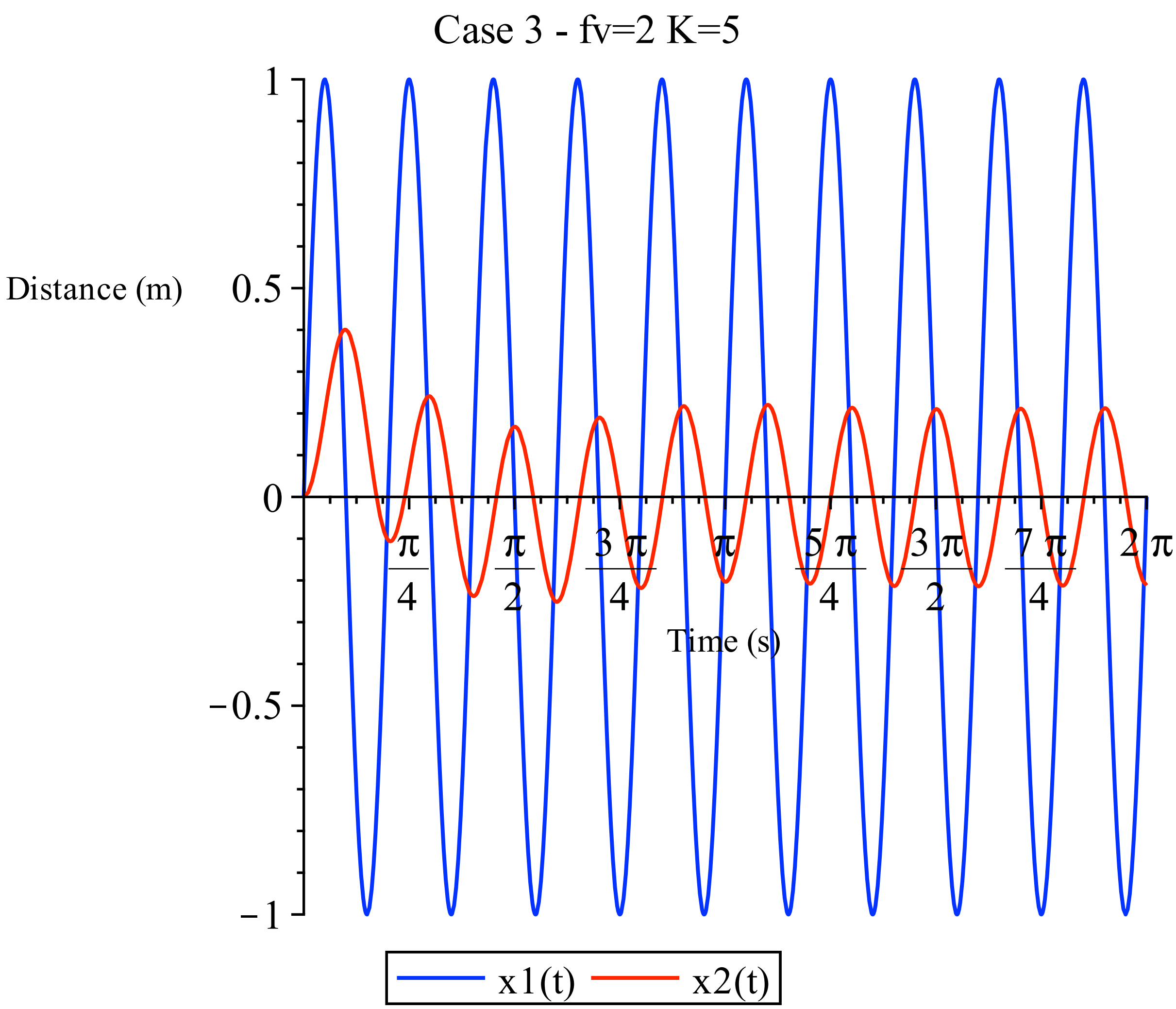
*Case 2*

> $plot([x1b, x2b], t = 0 .. 10\cdot \mathrm{Pi}, color = [\text{"Blue"}, \text{"Red"}], title = \text{"Case 2 - fv=6, K=5"}, legend = [\text{"x1(t)"}, \text{"x2(t)"}], labels = [\text{"Time (s)"}, \text{"Distance } (m)\text{"}]);$



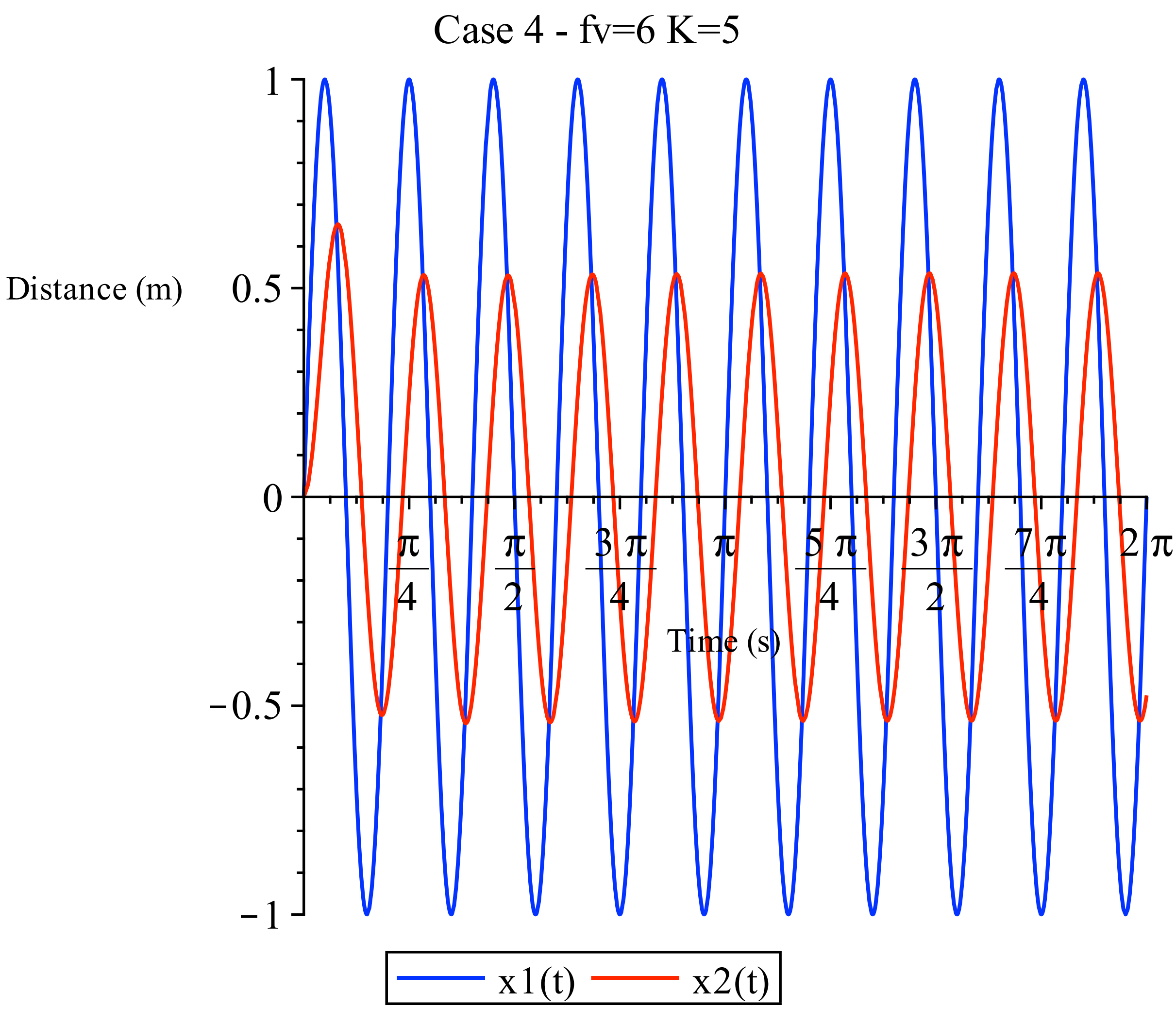

*Case 3*

> $plot([x1c, x2c], t = 0 .. 2\cdot \mathrm{Pi}, color = [\text{"Blue"}, \text{"Red"}], title = \text{"Case 3 - fv=2 K=5"}, legend = [\text{"x1(t)"}, \text{"x2(t)"}], labels = [\text{"Time (s)"}, \text{"Distance } (m)\text{"}]);$




*Case 4*

> *plot*( [*x1d*, *x2d*], *t* = 0 .. 2·Pi, *color* = ["Blue", "Red"], *title* = "Case 4 - fv=6 K=5", *legend* = ["x1(t)", "x2(t)"], *labels* = ["Time (s)", "Distance (*m*)"]);




# Discussion

It is clear from the above plots that in this system the output resembles the shape of the input, as they mimick each other in shape. However, it is apparent that larger damping coefficient make the output more closely resemble the input. In other words, larger damping coefficients correspond to a phase shift closer to 0 and gain closer to 1. In Case 2, as the damping coefficient is increased from Case 1 at the same frequency, the output "overshoots" the input by less and is shifted by a smaller amount. The same observation is made from Cae 4 to Case 2, where the increasing damping coefficient causes the output to "undershoot" by less. This observation can also be made emperically from the transfer function. As the damping coefficient increases to infinity the s terms in the numerator and the denominator dominate the other two terms. In other words, the limit of the transfer function as the damping coefficient approaches infinity is 1.

Case 3 and 4 have a higher frequency than Case 1 and 2. Case 3 and 4 also have a larger phase shift and smaller gain, as the output in Case 1 and 2 have magnitude larger than the input and Case 3 and 4 have magnitude smaller than the input. From these observations it can be concluded that the gain of the transfer funciton is inversely proportional to frequency (as frequency increases gain decreases) while the phase shift of the transfer funciton is proportional to the frequency (as frequency increases phase angle increases).

For more please reference my attached PDF on Sakai, which includes work and further discussion. Particularly this PDF contains plots from Matlab and discusses the impact of frequency on the transfer function.

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