HW #4 - Problem 1

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Problem 1A

R should **switch**. This would give him a $\mathbf{\frac{2}{3}}$ chance of guessing the correct coffee. If R chose not to switch his probability would only be $\frac{1}{3}$.

The intuition behind this answer is quite easy to follow. After R selects the first coffee his probability of guessing correctly is $\frac{1}{3}$. After an incorrect coffee is revealed the probability that his first guess is correct does not change. In other words, if R stays his chances of guessing the correct coffee is still $\frac{1}{3}$. However, after an incorrect choice is removed R has the option of selecting between his first choice and **not** his first choice. The probability of his first choice being incorrect is $\frac{2}{3}$, which corresponds to his probability of winning if he switches. A more formal explanation of why this is the case, using Bayes' Theorem, is shown below.

Without loss of generality let's say that R first selects coffee 1, and then W removes coffee 2. This holds because the probabilities that each coffee is correct is the same for each coffee and coffee 2 and 3 have the same probability, $\frac{1}{2}$, of being removed by W after the coffee 1 is selected.

$$\begin{aligned} P(\text{Coffee behind 1} \mid \text{Coffee 2 removed}) &= \frac{P(\text{ Coffee 2 removed} \mid \text{Coffee behind 1}) \text{ X } P\ (\text{Coffee behind 1})}{P(\text{Coffee 2 removed})} \\ &= \frac{1/2 * 1/3}{P(\text{Coffee 2 removed})} \end{aligned}$$

$$\begin{aligned} P(\text{Coffee 2 removed}) &= P(\text{1 selected}) * P(\text{2 removed } |\text{1 selected}) + \\ &\quad P(\text{2 selected}) * P(\text{2 removed } |\text{2 selected}) + \\ &\quad P(\text{3 selected}) * P(\text{2 removed } |\text{3 selected}) \\ &= \frac{1}{3}\frac{1}{2} + \frac{1}{3} * 0 + \frac{1}{3}\frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(\text{Coffee behind 1} \mid \text{Coffee 2 removed}) &= \frac{1/2 * 1/3}{P(\text{Coffee 2 removed})} \\ &= \frac{1/2 * 1/3}{1/2} \\ &= \frac{1}{3} \end{aligned}$$

As expected the probability of the first coffee being correct does not change after coffee 2 is revealed to be incorrect.

$$
\begin{aligned}
P(\text{Coffee behind 3} \mid \text{Coffee 2 removed}) &= \frac{P(\text{ Coffee 2 removed} \mid \text{Coffee behind 3}) \text{ X } P\text{ (Coffee behind 3)}}{P(\text{Coffee 2 removed})} \\
&= \frac{1*1/3}{1/2} \\
&= \frac{2}{3}
\end{aligned}
$$

The probability of coffee 3 being correct given coffee 1 is first selected and 2 was shown to be incorrect is $\frac{2}{3}$. Therefore, the chances of winning when switching from coffee 1 to coffee 3 is $\frac{2}{3}$.

Problem 1B

The ideal strategy is to first select coffee 1 (the coffee which R is least confident) and then switch regardless of which coffee is removed. This strategy will give R a $\mathbf{p_2} + \mathbf{p_3}$ chance of winning (guessing the correct coffee). This chance of winning is the same regardless of which coffee is removed after R's first guess of coffee 1.

The justification for this strategy is best seen with a decision tree. Let there be Coffee A, B, and C. Let Coffee A be the coffee that R first selects. Given this selection the decision tree that follows is given below,

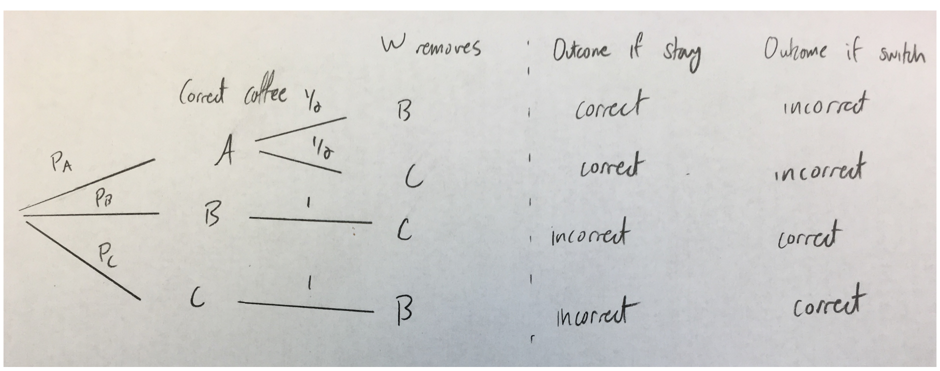

Figure 1: Decision Tree

From the decision tree above there are two possible paths to the correct outcome if a switch occurs. These occur with probability p_B and p_C respectively. Thus, the probability of being correct when switching is $p_B + p_C$ There are also two paths to correct outcomes if R stays. Each of these paths occur with probability $\frac{1}{2}p_A$. Thus, the probability of being correct when staying is $\frac{1}{2}p_A + \frac{1}{2}p_A = p_A$.

Since R has the freedom to chose the first coffee (A) he can achieve the following probabilities of winning if he stays: p_1, p_2, p_3. These correspond to choosing each of these coffees first and staying. Since $p_3 > p_2 > p_1$ the greatest chance R can have if he stays is p_3.

Again, since R has the freedom to choose the first coffee (A) then switch, he can achieve the following probabilities:

A (choice of first coffee)	probability of winning after switch
Coffee 1	$p_2 + p_3$
Coffee 2	$p_1 + p_3$
Coffee 3	$p_1 + p_2$

Again, since $p_3 > p_2 > p_1$ the greatest chance R can have if he switches is $p_2 + p_3$ - if he chooses coffee 1 first. Since $p_2 > 0$, it must be true that $p_2 + p_3 > p_3$. Therefore, the best strategy if R switches is better than the best strategy if R doesn't. This means R should pick coffee 1, then switch regardless, giving him a $p_2 + p_3$ probability of success.