CS330HW7

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Problem 2A

Show that if there exists a cut $(S, \bar{S})$, such that T does not contain any of its minimum cost edges, then T cannot be a minimum spanning tree.

Assume towards contradiction that T is a minimum spanning tree (MST) of the graph G. Let R be the cut $(S, \bar{S})$ for which T does not contain any of the cut's minimum cost edges. Let e' be one of the minimum edge in R. Also, let e be an edge in T and R. It is guaranteed that e must exist or T would have no connection between $(S, \bar{S})$ and thus could not be a spanning tree of G.

Let T' be the tree formed by a swap operation on the edges e' and e in the tree T. More specifically, generate the set of edges T_c by adding the edge e' to T. T_c must have n edges, since it has one more edge than T, which must have $n-1$ edges because it is a MST. Since T_c has n edges and is formed from adding an edge to an MST, it is guaranteed to have exactly one cycle.

Let T' be the tree formed by removing e from T_c. The cycle exists in T_c because there exist two edges in T_c across R - e and e'. This means that S can be connected to $\bar{S}$ through e then back to S through e'. Therefore, the action of removing e from T_c removes the cycle. Since there was only one cycle in T_c the result in removing one cycle from T_c must be a *tree*. Additionally, T' has all the same edges across all cuts as T, besides across the cut R. However, T' still has one and only one edge across this cut. Therefore, T' must be a spanning tree.

The *swap* operation can be defined more formally as:

$$T' = \{\forall x \in T + \{e'\} \mid x \neq e\}$$

Since e' is one of the minimum cost edge in R, not e, it can be established that $w(e) > w(e')$, where the notation $w(a)$ signifies the weight of an edge a. Additionally, the weight of T' is equal to the weight of T minus the weight of edge e plus the weight of edge e'. More formally,

$$w(T') = w(T) - w(e) + w(e')$$

Given $w(e) > w(e')$ or $w(e') - w(e) < 0$,

$$w(T') < w(T)$$

Therefore, T' is a spanning tree of G with a smaller weight than T. This means that T **cannot** be a minimum spanning tree because another, smaller spanning tree exists.

Problem 2B

Show that if for every cut $(S, \bar{S})$, T contains edges e that is one of the minimum cost edges in the cut $(S, \bar{S})$, the tree T must be a minimum spanning tree

In this problem it has to be shown that if for every cut $(S, \bar{S})$, T contains edges e that is one of the minimum cost edge in the cut, then T must be a MST. To do such, the tree T will be built by adding edges inductively, while maintaining at each step that the tree being formed is - 1. a subset of T and 2. a subset of an MST. Said differently, the tree F will be formed one edge at a time, growing in size from 0 to $k = n - 1$. It will be shown that once F is of size n it must be equal to T and an MST.

Before the proof by induction it must *first* be established that T must have $n - 1$ edges. Without this fact there is no way of proving that when the built set reaches $n - 1$ edges that it is equal to T. Let T have x edges. If $x \geq n$ then T would contain a cycle and not be a tree, which contradicts the problem statement. Therefore, $x < n$. If $x < n - 1$ then T would not connect all the vertices in G - since n vertices cannot be connected with less than $n - 1$ edges. If T does not connect all vertices in G then there must exist a cut around this unconnected vertex in which T has no edges. However, the problem statement says that T must contain a minimum edge for *all* cuts. Therefore, x cannot be less than $n - 1$. This leads to $n - 1 < x < n$ or $x = n - 1$.

Proof by Induction

Induction Hypothesis: Let F be a set of k $(0 \leq k < n - 1)$ edges in the graph G. F is a subset of T and a subset of a minimum spanning tree.
Base cases:

Let F be the empty set ($F = \{\}$). The empty set is a subset of all trees and minimal spanning trees.

Inductive Step:

Let the set F, of size k, be the set of all edges added inductively since the base case. Let R be any cut in G that does not contain the edges in set F. Since $k < n - 1$, this cut R must exist. In other words, since $k < n - 1$ there must be *at least 1* unconnected vertex in G. At the very least a cut can be formed around this unconnected vertex, which does not contain any edges in R.

Let e' be one of the minimal edges in R contained by T. Given the assumption T contains edges e that has one of the minimum cost edges in the cut $(S, \bar{S})$

for every cut $(S, \bar{S})$, e' must exist. Let F' be the set formed by adding e' to F. In other words,

$$F' = F + \{e'\}$$

Since F is a subset of T (by the Induction Hypothesis) and e is in T, by construction, F' must also be a subset of or identically T. Additionally, since e is a minimal edge in a cut that does not contain any edges in F, which is a subset of a minimal spanning tree (by the Induction Hypothesis), F' must be an MST directly from the **Key Property** (Key Lemma).

If $k < n-2$ then F' is a set of size $k < n-1$ and F' is a subset of T. If $k = n-2$ then F' is of size $n-1$ and thus must be T (since it has been shown above that T is of size $n-1$).

Therefore, given a set F, defined by the Induction Hypothesis, a set F' of $k+1$ edges can also be formed that is a subset of T and a subset of an MST *or* identically T and an MST.

It has been shown above that for any $k < n-2$, a set of size $k+1$ can be formed that is a subset of T and a subset of a MST. Additionally, it has been shown that for $k = n-1 < n-2$ a set of size $k+1 = n-1$ can be formed that is equal to T and an MST. Since T was constructed by adding all of its edges, there is no other possible way of forming T. Therefore, T must be a MST.

Another Approach

I just wanted to comment that the proof above is very similar to an approach that attempts to prove T is a MST by proving Prim's. The proof of Prim's algorithm is almost identical to the inductive proof given above. Let S be the subset of all possible trees that Prim's algorithm can produce. Prim's algorithm produces a MST if there exists a sequence of cuts in which does not contain the growing set F and in which there is a minimal edge across the cut. T contains the minimal edge for all such cuts. In other words the constraint necessary for Prim to build an MST is *weaker* than the constraint on T. Therefore, T must be in the set S of producible trees from Prim's and by consequence a MST.