Problem 1 : Forward Kinematics

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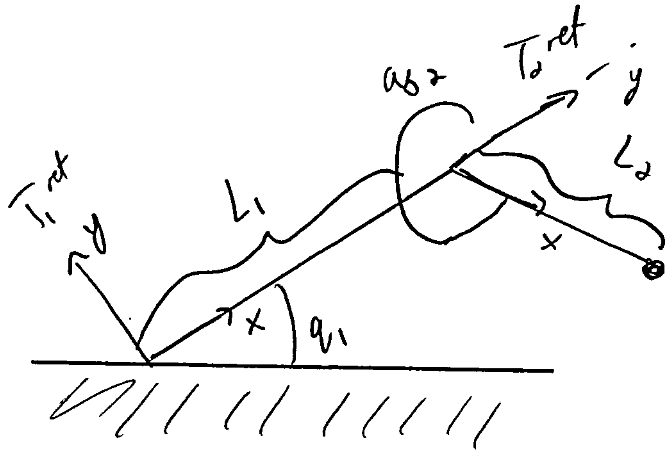
A1



q_1 : World $\rightarrow$ T_1^{ref}

q_2 : T_1^{ref} $\rightarrow$ T_2^{ref}

E.E location w.r.t T_2^{ref}

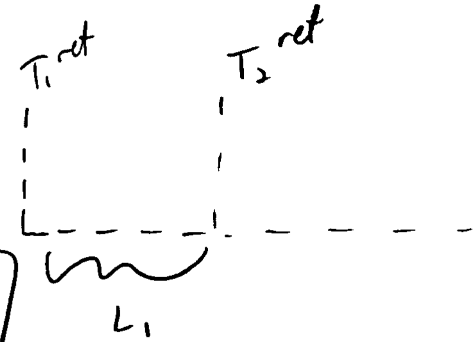
A. $\vec{x}(q_1, q_2) = T(q_1, q_2)\,\vec{p}$

conversion from T_2^{ref} to world reference frame after $R(q_2)$

$T_1^{ref} \rightarrow$ world

$$T_1(q_1) = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c_1 \equiv \cos q_1$
$s_1 \equiv \sin q_1$

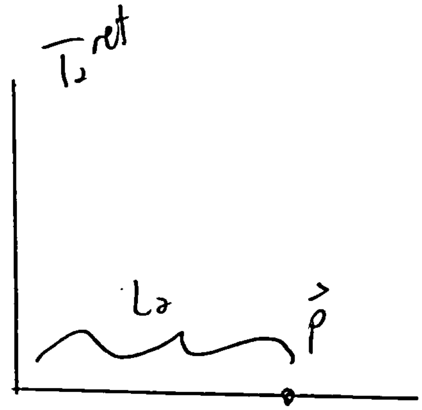


$$T_2(q_1, q_2) = T_1(q_1) \cdot T_{2\to 1}^{ref}\, R(q_2)$$

$$T_{2\to 1}^{ref} = \begin{bmatrix} 1 & 0 & L_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R(q_2) = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Point in T_2 w.r.t T_1

$c_2 \equiv \cos q_2$
$s_2 \equiv \sin q_2$



$$\vec{p} = \begin{bmatrix} L_2 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow$

Problem 1A cont

$$\vec{x}(q_1, q_2) = T_1(q_1)\, T_{2\to 1}^{ref}\, R(q_2) \begin{bmatrix} L_2 \\ 0 \\ 1 \end{bmatrix}$$

$$= T_1(q_1)\, T_{2\to 1}^{ref} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2 \\ 0 \\ 1 \end{bmatrix}$$

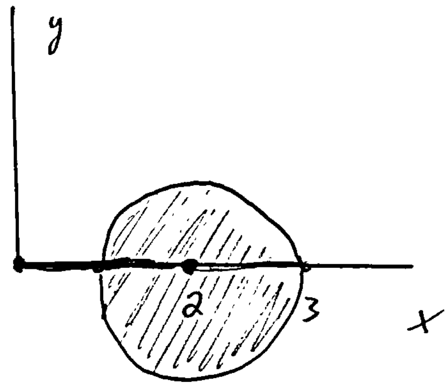
$$= T_1(q_1)\, T_{2\to 1}^{ref} \begin{bmatrix} L_2 c_2 \\ s_2 L_2 \\ 1 \end{bmatrix} = T_1(q_1) \begin{bmatrix} 1 & 0 & L_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2 c_2 \\ s_2 L_2 \\ 1 \end{bmatrix}$$

$$= T_1(q_1) \begin{bmatrix} L_2 c_2 + L_1 \\ s_2 L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2 c_2 + L_1 \\ s_2 L_2 \\ 1 \end{bmatrix}$$

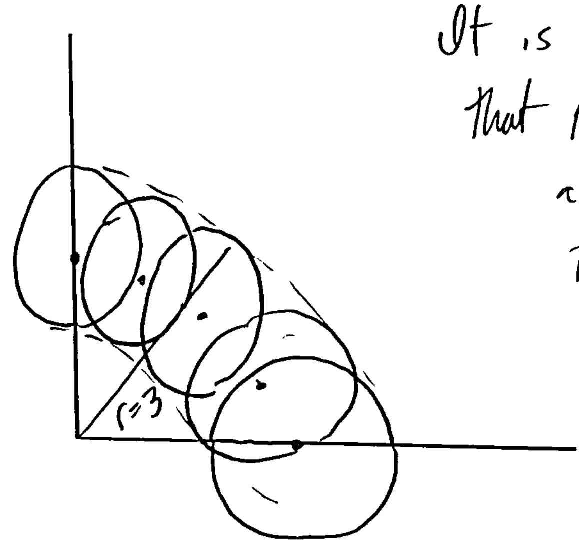
$$\boxed{\vec{x} = \begin{bmatrix} c_1 (L_2 c_2 + L_1) - s_1 s_2 L_2 \\ s_1 (L_2 c_2 + L_1) + c_1 s_2 L_2 \\ 1 \end{bmatrix}}$$

Problem 1B

Let $q_1 = 0$ and $q_2 \in [0°, 360°)$. This produces a workspace contained within a circle of radius 1 centered around the world point $(2,0)$



We can iterate this process for $q_1 \in (0, 360°]$ to get a better idea of the complete workspace.

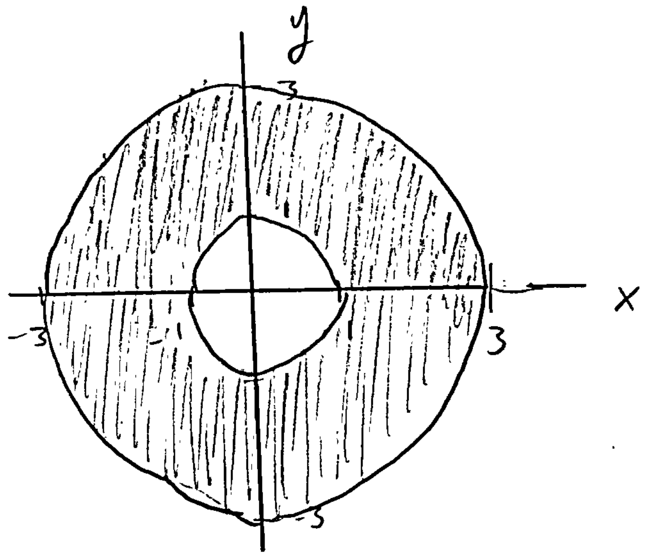


It is clear from this iterative process that the workspace is constrained by a circle of radius 3 around the origin. This makes sense, since when $q_2 = 0°$ the combined "length" of the two armatures is 3, which can be rotated anywhere is x, y space.

However, this is not the full picture. The circle of radius 1 around $(0,0)$ is not atainable.

Problem 1B cont

The final workspace is $L_2 - L_1 \le \sqrt{x^2+y^2} \le L_2 + L_1$



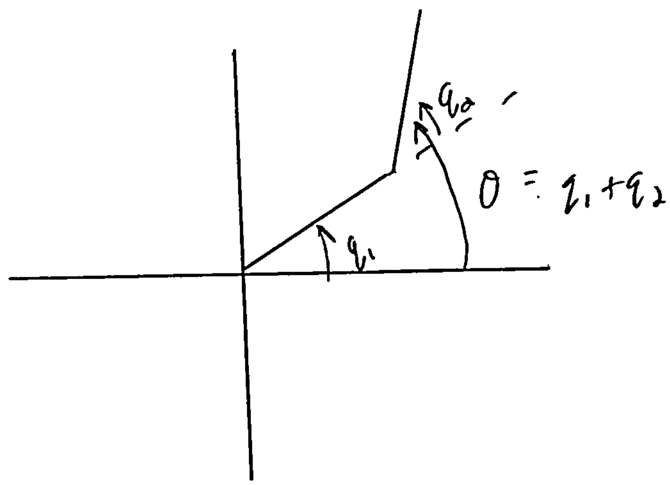
The workspace is given by the shaded region - between the circle with radius 1 and 3, both centered at the origin

In other words, $1 \le \sqrt{x^2+y^2} \le 3$ or $1 \le x^2+y^2 \le 9$

Problem 1C

θ = the angle of the second link in world coordinates

The angle of the second link in world coordinates is simply $q_1 + q_2$



$c_1 \equiv \cos q_1$
$s_1 \equiv \sin q_1$
$c_2 \equiv \cos q_2$
$s_2 \equiv \sin q_2$

From problem 1A
$$x = c_1(L_2 c_2 + L_1) - s_1 s_2 L_2$$
$$y = s_1(L_2 c_2 + L_1) + c_1 s_2 L_2$$

$\vec{x} = (x, y, \theta) \in SE(2)$

$$\vec{x} = \begin{bmatrix} c_1(L_2 c_2 + L_1) - s_1 s_2 L_2 \\ s_1(L_2 c_2 + L_1) + c_1 s_2 L_2 \\ q_1 + q_2 \end{bmatrix}$$

In homogeneous coordinates

$$\begin{bmatrix} c_1(L_2 c_2 + L_1) - s_1 s_2 L_2 \\ s_1(L_2 c_2 + L_1) + c_1 s_2 L_2 \\ q_1 + q_2 \\ 1 \end{bmatrix}$$

still a point

Problem 1D

The manifold of reachable values in the xy plane does not change, assuming L_2 is still 1 & L_1 is still 2. This means looking down on the xy axis, with z out of the page the workspace will look like a flat 'donut' centered at the origin (the space between a circle of radius 3 and a circle with radius 1).

The introduction of θ in the z dimension adds a bit of spatial complexity. This arises because there is not a one to one mapping between θ and a (x,y) point. This is shown below

$$q_1 = 0 \quad q_2 = 180^\circ. \Rightarrow \theta = 180^\circ \quad x = 1 \quad y = 0 \qquad q_1 = 180^\circ \quad q_2 = 0^\circ \Rightarrow \theta = 180^\circ \quad x = -3 \quad y = 0$$

Above it is shown each theta maps to a series of points in $\mathbb{R}^2$ space. Generally this produces a donut in x,y space that follows a spiraling pattern as θ ranges from 0 to 720 degrees. I used Matlab to demonstrate this visually.

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