Problem 1B

ms20

$$\begin{array}{cc} \checkmark x & \checkmark u \text{ center} \end{array}$$

$$A = \frac{\partial f}{\partial x}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$B = \frac{\partial f}{\partial u}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$$

PS 206

$$f_1 = \dot{p}_x = v\cos\theta$$

$$f_2 = \dot{p}_y = v\sin\theta$$

$$f_3 = \dot{\theta} = \frac{v}{L}\tan\phi$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial p_x} & \frac{\partial f_1}{\partial p_y} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial p_x} & \frac{\partial f_2}{\partial p_y} & \frac{\partial f_2}{\partial \theta} \\ \frac{\partial f_3}{\partial p_x} & \frac{\partial f_3}{\partial p_y} & \frac{\partial f_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -v\sin\theta \\ 0 & 0 & v\cos\theta \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f_1}{\partial \theta} = -v\sin\theta \qquad \frac{\partial f_2}{\partial \theta} = v\cos\theta$$

$$\frac{\partial f_3}{\partial \theta} = 0$$

$$\frac{\partial f}{\partial x}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A$$

$v = 1$

$\theta = 0$

$$\frac{\partial f}{\partial u} \begin{bmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial \phi} \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{L}\tan\phi & \frac{1}{L}\sec^2\phi \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/L \end{bmatrix}$$

$\phi = 0$

$$\dot{x} = Ax + Bu + c$$

drift

$$c = f\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix} 1\cos 0 \\ 1\sin 0 \\ \frac{1}{L}\tan 0 \end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Problem 1B cont

rus20

$$\dot{x} = Ax + Bu + c \quad \text{centered around } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v-1 \\ \phi \end{bmatrix}$$

$$\boxed{\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} v-1 \\ \phi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

As expected this linearization is only a good approximation near the centering values $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The best way to see this phenomenon is to expand the expression as follows.

$$\dot{x} = \begin{bmatrix} 0 \\ \theta \\ 0 \end{bmatrix} + \begin{bmatrix} v-1 \\ 0 \\ \phi/L \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ \theta \\ \phi/L \end{bmatrix}$$

$\dot{p}_x = v$: This approximation is solid when θ is small. When θ is small $v\cos\theta \approx v$. However, for larger values of θ such as $\theta = 90°$ this approximation is poor. The LTI approximation would have $\dot{p}_x = v$ for $\theta = 90°$ even though the true $\dot{p}_x$ value is $v\cos\theta$ or 0. To conclude, the LTI approximates $\dot{p}_x$ as v, independent of θ, which is only a good approximation when $\cos\theta \approx 1$.

$\dot{p}_y = \theta$: Like $\dot{p}_x$, this approximation only holds for small θ, and $v \approx 1$.

$\dot{P_y} \approx 0$: When θ is small $\sin\theta \approx \theta$ and if $v \approx 1$

then $v\sin\theta \approx \theta$ ← small angle approximation

$\dot{\theta} = \phi / L$: again, by small angle approximation $\frac{\tan\phi}{L} \approx \frac{\phi}{L}$ for small ϕ

Overall, this LTI approximation does not do well for values not around $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ & $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Problem 1B cont

rms20

Problem 1C

Since it is not specified I will assume b is a scalar quantity in the opposite direction of motion.

Controls are clear given the problem statement:

$$U = \begin{bmatrix} a \\ b \\ \dot{\phi} \end{bmatrix}$$

In order to characterize a system w/ acceleration the velocity & angular velocities are needed (as ~~well as position~~ as prior):

$$X = \begin{bmatrix} P_x \\ P_y \\ \theta \\ v \\ \dot{\theta} \end{bmatrix}$$

Equations of motion: $\dot{x} = f(x, v)$

$$\dot{P}_x = v\cos\theta$$

$$\dot{P}_y = v\sin\theta$$

$$\dot{\theta} = \frac{v}{L}\tan\phi$$

$$\dot{v} = a - b$$

$$\ddot{\theta} = \frac{1}{L}\left[(a-b)\tan\phi + v\dot{\phi}\sec^2\phi\right]$$

$$\dot{X} = \begin{bmatrix} \dot{P}_x \\ \dot{P}_y \\ \dot{\theta} \\ \dot{v} \\ \ddot{\theta} \end{bmatrix}$$

$\dot{P}_x$, $\dot{P}_y$, & $\dot{\theta}$ are given

$\dot{v}$ is acceleration

$$\dot{v} = a - b$$

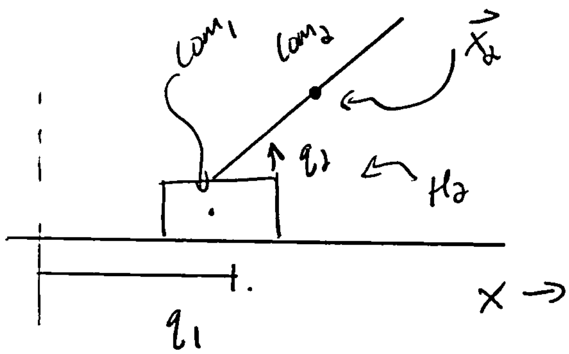
$$\dot{\theta} = \frac{v}{L}\tan\phi$$

$$\ddot{\theta} = \frac{1}{L}\left[\underbrace{\dot{v}}_{a-b}\tan\phi + v\dot{\phi}\sec^2\phi\right]$$

Equations of motions functions of x & u as desired

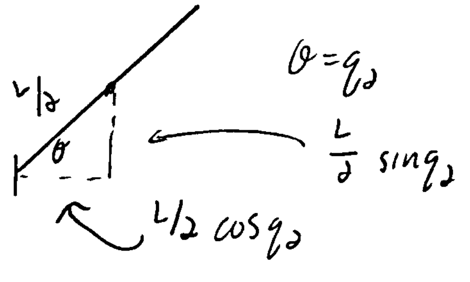
Problem 2

ns20



Assuming mass of cart, m_1, can be approximated as a point mass concentrated at q_1

SOH CAH TOA



$\theta = q_2$

com$_2$ located at $\vec{x}_2 = \begin{bmatrix} q_1 + \frac{L}{2}\cos q_2 \\ \frac{L}{2}\sin q_2 \end{bmatrix}$

$$KE_{cart} = \frac{1}{2} m_1 \dot{q}_1^2 + \cancel{\frac{1}{2} H_1 \omega_1^2}$$ zero, no angular rotation

$$KE_{cart} = \frac{1}{2} m_1 \dot{q}_1^2$$

Inertia of a pole rotated at its end

$$KE_{pole} = \underbrace{\frac{1}{2} m_2 \dot{x}_2^2} + \underbrace{\frac{1}{2} H_2 \dot{q}_2^2}$$

$$H_2 = \frac{1}{3} m_2 L^2 \qquad \dot{x}_2 = \begin{bmatrix} \dot{q}_1 - \dot{q}_2 \frac{L}{2}\sin q_2 \\ \frac{L}{2}\dot{q}_2 \cos q_2 \end{bmatrix}$$

$$\frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2} m_2 \left[\left(\dot{q}_1 - \dot{q}_2 \frac{L}{2}\sin q_2\right)^2 + \left(\frac{L}{2}\dot{q}_2 \cos q_2\right)^2 \right]$$

$$= \frac{1}{2} m_2 \left[\dot{q}_1^2 - L \dot{q}_2 \dot{q}_1 \sin q_2 + \frac{L^2}{4} \dot{q}_2^2 \left(\sin^2 q_2 + \cos^2 q_2\right) \right]$$

$$= \frac{1}{2} m_2 \left[\dot{q}_1^2 - L \dot{q}_2 \dot{q}_1 \sin q_2 + \frac{L^2}{4} \dot{q}_2^2 \right]$$

$$\frac{1}{2} H_2 \dot{q}_2^2 = \frac{1}{6} m_2 L^2 \dot{q}_2^2$$

$$KE_{TOT} = KE_{cart} + KE_{pole} = \frac{1}{2}(m_1 + m_2)\dot{q}_1^2 - \frac{L}{2} m_2 \dot{q}_2 \dot{q}_1 \sin q_2 + \overbrace{\frac{1}{8} L^2 \dot{q}_2^2 m_2 + \frac{1}{6} m_2 L^2 \dot{q}_2^2}^{7/24\, L^2 \dot{q}_2^2 m_2}$$

Problem 2

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$$KE = \frac{1}{2}(m_1+m_2)\dot{q}_1^2 - \frac{L}{2} m_2 \dot{q}_1 \dot{q}_2 \sin q_2 + \frac{7}{24} m_2 L^2 \dot{q}_2^2$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q} + \frac{\partial P}{\partial q} = \tau$$

$$\frac{\partial K}{\partial \dot{q}} = \begin{bmatrix} \partial K / \partial \dot{q}_1 \\ \partial K / \partial \dot{q}_2 \end{bmatrix}$$

$$\frac{\partial K}{\partial \dot{q}_1} = (m_1+m_2)\dot{q}_1 - \frac{L}{2} m_2 \dot{q}_2 \sin q_2$$

$$\frac{\partial K}{\partial \dot{q}_2} = \frac{7}{12} m_2 L^2 \dot{q}_2 - \frac{L}{2} m_2 \dot{q}_1 \sin q_2$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_1} = (m_1+m_2)\ddot{q}_1 - \frac{L}{2} m_2 \sin q_2 \ddot{q}_2 - \frac{L}{2} m_2 \dot{q}_2^2 \cos q_2$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_2} = \frac{7}{12} m_2 L^2 \ddot{q}_2 - \frac{L}{2} m_2 \ddot{q}_1 \sin q_2 - \frac{L}{2} m_2 \dot{q}_1 \dot{q}_2 \cos q_2$$

$$-\frac{\partial K}{\partial q} = \begin{bmatrix} \frac{\partial K}{\partial q_1} \\ \partial K / \partial q_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ -\frac{L}{2} m_2 \dot{q}_1 \dot{q}_2 \cos q_2 \end{bmatrix}$$

Now need to group terms for $B(q)\ddot{q}$ and $C(q,\dot{q})$

$\Rightarrow$

Problem 2

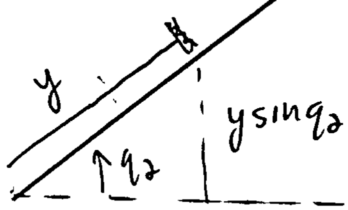
ms 20

$B(q)\ddot{q}$

$$= \begin{bmatrix} m_1+m_2 & -\frac{L}{2} m_2 \sin q_2 \\ -\frac{L}{2} m_2 \sin q_2 & \frac{7}{12} m_2 L^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} -\frac{L}{2} m_2 \dot{q}_2^2 \cos q_2 \\ -\frac{L}{2} m_2 \dot{q}_1 \dot{q}_2 \cos q_2 + \frac{L}{2} m_2 \dot{q}_1 \dot{q}_2 \cos q_2 \end{bmatrix} = \begin{bmatrix} -\frac{L}{2} m_2 \dot{q}_2^2 \cos q_2 \\ 0 \end{bmatrix}$$

Now need potential



$$\lambda = \frac{m_2}{L} \qquad \downarrow \lambda$$

$$P = \int_0^L \frac{m_2}{L} g y \sin q_2 \, dy$$

$$= \frac{m_2}{L} g \sin q_2 \int_0^L y \, dy$$

$$= \frac{m_2}{L} g \sin q_2 \left[\frac{1}{2} y^2 \right]\Big|_0^L$$

$$P = \frac{m_2}{2} g L \sin q_2$$

$$\frac{\partial P}{\partial q} = \begin{bmatrix} 0 \\ \frac{m_2}{2} g L \cos q_2 \end{bmatrix} = G(q)$$

$$\underbrace{\begin{bmatrix} m_1+m_2 & -\frac{L}{2} m_2 \sin q_2 \\ -\frac{L}{2} m_2 \sin q_2 & \frac{7}{12} m_2 L^2 \end{bmatrix}}_{B(q)} \ddot{q} + \underbrace{\begin{bmatrix} -\frac{L}{2} m_2 \dot{q}_2^2 \cos q_2 \\ 0 \end{bmatrix}}_{C(q,\dot{q})} + \underbrace{\begin{bmatrix} 0 \\ \frac{m_2}{2} g L \cos q_2 \end{bmatrix}}_{G(q)} = \tau$$

Problem 3A

rlls20

Need a 3rd order curve such that acceleration < a_{max}

This can be satisfied by an acceleration bounded Hermite curve.

$$\theta_{dest}(u(t)) = \underset{\text{start}}{\theta_s} + (\theta_{tgt} - \theta_s)(a + bu + cu^2 + du^3)$$ comes to a stop

Need $\theta_{st}(0) = \theta_s$ $\quad \theta_{dest}(1) = \theta_{tgt}$ $\quad \theta'_{dest}(0) = 0$ $\quad \theta'_{dest}(1) = 0$

↳ initially not moving

$\theta_{dest}(0) = \theta_s \Rightarrow \theta_s + (\theta_{tgt} - \theta_s)a = \theta_s \quad a = 0$

$\theta_{dest}(1) = \theta_{tgt} \Rightarrow \theta_s + (\theta_{tgt} - \theta_s)(b + c + d) \quad b + c + d = 1 \Rightarrow c + d = 1$

$\theta'_{dest}(0) = 0 \Rightarrow$ Define $u = t/t_f$ ← total time

$$\theta_{dest}(t) = \theta_s + (\theta_{tgt} - \theta_s)\left(b\left(\frac{t}{t_f}\right) + c\left(\frac{t}{t_f}\right)^2 + d\left(\frac{t}{t_f}\right)^3\right)$$

$$\theta'_{dest}(t) = (\theta_{tgt} - \theta_s)\left[\left(\frac{b}{t_f}\right) + 2c\frac{t}{t_f^2} + 3d\frac{t^2}{t_f^3}\right]$$

$\theta'_{dest}(0) = 0 \Rightarrow b = 0$

$\theta'_{dest}(1) = 0 \Rightarrow 2c + 3d = 0$

$2(1-d) + 3d = 0$

$2 - 2d + 3d = 0$

$d = -2$

$c = 3$

$$\theta_{dest}(u(t)) = \theta_s + (\theta_{tgt} - \theta_s)(3u^2 - 2u^3)$$

Problem 3A cont

$n > 20$

Need to bound acceleration

$$\theta_{dest}(t) = \theta_s + (\theta_{tgt} - \theta_s)\left(3\left(\frac{t}{t_f}\right)^2 - 2\left(\frac{t}{t_f}\right)^3\right)$$

$$\dot{\theta}_{dest}(t) = 6(\theta_{tgt} - \theta_s)\left[\frac{t}{t_f^2} - \frac{t^2}{t_f^3}\right]$$

$$\ddot{\theta}_{dest}(t) = 6(\theta_{tgt} - \theta_s)\underbrace{\left[\frac{1}{t_f^2} - \frac{2t}{t_f^3}\right]}_{\text{Max at the endpoints}} \leq a_{max}$$

$$\ddot{\theta}_{dest}(t)_{max} = \frac{6(\theta_{tgt} - \theta_s)}{t_f^2} \leq a_{max}$$

$$\Rightarrow t_f \geq \sqrt{\frac{6(\theta_{tgt} - \theta_s)}{a_{max}}}$$

Thus, for a given θ_{tgt} and θ_s,

$$\boxed{\theta_{dest}(t) = \theta_s + (\theta_{tgt} - \theta_s)\left(3\left(\frac{t}{t_f}\right)^2 - 2\left(\frac{t}{t_f}\right)^3\right) \quad \text{where } t_f = \sqrt{\frac{6(\theta_{tgt} - \theta_s)}{a_{max}}} \text{ and } t \in [0, t_f]}$$

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Problem 3B

To break as quickly as possible want to set deceleration at maximum

want to oppose velocity

$\ddot{\theta}(t) = b_{max}(-\text{sign}(\dot{\theta})) = a$

By necessity $\dot{\theta}(t) = at + \dot{\theta}(0)$

Then by necessity $\theta(t) = \frac{1}{2}at^2 + \dot{\theta}(0)t + \theta(0)$

Also need $\dot{\theta}(T) = 0 = aT + \dot{\theta}(0) \Rightarrow T = \frac{-\dot{\theta}(0)}{a}$

ending time

and start time + start

Given starting velocity $\dot{\theta}(0)$ and position $\theta(0)$ curve should be:

$$\theta(t) = \frac{1}{2} b_{max}(-\text{sign}(\dot{\theta}(0)))t^2 + \dot{\theta}(0)t + \theta(0)$$

$$t \in \left[0, \frac{-\dot{\theta}(0)}{b_{max}(-\text{sign}(\dot{\theta}(0)))}\right]$$

relative to when break is called

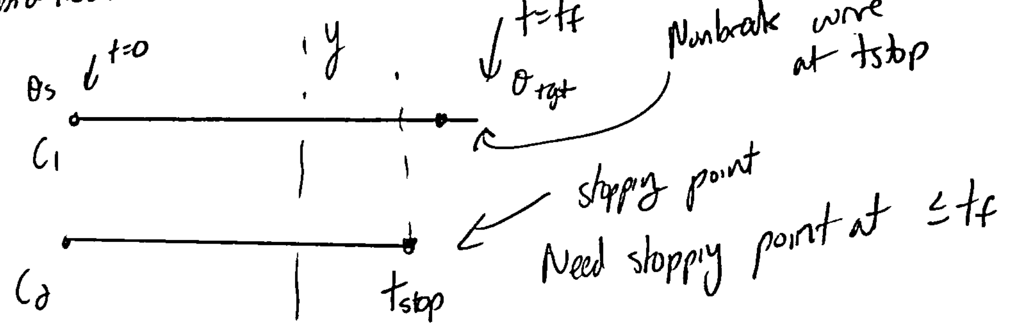
Problem 3C

Since the brake curve always decelerates at b_{max} if $b_{max} \geq a_{max}$ then the angle after break is called is guarenteed to be less than if break was not called. Imagine brake is called at time y (between 0 and t_f). For any time x between y and t_f, the
↳ the time the curve normally stops

breaked trajectory is guarenteed to be at a θ further from θ_{tgt} than a non-breaked curve. Since all angles between θ_s & θ_{tgt} are valid, it is sufficient to show the breaked curve stops before t_f if $b_{max} \geq a_{max}$ for $v \in [0,1]$
↳ anywhere on the curve

Non-break curve: C_1

Break curve: C_2



If θ on C_1 is valid for $t = t_{stop}$, C_2 must also be valid.

Now to prove breaking occurs before t_f...

Let's say break is called somewhere on the curve, that is $v \in [0,1]$
time break is called

The velocity at time of braking is $\dfrac{(\theta_{tgt} - \theta_s)(6v - 6v^2)}{t_f}$

Therefore the time of stopping $(v=0)$ is $\dfrac{(\theta_{tgt} - \theta_s)(6v - 6v^2)}{b_{max}\, t_f} + vt_f$

Problem 3c cont'd

Need $t_{stop} \le t_f$

$$\frac{(\theta_{tgt} - \theta_s)(6u - 6u^2)}{b_{max} t_f} + u t_f \le t_f$$

$$t_f = \sqrt{\frac{6(\theta_{tgt} - \theta_s)}{a_{max}}}$$

↳ from 3A

$$\frac{(\theta_{tgt} - \theta_s)(6u - 6u^2)}{b_{max}} \le t_f^2 (1 - u)$$

$$\frac{\cancel{(\theta_{tgt} - \theta_s)}(6u - 6u^2)}{b_{max}} \le 6\frac{\cancel{(\theta_{tgt} - \theta_s)}}{a_{max}}\cancel{(1-u)}$$

$$\frac{u}{b_{max}} \le \frac{1}{a_{max}}$$

This inequality must hold because $u \in [0,1]$ and $b_{max} > a_{max}$

$\boxtimes$