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ECE 383 / MEMS 442 Fall 2016

Ryan St Pierre

HW #1 : Linear Algebra & Transformations

Problem 1A

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Using $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\sin^2\theta + \cos^2\theta = 1$

$$R(\theta)^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

using $\cos(-x) = \cos(x)$ Even function

$\sin(-x) = -\sin(x)$ odd function

$$R(\theta)^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(-\theta) = R(\theta)^{-1}$$

Thus $R(\theta)^{-1} = R(\theta)^T = R(-\theta)$

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Problem 1B

$$R(\theta_1)\,R(\theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$

Note: $\sin(x \pm y) = \sin x\cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -(\sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_2+\theta_1) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$$

$\emptyset$

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Problem 2A

$$\| T\vec{x} - T\vec{y} \| = \| R(\theta)\vec{x} + \vec{t} - (R(\theta)\vec{y} + \vec{t})\|$$

$$= \| R(\theta)\vec{x} - R(\theta)\vec{y} \|$$

$$= \| R(\theta)(\vec{x} - \vec{y})\|$$

Vector norms are invariant under rotation

$$\| R(\theta)\vec{z} \| = \|\vec{z}\| \quad (13.15 \text{ of the text})$$

$$\| T\vec{x} - T\vec{y} \| = \| \vec{x} - \vec{y} \| \quad \boxtimes$$

I also proved $\|T\vec{x} - T\vec{y}\| = \|\vec{x} - \vec{y}\|$ without relying on 13.15 of the text. This is shown on the following pages →

Problem 2A cont

A) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

where $x_1, x_2, y_1, y_2, t_1, t_2 \in \mathbb{R}$:

$$T\vec{x} = R(\theta) \cdot \vec{x} + \vec{t}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$T\vec{x} = \begin{bmatrix} x_1 \cos\theta - x_2 \sin\theta + t_1 \\ x_1 \sin\theta + x_2 \cos\theta + t_2 \end{bmatrix}$$

Similarly $T\vec{y} = \begin{bmatrix} y_1 \cos\theta - y_2 \sin\theta + t_1 \\ y_1 \sin\theta + y_2 \cos\theta + t_2 \end{bmatrix}$

$$\|T\vec{x} - T\vec{y}\| = \left\| \begin{bmatrix} x_1\cos\theta - x_2 \sin\theta + t_1 - y_1 \cos\theta + y_2 \sin\theta - t_1 \\ x_1 \sin\theta + x_2 \cos\theta + t_2 - y_1 \sin\theta - y_2 \cos\theta - t_2 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} (x_1 - y_1)\cos\theta + (y_2 - x_2)\sin\theta \\ (x_2 - y_2)\cos\theta + (x_1 - y_1)\sin\theta \end{bmatrix} \right\|$$

$\|T\vec{x} - T\vec{y}\|^2 = \sqrt{\vec{h} \cdot \vec{h}}$ where $\vec{h} = \begin{bmatrix} (x_1 - y_1)\cos\theta + (y_2 - x_2)\sin\theta \\ (x_2 - y_2)\cos\theta + (x_1 - y_1)\sin\theta \end{bmatrix}$

$$\|T\vec{x} - T\vec{y}\|^2 = (x_1 - y_1)^2 \cos^2\theta + (x_1 - y_1)\cos\theta\,(y_2 - x_2)\sin\theta + (y_2 - x_2)^2 \sin^2\theta$$
$$+ (x_2 - y_2)^2 \cos^2\theta + (x_2 - y_2)\cos\theta\,(x_1 - y_1)\sin\theta + (x_1 - y_1)^2 \sin^2\theta$$

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Problem 2A cont

$(y_2 - x_2)^2 = (x_2 - y_2)^2$

$$\|T\vec{x} - T\vec{y}\|^2 = (x_1-y_1)^2\cos^2\theta + (x_1-y_1)\cos\theta\,(y_2-x_2)\sin\theta + (y_2-x_2)^2\sin^2\theta$$
$$+ (x_2-y_2)^2\cos^2\theta + (x_2-y_2)\cos\theta\,(x_1-y_1)\sin\theta + (x_1-y_1)^2\sin^2\theta$$

$$= (x_1-y_1)^2(\cancel{\cos^2\theta + \sin^2\theta}) + (y_2-x_2)^2(\cancel{\cos^2\theta + \sin^2\theta})$$
$$+ (x_1-y_1)(y_2-x_2)\cos\theta\sin\theta - (x_1-y_1)(y_2-x_2)\cos\theta\sin\theta$$

$$\|T\vec{x} - T\vec{y}\|^2 = (x_1-y_1)^2 + (y_2-x_2)^2$$

$$= \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$$

$$= \left\| \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} \right\|^2$$

$$\|T\vec{x} - T\vec{y}\|^2 = \|\vec{x} - \vec{y}\|^2$$

$$\|T\vec{x} - T\vec{y}\| = \|\vec{x} - \vec{y}\| \quad \boxtimes$$

Problem 2B

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$\vec{y} = T\vec{x} = R\vec{x} + \vec{t}$

T^{-1} is the transformation that has $x = T^{-1}\vec{y}$ hold

$\vec{y} = R\vec{x} + \vec{t}$

$R^{-1}\vec{y} = R^{-1}(R\vec{x} + \vec{t})$

$R^{-1}\vec{y} = \vec{x} + R^{-1}\vec{t}$ $\qquad [R^{-1} = R^T \quad (13.13)]$

$\vec{x} = R^T\vec{y} - R^T\vec{t}$

$= R'\vec{y} + \vec{t}\,'$ $\qquad$ where $R' = R^T$ and $\vec{t}\,' = -R^T\vec{t}$

$\vec{x} = T'\vec{y}$

where $T' = T^{-1}$ is a rigid tranform with parameters

∎

The above proof can also be made for the homogeneous representation.
This is shown in the following pages →

Also, R^T can also be achieved by negating theta due to the odd and even nature of sin & cos respectively

$$\boxed{\begin{aligned} \theta' &= -\theta \\ t' &= -R^T(\theta)\vec{t} \end{aligned}}$$

Problem 2B cont

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Show $T(\theta, t)^{-1} = T(\theta', \vec{t}')$ for some values of θ' & $\vec{t}'$

Let T be represented in the following $T(\theta, \vec{t}) = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$
where $\vec{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

Then $T^{-1} = \dfrac{1}{|T|} \overbrace{\operatorname{adj}(T)}^{\text{adjoint of } T}$

$$\text{Minors of } T = \begin{bmatrix} \cos\theta - 0t_y & \sin\theta - 0t_y & 0 - 0 \\ -\sin\theta - 0t_x & \cos\theta - 0t_x & 0 - 0 \\ -t_y\sin\theta - t_x\cos\theta & t_y\cos\theta - t_x\sin\theta & \cos^2\theta - -\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -t_y\sin\theta - t_x\cos\theta & t_y\cos\theta - t_x\sin\theta & 1 \end{bmatrix}$$

$$\text{Cofactor of } T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -t_y\sin\theta - t_x\cos\theta & t_x\sin\theta - t_y\cos\theta & 1 \end{bmatrix}$$

$$\operatorname{adj}(T) = \text{transpose of cofactor of } T = \begin{bmatrix} \cos\theta & \sin\theta & -(t_x\cos\theta + t_y\sin\theta) \\ -\sin\theta & \cos\theta & t_x\sin\theta - t_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$|T| = \cos\theta\,[\cos\theta - 0] - -\sin\theta\,(\sin\theta - 0) + t_x(0-0) = \cos^2\theta + \sin^2\theta = 1$$

Problem 2B cont

$$T^{-1} = \frac{1}{|T|} \text{adj}(T)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & -(t_x\cos\theta + t_y\sin\theta) \\ -\sin\theta & \cos\theta & t_x\sin\theta - t_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta' & -\sin\theta' & -(t_x\cos\theta + t_y\sin\theta \\ \sin\theta' & \cos\theta' & t_x\sin\theta - t_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

This follows the homogeneous form of a rigid transform with $\theta = \theta'$ and $\vec{t}\,' = \begin{bmatrix} -(t_x\cos\theta + t_y\sin\theta) \\ t_x\sin\theta - t_y\cos\theta \end{bmatrix} = -\begin{bmatrix} t_x\cos\theta + t_y\sin\theta \\ t_y\cos\theta - t_x\sin\theta \end{bmatrix}$

$$= -\begin{bmatrix} \vec{t}\cdot\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \\ \vec{t}\cdot\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} \end{bmatrix} = \underbrace{-\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{-R^T(\theta)} \vec{t}$$

$$\boxed{\begin{aligned} \theta' &= -\theta \\ \vec{t}\,' &= -R^T(\theta)\,\vec{t} \end{aligned}}$$

Problem 2c

$T_1 \vec{x} = R_1(\theta_1)\vec{x} + \vec{t}_1$

$T_2 \vec{x} = R_2(\theta_2)\vec{x} + \vec{t}_2$

$T\vec{x} = T_1(T_2 \vec{x})$ where $T = T_1 \circ T_2$

$= R_1(R_2 \vec{x} + \vec{t}_2) + \vec{t}_1$

$$T\vec{x} = \underbrace{R_1 R_2}_{R'} \vec{x} + \underbrace{R_1 \vec{t}_2 + \vec{t}_1}_{t'}$$

This is in the form of a rigid transformation with rotation matrix $R_1 R_2$ and translation $R_1 \vec{t}_2 + \vec{t}_1$

$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$ [section 13.2.2], thus $\theta = \theta_1 + \theta_2$

$$\boxed{\begin{aligned} \theta &= \theta_1 + \theta_2 \\ \vec{t} &= R_1(\theta_1)\vec{t}_2 + \vec{t}_1 \end{aligned}}$$

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Problem 3A

$$\check{y} = T(\theta, \vec{t})\,\vec{x}$$

$$= R(\theta)\,\vec{x} + \vec{t} \quad \text{where } \vec{t} \text{ is a vector in 2D space}$$

$$\check{y} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x} + \vec{t}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$. Then $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$

$$\vec{y} = \hat{T}(\theta, \vec{t}) \cdot \hat{x}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta + t_x \\ x_1\sin\theta + x_2\cos\theta + t_y \\ 1 \end{bmatrix}$$

$$\check{y} = \begin{bmatrix} (x_1\cos\theta - x_2\sin\theta + t_x)/1 \\ (x_1\sin\theta + x_2\cos\theta + t_y)/1 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta + t_x \\ x_1\sin\theta + x_2\cos\theta + t_y \end{bmatrix}$$

$$\check{y} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} + \vec{t_2} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x} + \vec{t} = \vec{y}$$

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Problem 3B

Prove $\hat{T}_1 \hat{T}_2 = \widehat{T_1 \circ T_2}$

$$\hat{T}_1 \cdot \hat{T}_2 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & t_{1x} \\ \sin\theta_1 & \cos\theta_1 & t_{1y} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & t_{2x} \\ \sin\theta_2 & \cos\theta_2 & t_{2y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ①\ \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & ②\ -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 & t_{2x}\cos\theta_1 - t_{2y}\sin\theta_1 + t_{1x} \\ ③\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & ④\ -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & t_{2x}\sin\theta_1 + t_{2y}\cos\theta_1 + t_{1y} \\ 0 & 0 & 1 \end{bmatrix}$$

① $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \frac{1}{2}\left[\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2)\right] - \frac{1}{2}\left[\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2)\right]$

$= \cos(\theta_1+\theta_2)$

② $-\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 = -\frac{1}{2}\left[\sin(\theta_1+\theta_2) - \sin(\theta_1-\theta_2)\right] - \frac{1}{2}\left[\sin(\theta_2+\theta_1) - \sin(\theta_2-\theta_1)\right]$

$= -\sin(\theta_1+\theta_2) + \frac{1}{2}\cancel{\sin(\theta_1-\theta_2)} - \frac{1}{2}\cancel{\sin(\theta_1-\theta_2)}$

$= -\sin(\theta_1+\theta_2)$

③ $\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \frac{1}{2}\left[\sin(\theta_1+\theta_2) - \cancel{\sin(\theta_2-\theta_1)}\right] + \frac{1}{2}\left[\sin(\theta_1+\theta_2) - \cancel{\sin(\theta_1-\theta_2)}\right]$

$= \sin(\theta_1+\theta_2)$

④ $-\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 = \frac{1}{2}\left[\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2)\right] - \frac{1}{2}\left[\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2)\right]$

$= \cos(\theta_1+\theta_2)$

$\rightarrow$

Problem 3B cont

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$$\hat{T}_1 \cdot \hat{T}_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & t_{2x}\cos\theta_1 - t_{2y}\sin\theta_1 + t_{1x} \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & t_{2x}\sin\theta_1 + t_{2y}\cos\theta_1 + t_{1y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1 \circ T_2 = T_1(T_2\vec{x}) = R_1(R_2\vec{x} + \vec{t}_2) + \vec{t}_1$$

$$= \underbrace{R_1 R_2}_{R(\theta_1+\theta_2)}\vec{x} + \underbrace{R_1 \vec{t}_2 + \vec{t}_1}_{\vec{t}\,'}$$

$$(T_1 \circ T_2)\vec{x} = R(\theta_1+\theta_2)\vec{x} + \vec{t}\,' \quad \text{where} \quad \vec{t}\,' = R_1\vec{t}_2 + \vec{t}_1$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} t_{2x} \\ t_{2y} \end{bmatrix} + \begin{bmatrix} t_{1x} \\ t_{1y} \end{bmatrix}$$

$$= \begin{bmatrix} t_{2x}\cos\theta_1 - t_{2y}\sin\theta_1 + t_{1x} \\ t_{2x}\sin\theta_1 + t_{2y}\cos\theta_1 + t_{2y} \end{bmatrix}$$

$T_1 \circ T_2$ is a rigid transform w/ parameters $\theta' = \theta_1+\theta_2$ & $\vec{t}\,' = R_1\vec{t}_2 + \vec{t}_1$. Writing a homogenous rigid transform with these parameters produces the following...

$$\widehat{T_1 \circ T_2} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & t_{2x}\cos\theta_1 - t_{2y}\sin\theta_1 + t_{1x} \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & t_{2x}\sin\theta_1 + t_{2y}\cos\theta_1 + t_{1y} \\ 0 & 0 & 1 \end{bmatrix} = \hat{T}_1 \cdot \hat{T}_2$$

Problem 3C

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$$T(\theta,\vec{t}) * \vec{d} = R(\theta)\cdot\vec{d} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$$

$$= \begin{bmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{bmatrix} = \check{y}$$

$$\vec{y} = \hat{T}(\theta,\vec{t})\cdot\hat{d} = \begin{bmatrix}\cos\theta & -\sin\theta & t_1 \\ \sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x \\ y \\ 0\end{bmatrix}$$

essentially "nullifies" the third column of $\hat{T}(\theta,\vec{t})$ which corresponds to the transnational component

$$= \begin{bmatrix}x\cos\theta - y\sin\theta + 0 \\ x\sin\theta + y\cos\theta + 0 \\ 0 + 0 + 0\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ \underline{0}\end{bmatrix}$$

* direction

The multiplication in homogeneous coordinates produces a <u>direction</u> in the $\begin{bmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{bmatrix}$ direction which is the same result as the star operator. Additionally, the multiplication in homogeneous coordinates produces a zero in the homogeneous coordinate, preserving the convention that directions have zero as their homogeneous coordinate.

Problem 4A

$$R_x(\theta)\,R_y(\theta) \overset{?}{=} R_y(\theta)\,R_x(\theta)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \overset{?}{=} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin^2\theta & \cos\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta & \cos^2\theta \end{bmatrix} \overset{?}{\neq} \begin{bmatrix} \cos\theta & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta\sin\theta & \cos^2\theta \end{bmatrix}$$

$$0 \neq \sin^2\theta \quad \forall\ \theta \in [0, 360°]$$

For example for $\theta = 45$...

$$0 \neq (\sin(45))^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

An example of $R_x(\theta)R_y(\theta) \neq R_y(\theta)R_x(\theta)$ is shown fully for $\theta = 90°$ on the next page

Problem 4A cont

Let R_1 be a $90°$ rotation with respect to the x-axis and let R_2 be a $90°$ rotation with respect to the y-axis.

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$R_1 R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2 R_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$\therefore$

$$R_1 R_2 \neq R_2 R_1$$

Problem 4B

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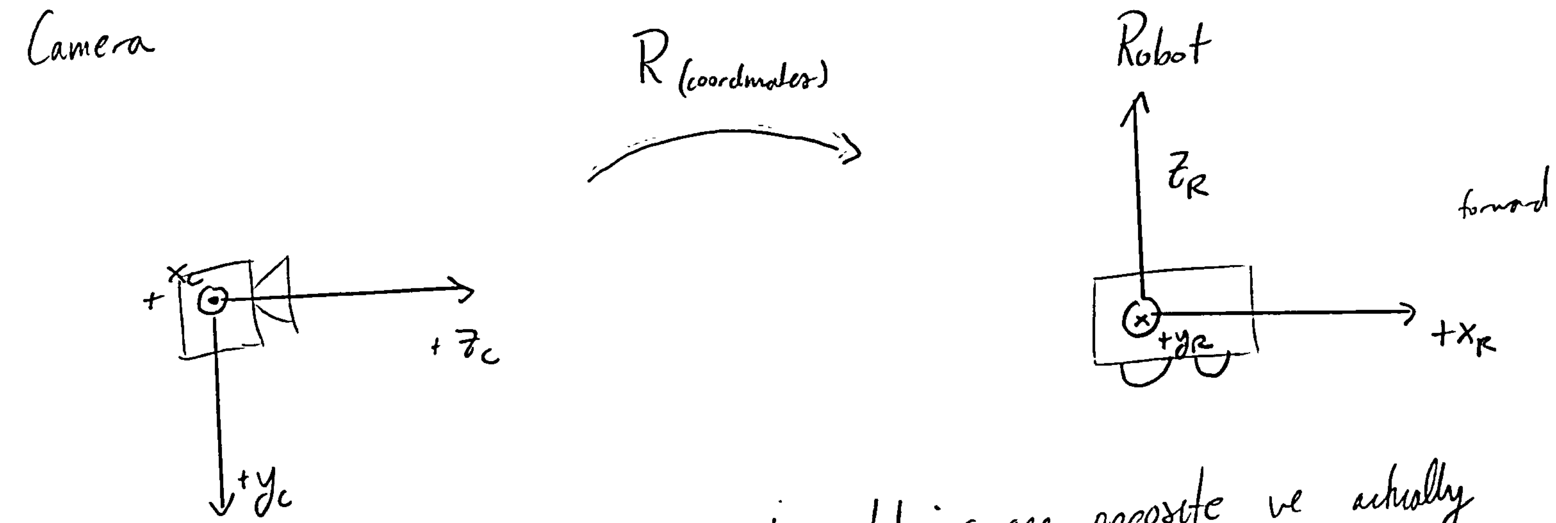

Since axis rotations (frame) and coordinate rotations are opposite we actually want to find the rotation R from robot frame to camera frame.

$$\boxed{R = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}$$

This is the rotation that will rotate robot coordinates to camera coordinates and give the result with respect to robot coordinates (which is what we desire)

This can be seen to be correct because $(x_x, x_y, x_z) = (0, -1, 0)$ gives the location of the new x-axis (x_c) with respect to the old (robot) reference frame. In other words $x_c = -y_R$. The same conclusions hold true for the new y & z -axes by looking at (y_x, y_y, y_z) and (z_x, z_y, z_z).

The same conclusions can be reached by finding the two rotation matrices that convert robot to camera frame (defining camera coordinates in the "original" robot frame). This is shown on the following page:

Problem 4B cont

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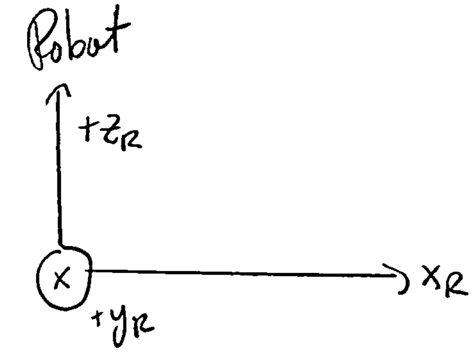
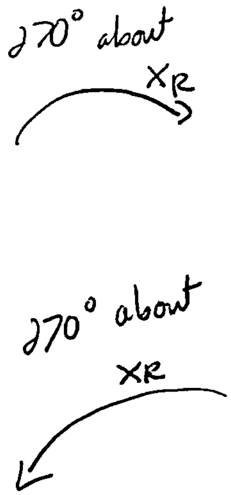
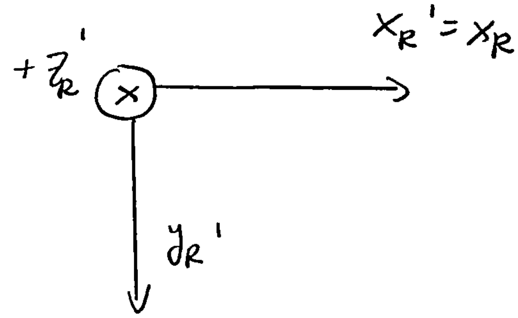
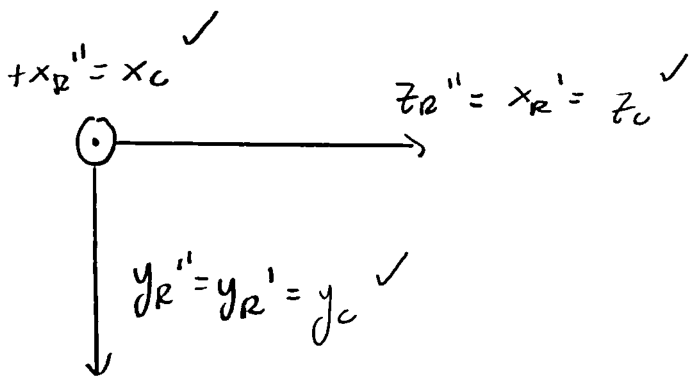








Thus the rotation is defined by $R_z(270°)\,R_x(270°)$

$$R_z(270°)\,R_x(270°) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = R'$$

Verification

$$\begin{array}{cccc} & R & \vec{x}_c & \vec{x}_R' \end{array}$$

$\vec{x}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ ✓

$\vec{x}_c = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ ✓

$\vec{x}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ✓

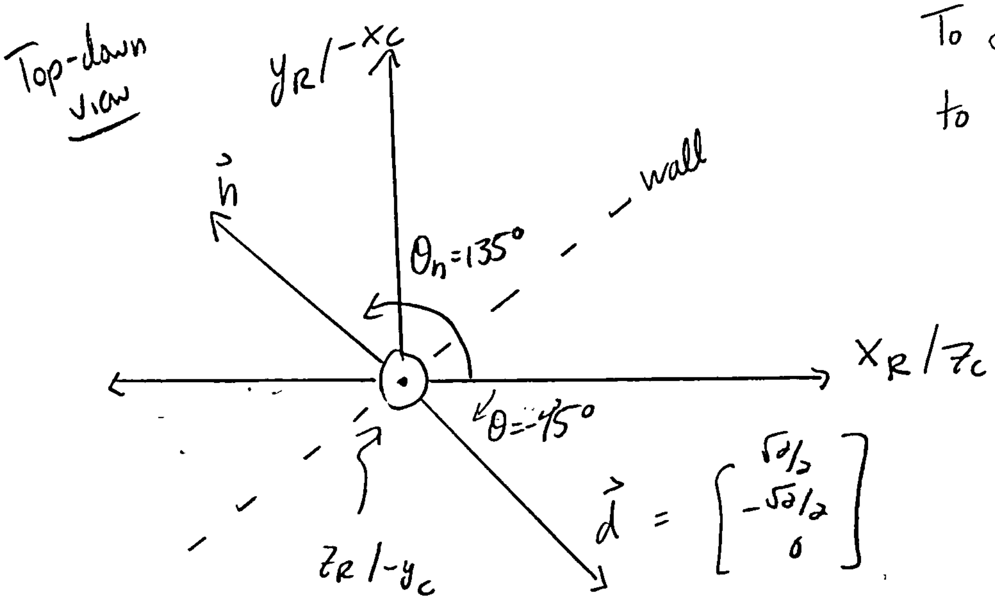
ns 20

Problem 4C

$(3cm, 0cm, 6cm)_R \rightarrow (1.5m, 1.2m, 0.06m)$

Camera pointing to wall whose outward normal direction has heading 135°

Handle direction first. Want "forward" pointing towards wall



$\vec{d} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix}$

To face the wall the robot has to turn −45° if the x_R / y_R plane ccw

$R_R = R_{z_R}(-45°)$

forward in the camera's frame

$\vec{d} = R_R R_C \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

R_C from 4B $= \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \frac{\sqrt{2}}{2} & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\sqrt{2}/2 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\sqrt{2}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix} \checkmark$$

Problem 4c cont

RAS 20

Now handle translation

Let $\vec{x}_i$ be the initial coordinates $(0.03m, 0m, 0.06m)$
and let $\vec{x}_f$ be the final coordinates $(1.5m, 1.2m, 0.06m)$

$$\vec{x}_f = T_R \vec{x}_i = R_R \vec{x}_i + \vec{t}$$

$$\begin{bmatrix} 1.5 \\ 1.2 \\ 0.06 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.03 \\ 0 \\ 0.06 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$\begin{bmatrix} 1.5 \\ 1.2 \\ 0.06 \end{bmatrix} = \begin{bmatrix} 0.03\sqrt{2}/2 + t_x \\ -0.03\sqrt{2}/2 + t_y \\ 0.06 + t_z \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

(1) $0.03\frac{\sqrt{2}}{2} + t_x = 1.5$

$t_x = 1.5 - 0.03\frac{\sqrt{2}}{2}$

(2) $1.2 = -0.03\frac{\sqrt{2}}{2} + t_y$

$t_y = 1.2 + 0.03\frac{\sqrt{2}}{2}$

(3) $0.06 = 0.06 + t_z \quad t_z = 0$

$$\vec{x}_f = \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \vec{x}_i + \underbrace{\begin{bmatrix} 1.5 - 0.03\frac{\sqrt{2}}{2} \\ 1.2 + 0.03\sqrt{2}/2 \\ 0 \end{bmatrix}}_{\vec{t}}$$

$$T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1.5 - 0.03\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1.2 + 0.03\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Verification

① Forward direction of the robot should have $\theta = -45^\circ$ after the transformation

x "= forward"

$$\vec{x}_f = T\vec{x}_i = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1.5 - 0.03\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1.2 + 0.03\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

direction

$$\vec{x}_f = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix} \qquad \theta = \tan^{-1}\left(\frac{-\sqrt{2}/2}{\sqrt{2}/2}\right) = \tan^{-1}(-1) = -45^\circ \checkmark$$

② Coordinate $(0.03, 0, 0.06)$ should map to $(1.5, 1.2, 0.06)$

$$\vec{x}_f = \begin{bmatrix} \frac{\sqrt{2}}{2} & \sqrt{2}/2 & 0 & 1.5 - 0.03\frac{\sqrt{2}}{2} \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 1.2 + 0.03\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.03 \\ 0 \\ 0.06 \\ 1 \end{bmatrix}$$

point

$$\vec{x}_f = \begin{bmatrix} 0.03\frac{\sqrt{2}}{2} + 1.5 - 0.03\frac{\sqrt{2}}{2} \\ -0.03\frac{\sqrt{2}}{2} + 1.2 + 0.03\frac{\sqrt{2}}{2} \\ 0.06 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.2 \\ 0.06 \\ 1 \end{bmatrix}$$

point

$= \checkmark$