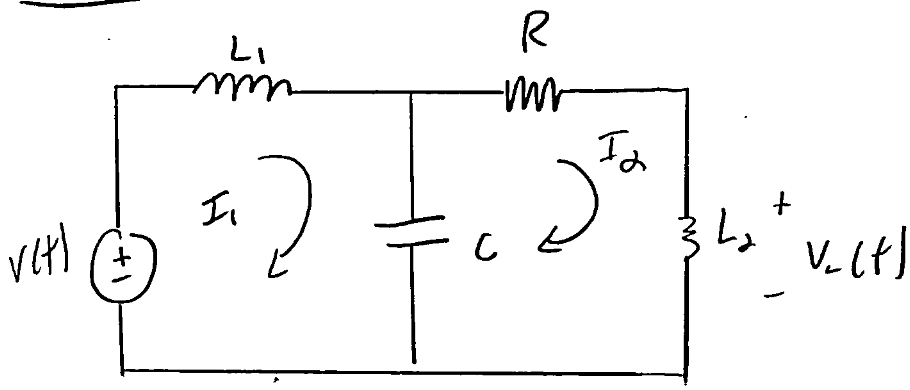
Problem 1A

HW #2

rasZO

Ryan St.Pierre
ME 344L



$$I_1 : \left(L_1 s + \frac{1}{cs}\right) I_1 - \left(\frac{1}{cs}\right) I_2 = V(s)$$

$$I_2 : \left(\frac{1}{cs} + R\right) I_2 - \frac{1}{cs} I_1 = -V_L(s)$$

Ohm's : $$V_L = I_2 L_2 s$$

From Maple

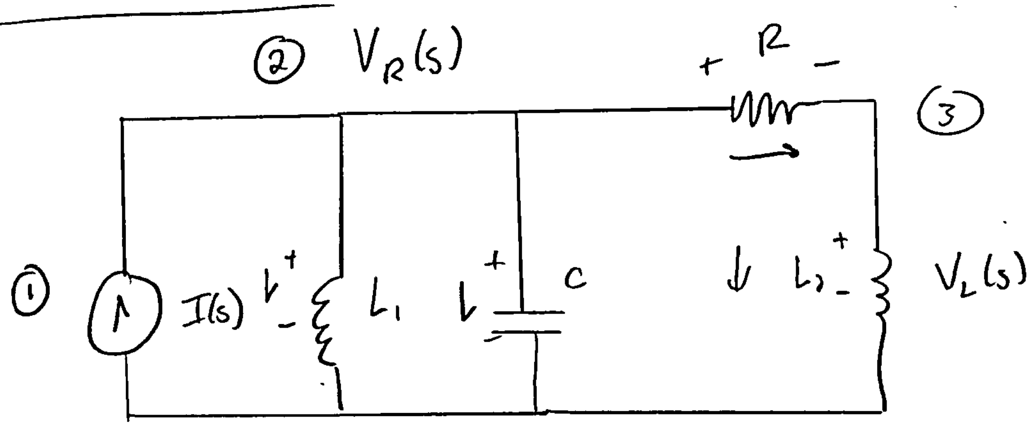
$$\boxed{\frac{V_L(s)}{V(s)} = \frac{L_2 s}{L_1 L_2 C s^3 + C L_1 R s^2 + L_1 s + L_2 s + R}}$$

For $R = 2$, $C = 1/6$, $L_1 = 2$, $L_2 = 3$

$$\boxed{\frac{V_L(s)}{V(s)} = \frac{9s}{3s^3 + 2s^2 + 15s + 6}}$$

Problem 1b

ras20



① $I_s(s) = G(s)\,V_s(s)$ where $G(s) = \dfrac{1}{L_1 s}$

$$I_s(s) = \frac{V_s(s)}{L_1 s}$$

② In / out

$$\frac{V_s}{L_1 s} = \frac{1}{L_1 s} V_R + C s V_R + \frac{1}{R}(V_R - V_L)$$

$$\frac{V_s}{L_1 s} = V_R\left(\frac{1}{L_1 s} + Cs + \frac{1}{R}\right) - \frac{1}{R} V_L$$

③ $$\frac{V_R - V_L}{R} = \frac{V_L}{L_2 s}$$

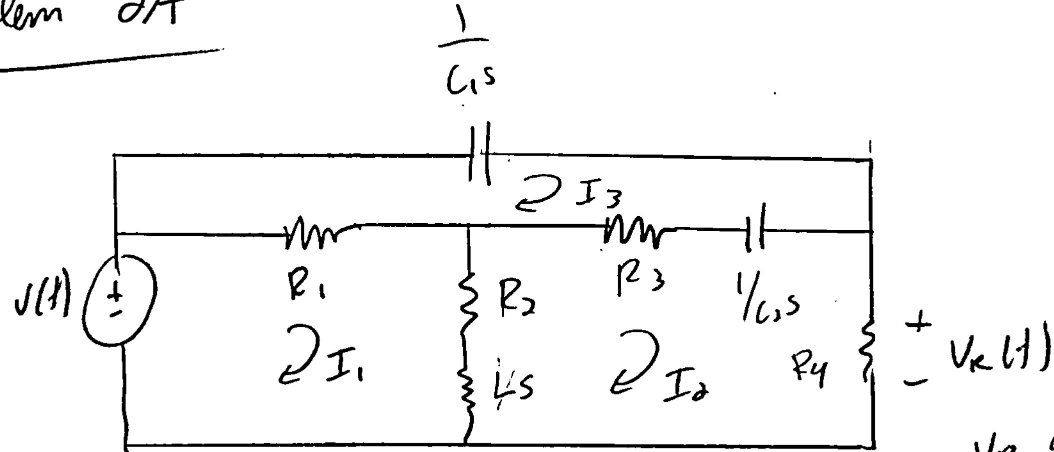
Maple produces the same result as 1a, that is ...

$$\boxed{\frac{V_L(s)}{V(s)} = \frac{L_2 s}{L_1 L_2 C s^3 + C L_1 R s^2 + (L_1 + L_2) s + R}} \Rightarrow \boxed{\frac{V_L(s)}{V(s)} = \frac{3s}{s^3 + \frac{2}{3}s^2 + 5s + 2}}$$

For $R = 2, C = 1/6, L_1 = 2, L_2 = 3$

Problem 2A

as20



" opposes passive directn "

① ignore R_4 because V_R accounts for this

I_1: $$(R_1 + R_2 + Ls)I_1 - (R_2 + Ls)I_2 - R_1 I_3 = V(s)$$

I_2: $$\left(R_3 + \frac{1}{C_2 s} + Ls + R_2\right)I_2 - (R_2 + Ls)I_1 - \left(R_3 + \frac{1}{C_2 s}\right)I_3 = -V_R$$

I_3: $$\left(\frac{1}{C_1 s} + R_1 + R_3\right)I_3 - R_1 I_1 - \left(R_3 + \frac{1}{C_2 s}\right)I_2 = 0$$

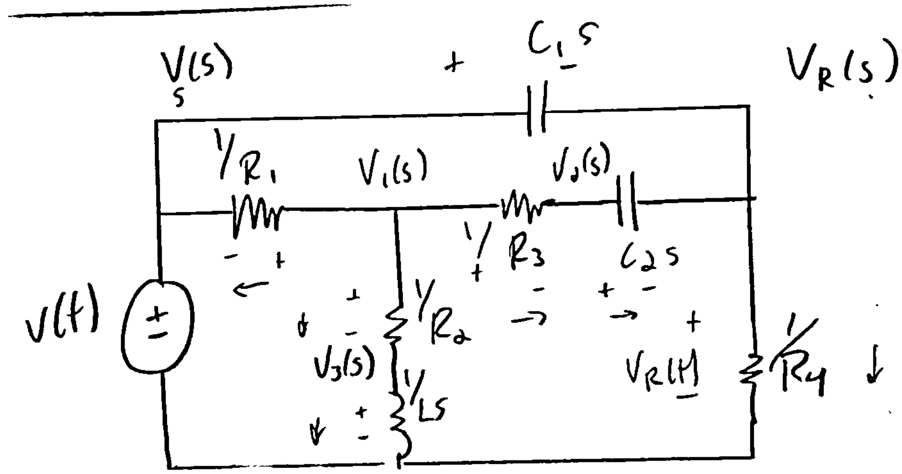
Solving into Maple with given C_1, C_2, R_1 .. etc values:

$$\frac{V_R(s)}{V(s)} = \frac{9s^3 + 21s^2 + 12s}{9s^3 + 27s^2 + 25s + 6}$$

ns 20

Problem 2b

V_1, V_2, V_3, V_R unknowns
→ need 4 equations



Use

$$\begin{bmatrix}\text{sum of impedances} \\ \text{at } n\end{bmatrix} - \sum_{n_i} \begin{matrix}\text{impendance} \\ \text{shared between} \\ n \text{ \& neighbor } n_i\end{matrix} = \begin{bmatrix}\text{current applied} \\ \text{at } n\end{bmatrix}$$

V_1 (1) $$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)V_1 - \frac{1}{R_1}V_s - \frac{1}{R_3}V_2 - \frac{1}{R_2}V_3 = 0$$

V_2 (2) $$\left(\frac{1}{R_3}+C_2 s\right)V_2 - \frac{1}{R_3}V_1 - C_2 s V_R = 0$$

V_3 (3) $$\left(\frac{1}{R_2}+\frac{1}{Ls}\right)V_3 - \frac{1}{R_2}V_1 = 0$$

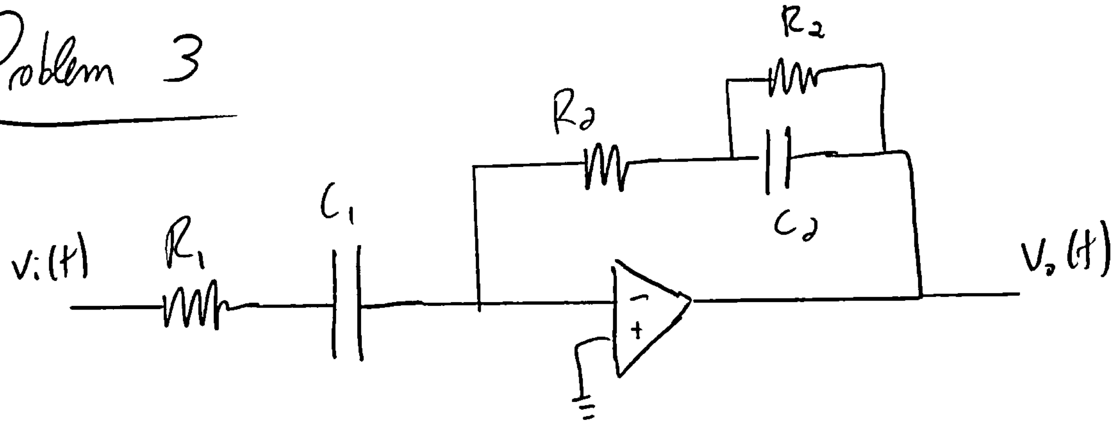
V_R (4) $$\left(C_1 s + C_2 s + \frac{1}{R_4}\right)V_R - C_1 s V_s - C_2 s V_2 = 0$$

Using Maple to solve this, and plugging in the correct values, for $C_1, C_2, R_1 \ldots$ etc results in the following transfer function

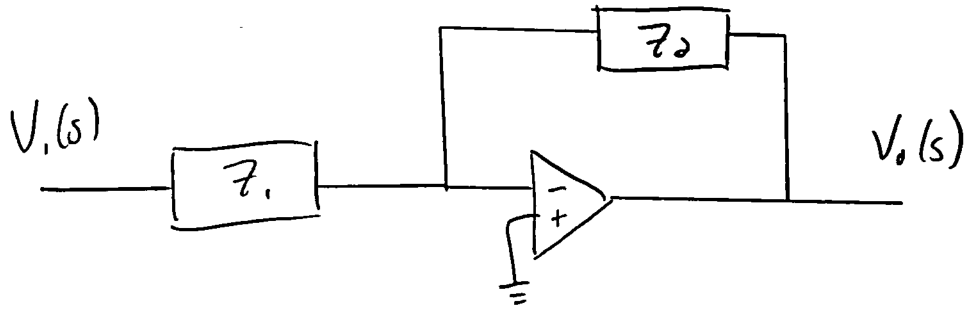
$$\boxed{\frac{V_R(s)}{V(s)} = \frac{9s^3+21s^2+12s}{9s^3+27s^2+25s+6}}$$

Problem 3

rw70



This can be simplified to the equivalent inverting amplifier



$$Z_1 = R_1(s) + C_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2 = R_2(s) + R_2(s) \| C_2(s) = R_2 + \frac{1}{C_2 s + \frac{1}{R_2}} = R_2 + \frac{R_2}{R_2 C_2 s + 1}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$ [given for inverting op amp]

$$= -\frac{R_2 + \frac{R_2}{R_2 C_2 s + 1}}{R_1 + \frac{1}{C_1 s}} \cdot \frac{C_1 s (R_2 C_2 s + 1)}{C_1 s (R_2 C_2 s + 1)}$$

$$= -\frac{R_2 C_1 s (R_2 C_2 s + 1) + R_2 C_1 s}{R_1 C_1 s (R_2 C_2 s + 1) + R_2 C_2 s + 1}$$

ras70

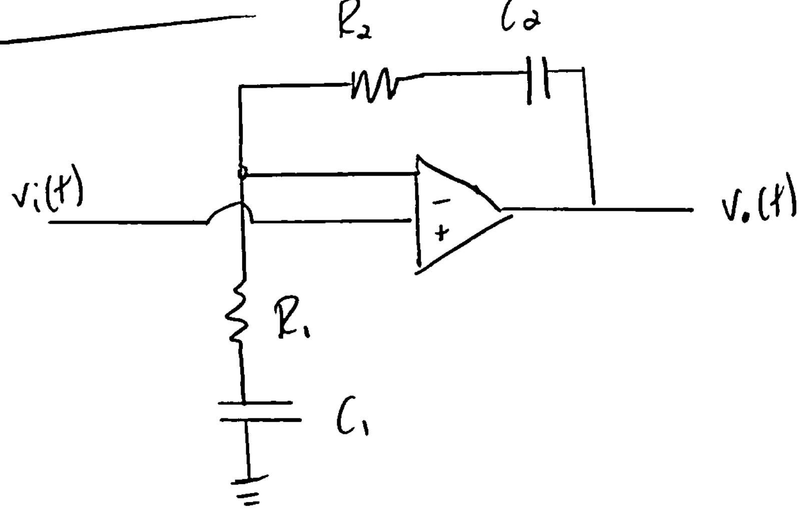
Problem 3 Cont

$$\frac{V_o(s)}{V_i(s)} = - \frac{R_2 [2C_1 s + R_2 C_1 C_2 s^2]}{R_1 R_2 C_1 C_2 s^2 + R_1 C_1 s + R_2 C_2 s + 1}$$

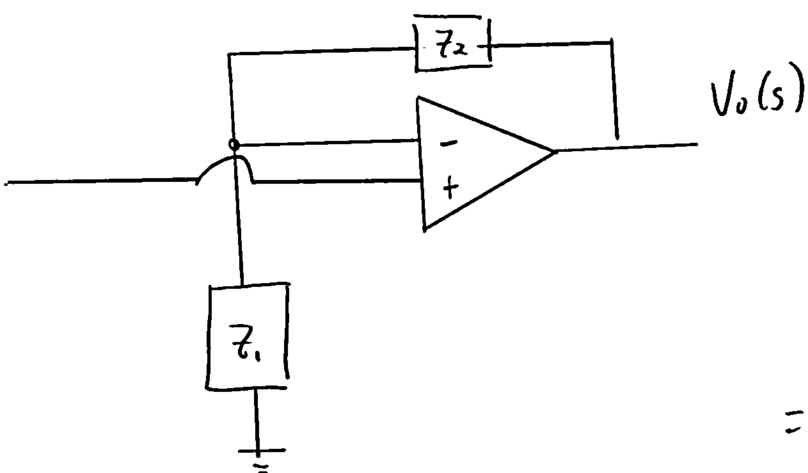
$$\frac{V_o(s)}{V_i(s)} = - R_2 C_1 s \cdot \frac{R_2 C_2 s + 2}{R_1 R_2 C_1 C_2 s^2 + R_1 C_1 s + R_2 C_2 s + 1}$$

ras70

Problem 4



This can be converted to the equivalent non-inverting amplifier



$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

$$= \frac{C_1C_2 s}{C_1C_2 s}\,\frac{R_1 + \frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}}{R_1 + \frac{1}{C_1 s}}$$

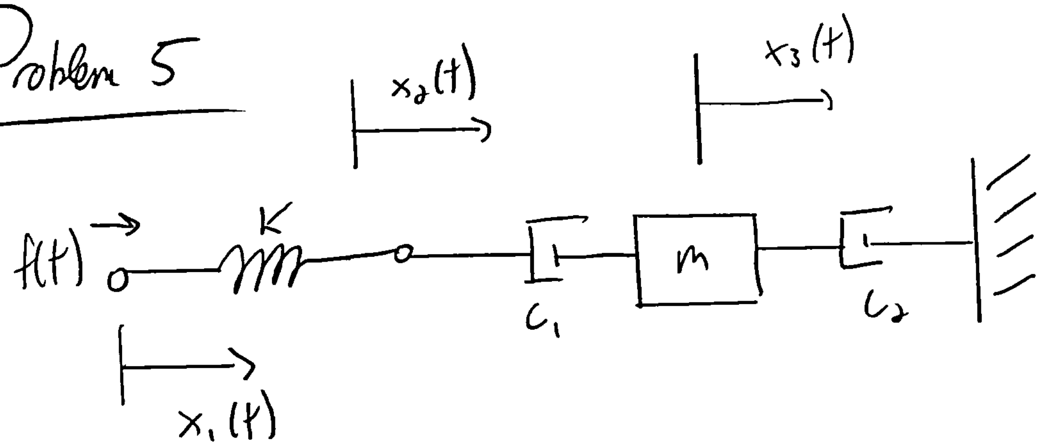
$$Z_1(s) = R_1(s) + C_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2(s) = R_2(s) + C_2(s) = R_2 + \frac{1}{C_2 s}$$

$$= \frac{R_1C_1C_2 s + C_2 + R_2C_1C_2 s + C_1}{R_1C_1C_2 s + C_2}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{C_1C_2 s(R_1+R_2) + C_1 + C_2}{C_2(R_1C_1 s + 1)}}$$

Problem 5



As suggested I will place a zero mass at x_2. However, I additionally need a zero mass at the coordinate x_1 (which I defined). I believe I need this because the force $f(t)$ acts on the spring, not directly at $x_2(t)$.

① $\begin{bmatrix} \text{Impedances at} \\ x_1 \end{bmatrix} X_1(s) - \begin{bmatrix} \text{impedance between} \\ x_1, x_2 \end{bmatrix} X_2(s) - \begin{bmatrix} \text{Impedance} \\ x_1, x_3 \end{bmatrix} X_3(s) = F(s)$

$$K X_1(s) - K X_2(s) = F(s)$$

② $-\begin{bmatrix} \text{Impedance} \\ x_1 / x_2 \end{bmatrix} X_1(s) + \begin{bmatrix} \text{Impedance at} \\ x_2 \end{bmatrix} X_2(s) - \begin{bmatrix} \text{Impedance} \\ x_2, x_3 \end{bmatrix} X_3(s) = 0$

$$-K X_1(s) + (K + c_1 s) X_2(s) - (c_1 s) X_3(s) = 0$$

③ $-\begin{bmatrix} \text{Impedance} \\ x_1 / x_3 \end{bmatrix} X_1(s) - \begin{bmatrix} \text{Impedance} \\ x_2 / x_3 \end{bmatrix} X_2(s) + \begin{bmatrix} \text{Impedance} \\ x_3 \end{bmatrix} X_3(s) = 0$

$$-(c_1 s) X_2(s) + (c_1 s + c_2 s + M s^2) X_3(s) = 0$$

$\rightarrow$

Problem 5 cont   ns 20

$$\begin{bmatrix} K & -K & 0 \\ -K & K+c_1 s & -c_1 s \\ 0 & -c_1 s & c_1 s + c_2 s + Ms^2 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \\ 0 \end{bmatrix}$$

Let $A = K$, $B = K + c_1 s$ $\quad C = -c_1 s \quad D = (c_1 + c_2)s + Ms^2$

$$\begin{bmatrix} A & -A & 0 \\ -A & B & C \\ 0 & C & D \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} A & -A & 0 \\ -A & B & C \\ 0 & C & D \end{bmatrix}^{-1} \begin{bmatrix} F(s) \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A & -A & 0 \\ -A & B & C \\ 0 & C & D \end{bmatrix}^{-1} \quad ?$$

Minors

$$\begin{bmatrix} BD - C^2 & -AD & -AC \\ -AD & AD & AC \\ -AC & AC & AB - A^2 \end{bmatrix} \rightarrow$$

Cofactors

$$\begin{bmatrix} BD - C^2 & AD & -AC \\ AD & AD & -AC \\ -AC & -AC & AB - A^2 \end{bmatrix}$$

Ajoint

$$\begin{bmatrix} BD - C^2 & AD & -AC \\ AD & AD & -AC \\ -AC & -AC & AB - A^2 \end{bmatrix} \longrightarrow$$

Problem 5 cont

ms20

$$\begin{bmatrix} A & -A & 0 \\ -A & B & C \\ 0 & C & D \end{bmatrix}^{-1} = \frac{1}{\Delta} \text{ adjoint}$$

$$F(s) \longrightarrow \boxed{\frac{K}{\Delta}} \longrightarrow X_2(s)$$

$$\Delta = A \begin{vmatrix} B & C \\ C & D \end{vmatrix} + A \begin{vmatrix} -A & C \\ 0 & D \end{vmatrix} + 0$$

$$= A(BD - C^2) + A(-AD)$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} BD-C^2 & AD & -AC \\ AD & AD & -AC \\ -AC & -AC & AB-A^2 \end{bmatrix} \begin{bmatrix} F(s) \\ 0 \\ 0 \end{bmatrix}$$

$$X_2(s) = \frac{1}{\Delta}(AD)\,F(s)$$

$$\frac{X_2(s)}{F(s)} = \frac{1}{\Delta}(AD) = \frac{AD}{A(BD-C^2) - A^2 D} = \frac{D}{BD - C^2 - AD}$$

$$= \frac{(c_1+c_2)s + Ms^2}{(K+c_1 s)(c_1+c_2)s + c_1^2 s^2 - K[(c_1+c_2)s + Ms^2]}$$

$$= \frac{c_1+c_2}{(K+c_1 s)(c_1+c_2) + c_1^2 s - K[(c_1+c_2) + Ms]}$$

$$= \frac{c_1 + c_2 + MKs}{K(c_1+c_2) + c_1 s(c_1+c_2) + c_1^2 s - K(c_1+c_2) - MKs}$$

Problem 5 cont

$$\frac{X_2(s)}{F(s)} = \frac{C_1 + C_2 + MKs}{C_1 s (C_1 + C_2) + C_1^2 s - MKs}$$

$$\frac{X_2(s)}{F(s)} = \frac{C_1 + C_2 + MKs}{s\left[C_1^2 + C_1 C_2 + C_1^2 - MK\right]} = \frac{C_1 + C_2 + MKs}{s\left[2C_1^2 + C_1 C_2 - MK\right]}$$

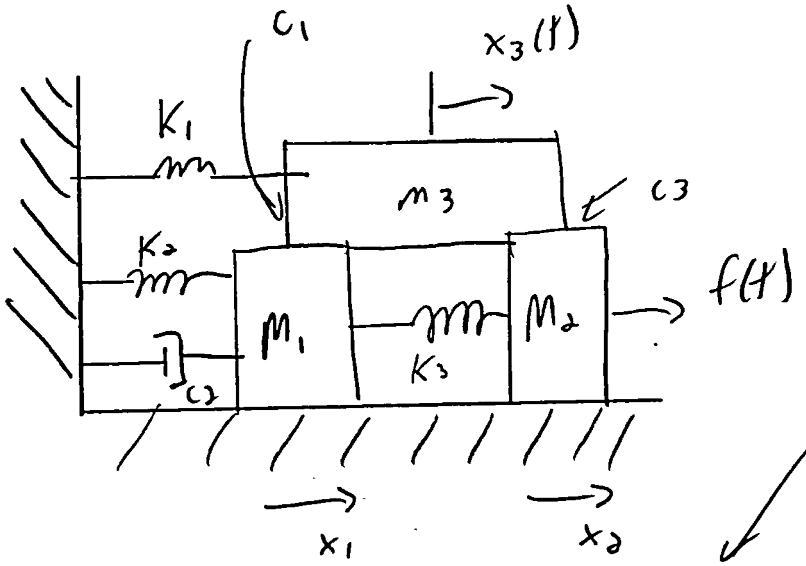
$$\frac{X_2(s)}{F(s)} = \frac{C_1 + C_2 + MKs}{s(2C_1^2 + C_1 C_2) - MK}$$

I have simplified this out fully. A cleaner representation is

$$\frac{X_2(s)}{F(s)} = \frac{K}{\Delta}$$

where $\Delta = \begin{vmatrix} K & -K & 0 \\ -K & K + C_1 s & -C_1 s \\ 0 & -C_1 s & C_1 s + C_2 s + Ms^2 \end{vmatrix}$

Problem 6



3 degrees of freedom

assume function of s

$$M_1: \left[M_1 s^2 + (C_1 + C_2)s + K_2 + K_3\right] X_1 - K_3 X_2 - C_1 X_3 = 0$$

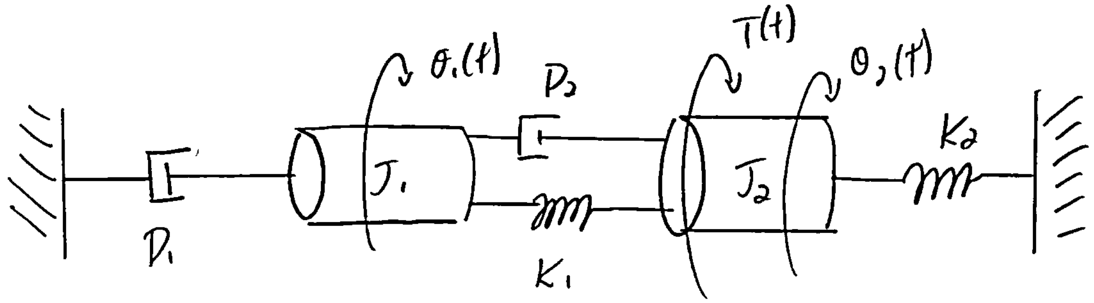
$$M_2: \left[M_2 s^2 + C_3 s + K_3\right] X_2 - K_3 X_1 - C_3 X_3 = F(s)$$

$$M_3 \quad \left[M_3 s^2 + (C_1 + C_3)s + K_1\right] X_3 - C_1 X_1 - C_3 X_2 = 0$$

I found the transfer function for this problem, as produced by Maple to be long. Thus, I have excluded it here. Please refer to my Maple scripts for the answer.

Problem 7

ans 70



2 degrees of freedom

As done in the book the equations of motion can be done by inspection in the following manner

(1) $\begin{bmatrix} \text{sum of impedances} \\ \text{connected to} \\ \text{motion at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of impedances} \\ \text{between } \theta_1 \text{ \& } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Torque} \\ \text{applied} \\ \text{to } \theta_1 \end{bmatrix}$

$$(J_1 s^2 + D_1 s + D_2 s + K_1)\theta_1(s) - (D_2 s + K_1)\theta_2(s) = 0$$

(2) $-\begin{bmatrix} \text{sum of impedences} \\ \text{between } \theta_1 \text{ \& } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{sum of impedances} \\ \text{connected to motion of } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Torque applied} \\ \text{at } \theta_2 \end{bmatrix}$

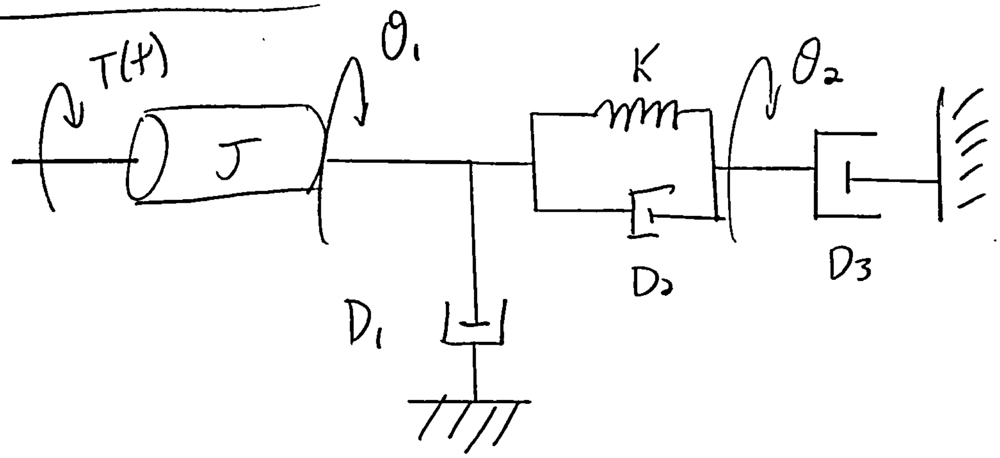
$$-(D_2 s + K_1)\theta_2(s) + (J_2 s^2 + D_2 s + K_1 + k_2)\theta_2(s) = T(s)$$

2 equations of motion are:

$$\boxed{\begin{aligned} &(J_1 s^2 + (D_1 + D_2)s + K_1)\theta_1(s) - (D_2 s + K_1)\theta_2(s) = 0 \\ &-(D_2 s + K_1)\theta_2(s) + (J_2 s^2 + D_2 s + K_1 + K_2)\theta_2(s) = T(s) \end{aligned}}$$

Problem 8

ras20



2 degrees of freedom - θ_1, θ_2 only coordinates needed to describe system

Using inspection method described in Problem 7

① $(Js^2 + D_1 s + D_2 s + K)\,\theta_1(s) - (D_2 s + K)\,\theta_2(s) = T(s)$

② $-(D_2 s + K)\,\theta_1(s) + (D_2 s + D_3 s + K)\,\theta_2(s) = 0$

From the pattern observed in the examples in the book I know the solution should be

$$\frac{\theta_2(s)}{T(s)} = \frac{D_2 s + K}{\Delta}$$

impedances between T & θ_2

where $\Delta = \begin{vmatrix} (Js^2 + (D_1 + D_2)s + K) & -(D_2 s + K) \\ -(D_2 s + K) & ((D_2 + D_3)s + K) \end{vmatrix}$

I'll solve the system to show why this is the case:

Let $A = (Js^2 + D_1 s + D_2 s + K)$

$B = -(D_2 s + K)$

$C = (D_2 s + D_3 s + K)$

The system can be represented by

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix} \rightarrow$$

Problem 8 cont

ras570

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1} \cdot \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} C & -B \\ -B & A \end{bmatrix} \begin{bmatrix} T(s) \\ 0 \end{bmatrix} \quad \text{where } \Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

$$\begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} C\,T(s) \\ -B\,T(s) \end{bmatrix}$$

$$\theta_2(s) = \frac{-B\,T(s)}{\Delta}$$

$$\frac{\theta_2(s)}{T(s)} = \frac{-B}{\Delta} = \frac{--(D_2 s + k)}{\Delta} = \frac{D_2 s + K}{\Delta}$$

$$\boxed{\frac{\theta_2(s)}{T(s)} = \frac{D_2 s + K}{\Delta}}$$

$$\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = (Js^2 + (D_1 + D_2)s + k)((D_2 + D_3)s + k)$$
$$- (D_2 s + k)^2$$

squared so I can drop the negative