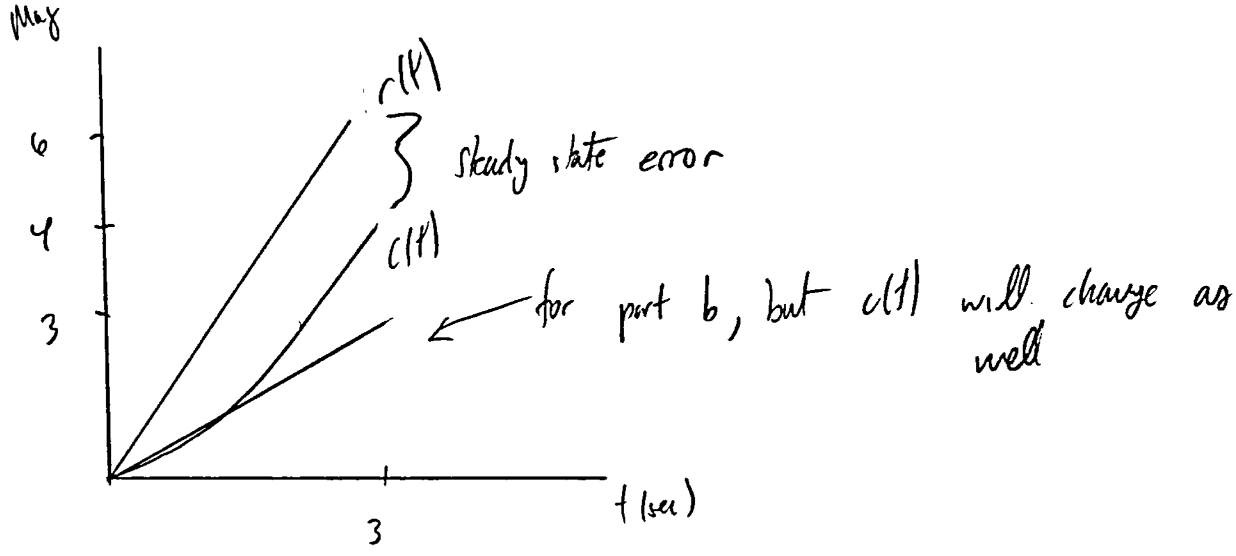
HW #6 Problem 1

Ryan St Pierre rs20



a) r(t) & c(t) are moving parallel with 2 units of separation

Steady state error = 2

b) Let $r_2(t)$ be $tu(t)$ and $r(t)$ be the input from problem a.

$r(t) = 2tu(t) = 2r_2(t)$

$c_2(t) = H\,r_2(t)$

$= H\left(\frac{1}{2}\right)r(t)$

$= \frac{1}{2}c(t)$

$e_2(t) = r_2(t) - c_2(t)$

$= \frac{1}{2}r(t) - \frac{1}{2}c(t)$

$= \frac{1}{2}\underbrace{(r(t) - c(t))}_{2}$

Steady state error = 1

$e_2(t) = 1$

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HW #5
Problem 2

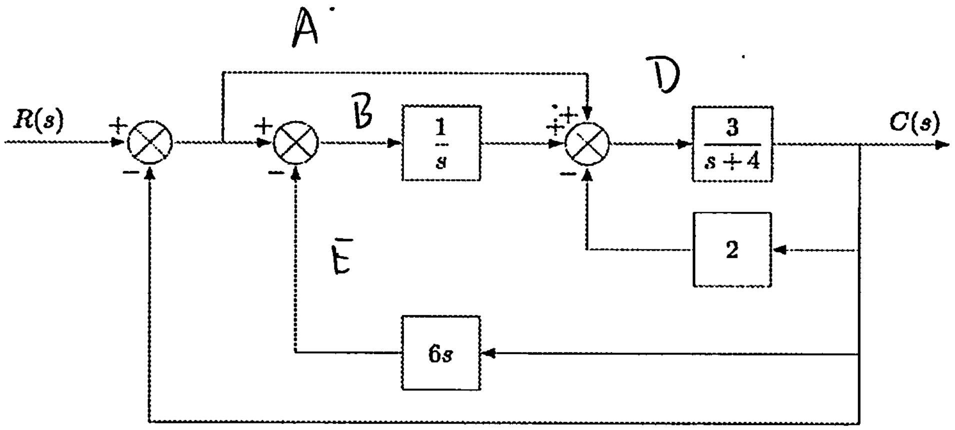
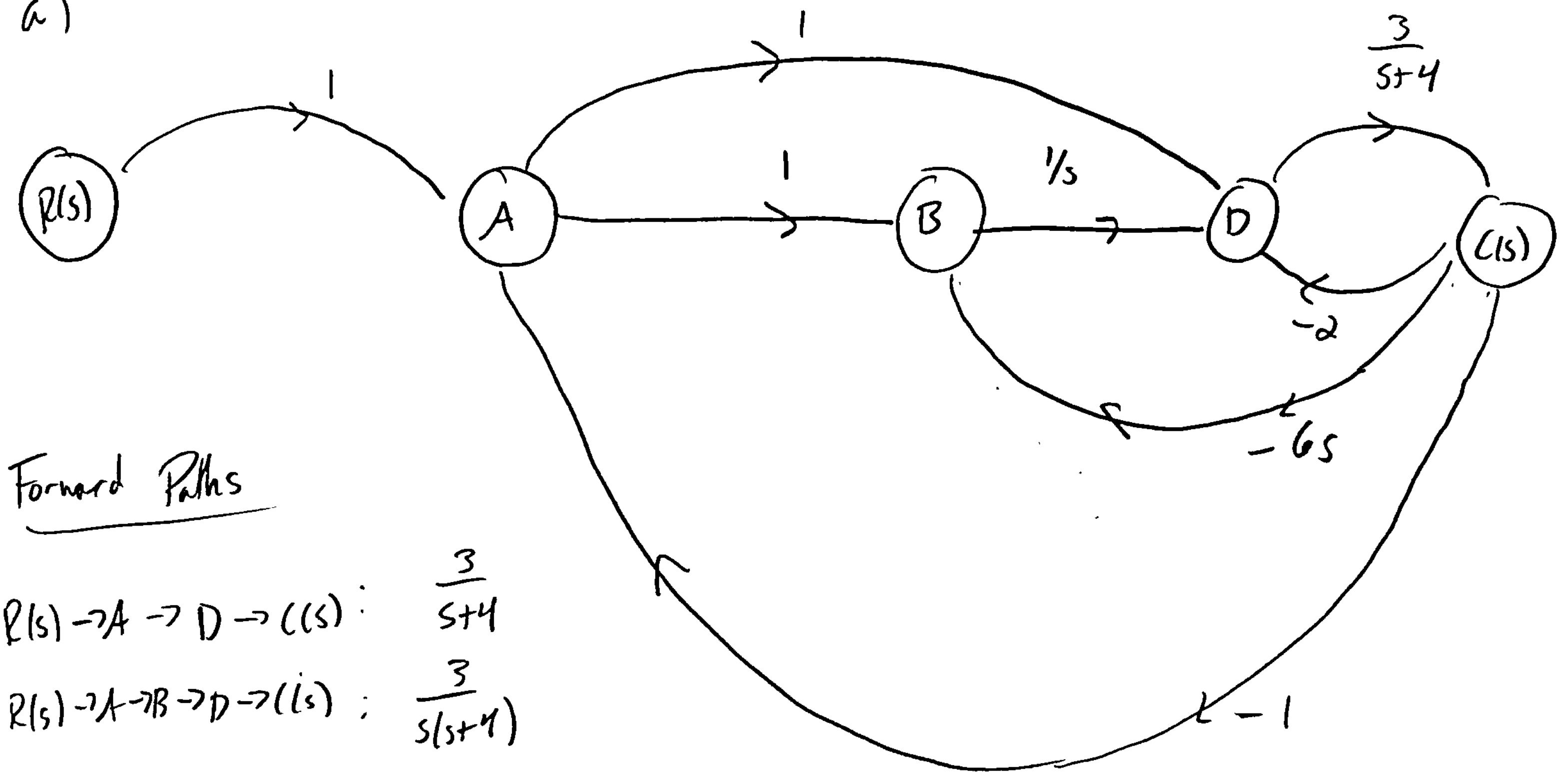


Figure 2: Block diagram for problem 2

a)



Forward Paths

$R(s) \to A \to D \to C(s)$: $\frac{3}{s+4}$

$R(s) \to A \to B \to D \to C(s)$: $\frac{3}{s(s+4)}$

Loop gains

$D \to C(s) \to D$: $\frac{-6}{s+4}$

$A \to D \to C(s) \to A$: $\frac{-3}{s+4}$

$B \to D \to C(s) \to B$: $\frac{-18}{s+4}$

Non-touching : All touching (all share D)

$A \to B \to D \to C(s) \to A$: $\frac{-3}{s(s+4)}$

Problem 2 cont

Also, all loop gains touch forward path.

Both forward paths contain D, all loop gains have D

Therefore $\Delta_K = 1$

$$\Delta = 1 - \sum \text{loop gains}$$

$$= 1 + \frac{6}{s+4} + \frac{18}{s+4} + \frac{3}{s(s+4)} + \frac{3}{s+4} \overset{\text{maple}}{=} 1 + \frac{27}{s+4} + \frac{3}{s(s+4)} = \frac{s^2+31s+3}{s(s+4)}$$

$\underline{K=1}$

$T_1 = \frac{3}{s+4}$ $\quad \Delta_1 = 1$

$$T(s) = \frac{\frac{3}{s+4}}{\Delta} + \frac{\frac{3}{s(s+4)}}{\Delta} \overset{\text{Maple}}{=} \frac{3(s+1)}{s^2+31s+3}$$

$\underline{K=2}$

$T_2 = \frac{3}{s(s+4)}$ $\quad \Delta_2 = 1$

$$\boxed{T(s) = \frac{3(s+1)}{s^2+31s+3}}$$

$$R(s) \longrightarrow \boxed{\frac{3(s+1)}{s^2+31s+3}} \longrightarrow C(s)$$

Step 1 → Check for stability

$$\begin{array}{c|cc} s^2 & 1 & 3 \\ s^1 & 31 & 0 \\ s^0 & 3 & 0 \end{array} \quad ✓$$

b) $E(s) = R(s) - C(s) = R(s) - T(s)R(s) = R(s)(1 - T(s))$

$$e_s(\infty) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\,\frac{1}{s}(1 - T(s)) = \lim_{s\to 0}(1 - T(s)) = 1 - \lim_{s\to 0} T(s)$$

$$\lim_{s\to 0} T(s) = \frac{3}{3} = 1 \qquad e_s(\infty) = 0 \qquad e_{ss}(\infty) = 10(0) = 0$$

Problem 2b cont

Input $10u(t)$ $\quad e_{ss}(\infty) = 0$ for $r(t) = 10u(t)$

unit ramp

incorporate gain below

$$e_r(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1}{s}(1 - T(s))$$

$$= \lim_{s \to 0} \frac{1}{s}\left(\frac{s^2 + 31s + 3}{s^2 + 31s + 3} - \frac{3s + 3}{s^2 + 31s + 3}\right)$$

$$= \lim_{s \to 0} \frac{1}{s} \frac{(s^2 + 28s)}{s^2 + 31s + 3} = \lim_{s \to 0} \frac{s + 28}{s^2 + 31s + 3} = \frac{28}{3}$$

Input $10tu(t)$ $\quad e_{ss}(\infty) = 10e_r(\infty) = \frac{10(28)}{3} = \frac{280}{3}$

$e_{ss}(\infty) = \frac{280}{3}$ for $r(t) = 10tu(t)$

Input $10u(t)t^2$ $\quad R(s) = \frac{10 \cdot 2}{s^3} = \frac{20}{s^3}$

finite

$$e_{ss} = \lim_{s \to 0} \frac{20}{s^2}\left[\frac{s^2 + 28s}{s^2 + 31s + 3}\right] = \lim_{s \to 0} \frac{20}{s}\left[\frac{s + 28}{s^2 + 31s + 3}\right] = \infty$$

$e_{ss}(\infty)$ for $r(t) = 10u(t)t^2 = \infty$

Problem 3

$$G(s) = \frac{K(s+7)}{s(s+5)(s+8)(s+12)}$$

Find $T(s)$

$H(s) = 1$

$$T(s) = \frac{G_N}{G_D + H(s)G_N}$$

$$= \frac{K(s+7)}{s(s+5)(s+8)(s+12) + K(s+7)} \xrightarrow[\text{maple. SCS}]{} = \frac{K(s+7)}{s^4+25s^3+196s^2+(K+480)s+7K}$$

Step 1: Analyze Stability

$$s^4 + 25s^3 + 196s^2 + (K+480)s + 7K$$

s^4	1	196	$7K$	
s^3	25	$K+480$	0	
s^2	① $\frac{884}{5} - \frac{K}{25}$	$7K$	0	
s^1	② $\frac{K^2+435K-2121600}{K-4420}$	0	0	
s^0	③ $7K$	0	0	

① $\frac{884}{5} - \frac{K}{25} > 0$

$K < 4420$

③ $K > 0$

② Denominator must be negative since $K < 4420$

Need numerator negative

$K^2 + 435K - 2121600 < 0$

$K = -1690.22$ ← K must be >0

$K = 1255.22$

For stability

$$K \in (0, 1255.22)$$

Problem 3a

Input = $1/10\, t$ ✓ ramp

Let r_y be $1/10\,t$ & c_y be output w/ $r(t) = r_y$

Let $r_0 = t$ & c_0 be output w/ $r(t) = r_0$

$$r_y = \frac{1}{10} r_0$$

$$c_y = T(s)\, r_y = \frac{1}{10} T(s)\, r_0 = \frac{1}{10} c_0$$

$$e_y = \frac{1}{10}(c_0 - r_0) = \frac{1}{10} e_r = 0.01 \Rightarrow e_r = 0.1$$

Desire $e_r(\infty) = 0.1$

$$e_r(\infty) = \lim_{s \to 0} \frac{1}{G(s)} = \frac{1}{\frac{7K}{480}} = \frac{480}{7K} = 0.1$$

open loop T.F

$$K = \frac{4800}{7}$$

$K \in (0, 1255.22)$ ✓

b) This is a type 1 system

$$K_v = \lim_{s \to 0} s\, G(s) = \lim_{s \to 0} \frac{7K}{480} = 10$$

$$K_v = 10$$

Error is $\frac{1}{K_v} = \frac{1}{10}$ ✓

↳ for standard ramp

Problem 3c

When the stability was checked it was found that $K \in (0, 1255.22)$ for stability.

Also $e_r(\infty) \propto \frac{1}{K}$. Thus, to minimize error we need to maximize K while maintaining stability $\Rightarrow K = 1255.22$.

$$e_r(\infty) = \frac{480}{7K} = 0.054692$$

$$e_y = \frac{1}{10} e_r = 0.0054692$$

Minimum error ≈ 0.0055

Problem 4

a) System Type = Type 1 because the input is a ramp & the position error is finite.

b) Assume $G(s)$ has no zeros

$G(s) = \frac{K}{s(s+a)}$ only has one pole so $T(s)$ is second order

integrator so type 1

$T(s) = \frac{K}{s(s+a)+K} = \frac{K}{s^2+as+K}$

input velocity $= \frac{dr(t)}{dt} = \frac{d}{dt}(u(t)t)$

$= u(t)$

$\mathcal{L}[u(t)] = 1$

$w_n = \sqrt{K} = 10 \Rightarrow K = 100$

$G(s) = \frac{100}{s(s+a)}$

$e_r(\infty) = \lim_{s\to 0} \frac{1}{sG(s)} = 0.01(1)$

$\lim_{s\to 0} \frac{1}{G(s)} = \frac{a}{K} = 0.01 \Rightarrow a = 0.01(K) = 1$

Test $T(s)$ for stability

$P = s^2 + s + 100$

$G(s) = \frac{100}{s(s+1)}$

s^2	1	100
s^1	1	0
s^0	100	0

✓

Problem 4c

$$T(s) = \frac{100}{s^2+s+100}$$

$$\boxed{\xi = \frac{1}{20}}$$

$$1 = 2\xi\omega_n$$

$$\xi = \frac{1}{2\omega_n} = \frac{1}{20}$$

Problem 5

$$G(s) = \frac{K(s+\alpha)}{s(s+\beta)}$$

$$\boxed{\beta = 1/5 \quad K = 9/5 \quad \alpha = \frac{10}{5}}$$

$$e_r(\infty) = \frac{1}{10} \Rightarrow K_v = 10$$

$$\lim_{s \to 0} sG(s) = K_v = 10$$

③ $\frac{\alpha K}{\beta} = 10 \Rightarrow \frac{2}{\beta} = 10 \quad \boxed{\beta = 1/5}$

$$T(s) = \frac{K(s+\alpha)}{s(s+\beta) + K(s+\alpha)} = \frac{K(s+\alpha)}{s^2 + (\beta+K)s + K\alpha}$$

$$\frac{-(\beta+K) \pm \sqrt{(\beta+K)^2 - 4K\alpha}}{2} = -1 \pm ji$$

①
$$-\frac{(\beta+K)}{2} = -1$$

$$\beta + K = 2$$

$$\frac{1}{5} + K = 2$$

$$\boxed{K = 9/5}$$

sub in for $\beta+K$

② $\frac{\sqrt{(\beta+K^2) - 4K\alpha}}{2} = \pm j$

$$\sqrt{(\beta+K)^2 - 4K\alpha} = \pm 2j$$

$$(\beta+K)^2 - 4K\alpha = -4$$

$$4 - 4K\alpha = -4$$

$$-4K\alpha = -8$$

$$K\alpha = 2$$

$$\frac{9}{5}\alpha = 2 \Rightarrow \boxed{\alpha = \frac{10}{9}}$$

Check for stability

$$T(s) = \frac{G_N}{G_D + H(s) G_N} = \frac{K(s+\alpha)}{s(s+\beta) + s(s+\alpha)}$$

$$\hookrightarrow 2s^2 + (\alpha+\beta)s$$

$$2s^2 + 1.31s$$

s^2	1	6
s^1	1.31	0
s^0	1.31	0

stable ✓

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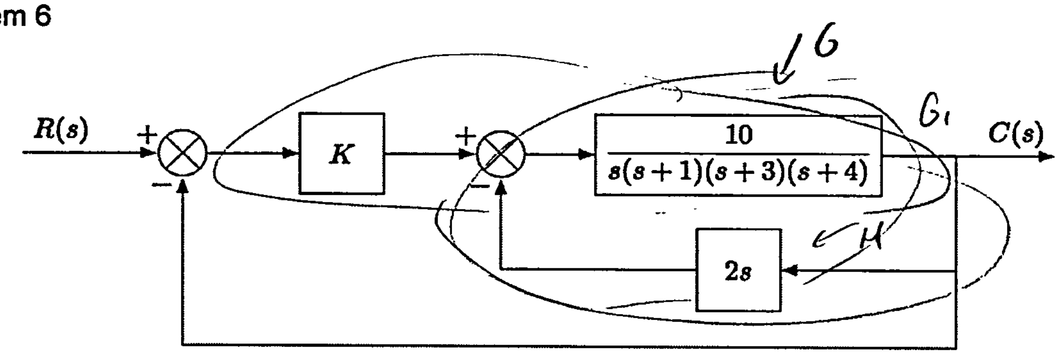
HW #6
Problem 6



a)

$$G(s) = \frac{G_{1N}}{G_{1D} + H\,G_{1N}} = \frac{10}{s(s+1)(s+3)(s+4)+20s}$$

$$\boxed{T(s) = \frac{10K}{s^4+8s^3+19s^2+32s+10K}}$$

$$G_2 = K\,G(s) = \frac{10K}{s(s+1)(s+3)(s+4)+20s}$$

$$T(s) = \frac{10K}{s(s+1)(s+3)(s+4)+20s+10K} = \frac{10K}{s^4+8s^3+19s^2+32s+10K}$$

Maple

Routh Table

s^4	1	19	10K
s^3	8	32	0
s^2	15	10K	0
s^1	(1) $32 - \frac{16K}{3}$	0	0
s^0	(2) 10K	0	0

(2) $10K > 0 \rightarrow K > 0$

(1) $32 - \frac{16K}{3} > 0$

$32 > \frac{16K}{3}$

$6 > K$

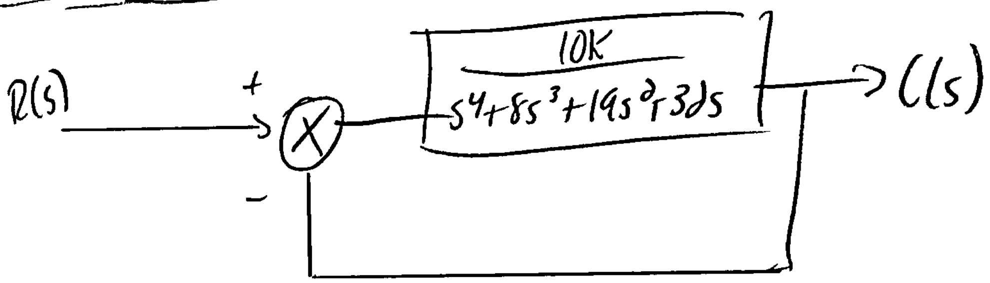
$K < 6$

$$\boxed{K \in (0,6)}$$

Problem 6b

b) from part a

$$\boxed{G(s) = \frac{10K}{s(s+1)(s+3)(s+4)+20s} = \frac{10K}{s^4+8s^3+19s^2+32s}}$$



c) $s^4+8s^3+19s^2+32s = s(s^3+8s^2+19s+32)$

one integrating factor → $\boxed{\text{Type 1}}$

d) $e_s(\infty) = \dfrac{1}{1+K_p} \Rightarrow e_s(\infty) = 0$

$K_p = \infty$

$\boxed{\text{0 steady state error}}$

e) $e_r(\infty) = \dfrac{1}{K_v} = \dfrac{16}{5K}$

Input $r_y = 5tu(t) = 5r_r$

$e_y = 5e_r = 5\dfrac{16}{5K} = \dfrac{16}{K}$

$K_v = \lim_{s\to 0} sG(s) = \dfrac{10k}{s^3+8s^2+19s+32} = \dfrac{10k}{32} = \dfrac{5K}{16}$

$\boxed{\text{Steady state error} = \dfrac{16}{K}}$

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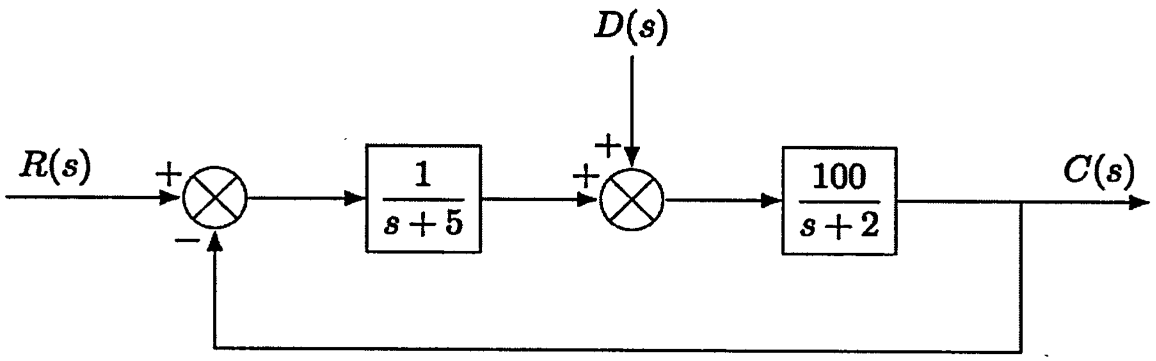
HW #6
Problem 7



$R(s) = 1/s$ (unit step)

$$e(\infty) = e_R(\infty) + e_D(\infty)$$

$$e_R(\infty) = \lim_{s\to 0} \frac{\cancel{s}}{1+G_1(s)G_2(s)} \cancel{R(s)}$$

check for stability!

$$G_1(s)G_2(s) = \frac{100}{(s+5)(s+2)}$$

$$e_R(\infty) = \lim_{s\to 0} \frac{1}{1+\frac{100}{(s+5)(s+2)}} = \lim_{s\to 0} \frac{(s+5)(s+2)}{(s+5)(s+2)+100} = \frac{10}{110} = \frac{1}{11}$$

$$e_D(\infty) = -\lim_{s\to 0} \frac{\cancel{s}\,G_2(s)}{1+G_1G_2} \cancel{D(s)}$$

$D(s) = 1/s$ (unit step)

$$= -\lim_{s\to 0} \frac{\frac{100}{s+2}}{1+\frac{100}{(s+5)(s+2)}} = \frac{-100(s+5)}{(s+5)(s+2)+100} = \frac{-500}{110} = -\frac{50}{11}$$

$$e_{ss}(\infty) = e_R(\infty) + e_D(\infty) = \frac{1}{11} - \frac{50}{11} = -\frac{49}{11}$$

$$e_{ss}(\infty) = \frac{-49}{11}$$

Problem 7 stability check

$$T(s) = \frac{100}{(s+5)(s+2)}$$ (ignoring disturbance)

$s^2 + 7s + 10$

s^2	1	10
s^1	7	0
s^0	10	0

✓ stable

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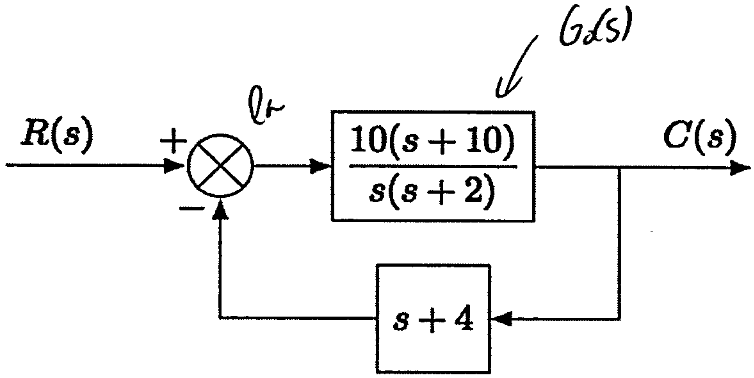
HW #6
Problem 8

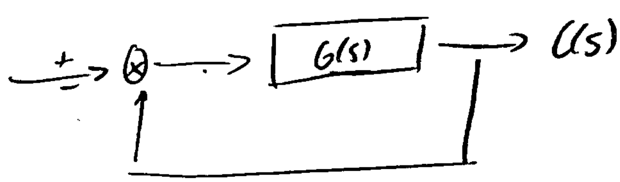


Check for stability

$$T(s) = \frac{G_N}{G_D + H(s)G_N} = \frac{10(s+10)}{s(s+2)+10(s+4)(s+10)} = \frac{10(s+10)}{11s^2+142s+400}$$

s^2	11	400
s^1	142	0
s^0	400	0

stable ✓



$$(s) = \frac{G_d(s)}{1+G_d(s)H(s) - G_d(s)} = \frac{\frac{10(s+10)}{s(s+2)}}{1 + \frac{10(s+10)}{s(s+2)}(s+4) - \frac{10(s+10)}{s(s+2)}} = \frac{10(s+10)}{s(s+2)+10(s+10)(s+4)-10(s+10)}$$

$$(s) = \frac{10(s+10)}{11s^2+132s+300}$$

:) $\boxed{\text{Type 0}}$ No integrating factor

Problem 8 cont

b) $K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{10(s+10)}{11s^2+132s+300} = \frac{100}{300} = 1/3$

$\boxed{K_p = 1/3}$

c) Type 0 → unit step for constant error

$\boxed{r(t) = u(t)}$

d) $e_s(\infty) = \frac{1}{1+K_p} = \frac{1}{1+1/3} = \frac{1}{4/3} = 3/4$ $\boxed{e_s(\infty) = 3/4}$

e) $E_a(s) = R(s) - \underbrace{(s+4)}_{H(s)} C(s)$

$E_a(s) = R(s) - H(s)C(s)$ $E_a G(s) = C(s)$

$E_a(s) + H(s)G(s)E_a = R(s)$ $R(s) = \frac{1}{s}$

$E_a(s) = \frac{R(s)}{1+H(s)G(s)}$

$e_a(\infty) = \lim_{s \to 0} s E_a(s) = \lim_{s \to 0} \frac{1}{1+H(s)G(s)} = \lim_{s \to 0} \frac{1}{1+\frac{(s+4)\cdot 10(s+10)}{s(s+2)}}$

$= \lim_{s \to 0} \frac{s(s+2)}{s(s+2)+10(s+4)(s+10)} = 0$ $\boxed{e_a(\infty) = 0}$

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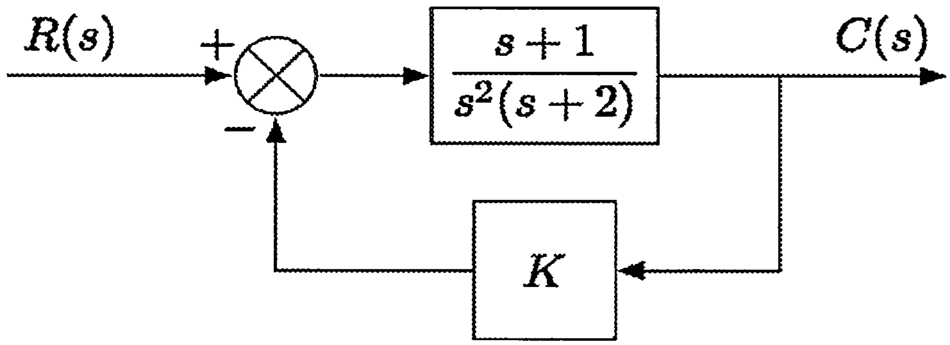
HW #6
Problem 9



1. Check for stability

$$T(s) = \frac{G_N}{G_D + H(s)G_N} = \frac{s+1}{s^2(s+2) + K(s+1)} = \frac{s+1}{s^3 + 2s^2 + Ks + K}$$

s^3	1	K
s^2	2	K
s^1	$K/2$	0
s^0	K	0

stable if $K > 0$

a) Find unity gain system

$$G(s) = \frac{\frac{s+1}{s^2(s+2)}}{1 + \frac{s+1}{s^2(s+2)}K - \frac{s+1}{s^2(s+2)}} = \frac{s+1}{s^2(s+2) + K(s+1) - (s+1)}$$

$$G(s) = \frac{s+1}{s^3 + 2s^2 + Ks - s + k - 1}$$

$$= \frac{s+1}{s^3 + 2s^2 + s(k-1) + (k-1)}$$

No integrating term

$\boxed{\text{Type } 0}$

Problem 9 cont

Type zero → step input

$$e_s(\infty) = \frac{1}{1+K_p} = \frac{0.1}{100} = \frac{1}{1000}$$

$$1+K_p = 1000$$

$$K_p = 999$$

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{s+1}{s^3+2s^2+s(k-1)+k-1} = \frac{1}{k-1} = 999$$

$$999k - 999 = 1$$

$$999k = 1000$$

$$k = \frac{1000}{999} = 1.001$$

$$\boxed{k = 1.001}$$

→ greater than zero so stable