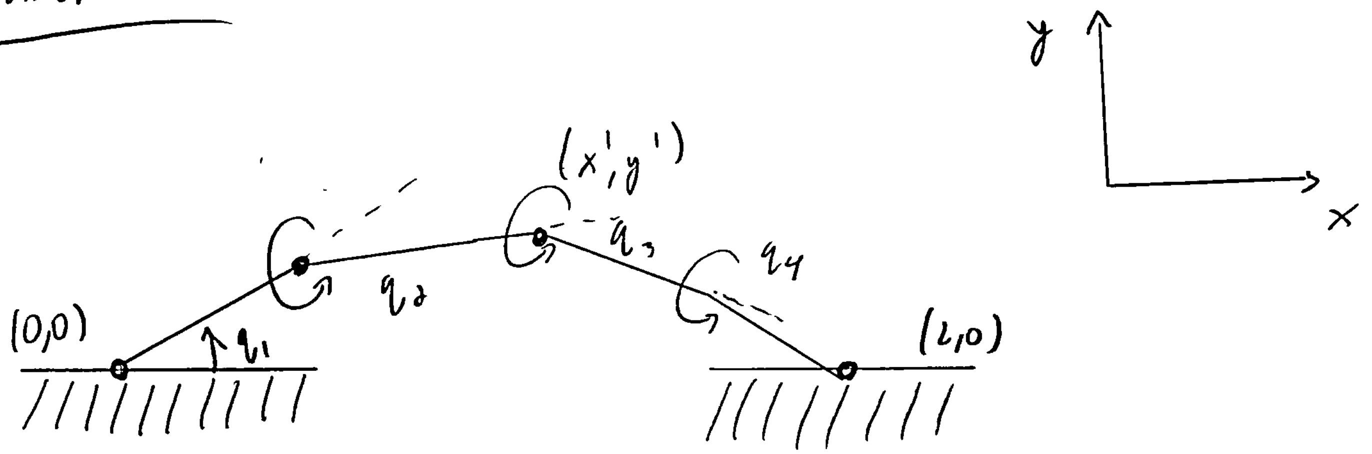
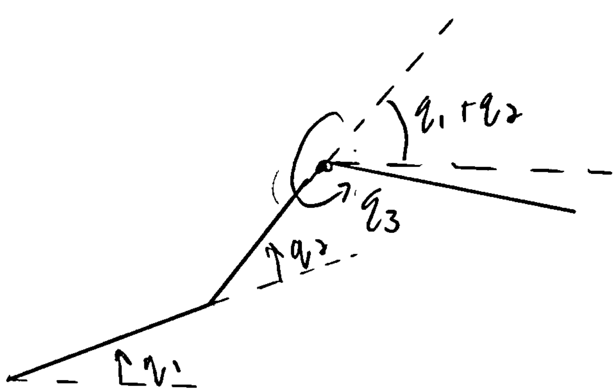
Problem 2A



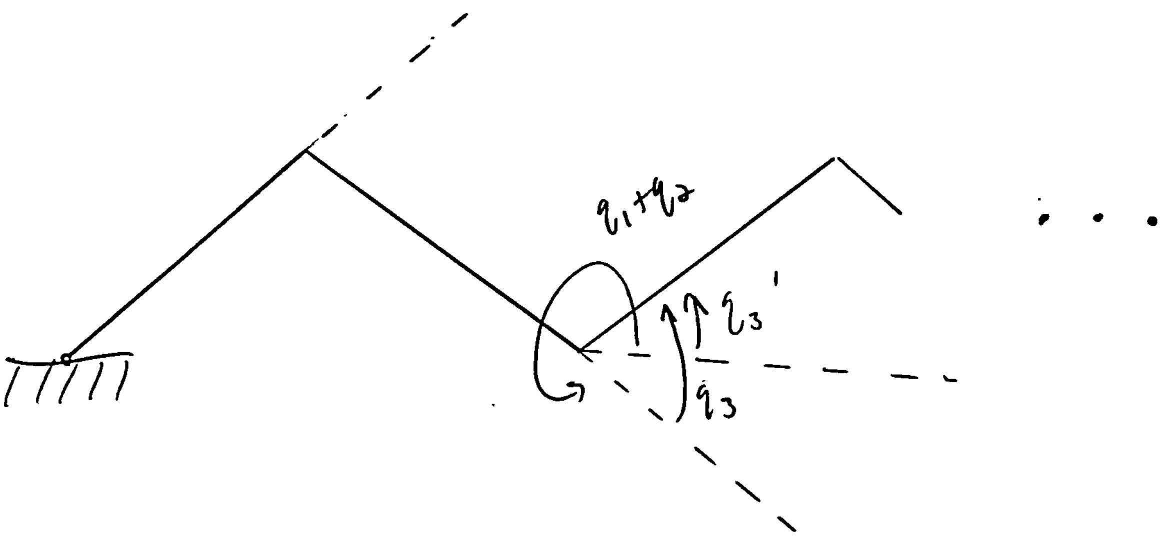
With the parameterization of q_1, q_2 we can solve for the location of the third joint. This is labeled (x', y') in the diagram above. Once this (x', y') location is calculated the problem is reduced to an inverse kinematics problem for the final two links (a 2R manipulator). However, assuming the 2RIK subfunction calculates the angles w.r.t the world xy coordinate system we must do a conversion from the solution to q_3 and q_4. In the diagram above q_3 is offset by $q_1 + q_2$. This correction needs to be made and is shown below



The returned answer of the subroutine 2RIK will return the angle $q_1 + q_2 + q_3$. Thus to find q_3 we need to take the solution and subtract q_1 and q_2

Problem 2A cont

However, the correction from q_3' to q_3 on the previous page only holds if $q_3' \geq q_1 + q_2$.
If this case does not hold a different conversion needs to be made. This is shown below.



Here, when $q_1 + q_2 > q_3'$ then $q_3 = 2\pi - (q_1 + q_2) + q_3'$

Thus, the "correction" from q_3' (defined w.r.t the world x-axis) to q_3, defined w.r.t the second armature is given as:

if $(q_3' \geq q_1 + q_2)$

$$q_3 = q_3' - (q_1 + q_2)$$

else

$$q_3 = 2\pi - (q_1 + q_2) + q_3'$$

$\longrightarrow$

The subroutine will return q_4 wrt the first armature, which in our case is the third armature. Thus, q_4 does not have to be changed.

Similar to Problem 1A the mapping of q_1, q_2 to x, y coordinates in the following way:

$$T_1(q_1) = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} c_1 \equiv \cos q_1 & c_2 \equiv \cos q_2 \\ s_1 \equiv \sin q_1 & s_2 \equiv \sin q_2 \end{matrix}$$

$$T_2(q_1, q_2) = T_1(q_1)\, T_{2\to 1}^{ret}\, R(q_2) \qquad R(q_2) = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{2\to 1}^{ret} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} L_1$$

$$\vec{x}(q_1, q_2) = T(q_1, q_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} L_2$$

$$\vec{x}(q_1, q_2) = T_1(q_1)\, T_{2\to 1}^{ret}\, R(q_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}(q_1, q_2) = \begin{bmatrix} c_1(c_2+1) - s_1 s_2 \\ s_1(c_2+1) + c_1 s_2 \\ 1 \end{bmatrix}$$

Now we have all the necessary building blocks to build a routine that calculates q_3, q_4 given q_1 and q_2

→

Problem 2A cont

$4RIK(q_1, q_2)$

$c_1 = \cos q_1$

$s_1 = \sin q_1$

$c_2 = \cos q_2$

$s_2 = \sin q_2$

$x' = c_1(c_2+1) - s_1 s_2$

$y' = s_1(c_2+1) + c_1 s_2$

* $(q_3', q_4) = 2RIK(1, 1, L-x', 0-y')$ **

if $(q_3' \geq q_1+q_2)$ $q_3 = q_3' - (q_1+q_2)$

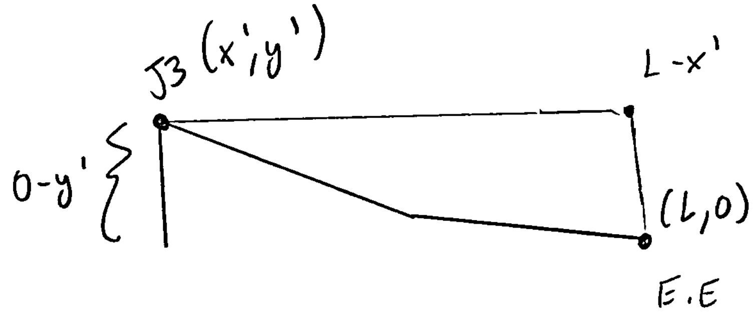
else $q_3 = 2\pi - (q_1+q_2) + q_3'$

return (q_3, q_4)

To the left is the routine to calculate all possible solutions for q_3 and q_4.
It takes in the parameters q_1, q_2 and returns the touple of solutions for (q_3, q_4)

* Here (q_3', q_4) are all possible solutions to the 2R IK problem

** $L-x'$ and $0-y'$ are the x and y distance respectively that the 2R IK kinematics must solve to get joint 3 to align the E.E at $(L,0)$



Problem 2A cont

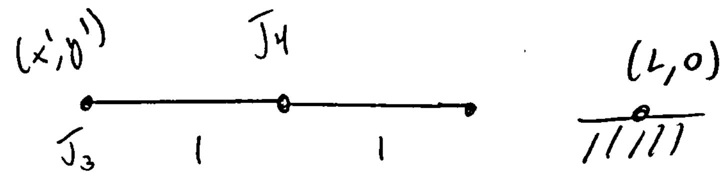
We will now consider the cases in which there are 0, 1, 2, or infinite solutions in the q_1, q_2 parameterization.

0 solutions

Since $L_3 = L_4$ the only case that can produce zero solutions is when the point $(L,0)$ is out of reach for the chosen q_1, q_2. The condition that must hold for this to be the case is given below.

$$x' = c_1(c_2+1) - s_1 s_2$$

$$y' = s_1(c_2+1) + c_1 s_2$$



$$\sqrt{(L-x')^2 + (-y')^2} > 2$$

$$(L-x')^2 + y'^2 > 4$$

$$\underbrace{(L - (c_1c_2 + c_1 - s_1s_2))^2 + (s_1c_2 + s_1 + c_1s_2)}_{G(L,q_1,q_2)} > 4$$

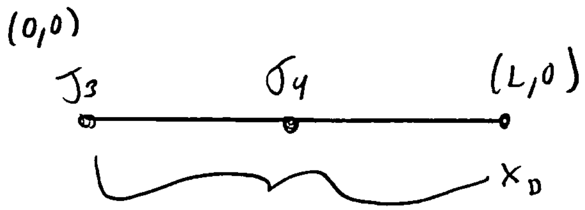
If we define G as the square of the distance between joint 3 (at a given q_1, q_2) and the point $(L,0)$, then the zero solution case is described by $G(L, q_1, q_2) > 4$ $\rightarrow$

Problem 2A cont

1 Solution

Intuitively, there is only 1 solution for (q_3, q_4) if q_1, q_2 are chosen such that $(L, 0)$ is just in reach. This is given by $G(L, q_1, q_2) = 4$.

Seen differently we can consider the case where armature 3 is along the x reference frame, without loss of generality. Given this we could calculate q_4 as follows.



$$\|x(q)\|^2 = \|x_D\|^2$$

$$\|x(q)\|^2 = (1 + c_4)^2 + (s_4)^2$$

$$= 1 + 2c_4 + c_4^2 + s_4^2$$

$$= 2 + 2c_4$$

$$c_4 = \frac{\|x_D\|^2 - 2}{2} = \frac{\|x_D\|^2}{2} - 1$$

If $\|x_D\| = 2$

then $c_4 = 1$ and

q_4 must be $0°$. With q_4 chosen q_3 must be chosen such that the armature points directly at $(L, 0)$, thus there is only one solution.

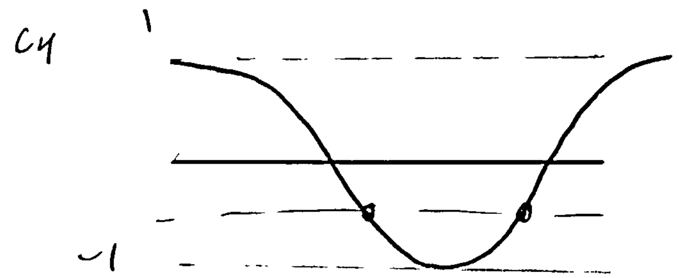
→

Problem 2A cont

2 Solutions

There are 2 solutions if $G(L, q_1, q_2) < 4$. Implicitely $G(L, q_1, q_2)$ is always larger than or equal to zero since it is defined as a distance.

$$c_4 = \frac{\|x_D\|^2}{2} - 1 \Rightarrow -1 \leq c_4 \leq 1$$



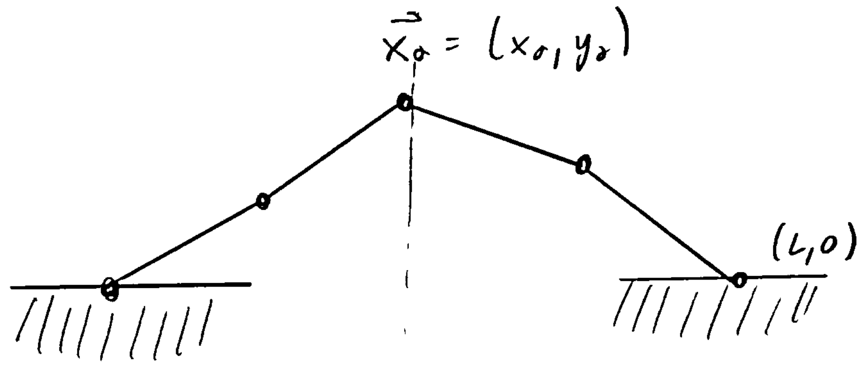
If c_4 falls between -1 and 1 2 q_4 values can be chosen.

∞ Solutions

There are infinite solutions if the third joint falls on $(L, 0)$ for the q_1 & q_2 values chosen. In this case q_4 can be made -180° and any q_3 can be chosen (since $L_3 = L_4$). This occurs when $c_1(c_2+1) - s_1 s_2 = L$ and $s_1(c_2+1) + c_1 s_2 = 0$

In the last couple of pages we have shown we can define a function G purely based on q_1, q_2, and L and use it to discern the size of the solution set

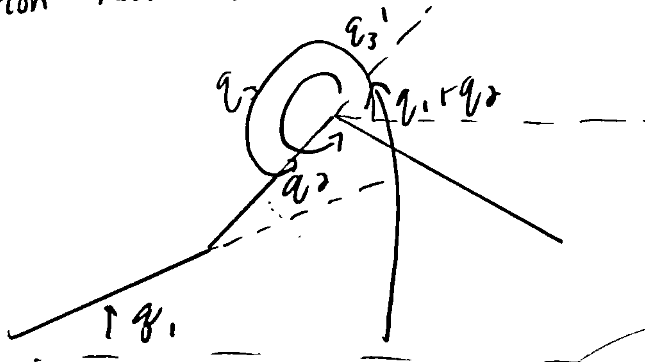
Problem 2B



Let $\vec{x}_2$ (the location of the third joint) be given by the point (x_2, y_2). We can divide this problem into 2 2RIK problems.

① $2RIK(1, 1, x_2, y_2)$ and ② $2RIK(1, 1, L - x_2, -y_2)$

Again, as in Problem 2A this will give us a q_3 with respect to the x-axis. However, we want it with respect to the second link. Thus, a conversion needs to be made.



solution from ②

$q_3 = q_3' - q_1 - q_2$ if $q_3' \geq q_1 + q_2$

$q_3 = 2\pi - (q_1 + q_2) + q_3'$ $\quad q_3' < q_1 + q_2$

(reference Problem 2A)

The routine to calculate $q_1, q_2, q_3, \& \ q_4$ given the position x_2, y_2 is given to the right

$4RIK(x_2, y_2, L)$

$(q_1, q_2) = 2RIK(1, 1, x_2, y_2)$

$(q_3', q_4) = 2RIK(1, 1, L - x_2, -y_2)$

if $(q_3' \geq q_1 + q_2)$ $\quad q_3 = q_3' - (q_1 + q_2)$

else $\quad q_3 = 2\pi - (q_1 + q_2) + q_3'$

return (q_1, q_2, q_3, q_4)

Problem 2B

We now consider the size of the solution set

0 solutions

This occurs $\vec{x}_2$ is out of range for q_1, q_2 or $\vec{x}_2$ is chosen such that ~~q_3 & q_4 cannot be chosen~~ to set to $(L, 0)$

(OR)

$$\|\vec{x}_2\| > 2 \quad \| \quad \left\| \vec{x}_2 - \begin{bmatrix} L \\ 0 \end{bmatrix} \right\| > 2$$

1 solution

This occurs when $\vec{x}_2$ is just in range of $(0,0)$ & $(L,0)$

$$\|\vec{x}_2\| = 2 \quad \& \quad \left\| \vec{x}_2 - \begin{bmatrix} L \\ 0 \end{bmatrix} \right\| = 2$$

2 solutions

When $\vec{x}_2$ is just in range of $(0,0)$ and within range from $(L,0)$

or

$\vec{x}_2$ is within range from $(0,0)$ and just in range to $(L,0)$

(or)

$$\left(\|\vec{x}_2\| = 2 \;\&\; \left\|\vec{x}_2 - \begin{bmatrix} L \\ 0 \end{bmatrix}\right\| < 2\right) \;\|\; \left(\|\vec{x}_2\| < 2 \;\&\; \left\|\vec{x}_2 - \begin{bmatrix} L \\ 0 \end{bmatrix}\right\| = 2\right)$$

Problem 2B cont

4 solutions

$\vec{x}_2$ within range from $(0,0)$ & $(L,0)$

$$\|\vec{x}_2\| < 2 \quad \& \quad \left\|\vec{x}_2 - \begin{bmatrix} L \\ 0 \end{bmatrix}\right\| < 2$$

∞ Solutions

$\vec{x}_2 = (0,0)$
or
$\vec{x}_2 = (L,0)$

q_1 can be any value between 0 & 360°

q_2 can be any value between 0 & 360°