Homework 5

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rs20

Question 1

$G(s) = \frac{K(s+1)}{s(s+2)(s+3)}$ $H(s) = 1$

Closed loop $Y(s) = \frac{G(s)}{1+H(s)G(s)} = \frac{\frac{K(s+1)}{s(s+2)(s+3)}}{1 + \frac{K(s+1)}{s(s+2)(s+3)}} = \frac{K(s+1)}{s(s+2)(s+3) + K(s+1)}$

$$Y(s) = \frac{K(s+1)}{s(s^2+5s+6)+Ks+K} = \frac{K(s+1)}{s^3+5s^2+(6+K)s+K}$$

Evaluate $s^3+5s^2+(6+K)s + K$

Routh Table

s^3	1	$6+K$
s^2	5	K
s	① $6+\frac{4}{5}K$	② 0
s^0	K	0

① $-\frac{\begin{vmatrix} 1 & 6+K \\ 5 & K \end{vmatrix}}{5} = \frac{-(K-30-5K)}{5} = \frac{30+4K}{5} = 6+\frac{4}{5}K$

② $-\frac{\begin{vmatrix} 6+K & 0 \\ K & 0 \end{vmatrix}}{5} = 0$

③ $-\frac{\begin{vmatrix} 5 & K \\ 6+\frac{4}{5}K & 0 \end{vmatrix}}{6+\frac{4}{5}K} = -\frac{-K(6+\frac{4}{5}K)}{6+\frac{4}{5}K} = K$

* Verified this w/ Routh table program

For stability Need all entries in first column positive

$6+\frac{4}{5}K>0$

$\frac{4}{5}K>-6$

$K>\frac{-6 \cdot 5}{4}$

$K>-\frac{15}{2}$

$K>0$

Must meet both constraints

$K>0$

$K \in (0,\infty)$

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HW #5 Problem 2

s^7	1	2	−1	−2
s^6	1	2	−1	−2
s^5	3	4	−1	0
s^4	1	−1	−3	0
s^3	7	8	0	0
s^2	−15	−21	0	0
s^1	−9	0	0	0
s^0	−21	0	0	0

root → s^6

was zeros → s^5

1 sign change

$s^6 + 2s^4 - s^2 - 2$ was root

After even (6th order) polynomial there is one sign change → 1 RHP

By symmetry 1 LHP. 4 remaining poles for even polynomial (4) on $j\omega$

must add up to 7

s^6 ....

1 RHP, 1 LHP, 4 $j\omega$ = 6 poles

Remaining pole on LHS because no sign switch $s^7 \to s^6$

2 LHP total

1 RHP

4 $j\omega$

	lhp	rhp	$j\omega$
even order 6	1	1	4
order 1	1	0	0
total	2	1	4

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Problem 3

$$G(s) = \frac{K}{(s+77)(s+27)(s+38)} \qquad H(s) = 1$$

$$Y(s) = \frac{G(s)}{1+H(s)G(s)} = \frac{\frac{K}{den}}{1+\frac{K}{den}} = \frac{K}{den + K} = \frac{K}{(s+77)(s+27)(s+38)+K}$$

$$= \frac{K}{s^3 + 142s^2 + 6031s + (79002+K)}$$

$$\frac{-\begin{vmatrix} 1 & 6031 \\ 142 & 79002+K \end{vmatrix}}{142} = -\frac{(79002+K - 142(6031))}{142} = \frac{777400-K}{142}$$

s^3	1	6031
s^2	142	$79002+K$
s^1	$\frac{777400-K}{142}$	0
s^0	$79002+K$	0

Verified using Matlab code

$$\frac{-\begin{vmatrix} 142 & 79002+K \\ \frac{777400-K}{142} & 0 \end{vmatrix}}{\frac{777400-K}{140}} = 79002+K$$

Stability: $K \in (-79002, 777400)$

For oscillation need $j\omega$ roots with multiplicity 1. To do so need row of zeros. $K = 777400 \Rightarrow$

s^3	1	6031
s^2	$\cancel{142}\ 1$	$\cancel{856402}\ 6031$
s^1	$\cancel{0}\ 1$	0
s	1	0

Produces root (s^2+6031) of original polynomial

Which has roots $\pm\sqrt{6031}\, i \rightarrow$ both of $j\omega$ $\longrightarrow$

Problem 3 cont

$K = 777400$ produces oscillations

Problem 4

$$G(s) = \frac{K}{(s+1)^3(s+4)} \qquad H(s) = 1$$

$$Y(s) = \frac{G(s)}{1+H(s)G(s)} = \frac{\frac{K}{(s+1)^3(s+4)}}{1+\frac{K}{(s+1)^3(s+4)}} = \frac{K}{(s+1)^3(s+4)+K}$$

$$= \frac{K}{(s^3+3s^2+3s+1)(s+4)+K} = \frac{K}{s^4+7s^3+15s^2+13s+4+K}$$

Analyze

$s^4+7s^3+15s^2+13s+4+K$

s^4	1	15	$4+K$
s^3	7	13	0
s^2	$\frac{92}{7}$	$4+K$	0
s^1	$\frac{250}{23}-\frac{49}{92}K = A$	0	0
s^0	$4+K$	0	0

$$-\frac{\begin{vmatrix} 1 & 15 \\ 7 & 13 \end{vmatrix}}{7} = \frac{92}{7}$$

$$-\frac{\begin{vmatrix} 1 & 4+K \\ 7 & 0 \end{vmatrix}}{7} = \frac{-(-7(4+K))}{7} = 4+K$$

$$-\frac{\begin{vmatrix} 7 & 13 \\ \frac{92}{7} & 4+K \end{vmatrix}}{\frac{92}{7}} = \frac{-\left(7(4+K)-\frac{92}{7}(13)\right)}{\frac{92}{7}} = \frac{250}{23}-\frac{49K}{92}$$

$$-\frac{\begin{vmatrix} \frac{92}{7} & 4+K \\ A & 0 \end{vmatrix}}{A} = 4+K$$

For closed loop stability need first column positive.

① $4+K>0$

$K>-4$

② $\frac{250}{23}-\frac{49}{92}K>0$

$\frac{49}{92}K<\frac{250}{23}$

$K<\frac{1000}{49}$

$K \in \left(-4, \frac{1000}{49}\right)$ for closed loop stability

Problem 4 cont

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Marginal stability → $j\omega$ poles w/ multiplicity 1
↳ Need row of zero

Let's try $K=-4$ first to get row of zeros in s^0 row

$K=-4$ Divide by $\frac{92}{7}$

$$\begin{array}{llll} s^4 & 1 & 15 & 0 \\ s^3 & 7 & 13 & 0 \\ s^2 & 1 & 0 & 0 \\ s^1 & 1 & 0 & 0 \\ s^0 & 0 & 0 & 0 \end{array}$$

Means s is a root

$s=0$

pole at zero, not on $j\omega$. Therefore $K=-4$ does not yield marginal stability

Try $\frac{250}{23} - \frac{49}{92}K = 0$ $K = \frac{1000}{49}$

$$-\frac{\begin{vmatrix} 1 & 15 \\ 7 & 13 \end{vmatrix}}{7} = \frac{92}{7}$$

$$-\frac{\begin{vmatrix} 1 & \frac{1196}{49} \\ 7 & 0 \end{vmatrix}}{7} = \frac{1196}{49}$$

$$\begin{array}{llll} s^4 & 1 & 15 & \frac{1196}{49} \\ s^3 & 7 & 13 & 0 \\ s^2 & \cancel{\frac{92}{7}}\ 7 & \cancel{\frac{1196}{49}}\ 13 & 0 \\ s^1 & 0 & 0 & 0 \leftarrow \end{array}$$

$7s^2+13=0$ is root

$s^2 = -\frac{13}{7}$

$s = \pm\sqrt{\frac{13}{7}}\, i$

$= j\omega$ $\omega = \sqrt{\frac{13}{7}}$

$$\boxed{\begin{array}{l}\text{Marginal stability.} \\ K = \frac{1000}{49} \\ \omega = \sqrt{\frac{13}{7}}\end{array}}$$

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Problem 5

$$G(s) = \frac{K}{s(s+1)(s+2)(s+6)}$$

use Maple to expand

$$Y(s) = \frac{\frac{K}{s(s+1)(s+2)(s+6)}}{1+\frac{K}{s(s+1)(s+2)(s+6)}} = \frac{K}{s(s+1)(s+2)(s+6)+K} = \frac{K}{s^4+9s^3+20s^2+12s+K}$$

a) Analyze $s^4+9s^3+20s^2+12s+K$

s^4	1	20	K
s^3	$\cancel{9}\ 3$	$\cancel{12}\ 4$	0
s^2	$+\frac{56}{3}$	K	0
s^1	$4-\frac{9}{56}K = A$	0	0
s^0	K	0	0

$$-\frac{\begin{vmatrix}1 & 20\\ 3 & 4\end{vmatrix}}{3} = -\frac{(4-60)}{3}$$

$$-\frac{\begin{vmatrix}1 & K\\ 3 & 0\end{vmatrix}}{3} = K$$

$$-\frac{\begin{vmatrix}3 & 4\\ +\frac{56}{3} & K\end{vmatrix}}{+\frac{56}{3}} = -\frac{\left(3K - \frac{56}{3}(4)\right)}{\frac{56}{3}}$$

$$-\frac{\begin{vmatrix}\frac{56}{3} & K\\ A & 0\end{vmatrix}}{A} = K$$

Closed loop stability

$K>0$ ← $K=0$ will produce $s=0$ root → produces constant so it won't decay to zero

$4-\frac{9}{56}K > 0$

$K < 4\left(\frac{56}{9}\right)$

$K < \frac{224}{9}$

Closed loop stability

$K \in \left(0, \frac{224}{9}\right)$

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Problem 5 cont

b)

Marginal stability: $K = \frac{224}{9}$

s^4	1	20	$\frac{224}{9}$
s^3	3	4	0
s^2	$\cancel{\frac{56}{3}}$ 3	$\cancel{\frac{224}{9}}$ 4	0
s^1	$\cancel{0}$ 1	0	0
s^0	1	0	0

$3s^2+4=0$ root

$s^2 = -\frac{4}{3}$

$s = \pm\sqrt{\frac{4}{3}}\, i$

Marginal stability $K = \frac{224}{9}$

c) Exact poles

$s^4 + 9s^3 + 20s^2 + 12s + K$ where $K = \frac{224}{9}$

(matlab)

Know $3s^2+4$ is a root

$$s^4 + 9s^3 + 20s^2 + 12s + \frac{224}{9} = \frac{1}{9}(3s^2+4)(3s^2+27s+56)$$

$\underline{3s^2+4}$

$3s^2 = -4$

$s^2 = -\frac{4}{3}$

$s = \pm\sqrt{4/3}\, i$

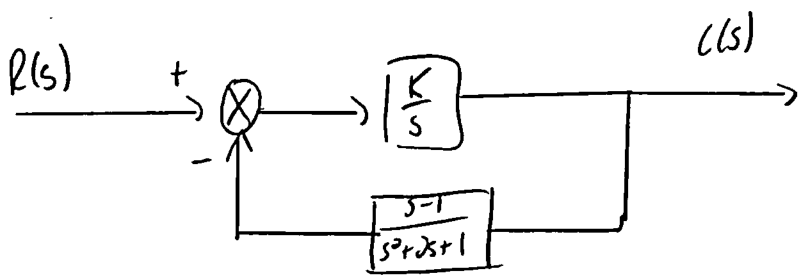
$\underline{3s^2+27s+56}$

$$\frac{-27 \pm \sqrt{27^2 - 4(56)(3)}}{3(2)} = -\frac{9}{2} \pm \frac{\sqrt{57}}{3(2)} = -3.2417$$
$$= -5.7831$$

Poles: $\pm\sqrt{4/3}\, i$, -3.24, -5.78

Problem 6

M3 20



$(s+1)^2$

$$Y(s) = \frac{G(s)}{1+H(s)G(s)} = \frac{\frac{K}{s}}{1+\frac{s-1}{s^2+2s+1}\frac{K}{s}} = \frac{K(s^2+2s+1)}{s(s^2+2s+1)+(s-1)K} = \frac{K(s+1)^2}{s^3+2s^2+s+Ks-K}$$

Analyze $s^3+2s^2+(1+K)s-K$

s^3	1	$1+K$
s^2	2	$-K$
s^1	$\frac{3}{2}K+1 \leftarrow A$	0
s^0	$-K$	0

& checked w/ my program

$$-\frac{\begin{vmatrix} 1 & 1+K \\ 2 & -K \end{vmatrix}}{2} = -\frac{(-K-2(1+K))}{2} = \frac{K+2+2K}{2} = \frac{3}{2}K+1$$

$$-\frac{\begin{vmatrix} 2 & -K \\ A & 0 \end{vmatrix}}{A} = -K$$

Stability

$-K>0$

$K<0$

$K \in \left(-\frac{3}{2}, 0\right)$

$\frac{3}{2}K+1>0$

$\frac{3}{2}K>-1$

$K>-\frac{3}{2}$

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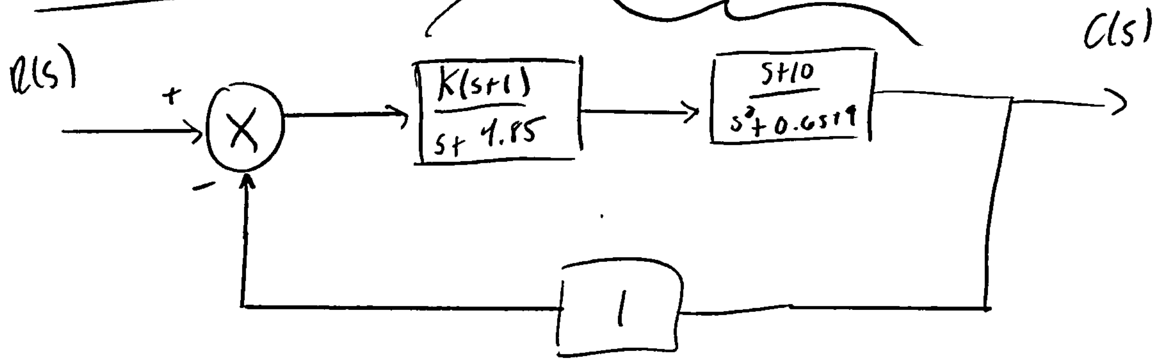
Problem 7

M270

$$\overbrace{\frac{K(s+1)(s+10)}{(s+4.85)(s^2+0.6s+9)}}$$



$$Y(s) = \frac{G(s)}{1+H(s)G(s)} = \frac{K(s+1)(s+10)}{(s+4.85)(s^2+0.6s+9) + K(s+1)(s+10)}$$

$$(s+4.85)(s^2+0.6s+9) + K(s+1)(s+10) = s^3 + (5.45+K)s^2 + (11.91+11K)s + 43.65+K$$

maple expand

s^3	1	$11.91+11K$
s^2	$5.45+K$	$43.65+K$
s^1	A	0
s^0	B	0

$$A = -\frac{\begin{vmatrix} 1 & 11.91+11K \\ 5.45+K & 43.65+K \end{vmatrix}}{5.45+K} = -\frac{(K+43.65 - (11.91+11K)(5.45+K))}{5.45+K}$$

$$B = -\frac{\begin{vmatrix} 5.45+K & 43.65+K \\ A & 0 \end{vmatrix}}{A} = 43.65+K$$

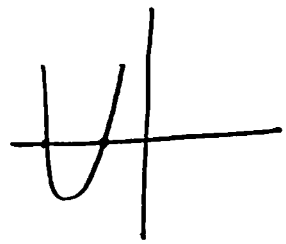
① $K > -5.45$

② $K > -43.65$ } redundant by ①

③/A Need $A > 0 \Rightarrow -\frac{(K+43.65-(11.91+11K)(5.45+K))}{5.45+K}$ $\longrightarrow$

$$A = \frac{11K^2 + 70.86K + 21.2595}{5.45 + K}$$ ← Maple

as 20



If $A > 0$

Case 1: num > 0, den > 0

$11K^2 + 70.86K + 21.2595 > 0$

Need to left & right of root

$r1 = -0.31547$

$r2 = -6.12635$

$K > -0.31547$ is tightest

$K > -5.45$

Case 2: num < 0, den < 0

$K < -6.12635$ is tightest but this contradicts ①

$K < -5.45$

$K > -0.31547$

Can the system ever be unstable for $K > 0$?

No. If $K > 0$, $A > 0$ & $B > 0$ Thus all entries in first column are positive → all poles in left half of plane

Prob 7 cont