CS330HW6

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Problem 2A

Prove that is a graph G does not have any negative cycles, for any vertex u there exists a shortest path from s to u that contains at most $n-1$ edges

Before the proof the following Lemma will be established:

Lemma I - A path with n or more edges contains at least one cycle

Let path P be a path comprised of k edges from graph G, where $k \geq n$ and n is the number of vertices in G. Let q be the number of vertices in path P. The path P contains $k+1$ vertices, since a path with a edges must contain $a+1$ vertices. It is shown below that $q > n$:

$$\begin{aligned} q &= k+1 \\ &\geq n+1 \\ q &> n \end{aligned}$$

Since G has n vertices and the path P has more than n vertices, at least one of the vertices in P must be a repeated vertex. Since the path P has a repeated vertex a cycle must exist. Thus, using the pigeonhole principle, it has been shown that path P or n or more edges must contain a cycle.

Proof

Let there be a graph G that contains the vertices s and u and has no negative cycles. Assume towards contradiction that there exists a path P with k edges such that P meets the following constraints:

1. P contains at least n edges ($k \geq n$).
2. P is a shortest path from s to u.
3. P contains the least number of edges of all shortest paths from s to u.

In other words, P is the path out of all the possible shortest paths, $S_1 ... S_z$, that contains the least number of edges. Since P has $k \geq n$ edges, P must contain a cycle C (Lemma I). Let w_c denote the weight of cycle C. That is, for cycle C, containing $(u_1, u_2, ..., u_l)$,

$$w_c = w[u_l, u_1] + \sum_{i=1}^{l-1} w[u_i, u_{i+1}]$$

Since G has no negative cycles $w_c \geq 0$.

Let $w[P]$ denote the weight of path P. That is,v let $w[P]$ denote the summation of all the weights of edges in the path P. Construct the set of edges P' that contains all edges in P except the edges in the cycle C. In other words,

$$P' = \{e \mid e \in P, e \notin C\}$$

Since P' contains all the edges of a path P, besides those in a cycle, it must also be a path. Additionally, the weight of P' is equal to the weight of P minus the weight of the cycle, or,

$$w[P'] = w[P] - w_c$$

.

Since $w_c \geq 0$, two cases need to be considered - when $w_c > 0$ and when $w_c = 0$.

Case I: $w_c > 0$

In this case C is a positive cycle. Therefore, $w[P'] < w[P]$. This means that path P' is a shorter path than path P. Hence, path P cannot be a shortest path and in this case, by contradiction, P cannot exist.

Case II: $w_c = 0$

In this case C is a zero cycle. Therefore, $w[P'] = w[P]$. This means P' is also a shortest path. Since a cycle must have a nonzero number of edges the number of edges in P' is less than that of P. Therefore, P is **not** the shortest path with the *least* number of edges. Thus, by contradiction P cannot exist.

It can be established that there must exist a path between s and u that contains $n-1$ or less edges. A path T with $n-1$ edges contains n vertices. These n vertices in T can be chosen to include all n vertices in G. If the vertices in T could not be chosen as the n vertices in G then G would not be connected. However, since G is connected this path T can be formed in this manner. Again since G is a connected graph, any path containing all n vertices can reach any vertex. Therefore, T must have a path from s to u, where s and u are both in G.

Since it has been shown through contradiction that no path with n or more edges can be the shortest path from s to u with the least number of edges, and a path from s to u with $n-1$ or less edges must exists, there must be a shortest path from s to u that contains at most $n-1$ edges.

Problem 2B

If the graph has a negative cycle $(u_1, u_2, ..., u_l)$ *prove that there exists a vertex* u *such that* $d[u, n] < d[u, n-1]$

Proof

Let graph G contain at least one negative cycle. Let one of these negative cycles be indicated by C. Without loss of generality let the vertices in C be labeled 1,2...l, where l is the number of vertices in the cycle C. In other words let C contain the vertices $u_1, u_2, ..., u_l$. Additionally, let w_c indicate the weight of the cycle. That is,

$$w_c = w[u_l, u_1] + \sum_{i=1}^{l-1} w[u_i, u_{i+1}]$$

Since C is a negative cycle $w_c < 0$.

The transition function of Bellman-Ford is given below:

$$d[u, i+1] = \min\{d[u, i], \min_{(v,u)\in E} d[v, i] + w[u, v]\} \tag{1}$$

The transition function given in Equation 1 states that the minimum path from s to u containing at most $i+1$ steps is the minimum of 1. the path to s using i steps or 2. the most optimal connection from a path that reaches a neighboring vertex of u in i steps. Since the transition function selects the most optimal option from a set of options, namely the minimum value of $d[u, i]$ and $d[v, i] + w[u, v]$ for all $(v, u) \in E$, it must be true that $d[u, i+1]$ is *at least* as optimal as any one of these options. In other words, any of these options can be worse than the optimal option selected, but no better. Therefore, it must follow that,

$$d[u, i+1] \leq d[v, i] + w[u, v] \tag{2}$$

where u is any vertex in G and v in Equation 2 is **any** vertex that satisfies $(v, u) \in E$.

Consider the edge (u_1, u_2) in C. Given Equation 2, $d[u_2, n] \leq d[u_1, n-1] + w[u_1, u_2]$ must hold. This can generalized to all of the edges in C in the following manner:

$$d[u_k, n] \leq \begin{cases} d[u_{k-1}, n-1] + w[u_{k-1}, u_k] & 2 \leq k \leq l \\ d[u_l, n-1] + w[u_l, u_1] & k = 1 \end{cases} \tag{3}$$

As stated above, Equation 3 above summarizes the result of applying Equation 2 to each edge in the cycle. The reason the $k = 1$ case is necessary is to account for the edge (u_l, u_1), in which the ordering of the vertices in the edge is non-increasing.

Next, an inequality can be built be summing $d[u, n]$ for all $u \in C$. In other words, an inequality can be build by summing all of the expressions in Equation 3 for $1 \leq k \leq l$. Therefore, given Equation 3 and 4 the following inequality must hold:

$$d[u_1, n] + \sum_{i=2}^{l} d[u_i, n] \leq d[u_l, n-1] + w[u_l, u_1] + \sum_{i=2}^{l} (d[u_{i-1}, n-1] + w[u_{i-1}, u_i]) \tag{4}$$

The following calculation can be done to reveal a dependency of w_c, the cycle weight, on the right side of the inequality:

$$\begin{aligned} w[u_l, u_1] + \textstyle\sum_{i=2}^{l} w[u_{i-1}, u_i] &= w[u_l, u_1] + \textstyle\sum_{i=1}^{l-1} + w[u_i, u_{i+1}] \\ &= w_c \end{aligned}$$

Given this revelation Equation 4 can be represented as:

$$d[u_1, n] + \sum_{i=2}^{l} d[u_i, n] \leq d[u_l, n-1] + \sum_{i=2}^{l} d[u_{i-1}, n-1] + w_c$$

Since $w_c < 0$,

$$d[u_1, n] + \sum_{i=2}^{l} d[u_i, n] < d[u_l, n-1] + \sum_{i=2}^{l} d[u_{i-1}, n-1]$$

Some bound manipulations of the above inequality yields the following:

$$\sum_{i=1}^{l} d[u_i, n] < \sum_{i=1}^{l} d[u_{i-1}, n-1]$$

In order for the above summation to hold there must be some k between 1 and l for which $d[u_k, n] < d[u_k, n-1]$. Therefore, it has been proven that there must exist some u for which $d[u_k, n] < d[u_k, n-1]$ is true.