# 1 General Expansions

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 + \frac{1}{6}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 - \frac{1}{6}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$f(x+2\Delta x) = f(x) + 2f'(x)\Delta x + 2f''(x)\Delta x^2 + \frac{4}{3}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$f(x-2\Delta x) = f(x) - 2f'(x)\Delta x + 2f''(x)\Delta x^2 - \frac{4}{3}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$f(x+3\Delta x) = f(x) + 3f'(x)\Delta x + \frac{9}{2}f''(x)\Delta x^2 + \frac{9}{2}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$f(x-3\Delta x) = f(x) - 3f'(x)\Delta x + \frac{9}{2}f''(x)\Delta x^2 - \frac{9}{2}f'''(x)\Delta x^3 + O(\Delta x^4)$$

# 2 Second-Order Differences

## 2.1 Centered Difference

$$f_{i+1} - f_{i-1} = 2f'(x)\Delta x + O(\Delta x^3) \rightarrow f'(x) = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2)$$
$$f_{i+1} + f_{i-1} = 2f(x) + f''(x)\Delta x^2 + O(\Delta x^4) \rightarrow f''(x) = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

## 2.2 Forward Difference

$$\Delta_1 = f_{i+1} - f_i = f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 + \frac{1}{6}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$\Delta_2 = f_{i+2} - f_i = 2f'(x)\Delta x + 2f''(x)\Delta x^2 + \frac{4}{3}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$\Delta_3 = f_{i+3} - f_i = 3f'(x)\Delta x + \frac{9}{2}f''(x)\Delta x^2 + \frac{9}{2}f'''(x)\Delta x^3 + O(\Delta x^4)$$

$$\begin{bmatrix} 1 & 1/2 & 1/6 \\ 2 & 2 & 4/3 \\ 3 & 9/2 & 9/2 \end{bmatrix} \begin{bmatrix} f'(x)\Delta x \\ f''(x)\Delta x^2 \\ f'''(x)\Delta x^3 \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} + O(\Delta x^4)$$

$$\begin{bmatrix} f'(x) \\ f''(x) \\ f'''(x) \end{bmatrix} = \begin{bmatrix} (3\Delta_1 - \frac{3}{2}\Delta_2 + \frac{1}{3}\Delta_3)/(\Delta x) \\ (-5\Delta_1 + 4\Delta_2 - \Delta_3)/(\Delta x^2) \\ (3\Delta_1 - 3\Delta_2 + \Delta_3)/(\Delta x^3) \end{bmatrix} + \begin{bmatrix} O(\Delta x^3) \\ O(\Delta x^2) \\ O(\Delta x) \end{bmatrix}$$

## 2.3 Backward Difference

$$\Delta_1 = f_i - f_{i-1} = -f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 - \frac{1}{6}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$\Delta_2 = f_i - f_{i-2} = -2f'(x)\Delta x + 2f''(x)\Delta x^2 - \frac{4}{3}f'''(x)\Delta x^3 + O(\Delta x^4)$$
$$\Delta_3 = f_i - f_{i-3} = -3f'(x)\Delta x + \frac{9}{2}f''(x)\Delta x^2 - \frac{9}{2}f'''(x)\Delta x^3 + O(\Delta x^4)$$

$$\begin{bmatrix} -1 & 1/2 & -1/6 \\ -2 & 2 & -4/3 \\ -3 & 9/2 & -9/2 \end{bmatrix} \begin{bmatrix} f'(x)\Delta x \\ f''(x)\Delta x^2 \\ f'''(x)\Delta x^3 \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} + O(\Delta x^4)$$

$$\begin{bmatrix} f'(x) \\ f''(x) \\ f'''(x) \end{bmatrix} = \begin{bmatrix} (-3\Delta_1 + \frac{3}{2}\Delta_2 - \frac{1}{3}\Delta_3)/(\Delta x) \\ (-5\Delta_1 + 4\Delta_2 - \Delta_3)/(\Delta x^2) \\ (-3\Delta_1 + 3\Delta_2 - \Delta_3)/(\Delta x^3) \end{bmatrix} + \begin{bmatrix} O(\Delta x^3) \\ O(\Delta x^2) \\ O(\Delta x) \end{bmatrix}$$