MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

Prepared by: Er. Ankit Kharel

Nepal college of information technology

PREDICATE LOGIC

1.QUANTFIERS

2.TYPES OF QUANTIFIERS

3.NEGATING QUANTIFIERS

4.TRANSLATION FROM ENGLISH

LIMITATION OF PROPOSITIONAL LOGIC:

Consider the following :

p: “All men are mortal”

q: “Ram is man”

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r: ∴ “Ram is mortal”

- No rule of propositional logic will allow us to conclude the truth of ‘r’.
- Therefore, We need more powerful type of logic called First order Logic or PREDICATE LOGIC.

- To understand predicate Logic we need to understand:
  a) Subject
  b) Predicates
  c) Quantifiers
  d) Domain(Universe of Discourse0

Consider an statement:

“Jose is Tall”

Subject

Tall(Jose)

Predicate

- Subject: “The **subject** is what (or whom) the sentence is about”

- Predicate: “**Predicate** refers to a property that the subject of a statement can have”

Consider an statement:

“X is greater than 3” (x>3)

Subject: Variable “x”

Predicate: Greater than 3

We can denote “x>3” as: P(x)

The statement P(x) becomes a proposition once the value has been assigned to the subject.

Example:

P(5): “5 is greater than 3”(TRUE)

P(2): “2 is greater than 3”(FALSE)

Q.1) Let Q(x) denotes the statement : "The word "x" contains the letter 'a' "
What are the Truth value of Q(ankit), Q(Logic), Q(nothing) ?

Solution

Q(x): " The word "x" contains the letter 'a' "

Q(ankit): ""The word "ankit" contains the letter 'a' " (TRUE)
Q(Logic): ""The word "Logic" contains the letter 'a' " (FALSE)
Q(nothing): ""The word "nothing" contains the letter 'a' " (FALSE)

Q.2) Let C(x, y) denotes the statement: "x is the capital of y"
What are the truth value of C(Kathmandu, Nepal), C(Texas, America) ?

Solution

C(x, y): "x is the capital of y"

C(Kathmandu, Nepal) : "Kathmandu is capital of Nepal" (TRUE)
C(Texas, America) : "Texas is capital of America" (FALSE)

Consider The Following:

P(x):"x is greater than 10"

Domain: All positive natural numbers.

Can we say the above statement is true for all values of x?

=No, because for x=1,2,3,4,5,6,7,8,9,10 above statement becomes FALSE.

So, We can say above statement as : **For some x, P(x) is TRUE.**

Consider The following:

Q(x): "x<x+1"

Domain : All positive natural numbers.

Can we say that Q(x) is TRUE For all values of x within our domain?

=Yes

So, we can say above statement as: **For all x, Q(x) is TRUE.**

QUANTIFIERS

In **predicate logic**, **predicates** are used alongside **quantifiers** to express the extent to which a **predicate** is true over a range of elements. Using **quantifiers** to create such propositions is called quantification.

1. UNIVERSAL QUANTIFICATION($\forall$):

"Every cat drinks milk"

CATS

X_1

X_2

X_3

Domain(Universe of Discourse)

Above statement is Equivalent to:

X_1 Drinks milk.

$\wedge$

X_2 Drinks milk.

$\wedge$

X_3 Drinks milk.

Milk(X_1)

$\wedge$

Milk(X_2)

$\wedge$

Milk(X_3)

$$\forall_x \text{ Milk}(X)$$

=For all X, Milk(X)

or

=For ever X, Milk(X)

1. UNIVERSAL QUANTIFICATION($\forall$):

The Universal Quantification of P(x) is:

"P(x) for all value of x in the Domain"

$$= \forall_x P(x)$$

We can also read $\forall_x P(x)$ as:

"For all P(x) " or "for every x, P(x)"

<u>Example:</u>

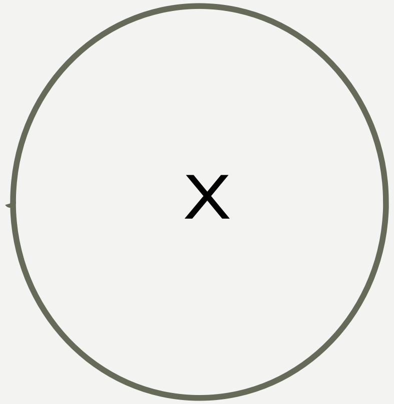
Q.1) "All student of BEIT takes course on Discrete Mathematics"

let,

D(x):"x takes course on Discrete Mathematics"

$$=\forall_x D(x)$$

<u>BEIT</u>



<u>Domain(Universe of Discourse)</u>

1. UNIVERSAL QUANTIFICATION($\forall$):

Example:

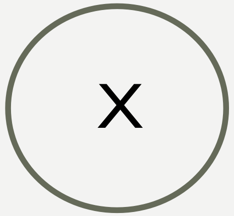
Q.2) ***"Every men are Mortal"***

let,

M(x):"x is mortal"

$=\forall_x M(x)$

MEN



Domain(Universe of Discourse)

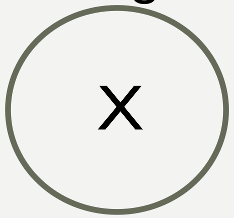
Q.3) "x+1>x"

let,

P(x):"x+1>x"

$=\forall_x P(x)$

+ve integers



Domain(Universe of Discourse)

1. UNIVERSAL QUANTIFICATION($\forall$):

Q.4) Let Q(x) be the statement "x<5". What is the truth value of Quantification, $\forall_x Q(X)$, where domain of discourse is all real numbers.

Solution

Q(x) is not True for every real number.

for instance,

Q(6)="6<5" is FALSE.

Thus , $\forall_x Q(x)$ is FALSE.

COUNTER EXAMPLE: An Element for which P(x) is False is called Counter Example of $\forall_x P(x)$.

Q.5) What is the truth value of $\forall_x P(x)$, where P(x) is the statement "$X^2 < 10$" and the domain consists of the positive integers not exceeding 4.

Solution: The statement $\forall_x P(x)$ is the same as the conjunction

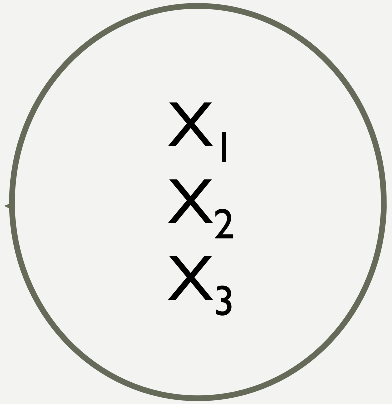
$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers 1, 2, 3, and 4.

Because P (4), which is the statement "42 < 10," is false, it follows that $\forall_x P(x)$ is false.

2. EXISTENTIAN QUANTIFICATION(∃):

"some lion drinks milk"

LION



Domain(Universe of Discourse)

Above statement is Equivalent to:

X_1 Drinks milk.
V
X_2 Drinks milk.
V
X_3 Drinks milk.

Milk(X_1)
V
Milk(X_2)
V
Milk(X_3)

$$\exists_x \text{Milk}(X)$$

=There exist an x in the domain such that Milk(X)
=There is at least one x such that Milk(x)
=for some x, Milk(x)

2. EXISTENTIAN QUANTIFICATION(∃):

Example:

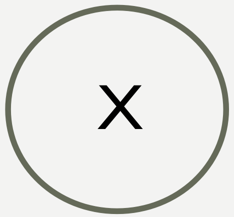
Q.1) ***"Some people are Bad"***

let,

B(x):"x is Bad"

$=\exists_x B(x)$

People



Domain(Universe of Discourse)

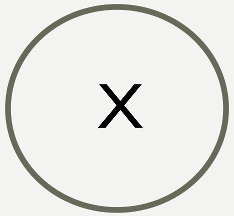
Q.2) ***"Some dogs are big"***

let,

D(x):"x is Big"

$=\exists_x D(x)$

DOGS



Domain(Universe of Discourse)

2. EXISTENTIAN QUANTIFICATION(∃):

Q.1) Let P (x) denote the statement "x > 3." What is the truth value of the quantification $\exists_x P(x)$, where the domain consists of all real numbers.

Solution:

Because "x > 3" is sometimes true—for instance, when x = 4—the existential quantification of P (x), which is $\exists_x P(x)$, is true.

Q.2) What is the truth value of $\exists_x P(x)$, where P(x) is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution:

Because the domain is {1, 2, 3, 4}, the proposition $\exists_x P(x)$ is the same as the disjunction P (1) ∨ P (2) ∨ P (3) ∨ P (4). Because P (4), which is the statement "42 > 10," is true, it follows that $\exists_x P(x)$ is true.

Statement	When True?	When False?
$\forall_x P(x)$	P(x) is true for every x.	There is an x for which P(x) is false
$\exists_x P(x)$	There is an x for which P(x) is true	P(x) is false for every x.

FREE & BOUND VARIABLES:

- When the variable is assigned a value or it is quantified it is called bound variable. If the variable is not bounded then it is called free variable.

- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

- Example:
  1. P(x, y) has two free variables x and y.
  2. P(2, y) has one bound variable 2 and one free variable y.
  3. ∀x P(x) has a bound variable x.
  4. ∀x P(x, y) has one bound variable x and one free variable y.

- Expression with no free variable is a proposition.

- Expression with at least one free variable is a predicate only.

NEGATING QUANTIFICATIONS:

1. Negating Universal Quantification:

   $\neg[\forall_x P(x)]$

   a) Negate the Proposition Function$[\neg P(x)]$

   b) Change to Existential Quantification

$$\boxed{\neg[\forall_x P(x)] = \exists_x \neg[\ P(x)]}$$

2. Negating Existential Quantification:

   $\neg[\exists_x P(x)]$

   a) Negate the Proposition Function$[\neg P(x)]$

   b) Change to Universal Quantification

$$\boxed{\neg[\exists_x P(x)] = \forall_x[\neg P(x)]}$$

De-Morgan's Law For Quantifiers

Negate The Following :

1. ***"Every student in BEIT has Taken Data mining***" [Domain: All BEIT student]

   <u>Solution</u>

   let, p(x): "x has taken Data Mining"

   $=\forall_x P(x)$

   Negation:

   $= \boldsymbol{\neg[\forall_x P(x)]}$

   $\boldsymbol{=\exists_x \neg[\, P(x)]}$

   "There is a student in BEIT who has not taken Data Mining"

2. "***There is a student in class who has long hair"*** [Domain: All BEIT student]

   <u>Solution</u>

   let, p(x): "x has long hair"

   $= \exists_{\mathbf{x}} P(x)$

   Negation:

   $= \boldsymbol{\neg[}\exists_{\mathbf{x}} P(x)\boldsymbol{]}$

   $\boldsymbol{=\forall_x \neg[\, P(x)]}$

   "All student in the class do not have long hair"

3. What are the negations of the statements:

a) $\forall_x(x^2 > x)$

Solution:

The negation of $\forall_x(x^2 > x)$ is,

$\neg\forall_x(x^2 > x)$, which is equivalent to

$\exists_x\neg(x^2 > x)$.

This can be rewritten as $\exists_x(x^2 \leq x)$.

b) $\exists_x(x^2 = 2)$

Solution:

The negation of $\exists_x(x^2 = 2)$is,

$\neg\exists_x(x^2 = 2)$, which is equivalent to

$\forall_x\neg(x^2 = 2)$.

This can be rewritten as $\forall_x(x^2 != 2)$.

TRANSLATING FROM ENGLISH:

1. Express the statement *"Every student in BEIT class has studied calculus"* using predicates and quantifiers.

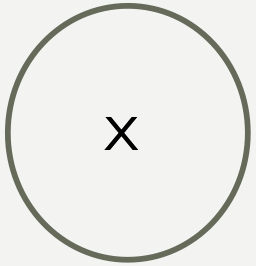
Solution:

First, we introduce a variable x so that our statement becomes *"For every student x in BEIT, x has studied calculus."*

Now, let C(x): *"x has studied calculus."*

Domain: BEIT

$= \forall_x C(x)$



DOMAIN: BEIT students

Domain: All people

Our statement becomes:

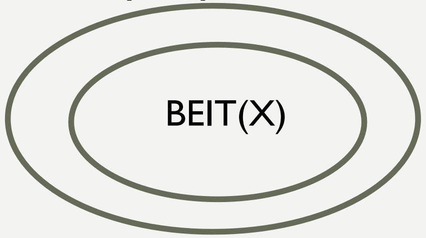
"For every person x, if person x is a student in BEIT, then x has studied calculus."

Now, let S(x) : *"x is a student in BEIT."*

C(x) : *"x has studied calculus"*

$= \forall_x [S(x) \rightarrow C(x)]$

all people



TRANSLATING FROM ENGLISH:

Domain: All people

Our statement becomes:

"For every person x, if person x is a student in BEIT, then x has studied calculus."

Now, let S(x) :*"x is a student in BEIT."*

C(x) :*"x has studied calculus"*

$$=\forall_x[\mathbf{S(x)} \rightarrow \mathbf{C(x)}]$$

Q(x, Calculus) :*"Student x has studied Calculus"*

$$=\forall_x[\mathbf{S(x)} \rightarrow \mathbf{Q(x, Calculus)}]$$

[Caution! Our statement cannot be expressed as $\forall_x[\mathbf{S(x)} \land \mathbf{C(x)}]$ because this statement says that all people are students in this class and have studied calculus!]

TRANSLATING FROM ENGLISH:

2. Express the statement *"Some student in this class has visited Jhapa"* using predicates and quantifiers.

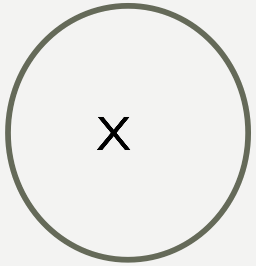
Solution:

First, we introduce a variable x so that our statement becomes *"There is a student x in this class that has visited Jhapa"*

Now, let p(x):*"x has visited Jhapa"*

Domain: BEIT

$$= \exists_x p(x)$$



DOMAIN: BEIT students

Domain: All people

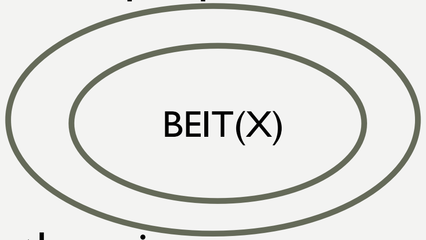
Our statement becomes:

"There is a person x, if person x is in this class, then x has studied calculus."

Now, let S(x) :*"x is a student in BEIT."*

p(x) :*"x has visited Jhapa"*

$$= \exists_x [S(x) \wedge p(x)]$$



Caution! Our statement cannot be expressed as $\exists_x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.

TRANSLATING FROM ENGLISH:

3. Express the statement *"Every Student in this class has visited Jhapa or Kathmandu"* using predicates and quantifiers.

Solution:

Now, let k(x): *"x has visited Kathmandu"*

J(x): *"x has visited Jhapa"*

Domain: BEIT

= $\forall_x[k(x) \lor j(x)]$

Domain: All people

Our statement becomes:

"for all person x, if person x is in this class, then x has visited Jhapa or Kathmandu."

Now, let S(x) : *"x is a student in BEIT."*

k(x): *"x has visited Kathmandu"*

J(x): *"x has visited Jhapa"*

= **$\forall_x [S(x) \rightarrow (k(x) \lor j(x))]$**

TRANSLATING FROM ENGLISH:

Consider these statement:

- *No professor are ignorant*
- *All ignorant people are vain*
- *Some professor are ignorant*

Let , P(x): *x is a Professor* , I(x): *x is ignorant* ,V(x): *x is vain*

Express above statement using quantifiers where domain consist of all people

a) No professor are ignorant

$$\forall_x[p(x) \rightarrow \neg q(x)]$$

b) All ignorant people are vain

$$\forall_x[q(x) \rightarrow r(x)]$$

c) Some professor are ignorant

$$\exists_x[p(x) \wedge q(x)]$$

TRANSLATING FROM ENGLISH:

Q. Let P (x) be the statement "x can speak Russian"

Q(x) be the statement "x knows the computer language C++."

Express each of these sentences in terms of P (x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

   =$\exists_x$ (P(x) /\ Q(x)).

a) There is a student at your school who can speak Russian but who doesn't know C++.

   = $\exists_x$ (P(x) /\ ¬ Q(x))

a) Every student at your school either can speak Russian or knows C++.

   = $\forall_x$ (P(x) V Q(x))

a) No student at your school can speak Russian or knows C++.

   = $\forall_x$ **[¬ P(x) ^ ¬ Q(x)]**

TRANSLATING FROM ENGLISH:

1. No one is sleeping.

    Negation of above: There is some who is sleeping

    $$=\exists_x[Person(x) \wedge sleeping(x)]$$

    Now , negate the predicate:

    $$=\neg\exists_x[Person(x) \wedge sleeping(x)]$$

2. Not everyone is sleeping.

    Negation of above: Everyone is sleeping.

    $$= \forall_x [Person(x) \rightarrow Sleeping(x)]$$

    Now , negate the predicate:

    $$=\neg\forall_x [Person(x) \rightarrow Sleeping(x)]$$

3. No one in this class is wearing glass and a cap.

    Negation of above statement: There is some one in this class who is wearing glass and a cap.

    $$=\exists_x[Glass(x) \wedge Cap(x)]$$

    Now, negate the predicate:

    $$=\neg\exists_x[Glass(x) \wedge Cap(x)]$$