MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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RULES OF INFERENCE

ARGUMENT:

- An argument is a sequence of proposition written as:

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline \therefore q_n \end{array} \qquad (P_1 \wedge P_2 \ldots\ldots P_n) \rightarrow Q \text{ is TAUTOLOGY}$$

- P_1, P_2 …..are called the Hypothesis or premises and the proposition Q is called Conclusion.
- The argument is valid provided that P_1, P_2 ……and P_n all are TRUE ,then Q also must be TRUE.
- This process of Drawing a conclusion from a sequence of proposition is called Deductive Reasoning.

RULES OF INFERENCE:

- If an argument consists of 10 different proposition variable then $2^{10}=1024$ combination are needed for Truth Table which is a tedious approach.
- Instead we can first establish the validity of some relatively simple arguments forms, called Rules of Inference.
- These rules then can be used to construct more complicated valid arguments.

1. MODUS PONENS:

- It states that if P and P→Q is TRUE then, we can infer Q is true.
- That is, (P^(P→Q))→Q is TAUTOLOGY

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Proof By Truth Table:

p	q	p→q	(p→q) ^p	((p→q) ^p)→q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

2. MODUS TOLLENS:

- It states that if P→Q and ¬Q is TRUE then, we can infer ¬P is true.
- That is, (¬q^(P→Q))→¬P is TAUTOLOGY

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Proof By Truth Table:

p	q	p→q	¬p	¬q	(p→q) ^¬q	(p→q) ^¬q)→¬p
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	F	T
F	F	T	T	T	T	T

3. HYPOTHETICAL SYLLOGISM:

- It states that if P→Q and Q→R is TRUE then, we can infer P→R is true.
- That is, ((P→Q)^(Q→R))→(P→R) is TAUTOLOGY

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

P	q	r	p→q	q→r	p→r	(p→q)^(q→r)	((p→q)^(q→r))→(p→r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

4. DISJUNCTIVE SYLLOGISM:

- It states that if PvQ and ¬P is TRUE then, we can infer Q is true.
- That is, (¬P^(PvQ))→Q is TAUTOLOGY

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Proof By Truth Table:

p	q	pVq	¬p	(pVq)^¬p	((pVq)^¬p)→q
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

5. ADDITION:

- It states that if P is TRUE then, PvQ will be TRUE.
- That is, P→(PvQ) is TAUTOLOGY

$$\frac{p}{\therefore p \vee q}$$

Proof By Truth Table:

p	q	pVq	P→(pVq)
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

6. SIMPLIFICATION:

- It states that if P^Q is TRUE then, P will be TRUE.
- That is, (P^Q)→P is TAUTOLOGY

$$\frac{p \wedge q}{\therefore p}$$

Proof By Truth Table:

p	q	p^q	(p^q)→p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

7. CONJUNCTION:

- It states that if P is TRUE and Q is TRUE then, P^Q will be TRUE.
- That is, (P)^(Q)→(P^Q) is TAUTOLOGY

$$\begin{array}{l} \quad p \\ \quad q \\ \hline \therefore p \wedge q \end{array}$$

Proof By Truth Table:

p	q	p^q	(p^q)→(p^q)
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

8. RESOLUTION:

- It states that if (PvQ) and (¬PvR) is TRUE then, we can infer (QvR) is true.
- That is,((PvQ)^(¬PvR))→(QvR) is TAUTOLOGY

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

p	q	r	¬p	pVq	¬pVr	qVr	((pVq)^(¬pVr))→(qVr)
T	T	T	F	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

RULES OF INFERENCE:

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Q.1) State which rule of inference is the basis of the following argument:

"It is below freezing now. Therefore, it is either below freezing or raining now."

<u>Solution:</u>

Let, p :"*It is below freezing now*"

q :"*It is raining now.*"

Then this argument is of the form:

$$\frac{p}{\therefore p \lor q}$$

This is an argument that uses the **addition rule.**

Q.2) State which rule of inference is the basis of the following argument:

"It is below freezing and raining now. Therefore, it is below freezing now."

<u>Solution:</u>

let, p:"*It is below freezing now*"

q :"*It is raining now*"

This argument is of the form:

$$\frac{p \land q}{\therefore p}$$

This argument uses the **simplification rule.**

Q.3) State which rule of inference is the basis of the following argument:

"If you have a current password, then you can log onto the network. You have a current password. Therefore, You can log onto the network."

<u>Solution:</u>

Let, p :"*you have a current password*"

q :"*you can log onto the network*"

Then this argument is of the form:

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

This is an argument that uses the **Modus Ponens rule.**

Q.4) State which rule of inference is the basis of the following argument:

"If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow"

<u>Solution</u>:

let, p:"*it rains today*"

q :"*We will have a barbecue today*"

r :"*we will have a barbecue tomorrow*"

This argument is of the form:

$$\begin{array}{l} p \rightarrow \neg q \\ \underline{\neg q \rightarrow r \qquad} \\ \therefore p \rightarrow r \end{array}$$

This argument uses the **Hypothetical rule.**

Q.5) Show that the premises: *"If I play football then I am tired the next day"*, *"I will take rest if I am tired"*, *"I did not take rest"* will lead to the conclusion *"I did not play football"*.

Solution:

Let, p: *"If I play football"*

q: *"I am tired"*

r: *"I will take rest "*

Hypothesis: i) $p \rightarrow q$

ii) $q \rightarrow r$

iii) $\neg r$

Conclusion: $\therefore \neg p$

s.n.	STEPS	REASONS
1.	$p \rightarrow q$	Given Hypothesis
2.	$q \rightarrow r$	Given Hypothesis
3.	$p \rightarrow r$	HYPOTHTICAL SYLLOGISM IN 1 & 2
4.	$\neg r$	Given Hypothesis
5.	$\neg p$	MODUS TOLLENSON 3 & 4

Q.6) Show that the premises *"It is not sunny this afternoon and it is colder than yesterday". "We will go swimming only if it is sunny." "If we do not go swimming, then we will take a canoe trip."* and *"If we take a canoe trip, then we will be home by sunset"* lead to the conclusion *"We will be home by sunset."*

<u>*Solution:*</u>

Let, p:"*It is sunny this afternoon*"

q:"*It is colder than yesterday*"

r:"*We will go swimming*"

s:"*We will take canoe trip*"

t:"*We will be home by sunset*"

Hypothesis: i) $\neg p \wedge q$

ii) $r \rightarrow p$

iii) $\neg r \rightarrow s$

iv) $s \rightarrow t$

Conclusion: $\therefore t$

s.n.	STEPS	REASONS
1.	$\neg p \wedge q$	Given Hypothesis
2.	$\neg p$	SIMPLIFICATION ON 1
3.	$r \rightarrow p$	Given Hypothesis
4.	$\neg r$	MODUS TOLLENS ON 2 & 3
5.	$\neg r \rightarrow s$	Given Hypothesis
6.	s	MODUS PONENS ON 4 & 5
7.	$s \rightarrow t$	Given Hypothesis
8.	t	MODUS PONENS ON 6 & 7

Q.7) Show that the premises *"If you send me an e-mail message, then I will finish writing the program", "If you do not send me an e-mail message, then I will go to sleep early," "If I go to sleep early, then I will wake up feeling refreshed"* lead to the conclusion *"If I do not finish writing the program, then I will wake up feeling refreshed."*

<u>Solution:</u>

Let, p:"*you send me an e-mail message*"
q: *"I will finish writing the program*"
r: *"I will go to sleep early*"
s:"*I will wake up feeling refreshed*"

Hypothesis: i) $p \rightarrow q$
ii) $\neg p \rightarrow r$
iii) $r \rightarrow s$

Conclusion: $\therefore \neg q \rightarrow s$

s.n.	STEPS	REASONS
1.	$p \rightarrow q$	Given Hypothesis
2.	$\neg q \rightarrow \neg p$	CONTRAPOSITIVE ON 1
3.	$\neg p \rightarrow r$	Given Hypothesis
4.	$\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5.	$r \rightarrow s$	Given Hypothesis
6.	$\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

Q.8) Show that the premises *"If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded,"* and *"The trophy was not awarded"* imply the conclusion *"It rained."*

Solution:

Let, p: *"It rains"*

q: *"It is foggy"*

r: *"The sailing race is held"*

s: *"Life saving demonstration is done"*

t: *"Trophy is awarded"*

Hypothesis: i) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$

ii) $r \rightarrow t$

iii) $\neg t$

Conclusion: $\therefore p$

s.n.	STEPS	REASONS
1.	$r \rightarrow t$	Given Hypothesis
2.	$\neg t$	Given Hypothesis
3.	$\neg r$	MODUS TOLLENS ON 1 & 2
4.	$(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Given Hypothesis
5.	$\neg r \vee \neg s$	Addition on 3
6.	$\neg(r \wedge s)$	DE-MORGAN'S LAW on 5
7	$\neg(\neg p \vee \neg q)$	MODUS TOLLENS ON 4 and 6
8	$p \wedge q$	DE-MORGAN'S LAW on 7
9	p	SIMPLIFICATION on 8

Q.9) Show that the premises *"If the interest rate drops , the housing market will improve", "The federal discount rate will drop or the housing market will not improve", "Interest rate will drop"* imply the conclusion *"The federal discount rate will drop"*

Solution:

Let, p: *"the interest rate drops "*
q: *"the housing market will improve"*
r: *"The federal discount rate will drop"*

Hypothesis: i) $p \rightarrow q$
ii) $r \vee \neg q$
iii) p

Conclusion: $\therefore r$

STEPS	REASONS
1. $p \rightarrow q$	Given Hypothesis
2. $\neg p \vee q$	Implication on 1
3. $r \vee \neg q$	Given Hypothesis
4. $\neg p \vee r$	Resolution From 2 and 3
5. p	Given hypothesis
6. r	From 4 and 5

Q.10) Show that the premises *"If my cheque book is in office, then I have paid my phone bill", "I was looking for phone bill at breakfast or I was looking for phone bill in my office", "If was looking for phone bill at breakfast then my cheque book is on breakfast table" , "If I was looking for phone bill in my office then my cheque book is in my office", "I have not paid my phone bill"* imply the conclusion *"My cheque book is on my breakfast table"*

Solution:

Let, p: *"my cheque book is in office"*
q: *"I have paid my phone bill"*
r: *"I was looking for phone bill at breakfast "*
s: *"I was looking for phone bill in my office"*
t: *"my cheque book is on breakfast table"*

Hypothesis: i) $p \rightarrow q$
ii) $r \vee s$
iii) $r \rightarrow t$
iv) $s \rightarrow p$
v) $\neg q$

Conclusion: $\therefore t$

Hypothesis: i) $p \rightarrow q$
ii) $r \vee s$
iii) $r \rightarrow t$
iv) $s \rightarrow p$
v) $\neg q$

Conclusion: $\therefore t$

STEPS	REASONS
1. $p \rightarrow q$	Given Hypothesis
2. $\neg q$	Given Hypothesis
3. $\neg p$	Modus Tollens on 1 and 2
4. $r \vee s$	Given Hypothesis
5. $r \rightarrow t$	Given Hypothesis
6. $\neg r \vee t$	Implication on 5
7. $s \vee t$	Resolution from 4 and 6
8. $s \rightarrow p$	Given Hypothesis
9. $\neg s \vee p$	Implication on 8
10. $t \vee p$	Resolution from 7 and 9
11. t	From 3 and 10

PROOF BY RESOLUTION:

- Propositional Resolution works only on expressions in *clausal form*. Before the rule can be applied, the premises and conclusions must be converted to this form.

A *literal* is either an atomic sentence or a negation of an atomic sentence. For example, if p is a logical constant, the following sentences are both literals.

$$p$$
$$\neg p$$

A *clausal sentence* is either a literal or a disjunction of literals. If p and q are logical constants, then the following are clausal sentences.

$$p$$
$$\neg p$$
$$\neg p \lor q$$

A *clause* is the set of literals in a clausal sentence. For example, the following sets are the clauses corresponding to the clausal sentences above.

$$\{p\}$$
$$\{\neg p\}$$
$$\{\neg p, q\}$$

CONVERTING TO CAUSAL FORM:

- Examples:
  1. $p \rightarrow q = \neg p \lor q$
  2. $p \leftrightarrow q = (\neg p \lor q) \wedge (\neg q \lor p)$-------{CNF}
  3. $\neg(p \wedge q) = \neg p \lor \neg q$
  4. $\neg(p \lor q) = \neg p \wedge \neg q$

CNF: CNF (Conjunctive normal form) if it is a $\wedge$(Conjunction) of $\vee$(Disjunction s) of literals (variables or their negation.)

DNF: DNF (Disjunctive normal form) if it is a $\vee$(Disjunction s) of $\wedge$(Conjunction) of literals (variables or their negation.)

Example:
1. $\neg p \lor \neg q$
2. $(p \wedge q) \lor (p \wedge r)$

Prove:

i) $P \rightarrow Q$

ii) $\neg P \rightarrow R$

iii) $R \rightarrow S$

$\therefore \neg Q \rightarrow S$

Solution:

The causal form of hypothesis and Conclusion are:

i) $\neg P \vee Q$

ii) $P \vee R$

iii) $\neg R \vee S$

$\therefore Q \vee S$

STEPS	REASONS
1. $\neg P \vee Q$	1. Given Hypothesis
2. $P \vee R$	2. Given Hypothesis
3. $Q \vee R = R \vee Q$	3. Using Resolution
4. $\neg R \vee S$	4. Given Hypothesis
5. $Q \vee S$	4. Using Resolution

Using Graphical Method:

Prove:

i) P→Q

ii) ¬P→R

iii) R→S

∴ ¬Q→S

STEP 1. The causal form of hypothesis and Conclusion are:

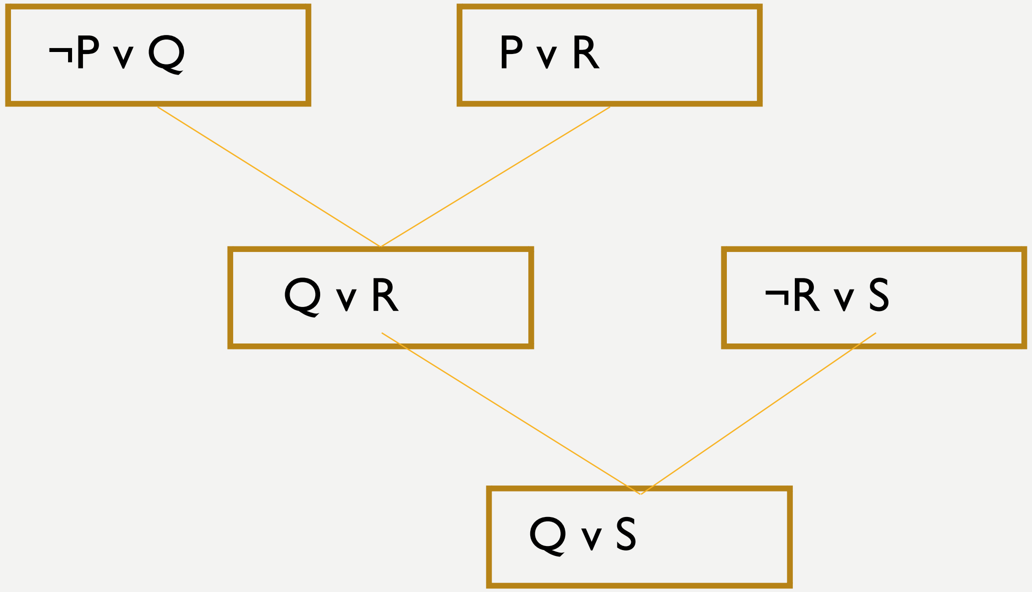
i) ¬P v Q

ii) P v R

iii) ¬R v S

∴ Q v S

STEP 2.



Using Graphical Method:

Prove:

i) P

ii) P→R

iii) R→S

∴ S

STEP 1. The causal form of hypothesis and Conclusion are:

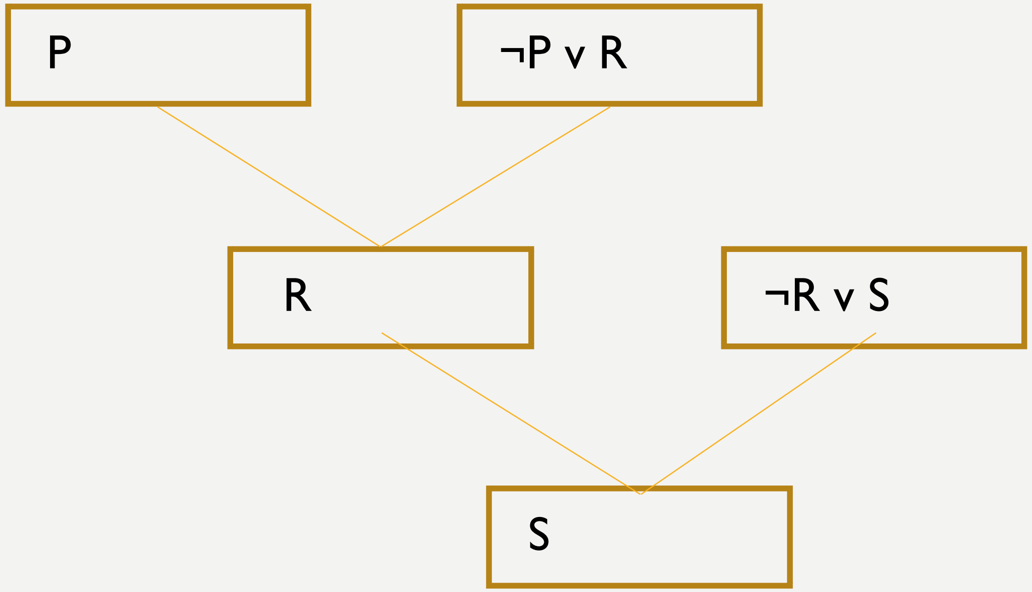
i) P

ii) ¬P v R

iii) ¬R v S

∴ S

STEP 2.



- Using resolution principle, prove that the hypotheses: "If today is Tuesday then I will have a test in Discreet Math or Microprocessor". "If my Microprocessor teacher is sick then I will not have a test in Microprocessor." "Today is Tuesday and my Microprocessor teacher is sick." lead to the conclusion that "I will have a test in Discrete Math"

Solution:

Let, p : "Today is Tuesday"

q : "I will have test in Discrete Math"

r : "I will have test in Microprocessor"

s : "My Microprocessor teacher is sick"

Hypothesis: i) $p \rightarrow (q \vee r)$

ii) $s \rightarrow \neg r$

iii) $p \wedge s$

Conclusion: q

The Causal Forms are:

Hypothesis:

i) $\neg p \vee (q \vee r)$

ii) $\neg s \vee \neg r$

iii) p

iv) s

Conclusion:

$\therefore q$

The Causal Forms are:

Hypothesis:

i) $\neg p \vee (q \vee r)$

ii) $\neg s \vee \neg r$

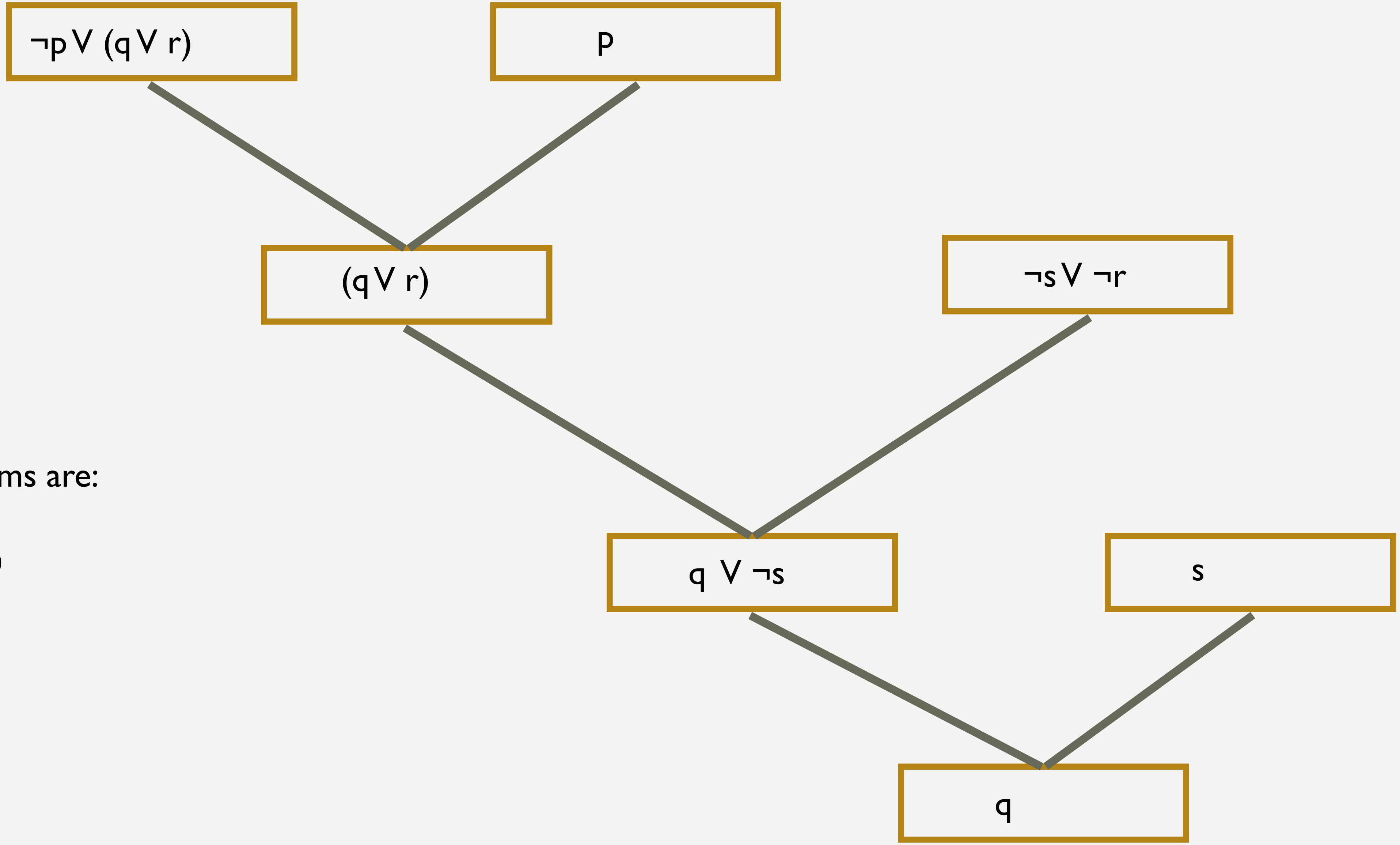
iii) p

iv) s

Conclusion:

$\therefore q$

STEPS	REASONS
1. $\neg p \vee (q \vee r)$	Given Hypothesis
2. p	Given Hypothesis
3. $q \vee r$	From 1 and 2
4. $\neg s \vee \neg r$	Given Hypothesis
5. $q \vee \neg s$	From 3 and 4
6. s	Given Hypothesis
7. q	From 5 and 6



The Causal Forms are:
Hypothesis:
i) ¬p V (q V r)
ii) ¬s V ¬r
iii) p
iv) s

Conclusion:
∴ q

SEMANTIC TABLEAUX:

- The formula is decomposed into its sub-formulas according to certain rules(α and β Rules), resulting a semantic tableau.

- Semantic Tableau is a binary tree constructed using semantic rules.

α Rule

a) P v Q
- P | Q

b)P ^ Q
- P
- Q

c)P→Q
- ¬ P | Q

d)P←→Q
- P, Q | ¬P, ¬Q

β Rule

a) ¬(¬P)
- P

b) ¬(P v Q)
- ¬P
- ¬Q

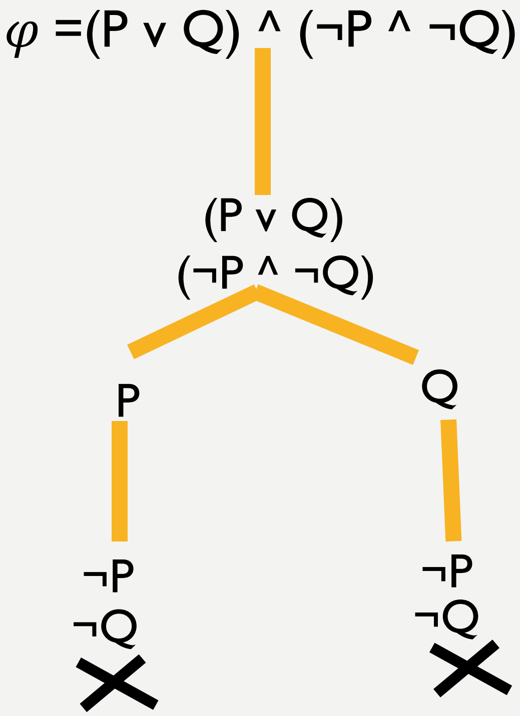
c) ¬(P ^ Q)
- ¬ P | ¬ Q

d) ¬(P→Q)
- P
- ¬Q

e) ¬(P←→Q)
- P, ¬ Q | ¬P, Q

SEMANTIC TABLEAUX:

- A finite set of formulas φ is satisfiable iff T(φ) is open.
- As a corollary, φ is contradictory (not satisfiable) iff T(φ) is closed

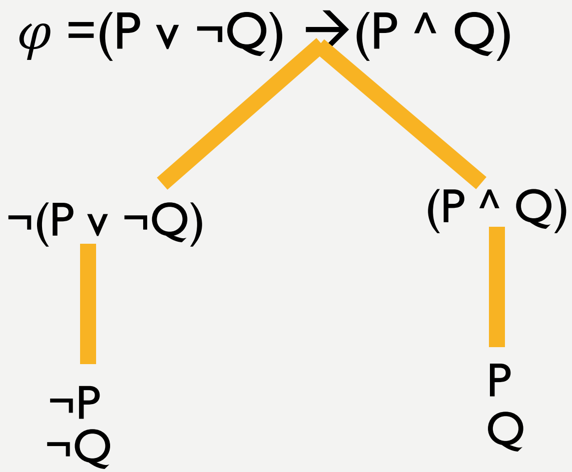


A=(P v Q)
B=(¬P ^ ¬Q)

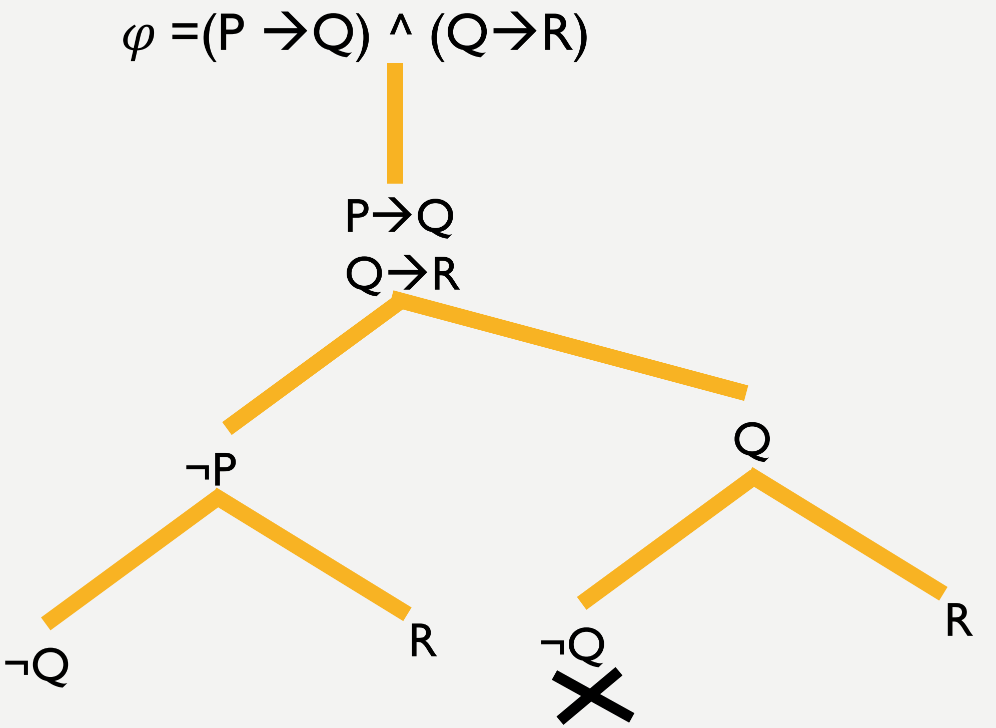
P	Q	¬P	¬Q	A	B	A^B
T	T	F	F	T	F	**F**
T	F	F	T	T	F	**F**
F	T	T	F	T	F	**F**
F	F	T	T	F	T	**F**

- When P and its negation ¬P appear on the same branch, a contradiction has been found and that branch is called closed.

SEMANTIC TABLEAUX:



SATISFIABLE
(There are open branches)

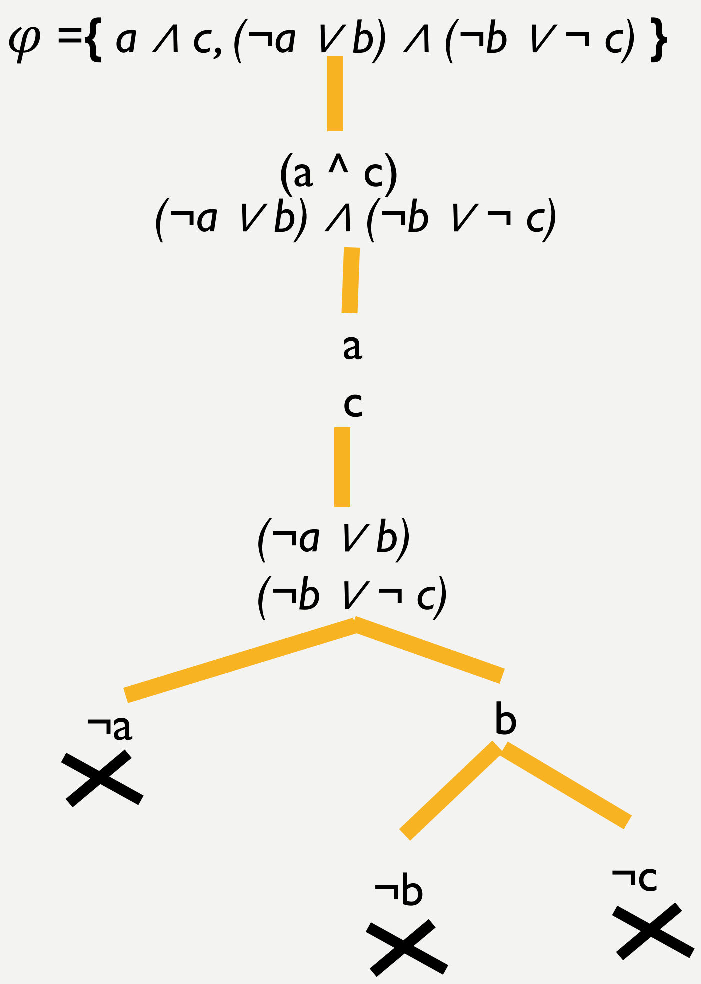


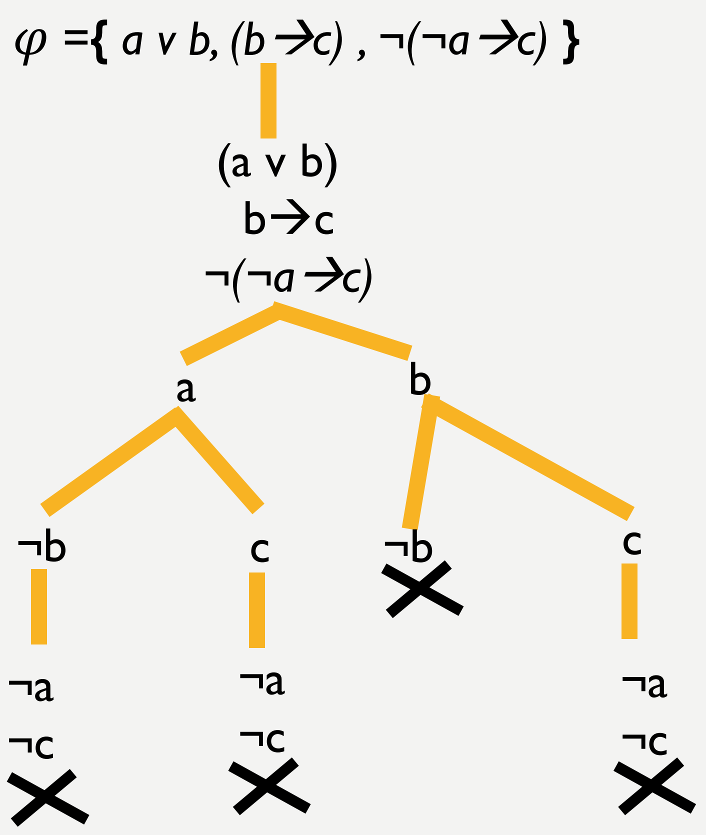
SATISFIABLE
(There are open branches)

P	Q	R	(P→Q)	(Q→R)	(P→Q)^(Q→R)
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

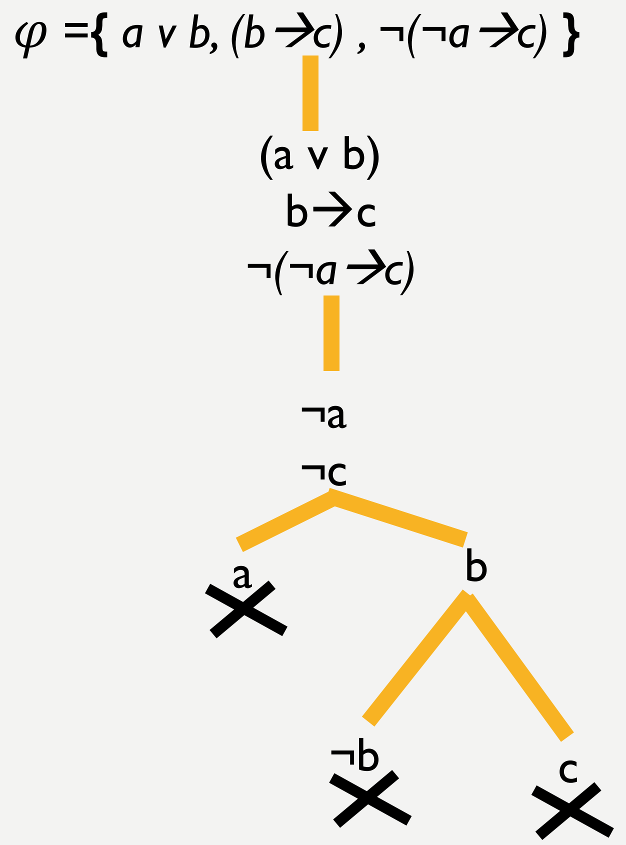
P	Q	¬Q	A=(PV¬Q)	B=(P^Q)	A→B
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

SEMANTIC TABLEAUX:





[Disjunction First Policy]



[Conjunction First Policy]

SEMANTIC TABLEAUX:

To prove the argument:

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ . \\ . \\ P_n \\ \hline \therefore Q \end{array}$$

$[(P_1 \wedge P_2, \ldots\ldots. \wedge P_n) \rightarrow Q]$ is Tautology.

- To prove the above argument we show that the set of premises along with negated conclusion is Unstisfiable i.e. we show semantic Tableau is Contradiction.

$$= [(P_1 \wedge P_2, \ldots\ldots. \wedge P_n) \rightarrow Q]$$
$$= \neg[(P_1 \wedge P_2, \ldots\ldots. \wedge P_n) \rightarrow Q]$$
$$= [P_1 \wedge P_2, \ldots\ldots.. \wedge P_n \wedge \neg Q]$$

- $\varphi = \{P_1, P_2, P_3, \ldots\ldots P_n, \neg Q\}$
- *If* $T(\varphi)$ is closed then the argument is valid else if $T(\varphi)$ is open the argument is invalid

Check if the following argument is valid or not.

$(C \wedge \neg G) \to P$

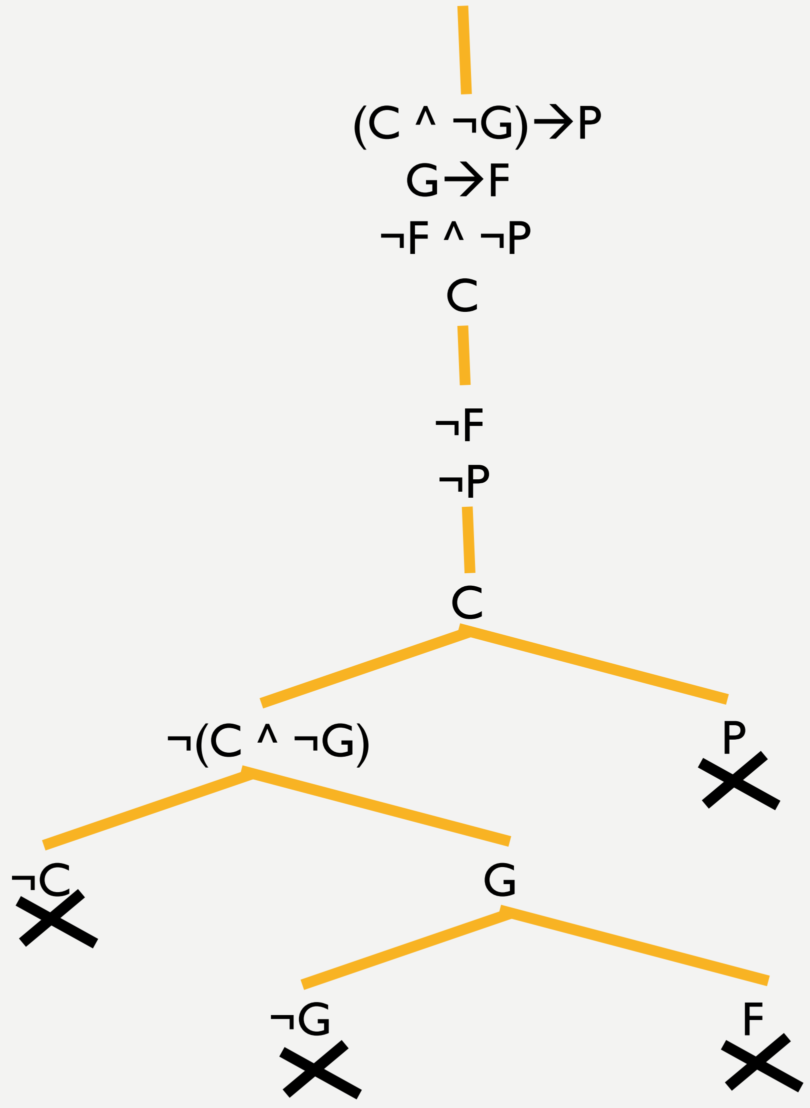
$G \to F$

$\neg F \wedge \neg P$

$\therefore \neg C$

$\varphi = \{(C \wedge \neg G) \to P,\ G \to F,\ \neg F \wedge \neg P,\ \neg(\neg C)\}$

$\varphi = \{(C \wedge \neg G) \to P,\ G \to F,\ \neg F \wedge \neg P,\ \neg(\neg C)\}$



The obtained Tableau is contradictory.Hence the argument is valid

i) $p \to q$
ii) $q \to r$
iii) $\neg r$
$\therefore \neg p$

Hypothesis: i) $(\neg p \vee \neg q) \to (r \wedge s)$
ii) $r \to t$
iii) $\neg t$
Conclusion: $\therefore p$

Hypothesis: i) $\neg p \wedge q$
ii) $r \to p$
iii) $\neg r \to s$
iv) $s \to t$
Conclusion: $\therefore t$

Hypothesis: i) $p \to q$
ii) $\neg p \to r$
iii) $r \to s$
Conclusion: $\therefore \neg q \to s$