Computer Graphics (L08)
EG678EX

2-D Algorithms

Translation

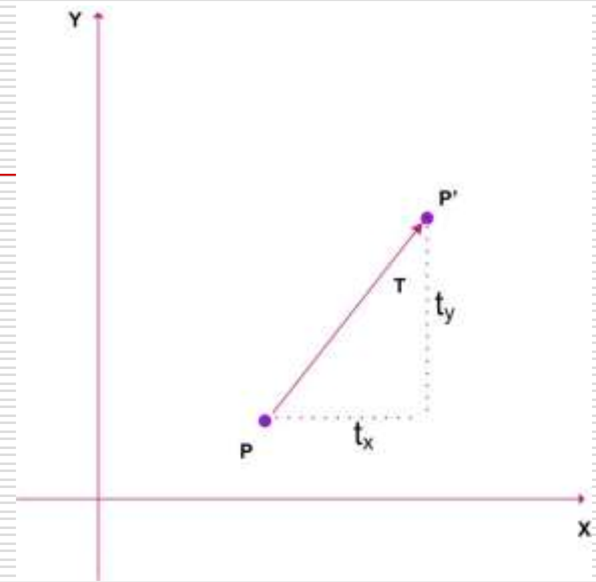
$$x' = x + t_x, \qquad y' = y + t_y$$

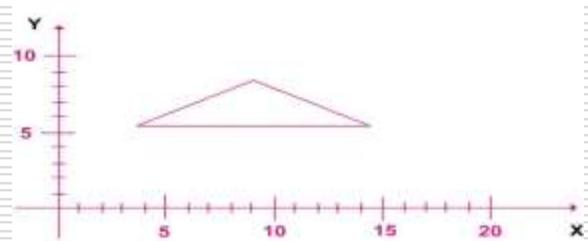
$$P = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad p' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T$$

In homogeneous representation if position P = (x,y) is translated to new position p’= (x’,y’) then:

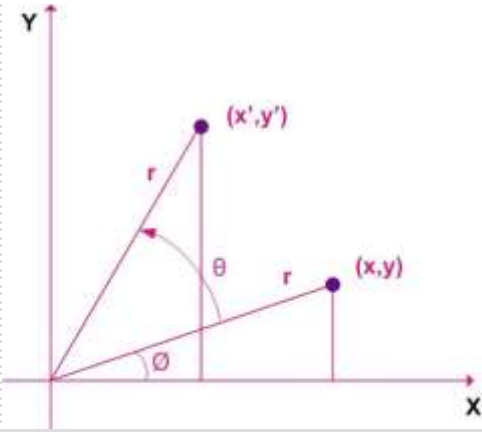
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = T(t_x, t_y).P$$





Rotation

$$x' = r\cos(\phi+\theta) = r\cos\phi.\cos\theta - r\sin\phi.\sin\theta$$
$$y' = r\sin(\phi+\theta) = r\cos\phi.\sin\theta + r\sin\phi.\cos\theta$$



$$x = r\cos\phi, \qquad y = r\sin\phi$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$P' = R.P$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

If Co-ordinates represented as row vector, Then:

$$P'^{T} = (R.P)^{T}$$
$$= \quad P^{T}.R^{T}$$

In homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

General Pivot Rotation

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$
$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$

Steps
1. Translate object so as to coincide pivot to origin
2. Rotate object about the origin
3. Translate object back so as to return pivot to original position

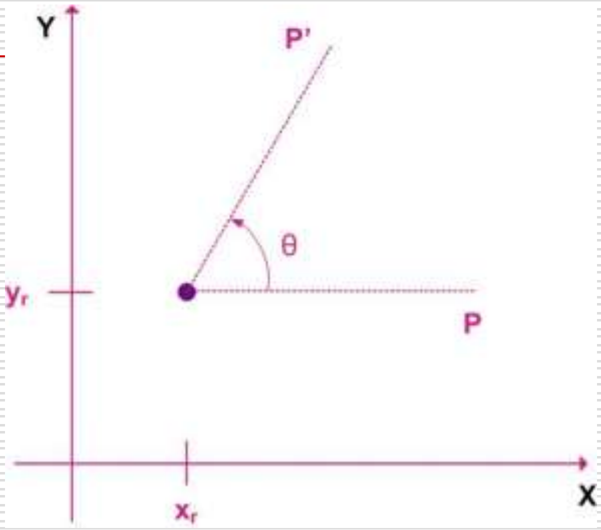
Composite Transformations

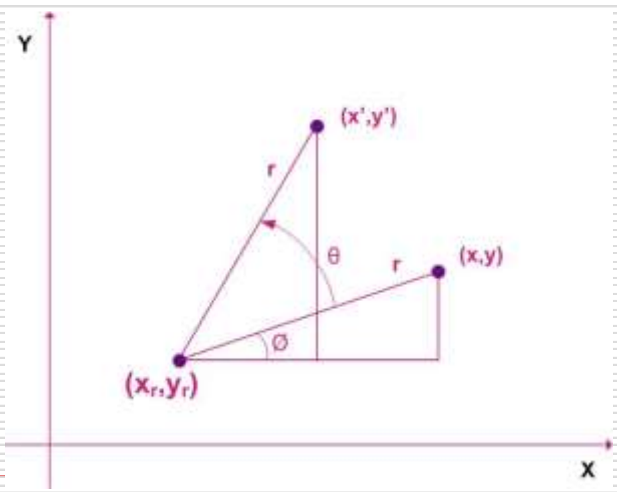
$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

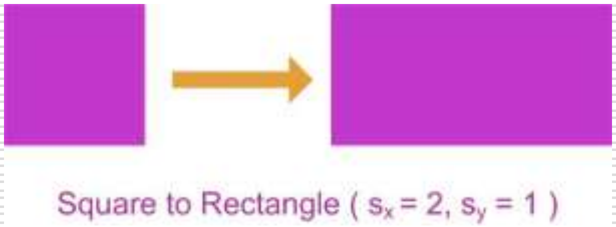
$$T(x_r, y_r).R(\theta).T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

$$T(-x_r, -y_r) = T^{-1}(x_r, y_r)$$





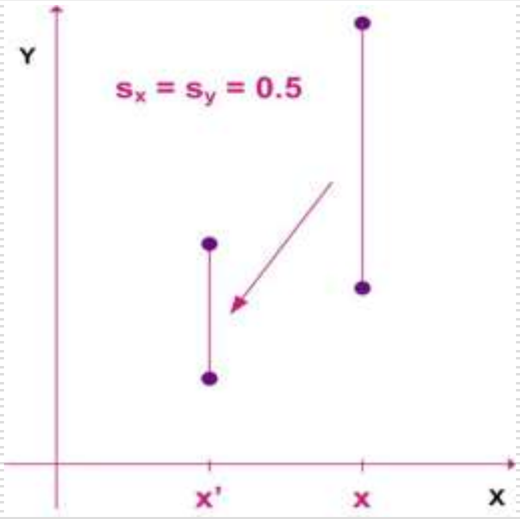
Scaling



$$x' = x.s_x, \qquad y' = y.s_y$$

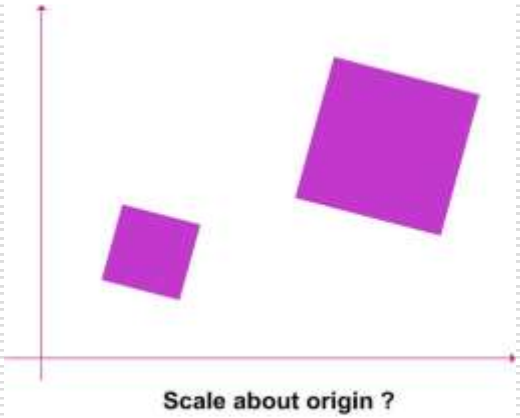
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S.P$$

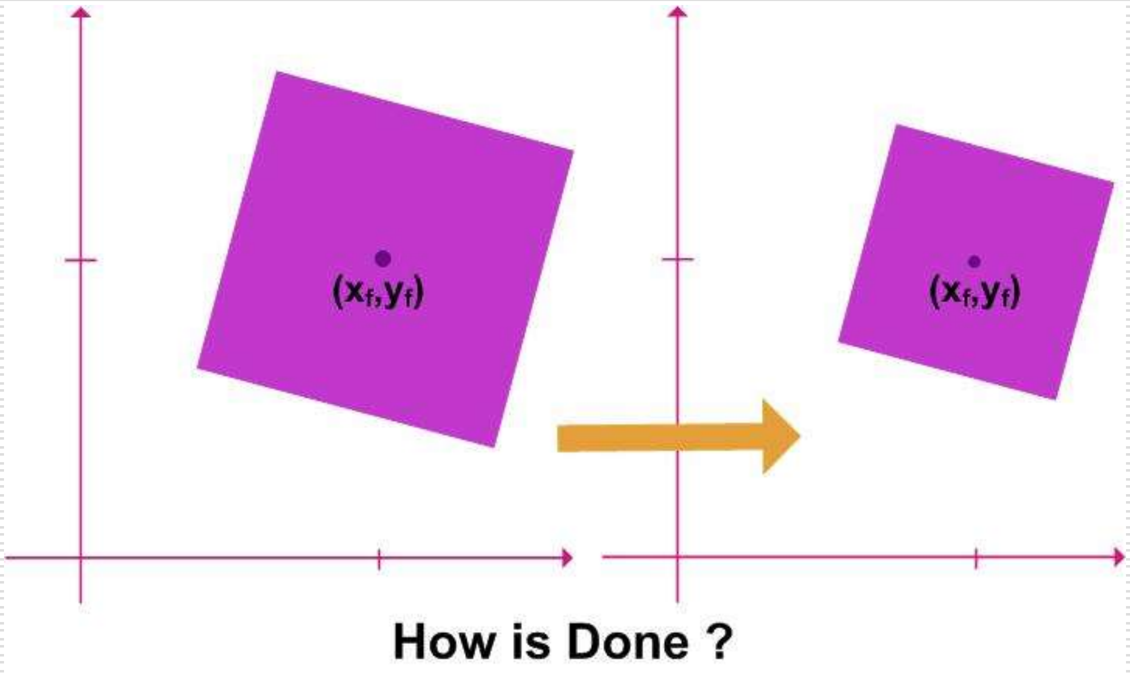


In homogeneous co-ordinate

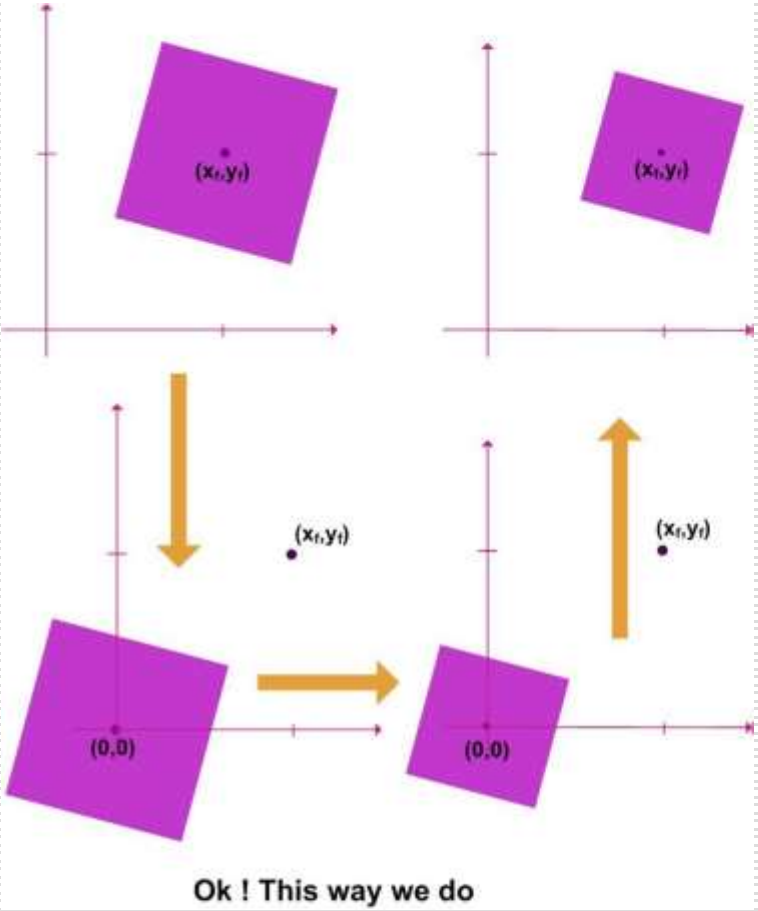
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = S(s_x, s_y).P$$

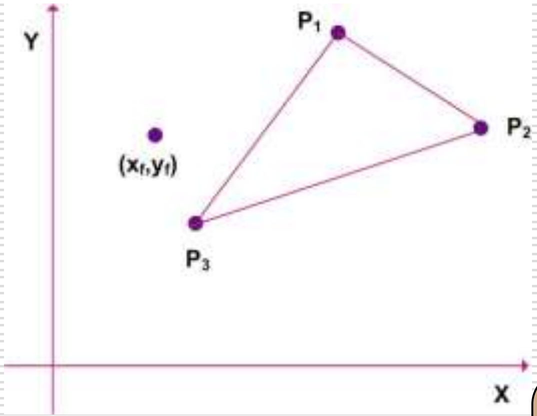


Fixed Point Scaling



Fixed Point Scaling





$$x' = x_f + (x - x_f).s_x,$$
$$y' = y_f + (y - y_f).s_y$$

$$x' = x.s_x + x_f(1 - s_x)$$
$$y' = y.s_y + y_f(1 - s_y)$$

Additive terms $x_f(1\text{-}s_x)$ and $y_f(1\text{-}s_y)$ are constant for all points in the object

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_f(1 - s_x) \\ y_f(1 - s_y) \end{bmatrix}$$

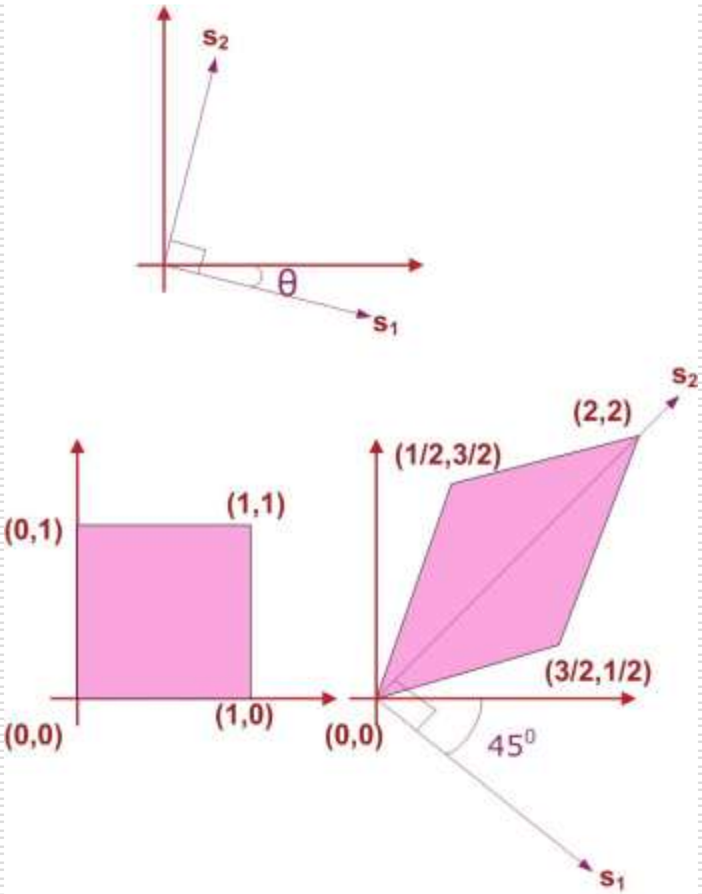
$$P' = S.P + C$$

General Scaling Directions

- Rotate the object so that x and y axes coincide with s_1 and s_2
- Apply scaling transformation in s_1 and s_2 direction
- Rotate the object in opposite direction to return points to the original orientations
- Example: square converted to parallelogram with $s_1=1$ and $s_2 = 2$ and $\theta = 45^0$ as shown in figure

$$R^{-1}(\theta).S(s_1, s_2).R(\theta)$$

$$= \begin{bmatrix} s_1 \cos^2\theta + s_2 \sin^2\theta & (s_2 - s_1)\cos\theta.\sin\theta & 0 \\ (s_2 - s_1)\cos\theta.\sin\theta & s_1 \sin^2\theta + s_2 \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composite Transformations

For Successive Translation vectors (tx1,ty1) and (tx2,ty2)

$$P' = T(t_{x2}, t_{y2}).\{T(t_{x1}, t_{y1}).P\}$$
$$= \{T(t_{x2}, t_{y2}).T(t_{x1}, t_{y1})\}.P$$

Successive Translations are additive

$$\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(t_{x2}, t_{y2}).T(t_{x1}, t_{y1}) = T(t_{x1} + t_{x2}, t_{y1} + t_{y2})$$

For Successive Rotations θ_1 and θ_2

$$P' = R(\theta_2).\{R(\theta_1).P\}$$
$$= \{R(\theta_2).R(\theta_1)\}.P$$

Successive Rotations are additive.

$$R(\theta_2).R(\theta_1) = R(\theta_1 + \theta_2)$$

!!!!!!! PROVE YOURSELF !!!!!!!!!

Composite Transformations

Successive Scaling are multiplicative

$$S(s_{x2}, s_{y2}).S(s_{x1}, s_{y1}) = S(s_{x1}.s_{x2}, s_{y1}.s_{y2})$$

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x1}.s_{x2} & 0 & 0 \\ 0 & s_{y1}.s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fixed Point Scaling

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(x_f, y_f).S(s_x, s_y).T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$$

Concatenation Properties

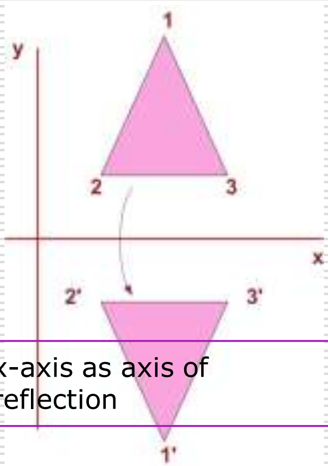
Matrix multiplication is associative, so

A.B.C = (A.B).C = A.(B.C)
Transformation product is not commutative, so

A.B ≠ B.A

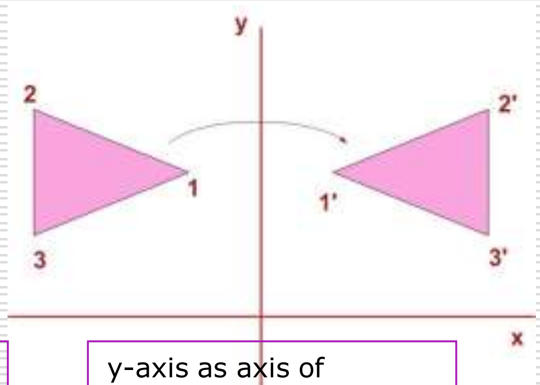
Reflection

- Transformation that produces mirror image of an object
- Mirror image is produced when rotated 180 degree about axis of reflection



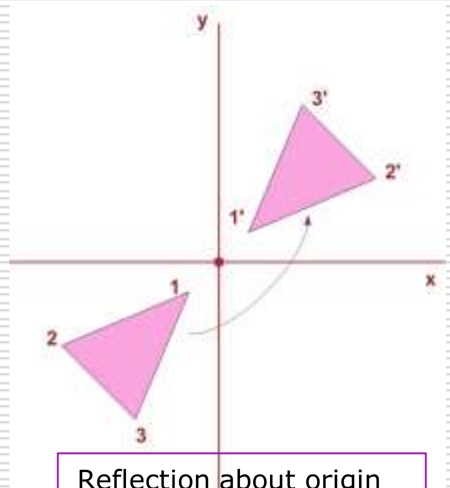
x-axis as axis of reflection

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



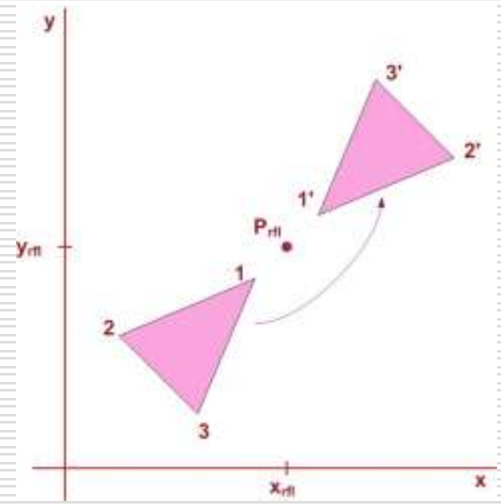
y-axis as axis of reflection

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



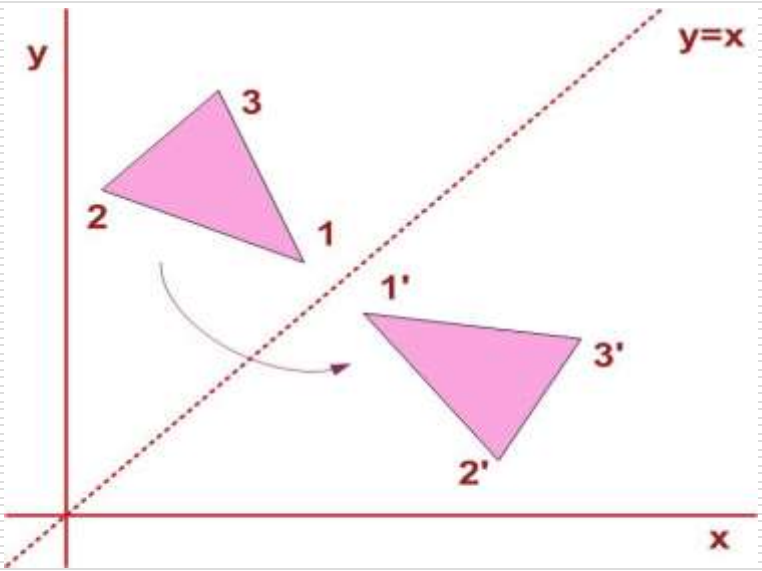
Reflection about origin

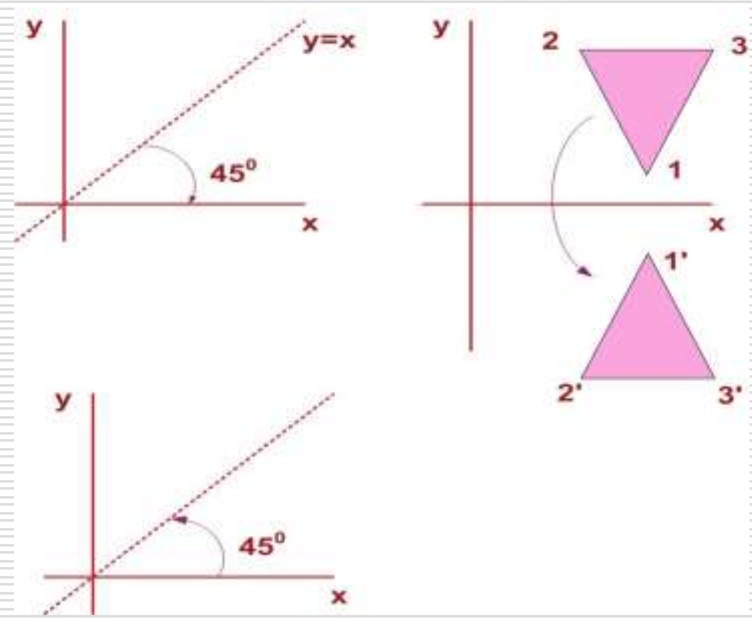
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection about fixed point

Reflection about a line





❑Reflection about a line y=x can be accomplished in the sequence of left→right→down in the second figure. The reflection matrix is as follows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

❑Similar sequence could be applied for line y=mx + b

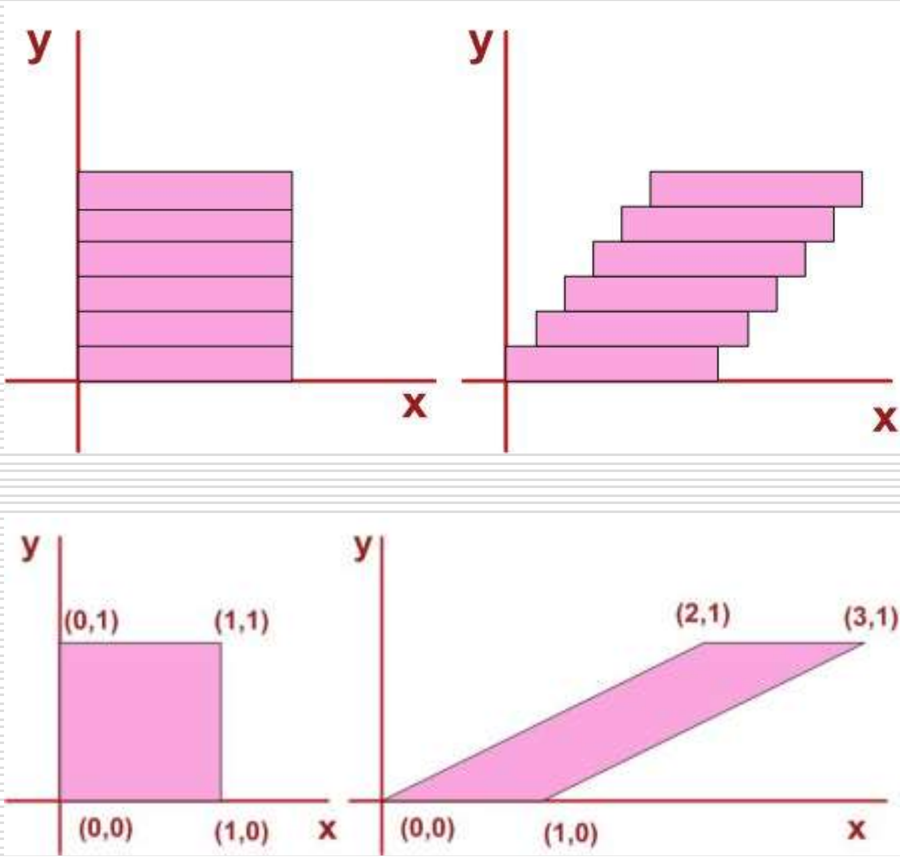
Shear

- Transformation that distorts the shape of an object such that Internal layers are shifted w.r.t each other.
- Mathematically: For fixed y, all points are shifted by fixed amount in the x-direction

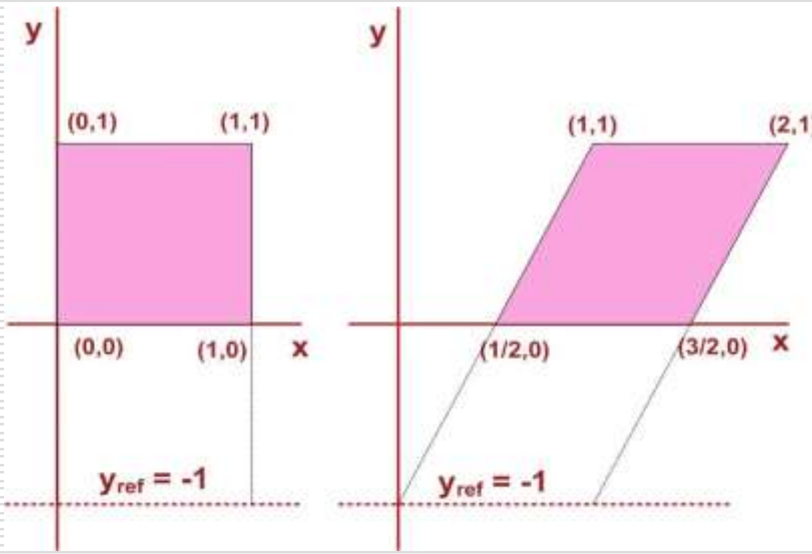
$$x' = x + sh_x . y$$
$$y' = y$$

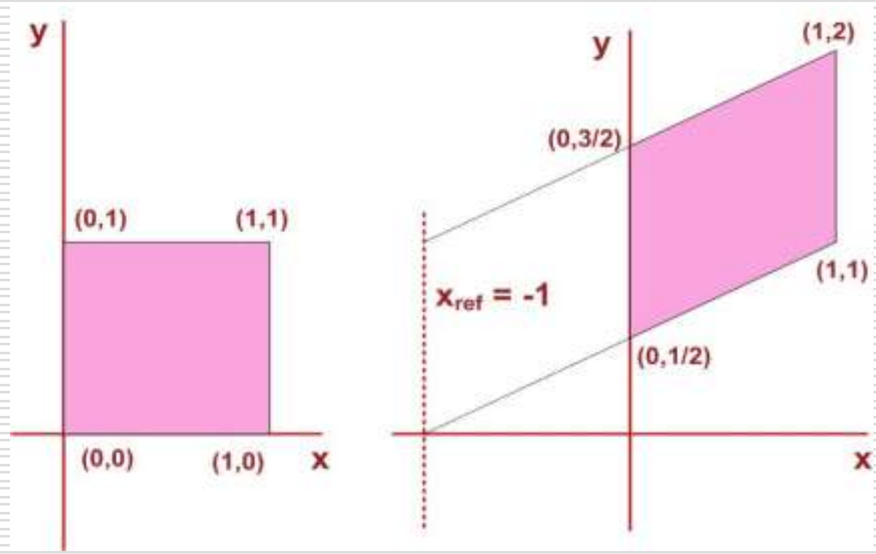
- All points on reference line (y=0 in figure) stays fixed under transformation
- Shear matrix in homogeneous co-ordinate:

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Shear





- Shear w.r.t y_{ref} = -1

  $x' = x + sh_x(y - y_{ref}), \quad y' = y$

$$\begin{bmatrix} 1 & sh_x & -sh_x.y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -(1/2).(-1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Shear w.r.t x_{ref} = -1

  $x' = x, \quad y' = y + sh_y(x - x_{ref})$

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y.x_{ref} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & -(1/2).(-1) \\ 0 & 0 & 1 \end{bmatrix}$$