MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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PREDICATE LOGIC

2.NESTED QUANTIFIERS

3.TRANSLATION FROM ENGLISH

1. NESTED QUANTIFIERS:

1. Consider the following :

"The sum of any two positive real number is positive"

This assertion can be restated as:

"for ever x and for every y , If x>0 and y>0, then x+y>0"

Let,

p(x, y) :"$(x>0)\wedge(y>0)\rightarrow(x+y)>0$

The given statement says that the sum of any two positive real number is positive, so we need two Universal quantifiers.

The Quantification is:

$$\boxed{\forall_x \forall_y [p(x , y)]}$$

1. NESTED QUANTIFIERS:

3. Consider the following :

“Some student in your class has taken some computer training course”

Restating above statement as:

“For some student x ,there exist a computer training course y such that x has taken y”

let, Q(x, y) :“Student x has taken training y”

The Quantification is:

$$\boxed{\exists_x \exists_y [Q(x , y)]}$$

1. NESTED QUANTIFIERS:

4. Consider the following :

"If a person is female and is a parent , then this person is someone's mother"

Restating above statement as:

"For every person x, if x is a female and person x is a parent , then there exist a person y such that person x is the mother of y.

let, $F(x)$:"x is female"

$P(x)$:" x is parent"

$M(x, y)$:"x is the mother of y"

The Quantification is:

$$= \forall_x[(F(x) \wedge P(x)) \rightarrow \exists_y M(x, y)]$$

$$= \forall_x \exists_y [(F(x) \wedge P(x)) \rightarrow M(x, y)]$$

1. NESTED QUANTIFIERS:

5. Consider the following :

"There is a man that has taken a flight on every airline in the world"

Restating above statement as:

"There is a man x, for all airlines a ,there exist a flight f such that x has taken flight f"

let, P(x, f) :"x has taken flight f"

Q(f, a) :" f is a flight on airline a"

The Quantification is:

$$=\exists_x \forall_a \exists_f[(P(x, f) \wedge Q(f, a)]$$

1. NESTED QUANTIFIERS:

Let,
L(x, y): "x loves y"

a) "Everyone Loves Somebody"
=For every person x, there exist person y such that x loves y
$=\forall_x \exists_y L(x, y)$

b) "Someone Loves Somebody"
=There exist some person x and some person y such that x loves y
$=\exists_x \exists_y L(x, y)$

c) "Someone is loved by everyone"
=There exist some person y for all x such that x loves y.
$=\exists_y \forall_x L(x, y)$

d) "Everybody Loves Everybody"
$= \forall_x \forall_y L(x, y)$

Q. Let $Q(x, y)$ denote "$x + y = 0$."
What are the truth values of the quantifications
a) $\exists_y \forall_x Q(x, y)$ and b) $\forall_x \exists_y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution:

a) The quantification $\exists_y \forall_x Q(x, y)$ denotes the proposition "There is a real number y such that for every real number x, $Q(x, y)$." No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists_y \forall_x Q(x, y)$ is false.

b) The quantification $\forall_x \exists_y Q(x, y)$ denotes the proposition "For every real number x there is a real number y such that $Q(x, y)$." Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall_x \exists_y Q(x, y)$ is true.

Q. Let $P(x, y)$ be the statement "$x + y = y + x$."
What are the truth values of the quantifications a) $\forall_x \forall_y P(x, y)$ and b) $\forall_y \forall_x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution:

a) The quantification $\forall_x \forall_y P(x, y)$ denotes the proposition "For all real numbers x, for all real numbers y, $x + y = y + x$." Because $P(x, y)$ is true for all real numbers x and y the proposition $\forall_x \forall_y P(x, y)$ is true.

b) The quantification $\forall_y \forall_x P(x, y)$ says "For all real numbers y, for all real numbers x, $x + y = y + x$." Because $P(x, y)$ is true for all real numbers x and y the proposition $\forall_y \forall_x P(x, y)$ is true.

Q. Let $Q(x, y)$ denote "$x + y = xy$."
What are the truth values of the quantifications
a) $\exists_x \exists_y Q(x, y)$ and b)$\exists_y \exists_x Q(x, y)$,domain for all variables consists of all positive real numbers?

Solution:

a) The quantification $\exists_x \exists_y Q(x, y)$ denotes the proposition "There is exist a number x such that for some number y, Q(x, y)." Q(x, y) is true for x=(0,2) and y= (0, 2). Hence, $\exists_x \exists_y Q(x, y)$ is TRUE

b) The quantification $\exists_y \exists_x Q(x, y)$ denotes the proposition "There is exist a number y such that for some number x, Q(x, y)." Q(x, y) is true for y=(0,2) and x= (0, 2). Hence, $\exists_y \exists_x Q(x, y)$ is TRUE

$\exists_y \forall_x Q(x, y)$!=$\forall_x \exists_y Q(x, y)$
$\forall_x \forall_y P(x, y)$ =$\forall_y \forall_x P(x, y)$
$\exists_x \exists_y P(x, y)$ =$\exists_y \exists_x P(x, y)$

2. NEGATING NESTED QUANTIFIERS:

- Statements involving nested quantifiers can be negated by successively applying the De-Morgan's rules for negating statements involving a single quantifier.

a) Express the negation of the statement $\forall_x \exists_y (xy = 1)$.

$=\neg[\forall_x \exists_y (xy = 1)]$

$=\exists_x \neg[\exists y (xy = 1)]$

$=\exists_x \forall_y \neg(xy = 1)$

$= \exists_x \forall_y (xy != 1)$

b) $\exists_x \exists_y P(x, y) \wedge \forall_x \forall_y Q(x, y)$

$= \neg[\exists_x \exists_y P(x, y) \wedge \forall_x \forall_y Q(x, y)]$

$=\neg[\exists_x \exists_y P(x, y)] \vee \neg[\forall_x \forall_y Q(x, y)]$

$= \forall_x \neg\exists_y P(x, y) \vee \exists_x \neg\forall_y Q(x, y)$

$= \forall_x \forall_y \neg P(x, y) \vee \exists_x \exists_y \neg Q(x, y)$

Statement	When True?	When False?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	P(x,y) is true for every pair x,y	There is a pair x,y for which P(x,y) is false
$\forall x \exists y P(x,y)$	For every x there is a y for which P(x,y) is true	There is an x such that P(x,y) is false for every y
$\exists x \forall y P(x,y)$	There is an x for which P(x,y) is true for every y	For every x there is a y for which P(x,y) is false
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which P(x,y) is true	P(x,y) is false for every pair x,y

$\forall x \exists y P(x, y)$
$= \neg[\forall x \exists y P(x, y)]$
$= \exists_x \neg \exists y P(x, y)$
$= \boldsymbol{\exists_x \forall_y \neg P(x, y)}$

$\exists_x \forall_y P(x, y)$
$= \neg[\exists_x \forall_y P(x, y)]$
$= \forall_x \neg \forall_y P(x, y)$
$= \boldsymbol{\forall_x \exists_y \neg P(x, y)}$

3. TRANSLATING FROM NESTED QUANTIFIERS INTO ENGLISH:

Q. Translate the statement:
$\forall_x[C(x) \lor \exists_y(C(y) \land F(x, y))]$ into English, where
C(x) is "x has a computer,"
F (x, y) is "x and y are friends," and the domain for both x and y consists of all students in your school.

<u>Solution:</u>

The statement says that "for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends."

In other words, "*Every student in your school has a computer or has a friend who has a computer*"

3. TRANSLATING FROM NESTED QUANTIFIERS INTO ENGLISH:

Q. Translate the statement:
$\forall_x[S(x) \vee \exists_y(S(y) \wedge F(x, y))]$ into English, where
$S(x)$ is "x uses snapchat"
$F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

<u>Solution:</u>

The statement says that "for every student x in your school, x either uses snapchat or there is a student y such that y uses snapchat and y and x are friends."

In other words, "*Every student in your school either uses snapchat or are friends with a student who uses snapchat*"

3. TRANSLATING FROM NESTED QUANTIFIERS INTO ENGLISH:

Q. Translate the statement

$\exists_x \forall_y \forall_z((F(x, y) \wedge F(x, z) \wedge (y \mathrel{!=} z)) \rightarrow \neg F(y,z))$

into English, where F (a, b) means a and b are friends and the domain for x, y, and z consists of all students in your school.

Solution:

We first examine the expression $(F(x, y) \wedge F(x, z) \wedge (y = z)) \rightarrow \neg F(y, z)$. This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends.

It follows that the original statement, which is triply quantified, says that "there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends. In other words,

"There is a student none of whose friends are also friends with each other."