MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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GRAPH THEORY

PLANAR GRAPHS:

1. A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a planar representation of the graph.

2. A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

3. A planar representation of a graph splits the plane into regions, including an unbounded region.

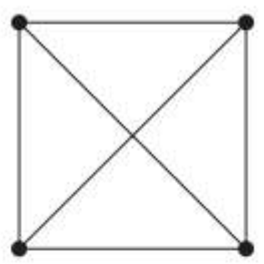


FIGURE 2 The Graph K_4.

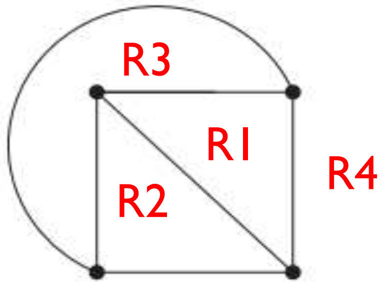


FIGURE 3 K_4 Drawn with No Crossings.

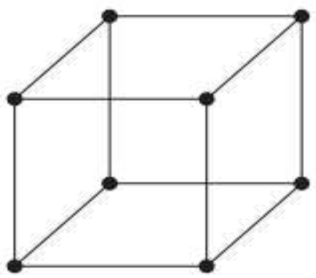


FIGURE 4 The Graph Q_3.

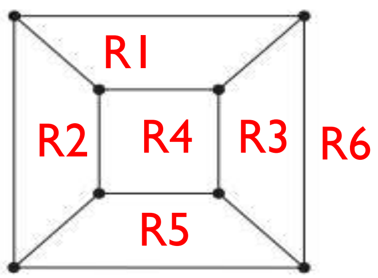


FIGURE 5 A Planar Representation of Q_3.

5. A complete graph of five vertices is non planar.

PLANAR GRAPHS:

1. **EULER'S FORMULA :** Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then r = e − v + 2.

Poof:
We will use Mathematical induction

(a) Base Case:

For n=1 i.e. For number of edge = 1.

$r_1 = e_1 - v_1 + 2$

$r_1 = 1-2+2$

$r_1 = 1$ (TRUE)

(b) Induction Hypothesis:

Assume the equation is true for n=k i.e. for number of edges = k

$r_k = e_k - v_k + 2$ is true for G_k

(c) Induction Step:

Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}.

PLANAR GRAPHS:

(c) Induction Step:

Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}.

There are two case to be considered:

i. Both a_{k+1} and b_{k+1} are already in G_k

$r_{k+1} = r_k+1$

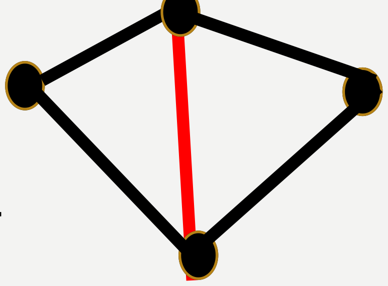
$e_{k+1} = e_k + 1$

$v_{k+1} = v_k$ {Because both vertices are in G_k}

$r_{k+1} = e_{k+1} - v_{k+1} + 2$

$r_k+1 = e_k + 1 - v_k + 2$

$\mathbf{r_k = e_k - v_k + 2}$ **which is true**



ii. One vertices is added to G_k

$r_{k+1} = r_k$

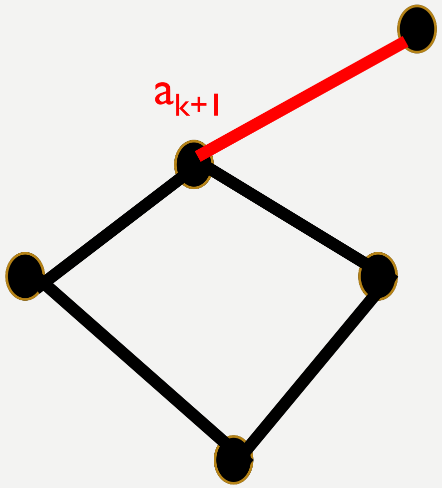
$e_{k+1} = e_k + 1$

$v_{k+1} = v_k +1$ {Because one vertices is added in G_k}

$r_{k+1} = e_{k+1} - v_{k+1} + 2$

$r_k = e_k + 1 - (v_k + 1) + 2$

$\mathbf{r_k = e_k - v_k + 2}$ **which is true**



Hence by induction method Euler's Formula is proved

PLANAR GRAPHS:

- 4 Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?
  Solution:

This graph has 20 vertices, each of degree 3, so $v = 20$

Now using Handshaking theorem ,

$$2e = (20*3)$$

$$e = 30$$

Consequently, from Euler's formula, the number of regions is

$$r = e - v + 2$$

$$r = 30 - 20 + 2$$

$$r = 12.$$

PLANAR GRAPHS:

Corollary of Euler's Theorem:

1. **If G is a connected planar simple graph with e edges and v vertices, where v ≥ 3, then e ≤ 3v − 6**

   Proof:

Let G(v, e) be the connected planar simple graph. A connected planar simple graph drawn in the plane divides the plane into regions, say **r** of them.

The degree of each region is at least three. (Because the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree two, or loops that could produce regions of degree one, are permitted.) Degree of region is defined to be the number of edges on the boundary of this region.

Sum of the degrees of the regions is exactly twice the number of edges in the graph, because each edge occurs on the boundary of a region exactly twice . Because each region has degree greater than or equal to three, it follows that

$$\mathbf{2e = \sum_{all\ regions\ R} deg(R) \geq 3r}$$

$$(2/3)e \geq r$$

Using r = e − v + 2 (Euler's formula),

we obtain v –e +r = 2

$$v - e + (2/3)e \geq 2$$

e ≤ 3v − 6. Hence proved

PLANAR GRAPHS:

- Show that K_5 is nonplanar using Corollary 1.

Solution:

The graph K_5 has five vertices and 10 edges.

However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because

$$10 \leq 3*5 - 6$$

$$10 \leq 9$$ which is false. Therefore, K5 is not planar.

2. If G is a connected planar simple graph, then G has a vertex of degree not exceeding five

Poof:

If G has at least three vertices, by Corollary 1 we know that

$$e \leq 3v - 6,$$

$$2e \leq 6v - 12.$$

$$2e + 12 \leq 6v$$

If the degree of every vertex were at least six, then because

$2e = \sum_{v \in V} \deg(v)$ (by the handshaking theorem), we would have

$$2e + 12 \geq 6v + 12.$$

But this contradicts the inequality $2e + 12 \leq 6v$. It follows that there must be a vertex with degree no greater than five.

PLANAR GRAPHS: