MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

Prepared by: Er. Ankit Kharel

Nepal college of information technology

- TAUTOLOGY
- CONTRADICTION
- CONTINGENCY
- PROPOSITIONAL SATISFIABILITY
- LOGICAL EQUIVALENCE

TAUTOLOGY:

- Compound proposition that is always TRUE , not matter what the truth values of the propositional variables that occur in it, is called TAUTOLOGY.

Examples:

a) p V ¬p

p	¬p	p ʌ ¬p
T	F	T
F	T	T

b) (p→q) v (q→p)

p	q	p→q	q→p	(p→q)V(q→p)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

CONTRADICTION:

- Compound proposition that is always FALSE, not matter what the truth values of the propositional variables that occur in it, is called CONTRADICTION.

Examples:

a) p ∧ ¬p

p	¬p	p ∧ ¬p
T	F	F
F	T	F

b) ¬(p^q) ←→ (q^p)

p	q	(p ∧ q)	¬(p^q)	(q^p)	¬(p^q) ←→ (q^p)
T	T	T	F	T	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	T	F	F

CONTINGENGY:

- Compound proposition that is neither a TAUTOLOGY or a CONTRADICTION

Examples:

a) $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	p→q	q→p	(p→q) ^ (q→p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

SATISFIABILITY:

- Compound proposition is satisfiable is there is at least one true value in its truth table.
- TAUTOLOGY is always satisfiable but satisfiable is not always TAUTOLOGY.

$(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

SATISFIABLE but not TAUTOLOGY

$(p \to q) \vee (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \vee (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

SATISFIABLE and alsoTAUTOLOGY

UNSATISFIABILITY:

- Compound proposition is unsatisfiable is there is no true value in its truth table.
- CONTRADICTION is always unsatisfiable.

¬(p^q) ←→ (q^p)

p	q	(p ^ q)	¬(p^q)	(q^p)	¬(p^q) ←→ (q^p)
T	T	T	F	T	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	T	F	F

VALID & INVALID:

VALID: Compound proposition always VALID when it is a TAUTOLOGY.

INVALID: Compound proposition always INVALID when it is either CONTRADICTION or CONTINGENCY.

SUMMARY

TAUTOLOGY
Always TRUE
Satisfiable
VALID

CONTRADICTION
Always FALSE
unsatisfiable
INVALID

CONTINGENCY
Sometimes TRUE or FALSE
Satisfiable
INVALID

LOGICAL EQUIVALENCES:

- Compound proposition ‘p’ and ‘q’ are logically equivalent is they have same Truth Values in all possible cases.
- Notation: $p \equiv q$ or $p \Leftrightarrow q$

Examples:

a) $\neg(p \vee q)$ and $(\neg p \wedge \neg q)$

p	q	(pVq)	¬(pVq)	¬p	¬q	(¬p^¬q)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Hence, $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$

Examples:

b) $(p \rightarrow q)$ and $(\neg p \lor q)$

p	q	$(p \rightarrow q)$	$\neg p$	$(\neg p \lor q)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence, $(p \rightarrow q) \Leftrightarrow (\neg p \lor q)$

IMPORTANT EQUIVALENCE:

Equivalences	Laws
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

p	T	p ^ T
T	T	T
F	T	F

p	F	p V T
T	F	T
F	F	F

1. Identity Laws

p	T	p V T
T	T	T
F	T	T

p	F	p ^ F
T	F	F
F	F	F

2. Domination Laws

EQUIVALENCE INVOLVING CONDITION:

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$ (Implication)

$p \to q \equiv \neg q \to \neg p$ (contra-positive)

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

1. (p→q)⇔ (¬p v q)

p	q	(p→q)	¬p	(¬pVq)
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

2. (p→q) ⇔ (¬q→ ¬p)

p	q	(p→q)	¬p	¬q	(¬q→ ¬p)
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

EQUIVALENCE INVOLVING BICONDITION:

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

1. (p←→q) ⇔ (p→q)^(q→p)

p	q	(p←→q)	(p→q)	(q→p)	(p→q)^(q→p)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Prove the following are logically equivalent by developing a series of logical equivalence.

1. $\neg(p \rightarrow q) \equiv (p \wedge \neg q)$

<u>solution:</u>

Taking LHS,

$= \neg(p \rightarrow q)$

$= \neg(\neg p \vee q)$-----------{$p \rightarrow q \equiv \neg p \vee q$}

$= \neg(\neg p) \wedge (\neg q)$------{De- Morgan's Law}

$= p \wedge \neg q$---------------{Double Negation Law}

Prove the following are logically equivalent by developing a series of logical equivalence.

1. ¬(pV(¬p^q)) ≡ (¬p^¬q)

<u>solution:</u>

Taking LHS,

=¬(pV(¬p^q))

=¬p ∧ ¬(¬p ∧ q) -------------------by the second De Morgan law

= ¬p ∧ [¬(¬p) ∨ ¬q] ---------------------by the first De Morgan law

= ¬p ∧ (p ∨ ¬q) -------------------------by the double negation law

= (¬p ∧ p) ∨ (¬p ∧ ¬q) ------------by the second distributive law

= F ∨ (¬p ∧ ¬q) -------------------------------------because ¬p ∧ p ≡ F

= (¬p ∧ ¬q) ∨ F---------- by the commutative law for disjunction

= ¬p ∧ ¬q --by the identity law for F