CHAPTER – 1

LINEAR ALGEBRA

INTRODUCTION

In linear algebra, there is the theory and application of system of linear equations, linear transformations, eigen value problems. These terms are arises in electrical networks, frameworks in mechanics, curve fitting and other optimization problems, system of differential equations.

This chapter makes the systematic use of vectors and matrices as well as determinants.

Determinants are introduced to solving the simultaneous linear equations. A large number of physical phenomenons are governed by linear differential equations which are solved by reducing them into a system of simultaneous linear equations.

Matrices originated as the stores of information but now a day it gives wide application. It plays the vital role in mathematics as well as network analysis, communication theory, structure, mechanism, Biology, Sociology, Economics, Psychology, and Statistics etc.

Determinants and matrices play the important role to providing the testing of consistency of system of linear equations.

MATRIX

A rectangular arrangement of numbers (real or complex), which are enclosed by a bracket ( ) or [ ], is said to be matrix.

It is denoted by capital letters A, B, C...... and the number are said to be elements of matrix and they are denoted by small letters a, b, c,.........

For example

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}$ 2. $B = \begin{pmatrix} a & 2 & 3 \\ o & b & 7 \end{pmatrix}$

3. $C = \begin{pmatrix} a & b & c \\ o & o & 2 \end{pmatrix}$ 4. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

Note:

In matrix, we get there is one or more horizontal rows or vertical columns or both.

SIZE OF A MATRIX

Let A be a matrix, having m'th row and n'th column, then the size (or order) of matrix A is m x n and denoted by $(A)_{m \times n}$.

For Example

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Here it has two row and three column, the its order is 2×3. That is

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$$

SPECIFICATION OF A MATRIX

Let A be a matrix and having size m × n, then it can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2j} \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mj} \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Also it is denoted by $A = (a_{ij})_{m \times n}$

Where a_{ij} is called i'th row and j^{th} column element of A.

For example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 1 & 2 \end{bmatrix}$

Here a_{23} = the element of second row and third column = 7 and $a_{32} = 1$.

SOME DEFINITIONS

Square matrix

A matrix A is said to be square matrix if it has same number of rows and columns.

If A has n number of row and n number of column, then it is said to be a square matrix A having size n. It is denoted by $(A)_{n\times n}$ be a n – square matrix.

For example

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ be a 3×3 square matrix.

2. $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ be a 2×2 square matrix.

Leading diagonal

Let A be a square matrix. Then a diagonal from top of left hand side to the bottom of right hand side of a matrix A is said to be leading diagonal or principal diagonal.

For example

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here the diagonal consisting a_{11}, a_{22}, a_{33} is said to be leading diagonal.

Row/column Matrix

A matrix A is said to be row/column matrix if it has only one row/column. It is also called row vector or column vector.

For example

1. $A = (1\ 2\ 3)$. This a row matrix

2. $B = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$. This is a column matrix.

Zero matrix

A matrix is said to be zero matrix if all elements of that matrix are zero and it is denoted by O.

For example

1. $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Identity matrix

A square matrix is said to be identity matrix if all elements in leading diagonal are one (or unit) and others are zero. It is denoted by I.

For example

1. $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, is an unit matrix of order 3 and is also denoted by I_3.

2. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is an unit matrix of order 2 and is also denoted by I_2.

3. $I = (1)$, is an unit matrix of order 1 and is also denoted by I_1.

Scalar matrix

A square matrix is said to be scalar matrix if all elements in leading diagonal are same number k and others are zero.

For example

1. $A = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}$. This is a scalar matrix of order 4×4.

2. $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This is a scalar matrix of order 3×3.

Note:

All identity matrix are scalar matrix but all scalar matrix are not identity matrix.

DIAGONAL MATRIX

A square matrix is said to be diagonal matrix if all elements in leading diagonal are not zero but others are zero.

For example

1. $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This is a diagonal matrix of order 3×3.

2. $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This is a diagonal matrix of size 4×4.

3. $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This is not a diagonal matrix.

Note:

All scalar and identity matrices are diagonal matrices but all diagonal matrices are not scalar as well as identity matrices.

Triangular matrix

A square matrix is said to be triangular matrix if all elements of its entries above or below or both the leading diagonal are zero.

For example

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 5 & 7 \end{bmatrix}$. This is triangular matrix and also said to be lower triangular matrix.

2. $B = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 8 \end{bmatrix}$. This is an upper triangular matrix.

Note:

All identity, scalar, diagonal matrix are triangular matrix.

Sub matrix

Let A and B are two matrices. Then B is said to be sub matrix of A if B is formed by deleting one or more rows or columns or both from A. It is denoted by $B \subseteq A$.

Example

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$

Here B is obtained from A by deleting second row and second column of A. Thus B is a sub matrix of A.

2. If $C = \begin{bmatrix} 1 & 3 \\ 7 & 8 \end{bmatrix}$, this is not a sub matrix of A.

EQUAL MATRIX

Let A and B are two matrices then A and B are said to be equal if they are of same order and their corresponding elements are equal. It is denoted by $A = B$. That is, if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$, then $A = B$ only when $m = p$ and $n = q$ and $a_{ij} = b_{ij}$ for all i and j.

For example

1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A = B$ only when $a = 1, b = 2, c = 3, d = 4$.

2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 4 & 6 \end{pmatrix}$

Here A is not equals to B, since each elements of A is not equals to corresponding element of B.

ALGEBRA OF MATRICES

1. Addition of matrices

Two matrices are said to be conformable for addition if they have the same order.

If A and B are two matrices of the same size, then their sum is denoted by A + B and defined by a matrix, whose elements are obtained by adding the corresponding elements of A and B.

That is, if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, then $A + B = C = (c_{ij})_{m\times n}$ (say) where $c_{ij} = a_{ij} + b_{ij}$

For example

1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Then $A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$

2. If $A = (1 \quad 2 \quad 3)$ and $B = (4 \quad 5 \quad 6)$

Then $A + B = (1 \quad 2 \quad 3) + (4 \quad 5 \quad 6) = (1+4 \quad 2+5 \quad 3+6)$

$A + B = (5 \quad 7 \quad 9)$

3. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 1 & 9 \end{pmatrix}$

Then A + B does not exist, because the size of A and B are not same.

Note:

The difference A – B is a matrix whose elements are obtained by subtracting the element of B from the corresponding elements of A.

For example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \end{pmatrix}$$

Then $$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1-2 & 2-4 & 3-6 \\ 4-2 & 5-1 & 6-3 \end{pmatrix}$$

$\therefore$ $$A - B = \begin{pmatrix} -1 & -2 & -3 \\ 2 & 4 & 3 \end{pmatrix}$$

2. *Scalar Multiplication*

Let A be a matrix, the scalar multiplication of A by a scalar λ denoted by λA is defined as a matrix which is obtained by multiplying each elements of A by λ. That is, if $A = (a_{ij})_{m\times n}$, then $\lambda A = (\lambda a_{ij})_{m\times n}$.

Example

1. Let $$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

   Then $$\lambda A = \lambda \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow \lambda A = \begin{pmatrix} \lambda & 2\lambda & 3\lambda \\ 4\lambda & 5\lambda & 6\lambda \end{pmatrix}$$

2. Let $$B = \begin{pmatrix} a & b & c \\ d & e & f \\ 1 & 3 & 5 \end{pmatrix}$$

   Then $$2B = 2\begin{pmatrix} a & b & c \\ d & e & f \\ 1 & 3 & 5 \end{pmatrix} \Rightarrow 2B = \begin{pmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2 & 6 & 10 \end{pmatrix}$$

3. *Matrix Multiplication*

Two matrices A and B are said to be conformable for the product AB if the number of columns in A is equal to the number of rows in B.

If the order of A and B are $m\times n$ and $p\times q$ respectively, then

(i) AB is defined if number of column in A is equals to number of row in B. That is if $n = p$.

(ii) BA is defined if number of columns in B is equals to number of rows in A. If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$, be two matrices, then AB exists and AB is defined as the matrix

$$C = (c_{ij})_{m\times p}, \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots\ldots + a_{in}\, b_{nj}$$

here c_{ij} is called the inner product of i^{th} row of A with j^{th} column of B.

That is, i, j^{th} element of (AB) is equal to the sum of the products of the elements of i^{th} row of A with the corresponding elements of j'th column of B.

For example

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2}$

Here AB exists, but BA doesnot exists (??).

Now AB is define and is of order 3×2 and $AB = C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$

We get c_{11} = sum of products of element of first row of A and corresponding elements of first column of B

$$c_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{12} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

$$= a_{11} b_{11} + a_{12} b_{12} + a_{13} b_{31}$$

Also, c_{12} = sum of products of elements of first row of A and corresponding elements of second column of B.

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32}$$

c_{21} = sum of products of elements of second row of A and first column of B

$$= \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

$$c_{21} = a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31}$$

c_{22} = sum of products of elements of second row of A and second column of B

$$= \begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$$

$$c_{32} = a_{21} b_{12} + a_{22} b_{22} + a_{23} b_{32}$$

c_{31} = sum of products of elements of third row of A and first column of B

$$= \begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

$$c_{21} = a_{31}\, b_{11} + a_{32}\, b_{21} + a_{33}\, b_{31}$$

and c_{32} = sum of products of elements of third row of A and second column of B.

$$= \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$$

$$c_{32} = a_{31}\, b_{12} + a_{32}\, b_{22} + a_{33}\, b_{32}$$

Thus we get

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

Note:

Matrix multiplication can be remember in this way.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \text{(Say)}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \text{(Say)}$$

$$\text{Then } AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}$$

Example

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$. Find AB and BA if exists.

Solution

We have $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$.

Here size of A and B is 2×3 and 2×2, hence AB does not exists, but BA exist and having size 2×3. Because number of column of B is equal to number of rows of A, and the size of BA will be 2×3.

$$\text{Then } BA = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3.1+4.4 & 3.2+4.5 & 3.3+4.6 \\ 7.1+9.4 & 7.2+9.5 & 7.3+9.6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 19 & 26 & 33 \\ 43 & 59 & 75 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Find $A^2 - 4A - 5I$, where I is the unit matrix of order 3.

Solution

$$\text{We have, } A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Then
$$A^2 = AA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Here $A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Therefore we get

$A^2 - 4A - 5I = O.$

Properties of matrices

1. ***Addition properties:***

   If A, B, C and O are matrices of same size, then

   (i) $A + B = B + A$ (Commutative law)

   (ii) $(A + B) + C = A + (B + C)$ (Associative law)

   (iii) $A + O = A$ (Existence of identity)

   (iv) $A + (-A) = 0$ (Existence of inverse)

2. ***Scalar multiplication properties:***

   Let A and B are two matrices of same size and let λ, μ are two scalars then

   (i) $\lambda(\mu A) = (\lambda\mu)A$ (Associative for scalars)

   (ii) $(\lambda + \mu)A = \lambda A + \mu A$ (Distributive for scalars)

   (iii) $\lambda(A + B) = \lambda A + \lambda B$ (Distributive for matrix)

   (iv) $IA = A$ (Identity)

3. ***Matrix multiplication properties:***

   Let A, B and C are there matrixes of specific size, then

   (i) $(AB)C = A(BC)$ (Associative law)

   (ii) $A(B + C) = AB + AC$ (Distributive law)

   (iii) $(A + B)C = AC + BC$ (Distributive law)

   (iv) $\lambda(AB) = (\lambda A)B = A(\lambda B)$, where λ is a scalar.

   (v) $AB \neq BA$ in general

   (vi) $AB = 0$ does not always implies either $A = 0$ or $B = 0$ or both are zero.

Note:

1. To prove $AB \neq BA$, in general. Let us take a counter example.

   Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$

   Then $AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2+4 & 1+8 \\ 0+6 & 0+12 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 4 \end{bmatrix}$

   also $BA = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2+0 & 4+3 \\ 2+0 & 4+12 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 2 & 16 \end{bmatrix}$

   Thus we get $AB \neq BA$ in this case.

2. We shall show that if $AB = 0$, it does not always imply that, $A = 0$ or $B = 0$ or both equal to zero.

   Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq O$

But $$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$ Hence proved.

EXERCISE 1.1

1. If $A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & 5 & -3 \end{bmatrix}$. Find $3A-4B$.

2. If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

   Show that $AB \neq BA$.

3. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$

   Find AB and explain why BA is not defined.

4. If $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$

   Verify $AB = BA = I$

5. Find $AB - BA$, where $A = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

   Find AB or BA whichever exists.

7. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$

   Show that $(AB)C = A(BC)$ and $A(B+C) = AB + AC$.

8. If $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $(2I - A)(10I - A) = 9I$

9. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. Find AB.

10. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 5A + 7I = 0$, where I is a unit matrix of size 2×2.

11. Let $A = \begin{bmatrix} -3 & 2 \\ 1 & -3 \\ -3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$

Show that AB and AC doesnot exists.

12. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, where $i^2 = -1$, then show that $(A + B)^2 = A^2 + B^2$

13. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$. Show that $A^3 - A^2 - 18A - 30I = 0$

14. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$. Show that $A^3 - 4A^2 - 3A + 11I = 0$, where I is the unit matrix of order 3.

ANSWERS

1. $\begin{bmatrix} -11 & 2 & -12 \\ 4 & -1 & -7 \\ 7 & -14 & 18 \end{bmatrix}$ 3. $AB = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$ 5. $\begin{bmatrix} 43 & 4 \\ 3 & -43 \end{bmatrix}$

6. $BA = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}$ 9. $\begin{bmatrix} -2 & -2 \\ 0 & -6 \end{bmatrix}$

TRANSPOSE OF A MATRIX

Let A be a matrix. Then transpose of A is denoted by A' or A^t and defined by changing the rows into corresponding columns (and columns into rows of A).

That is, if $A = (a_{ij})_{m \times n}$ be a matrix. Then $A^t = (a_{ji})_{n \times m}$.

For example

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$ then $A^t = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$

Properties of Transpose of a Matrix:

Let A and B are two matrices of specified order, then

(i) $(A^t)^t = A$

(ii) $(A + B)^t = A^t + B^t$

(iii) $(AB)^t = B^t A^t$

Proof:

(i) Let $A = (a_{ij})_{m\times n}$ be any matrix.

Here order of A^t is n×m. Also order of $(A^t)^t$ is m×n, which is also order of A. Thus we get, order of $(A^t)^t$ is equal to order of A. ....(1)

Also we have to show i,j'th element of $(A^t)^t$ is equals to i,j'th element of A.

Here, i,jth element of (A^t) is a_{ji}

and hence i, j'th element of $(A^t)^t$ is a_{ij}

which is i,j'th element of A ....(2)

Thus by definition, from (1) and (2), we get $(A^t)^t = A$.

(ii) Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, then A + B exists. We have to show $(A + B)^t = A^t + B^t$.

For this we have to show that size of $(A + B)^t$ is equals to size of $A^t + B^t$ and i, j'th element of $(A + B)^t$ is equals to i,j'th element of $A^t + B^t$.

Here we have size of A and B is m × n. Then size of (A + B) is also m × n. Thus size of $(A + B)^t$ is n×m.

Also size of A^t is n×m and size of B^t is n×m. Then size of $A^t + B^t$ is n × m thus we get,

Size of $(A + B)^t$ is equals to size of $A^t + B^t$ ....(1)

Let $A + B = C = (c_{ij})_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$.

Here i, j'th element of $(A + B)^t = C^t$ is

$$c_{ji} = a_{ji} + b_{ji} = (a_{ij})^t + (b_{ij})^t$$

$$= \text{i, j'th element of } A^t + \text{i,j}^{th} \text{ element of } B^t$$

$= i, j^{th}$ elements of $(A^t + B^t)$ ....(2)

Form (1) and (2), we get

$(A + B)^t = A^t + B^t.$

(iii) Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$. Then AB exists and having size $m \times p$. We have to show that $(AB)^t = B^tA^t$.

Here size of $(AB)^t$ is $p \times m$. The size of B^t is $p \times n$ and size of A^t is $n \times m$. Thus the size of B^tA^t is $p \times m$.

Therefore we get, size of $(AB)^t$ is equal to size of B^tA^t. ....(1)

Now we have to show that i, j^{th} element of $(AB)^t$ is equals to i, j^{th} element of B^tA^t.

$$\text{Let } AB = C = (c_{ij})_{m\times p}, \text{ where } c_{ij} = \sum_{r=1}^{n} a_{ir}\, b_{rj}$$

$$\text{Then i, } j^{th} \text{ element of } (AB)^t \text{ is } c_{ji} = \sum_{r=1}^{n} a_{jr}\, b_{ri} = \sum_{r=1}^{n} b_{ri}\, a_{jr}$$

Since all entries of A and B are lies in a field.

$$c_{ji} = \sum_{r=1}^{n} (b_{ir})^t\,(a_{rj})^t = \text{i, } j^{th} \text{ element of } B^t A^t$$

Thus we get,

i, j^{th} element of $(AB)^t$ is equals to i, j^{th} element of $B^t A^t$. .... (2)

Hence from equation (1) and (2), we get, $(AB)^t = B^tA^t$

This completes the proof.

Symmetric Matrix

A square matrix A is said to be symmetric if $A = A^t$. That is $a_{ij} = a_{ji}$ for all i and j.

For example

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$, this is a symmetric matrix. Since

$$A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

$\Rightarrow$ $A^t = A$

2. Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Here $A^t = A$. Then A is a symmetric matrix.

Skew Symmetric Matrix

A square matrix A is said to be skew symmetric if $A^t = -A$.

That is $A = (a_{ij})_{m \times m}$ is skew symmetric if $a_{ij} = -a_{ji}$ for all i and j.

This gives, $a_{ii} = -a_{ii}$

$\Rightarrow$ $2a_{ii} = 0$

$\Rightarrow$ $a_{ii} = 0$ for all i.

In skew symmetric matrix all elements in leading diagonal are zero.

For example

1. Let $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

Here $A^t = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

$\Rightarrow$ $A^t = -A$

This gives A is a skew symmetric matrix.

2. $A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 9 \\ -7 & -9 & 0 \end{bmatrix}$ is also skew symmetric matrix.

Complex matrix

A matrix is said to be complex matrix if its elements are complex numbers. For example:

$$A = \begin{bmatrix} 1+i & 2i & 0 \\ 2 & i+1 & 2i \\ 0 & 2 & 3i \end{bmatrix}$$

Conjugate matrix

Let $A = (a_{ij})_{m \times n}$ be a complex matrix then its conjugate matrix is denoted by $\overline{A}$ and defined by $\overline{A} = (\overline{a_{ij}})_{m \times n}$.

Example:

Let $A = \begin{pmatrix} 1+i & 2i & 1 \\ 2 & 1-2i & 3+2i \end{pmatrix}$ be a complex matrix, then its conjugate matrix is $\overline{A} = \begin{pmatrix} 1-i & -2i & 1 \\ 2 & 1+2i & 3-2i \end{pmatrix}$.

Properties of conjugate matrix

Let A and B are two complex matrix then following are holds

i) $\overline{\overline{A}} = A$

ii) $(\overline{\lambda A}) = \overline{\lambda}\,\overline{A}$, where λ is a complex number.

iii) $\overline{AB} = \overline{A}\,\overline{B}$ iv) $\overline{A+B} = \overline{A} + \overline{B}$

Hermitian matrix

A square complex matrix A is said to be hermitian if $A^t = \overline{A}$. That is $(\overline{A})^t = A$. It is denoted by $A^* = A$, where $A^* = (\overline{A})^t$.

Skew-hermitian matrix

A square complex matrix A is said to be skew hermitian if $A^* = -A$.

Properties of hermitian matrix

Let A and B are two complex matrix with complex number λ, then

i) $(A^*)^* = A$

ii) $(A + B)^* = A^* + B^*$

iii) $(\lambda A)^* = \overline{\lambda} A^*$

iv) $(AB)^* = B^*A^*$, whenever AB exists.

Unitary matrix

A square complex matrix A is said to be unitary of $A^*A = I$.

Orthogonal Matrix

A square matrix A is said to be an orthogonal matrix if $AA^t = I = A^tA$.

Example

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$ is an orthogonal matrix.

Theorem:

For any two orthogonal matrices A and B, show that AB is an orthogonal matrix.

Proof:

Let A and B are orthogonal matrix. Then we have to show AB is also orthogonal.

Here, $AA^t = I$ and $BB^t = I$.

Also we get, $(AB)(AB)^t = (AB)(B^t A^t) = A(B B^t) A^t$

[Since matrix is associative]

$$= A I A^t$$

$$= AA^t \qquad \text{[Since AI = A]}$$

$$= I$$

Thus $(AB)(AB)^t = I$, hence by definition AB is also orthogonal matrix.

Idempotent Matrix

A square matrix A is said to be idempotent matrix if $A^2 = A$.

For example

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$ is an idempotent matrix.

Nilpotent Matrix

A square matrix A is said to be nilpotent matrix of order n if $A^n = 0$ but $A^{n-1} \neq 0$.

Example

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of order 3.

Involuntary matrix

A square matrix A is said to be involuntary matrix if $A^2 = I$.

EXERCISE 1.2

1. If $A = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}, D = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}$

   Calculate the following products or give reasons why they are not defined.

   (i) BA, A^tB, AB (iii) A^tD, A^tD^t, DA, AD

   (ii) C^2, C^tC, CC^t (iv) BB^t, B^tB, BB^tB

2. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Verify that $(AB)^t = B^tA^t$.

3. If $A = \begin{pmatrix} 2 & 3 \\ 5 & -7 \end{pmatrix}$, show that $(A^2)^t = (A^t)^2$

4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Verify $(AB)^t = B^t A^t$.

5. Find A^*, where $A = \begin{pmatrix} 1+i & 2 & 3i \\ 2+i & i & 3 \end{pmatrix}$

6. Show that the matrix $\begin{pmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{pmatrix}$ is hermitian

7. Show that the square matrix $A = \begin{pmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{pmatrix}$ is skew hermitain matrix.

8. If $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ then show that $A + A^t$ is symmetric and $A - A^t$ is skew symmetric matrix.

9. Prove that the following:
   i) The sum of two symmetric matrix is symmetric.
   ii) The sum of two skew-symmetric matrix is skew-symmetric.
   iii) If A is a square matrix then $A + A^t$ is symmetric and $A - A^t$ skew symmetric.
   iv) Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

   [Hint: $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$]

ANSWERS

1. (i) Undefined, (2 8), undefined. (ii) $\begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}$, C^2, C^2

(iii) Undefined, (16), (16), $\begin{bmatrix} 4 & 3 & 0 \\ 16 & 12 & 0 \\ 12 & 9 & 0 \end{bmatrix}$

(iv) $\begin{pmatrix} 13 & -6 & -3 \\ -6 & 4 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}, \begin{pmatrix} 26 & -54 \\ -12 & 28 \\ -6 & 14 \end{pmatrix}$ 5. $\begin{pmatrix} 1-i & 2-i \\ 2 & -i \\ -3i & 3 \end{pmatrix}$

DETERMINANT

A determinant is an algebraic expression written in convenient and concise form. Also determinant is a function of a square matrix and defined by

1. If $A = (a_{11})$ be a 1×1 square matrix and the determinant function of A is denoted by $|A|$ and defined by $|A| = |a_{11}| = a_{11}$.

2. If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 square matrix and the determinant function of A is denoted by $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and defined by

   $|A| = a_{11}\, a_{22} - a_{21}\, a_{12}$

3. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 square matrix and the determinant function of A is denoted

   by $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and defined by

   $$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

4. The determinant of the n'th order is denoted by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

Which is a block of n×n elements arranged in the form of a solid square along n rows and n columns and is bounded by two vertical lines.

Here the elements a_{11}, a_{22}, a_{33}, ..., a_{nn} are called the diagonal elements. The diagonal through the left hand top corner along which the diagonal elements lie is called the leading or principal diagonal. Also a_{11}, a_{22} a_{33}, ..., a_{nn} is called the leading term (or elements of principal diagonal).

MINORS AND COFACTORS

Minors:

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then the minor of a_{ij} is denoted by M_{ij} and defined by the determinant of sub matrix of A by omitting i'th row and j'th column of A.

For example

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\text{Then minor of } 8 = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6$$

$$\text{also minor of } 6 = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6.$$

Cofactor:

Let $A = (a_{ij})_{n \times n}$ be a square matrix, then cofactor of a_{ij} is denoted by A_{ij} and defined by $A_{ij} = (-1)^{i+j} M_{ij}$

For example

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Then cofactor of a_{11} (= 1) is $A_{11} = (-1)^{1+1} M_{11}$

$$= (-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48$$

$$A_{11} = -3$$

Also cofactor of a_{23} (= 6) is

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -(8-14) = 6$$

Similarly others.

EXPRESSION OF DETERMINANT

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$ (expanding by first row)

or, $|A| = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$ (expanding by second row)

or, $|A| = a_{31} A_{31} + a_{32} A_{32} + a_{33} A_{33}$ (expanding by third row)

or, $|A| = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$ (expanding by first column)

or, $|A| = a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32}$ (expanding by second column)

or, $|A| = a_{13} A_{13} + a_{23} A_{23} + a_{33} A_{33}$ (expanding by third column)

These expansions are called Laplace's expansions and the technique can be used for the determinant of any order.

Example

Find the determinant $\Delta = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix}$ by expanding (i) from 2nd row (ii) from first column.

Solution

(i) Expanding from second row

$$\Delta = 1\left(-\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}\right) + 3\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} + 2\left(-\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right)$$

$$\therefore \Delta = -(4-2) + 3(2-4) - 2(1-4) = -2-6+6$$

$$\Delta = -2$$

(ii) Expanding from first column, we get

$$\Delta = 1\begin{vmatrix} 8 & 2 \\ 1 & 2 \end{vmatrix} + 1\left(-\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}\right) + 2\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}$$

$$= 6-2-(4-2)+2(4-6) = 4-2-4$$

$$\Delta = -2$$

PROPERTIES OF DETERMINANTS

The following properties of determinant hold good for any order but we shall verify these properties for determinant of third order only.

1. If a determinant has a row or column of zeros, then its value is zero.

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$

Expanding from third column

$$= 0\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - 0\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + 0\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = 0.$$

2. If two rows or columns of a determinant are equal, then its value is zero.

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{11} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix}$$

In which first and third columns are identical, we shall show that $\Delta = 0$.

$$\text{Now, } \Delta = a_{11}\begin{vmatrix} a_{22} & a_{21} \\ a_{32} & a_{31} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{21} \\ a_{31} & a_{31} \end{vmatrix} + a_{11}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{31} - a_{32}a_{21}) - a_{12}(a_{21}a_{31} - a_{31}a_{21}) + a_{11}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}\, a_{22}\, a_{31} - a_{11}\, a_{32}\, a_{21} + a_{11}\, a_{21}\, a_{32} - a_{11}\, a_{31}\, a_{22}$$

$$\Delta = 0$$

3. The determinant of identity matrix is one.

$$\text{Let} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Then} \quad |I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1\,(1 - 0) - 0 + 0$$

$$|\,I\,| = 1$$

4. The determinant of a triangular matrix is the product of diagonal (leading) elements.

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding from first row

$$= a_{11}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11}\,(a_{22}\, a_{33} - 0)$$

$$|A| = a_{11}\, a_{22}\, a_{33}$$

5. Let A be a square matrix, then $|A| = |A^t|$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$ ....(1)

and $A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Then $|A^t| = a_{11}(a_{22}a_{33} - a_{23}a_{31}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$ ....(2)

From (1) and (2), we get

$$|A| = |A^t|.$$

Thus in a determinant if rows are changed into corresponding columns the value of the determinant remains unaltered.

6. If two successive lines (rows or columns) of a determinant are interchanged, the determinant retains its numerical value but changes in sign.

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Let $|B|$ be the determinant obtained by $|A|$ by interchanging its second and third rows, then

$$|B| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$$

We get $|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

and
$$\begin{aligned} |B| &= a_{11}(a_{32}a_{23} - a_{22}a_{33}) - a_{12}(a_{31}a_{23} - a_{21}a_{33}) + a_{13}(a_{31}a_{22} - a_{21}a_{32}) \\ &= -[a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})] \\ |B| &= -|A| \end{aligned}$$

Note:

If any line of a determinant $|A|$ is passed over two parallel lines to get a determinant $|B|$, then

$$|B| = (-1)^2 |A|$$

Let $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Let us pass the fourth column over the third and second columns to get

$$|B| = \begin{vmatrix} a_{11} & a_{14} & a_{12} & a_{13} \\ a_{21} & a_{24} & a_{22} & a_{23} \\ a_{31} & a_{34} & a_{32} & a_{33} \\ a_{41} & a_{44} & a_{42} & a_{43} \end{vmatrix}$$

Here we get $|B|$ from $|A|$ by applying the following two steps:

(i) Interchanging fourth and third columns.

(ii) Interchanging the new third column and the second column.

Since both interchange gives one – one negative sign, then

$|B| = (-1)^2 \; |A|$

Thus we get, in general, if a line of a determinant $|A|$ is passed over r parallel lines, then the new determinant $|B|$ is given by $|B| = (-1)^r |A|$.

7. If all the elements of a line (row or column) of a determinant are multiplied by a scalar k, then the whole determinant gets multiplied by that factor.

Let $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let $|B|$ be the determinant obtained from $|A|$ by multiplying each elements of first row by k, then

$$|B| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding $|B|$ by first row, we get

$$|B| = ka_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - ka_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + ka_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|B| = k\,\{a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}\,(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}\,(a_{21}\,a_{32} - a_{31}\,a_{22})\}$$

$$|B| = k\,|A|$$

8. If each element of a line of a determinant is the sum of n terms, the determinants can be expressed as the sum of n determinants of the same order.

Let $|A| = \begin{vmatrix} a_{11}+a'_{11} & a_{12} & a_{13} \\ a_{21}+a'_{21} & a_{22} & a_{23} \\ a_{31}+a'_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding from column one

$$= (a_{11}+a'_{11})\begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - (a_{21}+a'_{21})\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ (a_{31}+a'_{31})\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$+ a'_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a'_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a'_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a'_{11} & a_{12} & a_{13} \\ a'_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{vmatrix}$$

Thus we get $|A|$ can be expressed as the sum of two determinants of the third order.

In general, if each element of one row (or column) consists of p terms, second row (or column) consists of q terms, third row (or column) consists of r terms and so on, the determinant can be expressed as a sum of $p \times q \times r \times$........ determinants.

9. If to each element of a line (row or column) of a determinant be added equi – multiples of the corresponding elements of one or more parallel lines, then the determinant remains unaltered.

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Let $|B|$ be the determinant obtained from $|A|$ by adding to each element of first column p times the corresponding element of second column, then

$$|B| = \begin{vmatrix} a_{11} + pa_{12} & a_{12} & a_{13} \\ a_{21} + pa_{22} & a_{22} & a_{23} \\ a_{31} + pa_{23} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} pa_{12} & a_{12} & a_{13} \\ pa_{22} & a_{22} & a_{23} \\ pa_{23} & a_{32} & a_{33} \end{vmatrix} \quad \text{(by property 8)}$$

$$= |A| + p\begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{23} & a_{32} & a_{33} \end{vmatrix} \quad \text{(by property 2)}$$

$$|B| = |A| + p.\,0$$

$$\therefore \quad |B| = |A|$$

10. The determinant of a product of two matrices is equal to the product of their determinants. That is, $|AB| = |A|\,|B|$.

Examples

1. Evaluate: $\begin{vmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{vmatrix}$**, where w is a cube root of unity.**

Solution

Here $\begin{vmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{vmatrix}$

Apply $C_1 : C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+w+w^2 & w & w^2 \\ 1+w+w^2 & 1 & w \\ 1+w+w^2 & w^2 & 1 \end{vmatrix} = (1 + w + w^2)\begin{vmatrix} 1 & w & w^2 \\ 1 & 1 & w \\ 1 & w2 & 1 \end{vmatrix}$$

$$= 0\begin{vmatrix} 1 & w & w^2 \\ 1 & 1 & w \\ 1 & w^2 & 1 \end{vmatrix} = 0 \qquad [\text{Since } 1 + w + w^2 = 0.]$$

2. **Show that** $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = 0.$

Solution

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$$

Apply $C_4{:}C_4 - C_3$, $C_3{:}C_3 - C_2$ $C_2{:}C_2 - C_1$

$$= \begin{vmatrix} 1 & 3 & 5 & 7 \\ 4 & 5 & 7 & 9 \\ 9 & 7 & 9 & 11 \\ 16 & 9 & 11 & 13 \end{vmatrix}$$

Apply $C_4{:}C_4 - C_3$, $C_3{:}C_3 - C_2$, we get

$$= \begin{vmatrix} 1 & 3 & 2 & 2 \\ 4 & 5 & 2 & 2 \\ 9 & 7 & 2 & 2 \\ 16 & 9 & 2 & 2 \end{vmatrix} = 0, \text{ since } C_3 \text{ and } C_4 \text{ are identical.}$$

3. **Solve** $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

Solution

We have

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Apply $C_3{:}\ C_3 - C_2 - C_1$ we get

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 1 \\ x-4 & 2x-9 & -3 \\ x-8 & 2x-27 & -29 \end{vmatrix} = 0$$

Apply $C_2: C_2 - 2C_1$ we get

$$\Rightarrow \begin{vmatrix} x-2 & 1 & 1 \\ x-4 & -1 & -3 \\ x-8 & -11 & -29 \end{vmatrix} = 0$$

Apply $R_2: R_2 - R_1$, $R_3: R_3 - R_1$ we get

$$\Rightarrow \begin{vmatrix} x-2 & 1 & 1 \\ -2 & -2 & -4 \\ -6 & -12 & -30 \end{vmatrix} = 0$$

$$\Rightarrow 12 \begin{vmatrix} x-2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 5 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)(5-4) - 1(5-2) + 1(2-1) = 0$

$\Rightarrow \quad (x-2) - 3 + 1 = 0$

$\Rightarrow \quad x - 2 - 2 = 0$

$\therefore \quad x = 4$

4. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ b+c & c+d & d+a & a+b \\ d & a & b & c \end{vmatrix} = 0$

Solution

We have

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ b+c & c+d & d+a & a+b \\ d & a & b & c \end{vmatrix}$$

Apply $C_2 : C_2 - C_1$, $C_3 : C_3 - C_1$, $C_4 : C_4 - C_1$ we get

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ b+c & d-b & a+d-b-c & a-c \\ d & a-d & b-d & c-d \end{vmatrix}$$

Expanding from R_1,

$$= \begin{vmatrix} b-a & c-a & d-a \\ d-b & a+d-b-c & a-c \\ a-d & b-d & c-d \end{vmatrix}$$

Apply $R_1 : R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 0 & 0 & 0 \\ d-b & a+d-b-c & a-c \\ a-d & b-d & c-d \end{vmatrix} = 0$$

Thus we get $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ b+c & c+d & d+a & a+b \\ d & a & b & c \end{vmatrix} = 0.$

Note:

Apply $R_2 : R_2 + R_3 + R_4$ and take $(a + b + c + d)$ common from second row, then we get R_1 and R_2 are identical. Then required determinant equal to zero.

5. Show that $\begin{vmatrix} \mathbf{a+b+2c} & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{b+c+2a} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a+2b} \end{vmatrix} = \mathbf{2(a+b+c)^3}$

Solution

We have,

$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$, apply, $C_1 : C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Apply $R_2 : R_2 - R_1, R_3 : R_3 - R_1$, we get,

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)\ 1(a+b+c)\ (a+b+c)$$
$$= 2(a+b+c)^3$$

$\therefore$ We get, $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

6. **Show that**

$$\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{x^2} & \mathbf{y^2} & \mathbf{z^2} \\ \mathbf{yz} & \mathbf{zx} & \mathbf{xy} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{x^2} & \mathbf{y^2} & \mathbf{z^2} \\ \mathbf{x^3} & \mathbf{y^3} & \mathbf{z^3} \end{vmatrix}$$

$$= \mathbf{(y-z)(z-x)(x-y)(yz+zx+xy)}$$

Proof:

We have

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$$

Taking xyz common from R_3.

$$= xyz\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} x^2 & y^2 & z^2 \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$. Hence proved.

Again, we have to show

$$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy)$$

Here $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

Apply $C_2 : C_2 - C_1,\ C_3 : C_3 - C_1$, we get

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \\ x^3 & (y-x)(y^2+xy+x^2) & (z-x)(z^2+xz+x^2) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & y+x & z+x \\ x^3 & y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}$$

Expanding from C_1, we get

$$= (y-x)\,(z-x)\begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}$$

$$\begin{aligned}
&= (y-x)(z-x)\,[(y+x)\,(z^2+xz+x^2)-(z+x)(y^2+xy+x^2)] \\
&= (y-x)(z-x)(yz^2+xyz+yx^2+xz^2+x^2z+x^3 \\
&\quad - zy^2-xyz-zx^2-xy^2-x^2y-x^3) \\
&= (y-x)\,(z-x)\,(yz^2+xz^2-zy^2-xy^2) \\
&= (y-x)\,(z-x)\,(yz^2-zy^2+xz^2-xy^2) \\
&= (y-x)\,(z-x)\,[zy\,(z-y)+x(z^2-y^2)] \\
&= (y-x)\,(z-x)\,[zy\,(z-y)+x(z-y)(z+y)] \\
&= (y-x)\,(z-x)\,(z-y)\,(zy+xz+xy)
\end{aligned}$$

Thus we get,

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(zy+xz+xy)$$

This completes the proof.

7. **Show that** $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

Solution

We have

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Apply. $C_2 : C_2 - C_1$, $C_3 : C_3 - C_1$ We get,

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & b^2-b^2 \\ c^2 & c^2-c^2 & (a+b)^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a-b-c)(a+b+c) \\ b^2 & (c+a-b)(a+b+c) & 0 \\ c^2 & 0 & (a+b-c)(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Apply $R_1 : R_1 - R_2 - R_3$

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Expanding from R_1

$$= (a+b+c)^2\,[2bc(c+a-b)(a+b-c)$$
$$+ 2cb^2(a+b-c) - 2b\{-c^2(c+a-b)\}]$$
$$= 2(a+b+c)^2\,[bc(ac+bc-c^2+a^2+ab-ac-ab-b^2+bc)$$
$$+ b^2ac + b^3c - b^2c^2 + bc^3 + abc^2 - b^2c^2]$$

$$= 2(a+b+c)^2\,(abc^2 + b^2c^2 - bc^3 + a^2bc - abc^2 - b^3c$$
$$+\, b^2c^2 + b^2ac + b^3c - b^2c^2 + bc^3 + abc^2 - b^2c^2)$$
$$= 2(a+b+c)^2\,(abc^2 + a^2bc + b^2ac)$$
$$= 2abc(a+b+c)^3. \text{ Hence Proved}$$

8. **Show that**

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \end{vmatrix}^2 = \begin{vmatrix} \mathbf{2bc - a^2} & \mathbf{c^2} & \mathbf{b^2} \\ \mathbf{c^2} & \mathbf{2ca - b^2} & \mathbf{a^2} \\ \mathbf{b^2} & \mathbf{a^2} & \mathbf{2ab - c^2} \end{vmatrix}$$

Solution

We have
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \left|\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{bmatrix}\right|$$

$$= \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}. \text{ Hence proved.}$$

EXERCISE 1.3

1. Without expanding, show that each of the following determinants vanishes.

(i) $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

(iii) $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$ (iv) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

(v) $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

(vii) $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$ (viii) $\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + acd \\ 1 & c & c^2 & c^3 + abd \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}$

ix) $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ac & a+c \\ a^2b^2 & ab & a+b \end{vmatrix}$ x) $\begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{vmatrix}$

2. Show that

(i) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-z)\,(x-y)\,(z-x)$

(ii) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(iii) $\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$

(iv) $\begin{vmatrix} b+c & a & b \\ a+c & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)\,(a-c)^2$

(v) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = x^3\,(4+x)$

(vi) $\begin{vmatrix} a & x & x & x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix} = (a+3x)\,(a-x)^3$

(vii) $\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} = 1+a+b+c+d$

(viii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(ix) $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = (x-2y+z)^2$

(x) $\begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix} = -(a-b)^4$

(xi) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (b-c)(c-a)(a-b)$

(xii) $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

(xiii) $\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$

(xiv) $\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix} = 1+a^2+b^2+c^2+d^2$

(xv) $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$

$$=\lambda^3(a^2+b^2+c^2+d^2+\lambda)$$

(xvi) $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

(xvii) $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c)$

xviii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

xix) $\begin{vmatrix} bcd & 1 & a & a^2 \\ acd & 1 & b & b^2 \\ abd & 1 & c & c^2 \\ abc & 1 & d & d^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

xx) $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = 4\,a^2b^2c^2.$

xxi) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4\,a^2b^2c^2.$

xxii) $\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix} = x^2(6+x)$

xxiii) $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$

xxiv) $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

xxv) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

3. Solve the following equations

(i) $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$

4. If $a+b+c=0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

5. Show that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$$= (a^3+b^3+c^3-3abc)^2$$

6. Show that

$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$ as the product of two determinants.

7. Show that the determinant

$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$ is a perfect square and find the value.

ANSWER

3. (i) $0, -\frac{1}{2}$ (ii) $-\frac{11}{97}$ 4. $0, \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$

COFACTOR MATRIX

Let $A = (a_{ij})_{n \times n}$ be a square matrix, then its cofactor matrix is denoted by $(A_{ij})_{n \times n}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Adjoint of a square matrix

Let $A = (a_{ij})_{n \times n}$ be a square matrix, then adjoint of A is denoted by adjA and defined by the transpose of cofactor matrix. That is, adj $A = (A_{ji})_{n \times n}$

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find its cofactor matrix as well as adjoint matrix.

Solution

We have
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Here Cofactor of a_{11} (= 1) is $A_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$

Cofactor of a_{12} (= 2) is $A_{12} = -\begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} = -9$

Cofactor of a_{13} (= 1) is $A_{13} = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$

Cofactor of a_{21} (= 3) is $A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -5$

Cofactor of a_{22} (= 2) is $A_{22} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$

Cofactor of a_{23} (= 1) is $A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1$

Cofactor of a_{31} (= 0) is $A_{31} = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$

Cofactor of a_{32} (= 1) is $A_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 2$

Cofactor of a_{33} (= 3) is $A_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$

Thus we get cofactor matrix of A is

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 5 & -9 & 3 \\ -5 & 3 & -1 \\ 0 & 2 & -4 \end{bmatrix}$$

Then, by definition

$$\text{adj } A = \begin{bmatrix} 5 & -9 & 3 \\ -5 & 3 & -1 \\ 0 & 2 & -4 \end{bmatrix}^t$$

$$\therefore \text{adj } A = \begin{bmatrix} 5 & -5 & 0 \\ -9 & 3 & 2 \\ 3 & -1 & -4 \end{bmatrix}$$

Theorem

If A is an $n \times n$ square matrix, then $(\text{adj } A)\, A = A\, (\text{adj } A) = |A|\, I$ where I be an identity matrix of order n.

Proof:

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\text{and adj } A = \text{ transpose of } \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

$$\Rightarrow \quad \text{adj} A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Then i, jth element of $A(\text{adj } A) = a_{i1} A_{j1} + a_{i2} A_{j2} + a_{i3} Aj_3 + \ldots\ldots + a_{in} A_{jn}$

$$= \begin{cases} |A| & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Thus we get, in the product $A\,(\text{adj} A)$, each diagonal element is $|A|$ and each non-diagonal element is 0.

$$\therefore A(\text{adj } A) = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\therefore A(\text{adj}A) = |A|\, I$$

Similarly we get, $(\text{adj } A)\, A = |A|\, I$

Therefore $A(\text{adj}A) = (\text{adj}A)\, A = |A|\, I$

This completes the proof.

Singular and non singular matrix

A square matrix A is said to be singular matrix if $|A| = 0$ otherwise non-singular.

For example

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 5 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 0 & 5 \end{vmatrix}$

$|A| = 0$

Thus we get A is a singular matrix.

INVERSE OF A MATRIX

Let A and B are two square matrix if size $n \times n$, then B is said to be inverse (or reciprocal) of A if $AB = BA = I$. Here A is said to be invertible.

Note:

1. *Only non-singular square matrices can be invertible.*
2. *If A is inverse of B, then B is also inverse of A.*
3. *The inverse of A is denoted by A^{-1}. That is, $B = A^{-1}$.*

Theorem:

The inverse of a matrix if it exists, is unique.

Proof:

Let A be a given matrix. Let us suppose B and C are inverse of A. We have to show that $B = C$.

Here

$AB = BA = I$ ....(1)

and $AC = CA = I$ ....(2)

$CAB = C(AB) = CI = C$ ....(3)

and $CAB = (CA)B = IB = B$ ....(4)

From (3) and (4) we get

$$CAB = CAB \quad \therefore C = B$$

Hence the inverse of a matrix if exists, is unique.

Theorem:

The necessary and sufficient condition for a square matrix A to possess inverse is that $|A| \neq 0$ (i.e., A is non – singular)

Proof:

"$\Rightarrow$" (The necessary condition)

Let A^{-1} exists, then we have to show $|A| \neq 0$. let B is the inverse of A, then, $AB = I$

Taking determinant on both sides

$$|AB| = |I|$$

$$\Rightarrow \quad |A|\,|B| = 1$$

This gives $|A| \neq 0$ and $|B| \neq 0$. Hence A is non – singular.

"$\Leftarrow$" (Sufficient Condition)

Let $|A| \neq 0$, then we have to show that A^{-1} exists.

Let us define another matrix

$$B = \frac{\text{adj}A}{|A|}, \text{ which is defined.}$$

$$\text{Here, } AB = \frac{A\ \text{adj}A}{|A|} = \frac{|A|\ I}{|A|} = I$$

$$\text{and } BA = \frac{\text{adj}A}{|A|}\ A = \frac{|A|\ I}{|A|} = I$$

This gives $B = \dfrac{\text{adj}A}{|A|}$ is the inverse of A.

i.e.,

$$A^{-1} = B = \frac{\text{adj}A}{|A|}$$

This completes the proof.

Note:

From this theorem, we infer that A has inverse if and only if $|A| \neq 0$ *and* $A^{-1} = \dfrac{AdjA}{|A|}$.

Theorem

Let A and B are two matrices of specified order, then $(AB)^{-1} = B^{-1}A^{-1}$, where A and B are non–singular matrix.

Proof:

Let A and B are two non–singular matrix.

Then $|AB| = |A|\,|B| \neq 0$, Since $|A| \neq 0$ and $|B| \neq 0$.

Thus we get AB is a non singular matrix. Then $(AB)^{-1}$ exists.

Also we have to show that

$$(AB)^{-1} = B^{-1} A^{-1}.$$

Here, $(AB)(B^{-1} A^{-1}) = A(B B^{-1}) A^{-1}$ (since matrix is associative)

$$= AIA^{-1} = AA^{-1} = I$$

$$\therefore (AB)(B^{-1} A^{-1}) = I \qquad \ldots\ldots\ldots\ldots(1)$$

and $(B^{-1} A^{-1})(AB) = B^{-1}(A^{-1} A) B = B^{-1} I B$

$$= B^{-1} B = I$$

$$\therefore (B^{-1} A^{-1})(AB) = I \qquad \ldots\ldots\ldots\ldots(2)$$

Thus from equation (1) and (2), we get

$(AB)^{-1} = B^{-1} A^{-1}$. This completes the proof.

Theorem

Let A be a square matrix, then $(A^{-1})^t = (A^t)^{-1}$, when A^{-1} exists.

Proof:

We have A^{-1} exists. Then $|A| \neq 0$. Also we know $|A^t| = |A| \neq 0$. This gives A^t is an non singular matrix. We have to show that

$$(A^{-1})^t = (A^t)^{-1}$$

Here $(AA^{-1})^t = (I)^t = I$

$\Rightarrow$ $A^t(A^{-1})^t = I$, by definition of transpose.

This gives $(A^{-1})^t$ is the inverse of A^t.

i.e., $(A^t)^{-1} = (A^{-1})^t$. This completes the proof.

Examples

1. **Find A^{-1} if exists, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.**

$\begin{bmatrix} 4 & \cancel{-2} \\ -3 & 1 \end{bmatrix}$?

$= \begin{bmatrix} 4 & \cancel{-3} \\ -2 & 1 \end{bmatrix}$

Solution

We have $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Thus A is a non singular matrix. Then A^{-1} exists where

$$A^{-1} = \frac{\text{adj}A}{|A|} \qquad \ldots\ldots\ldots\ldots(1)$$

For adjoint of A.

Cofactor of $a_{11} = A_{11} = 4$

Cofactor $a_{12} = A_{12} = -3$

Cofactor $a_{21} = A_{21} = -2$

Cofactor of $a_{22} = A_{22} = 1$

$$\text{Thus cofactor matrix is} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\text{Then, adj}A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

From equation (1), we get

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

2. Find A^{-1} is exists, where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Solution

We have
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Then
$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$$
$$= 1\,(2-3) - 2\,(4-0) + 1\,(2-0)$$
$$= -1-8+2 = -7$$

$\therefore |A| = -7 \neq 0$, this gives A is non singular then A^{-1} exists.

$$\text{where} \quad A^{-1} = \frac{\text{adj}A}{|A|} \qquad \text{............(1)}$$

For adj A,

Cofactor of $(a_{11} = 1) = A_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$

Cofactor of $(a_{12} = 2) = A_{12} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4$

Cofactor of $(a_{13} = 1) = A_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$

Cofactor of $(a_{21} = 2) = A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$

$$\text{Cofactor of } (a_{22} = 1) = A_{22} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\text{Cofactor of } (a_{23} = 3) = A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1$$

$$\text{Cofactor of } (a_{31} = 0) = A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

$$\text{Cofactor of } (a_{32} = 1) = A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$$

$$\text{Cofactor of } (a_{33} = 2) = A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\text{Thus cofactor matrix is } = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 2 \\ -3 & 2 & -1 \\ 5 & -1 & -3 \end{bmatrix}$$

$$\text{Then, } \operatorname{adj}A = \begin{bmatrix} -1 & -4 & 2 \\ -3 & 2 & -1 \\ 5 & -1 & -3 \end{bmatrix}^t$$

$$\operatorname{adj}A = \begin{bmatrix} -1 & -3 & 5 \\ -4 & 2 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

$$\text{From equation (1), } A^{-1} = \frac{1}{-7}\begin{bmatrix} -1 & -3 & 5 \\ -4 & 2 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

EXERCISE 1.4

1. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, Show that its adjoint is itself.

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$. Find adjA and show that A (adjA) = adjA)A = |A| I

3. Find the inverse of each of the following matrices if exists:

(i) $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -1 & 5 \\ 2 & 6 & 4 \\ 5 & 5 & 9 \end{bmatrix}$ (vi) $\begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$

(vii) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

6. Find A, where $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

ANSWERS

3. (i) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{2} & -\frac{3}{4} & \frac{11}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{5}{4} \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{-3}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{5}{4} \\ \frac{5}{4} & \frac{1}{4} & \frac{-13}{4} \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

(v) Singular

(vi) $\begin{bmatrix} 29 & -11 & 10 \\ -160 & 61 & -55 \\ 55 & -21 & 19 \end{bmatrix}$

(vii) $\frac{A}{9}$

(viii) $\begin{bmatrix} 1 & -8 & 31 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

SYSTEM OF LINEAR EQUATIONS

A system of linear (m) equations in n unknown variables $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$ is said to be coefficient matrix and

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & : & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

is said to be augmented matrix. If all b_i are zero, then the system of linear equations is called a homogeneous system and if at least one b_i is not zero, then that system of linear equations is said to be non homogeneous system.

GAUSS ELIMINATION

This method is a standard method for solving system of linear equations. It is a systematic elimination process, a method of great importance that works in practice and is reasonable with respect to computing time and storage demand.

In Gauss elimination, there are first two of three operations, which are Elementary operations for equations:

(i) Interchange of two equations,

(ii) Addition of a constant multiple of one equation to another equation.

(iii) Multiplication of an equation by a non zero constant c.

Example

Solve the system of linear equations

$$\begin{aligned} -x + y + 2z &= 2 \\ 3x - y + z &= 6 \\ -x + 3y + 4z &= 4 \end{aligned}$$

Solution

We have, given system of equation is

$$\begin{aligned} -x + y + 2z &= 2 \\ 3x - y + z &= 6 \\ -x + 3y + 4z &= 4 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 2 & : & 2 \\ 3 & -1 & 1 & : & 6 \\ -1 & 3 & 4 & : & 4 \end{bmatrix}$$

To eliminate x from the second and third equations apply in $R_2 : R_2 + 3R_1$ and $R_3 : R_3 - R_1$

$$\begin{aligned} -x + y + 2z &= 2 \\ 2y + 7z &= 12 \\ 2y + 2z &= 2 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 2 & : & 2 \\ 0 & 2 & 7 & : & 12 \\ 0 & 2 & 2 & : & 2 \end{bmatrix}$$

To eliminate y from third equation and also apply in $R_3 : R_3 - R_2$ we get

$$\begin{aligned} -x + y + 2z &= 2 \\ 2y + 7z &= 12 \\ -5z &= -10 \end{aligned} \qquad \begin{bmatrix} -1 & 1 & 2 & : & 2 \\ 0 & 2 & 7 & : & 12 \\ 0 & 0 & -5 & : & 10 \end{bmatrix}$$

From last equation, we get

$$z = 2$$

From second equation, putting $z = 2$,

$$\begin{aligned} 2y + 14 &= 12 \\ 2y &= -2 \\ \Rightarrow \quad y &= -1 \end{aligned}$$

From first equation, we get

$$\begin{aligned} -x + y + 2z &= 2 \\ -x + (-1) + 2(2) &= 2 \\ -x - 1 + 4 &= 2 \\ \Rightarrow \quad x &= 1 \end{aligned}$$

Thus we get the required solution of the system of linear equation is $x = 1$, $y = -1$ and $z = 2$.

3. Echelon form

In this method, the form of the coefficient matrix in augmented matrix of the given system of equations is reduced to triangular matrix by eliminating Rows operations.

Example

1. **Solve**

$0.4x + 1.2y = -2$

$1.7x - 3.2y = 8.1$

by using Gauss – elimination method.

Solution

We have given system of equation is

$0.4x + 1.2y = -2$

$1.7x - 3.2y = 8.1$

Its augmented matrix is

$$\begin{bmatrix} 0.4 & 1.2 & : & -2 \\ 1.7 & -3.2 & : & 8.1 \end{bmatrix}$$

Apply in R_2: $0.4\,R_2 - (1.7)\,R_1$ we get

$$\begin{bmatrix} 0.4 & 1.2 & : & -2 \\ 0 & -3.32 & : & 6.64 \end{bmatrix}$$

From 2nd row we get

$$-3.32y = 6.64$$

$$\Rightarrow \quad y = -2$$

And from first row we get

$$0.4x + 1.2y = -2$$

$$\Rightarrow \quad 0.4x + (1.2)(-2) = -2$$

$$0.4x - 2.4 = -2$$

$$0.4x = 0.4$$

$$x = 1$$

Thus we get required solution of given system of equation is $x = 1$, $y = -2$.

2. **Solve by using Gauss elimination method**

$$7y + 3z = -12$$

$$2x + 8y + z = 0$$

$$-5x + 2y - 9z = 26$$

Solution

The given system of linear equation can be written as

$$2x + 8y + z = 0$$

$$7y + 3z = -12$$

$$-5x + 2y - 9z = 26$$

Its augmented matrix is

$$\begin{bmatrix} 2 & 8 & 1 & : & 0 \\ 0 & 7 & 3 & : & -12 \\ -5 & 2 & -9 & : & 26 \end{bmatrix}$$

Apply $R_3 : 2R_3 + 5R_1$, we get

$$\begin{bmatrix} 2 & 8 & 1 & : & 0 \\ 0 & 7 & 3 & : & -12 \\ 0 & 44 & -13 & : & 52 \end{bmatrix}$$

Apply $R_3 : 7R_3 - 44R_2$, we get

$$\begin{bmatrix} 2 & 8 & 1 & : & 0 \\ 0 & 7 & 3 & : & -12 \\ 0 & 0 & -223 & : & 892 \end{bmatrix}$$

From R_3, $-223z = 892 \Rightarrow z = -4$

From R_2, $7y + 3z = -12 \Rightarrow 7y-12=-12 \Rightarrow y=0$

From R_1, $2x + 8y + z = 0 \Rightarrow 2x-4 = 0$

$\Rightarrow x = 2$

Thus required solution of the given system of linear equation is $x = 2, y = 0, z = -4$.

3. **Solve**

$$\begin{aligned} 4x + y &= 4 \\ 5x-3y + z &= 2 \\ -9x + 2y-z &= 5 \end{aligned}$$

by using Gauss elimination method.

Solution

We have given system of linear equation is

$$\begin{aligned} 4x + y &= 4 \\ 5x-3y + z &= 2 \\ -9x + 2y-z &= 5 \end{aligned}$$

Its augmented matrix is

$$\begin{bmatrix} 4 & 1 & 0 & : & 4 \\ 5 & -3 & 1 & : & 2 \\ -9 & 2 & -1 & : & 5 \end{bmatrix}$$, apply R_2: $4R_2 - 5R_1$, R_3: $4R_3 + 9R_1$, we get

$$\begin{bmatrix} 4 & 1 & 0 & : & 4 \\ 0 & -17 & 4 & : & -12 \\ 0 & 17 & -4 & : & 56 \end{bmatrix}$$, Apply $R_3 : R_3 + R_2$, We get

$$\begin{bmatrix} 4 & 1 & 0 & : & 4 \\ 0 & -17 & 4 & : & -12 \\ 0 & 0 & 0 & : & 44 \end{bmatrix}$$

From R_3, we get

$0.z = 44$

which is not possible. Thus, we get the solution of given system of equation does not exists.

4. Solve

$$4x - 8y + 3z = 16$$
$$-x + 2y - 5z = -21$$
$$3x - 6y + z = 7$$

by using Gauss elimination method.

Solution

We have given system of equation is

$$4x - 8y + 3z = 16$$
$$-x + 2y - 5z = -21$$
$$3x - 6y + z = 7$$

Its augmented matrix is

$$\begin{bmatrix} 4 & -8 & 3 & : & 16 \\ -1 & 2 & -5 & : & -21 \\ 3 & -6 & 1 & : & 7 \end{bmatrix}$$

Apply $R_2: 4R_2 + R_1,\ R_3: 4R_3 - 3R_1$, we get

$$\begin{bmatrix} 4 & -8 & 3 & : & 16 \\ 0 & 0 & -17 & : & -68 \\ 0 & 0 & -5 & : & -20 \end{bmatrix}$$

From R_3, we get

$$-5z = -20 \Rightarrow z = 4$$

From R_2, we get

$$0.y - 17z = -68$$
$$0.y = 17z - 68$$
$$0.y = 0$$

This gives y is free.

Also, from R_1, we get

$$4x - 8y + 3z = 16$$
$$4x - 8y + 12 = 16$$
$$4x = 4 + 8y$$
$$x = 1 + 2y$$

Thus required solution of the given system of linear equation is,

$$x = 1 + 2y, \; y = y, \; z = 4$$

This gives, given system of linear equation has infinite solutions.

5. **Solve**

$$12x - 26y + 34z = 18$$
$$-30x + 65y - 85z = -46$$, by using Gauss elimination method.

Solution

We have, given system of linear equation is

$$12x - 26y + 34z = 18$$
$$-30x + 65y - 85z = -46.$$

This system of equations can be written as,

$$12x - 26y + 34z = 18$$
$$-30x + 65y - 85z = -46$$
$$0x + 0y + 0z = 0$$

Its augmented matrix is

$$\begin{bmatrix} 12 & -26 & 34 & :18 \\ -30 & 65 & -85 & :-46 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

Apply $R_2 : 2R_2 + 5R_1$, we get

$$\begin{bmatrix} 12 & -26 & 34 & :18 \\ 0 & 0 & 0 & :-2 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

From R_3, We get,

$$0z = 0$$

This gives a is free.

From R_2, we get

$$0.y + 0.z = -2$$

which is not possible. Thus, solution of given system of equations does not exists.

6. **Solve**

$$2w - 2x + 4y = 0$$
$$-3w + 3x - 6y + 5z = 15$$
$$w - x + 2y = 0$$

by using Gauss elimination method.

Solution

We have given system of linear equation is

$$2w - 2x + 4y = 0$$
$$-3w + 3x - 6y + 5z = 15$$

$$w - x + 2y = 0$$

Its augmented matrix is

$$\begin{bmatrix} 2 & -2 & 4 & 0 & : & 0 \\ -3 & 3 & -6 & 5 & : & 15 \\ 1 & -1 & 2 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Apply $R_2 : 2R_2 + 3R_1$

$R_3 : 2R_3 - R_1$, we get

$$\begin{bmatrix} -2 & -2 & 4 & 0 & : & 0 \\ 0 & 0 & 0 & 10 & : & 30 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

From R_2, we get $10z = 30$

$\Rightarrow$ $z = 3$

From R_3, we get, $0y = 0$

This gives y is free.

From R_4, we get

$$0x = 0$$

This gives x is free.

From R_1, we get,

$$2w - 2x + 4y = 0$$
$$w = x - 2y$$

Thus required solution of given system of linear equation is

$$w = x - 2y, \quad x = x, \quad y = y, \quad z = 3$$

It has infinite solutions.

7. Solve the given system of linear equations by using Gauss elimination method:

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$2x + 3y + 4z = 7$$

Solution

The augmented matrix of the given system of equation is

$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$, apply $R_2 : R_2 - R_1$, $R_3 : R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$, apply R_3: $R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From R_3, we get $0z = 0 \Rightarrow z$ is free.

From R_2, we get, $y + 2z = 1$

$$\Rightarrow y = 1 - 2z$$

From R_1, we get

$$x + y + z = 3$$

$$\Rightarrow x = 3 - y - z = 2 + z$$

Thus required solution of the given system of equation is $(z + 2, 1 - 2z, z)$

8. Solve the following system of linear equation by using Gauss elimination method:

$x + y + z = 6$

$x - y + z = 2$

$2x - y + 3z = 9$

Solution

The augmented matrix of the given system of liner equation is

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 9 \end{bmatrix}$$

Apply R_2: $R_2 - R_1$

R_3: $R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

Apply R_3: $2R_3 - 3R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Form R_3, we get $z = 3$

Form R_2, we get $y = 2$

Form R_1 we get, $x + y + z = 6$

$x = 1$.

Thus, we get required solution of the given system of equation is

$$(x, y, z) = (1, 2, 3).$$

EXERCISE 1.5

Solve the following system of linear equations by using Gauss elimination method:

1. $6x + 4y = 2$
   $3x - 5y = -34$
2. $3x - 0.5y = 0.6$
   $1.5x + 4.5y = 6$
3. $x + y - z = 9$
   $8y + 6z = -6$
   $-2x + 4y - 6z = 40$
4. $13x + 12y = -6$
   $-4x + 7y = -73$
   $11x - 13y = 157$
5. $4y + 3z = 8$
   $2x - z = 2$
   $3x + 2y = 5$
6. $7x - 4y - 2z = -6$
   $16x + 2y + z = 3$
7. $x - xy + 3z = 11$
   $3x + y - z = 2$
   $5x + 3y + 2z = 3$
8. $2x + 3y + 4z = 20$
   $3x + 4y + 5z = 26$
   $3x + 5y + 6z = 31$
9. $3x - y + z = -2$
   $x + 5y + 2z = 6$
   $2x + 3y + z = 0$
10. $5x + 5y - 10z = 0$
    $2w - 3x - 3y + 6z = 2$
    $4w + x + y - 2z = 4$
11. $10x + 4y - 2z = -4$
    $-3w - 17x + y + 2z = 2$
    $w + x + y = 6$
    $8w - 34x + 16y - 10z = 4$

ANSWERS

1. $x = -3, y = 5$
2. $x = 0.4, y = 1.2$
3. $x = 1, y = 3, z = -5$
4. $x = 6, y = -7$
5. No solution
6. $x = 0, z = 3 - 2y$
7. $x = 2, y = -3, z = 1$
8. $x = 1, y = 2, z = 3$
9. $x = -2, y = 0, z = 4$
10. $w = 1, y = 2z - x$
11. $w = 4, x = 0, y = 2, z = 6$

CRAMER'S RULE

Let

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

be a system of linear equations with x_1, x_2 and x_3 are variables, where as a_{ij} and b_i are constants.

Then $$x_1 = \frac{D_1}{D} \quad \text{for } D \neq 0$$

$$x_2 = \frac{D_2}{D} \qquad \text{for } D \neq 0$$

and $$x_3 = \frac{D_3}{D} \qquad \text{for } D \neq 0$$

where, $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$, $D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

similarly for n variables of the system of linear equations.

Example

1. **Solve by using cramer's rule**

$$\mathbf{x + 2y + 3z = 20}$$
$$\mathbf{7x + 3y + z = 13}$$
$$\mathbf{x + 6y + 2z = 0}$$

Solution

We have given system of linear equation is

$$x + 2y + 3z = 20$$
$$7x + 3y + z = 13$$
$$x + 6y + 2z = 0$$

Here, $$D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 1(6-6) - 2(14-1) + 3(42-3)$$

$$= -26 + 117$$
$$D = 91 \neq 0$$

Also, $$D_1 = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} = 20(6-6) - 2(26) + 3(78)$$

$$= -52 + 234$$
$$D_1 = 182$$

and $$D_2 = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1(26) - 20\,(14-1) + 3(-13)$$

$$= 26 - 260 - 39 = 26 - 299 = -273$$

$$D_3 = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} = 1(0-78) - 2(-13) + 20(42-3)$$

$$D_3 = -78 + 26 + 780 = 728$$

Thus by cramer's rule

$$x = \frac{D_1}{D} = \frac{182}{91} = 2$$

$$y = \frac{D_2}{D} = \frac{-273}{91} = -3$$

$$z = \frac{D_3}{D} = \frac{728}{91} = 8$$

Thus required solution of given system of equation is

$$x = 2, y = -3, z = 8.$$

2. By using cramer's rule, solve the following system of equations:

$$-x + 3y - 2z = 7$$
$$3x + 0y + 3z = -3$$
$$2x + y + 2z = -1$$

Solution

We know, by cramer's rule

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D} \text{ for } D \neq 0.$$

where $D = \begin{vmatrix} -1 & 3 & -2 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{vmatrix} = -1(-3) - 3(0) - 2(3) = 3 - 6 = -3 \neq 0$

$$D_1 = \begin{vmatrix} 7 & 3 & -2 \\ -3 & 0 & 3 \\ -1 & 1 & 2 \end{vmatrix} = 7(-3) - 3(-6+3) - 2(-3)$$

$$D_1 = -6$$

Also, $D_2 = \begin{vmatrix} -1 & 7 & -2 \\ 3 & -3 & 3 \\ 2 & -1 & 2 \end{vmatrix} = -1(-6+3) - 7(6-6) - 2(-3+6)$

$$= 3 - 6 = -3.$$

and $D_3 = \begin{vmatrix} -1 & 3 & 7 \\ 3 & 0 & -3 \\ 2 & 1 & -1 \end{vmatrix} = -1(3) - 3(-3+6) + 7(3) = -3 - 9 + 21$

$$D_3 = 9$$

Thus we get, $x = 2, y = 1, z = -3$.

This is the required solution of the given system of equations.

EXERCISE 1.6

Solve by using cramers rule of the following system of linear equations

1) $5x - 3y = 37$
   $-2x + 7y = -38$

2) $3x + 7y + 8z = -13$
   $2x + 9z = -5$
   $-4x + y - 26z = 2$

3) $x + y + z = 0$
   $2x + 5y + 3z = 1$
   $-x + 2y + z = 2$

4) $3x + y + 2z = 3$
   $2x - 3y - z = -3$
   $x + 2y + z = 4$

5) $x + y + z = 1$
   $2x + 3y + 2z = 2$
   $3x + 3y + 4z = 1$

6) $x + 2y + 3z = 6$
   $2x + 4y + z = 7$
   $3x + 2y + 9z = 14$

7) $x + 3y + 6z = 2$
   $3x - y + 4z = 9$
   $x - 4y + 2z = 7$

ANSWER

1) $x = 5, y = -4$
2) $x = -7, y = 0, z = 1$
3) $x = -1, y = 0, z = 1$
4) $x = 1, y = 2, z = -1$
5) $x = 3, y = 0, z = -2$
6) $x = y = z = 1$
7) $x = 2, y = -1, z = \frac{1}{2}$.

LINEAR DEPENDENT AND INDEPENDENT

Let $\{u_1, u_2, \ldots, u_n\}$ be a set of n vectors, then the linear combination of these vectors is an expression of the form

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = \sum_{i=1}^{n} a_i u_i, \text{ where all } a_i \text{ are any scalar.}$$

The set of n vectors $\{u_1, u_2, \ldots, u_n\}$ is said to be linearly independent if $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$ implies all $a_i = 0$.

Also the set of n vectors $\{u_1, u_2, \ldots, u_n\}$ are said to be linearly dependent if, $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$ implies at least one $a_i \neq 0$.

Example

1. If $u = (7, -3, 11, -6)$, $v = (-56, 24, -88, 48)$. Check $\{u, v\}$ are dependent or independent.

Solution

Let $au + bv = 0$, where a and b are scalar.

$\Rightarrow \quad a(7, -3, 11, -6) + b(-56, 24, -88, 48) = 0$

$\Rightarrow \quad (7a - 56b, -3a + 24b, 11a - 88b, -6a + 48b) = 0$

This gives,

$7a - 56b = 0 \Rightarrow \quad a - 8b = 0$

$-3a + 24b = 0 \quad \Rightarrow \quad a - 8b = 0$

$11a - 88b = 0 \Rightarrow \quad a - 8b = 0$

$-6a + 48b = 0 \quad \Rightarrow \quad a - 8b = 0$

These four equations gives $a = 8b$. Thus this equation doesn't always implies a and b are zero.

Hence we get, $au + bv = 0$ doesnot always implies $a = 0$ and $b = 0$. Therefore u and v are linearly dependent.

2. **If $u = (1, -1, 1)$, $v = (1, 1, -1)$, $w = (-1, 1, 1)$, $x = (0, 1, 0)$, check $\{u, v, w, x\}$ are dependent or independent.**

Solution

Let $au + bv + cw + dx = 0$, where a, b, c, d are scalar.

This gives $a(1, -1, 1) + b(1, 1, -1) + c(-1, 1, 1) + d(0, 1, 0) = 0$

$\Rightarrow \quad (a, -a, a) + (b, b, -b) + (-c, c, c) + (0, d, 0) = (0, 0, 0)$

$\Rightarrow \quad (a + b - c, -a + b + c + d, a - b + c) = (0, 0, 0)$

This gives $\quad a + b - c = 0 \quad \text{...........(i)}$

$-a + b + c + d = 0 \quad \text{...........(ii)}$

$a - b + c = 0 \quad \text{...........(iii)}$

From (i) and (iii), we get

$2a = 0$ (by adding)

$\Rightarrow \quad a = 0$

Putting value $a = 0$ in equation (i), (ii), (iii) we get,

$b - c = 0$

$b + c + d = 0$

$-b + c = 0 \quad \Rightarrow \quad b = c$ and $d = -2c$

This gives b, c and d are not zero always.

Thus given $\{u, v, w, x\}$ are linearly dependent.

3. **Check dependent and independent of $u = (1, 2, 3)$, $v = (0, 0, 0)$, $w = (5, 5, 1)$.**

Solution

Let $au + bv + cw = 0$

$\Rightarrow \quad a(1,2,3) + b(0, 0, 0) + c(5, 5, 1) = 0$

$\Rightarrow \quad (a, 2a, 3a) + (5c, 5c, c) = 0$

$\Rightarrow \quad (a + 5c, 2a + 5c, 3a + c) = 0$

This gives
$$a + 5c = 0$$
$$2a + 5c = 0$$
$$3a + c = 0$$

We get, $a = 0 = c$ but b is not always zero. Hence the given {u, v, w} are dependent.

4. Check dependent or independent of

$$u = \left(\frac{1}{4}, 0, -\frac{1}{4}\right),\ v = \left(0, \frac{1}{2}, -\frac{1}{2}\right),\ w = \left(-\frac{1}{3}, -\frac{1}{3}, 0\right)$$

Solution

Let $au + bv + cw = 0$, where a, b, c are scalar.

$$\Rightarrow \quad a\left(\frac{1}{4}, 0, -\frac{1}{4}\right) + b\left(0, \frac{1}{2}, -\frac{1}{2}\right) + c\left(-\frac{1}{3}, -\frac{1}{3}, 0\right) = 0$$

$$\Rightarrow \quad \left(\frac{1}{4}a, 0, -\frac{1}{4}a\right) + \left(0, \frac{1}{2}b, -\frac{1}{2}b\right) + \left(-\frac{c}{3}, -\frac{c}{3}, 0\right) = 0$$

$$\Rightarrow \quad \left(\frac{a}{4} - \frac{c}{3}, \frac{b}{2} - \frac{c}{3}, -\frac{a}{4} - \frac{b}{2}\right) = 0 = (0, 0, 0)$$

This gives,

$$\frac{a}{4} - \frac{c}{3} = 0 \Rightarrow a = \frac{4}{3}c, \quad \text{.......... (i)}$$

$$\frac{b}{2} - \frac{c}{3} = 0 \Rightarrow b = \frac{2}{3}c \quad \text{..........(ii)}$$

and $-\frac{a}{4} - \frac{b}{2} = 0 \Rightarrow a + 2b = 0 \Rightarrow 2b = \frac{-4}{3}c$ .... (iii)

From (ii) and (iii) we get, $\frac{4}{3}c = \frac{-4}{3}c$

$$\Rightarrow \quad \frac{8}{3}c = 0 \quad \Rightarrow \quad c = 0$$

This gives $b = 0$ and $a = 0$

Hence by definition, we get {u, v, w} are linearly independent.

EXERCISE 1.7

A. Are the following sets of vectors linearly independent or dependent ?

1. (1, 0, 0), (1, 1, 0), (1, 1, 1)
2. (– 1, 5, 0), (16, 8, – 3), (– 64, 56, 9)
3. (2, – 4), (1, 9), (3, 5)
4. (1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 0, 8)

B. Show that the following sets of vectors are linearly independent:

1. (1, 2), (1, 3)
2. (1, 1, 1), (1, – 1, 0), (0, 1, 1)

3. $(2, 3, 5), (4, 9, 11)$
4. $(1, 1, 2), (3, 1, 2), (0, 1, 4)$
5. $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

C. Check dependent or independent of the following:

1. $(0, 1, 0), (0, 0, 1), (1, 1, 1)$
2. $(1, 0, 1), (1, 1, 0), (-1, 0, -1)$
3. $(1, 2, -1), (2, 3, 0), (0, 0, 0)$
4. $(2, 1, 1), (3, -2, 2), (-1, 2, -1)$

ANSWERS

A.
1. Linearly independent
2. Linearly independent
3. Linearly dependent
4. Linearly independent

C.
1. Linearly independent
2. Linearly dependent
3. Linearly dependent
4. Linearly independent

RANK OF MATRIX

Let $A = (a_{ij})_{m\times n}$ be a matrix, then the positive number r is said to be rank of A if r is the largest number of independent row vectors or column vectors of a matrix. It is denoted by $\rho(A) = r$.

Note:

(i) *If $A = 0$, then rank of $A = 0$* *(ii) If $\rho(A) = r$, then $\rho(A^t) = r$.*

(iii) *$\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.*

Example

Find rank of
$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Solution

We have,
$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}_{3\times 2}$$

This matrix will have maximum rank three. To check this, here

$$R_1 = (3, 0, 2, 2)$$
$$R_2 = (-6, 42, 24, 54)$$
$$R_3 = (21, -21, 0, -15)$$

Let $aR_1 + bR_2 + cR_3 = 0$

$\Rightarrow$ $a(3,0,2,2) + b(-6,42,24,54) + c(21,-21,0,-15) = 0$

$\Rightarrow$ $(3a - 6b + 21c, 42b - 21c, 2a + 24b, 2a + 54b - 15c) = 0 = (0,0,0,0)$

This gives,

$3a - 6b + 21c = 0 \Rightarrow a - 2b + 7c = 0$ ....(i)

$42b - 21c = 0 \Rightarrow 2b - c = 0$ ....(ii)

$2a + 24b = 0 \Rightarrow a + 12b = 0$ ....(iii)

$2a + 54b - 15c = 0$ ....(iv)

From (i) and (ii)

$a - 2b + 14b = 0 \Rightarrow a + 12b = 0$ ....(v)

From (iii) and (v), we get, $a = -12b$

This gives a and b are not always zero; thus by definition $\{R_1, R_2, R_3\}$ is linearly dependent.

Again, Let $aR_1 + bR_2 = 0$

$\Rightarrow$ $a(3,0,2,2) + b(-6,42,24,54) = 0$

$\Rightarrow$ $(3a - 6b, 42b, 2a + 24b, 2a + 54b) = 0 = (0,0,0,0)$

This gives, $3a - 6b = 0 \Rightarrow a = 2b$ ....(vi)

$42b = 0 \Rightarrow b = 0$ ....(vii)

$2a + 24b = 0$ ....(viii)

$2a + 54b = 0$ ....(ix)

From (vi) and (vii), we get $a = 0, b = 0$. Therefore $\{R_1, R_2\}$ is linearly independent. Thus by definition $\rho(A) = 2$.

Note:

The rank of a matrix is invariance under the elementary row operation. This gives row – equivalent matrices have the same rank. This shows that to determine the rank of A, we can reduce A to echelon form by using technique of Gauss elimination method.

Example

Find the rank of $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Solution

We have $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Apply $R_2 : R_2 + 2R_1, R_3 : R_3 - 7R_1$, we get

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}, \text{ apply } R_3 : 2R_3 + R_2, \text{ we get}$$

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix has two non zero rows. So it will have maximum rank 2.

Here, $R_1 = (3, 0, 2, 2), R_2 = (0, 42, 28, 58)$

Let $aR_1 + bR_2 = 0$

$\Rightarrow$ $a(3, 0, 2, 2) + b(0, 42, 28, 58) = 0$

$\Rightarrow$ $(3a, 42b, 2a + 28b, 2a + 58b) = 0 = (0, 0, 0, 0)$

This gives

$$\begin{aligned} 3a &= 0 \\ 42b &= 0 \\ 2a + 28b &= 0 \quad \Rightarrow \quad a = 0 = b \\ 2a + 58b &= 0 \end{aligned}$$

Hence $\{R_1, R_2\}$ is linearly independent. Thus rank of $A = 2$.

Note:

Let $A = (a_{ij})_{m\times n}$ be a matrix. Then a positive number $r \le \min\{m, n\}$ is said to be rank of A if there exists a nonzero minor of A having order r and all minor of A having order r+1 is equals to zero.

Example

1. Find rank of $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution

We have $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be a 2×1 matrix. Its rank is $\rho(A) \le 1$.

Here minor of A having order 1 is $|1| = 1 \neq 0$

but minor of A having order 2×2 is

$$\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

Hence by definition $\rho(A) = 1$.

2. **Find rank of** $\mathbf{A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \end{bmatrix}}$

Solution

We have $$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \end{bmatrix}$$

By definition, its rank is less than or equal to 2.

Here minor of A having order 2×2 is

$$= \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 \neq 0$$

But minor of A having order 3×3 is

$$\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Thus by definition $\rho(A) = 2$.

3. **Find rank of** $\mathbf{A = \begin{bmatrix} 2 & 0 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix}}$

Solution

We have, $$A = \begin{bmatrix} 2 & 0 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix}$$

Its rank is less than or equals to 3. Here minor of A having order 3×3 is

$$= \begin{vmatrix} 2 & 0 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{vmatrix} = 2(12 - 12) - 0\,(12 - 12) + 7\,(6 - 6)$$

$$= 0$$

Also minor of A having order 2×2 is

$$= \begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} = 6 \neq 0$$

Hence by definition $\rho(A) = 2$.

4. **Find rank of the matrix** $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$.

Solution

We have given square matrix

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

Apply R_3. $R_3 - R_1$, $R_4 : R_4 - R_1$, we get

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

Apply R_3: $3R_3 - 4R_2$, $R_4 : R_4 - 2R_2$, we get

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives $\{R_1, R_2, R_3\}$ are linearly independent. Thus by definition rank of $A = 3$.

5. **Find rank of the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 2 \\ 6 & 9 & 3 & 3 \end{bmatrix}$

Solution

We have, given square matrix

$$\begin{bmatrix} 1 & 2 & 0 & 6 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 2 \\ 6 & 9 & 3 & 3 \end{bmatrix}$$

Apply $R_2 : R_2 - 2R_1$, $R_3 : R_3 - 3R_1$, $R_4 : R_4 - 6R_1$, we get

$$\begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & -1 & 1 & -11 \\ 0 & -2 & 2 & -16 \\ 0 & -3 & 3 & -33 \end{bmatrix}$$

Apply $R_3 : R_3 - 2R_2$, $R_4 : R_4 - 3R_2$, we get

$$\begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & -1 & 1 & -11 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This given $\{R_1, R_2, R_3\}$ are linearly independent. Thus required rank of the matrix A is 3.

6. Find rank of the matrix $\mathbf{A} = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & -3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Solution

We have given matrix is

$\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & -3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$, apply R_3: $R_3 + R_1$, we get

$$\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

This gives $\{R_1, R_2, R_3\}$ are linearly independent. So rank of the given matrix is 3.

7. Determine the rank of the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution

We have given matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Apply $R_2 : R_2 - R_4$, $R_3 : R_3 - 3R_4$, we get

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 8 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}, \text{ apply } R_1 : 2R_1 + R_3,\ R_2 : 2R_2 - R_3, \text{ we get}$$

$$\begin{bmatrix} 0 & 0 & 14 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 8 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}, \text{ apply } R_1 : R_1 + 7R_2, \text{ we get}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 8 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

This given { R_2, R_3, R_4} are linearly independent. Thus by definition rank of A = 3.

8. Find rank of A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} \\ \mathbf{4} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Solution

We have, $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

Apply, $R_2 : R_2 - 4R_1$, $R_3 : R_3 - 3R_1$, $R_4 : R_4 - R_1$, we get,

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \text{ apply } R_3 : R_3 - R_2, \text{ we get,}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \text{ apply } R_4 : 2R_4 + R_3, \text{ we get,}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This given $\{R_1, R_2, R_3\}$ are linearly independent. Thus rank of $A = 3$.

EXERCISE 1.8

A. Find rank of the following matrices

1. $A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$

2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

5. $A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{pmatrix}$

6. $A = \begin{bmatrix} 8 & -3 & 7 \\ -20 & -17 & -15 \\ 11 & 2 & 9 \end{bmatrix}$

7. $A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 10 & 0 & 14 \end{bmatrix}$

8. $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

9. $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

10. $A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

13. $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ 16. $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$

17. $\begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix}$ 18. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

19. $\begin{bmatrix} 4 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ 20. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

B. Find rank of AB, where

1. $A = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $B = (0 \ \ 3 \ \ 4)$

2. $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}, B = \begin{pmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$

ANSWERS

A.	1. 1	2. 2	3. 1	4. 2
	5. 2	6. 2	7. 2	8. 2
	9. 2	10. 4	11. 2	12. 2
	13. 2	14. 3	15. 3	16. 3
	17. 2	18. 2	19. 4	20. 2.
B.	1. 1	2. 2		

CONSISTENCY OF A SYSTEM OF LINEAR EQUATION

Let

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1)$$

be a system of linear equations. And its coefficient matrix is

$$\text{(Say) } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and Augmented matrix is}$$

$$\text{(Say) } B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Here the system of linear equation (1) is consistence if at least one set of values of $x_1, x_2, \ldots, x_n$ can be found to satisfy all the equations and inconsistency if no such set of solution exists.

Also the system of linear equations are consistence if the rank of the coefficient matrix is equals to rank of augmented matrix.

Note:

1. If rank of augmented matrix = Rank of coefficient matrix = number of independent variables; in the system of linear equation, then this system has unique solution.

2. If rank of augmented matrix = Rank of coefficient matrix ≠ number of independent variables, in the system of linear equation, then this system has infinite solution.

Example

1. Show that the given system of linear equation

$$\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0 \end{aligned}$$ **is consistence and solve it.**

Solution

We have, given system of linear equation is

$$\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0 \end{aligned}$$

Its coefficient matrix is $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$

and augmented matrix is $B = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$ 3×4

We have to show $\rho(A) = \rho(B)$.

For $\rho(A)$:

Minor of having order 3×3 is $\begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix}$

$$= 2(-8-1) + 1(4-3) + 3(-1-6)$$
$$= -18 + 1 - 21 = -38 \neq 0$$

But minor of A having order 4×4 is

$$= \begin{vmatrix} 2 & -1 & 3 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Thus by definition $\rho(A) = 3$.

For $\rho(B)$:

Minor of B having order 3×3 is

$$= \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = -38 \neq 0$$

But minor of B having order 4×4 is

$$\begin{vmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Thus by definition $\rho(B) = 3$.

Hence we get $\rho(A) = \rho(B) = 3$. Thus the given system of linear equation is consistence.

For Solve:

We have Augmented matrix is $\begin{bmatrix} 2 & -1 & 3 & : & 8 \\ -1 & 2 & 1 & : & 4 \\ 3 & 1 & -4 & : & 0 \end{bmatrix}$

Apply $R_2 : 2R_2 + R_1$, $R_3 : 2R_3 - 3R_1$, we get

$$= \begin{bmatrix} 2 & -1 & 3 & : & 8 \\ 0 & 3 & 5 & : & 16 \\ 0 & 5 & -17 & : & -24 \end{bmatrix}$$

Apply $R_3 : 3R_3 - 5R_2$, we get

$$= \begin{bmatrix} 2 & -1 & 3 & : & 8 \\ 0 & 3 & 5 & : & 16 \\ 0 & 0 & -76 & : & -152 \end{bmatrix}$$

From R_3, we get

$$-76z = -152$$

$$\therefore \quad z = 2$$

From R_2, we get,

$$3y + 5z = 16$$
$$3y = 16 - 5z = 16 - 5 \times 2$$
$$3y = 6,$$
$$\therefore \quad y = 2$$

From R_1, we get,

$$2x - y + 3z = 8$$
$$\Rightarrow \quad 2x = 8 + y - 3z$$
$$2x = 8 + 2 - 6$$
$$\therefore \quad x = 2$$

Thus required solution of given system of equation is

$$x = 2, y = 2, \quad z = 2.$$

2. Test consistency of the system of equation

$x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + 9z = 6$ and solve it.

Solution

The coefficient matrix and augmented matrix of the given system of equation are

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix} \text{ respectively.}$$

We know given system of equation is consistency if rank of coefficient matrix is equals to rank of augmented matrix.

For ρ (A)

We have $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Apply $R_2 : R_2 - R_1$, $R_3 : R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix}, \text{ apply } R_3 : R_3 - 3R_2, \text{ we get}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

This given $\{R_1, R_2, R_3\}$ are linearly independent. Thus $\rho(A) = 3$.

For $\rho(B)$.

We have, $B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}$, apply $R_2 : R_2 - R_1, R_3 : R_3 - R_1$, we get,

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 4 \end{bmatrix}, \text{ apply } R_3 : R_3 - 3R_2, \text{ we get}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

This given $\{R_1, R_2, R_3\}$ are linearly independent. Thus $\rho(B) = 3$.

Hence, we get $\rho(A) = \rho(B)$, so the given system of linear equation is consistence.

For Solution:

We have $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$

From R_3, we get, $z = -1$

From R_2, we get, $y + 2z = 2$, $y = 4$

From R_1, we get, $x + y + z = 2$, $x = -1$

Thus required solution of the given system of equation is $(-1, 4, -1)$.

3. **Check consistency of the system of equations**

$\mathbf{5x + 3y + 7z = 4}$

$\mathbf{3x + 26y + 2z = 9}$

$\mathbf{7x + 2y + 10z = 5}$ **and solve it.**

Solution

Here augmented matrix of the given system of equation is

$$B = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix} \text{ and coefficient matrix is } A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

We know, given system of equation is consistence if $\rho(A) = \rho(B)$.

For ρ (B):

$$\text{We have, } B = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

Apply $R_2 : 5R_2 - 3R_1$, $R_3 : 5R_3 - 7R_1$, we get,

$$\begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Apply $R_3 , R_2 + 11R_3$, we get,

$$\begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This give $\{R_1, R_2\}$ are linearly independent. Thus $\rho(B) = 2$.

For ρ (A):

$$\text{We have } A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Apply $R_2 : 5R_2 - 3R_1$, $R_3 : 5R_3 - 7R_1$, we get,

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix}, \text{ apply, } R_3 : 11R_3 + R_2, \text{ we get,}$$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives $\{R_1, R_2\}$ are linearly independent.

So, $\rho(A) = 2$.

Thus we get $\rho(A) = \rho(B)$, hence the given system of equation is consistence.

For solution

$$\text{We have } \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From R_3, we get $oz = 0$ implies $z = z$.
Form R_2 we get,

$$121y - 11z = 33$$

$$11y = 3 + z \quad \Rightarrow y = \frac{3+z}{11}$$

From R_1, we get, $5x + 3y + 7z = 4$

$$5x = 4 - 3\left(\frac{3+z}{11}\right) - 7z = \frac{44 - 9 - 3z - 77z}{11}$$

$$\therefore \quad x = \frac{35 - 80z}{55} \quad \Rightarrow x = \frac{7 - 16z}{11}$$

Thus we get, required solution of the given system of equation is $\left(\frac{7-16z}{11}, \frac{z+3}{11}, z\right)$.

EXERCISE 1.9

Check following system of linear equations is consistence or not; if consistence solve it:

1. $x + y + z = -3$
   $3x + y - 2z = -2$
   $2x + 4y + 7z = 7$

2. $3x - 4y = 2$
   $-x + 3y = 1$

3. $x + y + z = 6$
   $x + 7y + 3z = 10$
   $x + 2y + 4z = 1$

4. $x + y + z = 3$
   $x + 2y + 3z = 4$
   $2x + 3y + 4z = 9$

5. $x - y + 2z = 4$
   $3x + y + 4z = 6$
   $x + y + z = 1$

6. $2x + 5y + 6z = 13$
   $3x + y - 4z = 0$
   $x - 3y - 8z = -10$

7. $x + y + z = 8$
   $x - y + z = 6$
   $3x + 5y - 7z = 14$

8. $x + y + z = 6$
   $x - y + z = 5$
   $3x + y + z = 8$

9. $x + 2y + 3z = 1$
   $2x + 3y + 2z = 2$
   $2x + 3y + 4z = 1$

10. $x + 2y - z = 3$
    $3x - y + z = 1$
    $2x - 2y + 3z = 2$

11. $2x - 3y + 7z = 5$
    $3x + y - 3z = 13$
    $2x + 19y - 47z = 32$

12. $4x - 2y + 6z = 8$
    $x + y - 3z = -1$

ANSWERS

1.	Inconsistence	2.	$x = 2, y = 1$
3.	$x = -7, y = 22, z = -9$	4.	Inconsistence
5.	$x = \frac{5-3z}{2}, y = \frac{-3}{2} + \frac{z}{2}, z$	6.	$x = -1 + 2z, y = 3 - 2z, z$
7.	$x = 5, y = \frac{5}{3}, z = \frac{4}{3}$	8.	$x = 1, y = 2, z = 3$
9.	$x = -\frac{3}{2}, y = 2, z = -\frac{1}{2}$	10.	$x = -1, y = 4, z = 4$
11.	Inconsistence	12.	$x = 1, y = 2z - 2, z = z.$

FIELDS

Let F be a non empty set with two operations addition ' + ' and multiplication '.', is said to be field if for all u, v, w $\in$ F then

1. $u + v \in F$ and $u.v \in F$ (Closure property)
2. $u + v = v + u$ and $uv = vu$ (Commutative property)
3. $u + (v + w) = (u + v) + w$ and $u(v.w)=(u.v)w$ (Associative property)
4. $u(v + w) = uv + uw$ (Distributive property)
5. For all $u\in F$, therexists $o\in F$ such that $u + 0 = u$
   And for al $u\in F$, therexists $1\in F$ such that $u. 1 = u$ (Existence of identity)
6. For all $u\in F$, therexists $-u\in F$ such that $u + (-u) = 0$
   And for all $u\in F$, therexists $\frac{1}{u} \in F$ such that $u.\frac{1}{u} = 1.$

[Existence of inverse]

Here the set F, together with the two operations ' + ' and '.' and satisfying above axioms (properties) is called a field.

For example

1. The set of real number **R** is a field.
2. The set of complex number **C** is a field.
3. The set of integer, Natural numbers are not a field.

VECTOR SPACE

Let $V = \{u_1\ u_2, u_3 \ldots\}$ be a non empty set of vectors, then V is said to be vector space over a field F if it satisfy following properties:

A. Vector Addition:
   1. For all $u_1, u_2 \in V$, then $u_1 + u_2 \in V$.
   2. For all $u_1, u_2 \in V$, then $u_1 + u_2 = u_2 + u_1$
   3. For all $u_1, u_2, u_3 \in V$, then $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

4. For all $u_1 \in V$, there exist null vector $O \in V$ such that $u_1 + O = u_1$
5. For all $u_1 \in V$, there exists $-u_1 \in V$ such that $u_1 + (-u_1) = O$.

B. Vector multiplication by scalar:

1. For all $u_1 \in V$, $a \in F$, then $au_1 \in V$
2. For all $u_1, u_2 \in V$, then, $a(u_1 + u_2) = au_1 + au_2$, for $a \in F$.
3. For all $a, b \in F$ and $u_1 \in V$, then, $(a + b)u_1 = au_1 + bu_1$
4. For all $u_1 \in V$, $a, b \in F$, then, $a(bu_1) = (ab)u_1 = b(au_1)$
5. For all $u_1 \in V$, $1 \in F$, then $1.u_1 = u_1$.

For example

1. The set of real number is a vector space.
2. The set of complex number is a vector space over a field F as a real or complex number.
3. The set of integer is a vector space but the set of Natural number is not a vector space.
4. The set of all $n \times m$ matrices is also a vector space over a field F as a veal or complex number,
5. The set of all $\mathbf{R}^n$ space is also vector space.

VECTOR SUBSPACE

Let V be a vector space over a field F. Then a non empty subset U of V is said to be vector subspace of V if U itself is a vector space over the same field F.

"OR"

Let V be a vector space over a field F, then a non empty space U of V is said to b vector subspace of V if for all $u, v \in U$ and $\alpha \in F$, then

(i) $u + v \in U$

(ii) $\alpha u \in U$.

"OR"

Let V be a vector space over a field F, then a non empty sublet U of V is said to be vector subspace of V if for all $u, v \in V$. $a, b \in F$ then $au + bv \in U$.

Note:

The intersection of any two subspace of a vector space is also a vector space but the union of any two subspace of a vector space is not always a vector space.

Examples

1. Let $V = R^3$ be a vector space and $W = \{x, y, z\}: x + y = 0\}$
Show that W is a vector subspace of V.

Solution

We have $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be a vector space over a field F.

We have $W = \{x, y, z) : x + y = 0\}$.

Clearly W is a subset of V.

Let $u = (x_1, y_1, z_1),\ v = (x_2, y_2, z_2) \in W$, a, $b \in F$

Then $au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$ ....(1)

Here, $ax_1 + bx_2 + ay_1 + by_2$

$= a(x_1 + y_1) + b(x_2 + y_2)$

$= a.0 + b.0$ since u, $v \in W$

$= 0$

Then from equation (1), we get

$au + bv \in W$.

Thus W is a vector subspace of a vector space V.

2. Let V = set of all 2 × 2 matrices, be a vector space and

$$W = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,b,c \in R\right\}$$

Show that W is a vector subspace of V.

Solution

We have V = set of all 2 × 2 matrices

$W = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R\right\}$ be a non empty subset of V.

Let $u = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix},\ v = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in W$ and x, $y \in F$

Then, xu + yv

$$= x\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} + y\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} xa_1 & xb_1 \\ 0 & xc_1 \end{pmatrix} + \begin{pmatrix} a_2y & yb_2 \\ 0 & yc_2 \end{pmatrix}$$

$$xu + yv = \begin{pmatrix} xa_1 + a_2y & xb_1 + yb_2 \\ 0 & xc_1 + yc_2 \end{pmatrix} \quad \text{...........(1)}$$

Here $xa_1 + a_2y \in R,\ xb_1 + yb_2 \in R,\ xc_1 + yc_2 \in R$

Then from equation (1), we get, $xu + yv \in W$.

Hence W is a vector subspace of V.

Generator or Span:

Let V be a vector space over a field F. Then $\{u_1, u_2, \ldots, u_n\}$ are said to be generator of V if every element of V can be expressed as a linear combination of $\{u_1, u_2 \ldots u_n\}$. The set of linear combinations of $\{u_1, u_2 \ldots, u_n\}$ is also said to be linear hull of the given set of vectors and denoted by $L(u_1, u_2, \ldots, u_n)$.

Basis and Dimension:

Let V be a vector space over a field F. Then $\{u_1, u_2 \ldots, u_n\}$ of V are said to be basis of V if they are linearly independent and generator of V.

Let V be a vector space over a field F, then $\{u_1, u_2, \ldots, u_n\}$ are basis of V, then dimension of V is n(i.e. the number of basis element) and it is denoted by dim V = n.

Example

1. **Let $V = R^3$ be a vector space. Show that {(1,0,0), (0,1,0), (0,0,1)} are basis of V.**

Proof:

We have $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be a vector space.

We have to show {(1, 0, 0), (0, 1, 0), (0, 0, 1)} be a basis of V. For this we have to show that they are linearly independent and they generate V.

For linear independent:

Let $\quad a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = 0$

$\Rightarrow \quad (a, 0, 0) + (0, b, 0) + (0, 0, c) = 0$

$\Rightarrow \quad (a, b, c) = 0 = (0, 0, 0)$

This gives a = 0, b = 0, c = 0.

So {1, 0, 0), (0, 1, 0), (0, 0, 1)} are linearly independent. ....(1)

For Span:

Let $(x, y, z) \in V$, then we have to show that (x, y, z) can be written as the linear combination of (1, 0, 0), (0, 1, 0), (0, 0, 1).

Clearly (x, y, z) = x (1, 0, 0) + y(0, 1, 0) + z(0, 0, 1).

Here (x, y, z) can be written as the linear combination of (1, 0, 0), (0, 1, 0), (0, 0, 1) ....(2)

Hence from (1) and (2), we get

{(1, 0, 0), (0, 1, 0), (0, 0, 1)} be a basis of V.

2. **Let V = set of all 2 × 2 matrices. Show that**

$$\mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ **be the basis of V.**

Solution

We have V be a set of 2×2 matrices, which is a vector space over a field F. we have to show that {u, v, w, x} are basis of V.

First we have to show that {u, v, w, x} are linearly independent.

Let $au + bv + cw + dx = 0$

$$\Rightarrow \quad a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

$$\Rightarrow \quad \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \mathbf{0}$$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives, $a = 0, b = 0, c = 0$ and $d = 0$.

Hence by definition {u, v, w, x} are linearly independent. ....(1)

Again we have to show that they generate V.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$, then clearly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = au + bv + cw + dx$$

This gives every element of V can be written as the linear combination of {u, v, w, x}. Thus by definition {u, v, w, x} generate V. ....(2)

From (1) and (2) we get {u, v, w, x} be a basis of V.

Linear transformation:

Let U and V be two vector space over a same field F. A transformation $T : U \rightarrow V$ is said to be linear transformation if

1. For any $u_1, u_2 \in V$, then $T(u_1 + u_2) = T(u_1) + T(u_2)$
2. For any $a \in F$, then $T(au_1) = a\,T(u_1)$

Note:

1. *A transformation $T: U \rightarrow V$ is a linear transformation if $T(au + bv) = aT(u) + b\,T(v)$ for all $a, b \in F$; $u, v \in U$.*
2. *A linear transformation T from a vector space V to itself is known as Linear operator.*

Example

1. **Let $T: R^2 \rightarrow R^2$ be defined by $T(x, y) = (x + y, x - y)$. Check T is linear or not.**

Solution

We have $T : R^2 \to R^2$ be defined by $T(x,y) = (x + y, x - y)$

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$; a, b$\in$F

Then, T (au + bv)

$$= T\{a(x_1, y_1) + b(x_2, y_2)\}$$
$$= T(ax_1 + bx_2, ay_1 + by_2)$$
$$= (ax_1 + bx_2 + ay_1 + by_2, ax_1 + bx_2 - ay_1 - by_2)$$
$$= (a(x_1 + y_1) + b(x_2 + y_2), a(x_1 - y_1) + b(x_2 - y_2))$$
$$= (a(x_1 + y_1), a(x_1 - y_1)) + (b(x_2 + y_2), b(x_2 - y_2))$$
$$= a(x_1 + y_1, x_1 - y_1) + b(x_2 + y_2, x_2 - y_2)$$
$$= a\,T(x_1, y_1) + bT(x_2, y_2)$$

$\therefore$ T (au + bv) = aT(u) + b T(v)

Hence by definition T is a linear transformation.

2. **Let $T : R^2 \to R$ be defined by $T(x, y) = |x + y|$, check T is linear or not.**

Solution

We have $T : R^2 \to R$ be defined by

$$T(x,y) = |x + y|$$

Let $u = (x_1, y_1)$, $v = (x_2, y_2) \in R^2$, a, b$\in$F then

$$T(au + bv) = T[a(x_1, y_1) + b(x_2, y_2)] = T(ax_1 + bx_2, ay_1 + by_2)$$
$$= |ax_1 + bx_2 + ay_1 + by_2| = |a(x_1 + y_1) + b(x_2 + y_2)|$$
$$\neq a|x_1 + y_1| + b|x_2 + y_2| = a\,T(u) + bT(v)$$

Thus we get, T (au + bv) $\neq$ a T(u) + b T(v)

Hence T is not a linear transformation.

3. **Let $T : R^3 \to R^2$ be defined by $T(x, y, z) = (x + y + z, 0)$. Check this transformation T is linear or not.**

Solution

We have a transformation

$T : R^3 \to R^2$ be defined by $T(x, y, z) = (x + y + z, 0)$

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in R^3$; a, b $\in$F

$$\text{Then, } T(au + bv) = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$$
$$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$
$$= (ax_1 + bx_2 + ay_1 + by_2 + az_1 + bz_2, 0)$$
$$= (\{a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2)\}, 0)$$
$$= (a(x_1 + y_1 + z_1), 0) + (b(x_2 + y_2 + z_2), 0)$$

$$= a(x_1 + y_1 + z_2, 0) + b(x_2 + y_2 + z_2, 0).$$
$$= aT(x_1, y_1, z_1) + b\,T(x_2, y_2, z_2)$$

$\therefore\ T(au + bv) = a\,T(u) + b\,T(v)$

Thus by definition T is a linear transformation.

Note:

Let $X = R^n$, $Y = R^m$, then a matrix $A = (a_{ij})_{m \times n}$ gives the transformation of R^n into R^m by $y = Ax$, since $A(u + x) = Au + Ax$ and $A(cx) = cAx$, this transformation is also a linear transformation.

EXERCISE 1.10

1. Let $v = R^2$ be a vector space. Show that $W = \{(x, y) : x + 2y = 0\}$ is a vector subspace of V.
2. Let $V = R^3$ be a vector space. Show that
   (i) $W = \{(x, y, z) : x + 2y + z = 0\}$
   (ii) $W = \{(x, y, z) : 2x + y + 2z = 0\}$
   are vector subspace of V.
3. Let $V = R^3$ be a vector space. Let
   (i) $W_1 = \{(x, 0, z) : x, z \in R\}$
   (ii) $W_2 = \{(0, y, z) :, y, z \in R\}$
   Show that W_1, W_2 and $W_1 \cap W_2$ are vector subspace of V.
4. Let V = set of all 2×2 matrices, be a vector space. Let.
   (i) $W_1 = \left\{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in R\right\}$
   (ii) $W_2 = \left\{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} : c, d \in R\right\}$
   Show that W_1, W_2 are vector subspace of V.
5. Check which of the following forms a basis of R^3.
   (i) (1, 1, 1), (1, 3, 2), (– 1, 0, 1)
   (ii) (1, 2, 1), (2, 1, 0), (1, – 1, 2)
   (iii) (1, 1, 0), (1, 0, 1)
   (iv) (1, 1, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4)
6. Check the following transformation are linear or not.
   (i) $T : R^2 \to R$ be defined by $T(x, y) = x + y$.
   (ii) $T : R^2 \to R^3$ be defined by $T(x, y) = (x, y, xy)$.
   (iii) $T : R^2 \to R^2$ be defined by $T(x, y) = (x + 3, y)$.
   (iv) $T : R \to R$ be defined by $T(x) = x + 4$.
   (v) $T : R^2 \to R^2$ be defined by $T(x, y) = (x, -2y)$.

ANSWERS

5. (i) basis (iii) not a basis
   (ii) basis (iv) not a basis
6. (i) Linear transformation
   (ii) not linear (iv) not linear
   (iii) not linear (v) Linear

EIGEN VALUE AND EIGEN VECTOR

Let V be a vector space over a field F. Let T be a linear operator on V. Then a scalar $\lambda \in F$ is said to be eigen value of T if therexists a non zero vector $v \in V$, such that $T(v) = \lambda v$.

Where v be the eigen vector of T associated with eigen value λ.

Note:

1. *The collection of all eigen vector $v \in V$ such that $T(v) = \lambda v$ is called the eigen space.*
2. *Eigen value and eigen vectors are known as characteristics values (or Latent root or proper values or spectral values) and characteristics vectors (or latent vectors or proper vectors or spectral vectors) respectively.*
3. *We know if T be a linear operator on V, then the square matrix A is formed by the linear operator such that $Y = AX$.*
   Then a number λ is eigen value of A if $AX = \lambda X$ for all $X \neq 0$ where X is said to be eigen vector of A associated with eigen value λ.
4. *We have,* $AX = \lambda X$
   $\Rightarrow \quad AX - \lambda X = 0$
   $\Rightarrow \quad (A - \lambda I)\, X = 0$

This shows that $A - \lambda I$ is singular if $X \neq 0$ and non – singular if $X = 0$. But we have $X \neq 0$, then $A - \lambda I = 0 \Rightarrow |A - \lambda I| = 0$

This is said to be characteristics equation.

CAYLEY HAMILTON THEOREM

Every square matrix satisfies its characteristics equation. i.e., if the characteristics equation of the nth order square matrix A is $|A - \lambda I| = 0$

$$\Rightarrow (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots\ldots + k_n = 0$$

Then $(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots + k_n I = 0.$

Proof:

Let $P = \text{adj}(A - \lambda I)$. The elements of P are polynomials in λ of degree $n - 1$ or less because the elements of $A - \lambda I$ are at most of first degree in λ. Thus, we write $P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \ldots\ldots + P_{n-1}\lambda + P_n$

Also, we know, $B(\text{adj } B) = |B|\, I$. Thus, $(A - \lambda I)P = |A - \lambda I|\, I$

This implies, $(A - \lambda I)\,(P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots + P_{n-1}\lambda + P_n)$

$$= ((-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_{n-1}\lambda + k_n)\, I$$

Equating co – efficient of like powers of λ on both sides, we get

$$\begin{aligned} -P_1 &= (-1)^n I \\ AP_1 - P_2 &= k_1 I \\ AP_2 - P_3 &= k_2 I \\ \ldots\ldots\ldots\ldots &\ \ \ldots\ldots\ldots \\ AP_{n-2} - P_{n-1} &= k_{n-2} I \\ AP_{n-1} - P_n &= k_{n-1} I \\ AP_n &= k_n I \end{aligned}$$

Pre – multiplying these equation by $A^n, A^{n-1}, \ldots, A, I$ respectively and adding, we get

$$0 = (-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots + k_{n-1} A + k_n I$$

$$\Rightarrow (-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots + k_{n-1} A + k_n I = 0 \quad \ldots\ldots (1)$$

This completes the proof.

Note:

(1) Multiplying equation (1) by A^{-1}, we get

$$(-1)^n A^{n-1} + k_1 A^{n-2} + k_2 A^{n-3} + \ldots + k_{n-1} I + k_n A^{-1} = 0$$

Thus, $$A^{-1} = -\frac{1}{k_n}\,[(-1)^n A^{n-1} + k_2 A^{n-2} + \ldots + k_{n-1} I]$$

This gives Cayley – Hamilton theorem gives another method for computing the inverse of a matrix. Since this method express the inverse of a matrix of order n in terms of (n – 1) power of A. This method is suitable for computing inverse of large matrices.

(2) If m be a positive integer such that m > n then multiplying equation (1) by A^{m-n}, we get

$$(-1)^n A^m + k_1 A^{m-1} + k_2 A^{m-2} + \ldots + k_{n-1} A^{m-n+1} + k_n A^{m-n} = 0$$

Shows that any positive integral power $A^m (m > n)$ of A is linearly expressible in terms of those of lower degree.

Example

1. Verify Cayley – Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ **and then evaluate A^{-1}.**

Solution

The characteristics equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley – Hamilton theorem, we have to show that

$$A^3 - 6A^2 + 9A - 4I = 0$$

Here,

$$A^2 = AA = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\text{Also } A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^3 = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Then

$A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ We get, $A^3 - 6A^2 + 9A - 4I = \mathbf{0}$

This verifies cayley – Hamilton theorem

Multiplying both sides by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

Note:

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Note:

1. *The matrix B which diagonalise A is called the modal matrix of A and is obtained by grouping the eigen vectors of A into a square matrix.*
2. *If X is an eigen vector of a matrix A associated with eigen value λ, so is kX with any $k \neq 0$.*

Example

1. **Reduce the matrix** $\mathbf{A} = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ **to the diagonal form.**

Solution

We have characteristics equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 + 5\lambda + 5 = 0$$

$$\Rightarrow \lambda^2(\lambda - 1) - 5(\lambda - 1) = 0$$

$$\Rightarrow (\lambda^2 - 5)(\lambda - 1) = 0$$

$$\therefore \lambda = 1, \pm\sqrt{5}$$

When $\lambda = 1$

$$\Rightarrow AX_1 = \lambda X_1$$

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

This gives, $-2x_1 + 2y_1 - 2z_1 = 0$

$$x_1 + y_1 + z_1 = 0$$

$$x_1 - y_1 + z_1 = 0$$

After solving we get eigen vector $(1, 0, -1)$

Similarly when $\lambda = \sqrt{5}$, the eigen vector $(\sqrt{5} - 1, 1, -1)$ and when $\lambda = -\sqrt{5}$, the eigen vector $(\sqrt{5} + 1, -1, 1)$.

Then modal matrix

$$B = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$ and diagonal matrix is

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$ which is obtained by diagonalising A.

2. Find eigen value and vector of $\mathbf{A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}}$.

Solution

We have $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Its characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \left| \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 7\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\therefore \lambda = 1, 6$$

Thus we get, eigen value of A are 1 and 6.

For eigen vector

At $\lambda = 1$, $\quad AX = \lambda X$

$$\Rightarrow \quad \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5x + 4y \\ x + 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This gives, $\quad 5x + 4y = x \Rightarrow 4x + 4y = 0$

and $\quad x + 2y = y \Rightarrow x + y = 0$

We get $\quad x + y = 0 \Rightarrow x = -y$

Thus required eigen vector of A associated with eigen value is $\begin{bmatrix} x \\ -x \end{bmatrix}$ or $(x, -x)$.

$$\text{At } \lambda = 6, \qquad AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 6\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5x + 4y \\ x + 2y \end{bmatrix} = \begin{bmatrix} 6x \\ 6y \end{bmatrix}$$

$$\Rightarrow 5x + 4y = 6x \Rightarrow x - 4y = 0$$

$$\text{and } x + 2y = 6y \Rightarrow x - 4y = 0 \Rightarrow x = 4y$$

Thus required eigen vector associated with eigen value $\lambda = 6$ is $\begin{bmatrix} 4x \\ x \end{bmatrix}$ or $(\cancel{4x}, x)$. $(x, x/4)$

3. **Find eigen value and vector of** $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution

Let λ and X are eigen value and vector of A. Then its characteristics equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)\{(1-\lambda)(-\lambda) - 12\} - 2(-2\lambda - 6) - 3(-4 + 1 - \lambda) = 0$$

$$\Rightarrow (2+\lambda)\{(1-\lambda)\lambda + 12\} + 2(2\lambda + 6) + 3(3 + \lambda) = 0$$

$$\Rightarrow (2+\lambda)(\lambda - \lambda^2 + 12) + 7\lambda + 21 = 0$$

$$\Rightarrow 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 7\lambda + 21 = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow \lambda^3 + 3\lambda^2 - 2\lambda^2 - 6\lambda - 15\lambda - 45 = 0$$

$$\Rightarrow \lambda^2(\lambda + 3) - 2\lambda(\lambda + 3) - 15(\lambda + 3) = 0$$

$$\Rightarrow (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$$

This gives $\lambda = -3$ and $\lambda = 5$

Thus required eigen value of A are $\lambda = -3, 5$.

For eigen vector, At, $\lambda = -3$, $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -3\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2x+2y-3z \\ 2x+y-6z \\ -x-2y \end{bmatrix} = \begin{bmatrix} -3x \\ -3y \\ -3z \end{bmatrix}$$

This gives,
$$-2x+2y-3z = -3x \Rightarrow x+2y-3z=0$$
$$2x+y-6z = -3y \Rightarrow 2x+4y-6z=0$$
$$-x-2y = -3z \Rightarrow x+2y-3z=0$$

These three equations gives,
$$x+2y-3z = 0$$
$$\Rightarrow \quad x = 2y-3z$$

Thus required eigen vector of a associated with eigen value $\lambda = -3$ is $(2y-3z, y, z)$.

At $\lambda = 5$, $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} -2x+2y-3z \\ 2x+y-6z \\ -x-2y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \\ 5z \end{bmatrix}$$

This gives,
$$-2x+2y-3z = 5x \Rightarrow -7x+2y-3z=0 \quad \text{............(i)}$$
$$2x+y-6z = 5y \Rightarrow 2x-4y-6z=0 \quad \text{............(ii)}$$
$$-x-2y = 5z \Rightarrow x+2y+5z=0 \quad \text{............(iii)}$$

Solving these equations, we get
$x=1, y=2, z=-1$

Hence the required eigen vector of A associated with eigen value $\lambda = 5$ is $(1, 2, -1)$.

4. Find eigen value of $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}$

Solution

Let λ be eigen vector value of A. Then its characteristics equation is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)\{(a-\lambda)^2-1\} - 1(a-\lambda) = 0$$
$$\Rightarrow (a-\lambda)\{(a-\lambda)^2-1-1\} = 0$$

This gives $a-\lambda = 0 \Rightarrow \lambda = a$

And $(a-\lambda)^2 - 2 = 0$
$$\Rightarrow (a-\lambda)^2 = 2$$

$$\Rightarrow \quad a-\lambda = \pm\sqrt{2}$$
$$\Rightarrow \quad \lambda = a\pm\sqrt{2}$$

Thus required eigen value of A are $\lambda = a$ and $a \pm \sqrt{2}$.

5. **Find eigen value as well as eigen vector of** $A = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Solution

Let λ be eigen value of A. Then its characteristics equation is

$$|A-\lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 0-\lambda & 7 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

This gives $\quad \lambda^2(2+\lambda) = 0$

$\Rightarrow$ $\lambda = 0$ and $\lambda = -2$. This gives required eigen values are $\lambda = 0$ (which is not possible) and $\lambda = -2$.

For eigen vector at $\lambda = -2$,

$$AX = \lambda X$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 7y \\ 0 \\ -2z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}$$

This gives $\quad -7y = -2x \Rightarrow x = 0$

$$-2y = 0 \Rightarrow y = 0$$
$$-2z = -2z \Rightarrow z \text{ if free.}$$

Thus required eigen vector is $(0, 0, z)$ at $\lambda = -2$.

6. **Find eigen value and eigen vector of** $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

Solution

Let λ be eigen value of $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

Then by definition

$AX = \lambda X$

where its characterististies equation is

$$|A-\lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + 9 = 0$$

$$\Rightarrow \quad \lambda^2 = -9$$

$$\lambda = \pm i3$$

This gives eigen values are imaginary thus eigen value as well as eigen vectors does not exists, since a transformation A is defined on a real field.

7. Find eigen value and vector of $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{a} \\ \mathbf{a} & \mathbf{0} \end{bmatrix}$.

Solution

Let λ be eigen value of A. Then $AX = \lambda X$ for $X \neq 0$

Its characteristics equation is

$$|A-\lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -\lambda & a \\ a & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - a^2 = 0 \quad \Rightarrow \quad \lambda = \pm a$$

Thus required eigen value of A are $\lambda = \pm a$.

For eigen vector,

At $\lambda = a$

$$\Rightarrow \quad AX = \lambda X$$

$$\Rightarrow \quad \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} ay \\ ax \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}$$

$$\Rightarrow \quad ay = ax \quad \Rightarrow \quad x = y$$

$$ax = ay$$

Thus required eigen vector at $\lambda = a$ are (x, x)

$$\text{At } \lambda = -a, \quad AX = \lambda X$$

$$\Rightarrow \quad \begin{bmatrix} ay \\ ax \end{bmatrix} = -a \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad ay = -ax \Rightarrow \quad y = -x$$

$$ax = -ay \Rightarrow \quad x = -y$$

This gives the required eigen vector of A associated with eigen valve $\lambda = -a$ is $(x, -x)$.

8. **Find the eigen value and vector of the square matrix $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.**

Solution

Let λ and X are eigen value and vector of a square matrix A such that

$$AX = \lambda X$$

Then its characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 8-\lambda & 0 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda (8-\lambda)(6-\lambda) = 0$$

This gives, $\lambda = 4, 8, 6$.

Thus required eigen value are $\{4, 8, 6\}$.

For eigen vector:

At $\lambda = 4$, $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x \\ 8y \\ 6z \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \\ 4z \end{bmatrix}$$

This gives ,

$4x = 4x$ implies x is free

$8y = 4y$ implies $y = 0$

$6z = 4z$ implies $z = 0$

Thus required eigen vector at $\lambda = 4$ is $(x, 0, 0)$.

At $\lambda = 8$,

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 4x \\ 8y \\ 6z \end{bmatrix} = \begin{bmatrix} 8x \\ 8y \\ 8z \end{bmatrix}$$

This given $x = 0$, y is free, $z = 0$.

Thus required eigen vector at $\lambda = 8$ is $(0, y, 0)$.

At $\lambda = 6$.

$$AX = \lambda X$$

$$\begin{bmatrix} 4x \\ 8y \\ 6z \end{bmatrix} = \begin{bmatrix} 6x \\ 6y \\ 6z \end{bmatrix}$$

This given $x = 0$, $y = 0$ and z is free.
Thus required eigen vector at $\lambda = 6$ is $(0, 0, z)$.

9. Find eigen values and vectors of the square matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

Solution

Let λ and X are eigen value and eigen vectors of a matrix A respectively such that $AX = \lambda X$. Its characteristics equation is

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

This gives $(3 - \lambda)(5 - \lambda)(7 - \lambda) = 0$ implies $\lambda = 3,5,7$.
Thus we get required eigen value of A are $\{3,5,7\}$
For eigen vector:

At $\lambda = 3$, $AX = \lambda X$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x \\ 5y \\ 7z \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \\ 3z \end{bmatrix}$$

This gives $x = x$, $y = 0$, $z = 0$. Thus required eigen vectors of A at $\lambda = 3$ is $(x, 0, 0)$.

At $\lambda = 5$, $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 3x \\ 5y \\ 7z \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \\ 5z \end{bmatrix}$$

This gives $x = 0$, $y = y$, $z = 0$. Thus required eigen vectors of A at $\lambda = 5$ is $(0, y, 0)$

At $\lambda = 7$, $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 3x \\ 5y \\ 7z \end{bmatrix} = \begin{bmatrix} 7x \\ 7y \\ 7z \end{bmatrix}$$

This gives $x = 0$, $y = 0$, $z = z$. Thus required eigen vectors of A at

$\lambda = 7$ is (0, 0, z).

10. Find eigen values and vectors of the square matrix $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Solution

Let λ and X are eigen value and eigen vectors respectively of a matrix A such that $AX = \lambda X$. Its characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(1-\lambda)(-1-\lambda) - 3] + 2[(-1-\pi - 1] + 2[3 - (1-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)(-1+\lambda^2 - 3) - 4 - 2\lambda + 4 + 2\lambda = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 4) = 0$$

$$\therefore \lambda = 2, -2$$

This gives required eigen value of A are $\{2, -2\}$.

For eigen vector:

At $\lambda = 2$, $AX = \lambda X$

$$\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x - 2y + 2z \\ x + y + z \\ x + 3y - z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

This gives $-2y + 2z = 0 \Rightarrow y = z$

$x - y + z = 0 \Rightarrow x = 0$

$x + 3y - 3z = 0 \Rightarrow y = z$

Thus we get the required eigen vector of A at $\lambda = 2$ is (0, y, y).

At $\lambda = -2$, $AX = \lambda X$

$$\begin{bmatrix} 2x - 2y + 2z \\ x + y + z \\ x + 3y - z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}$$

This gives $4x - 2y + 2z = 0$ ---------- (i)

$x + 3y + z = 0$ ---------- (ii)

$$x + 3y + z = 0 \quad \text{------------- (iii)}$$

Solving (i) and (ii),

$$2x - 8y = 0$$
$$x = 4y$$

From (iii), $7y = -z$

$$y = -\frac{z}{7}$$

Thus we get required eigen vector at $\lambda = -2$ is $\left(x, \frac{x}{4}, -\frac{7x}{4}\right)$.

EXERCISE 1.11

1. Find eigen value and eigen vector of the following matrices:

(i) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}$ (iii) $\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix}$

(iv) $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$

(vii) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (viii) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

2. Find eigen value as well as eigen vector of the following

(i) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(xi) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (xii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(xiii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (xiv) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

3. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

4. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

Show that the equation is satisfied by A.

5. Using cayley Hamiltan theorem find the inverse of

i. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ ii. $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

6. Diogonalise the following matrices and obtain the modal matrix in each case:

i. $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ ii. $\begin{bmatrix} 9 & -1 & -9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$

ANSWERS

1. (i) $\lambda = 1, X = (x,0)$
$\lambda = -2, X = (0, x)$

(ii) $\lambda = 4, X = (x, 0)$
$\lambda = -6, X = (0, y)$

(iii) $\lambda = -8, X = (2x, 9x)$
$\lambda = 6, X = (x, x)$

(iv) $\lambda = -5, X = (x, -2x)$
$\lambda = 5, X = (2x, x)$

(v) $\lambda = 1, X = (x, 0)$
$\lambda = 3, X = (x, x)$

(vi) $\lambda = -3, X = (x, -x)$
$\lambda = 2, X = (3x, 2x)$

(vii) $\lambda = 1, X = (x, -x)$
$\lambda = 3, X = (x, x)$

viii. $\lambda = \{1,6\}, X_1 = (x, -x), X_2 = (4x, x)$

2. (i) $\lambda = 3, X = (x, 0, 0)$
$\lambda = -8, X = (0, y, 0)$
$\lambda = 4, X = (0, 0, z)$

(ii) $\lambda = 1, X = (7x, -4x, 2x)$
$\lambda = 3, X = (x, 0, 0)$
$\lambda = 4, X = (5x, x, 0)$

(iii) $\lambda = \frac{1}{2}, X = (0, y, 0)$
$\lambda = 3, X = (x, 0, -x)$

(iv) $\lambda = 1, X = (x, x, z)$
$\lambda = -1, X = (x, -x, 0)$

(v) $\lambda = 2, X = (x, 0, 0)$
$\lambda = -2, X = (x, -12x, 4x)$

(vi) $\lambda = 2, X = (x, -x, x)$
$\lambda = 4, X = (x, x, 0)$

(vii) $\lambda = 2, X = (x, 0, 0)$
$\lambda = 4, X = (0, y, 0)$
$\lambda = 3, X = (0, 0, z)$

(viii) $\lambda = 2, X = (x, -x, 0)$
$\lambda = 3, X = (x, 0, 0)$
$\lambda = 5, X = (x, 2x, x)$

(ix) $\lambda = 1, X = (x, 0, 0)$

(xi) $\lambda = 2, X = (x, 0, -2x)$
$\lambda = 2, X = (x, 2x, 0)$
$\lambda = 8, X = (2x, -x, x)$

(x) $\lambda = 0, X = (x, 2x, 2x)$
$\lambda = 3, X = (2x, x, -2x)$
$\lambda = 15, X = (2x, -2x, x)$

(xii) $\lambda = 1, X = (x, 0, x)$
$\lambda = 2, X = (x, 0, -x)$
$\lambda = 3, X = (0, x, 0)$

(xiii) $\lambda = 1, X = (x, -2x, x)$
$\lambda = 3, X = (x, x, 0)$

(xiv) $\lambda = 2, X = (x, -x, 0)$
$\lambda = 3, X = (x, 0, 0)$
$\lambda = 5, X = (2x, 0, x)$

3. $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0;\ \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ -6 & 1 & -10 \end{bmatrix}$

4. $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

5. i. $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ ii. $\frac{1}{130}\begin{bmatrix} 4 & 20 & -9 \\ -42 & 50 & -3 \\ 20 & -30 & 20 \end{bmatrix}$

6. i. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ ii. $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix}$

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