MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE

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GRAPH THEORY

REPRESENTING GRAPHS:

1. **Adjacency List:**

   One way to represent a graph is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph

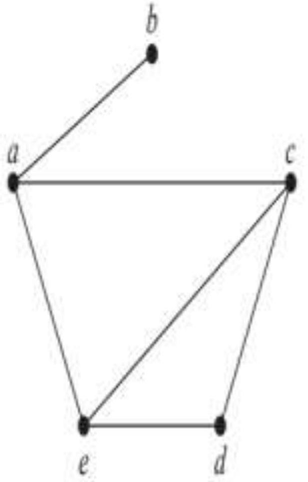


FIGURE 1 A Simple Graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	*b, c, e*
b	*a*
c	*a, d, e*
d	*c, e*
e	*a, c, d*

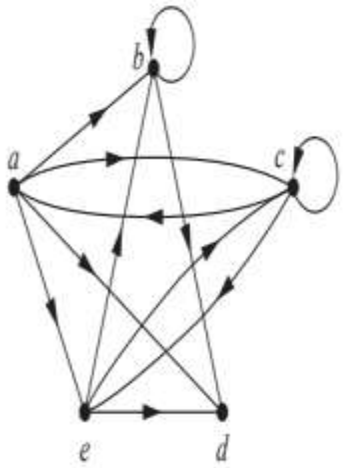


FIGURE 2 A Directed Graph.

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	*b, c, d, e*
b	*b, d*
c	*a, c, e*
d	
e	*b, c, d*

REPRESENTING GRAPHS:

2. Adjacency Matrix:

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The **adjacency matrix** A (or AG) of G, with respect to this listing of the vertices, is the n x n zero–one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{vi, vj\} \text{ is an edge of G,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

REPRESENTING GRAPHS:

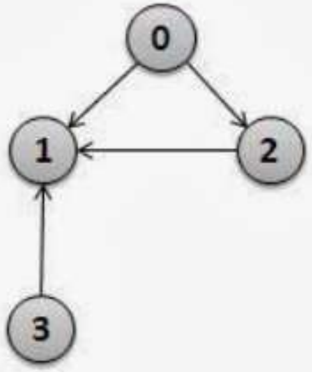
Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. When multiple edges connecting the same pair of vertices vi and vj , or multiple loops at the same vertex, are present, the adjacency matrix is no longer a zero–one matrix, because the (i, j)th entry of this matrix equals the number of edges that are associated to {vi, vj }. All undirected graphs, including multigraphs and pseudo graphs, have symmetric adjacency matrices.

EXAMPLE 5 Use an adjacency matrix to represent the pseudograph shown in Figure

Solution: The adjacency matrix using the ordering of vertices a, b, c, d is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

REPRESENTING GRAPHS:



	0	1	2	3
0	0	1	1	0
1	0	0	0	0
2	0	1	0	0
3	0	1	0	0

Adjacency Matrix Representation of Directed Graph

The matrix for a directed graph G = (V , E) has a 1 in its (i, j)th position if there is an edge from vi to vj , where v1, v2,..., vn is an arbitrary listing of the vertices of the directed graph. In other words, if A = [aij] is the adjacency matrix for the directed graph with respect to this listing of the vertices, then

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from vj to vi when there is an edge from vi to vj .

REPRESENTING GRAPHS:

Draw the graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

Since the matrix is not symmetric, we need directed edges.

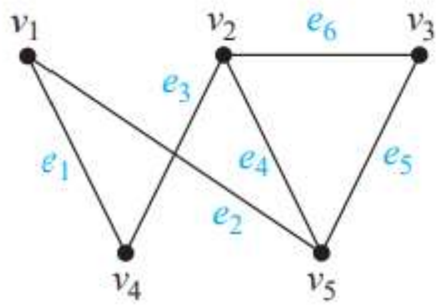
REPRESENTING GRAPHS:

3. Incidence Matrices:

Another common way to represent graphs is to use **incidence matrices**. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Represent the graph shown in Figure 6 with an incidence matrix.

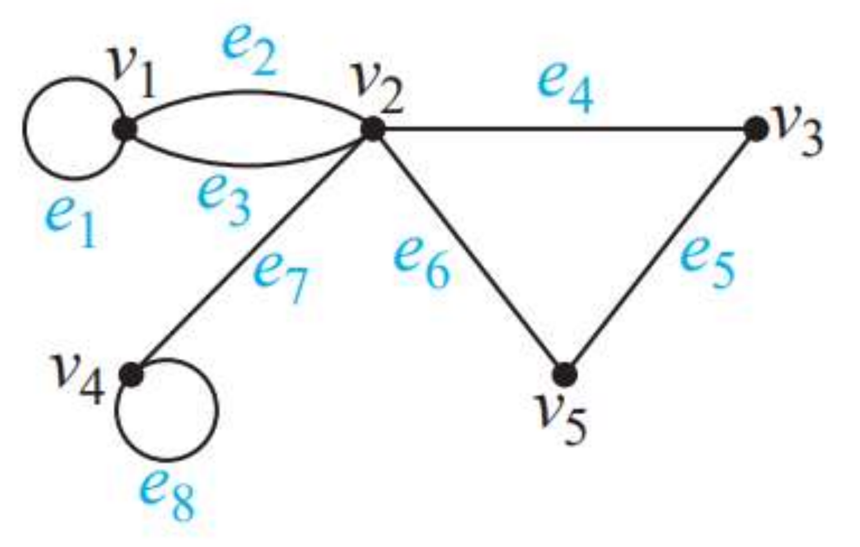


Solution: The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

REPRESENTING GRAPHS:

3. **Incidence Matrices:**



Solution: The incidence matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

CONNECTIVITY:

Connectivity is a basic concept of graph theory. It defines whether a graph is connected or disconnected. Without connectivity, it is not possible to traverse a graph from one vertex to another vertex.

1. **WALK:** A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices such that each edge is incident with the vertices preceding and following it.

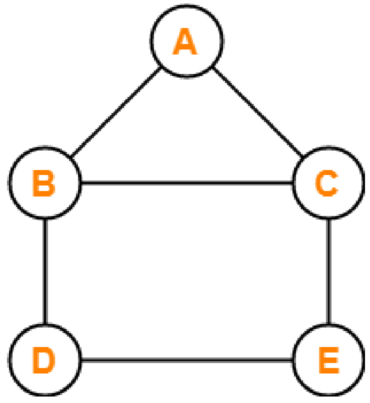
   Open Walk: a walk is called as an Open walk if Length of the walk is greater than zero and the vertices at which the walk starts and ends are different.

   Closed Walk: a walk is called as an Closed walk if Length of the walk is greater than zero and the vertices at which the walk starts and ends are same.

- If length of walk i= 0, then it is called as a Trivial Walk
- Both vertices and edges can repeat in a walk whether it is an open walk or a closed walk.

In this graph, few examples of walk are-

- a , b , c , e , d (Length = 4)
- d , b , a , c , e , d , e , c (Length = 7)
- e , c , b , a , c , e , d (Length = 6)



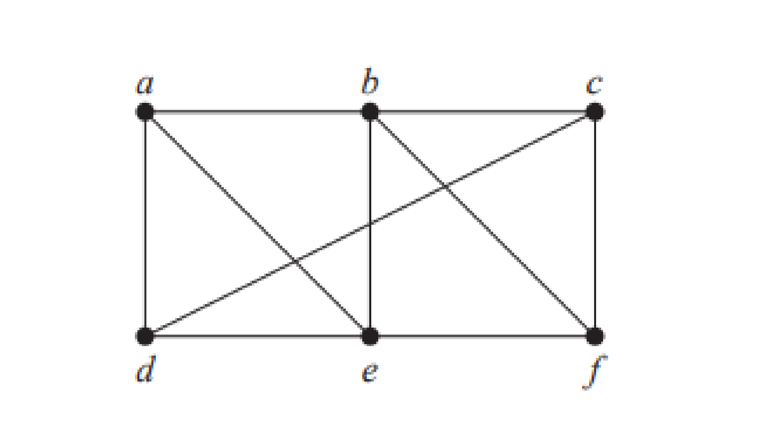
CONNECTIVITY:

2. **TRIAL:** A trail is defined as an open walk in which Vertices may repeat but edges are not allowed to repeat.

3. **CIRCUIT:** A circuit is defined as a closed walk in which Vertices may repeat but edges are not allowed to repeat.(Closed trial)

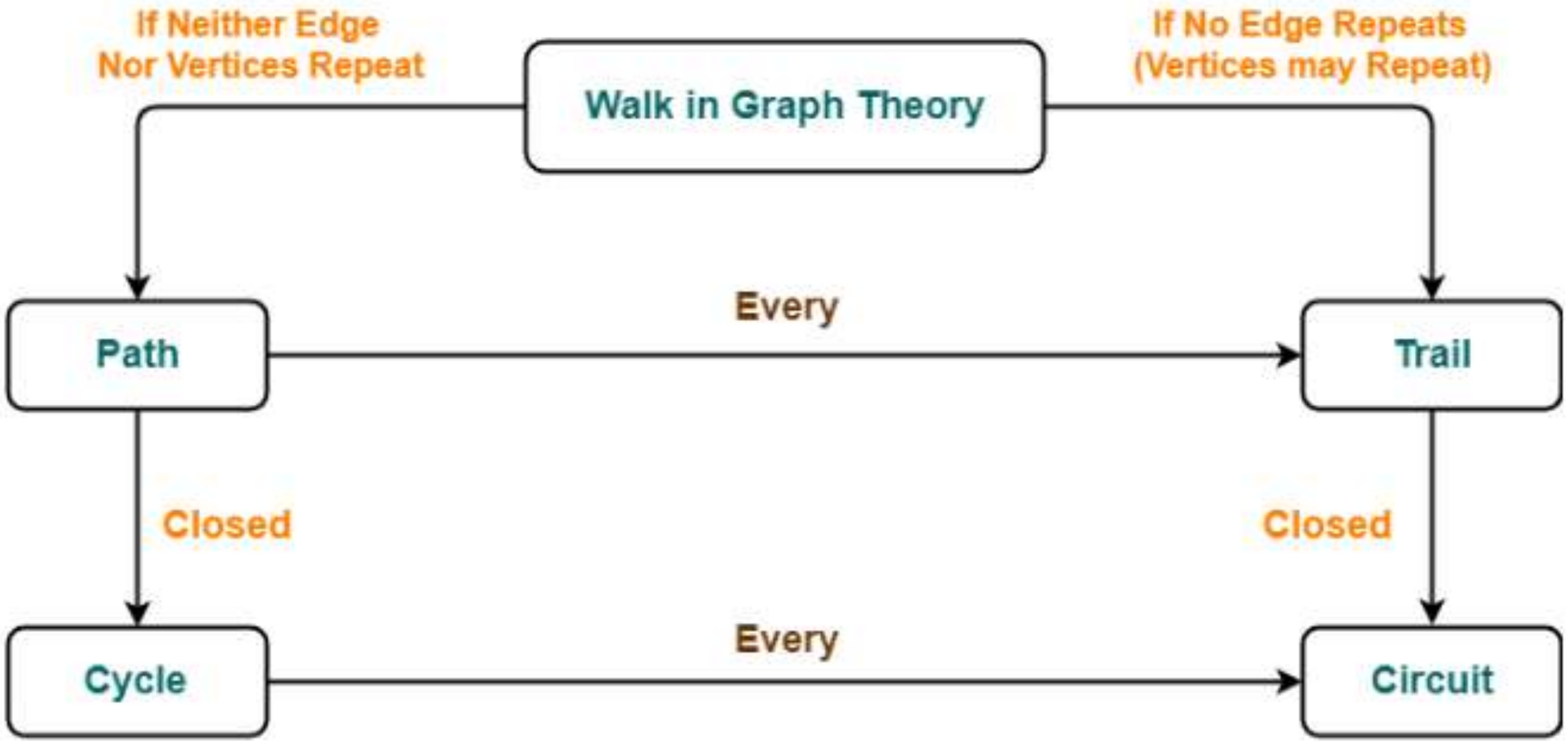
4. **PATH:** A path is defined as an open walk in which neither vertices are allowed to repeat nor edges are allowed to repeat.

5. **CYCLE:** A cycle is defined as a closed walk in which neither vertices (except possibly the starting and ending vertices) are allowed to repeat nor edges are allowed to repeat.(Closed Path)



Important Chart-

The following chart summarizes the above definitions and is helpful in remembering them-



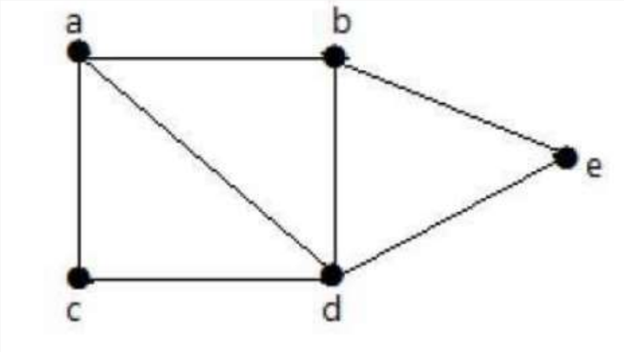
Important Chart to Remember

CONNECTEDNESS IN UNDIRECTED GRAPHS:

- A graph is said to be **connected if there is a path between every pair of vertex**. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph

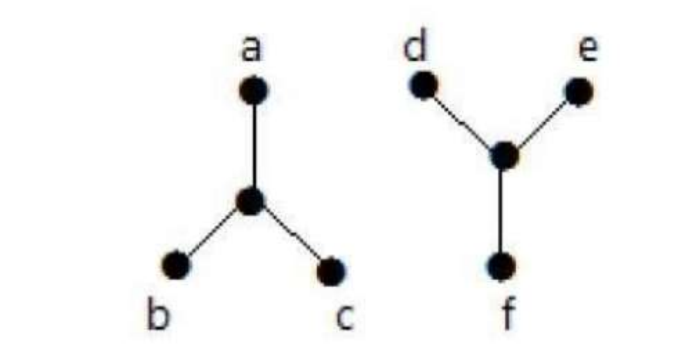
Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Example 2

In the following example, traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.



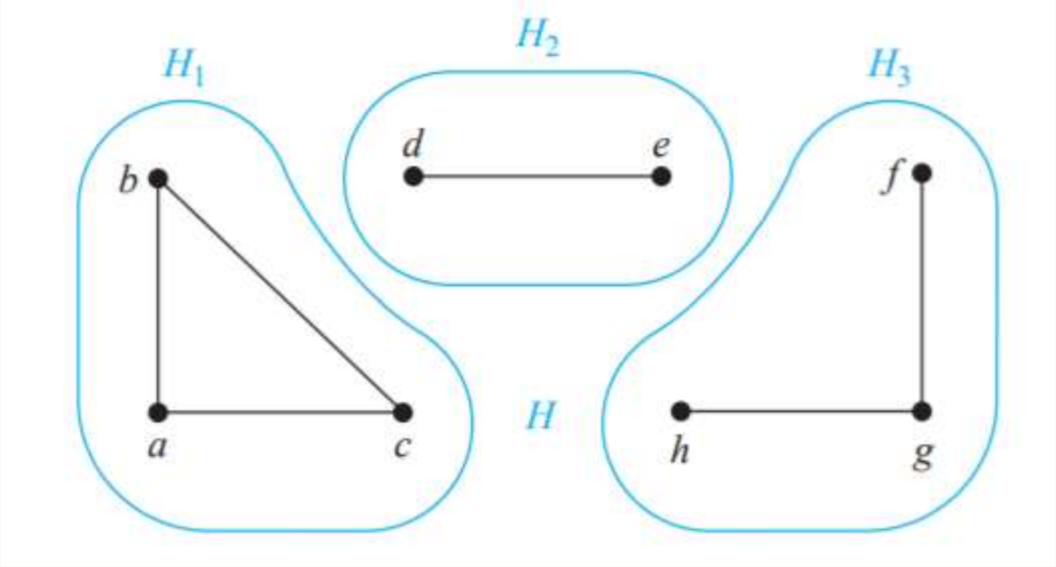
CONNECTEDNESS IN UNDIRECTED GRAPHS:

- **CONNECTED COMPONENTS:** A connected component of a graph G is a maximal connected subgraph of G.

- What are the connected components of the graph H shown in below figure?

  Solution:

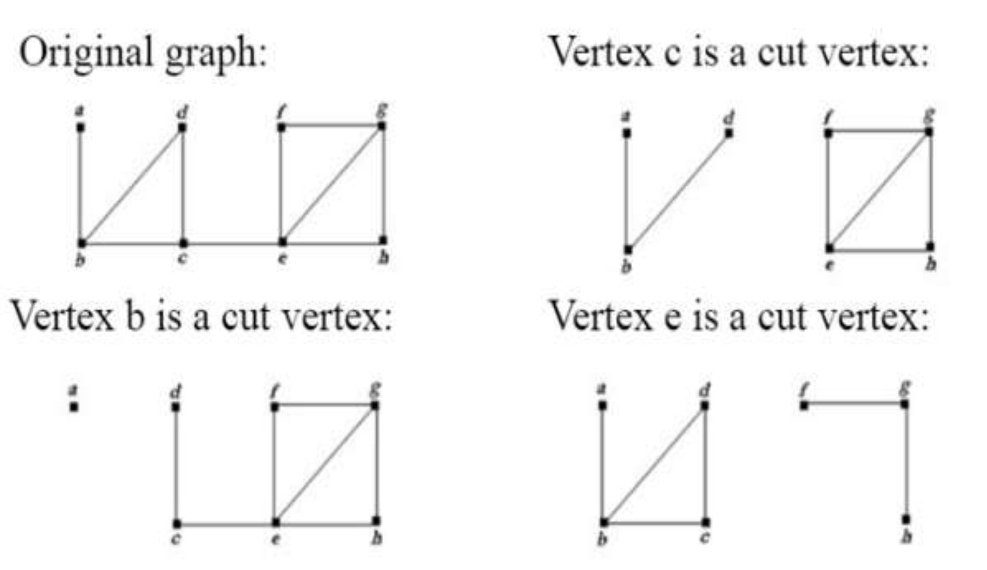
  The graph H is the union of three disjoint connected subgraphs H1, H2, and H3, shown in Figure. These three subgraphs are the connected components of H.



CONNECTEDNESS IN UNDIRECTED GRAPHS:

1. **CUT VERTICES:** Sometimes the removal from a graph of a vertex and all incident edges produces a subgraph with more connected components or disconnects the Graph. Such vertices are called cut vertices(or articulation points).

2. **CUT EDGES:** Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a cut edge or bridge. A cut edge 'e' must not be the part of any cycle in G.



- Find the cut vertices and cut edges in the graph G shown in above Figure.

  Solution:

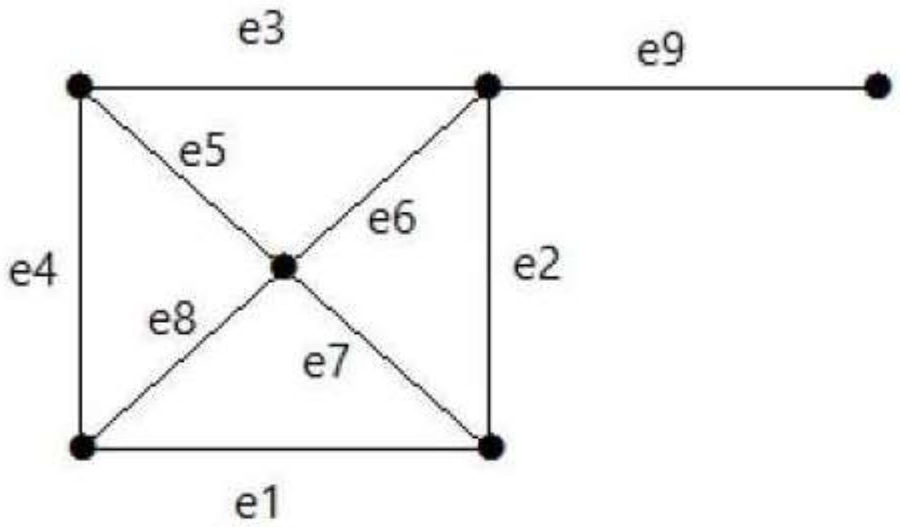
  The cut vertices of G are **b**, **c**, and **e**. The removal of one of these vertices (and its adjacent edges) disconnects the graph.

  The cut edges are {a, b} and {c, e}. Removing either one of these edges disconnects G.

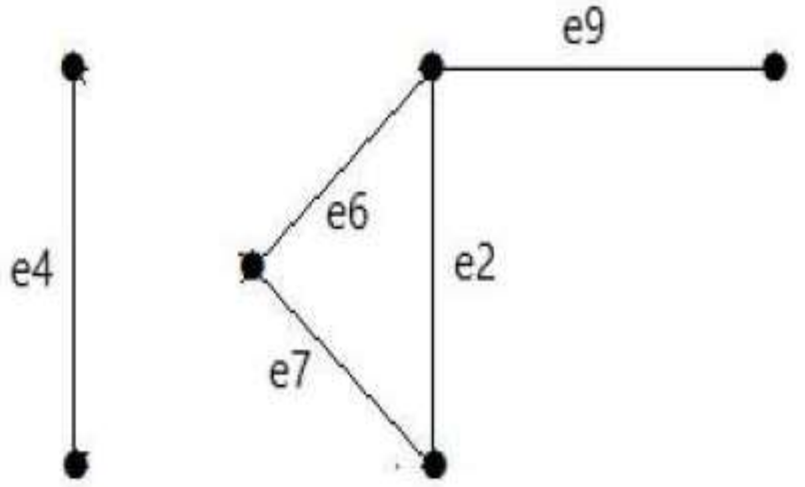
CONNECTEDNESS IN UNDIRECTED GRAPHS:

3. **Cut Set of a Graph:** Let 'G'= (V, E) be a connected graph. A subset E' of E is called a cut edge set of G if deletion of all the edges of E' from G makes G disconnect. A subset V' of V is called a cut vertex set of G if deletion of all the vertex of V' from G makes G disconnect.

Take a look at the following graph. Its cut set is E1 = {e1, e3, e5, e8}.



After removing the cut set E1 from the graph, it would appear as follows –



Similarly there are other cut sets that can disconnect the graph-

E3 = {e9} – Smallest cut set of the graph.

E4 = {e3, e4, e5}

E5 = {e1, e7, e2}

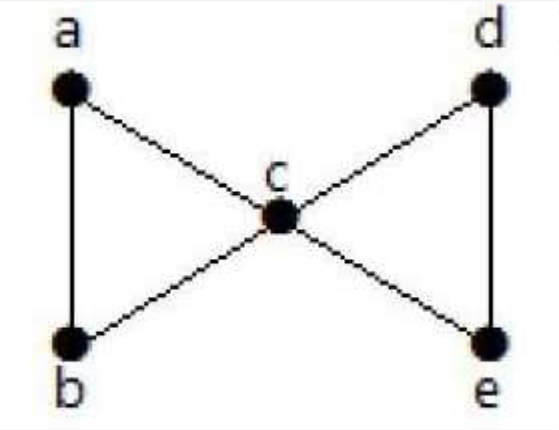
CONNECTEDNESS IN UNDIRECTED GRAPHS:

- Not all graphs have cut vertices. For example, the complete graph K_n, where $n \geq 3$, has no cut vertices. When you remove a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph. Connected graphs without cut vertices are called **non separable graphs.**

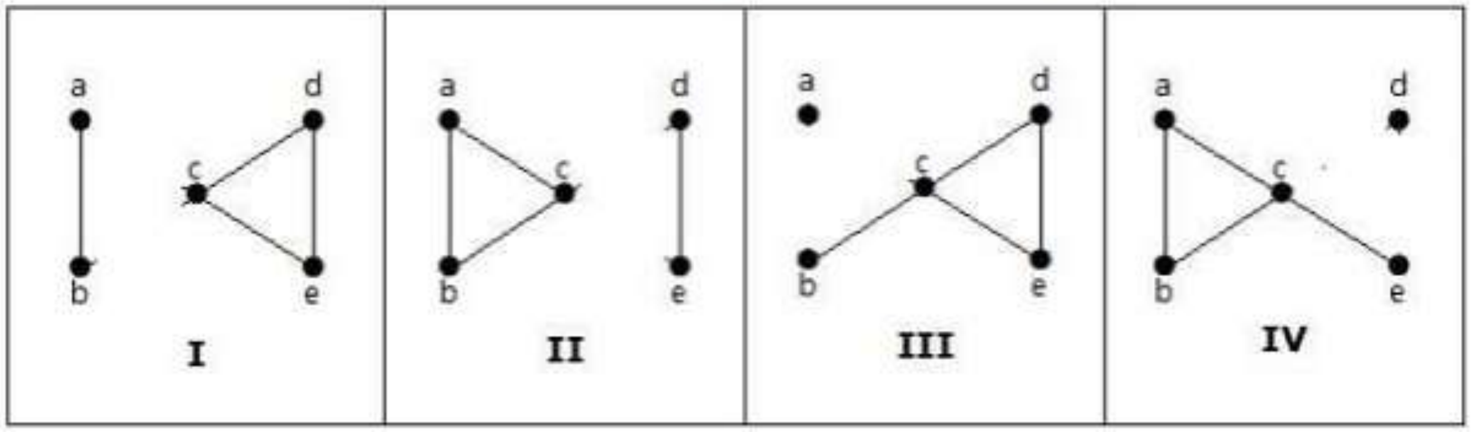
- **Edge Connectivity:** Let 'G' be a connected graph. The minimum number of edges whose removal makes 'G' disconnected is called edge connectivity of G.(minimum cut edge set of a graph)

  Notation − $\lambda(G)$

  Take a look at the following graph. By removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity ($\lambda(G)$) is 2. Therefore the above graph is a **2-edge-connected graph**.



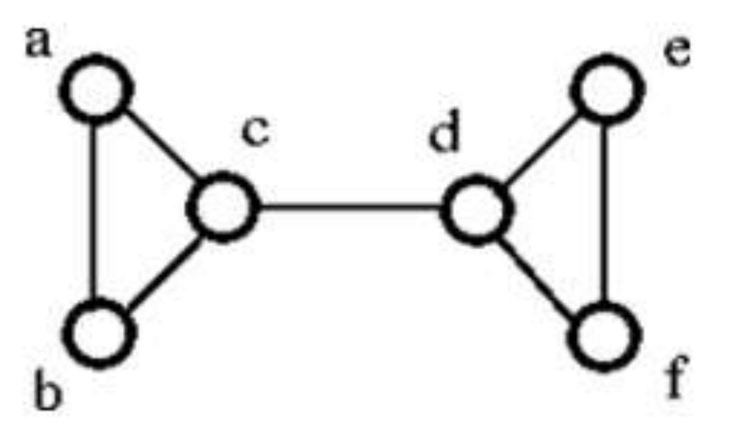
Here are the four ways to disconnect the graph by removing two edges −



CONNECTEDNESS IN UNDIRECTED GRAPHS:

- **Vertex Connectivity:** The connectivity (or vertex connectivity) of a connected graph G is the minimum number of vertices whose removal makes G disconnects or reduces to a trivial graph. To remove a vertex we must also remove the edges incident to it. (minimum cut vertex set of a graph)

  Notation − K(G)



The above graph G can be disconnected by removal of the single vertex either 'c' or 'd'. Hence, its vertex connectivity, K(G) is 1. Therefore, it is a 1-connected graph.

AN INEQUALITY FOR VERTEX CONNECTIVITY AND EDGE CONNECTIVITY

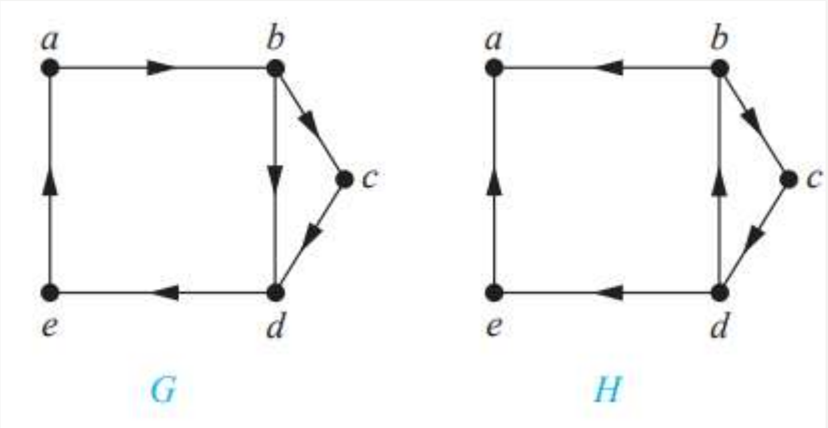
$$\kappa(G) \le \lambda(G) \le \min_{v\in V} \deg(v).$$

CONNECTEDNESS IN DIRECTED GRAPHS:

- There are two notions of connectedness in directed graphs, depending on whether the directions of the edges are considered.

- A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

- A directed graph is **weakly connected** if there is a path between every two vertices in the underlying undirected graph



- Graph G is strongly connected because there is a path between any two vertices in this directed graph.

- The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H

CONNECTEDNESS IN DIRECTED GRAPHS:

- A **strongly connected component** (**SCC**) of a directed graph is a maximal strongly connected subgraph. For example, there are 3 SCCs in the following graph.