The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	S_1	S_2	b	Ratio
	1	-140	-120	0	0	0	
S_1	0	2	4	1	0	1400	$\frac{1400}{2} = 700$
$\leftarrow S_2$	0	4	3	0	1	1500	$\frac{1500}{4} = 375$

Basic variables: $S_1 = 1400$, $S_2 = 1500$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_3, pivot element: 4

Applying $R_1 \to R_1 + 35R_3$, $R_2 \to R_2 - \frac{1}{2}R_3$

B	Z	x_1	$x_2\downarrow$	S_1	S_2	b	Ratio
	1	0	-15	0	35	52500	
$\leftarrow S_1$	0	0	$\frac{5}{2}$	1	$\frac{-1}{2}$	650	$\frac{650}{5/2} = 260$
x	0	4	3	0	1	1500	$\frac{1500}{3} = 500$

Basic variables: $S_1 = 650$, $x_1 = 1500$

Non-basic variables: $x_2 = S_2 = 0$

For these values, $Z = 210000$

Pivot column: column of x_2, pivot row: R_2, pivot element: $\frac{5}{2}$

Applying $R_1 \to R_1 + 6R_2$, $R_3 \to R_3 - \frac{6}{3}R_2$

B	Z	x_1	x_2	S_1	S_2	b
	1	0	0	6	32	56400
x_2	0	0	$\frac{5}{2}$	1	$\frac{-1}{2}$	650
x_1	0	1	0	$\frac{-3}{10}$	4	180

Since there is no negative number in first row, so the simplex process is completed.

From last table,

$x_1 = 180$ and $\frac{5}{2}x_2 = 650$

i.e., $x_1 = 180$ and $x_2 = 260$

$\therefore$ Max. $Z = 56400$ at $(x_1, x_2) = 56400$.

Exercise 3.1

1. By using the simplex method, solve the following LPP.

   a. Maximize $Z = 25x_1 + 30x_2$ subject to the constraints
   $20x_1 + 30x_2 \le 690$ $\quad x_1 + 4x_2 \le 120$ $\quad x_1, x_2 \ge 0$

   b. Maximize $Z = 3x_1 + x_2$ subject to the constraints
   $2x_1 + x_2 \le 6$ $\quad x_1 + 3x_2 \le 9$ $\quad x_1, x_2 \ge 0$

c. Minimize $Z = 5x_1 + 7x_2$ subject to the constraints

$2x_1 + 3x_2 \le 6$ $\quad 2x_1 + 5x_2 \le 27$

$3x_1 - x_2 \le 15$ $\quad x_1 \ge 0$

$-x_1 + x_2 \le 4$ $\quad x_2 \ge 0$

d. Maximize $Z = 2x_1 + 2x_2$ subject to the constraints

$x_1 + 2x_2 \le 20$ $\quad 2x_1 + x_2 \le 20$ $\quad x_1 \ge 0$ $\quad x_2 \ge 0$

e. Maximize $Z = 30x_1 + 20x_2$ subject to the constraints

$-x_1 + x_2 \le 5$ $\quad 2x_1 + x_2 \le 10$ $\quad x_1, x_2 \ge 0$

f. Minimize $Z = 5x_1 - 20x_2$ subject to the constraints

$-2x_1 + 10x_2 \le 5$ $\quad 2x_1 + 5x_2 \le 10$ $\quad x_1, x_2 \ge 0$

g. Maximize $Z = 300x_1 - 500x_2$ subject to the constraints

$2x_1 + 8x_2 \le 60$ $\quad 2x_1 + x_2 \le 30$ $\quad 4x_1 + 4x_2 \le 60$ $\quad x_1, x_2 \ge 0$

h. Maximize $Z = 90x_1 + 50x_2$ subject to the constraints

$x_1 + 3x_2 \le 18$ $\quad x_1 + x_2 \le 10$ $\quad 3x_1 + x_2 \le 24$ $\quad x_1, x_2, x_3 \ge 0$

i. Maximize $Z = 5x_1 + 3x_2$ subject to the constraints

$4x_1 + 2x_2 \le 10$ $\quad 2x_1 + 2x_2 \le 8$ $\quad x_1, x_2 \ge 0$

2. Solve the following LPP by using simplex for three decision variables.

a. Maximize $Z = 4x_1 + x_2 + 2x_3$ subject to the constraints

$x_1 + x_2 + x_3 \le 1$ $\quad x_1 + x_2 - x_3 \le 0$ $\quad x_1, x_2, x_3 \ge 0$

b. Maximize $Z = 4x_1 - 10x_2 - 20x_3$ subject to the constraints

$3x_1 + 4x_2 + 5x_3 \le 60$ $\quad 2x_1 + x_2 \le 20$ $\quad 2x_1 + 3x_3 \le 30$ $\quad x_1, x_2 \ge 0$

c. Maximize $Z = 3x_1 + 2x_2 + x_3$ subject to the constraints

$4x_1 + x_2 + x_3 \le 30$

$2x_1 + 3x_2 + x_3 \le 60$

$x_1 + 2x_2 + 3x_3 \le 40$

$x_1, x_2, x_3 \ge 0$

d. Maximize $Z = 3x_1 + 4x_2 + x_3$ subject to the constraints

$x_1 + 2x_2 + 3x_3 \le 90$ $\quad 3x_1 + x_2 + 2x_3 \le 80$ $\quad 2x_1 + x_2 + x_3 \le 60$ $\quad x_1, x_2, x_3 \ge 0$

e. Max $Z = 3x_1 + 5x_2 + 4x_3$ subject to the constraint

$2x_1 + 3x_3 \le 8$ $\quad 5x_1 + 2x_2 + 2x_3 \le 10$ $\quad 5x_2 + 4x_3 \le 15$ $\quad x_1, x_2, x_3 \ge 0$

3. Modeling by linear programming problem

a. A firm produces two products P and Q. Daily production upper limit is 600 units for total production. The least production is 300 units per day. The machine hours consumption for P is 6 per unit and for Q 2 per Q. The 1200 machines are used daily. Manufacturing cost per P and Q are respectively Rs 50 and Rs 20. Find the optimal solution by using simplex method.

b. A company manufactures two products A and B. Both products are processed on two machines m_1 and m_2.

	M_1	M_2
A	6 hrs/units	2 hrs/units
B	4 hrs/units	4 hrs/units
Availability	7200 hrs/month	4000 hrs/month

The profit per unit for A is Rs 100 and for is Rs 80. Find out the monthly production of A and B to maximize the profit by simplex method.

Answers

1. a. Max $Z = 3000$ at $(120, 15)$
   b. Max $Z = 9$ at $(x_1, x_2) = (3, 0)$
   c. Min $Z = 14$ at $(0, 2)$
   d. Max $Z = \frac{80}{3}$ at $\left(\frac{20}{3}, \frac{20}{3}\right)$
   e. Max $Z = \frac{550}{3}$ for $(x_1, x_2) = \left(\frac{5}{3}, \frac{20}{3}\right)$
   f. Min $Z = -10$ for $(x_1, x_2) = \left(0, \frac{1}{2}\right)$
   g. Max $Z = 4500$ for $(x_1, x_2) = (15, 0)$
   h. Max $Z = 780$ at $(x_1, x_2) = (7, 3)$
   i. Max $Z = 14$ at $(x_1, x_2) = (1, 3)$
2. a. Max $Z = 3$ at $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$
   b. Max $Z = 40$ at $(10, 0, 0)$
   c. Max $Z = 42$ for $(x_1, x_2, x_3) = (3, 18, 0)$
   d. Max $Z = 190$ for $(x_1, x_2, x_3) = (10, 40, 0)$
   e. Max $Z = \frac{87}{5}$ at $(x_1, x_2, x_3) = \left(\frac{4}{5}, 3, 0\right)$
3. a. Min $Z = 10{,}500$ for $(x_1, x_2) = (150, 150)$
   b. Max $Z = $ Rs 128000 for $(x_1, x_2) = (800, 600)$

3.7 Duality in Linear Programming Problems

For every linear programming problems there exists a dual problem known as dual LPP. The initial problem is called the primal. The dual is formed with the same data of primal LPP but in different variables. These two types of problems are closely related and the solution of dual gives the solution of primal and vice versa. If the primal has m constraints and n variables, then the dual contains n constraints and m variables.

Duality in linear programming has many practical applications.

1. If a model has a large number of constraints with few variables, then such problems can be converted into its dual with fewer constraints and more variables. Thus the computation will be considerably easy because the steps will be reduced when it is solved through the duality process.
2. Since duality is an alternative approach to solve the model, it can be used to verify the results.

3.7.1 Formation of dual LPP from given primal LPP

1. The dual of the given primal is obtained as follows.
   i. If the primal LPP is maximization problem, then its dual will be minimization problem and vice-versa.
   ii. The constraint coefficient matrix of the primal is transposed to get constraint coefficient matrix of dual.
   iii. The bounding constants b_i's of primal become cost coefficient of objective function of the dual and cost coefficient c_i's of primal LPP are bounding constants of the constraints of the dual LPP.
   iv. The inequality signs of the constraints in the dual will be reversed.

Applying $R_1 \to R_1 + (M-1)R_4$, $R_2 \to R_2 - R_4$

B	Z	x_1	x_2	x_3	S_1	S_2	a_2	b
	1	3	0	0	0	1	M − 1	10
S_1	0	2	0	0	1	1	− 1	15
x_3	0	1	0	1	0	0	0	5
x_2	0	− 1	1	0	0	−1	1	5

Since there is no negative numbers in the first row, so the optimal solution is obtained from table $x_1 = 0, x_2 = 5,\ x_3 = 5$ and Max $Z = 10$.

Exercise 3.2

1. Write the dual to the following problems.
   a. Maximize $Z = x_1 - x_2 + 3x_3$ subject to the constraints
      $x_1 + x_2 + x_3 \le 10$
      $2x_1 - x_2 - x_3 \le 2$
      $2x_1 - 2x_2 - 3x_3 \le 6$
      $x_1,\ x_2,\ x_3 \ge 0$
   b. Maximize $Z = 3x_1 + 4x_2 + x_3$ subject to the constraints
      $x_1 + 2x_2 + 3x_3 \le 90$
      $2x_1 + x_2 + x_3 \le 60$
      $3x_1 + x_2 + 2x_3 \le 80$
      $x_1,\ x_2,\ x_3 \ge 0$
2. The primal LPP are given below. Construct the dual problem and solve by using simplex method.
   a. Minimize $Z = 8x_1 + 9x_2$ subject to
      $x_1 + x_2 \ge 5$ $\quad 3x_1 + x_2 \ge 21$ $\quad x_1 \ge 0$ $\quad x_2 \ge 0$
   b. Minimize $Z = 3x_1 + 2x_2$ subject to constraints
      $7x_1 + 2x_2 \ge 30$ $\quad 5x_1 + 4x_2 \ge 20$ $\quad 2x_1 + 8x_2 \ge 16$ $\quad x_1, x_2 \ge 0$
   c. Minimize $Z = 21x_1 + 50x_2$ subject to the constraints
      $2x_1 + 5x_2 \ge 12$ $\quad 3x_1 + 7x_2 \ge 17$ $\quad x_1, x_2 \ge 0$
   d. Minimize $Z = 20x_1 + 30x_2$ subject to
      $x_1 + 4x_2 \ge 8$ $\quad x_1 + x_2 \ge 5$ $\quad 2x_1 + x_2 \ge 7$ $\quad x_1 \ge 0$ $\quad x_2 \ge 0$
   e. Minimize $Z = 8x_1 + 9x_2$ subject to constraints
      $x_1 + 3x_2 \ge 4$ $\quad 2x_1 + x_2 \ge 5$ $\quad x_1, x_2 \ge 0$
   f. Minimize $Z = 2x_1 + 9x_2 + x_3$ subject to constraints
      $x_1 + 4x_2 + 2x_3 \ge 5$ $\quad 3x_1 + x_2 + 2x_3 \ge 4$ $\quad x_1, x_2, , x_3 \ge 0$
3. Solve the following LPP by Big M method.
   a. Maximize $Z = 5x_1 - 2x_2$ subject to the constraints
      $3x_1 - 4x_2 \le 2,\ x_1 + 2x_2 \ge 4,\ x_2 \le 4,\ x_1, x_2 \ge 0.$
   b. Maximize $Z = -3x_1 + 7x_2$ subject to the constraints
      $2x_1 + 3x_2 \le 5,\ 5x_1 + 2x_2 \ge 3,\ x_2 \le 1,\ x_1, x_2 \ge 0.$
   c. Maximize $Z = 4x_1 + 2x_2$ subject to the constraints
      $3x_1 + x_2 \le 27,\ x_1 + x_2 \ge 21,\ x_1, x_2 \ge 0.$
   d. Minimize $Z = 12x_1 + 20x_2$ subject to the constraints
      $6x_1 + 8x_2 \ge 100,\ 7x_1 + 12x_2 \ge 120,\ x_1, x_2 \ge 0.$
   e. Maximize $Z = -2x_1 - x_2$ subject to the constraints
      $3x_1 + x_2 = 3,\ 4x_1 + 3x_2 \ge 6,\ x_1 + 2x_2 \le 4,\ x_1, x_2 \ge 0.$

4. One unit of product A contributes Rs 7 and requires 3 unit of raw materials and 2 hrs of labour. One unit of product B contributes Rs 5 and requires one unit of raw material and one hour labour. Availability of raw materials at present is 48 units and there are 40 hours of labour.
   a. Formulate this problem as a LPP.
   b. Write its dual.

5. A company produces three products P, Q and R from three raw materials A, B and C. One unit of product P requires 2 units of A and 3 units of B. A units of product Q requires 2 units of B and 4 units of C. The company has 8 units of A. 10 units of material B and 15 units of material C available to it. Profits per units of product P, Q and R are Rs 3, Rs 5 and Rs 4 respectively.
   a. Formulate this problem as an LPP.
   b. How many units of each product should be produced to maximize profit.
   c. Write the dual of this problem.

6. The XYZ Plastic Company has received a government contract to produce different plastic values. The values must be highly pressure and heat resistant and the company has developed a three stage production process that will provide the values with the necessary properties involving work in three different chambers.

Chamber 1 provides the necessary pressure resistant and can process values for 1200 minutes each week. Chamber 2 provides heat resistance and can process values for 900 minutes per week. Chamber 3 tests the values and can work 1300 minutes per week. The three value types and the time in minutes required in each chamber are

Value type	Time required in		
	Chamber 1	Chamber 2	Chamber 3
A	5	7	4
B	3	2	10
C	2	4	5

The government will buy all values that can be produced and the company will receive the following profit margins on each value.

Value A : Rs 15

Value B : Rs 13.50

Value C : Rs 10

How many values of each type should the company produce each week in order to maximum profits? Write the dual of the given LPP and give its economic interpretation.

Answers

1. a. Minimize $\bar{Z} = 10y_1 + 2y_2 + 6y_3$; Subject to $y_1 + 2y_2 + 2y_3 \geq 1$, $y_1 - y_2 - 2y_3 \geq -1$, $y_1 - y_2 - 3y_2 \geq 3$, $y_1, y_2, y_3 \geq 0$

   b. Minimize $\bar{Z} = 90y_1 + 60y_2 + 80y_3$; Subject to constraints $y_1 + 2y_2 + 3y_3 \geq 3$, $2y_1 + y_2 + y_3 \geq 4$, $3y_1 + y_2 + 2y_2 \geq 1$, $y_1, y_2, y_3 \geq 0$

2. a. Max$\bar{Z}$ = 56 at $(0, \frac{8}{3})$, Min Z = 56 at (7, 0)   b. Max Z = 14 at $(y_1, y_2, y_3) = \left(\frac{5}{13}, 0, \frac{2}{13}\right)$

   c. Max Z' = 121 at $y_1 = 3$, $y_2 = 5$, Min Z = 121 at $x_1 = 1$, $x_2 = 2$

   d. Min Z = 110 at $(x_1\ x_2) = (4, 1)$

   e. Maximum = 23 at (2, 3) and minimum = 23 at (11/5, 3/5)

   f. Min Z $= \frac{5}{2}$ for $(x_1\ x_2\ x_3) = \left(0\ ,\ 0\ \frac{5}{2}\right)$

3. a. Z = 22 at $x_1 = 6$ and $x_2 = 4$ b. Z = 4 at $x_1 = 1$ and $x_2 = 1$

   c. Z = 54 at $x_1 = 0$ and $x_2 = 27$ d. Z = 205 at $x_1 = 15$ and $x_2 = \frac{5}{4}$

   e. $Z = -\frac{12}{5}$ at $x_1 = \frac{3}{5}$ and $x_2 = \frac{6}{5}$

4. a. Maximize $Z = 7x_1 + 5x_2$; subject to $3x_1 + x_2 \leq 48$, $2x_1 + x_2 \leq 40$, $x_1, x_2 \geq 0$; As Maximize Z = Rs 200 at (0, 40)

   b. Dual: Minimize $Z = 48y_1 + 40y_2$; subject to $3y_1 + 2y_1 \geq 7$, $y_1 + y_2 \geq 5$, $y_1, y_2 \geq 0$; Min Z = Rs 200 at (0, 5)

5. Dual: Min $Z = 8y_1 + 10y_2 + 15y_3$ subject to $2y_1 + 5y_2 \geq 3$, $2y_2 + 5y_3 \geq 5$, $3y_1 + 2y_2 + 4y_3 \geq 4$, $y_1, y_2, y_3 \geq 0$

6. Dual Min $Z = 1200y_1 + 900y_2 + 1300y_3$; subject to $5y_1 + 7y_2 + 4y_3 \geq 15$, $3y_1 + 2y_2 + 10y_3 \geq 13.5$, $2y_1 + 4y_2 + 5y_3 \geq 10$, $y_1, y_2, y_3 \geq 0$; Max Z = Rs $\frac{58125}{31}$ at $(x_1\ x_2\ x_3) = \left(\frac{22400}{217}, \frac{2750}{31}, 0\right)$;

   Min $Z = \frac{385650}{31}$ at $(y_1\ y_2\ y_3) = \left(0, \frac{-41}{31}, \frac{-645}{62}\right)$