Unit 4

Vector Algebra

Pre-requisite knowledge

Before starting this unit, students are expected to have fundamental concepts and evaluation skills on

- vector in two and three dimension
- geometrical representation of vector
- vector and position vector in two and three dimension
- standard unit vectors
- addition rules of vectors
- scalar product of two vectors and physical interpretation
- vector product of two vectors and its physical and geometrical interpretations
- projection of vectors

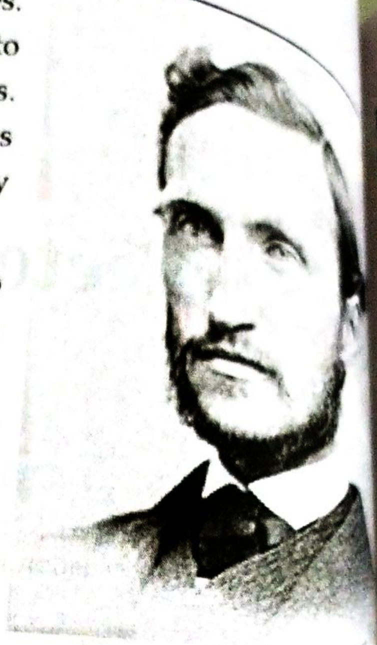
Expected learning outcomes

After completion of this unit, student will develop sufficient knowledge and evaluation skills on

- scalar triple product of three and four vectors.
- vector product of two and three vectors.
- physical interpretation of products.
- compute volume of parallelepiped.
- coplanarity of three and four vectors.
- linear relation of coplanar vectors.
- reciprocal system of vectors.

4.0 Introduction

Aristotle used vectors to describe the effects of forces. Descrates gave the idea of resolving vectors into geometric components parallel to the coordinate axes. Vector algebra was developed by Jusiah Willard Gibbs (1839-1903) a mathematical physicist at Yale University and English mathematical physicist Oliver Heavside.



Jusiah Willard Gibbs (1839-1903)

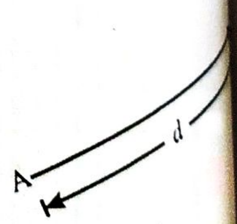
Historically vectors were conceived in mathematics to describe the geometry of physical objects and in physics (specially in mechanics) for the quantities that bear two characteristics magnitude and direction. The fundamentals of such quantities are displacement, force and velocity. Such quantities can not be completely described with a single characteristics is known as scalar.

Thus, in classifying the physical quantities broadly, we have two types of quantities:

i. **Scalar quantity:** The quantities that can be described by single characteristics a magn (number).

ii. **Vector quantity:** The quantities that can be describe by two characteristics magnitude direction. The magnitude expressed in positive real numbers and direction in term orientation (or location) in terms of coordinates.

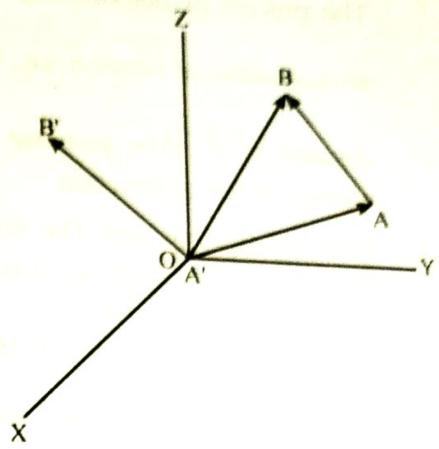
In order to represent a vector quantity physically we use directed line segment whose le gives the magnitude and the direction is given by the orientation from initial to terminal p The idea of representing a vector by a line segment gives the geometrical intuition of a vecto The vector alongside has initial point A and terminal point B and written as $\overrightarrow{AB}$ whose magnitude is the length $d = |\overrightarrow{AB}|$ of line segment $\overrightarrow{AB}$. Thus, geometrically a vector can be represented by a line segment which has an initial and a terminal point.



Mathematically, we shall study vectors as a directed line segment by fitting the vector (segment) into two dimensional or three dimensional coordinate system depending on consideration of vector in plane or space. As origin is always initial point in coordinate syste so it will also serve as the initial point for a vector and the point with given coordinate terminal point. Thus, the coordinate of a point always describes the terminal point as a vec whose initial point is origin, so the coordinate of a point in plane or space is known as positi vector.

While studying a vector in mathematics the coordinate of a point (position vector) is sufficie to describe a vector.

If initial point of a vector is any other point in plane or space we can still describe the vector in terms of vectors whose initial points are again at origin. The idea of expressing given vector with initial point other than origin in terms of vectors with initial point at origin is given by triangle law of vector addition.

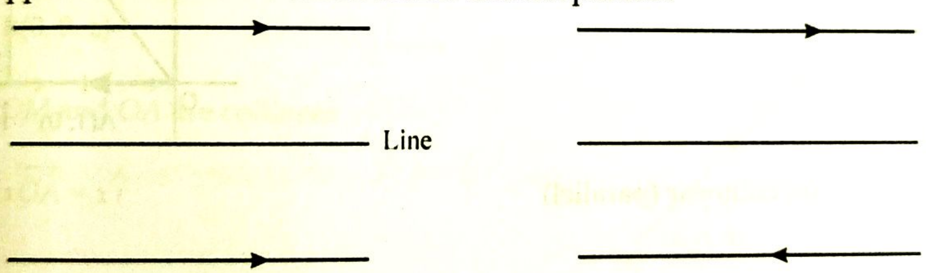


$\overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OB}$

We can consider an equivalent vector $\overrightarrow{A'B'}$ of $\overrightarrow{AB}$ through O which has same direction and magnitude and A coinciding with O. this vector $\overrightarrow{A'B'} = \overrightarrow{OB'}$ through O is known as localized vector of AB.

4.1 Fundamental Vectors

1. **Null vector:** A vector with initial and terminating points same i.e., having zero length.
2. **Collinear vectors:** Two vectors that are parallel to same line. Collinear vectors are also called parallel vectors. Collinear vectors with same direction are called like and parallel and opposite direction are called unlike and antiparallel.



Collinear (like and parallel) Collinear (unlike and antiparallel)

Two vectors along same line in same or opposite direction are parallel (collinear). Parallel and antiparallel (collinear) vectors have same or exact opposite direction and magnitude differ by constant. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ are collinear (parallel or antiparallel) then $\vec{a} = k\vec{b}$ for some constant (scalar) k and (i) k is positive for like vectors (ii) k is negative for unlike vectors (iii) if $|k| > 1$, then $\vec{b}$ is smaller than $\vec{a}$ (iv) if $0 < |k| < 1$, then $\vec{b}$ is greater than $\vec{a}$.

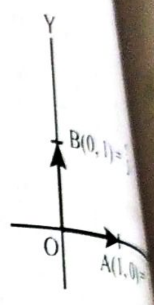
3. **Unit vector:** A vector having magnitude unity i.e., 1 is called unit vector. Mathematically, a unit vector is used to express the direction of a vector.

If $\vec{a}$ is any vector, then the unit vector along the direction of vector $\vec{a}$ is denoted by $\hat{a}$ and is defined by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \text{ as } |\hat{a}| = \left|\frac{\vec{a}}{|\vec{a}|}\right| = \frac{|\vec{a}|}{|\vec{a}|} = 1$$

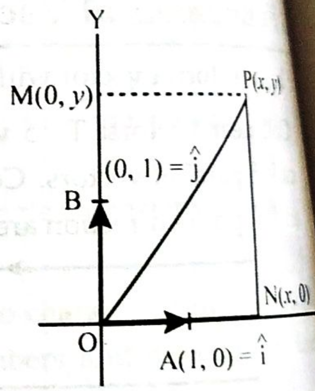
The process of determining unit vector $\hat{a}$ along the direction of vector $\vec{a}$ is known as normalization of vector $\vec{a}$. i.e., by the normalization of vector $\vec{a}$, we mean computing $\hat{a}$ denoted by $\hat{a}$. The purpose of unit vector is just an algebraic representation of direction.

4. **Standard unit vectors:** The unit vectors along the coordinate axes are called standard unit vectors.



i. In plane the position vectors A(1, 0) and B(0, 1) i.e. $\overrightarrow{OA}$ = A(1, 0) and $\overrightarrow{OB}$ = B(0, 1) are standard unit vectors along x-axis and y-axis respectively. They are denoted by $\hat{i}$ = A(1, 0) = $\overrightarrow{OA}$ and $\hat{j}$ = B(1, 0) = $\overrightarrow{OB}$.

If $\overrightarrow{OP} = (x, y)$ is any vector in plane.



$\overrightarrow{OA} = A(1, 0) = \hat{i}$

$\overrightarrow{OB} = B(0, 1) = \hat{j}$

$\overrightarrow{ON} = N(x, 0)$

$\overrightarrow{OM} = M(0, y)$

Here, $\overrightarrow{ON}$ and $\overrightarrow{OA}$ are collinear (parallel)

$\therefore\ \overrightarrow{ON} = x\overrightarrow{OA} = x\hat{i}$

Also, $\overrightarrow{OM}$ and $\overrightarrow{OB}$ are collinear

$\therefore\ \overrightarrow{OM} = y\overrightarrow{OB} = y\hat{j} = \overrightarrow{NP}$

By triangle law of vector addition in ΔOPN

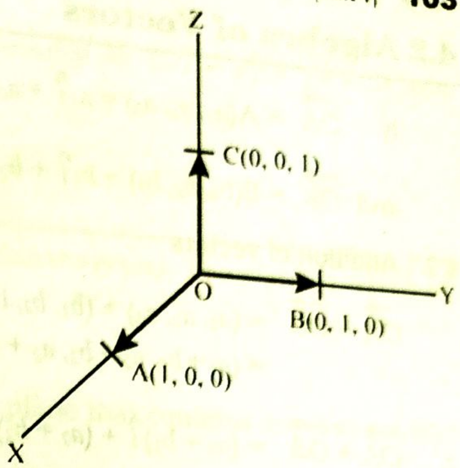
$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP}$

$\overrightarrow{OP} = P(x, y) = x\hat{i} + y\hat{j}$

Clearly, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

Thus, every plane vector can be represented by a linear combination of vectors parallel to x-axis $(\hat{i})$ and y-axis $(\hat{j})$

ii. In three dimensional space the position vectors A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) i.e., $\overrightarrow{OA} = A(1, 0, 0)$, $\overrightarrow{OB} = B(0, 1, 0)$ and $\overrightarrow{OC} = C(0, 0, 1)$ are standard unit vectors along x-axis, y-axis and z-axis respectively. They are denoted by



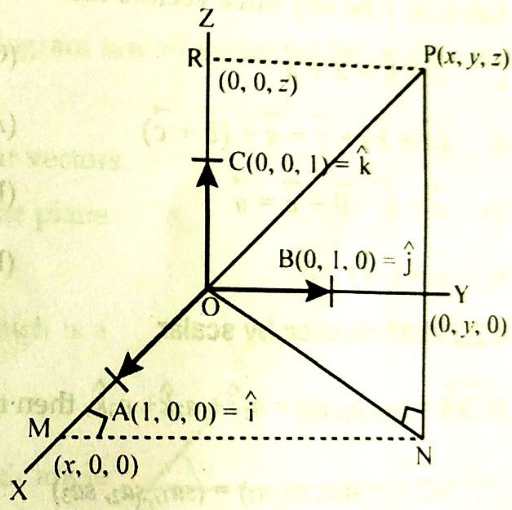
$\overrightarrow{OA} = A(1, 0, 0) = \hat{i}$

$\overrightarrow{OB} = B(0, 1, 0) = \hat{j}$

$\overrightarrow{OC} = B(0, 0, 1) = \hat{k}$

If $\overrightarrow{OP} = P(x, y, z)$ is any vector in space.

From figure,



$\overrightarrow{OM} = M(x, 0, 0)$

$\overrightarrow{OQ} = Q(0, y, 0)$

$\overrightarrow{OR} = R(0, 0, z)$

Here, $\overrightarrow{OM}$ and $\overrightarrow{OA}$ are collinear

$\therefore\ \overrightarrow{OM} = x\overrightarrow{OA} = x\hat{i}$

Similarly,

$\overrightarrow{OQ} = y\overrightarrow{OB} = y\hat{j}$ and $\overrightarrow{OR} = z\overrightarrow{OC} = z\hat{k}$

In ΔOMN, $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN} = \overrightarrow{OM} + \overrightarrow{OQ}$

$\therefore\ \overrightarrow{ON} = x\hat{i} + y\hat{j}$

Also, in ΔONP,

$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{ON} + \overrightarrow{OR}$

$\therefore\ \overrightarrow{OP} = P(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$

Clearly, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

Thus, every space vector can be represented by a linear combination of vectors parallel to x-axis ($\hat{i}$) and y-axis ($\hat{j}$) and z-axis ($\hat{k}$).

4.2 Algebra of Vectors

If $\overrightarrow{OA} = A(a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\overrightarrow{OB} = B(b_1, b_2, b_3) = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

4.2.1 Addition of vectors

$$\overrightarrow{OA} + \overrightarrow{OB} = (a_1, a_2, a_3) + (b_1, b_2, b_3)$$
$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

$$\therefore \quad \overrightarrow{OA} + \overrightarrow{OB} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

Which shows that the addition of vectors simply amounts to the addition of coordinates.

Properties of addition of vectors

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors then

i. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative law)

ii. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative law)

iii. $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$ (Existence of zero or identity vector)

iv. $\vec{a} + (-\vec{a}) = \vec{0}$ (Existence of additive inverse)

4.2.2 Multiplication by scalar

If $\overrightarrow{OA} = (a_1, a_2, a_3) = a_1\hat{j} + a_2\hat{k} + a_3\hat{k}$, then multiplication by scalar (scaling factor) s is given by

$$s\overrightarrow{OA} = s(a_1, a_2, a_3) = (sa_1, sa_2, sa_3)$$

$$\therefore \quad s\overrightarrow{OA} = sa_1\hat{i} + sa_2\hat{j} + sa_3\hat{k}$$

Which shows that the multiplication by scalar is same as the multiplication of each of coordinates by the scalars.

Properties of scalar multiplication

Let $\vec{a}$ and $\vec{b}$ be any two vectors and s, s_1, s_2 be any scalars, then

i. $s(\vec{a} + \vec{b}) = s\vec{a} + s\vec{b}$

ii. $(s_1 + s_2)\vec{a} = s_1\vec{a} + s_2\vec{a}$

iii. $s_1s_2(\vec{a}) = s_1(s_2\vec{a}) = s_2(s_1\vec{a})$

iv. $1\vec{a} = \vec{a}$

v. $0\vec{a} = \vec{0}$

vi. $(-1)\vec{a} = -\vec{a}$

Remark

If $\overrightarrow{OP} = x\hat{i} + y\hat{j} + zk$ then we write $\overrightarrow{OP} = \Sigma x\hat{i}$, the summation $\Sigma x\hat{i} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\Sigma x^2}$, $\Sigma x^2 = x^2 + y^2 + z^2$.

4.3 Coplanar Vectors

A set of vectors in same or parallel planes are said to be coplanar vectors. The plane to which all vectors lie is said to be common plane.

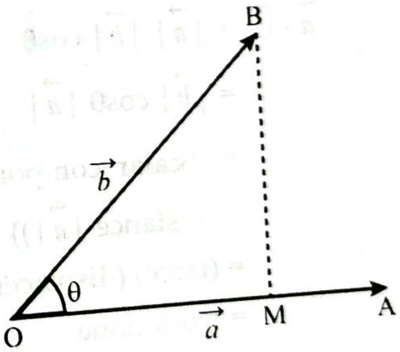
Remarks

1. Two vectors are always coplanar as we can imagine a plane that contains any two vectors under consideration.
2. If any two vectors are coplanar then the sum of two vectors can be determined by triangle law or parallelogram law of vector addition. Thus, any three vectors will be coplanar if they satisfy triangle law of vector addition.

   i.e., if any three vectors follow triangle or parallelogram law of vector addition then the vectors are coplanar.

   Mathematically, if $\vec{a} = \vec{b} + \vec{c}$ then $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.
3. The sum of two vectors is again a vector lying in the plane of constituent vectors.

   i.e. if $\vec{a}$ is vector sum of $\vec{b}$ and $\vec{c}$ then $\vec{a} = \vec{b} + \vec{c}$, which is a vector that again lies in the plane of $\vec{b}$ and $\vec{c}$.



4. Projection of a vector $\overrightarrow{OA} = \vec{a}$ on vector $\overrightarrow{OB} = \vec{b}$ with angle θ between them is (projection of $\vec{a}$ on $\vec{b}$), OM = $|\vec{a}|\cos\theta$

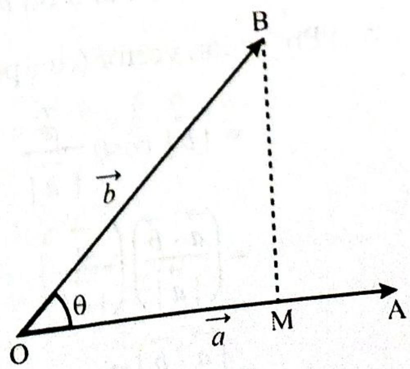
4.4 Product of Two Vectors

Two vectors can be combined as product in two ways such that (i) the result of product is a scalar quantity and (ii) the result of product is a vector quantity.

4.4.1 Scalar product

The scalar product of two vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ meeting at a point O is denoted by $\vec{a}\cdot\vec{b}$ and is defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$



A scalar product is also known as dot product.

Remark

If $\vec{a} = \vec{b}$ then $\theta = 0$ then $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{a}$

i.e., $\vec{a} \cdot \vec{a} = |\vec{a}|\ |\vec{a}| \cos 0$

$\therefore \quad |\vec{a}|^2 = \vec{a} \cdot \vec{a}$

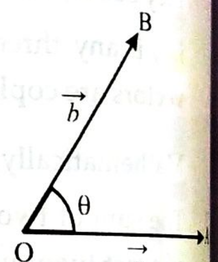
i.e., $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

4.4.2 Geometrical interpretation

$$\begin{aligned}\vec{a} \cdot \vec{b} &= |\vec{a}|\ |\vec{b}| \cos\theta \\ &= |\vec{a}|\ |OM| \\ &= |\vec{a}| \text{ projection of } \vec{a} \text{ on } \vec{b}\end{aligned}$$

Thus, geometrically the scalar product of two vectors $\vec{a}$ and $\vec{b}$ is product of length of $\vec{a}$ and projection of $\vec{b}$ on $\vec{a}$.

4.4.3 Physical interpretation



If $\vec{b}$ be a force acting on a body at O and it displaces the body through distance $|\vec{a}|$ in the direction of vector $\vec{a}$ then,

$$\begin{aligned}\vec{a} \cdot \vec{b} &= |\vec{a}|\ |\vec{b}| \cos\theta \\ &= |\vec{b}| \cos\theta\ |\vec{a}|\end{aligned}$$

= (Scalar component of force $\vec{b}$ along the direction of $\vec{a}$) (displacement through distance $|\vec{a}|$))

= (force) (displacement)

= work done

Thus, physically the scalar product of $\vec{a}$ and $\vec{b}$ gives the work done by component of force along the direction of displacement in moving a body through the distance $|\vec{a}|$.

4.4.4 Projection of vector in vector form

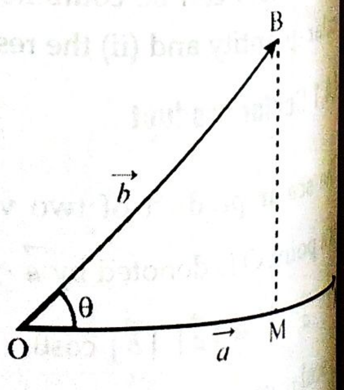
The projection of $\vec{b}$ on $\vec{a}$ = OM = $|\vec{b}| \cos\theta$

$\therefore$ Projection vector (component of $\vec{b}$ along $\vec{a}$)

$$= |\vec{b}| \cos\theta \frac{\vec{a}}{|\vec{a}|}$$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right)\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\right)\vec{a}$$



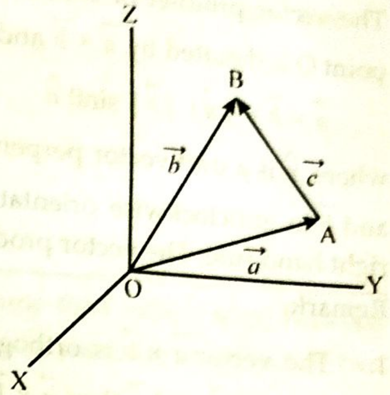
4.4.5 Scalar product of vectors in terms of coordinate (computational formula)

Let $\overrightarrow{OA} = \vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\overrightarrow{OB} = \vec{b} = (b_1, b_2, b_3) = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then, $\vec{c} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$



Also, using cosine law in triangle OAB,

i.e., $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}|\cos\theta$

$\therefore \quad |\vec{a}|\,|\vec{b}|\cos\theta = \dfrac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2}$

$= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - \dfrac{\{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a^3)^2\}}{2}$

$= \dfrac{2(a_1b_1 + a_2b_2 + a_3b_3)}{2}$

$\therefore \quad \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

4.4.6 Angle between two vectors

For $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ we have

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

$\therefore \quad \cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{a}|\,|\vec{b}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

4.4.7 Orthogonal (or perpendicular) vectors

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and if $\vec{a}$ and $\vec{b}$ are perpendicular, then $\theta = \frac{\pi}{2}$.

$\therefore \quad \cos\frac{\pi}{2} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{a}|\,|\vec{b}|}$

$\therefore \quad a_1b_1 + a_2b_2 + a_3b_3 = 0$

Thus, two non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) if and only if $\vec{a}\cdot\vec{b} = 0$.

Remark: Since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x, y and z-axis, so $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

Some properties of scalar product

If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b}\ (b_1, b_2, b_3)$ then

i. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

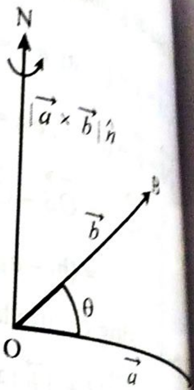
ii. $(c\vec{a})\cdot\vec{b} = \vec{a}(c\,\vec{b}) = c(\vec{a}\cdot\vec{b})$

iii. $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ and $(\vec{a} + \vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

4.4.8 Vector product of two vectors

The vector product of two vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ meeting at a point O is denoted by $\vec{a} \times \vec{b}$ and defined by

$$\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}|\ \sin\theta\ \hat{n}$$



where, $\hat{n}$ is a unit vector perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$ in anticlockwise orientation from $\vec{a}$ towards $\vec{b}$ determined by right hand rule. The vector product is also known as cross product.

Remark

1. The vector $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$. If $\hat{n}$ is unit orthogonal (normal) vector to plane of $\vec{a}$ and $\vec{b}$ then $\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|\hat{n}$.
2. If $\theta = 0$ or π then $\vec{a} \times \vec{b} = |\vec{a}\ |\vec{b}|\sin\theta\ \hat{n} = 0$.

   Thus, non zero vectors $\vec{a}$ and $\vec{b}$ are parallel (collinear) if and only if $\vec{a} \times \vec{b} = 0$.
3. $\vec{a} \times \vec{b} = |\vec{a}\ |\vec{b}|\sin\theta\ \hat{n}$

   $$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\ |\vec{b}|}, \text{ as } |\hat{n}| = 1.$$
4. $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}|\sin\theta\ \hat{n}$

   Where θ is anticlockwise (right hand rule) orientation. If we take $\vec{b} \times \vec{a}$ then θ will be clockwise orientation, hence will be negative, i.e.,

   $$\vec{b} \times \vec{a} = |\vec{a}|\ |\vec{b}|\ \sin(-\theta)\hat{n} = -|\vec{a}|\ |b|\ \sin\theta\ \hat{n} = -(a \times \vec{b})$$

   $$\therefore\ \vec{b} \times \vec{a} = -(a \times \vec{b})$$

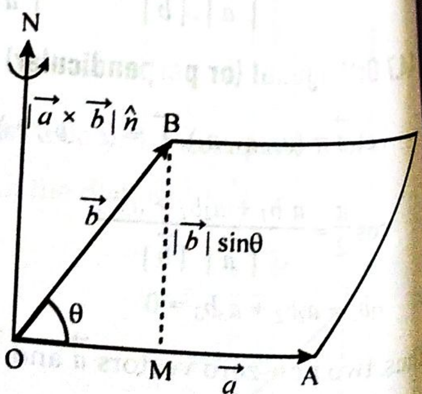
4.4.9 Geometrical interpretation of $\vec{a} \times \vec{b}$

We have

$$\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}|\sin\theta\ \hat{n}$$

Taking magnitude of vectors on both sides

$$\begin{aligned} |\vec{a} \times \vec{b}| &= |\vec{a}|\ |\vec{b}|\,|\sin\theta|\ |\hat{n}| \\ &= |\vec{a}|\ |\vec{b}|\ \sin\theta \\ &= \text{(Length of base) (Height of parallelogram)} \\ &= \text{Area of parallelogram determined by } \vec{a} \text{ and } \vec{b}. \end{aligned}$$



Thus, $\vec{a} \times \vec{b}$ is a vector which acts along the normal of plane of $\vec{a}$ and $\vec{b}$ and magnitude given by the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

Remarks: The area of triangle determined by $\vec{a}$ and $\vec{b}$ is $A = \frac{1}{2}|\vec{a} \times \vec{b}|$

Evaluation of $\vec{a} \times \vec{b}$ in terms of coordinates

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then applying the distributive law we can show that

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4.5 Product of Three Vectors

It is a simple fact that we cannot apply two operations at the same time (single step) between three members of a system (in this case three vectors). But we can measure the effect of two operations in two steps.

We will combine two out of three vectors to construct a new vector and apply the result to the third vector in turn.

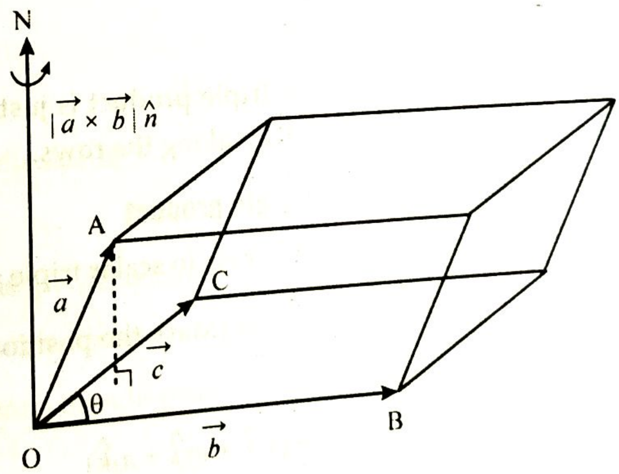
Thus, if we have three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and we need the result of product of these three vectors we first construct $\vec{b} \times \vec{c}$ from $\vec{b}$ and $\vec{c}$ and take the product with vector $\vec{a}$. We see that taking product of three vectors is in fact again product of two vectors.

4.5.1 Scalar product of three vectors

Let $\vec{a}, \vec{b}, \vec{c}$ be any three non zero vectors then the scalar product of $\vec{a}$ with $\vec{b} \times \vec{c}$ i.e., $\vec{a} \cdot (\vec{b} \times \vec{c})$ is defined to be the scalar product of three vectors. It is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$ i.e., $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$. It is also known as scalar triple product.

4.5.2 Physical interpretation of scalar triple product

Consider any three vectors $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$ meeting at a point O. The vector $\vec{b} \times \vec{c}$ is a vector perpendicular (normal) to the plane of $\vec{b}$ and $\vec{c}$ whose magnitude is given by the area of parallelogram with adjacent sides $\vec{b}$ and $\vec{c}$ as shown in figure.



If $\vec{b} \times \vec{c}$ makes angle θ with $\vec{a}$, then

$$[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= |\vec{a}|\ |\vec{b} \times \vec{c}|\ \cos\theta$$

$$= |\vec{b} \times \vec{c}|\ |\vec{a}|\ \cos\theta$$

$$= |\vec{b} \times \vec{c}|\ (ON)$$

$= |\vec{b} \times \vec{c}|$ (AM)

= (Area of base parallelogram) (Height of parallelopiped)

= Volume of parallelopiped whose adjacent sides are $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Thus, the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ gives the volume of parallelopiped with adjacent $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Remark: The volume of tetrahedron with sides $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$.

4.5.3 Evaluation expression of scalar triple product in terms of coordinates

Let $\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = (b_1, b_2, b_3) = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = (c_1, c_2, c_3) = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

We have, $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$ ....(1)

Now, $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)$

Thus, $\vec{a} \cdot (\vec{b} \times \vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)\}$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$\therefore\ [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Hence, evaluation of scalar triple product is just the evaluation of determinants of order 3×3 taking the coordinates in order along the rows.

4.5.4 Properties of scalar triple product

1. The position of dot and cross in scalar triple product can be interchanged

   i.e., $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ (Mark the position of bracket)

   Proof

   Let $\vec{a} = (a_1, a_2, a_3) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

   $\vec{b} = (b_1, b_2, b_3) = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

   $\vec{c} = (c_1, c_2, c_3) = (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

Then, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad R_1 \leftrightarrow R_3$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad R_2 \leftrightarrow R_3$$

$$= \vec{c}\cdot(\vec{a}\times\vec{b})$$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ as we know $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$.

The value of scalar triple-product is same in any cyclic order.

i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

From property (1), we have

$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{a})\cdot\vec{c} = \vec{c}\cdot(\vec{a}\times\vec{b})$

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{c}\ \vec{a}\ \vec{b}]$

Similarly, we can prove $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]$

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

For any three non zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ if and only if $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Proof:

Let $\vec{a}, \vec{b}, \vec{c}$ be any three non zero vectors such that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, then to prove $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

We have,

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$, i.e., $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ ....(1)

Equation (1) shows that vector $\vec{a}$ is perpendicular to $\vec{b}\times\vec{c}$. But $\vec{b}\times\vec{c}$ is perpendicular to the vectors $\vec{b}$ and $\vec{c}$.

Thus, we see that the same vector $\vec{b}\times\vec{c}$ is perpendicular to vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, so the vectors must lie in same plane or parallel planes. Hence $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Again, let $\vec{a}$, $\vec{b}$ and $\vec{c}$ non zero coplanar vectors, then to prove $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

We have $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$ ....(2)

We know that $\vec{b}\times\vec{c}$ is perpendicular (normal) to the vectors $\vec{b}$ and $\vec{c}$. Here all vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, so $\vec{b}\times\vec{c}$ must also be perpendicular to $\vec{a}$. Thus the vectors $\vec{a}$ and $(\vec{b}\times\vec{c})$ are perpendicular to each other.

$\therefore \vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

4. For any three non zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, if any two of the vectors are equal.

Proof

Let $\vec{a}, \vec{b}, \vec{c}$ be any three non zero vectors such that

$\vec{b} = \vec{c}$

$$\begin{aligned}\text{Then } [\vec{a}\ \vec{b}\ \vec{c}] &= \vec{a} \cdot (\vec{b} \times \vec{c}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{b}) \\ &= \vec{a} \cdot 0 \\ &= 0\end{aligned}$$

5. For any three non zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ if any two of them are parallel.

Proof

Let $\vec{a}, \vec{b}, \vec{c}$ be any three non zero vectors such that $\vec{b}$ and $\vec{c}$ are parallel.

$\therefore$ $\vec{b} = m\vec{c}$

$$\begin{aligned}\text{Then } [\vec{a}\ \vec{b}\ \vec{c}] &= \vec{a} \cdot (\vec{b} \times \vec{c}) \\ &= \vec{a} \cdot (m\vec{c} \times \vec{c}) \\ &= m\{\vec{a}(\vec{c} \times \vec{c})\} \\ &= m \cdot (0) \\ &= 0\end{aligned}$$

Remarks

1. The scalar triple product is also known as box product.
2. For any vector $\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ which shows that any space vector $\vec{a}$ can be expressed as linear combination of three non coplanar vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

4.5.5 Vector product of three vectors

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three non zero vectors. The vector product of $\vec{a}$ with $\vec{b} \times \vec{c}$ i.e., $\vec{a} \times (\vec{b} \times \vec{c})$ is defined as vector product of three vectors.

4.5.6 Geometrical interpretation of $\vec{a} \times (\vec{b} \times \vec{c})$

We have

$\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $\vec{a}$ and $(\vec{b} \times \vec{c})$.

Also, $\vec{b} \times \vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}$.

Thus, the same vector $\vec{b} \times \vec{c}$ is perpendicular to vectors $\vec{a} \times (\vec{b} \times \vec{c})$, $\vec{b}$ and $\vec{c}$.

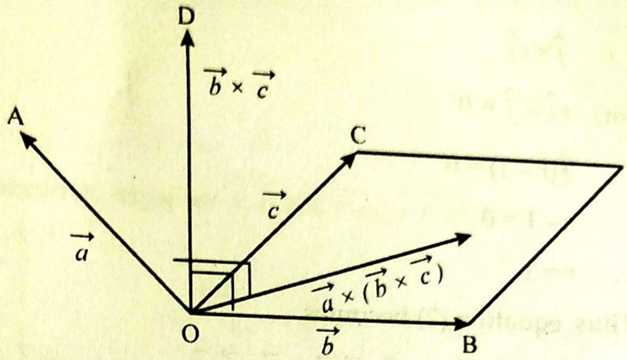
So, the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is

i. coplanar with $\vec{b}$ and $\vec{c}$.

ii. perpendicular to $\vec{a}$.

In the figure, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{b} \times \vec{c}$, and $\vec{b} \times \vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}$.

$\vec{a} \times (\vec{b} \times \vec{c})$ perpendicular to $\vec{a}$ and $\vec{b} \times \vec{c}$, so $\vec{a} \times (\vec{b} \times \vec{c})$ lies in the plane of $\vec{b}$ and $\vec{c}$ and is perpendicular to $\vec{a}$ and $\vec{b} \times \vec{c}$.



Remark: Vector method (condition) for coplanar vectors

Any three vectors that satisfy triangle law of vector addition, they are coplanar.

i.e., if $\vec{a} = x\vec{b} + y\vec{c}$ for some scalars x and y then, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar vector.

We see that, if any of the three vectors can be written as linear combination of other two, then the vectors are coplanar (dependent vectors).

4.5.7 Evaluation expression of $\vec{a} \times (\vec{b} \times \vec{c})$

As we have seen $\vec{a} \times (\vec{b} \times \vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$, so by the vector condition for coplanar vectors, we can write,

$$\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c} \quad \ldots(1)$$

Where, x and y are to be determined.

Also, the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $\vec{a}$, so taking dot product on both sides of (1)

$$\vec{a} \cdot \{\vec{a} \times (\vec{b} \times \vec{c})\} = x(\vec{a} \cdot \vec{b}) + y(\vec{a} \cdot \vec{c})$$

$$\therefore \; x(\vec{a} \cdot \vec{b}) + y(\vec{a} \cdot \vec{c}) = 0$$

$$\text{or, } x(\vec{a} \cdot \vec{b}) = -y(\vec{a} \cdot \vec{c})$$

$$\therefore \; \frac{x}{\vec{a} \cdot \vec{c}} = \frac{y}{-(\vec{a} \cdot \vec{b})} = r \text{ (ratio of constants)}$$

$$\therefore \; x = r(\vec{a} \cdot \vec{c}), \; y = -r(\vec{a} \cdot \vec{b})$$

Substituting x and y in (1), we have

$$\vec{a} \times (\vec{b} \times \vec{c}) = r(\vec{a} \cdot \vec{c})\vec{b} - r(\vec{a} \cdot \vec{b})\vec{c} \quad \ldots(2)$$

Equation (2) is an identity, i.e., satisfied by all values of $\vec{a}$, $\vec{b}$ and $\vec{c}$. Since the constant r is not zero. We need to choose $\vec{a}$, $\vec{b}$, $\vec{c}$ such that $\vec{a} \times (\vec{b} \times \vec{c})$ is non zero.

Taking $\vec{a} = \hat{i}, \vec{b} = \hat{j}$ and $\vec{c} = \hat{i}$, in (2)

$$\hat{i} \times (\hat{j} \times \hat{i}) = r(\hat{i} \cdot \hat{i})\hat{j} - r(\hat{i} \cdot \hat{j})\hat{i}$$

$$\hat{i} \times (-\hat{k}) = r(1)\hat{j}$$

$\therefore \quad \hat{j} = r\hat{j}$

or, $\quad r\hat{j} - \hat{j} = 0$

$\quad \hat{j}(r-1) = 0$

$\therefore \quad r - 1 = 0$

$\therefore \quad r = 1$

Thus, equation (2) becomes

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Which is the evaluation formula for $\vec{a} \times (\vec{b} \times \vec{c})$.

Remark

1. The expression for $(\vec{a} \times \vec{b}) \times \vec{c}$

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b})$$
$$= -\{\vec{a}(\vec{c} \cdot \vec{b}) - \vec{b}(\vec{c} \cdot \vec{a})\}$$
$$= \vec{b}(\vec{c} \cdot \vec{a}) - \vec{a}(\vec{c} \cdot \vec{b})$$
$$= \vec{b}(\vec{a} \cdot \vec{c}) - \vec{a}(\vec{b} \cdot \vec{c})$$

$\therefore \quad (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

Which shows that $(\vec{a} \times \vec{b}) \times \vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$.

2. Since $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

Which shows that $(\vec{a} \times \vec{b}) \times \vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c}$ are different vectors lying in the planes of $\vec{b}$ and $\vec{c}$ and $\vec{a}$ and $\vec{b}$ respectively.

3. If we take $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

We can prove $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

by computing $(\vec{b} \times \vec{c})$ and $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of coordinates of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Example 1. Find the volume of parallelopiped whose concurrent edges are the vectors with position vectors (1, 2, 3), (3, 7, 4) and (1, –5, 3).

Solution

We have the position vectors of the concurrent edges of a parallelopied.

$\vec{OA} = \vec{a} = (1, 2, 3) = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{OB} = \vec{b} = (3, 7, 4) = 3\hat{i} + 7\hat{j} + 4\hat{k}$

$\vec{OC} = \vec{c} = (1, -5, 3) = \hat{i} - 5\hat{j} + 3\hat{k}$

We know, the value of parallelopiped whose concurrent edges are $\vec{a}, \vec{b}$ and $\vec{c}$ is $[\vec{a}\ \vec{b}\ \vec{c}]$.

$$\text{Now, } [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 7 & -4 \\ 0 & -7 & 0 \end{vmatrix} \quad R_3 \leftrightarrow R_3 - R_1$$

Expanding along R_3 $= 0 - (-7)(-4 - 9)$

$= -91$ (volume is never negative)

∴ Volume of parallelopiped $[\vec{a}\ \vec{b}\ \vec{c}] = 91$ cubic units.

Example 2. Show that the four points with position vectors (4, 5, 1), (0, –1, –1), (3, 9, 4) and (–4, 4, 4) are coplanar.

Solution

Let $\vec{OA} = (4, 5, 1) = 4\hat{i} + 5\hat{j} + \hat{k}$

$\vec{OB} = (0, -1, -1) = 0\hat{i} - \hat{j} - \hat{k}$

$\vec{OC} = (3, 9, 4) = 3\hat{i} + 9\hat{j} + 4\hat{k}$

$\vec{OD} = (-4, 4, 4) = -4\hat{i} + 4\hat{j} + 4\hat{k}$

We know that

$\vec{AB} = \vec{a} = \vec{OB} - \vec{OA} = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\vec{BC} = \vec{b} = \vec{OC} - \vec{OB} = 3\hat{i} + 10\hat{j} + 5\hat{k}$

$\vec{CD} = \vec{c} = \vec{OD} - \vec{OC} = -7\hat{i} - 5\hat{j} + 0\hat{k}$

Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -4 & -6 & -2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -2 \\ -7 & -5 & 5 \\ -7 & -5 & 0 \end{vmatrix} \quad \begin{matrix} c_1 \to c_1 - 2c_3 \\ c_2 \to c_2 - 3c_3 \end{matrix}$$

Expanding along R_1 $= 0 - 0 + (-2)\{35 - 35\}$

$= 0$

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$, so the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ and hence given four points are coplanar.

Example 3. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ show that $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

Solution

We know that

$$\begin{aligned}
[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] &= (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\} \\
&= (\vec{a}+\vec{b})\cdot\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\} \\
&= (\vec{a}+\vec{b})\cdot\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\} \qquad [\because \vec{c}\times\vec{c}=0] \\
&= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a}) \\
&= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]; \text{ since scalar triple product is zero if any two vectors are equal} \\
&= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}] \\
&= 2[\vec{a}\ \vec{b}\ \vec{c}]
\end{aligned}$$

$\therefore \quad [\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$.

Example 4. Prove that $b^2\vec{a} = (\vec{a}\cdot\vec{b})\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})$

Solution

Taking R.H.S.

$$\begin{aligned}
(\vec{a}\cdot\vec{b})\vec{b}+\vec{b}\times(\vec{a}\times\vec{b}) &= (\vec{a}\cdot\vec{b})\vec{b}+\vec{a}(\vec{b}\cdot\vec{b})-\vec{b}(\vec{a}\cdot\vec{b}) \\
&= \vec{a}(\vec{b}\cdot\vec{b}) \\
&= \vec{a}\, b^2, \text{ where } b^2=\vec{b}\cdot\vec{b}
\end{aligned}$$

$\therefore \quad (\vec{a}\cdot\vec{b})\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})= b^2$a.

Example 5. If $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$, show that $\vec{a}$ and $\vec{c}$ are collinear.

Solution

We are given

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c} \qquad \ldots.(1)$$

Now, $\quad \vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}) \qquad \ldots.(2)$

and $\quad (\vec{a}\times\vec{b})\vec{c}=\vec{b}(\vec{a}\cdot\vec{c})-\vec{a}(\vec{b}\cdot\vec{c}) \qquad \ldots.(3)$

From (1), (2) and (3), we have

$$\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{a}(\vec{b}\cdot\vec{c})$$

$$\therefore \quad \vec{a}=\frac{(\vec{a}\cdot\vec{b})}{(\vec{b}\cdot\vec{c})}\vec{c}$$

Which shows that $\vec{a}$ is a scalar multiple of $\vec{c}$, so they are collinear vectors.

Example 6. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{2}\vec{b}$ and $\vec{b}$, and $\vec{c}$ are not parallel. Find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$.

Solution

Given $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{b}$ and $\vec{c}$ are not parallel i.e., $\vec{b}\neq m\vec{c}$ for any number m. Also

$$\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{2}\vec{b}$$

$$\therefore \quad \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\frac{1}{2}\vec{b}$$

Which is an identity, so equating coefficients of $\vec{b}$ and $\vec{c}$ on both sides.

$\vec{a}\cdot\vec{c}=\frac{1}{2}$ and $\vec{a}\cdot\vec{b}=0$

If θ is angle between $\vec{a}$ and $\vec{c}$ and β is angle between $\vec{a}$ and $\vec{b}$.

$|\vec{a}||\vec{c}|\cos\theta=\frac{1}{2}$ and $|\vec{a}|\,|\vec{b}|\cos\beta=0$

$\cos\theta=\cos\frac{\pi}{3}$ and $\cos\beta=\cos\frac{\pi}{2}$

$\theta=\frac{\pi}{3}$ and $\beta=\frac{\pi}{2}$.

Thus, the angles between $\vec{a}$ and $\vec{c}$ is $\theta=\frac{\pi}{3}$ and angle between $\vec{a}$ and $\vec{b}$ is $\beta=\frac{\pi}{2}$.

Exercise 4.1

1. If $\vec{a}=(\hat{i}-2\hat{j}+3\hat{k})$, $\vec{b}=(2\hat{i}+\hat{j}-\hat{k})$ and $\vec{c}=(\hat{j}+\hat{k})$ find $[\vec{a}\,\vec{b}\,\vec{c}]$.
2. Find the scalar triple product of $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.
3. Find the volume of parallelopiped whose concurrent edges are given by:
   i. $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+2\hat{k}$, $\vec{c}=2\hat{i}+\hat{j}-\hat{k}$
   ii. $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$, $\vec{c}=\hat{i}+2\hat{j}-\hat{k}$
   iii. $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=3\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+3\hat{k}$.
4. Show that four points with given position vectors are coplanar.
   $-\hat{i}+2\hat{j}-4\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $6\hat{i}+2\hat{j}-\hat{k}$, $12\hat{i}-\hat{j}-3\hat{k}$
5. Find the value of λ such that the vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\lambda\hat{j}-3\hat{k}$, $\vec{c}=3\hat{i}-4\hat{j}+5\hat{k}$ are coplanar.
6. If the four points $2\hat{i}+3\hat{j}-\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$, $\lambda\hat{i}+4\hat{j}-2\hat{k}$ and $\hat{i}-6\hat{j}+6\hat{k}$ are coplanar find the value of λ.
7. For $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=\hat{i}+2\hat{j}-\hat{k}$, verify that $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$.
   $\vec{b}\times\vec{c}=l\vec{a}+m\vec{b}+n\vec{c}$
8. Show that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.
9. If $\vec{a},\vec{b},\vec{c}$ are three non coplanar vectors then express $\vec{b}\times\vec{c}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.
10. Using vector product of three vectors and vector sum condition of coplanarity, show that $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar if $\vec{a},\vec{b},\vec{c}$ are coplanar vectors. [Hint: Show that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$]

Answers

1. 12	2. 7	3. i. 14 units	ii. 4 units	iii. 56 units
5. 2	6. 3			

$\vec{b}\times\vec{c}\cdot\vec{b}\times\vec{c}=l[\vec{a}\,\vec{b}\,\vec{c}]$

4.6 Product of Four Vectors

We can define vector and scalar product of four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ taking in pair.

4.6.1 Scalar product of four vectors

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be any four vectors. The scalar product of the vectors $(\vec{a} \times \vec{b})$ and $(\vec{c} \times \vec{d})$ i.e. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is defined as scalar product of four vectors.

4.6.2 Evaluation of scalar product of four vectors

Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors.

We have, $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{m} \cdot (\vec{c} \times \vec{d})$, where $\vec{m} = \vec{a} \times \vec{b}$

$$= \{\vec{m} \times \vec{c}\} \cdot \vec{d}$$

$$= \{(\vec{a} \times \vec{b}) \times \vec{c}\} \cdot \vec{d}$$

$$= \{\vec{b}(\vec{a} \cdot \vec{c}) - \vec{a}(\vec{b} \cdot \vec{c})\} \cdot \vec{d} = (\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

$$\therefore \quad (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

4.6.3 Vector product of four vectors

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be any four non zero vectors then the vector product of $(\vec{a} \times \vec{b})$ with $(\vec{c} \times \vec{d})$, i.e. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is defined to be the vector product of four vectors.

4.6.4 Geometrical interpretation of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

i. The vectors $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is perpendicular to the vectors $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$.

ii. The vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$.

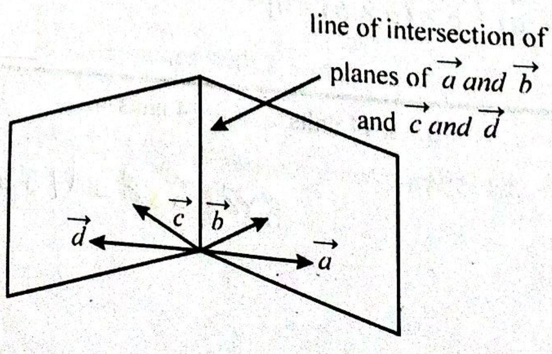
ii. The vector $\vec{c} \times \vec{d}$ is perpendicular to $\vec{c}$ and $\vec{d}$.

Thus

a. From (i) and (ii) we see that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is coplanar with $\vec{a}$ and $\vec{b}$ as same vector $\vec{a} \times \vec{b}$ is perpendicular to $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and $\vec{a}$ and $\vec{b}$.

b. From (i) and (iii), we see that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is coplanar with $\vec{c}$ and $\vec{d}$ as same vector $\vec{c} \times \vec{d}$ is perpendicular to $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and $\vec{c}$ and $\vec{d}$.

Hence, from (a) and (b) we conclude that the vector $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is a vector parallel to the line of intersection of planes of $\vec{a}$ and $\vec{b}$ and $\vec{c}$ and $\vec{d}$.



4.5 Evaluation of vector products of four vectors $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be any four non zero vectors. Let $\vec{m} = \vec{a} \times \vec{b}$ then

$$
\begin{aligned}
(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \vec{m} \times (\vec{c} \times \vec{d}) \\
&= \vec{c}(\vec{m} \cdot \vec{d}) - \vec{d}(\vec{m} \cdot \vec{c}) \\
&= \vec{c}\{(\vec{a} \times \vec{b}) \cdot \vec{d}\} - \vec{d}\{((\vec{a} \times \vec{b}) \cdot \vec{c}\} \\
(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \vec{c}[\vec{a}\ \vec{b}\ \vec{d}] - \vec{d}[\vec{a}\ \vec{b}\ \vec{c}] \qquad \ldots (1)
\end{aligned}
$$

Again, let $\vec{n} = \vec{c} \times \vec{d}$

$$
\begin{aligned}
(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= (\vec{a} \times \vec{b}) \times \vec{n} \\
&= \vec{b}(\vec{a} \cdot \vec{n}) - \vec{a}\,(\vec{b} \cdot \vec{n}) \\
&= \vec{b}\{\vec{a}\,(\vec{c} \times \vec{d})\} - \vec{a}\{\vec{b}\,(\vec{c} \times \vec{d})\} \\
\therefore (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \vec{b}[\vec{a}\ \vec{c}\ \vec{d}] - \vec{a}[\vec{b}\ \vec{c}\ \vec{d}] \qquad \ldots (2)
\end{aligned}
$$

Any one of equations (1) and (2) gives the evaluation of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$. We use (1) or (2) depending on the result required.

Remarks

1. Equation (1) and (2) show that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is coplanar with the vectors $\vec{a}$ and $\vec{b}$ and also with $\vec{c}$ and $\vec{d}$. Thus only possible case for these two conditions to satisfy is that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ must be parallel with the line of intersection of planes of $\vec{a}$ and $\vec{b}$ and $\vec{c}$ and $\vec{d}$ which is coplanar with both of the planes.

2. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the coplanar then each of the scalar triple products in (1) and (2) will be zero. So, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0$.

   Thus, $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ will be coplanar if $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0$.

3. If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors, then $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, so from equation (1) and (2), we have

$$\vec{c}[\vec{a}\ \vec{b}\ \vec{d}] - \vec{d}\,[\vec{a}\ \vec{b}\ \vec{c}] = \vec{b}[\vec{a}\ \vec{c}\ \vec{d}] - \vec{a}\,[\vec{b}\ \vec{c}\ \vec{d}]$$

or, $$\vec{d}\,[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\,[\vec{b}\ \vec{c}\ \vec{d}] - \vec{b}[\vec{a}\ \vec{c}\ \vec{d}] + \vec{c}[\vec{a}\ \vec{b}\ \vec{d}]$$

$$\therefore \vec{d} = \frac{[\vec{b}\ \vec{c}\ \vec{d}]}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{a} + \frac{[\vec{c}\ \vec{a}\ \vec{d}]}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{b} + \frac{[\vec{a}\ \vec{b}\ \vec{d}]}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{c}$$

Which shows that if $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors, then any space vector $\vec{d}$ can be written as liner combination of $\vec{a}, \vec{b}, \vec{c}$.

We have been using this fact for $\hat{i}, \hat{j}$ and $\vec{k}$ as any vector $\vec{a} = (a_1, a_2, a_3)$ can be written as

... are non coplanar vectors.

4.7 Reciprocal System of Vectors

Let $\vec{a}, \vec{b}, \vec{c}$ be any three non coplanar vectors, i.e., $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$.

The vectors defined by

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \text{ which is perpendicular to } \vec{b} \text{ and } \vec{c}.$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \text{ which is perpendicular to } \vec{c} \text{ and } \vec{a}.$$

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \text{ which is perpendicular to } \vec{a} \text{ and } \vec{b}.$$

are called reciprocal system of vectors corresponding to $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

4.7.1 Properties of reciprocal system of vectors

1. $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$

Proof

We have, $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$\therefore \quad \vec{a} \cdot \vec{a}' = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 1$$

Similarly, we can prove $\vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$

2. $\vec{a} \cdot \vec{b}' = \vec{b} \cdot \vec{c}' = \vec{c} \cdot \vec{a}' = 0$

Proof

We have, $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$\therefore \quad \vec{a} \cdot \vec{b}' = \frac{\vec{a} \cdot (\vec{c} \times \vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{[\vec{a}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{0}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$\vec{a} \cdot \vec{b} = 0$

Similarly, we can prove $\vec{b} \cdot \vec{c}' = \vec{c} \cdot \vec{a}' = 0$.

3. If $\vec{a}', \vec{b}', \vec{c}'$ is a reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ then $\vec{a}', \vec{b}', \vec{c}'$ is also reciprocal system to $\vec{a}, \vec{b}, \vec{c}$.

Proof

Let $\vec{a}', \vec{b}', \vec{c}'$ be the reciprocal system of vectors corresponding to $\vec{a}, \vec{b}$ and $\vec{c}$.

By definition.

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

To show $\vec{a}, \vec{b}, \vec{c}$ are reciprocal system of $\vec{a}', \vec{b}', \vec{c}'$, we need to show that,

$$\vec{a} = \frac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\ \vec{b} = \frac{\vec{c}' \times \vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\ \vec{c} = \frac{\vec{a}' \times \vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$$

Now,

$$\vec{b}' \times \vec{c}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} \times \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{\vec{a}\,[\vec{c}\ \vec{a}\ \vec{b}] - \vec{b}\,[\vec{c}\ \vec{a}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{\vec{a}\,[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}, \qquad \because [\vec{c}\,\vec{a}\,\vec{a}] = 0 \text{ and } [\vec{c}\,\vec{a}\,\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]$$

$$\therefore\ \vec{b}' \times \vec{c}' = \frac{\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} \quad \ldots.(1)$$

Again,

$$\vec{a}' \cdot (\vec{b}' \times \vec{c}') = \frac{(\vec{b} \times \vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]} \cdot \frac{\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{[\vec{b}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$\therefore\ \vec{a}' \cdot (\vec{b}' \times \vec{c}') = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]} \quad \ldots.(2)$$

Dividing (1) by (2)

$$\frac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']} = \vec{a}$$

$$\therefore\ \vec{a} = \frac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$$

Which shows that $\vec{a}'$ is reciprocal of $\vec{a}$. Similarly, we can prove for $\vec{b}'$ and $\vec{c}'$.

Remarks

1. From equation (2), we see that

$$[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a}'\ \vec{b}'\ \vec{c}'] = 1$$

2. The reciprocal system of $\hat{i}, \hat{j}, \hat{k}$ are $\hat{i}, \hat{j}, \hat{k}$ (verify!).

Example 7. Show that $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$.

Solution

We have

$$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\} \quad \ldots(1)$$

Now, $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = \vec{c}[\vec{b}\ \vec{c}\ \vec{a}] - \vec{b}[\vec{c}\ \vec{c}\ \vec{a}]$

$= \vec{c}[\vec{a}\ \vec{b}\ \vec{c}] - 0$

$\therefore\ (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = \vec{c}[\vec{a}\ \vec{b}\ \vec{c}]$

Substituting in (1),

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\} &= (\vec{a}\times\vec{b})\cdot\{\vec{c}\,[\vec{a}\ \vec{b}\ \vec{c}]\}\\ &= [\vec{a}\ \vec{b}\ \vec{c}]\,\{(\vec{a}\times\vec{b})\cdot\vec{c}\}\\ &= [\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a}\ \vec{b}\ \vec{c}]\\ &= [\vec{a}\ \vec{b}\ \vec{c}]^2\end{aligned}$$

$\therefore\ (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\} = [\vec{a}\ \vec{b}\ \vec{c}]^2.$

Remark

1. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, then $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = 0$, shows that $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are coplanar vectors.
2. If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar, then $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are also non coplanar.
   i.e., $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2 \neq 0$.
   So, $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are non coplanar.

Example 8. Prove that $[\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}] = [\vec{a}\ \vec{b}\ \vec{e}]\,[\vec{f}\ \vec{c}\ \vec{d}] - [\vec{a}\ \vec{b}\ \vec{f}]\,[\vec{e}\ \vec{c}\ \vec{d}]$.

Solution

We have

$$\begin{aligned}[\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}] &= (\vec{a}\times\vec{b})\cdot\{(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})\}\\ &= (\vec{a}\times\vec{b})\cdot\{\vec{e}[\vec{c}\,\vec{d}\,\vec{f}] - \vec{f}[\vec{c}\ \vec{d}\ \vec{e}]\\ &= \{(\vec{a}\times\vec{b})\cdot\vec{e}\}\,[\vec{f}\ \vec{c}\ \vec{d}] - \{(\vec{a}\times\vec{b})\cdot\vec{f}\}\,[\vec{e}\ \vec{c}\ \vec{d}]\\ &= [\vec{a}\ \vec{b}\ \vec{e}]\,[\vec{f}\ \vec{c}\ \vec{d}] - [\vec{a}\ \vec{b}\ \vec{f}]\,[\vec{e}\ \vec{c}\ \vec{d}]\end{aligned}$$

$\therefore\ [\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}] = [\vec{a}\ \vec{b}\ \vec{e}]\,[\vec{f}\ \vec{c}\ \vec{d}] - [\vec{a}\ \vec{b}\ \vec{f}]\,[\vec{e}\ \vec{c}\ \vec{d}]$

Example 9. Show that $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0.$

Solution

We have,

$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})=(\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d})-(\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})$ ....(1)

$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=(\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$ ....(2)

and $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$ ....(3)

Adding (1), (2) and (3)

$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\cdot\vec{a})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})=0$, note that $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$.

$\therefore\ (\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0.$

Example 10. Show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})=-2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}.$

Solution

We have,

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{c}[\vec{a}\ \vec{b}\ \vec{d}]-\vec{d}[\vec{a}\ \vec{b}\ \vec{c}]$ ....(1)

$(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})=\vec{a}[\vec{b}\ \vec{c}\ \vec{d}]-\vec{d}[\vec{b}\ \vec{c}\ \vec{a}]$ ....(2)

$(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})=\vec{a}[\vec{c}\ \vec{b}\ \vec{d}]-\vec{c}[\vec{a}\ \vec{b}\ \vec{d}]$

i.e., $(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})=-\vec{a}[\vec{b}\ \vec{c}\ \vec{d}]-\vec{c}[\vec{a}\ \vec{b}\ \vec{d}]$ ....(3)

Adding (1), (2) and (3),

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})=-2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}.$

Example 11. Find the set of reciprocal system of vectors for $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}-2\hat{k}$, $\vec{c}=-\hat{i}+2\hat{j}+2\hat{k}$.

Solution

Given, $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}-2\hat{k}$, $\vec{c}=-\hat{i}+2\hat{j}+2\hat{k}$.

Now, $\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix}$

$=\hat{i}(-2+4)-\hat{j}(2-2)+\hat{k}(2-1)$

$\therefore\ \vec{b}\times\vec{c}=2\hat{i}+0\hat{j}+\hat{k}$ ....(1)

Also, $[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})$

$=(2\hat{i}+3\hat{j}-\hat{k})\cdot(2\hat{i}+0\hat{j}+\hat{k})=4-1$

$\therefore\ [\vec{a}\ \vec{b}\ \vec{c}]=3$ ....(2)

Again, $\vec{c}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 2\\ 2 & 3 & -1\end{vmatrix}$

$=\hat{i}(-2-6)-\hat{j}(1-4)+\hat{k}(-3-4)$

$\therefore\ \vec{c}\times\vec{a}=-8\hat{i}+3\hat{j}-7\hat{k}$ ....(3)

And, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ 1 & -1 & -2\end{vmatrix}$

$=\hat{i}(-6-1)-\hat{j}(-4+1)+\hat{k}(-2-3)$

$\therefore\ \vec{a}\times\vec{b}=-7\hat{i}+3\hat{j}-5\hat{k}$ ....(4)

For reciprocal of $\vec{a}$

From (1) and (2),

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{1}{3}(2\hat{i} + \hat{k}) = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{k}$$

For reciprocal of $\vec{b}$

From (3) and (2)

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{1}{3}(-8\hat{i} + 3\hat{j} - 7\hat{k}) = \left(\frac{-8}{3}\right)\hat{i} + \hat{j} - \frac{7}{3}\hat{k}$$

For reciprocal of $\vec{c}$

From (2) and (4)

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{1}{3}(-7\hat{i} + 3\hat{j} - 5\hat{k}) = \frac{-7}{3}\hat{i} + \hat{j} - \frac{5}{3}\hat{k}.$$

Exercise 4.2

1. For the vectors $\vec{a} = -\hat{i} + 4\hat{j} - 3\hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{c} = 3\hat{i} + 8\hat{j} - 5\hat{k}$, $\vec{d} = -3\hat{i} + 2\hat{j} + \hat{k}$, verify that:
   a. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{c}[\vec{a}\ \vec{b}\ \vec{d}] - \vec{d}\,[\vec{a}\ \vec{b}\ \vec{c}]$
   b. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$
2. Show that $[\vec{a} \times \vec{b}\ \ \vec{c} \times \vec{d}\ \ \vec{e} \times \vec{f}] = [\vec{a}\ \vec{b}\ \vec{d}]\,[\vec{c}\ \vec{e}\ \vec{f}] - [\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{d}\ \vec{e}\ \vec{f}]$.
3. Prove that $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) + (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{c}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{c})$
4. For any four vectors show that $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$.
5. Prove that $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})] = (\vec{b} \cdot \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{a} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - (\vec{a} \cdot \vec{b})(\vec{c} \times \vec{d})$.
6. Prove that $2(\vec{c} \times \vec{d}) \times (\vec{a} \times \vec{b}) = \begin{vmatrix} \vec{a} & \vec{b} & -\vec{c} & -\vec{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$.
7. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar and $\vec{a}$ is not parallel to $\vec{b}$, prove that:
   $$\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix} \vec{c} = \begin{vmatrix} \vec{c} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{c} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix} \vec{a} + \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{c} \cdot \vec{b} \end{vmatrix} \vec{b}.$$
8. Find set of reciprocal system of vectors:
   i. $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{c} = \hat{i} - \hat{j} - 2\hat{k}$.
   ii. $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{c} = 3\hat{i} - 4\hat{j} + 2\hat{k}$.
9. Show that $(\vec{a} + \vec{b}) \cdot \vec{a}' + (\vec{b} + \vec{c}) \cdot \vec{b}' + (\vec{c} + \vec{a}) \cdot \vec{c}' = 3$.

Answers

8. i. $\vec{a}' = -\hat{i} + \hat{j} - \hat{k}$, $\vec{b}' = \frac{2}{5}\hat{i} + \frac{1}{5}\hat{k}$, $\vec{c}' = \frac{-4}{5}\hat{i} + \hat{j} - \frac{7}{5}\hat{k}$
   ii. $\vec{a}' = -\frac{8}{35}\hat{i} + \frac{1}{5}\hat{j} + \frac{2}{35}\hat{k}$, $\vec{b}' = \frac{2}{35}\hat{i} - \frac{1}{5}\hat{j} - \frac{17}{35}\hat{k}$, $\vec{c}' = \frac{1}{5}\hat{i} - \frac{1}{5}\hat{j} + \frac{1}{5}\hat{k}$.