# Summary Robot Dynamics

Adrian Hartmann

HS 2020 - Prof. Hutter

## 0 Basics

Vector from point B to P in frame $\mathcal{A}$: ${}_{\mathcal{A}}r_{BP}$

Reference coordinate System $\mathcal{A}$: $(e_x^{\mathcal{A}}, e_y^{\mathcal{A}}, e_z^{\mathcal{A}})$

**Cartesian Coordinates**

Stacked parameters of Position: $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Postion Vector: ${}_{\mathcal{A}}r = x e_x^{\mathcal{A}} + y e_y^{\mathcal{A}} + z e_z^{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

**Cylindric Coordinates**

$$\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}; \quad {}_{\mathcal{A}}r = \begin{pmatrix} \rho\, cos\theta \\ \rho\, sin\theta \\ z \end{pmatrix}$$

**Spherical Coordinates**

$$\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}; \quad {}_{\mathcal{A}}r = \begin{pmatrix} r\, cos\theta\, sin\phi \\ r\, sin\theta\, sin\phi \\ r\, cos\phi \end{pmatrix}$$

## 1 Kinematics

### 1.1 Linear Velocity

The Velocity of point B relative to point A is given by: ${}_{}\dot{r}_{AB}$. There exists a linear mapping $E_p(\chi)$ between velocities $\dot{r}$ and the derivatives of the representation $\dot{\chi}_P$:

$$\dot{r} = E_P(\chi_P)\dot{\chi}_P$$

$$\dot{\chi}_P = E_P^{-1}(\chi_P)\dot{r}$$

For **Cartesian** this is the identity: $E_{Pc}(\chi_{Pc}) = E_{Pc}^{-1}(\chi_{Pc}) = \mathbb{1}$

In **Cylindrical** we get:

$$E_{Pz} = \frac{\partial r(\chi)}{\partial \chi} = \begin{pmatrix} c\theta & -\rho\, s\theta & 0 \\ s\theta & \rho\, c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad E_{Pz}^{-1} = \begin{pmatrix} c\theta & s\theta & 0 \\ -\frac{1}{\rho} s\theta & \frac{1}{\rho} c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For **Spherical**:

$$E_{Ps} = \begin{pmatrix} c\theta\, s\phi & -r\, s\theta\, s\phi & r\, c\theta\, c\phi \\ s\theta\, s\phi & r\, c\theta\, s\phi & r\, s\theta\, c\phi \\ c\phi & 0 & -r\, s\phi \end{pmatrix}$$

$$E_{Ps}^{-1} = \begin{pmatrix} c\theta\, s\phi & s\theta s\phi & c\phi \\ -s\theta/(r\, s\phi) & c\theta/(r\, s\phi) & 0 \\ \frac{1}{r} c\theta\, c\phi & \frac{1}{r} s\theta\, c\phi & -\frac{1}{r} s\phi \end{pmatrix}$$

### 1.2 Rotation

Orientation of frame $\mathcal{B}$ with reference to frame $\mathcal{A}$:

$$\phi_{\mathcal{AB}} \in SO(3)$$

**Important:** There is no numerical equivalent to a position such as "angular position". Instead the orientation can be parametrized in several ways.

**Passive Rotation:** Passive rotations are transformations between different coordinate frames. ${}_{\mathcal{A}}u = C_{\mathcal{AB}} \cdot {}_{\mathcal{B}}u$

**Active Rotations:** Active rotations contain an operator (i.e. $R \in \mathbb{R}^{3\times 3}$) to rotate a vector in the same frame. ${}_{\mathcal{A}}v = R \cdot {}_{\mathcal{A}}u$

#### 1.2.1 Rotation Matrix

Mapping Coordinates P from frame $\mathcal{B}$ to $\mathcal{A}$:

$$\begin{aligned} {}_{\mathcal{A}}r_{AP} &= \begin{pmatrix} {}_{\mathcal{A}}e_x^{\mathcal{B}} & {}_{\mathcal{A}}e_y^{\mathcal{B}} & {}_{\mathcal{A}}e_z^{\mathcal{B}} \end{pmatrix} \cdot {}_{\mathcal{B}}r_{AP} \\ &= C_{\mathcal{AB}} \cdot {}_{\mathcal{B}}r_{AP} \end{aligned}$$

This matrix is orthogonal: $C_{\mathcal{BA}} = C_{\mathcal{AB}}^{-1} = C_{\mathcal{AB}}^T$

#### 1.2.2 Elementary Rotations

| Around X-Axis | Around Y-Axis | Around Z-Axis |
|---|---|---|
| $\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{pmatrix}$ | $\begin{pmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -sin\phi & 0 & cos\phi \end{pmatrix}$ | $\begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |

#### 1.2.3 Homogeneous Transformation

Combined Translation and Rotation for frames with a offset. The vector ${}_{\mathcal{A}}r_{AB}$ points from the origin of frame $\mathcal{A}$ to the origin of the $\mathcal{B}$ in frame $\mathcal{A}$. $T_{\mathcal{AB}}$ transforms a **point P** from frame $\mathcal{B}$ to $\mathcal{A}$.

$$\begin{pmatrix} {}_{\mathcal{A}}r_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} C_{\mathcal{AB}} & {}_{\mathcal{A}}r_{AB} \\ 0_{1\times 3} & 1 \end{bmatrix}}_{T_{\mathcal{AB}}} \begin{pmatrix} {}_{\mathcal{B}}r_{BP} \\ 1 \end{pmatrix}$$

With inverse: $T_{\mathcal{AB}}^{-1} = \begin{bmatrix} C_{\mathcal{AB}}^T & \overbrace{-C_{\mathcal{AB}}^T\, {}_{\mathcal{A}}r_{AB}}^{{}_{\mathcal{B}}r_{BA}} \\ 0_{1\times 3} & 1 \end{bmatrix}$

### 1.3 Representation of Rotations

A $3 \times 3$ rotation matrix contains 9 parameters. These parameters, however, aren't independent. With the orthogonality condition only 3 parameters must be defined, such as **Euler Angles**.

#### 1.3.1 Euler Angles

Euler Angles describe a rotation as a sequence of 3 elementary rotations. When the first and the last rotations are made around the same axis, the parametrisation is called **proper Euler angles**. For three different Axis we typically refer to *Tait-Bryan*, *Cardan* or *roll-pitch-yaw* angles.

#### 1.3.2 Angle Axis

$\chi_{R,\text{AngleAxis}} = \begin{pmatrix} \theta \\ \vec{n} \end{pmatrix}$ **Rotation Vector:** $\varphi = \theta \cdot \vec{n}$

#### 1.3.3 Unit Quaternion

A non-minimal representation which doesn't suffer from singularity is provided by unit quaternions, aka **Euler parameters**.
Considering a rotational vector $\varphi \in \mathbb{R}^3$, a unit quaternion is defined by:

$$\chi_{R,quat} = \xi = \begin{pmatrix} \xi_0 \\ \check{\xi} \end{pmatrix} = \begin{pmatrix} cos(\frac{\theta}{2}) \\ sin(\frac{\theta}{2})\vec{n} \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

The first parameter $\xi_0$ is called the real part of the quaternion, the latter $\check{\xi}$ the imaginary part. The **unit** quaternion fulfils the constraint:

$$\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

Given a rotation matrix with entries from $c_{11}$ up to $c_{44}$, the corresponding quaternion are:

$$\chi_{R,quat} = \xi_{\mathcal{AD}} = \frac{1}{2} \begin{pmatrix} \sqrt{c_{11}+c_{22}+c_{33}+1} \\ sgn(c_{32}-c_{23})\sqrt{c_{11}-c_{22}-c_{33}+1} \\ sgn(c_{13}-c_{31})\sqrt{c_{22}-c_{33}-c_{11}+1} \\ sgn(c_{21}-c_{12})\sqrt{c_{33}-c_{11}-c_{22}+1} \end{pmatrix}$$

with $sgn(x) = 1$ for $x \geq 0$ and $sgn(x) = -1$ for $x < 0$

The Corresponding Rotation Matrix can be calculated with:

$$\begin{aligned} C_{\mathcal{AD}} &= \mathbb{1}_{3\times 3} + 2\xi_0\,[\check{\xi}]_\times + 2[\check{\xi}]_\times^2 \\ &= (2\xi_0^2 - 1)\,\mathbb{1}_{3\times 3} + 2\xi_0[\check{\xi}]_\times + 2\check{\xi}\check{\xi}^T \\ &= \begin{pmatrix} \xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2 & 2\xi_1\xi_2-2\xi_0\xi_3 & 2\xi_0\xi_2+2\xi_1\xi_3 \\ 2\xi_0\xi_3+2\xi_1\xi_2 & \xi_0^2-\xi_1^2+\xi_2^2-\xi_3^2 & 2\xi_2\xi_3-2\xi_0\xi_1 \\ 2\xi_1\xi_3-2\xi_0\xi_2 & 2\xi_0\xi_1-2\xi_2\xi_3 & \xi_0^2-\xi_1^2-\xi_2^2+\xi_3^2 \end{pmatrix} \end{aligned}$$

It holds that:

$$\xi = \begin{pmatrix} \xi_0 \\ \check{\xi} \end{pmatrix} \xrightarrow{inverse} \xi^{-1} = \xi^T = \begin{pmatrix} \xi_0 \\ -\check{\xi} \end{pmatrix}$$

Left Multiplication:

$$\xi_{\mathcal{AB}} \otimes \xi_{\mathcal{BC}} = \underbrace{\begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{pmatrix}_{\mathcal{AB}}}_{M_l(\xi_{\mathcal{AB}})} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_{\mathcal{BC}}$$

There exists a similar Matrix $M_r$ for right multiplications.

Direct way of rotating a Vector: (with $p({}_{\mathcal{B}}r) = \begin{pmatrix} 0 \\ {}_{\mathcal{B}}r \end{pmatrix}$)

$$p({}_{\mathcal{A}}r) = \xi_{\mathcal{AB}} \otimes p({}_{\mathcal{B}}r) \otimes \xi_{\mathcal{AB}}^T = M_l(\xi_{\mathcal{AB}})M_r(\xi_{\mathcal{AB}}^T)p({}_{\mathcal{B}}r)$$

### 1.4 Angular Velocity

Consider Frame $\mathcal{B}$ which is moving with respect to fixed Frame $\mathcal{A}$. The angular velocity of the rotation of $\mathcal{B}$ w.r.t $\mathcal{A}$ is ${}_{\mathcal{A}}\omega_{\mathcal{AB}}$.
It can be shown that:

$$[{}_{\mathcal{A}}\omega_{\mathcal{AB}}]_\times = \dot{C}_{\mathcal{AB}} \cdot C_{\mathcal{AB}}^T$$

with $[{}_{\mathcal{A}}\omega_{\mathcal{AB}}]_\times = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$ and ${}_{\mathcal{A}}\omega_{\mathcal{AB}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

Angular Velocities can be transformed as normal vectors:

$${}_{\mathcal{B}}\omega_{\mathcal{AB}} = C_{\mathcal{BA}} \cdot {}_{\mathcal{A}}\omega_{\mathcal{AB}}$$

the product matrix is transformed by:

$$[{}_{\mathcal{B}}\omega_{\mathcal{AB}}]_\times = C_{\mathcal{BA}} \cdot [{}_{\mathcal{A}}\omega_{\mathcal{AB}}]_\times \cdot C_{\mathcal{AB}}$$

Angular Velocities in the same Frame can be added:

$${}_{\mathcal{I}}\omega_{\mathcal{IE}} = {}_{\mathcal{I}}\omega_{\mathcal{I}0} + \dots + {}_{\mathcal{I}}\omega_{n\mathcal{E}}$$

#### 1.4.1 Time Derivative of Rotation Parameters

Similarly to the linear velocity we can define:

$${}_{\mathcal{A}}\omega_{\mathcal{AB}} = E_R(\chi_R) \cdot \dot{\chi}_R$$

i.e. quaternions: $E_R = 2H(\xi) = 2\left[-\check{\xi} \quad [\check{\xi}]_\times + \xi_0 \mathbb{1}_{3\times 3}\right] \in \mathbb{R}^{3\times 4}$

### 1.5 Velocity in Moving Bodies

| | |
|---|---|
| $v_P$ | the absolute Velocity of point P |
| $a_P = \dot{v}_P$ | (absolute) acceleration of P |
| $\Omega_{\mathcal{B}} = \omega_{\mathcal{AB}}$ | (absolute) angular velocity of body B |
| $\Psi_{\mathcal{B}} = \dot{\Omega}_{\mathcal{B}}$ | (absolute) angular acceleration of body B |

We can write the Position P as:

$${}_{\mathcal{A}}r_{AP} = {}_{\mathcal{A}}r_{AB} + C_{\mathcal{AB}} \cdot {}_{\mathcal{B}}r_{BP}$$

Which can be differentiated to:

$${}_{\mathcal{A}}\dot{r}_{AP} = {}_{\mathcal{A}}\dot{r}_{AB} + {}_{\mathcal{A}}\omega_{\mathcal{AB}} \times {}_{\mathcal{A}}r_{BP}$$

This is the famous **Rigid Body Formulation** for velocities:

$$v_P = v_B + \Omega \times r_{BP}$$

For Accelerations: $a_P = a_B + \Psi \times r_{BP} + \Omega \times (\Omega \times r_{BP})$

In case a moving system $\mathcal{B}$ is used for representation, the **Euler Differentiation rule** must be applied (with a non moving system $\mathcal{A}$):

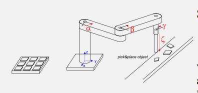
$$\begin{aligned} {}_{\mathcal{B}}v_P &= C_{\mathcal{BA}} \cdot \frac{d}{dt}(C_{\mathcal{AB}} \cdot {}_{\mathcal{B}}r_{AP}) \\ &= {}_{\mathcal{B}}\dot{r}_{AP} + {}_{\mathcal{B}}\omega_{\mathcal{AB}} \times {}_{\mathcal{B}}r_{AP} \end{aligned}$$

### 1.6 Kinematics of Systems of Bodies

#### 1.6.1 Generalized Coordinates and Joint Configurations

The configuration of a root such as a manipulator can be described by the **generalized coordinate vector**: $q = \begin{pmatrix} q_1 & \dots & q_n \end{pmatrix}^T$ The choice of q isn't unique, but it has to completely describe the configuration of the system (q const. $\Rightarrow$ robot can't move).

Example 2.8.1: Generalized Coordinates and Joint Configuration




The generalized coordinates of a SCARA robot arm are:

$$\mathbf{q} = \begin{pmatrix} \alpha & \beta & \gamma & \zeta \end{pmatrix}^T, \qquad (2.132)$$

with the rotation angles $\alpha$, $\beta$, and $\gamma$ around the global vertical axis and the linear displacement $\zeta$.

#### 1.6.2 Task-Space Coordinates

**End-Effector Configuration Parameters**
The position $r_e \in \mathbb{R}^3$ and rotation $\phi_e \in SO(3)$ of a frame w.r.t a base can be parametrized by:

$$\chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_m \end{pmatrix} \in \mathbb{R}^m$$

**Operational Space Coordinates** The end-effector operates in the operational space, which depends on the geometry and structure of the arm. It can be described with:

$$\chi_o = \begin{pmatrix} \chi_{oP} \\ \chi_{oR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{m_o} \end{pmatrix}$$

where $\chi_1 \dots \chi_{m_o}$ are *independent* operational space coordinates. It can be seen as a **minimal selection** of the above end-effector parameters.

#### 1.6.3 Forward Kinematics

Forward kinematics map from joint coordinates $q$ to the end-effector configuration $\chi_e$: $\chi_e = \chi_e(q)$
This relation can be obtained through the transformations of each link:

$$T_{\mathcal{IE}}(q) = T_{\mathcal{I}0} \left(\prod_{k=1}^{n_j} T_{k-1,k}\right) T_{n_j\mathcal{E}} = \begin{bmatrix} C_{\mathcal{IE}}(q) & {}_{\mathcal{I}}r_{IE}(q) \\ 0_{1\times 3} & 1 \end{bmatrix}$$

#### 1.6.4 Differential Kinematics and Analytical Jacobian

To linearise the forward kinematics we use a first order approximation:

$$\Delta\chi_e \approx J_{eA}(q)\,\Delta q, \quad J_{eA}(q) = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \dots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \dots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$$

It results in an exact relation between velocities:

$$\dot{\chi}_e = J_{eA}(q)\dot{q}$$

Literature often talk about **position** and **rotation** Jacobians:

$$J_{eA} = \begin{bmatrix} J_{eA_P} \\ J_{eA_R} \end{bmatrix} = \begin{bmatrix} \frac{\partial \chi_{eP}}{\partial q} \\ \frac{\partial \chi_{eR}}{\partial q} \end{bmatrix}$$

### 1.6.5 Geometric or Basic Jacobian

The geometric or basic Jacobian relates the generalized velocity $\dot{q}$ to the end-effector velocity (linear $v_e$ and angular $\omega_e$):

$$w_e = \begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = J_{e0}(q) \cdot \dot{q}$$

Note: In the most general cases $J_{e0}$ has dimension $6 \times n_j$ and has Frame $\mathcal{A}$ as a basis (like the velocity).

From the velocities $w_{\mathcal{C}} = w_{\mathcal{B}} + w_{\mathcal{BC}}$ we can derive, that geometric Jacobians can simply be added (in the same reference):

$$_{\mathcal{A}}J_{\mathcal{C}} = {_{\mathcal{A}}J_{\mathcal{B}}} + {_{\mathcal{A}}J_{\mathcal{BC}}}$$

**Geometric Jacobian:**

$$_{\mathcal{I}}J_{e0} = \begin{bmatrix} _{\mathcal{I}}J_{e0_P} \\ _{\mathcal{I}}J_{e0_R} \end{bmatrix} = \begin{bmatrix} _{\mathcal{I}}n_1 \times {_{\mathcal{I}}r_{1(n+1)}} & \cdots & _{\mathcal{I}}n_n \times {_{\mathcal{I}}r_{n(n+1)}} \\ _{\mathcal{I}}n_1 & \cdots & _{\mathcal{I}}n_n \end{bmatrix}$$

where $n_k$ represents the rotation axis of joint k such that:

$$\omega_{(k-1)k} = n_k \dot{q}_k$$

and $r_{1(n+1)} \cdots r_{n(n+1)}$ represent the position vector from the joint $1 \dots n$ to the end-effector.
Don't forget to transform to inertial frame!

For **prismatic Joints** the Position part $(n \times r)$ is an unit-vector $\xi_k$ in joint direction. The Rotational part $(n_i)$ is obviously zero.

Mapping from Analytic to Geometric Jacobian, it holds that:

$$J_{e0}(q) = E_e(\chi) J_{eA}(q)$$

with $E_e(\chi)$ containing $E_P$ and $E_R$ as diagonal elements.

## 1.7 Kinematic Control Methods

### 1.7.1 Inverse Differential Kinematics

The Jacobian $J_{e0}(q)$ performs a simple mapping from joint space to end-effector velocity.

$$w_e = J_{e0}\dot{q}$$

To solve the inverse problem, we use take the **pseudo-inverse** $J_{e0}^+$ of the Jacobian.

$$\dot{q} = J_{e0}^+ \cdot w_e^*$$

By taking the Moore-Penrose pseudo inverse, the solution $\dot{q} = J_{e0}^+ \cdot w_e^*$ minimises the least square error $||w_e^* - J_{e0}\dot{q}||^2$

**Moore-Penrose Inverse**

$$A^+ = A^T(AA^T)^{-1} \text{ right inverse (full row rank)}$$

$$A^+ = (A^TA)^{-1}A^T \text{ left inverse (full col rank)}$$

Note: For close to singular configurations, $J_{e0}$ becomes badly conditioned, what causes large joint velocities for just a small end-effector velocity. This can be handled by using a **damped solution**. $\dot{q} = J_{e0}^T(J_{e0}J_{e0}^T + \lambda^2 \mathbb{1})^{-1} w_e^*$

**Redundancy**
For a Robot that has more joints than DOF $(rank(J_{e0}) < n)$, the configuration is called *redundant*. Like previous, we can take the pseudo inverse:

$$\dot{q} = J_{e0}^T(J_{e0}J_{e0}^T)^{-1} \cdot w_e^* = J_{e0}^+ \cdot w_e^*$$

redundancy implies, that there are infinite additional solutions:

$$\dot{q} = J_{e0}^+ \cdot w_e^* + N\dot{q}_0$$

with $N = \mathcal{N}(J_{e0})$ as null-space projection matrix, fulfilling $J_{e0}N = 0$. Thus, we can choose arbitrary $\dot{q}_0$ without changing the velocity $w_e^*$.
The simplest method for the **null-space projection** is:

$$N = \mathbb{1} - J_{e0}^+ J_{e0}$$

### 1.7.2 Multi-task Inverse Differential Kinematic Control

For multiple tasks (same priority) $task_i := \{J_i, w_i^*\}$ we can calculate the velocity:

$$\dot{q} = \underbrace{\begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^+}_{\bar{J}} \cdot \underbrace{\begin{bmatrix} w_1^* \\ \vdots \\ w_{n_t}^* \end{bmatrix}}_{\bar{w}}$$

For weighted tasks we could use a weighted pseudo inverse: $\bar{J}^{+W} = (\bar{J}^TW\bar{J})^{-1}\bar{J}^TW$ with weight $W = diag(w_1, \dots, w_m)$

**Multitask Prioritisation**
An approach for prioritisationing tasks (descending priority) is to use consecutive null-space projections.

Using the solution for task 1 $\dot{q} = J_1^+ w_1^* + N_1\dot{q}_0$, we can derive a term for $q_0$:

$$w_2 = J_2\dot{q} = J_2(J_1^+ w_1^* + N_1\dot{q}_0)$$

$$\iff \dot{q}_0 = (J_2N_1)^+(w_2^* - J_2J_1^+w_1^*)$$

Substitution in the first solution for task 1 gives:

$$\dot{q} = J_1^+ w_1^* + N_1(J_2N_1)^+(w_2^* - J_2J_1^+w_1^*)$$

For $n_t$ tasks this can be written recursively:

$$\dot{q} = \sum_{i=1}^{n_t} \bar{N}_i \dot{q}_i \text{ with } \dot{q}_i = (J_i\bar{N}_i)^+ \left( w_i^* - J_i \sum_{k=1}^{i-1} \bar{N}_k \dot{q}_k \right)$$

with $\bar{N}_i$ the null space projection of the stacked J, $\bar{J}_i = [J_1^T \dots J_{i-1}^T]^T$

### 1.7.3 Inverse Kinematics (Numerical Solution)

The goal of inverse Kinematics is to find the joint configuration for a given end-effector configuration $\chi_e^*$: $q = q(\chi_e^*)$
We can solve this problem iteratively by using: $\Delta\chi_e = J_{eA}\Delta q$

**Algorithmus 1 :** Numerical Inverse Kinematics

$q \leftarrow q^0$ ; `// Start Configuration`
**while** $||\chi_e^* - \chi_e(q)|| > tol$ **do**
  $J_{eA} \leftarrow J_{eA}(q) = \frac{\partial \chi_e}{\partial q}(q)$ ; `// Evaluate (local) Jacobian`
  $J_{eA}^+ \leftarrow (J_{eA})^+$ ; `// Calculate Pseudo Inverse`
  $\Delta\chi_e \leftarrow \chi_e^* - \chi_e(q)$ ; `// Find Error Vector`
  $q \leftarrow q + J_{eA}^+\Delta\chi_e$; `// Update generalized Coordinates`
**end**

To reduce the inaccuracy for large steps $\Delta\chi_e^i$ we could simply scale each step: $q \leftarrow q + kJ_{eA}^+\Delta\chi_e, \quad 0 < k < 1$
For badly conditioned (singular) Jacobians, we use either the damped inverse or use the Jacobi-transposed method:

$$q \leftarrow q + \alpha J_{eA}^T \Delta\chi_e$$

For small enough $\alpha$ convergence can be guaranteed.

**Shortest Path rotation**
For a straight rotation along the "shortest path", we rotate along the rotation vector $\Delta\varphi$.
The rotation Matrix is given by:

$$C_{\mathcal{AB}}(\Delta\varphi) = C_{\mathcal{IA}}(\varphi^t)^T C_{\mathcal{IB}}(\varphi^*)$$

(Note that $\Delta\varphi \neq \varphi^* - \varphi^t$)

The rotation vector is the same in both frames A & B.

$$_{\mathcal{A}}\Delta\varphi = {_{\mathcal{B}}\Delta\varphi} = \text{rotVec}(C_{\mathcal{AB}})$$

Instead of mapping this vector into $\mathcal{I}$, we can derive it directly:

$$_{\mathcal{I}}\Delta\varphi = \text{rotVec}(C_{\mathcal{IB}}C_{\mathcal{IA}}^T)$$

Now we can change the update step 6 of the algorithm to:

$$q \leftarrow q + k_{p_R} {_{\mathcal{I}}J_{e0_R}^+} {_{\mathcal{I}}\Delta\varphi}$$

### 1.7.4 Trajectory Control

Pure inverse differential kinematics often drift away from the predefined path. Hence, we introduce a feedback.
For predefined position $r_e^*(t)$ and velocity $\dot{r}_e^*(t)$:

$$\dot{q}^* = J_{e0_P}^+(q^t) \cdot (\dot{r}_e^*(t) + k_{p_P}\Delta r_e^t) \quad \text{with } \Delta r_e^t = r_e^*(t) - r_e(q^t)$$

Similar for orientation $\chi_R^*(t)$ and angular velocity $\omega^*(t)$ :

$$\dot{q}^* = J_{e0_R}^+(q^t) \cdot (\omega_e^*(t) + k_{p_R}\Delta\varphi) \quad \text{with } \Delta\varphi \text{ from above.}$$

# 2 Dynamics

We formulate multi-body dynamics as:

$$M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau + J_c(q)^T F_c$$

consisting of the following elements:

| | |
|---|---|
| $M(q)$ | Generalized mass (or inertia) matrix (orthogonal) |
| $q, \dot{q}, \ddot{q}$ | Generalized position, velocity and acceleration vectors |
| $b(q,\dot{q})$ | Coriolis and centrifugal terms |
| $g(q)$ | Gravitational terms |
| $\tau$ | External generalized forces |
| $F_c$ | External cartesian forces (e.g. from contacts) |
| $J_c(q)$ | Geometric Jacobian corresponding to external forces |

## 2.1 Principle of virtual Work

$$\delta W \int_{\mathcal{B}} \delta r^T \cdot (\ddot{r}dm - dF_{ext}) = 0, \quad \forall \delta r$$

| | |
|---|---|
| $dm$ | infinitesemal mass element |
| $dF_{ext}$ | external Forces acting on element dm |
| $\ddot{r}$ | acceleration of element dm |
| $\delta r$ | virtual displacement of dm |
| $\mathcal{B}$ | Body System containing infinitesemal particles dm |

## 2.2 Newton-Euler Method

$$m \cdot \ddot{x} = \sum F_i \quad \text{and} \quad \Theta \cdot \ddot{\varphi} = \sum T_i$$

For Multi-Body System we need to cut every joint free and introduce constraining forces for every piece. This results in a system of equations with additional kinematic constraints.

## 2.3 Lagrange Method

The **Lagrangian Function** is for mechanical systems exactly the difference between the total kinetic energy $\mathcal{T}$ and the total potential energy $\mathcal{U}$.

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

**Euler-Lagrange (of the second kind):** $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \tau$
The **Hamiltonian** states the total energy: $\mathcal{H} = \mathcal{T} + \mathcal{U}$

### 2.3.1 Kinetic Energy

The kinetic energy is defined as (recall the basic formulas $E = \frac{1}{2}mv^2$ & $\frac{1}{2}J\omega^2$):

$$\mathcal{T} = \sum_{i=1}^{n_b} \left( \frac{1}{2} m_i {_{\mathcal{A}}\dot{r}_{S_i}^T} {_{\mathcal{A}}\dot{r}_{S_i}} + \frac{1}{2} {_{\mathcal{B}}\Omega_{S_i}^T} \cdot {_{\mathcal{B}}\Theta_{S_i}} \cdot {_{\mathcal{B}}\Omega_{S_i}} \right)$$

With the Jacobian relations $\dot{r}_{S_i} = J_{S_i}\dot{q}, \quad \Omega_{S_i} = J_{R_i}\dot{q}$ we can rewrite this:

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^T \underbrace{\left( \sum_{i=1}^{n_b} (J_{S_i}^T m J_{S_i} + J_{R_i}^T \Theta_{S_i} J_{R_i}) \right)}_{M(q)} \dot{q}$$

### 2.3.2 Potential Energy

Knowing $r_{S_i}$ to the Center of Mass of each body, we can calculate gravitational forces (Zero energy level can be choosen arbitrarily):

$$F_{g_i} = m_i \cdot g \cdot {_{\mathcal{I}}e_g} \Rightarrow \mathcal{U}_g = -\sum_{i=1}^{n_b} r_{S_i}^T F_{g_i}$$

Potential energy for elastic elements: $\mathcal{U}_{E_j} = \frac{1}{2}k_j \underbrace{(d(q) - d_0)}_{\text{deflection}}^2$

with i.e, the current length of the spring $d(q)$ & resting length $d_0$.

## 2.4 Projected Euler Method

Rotate Inertia Tensor: $_{\mathcal{I}}\Theta = C_{\mathcal{IB}} \cdot {_{\mathcal{B}}\Theta} \cdot C_{\mathcal{IB}}^T$

$$M = \sum_{i=1}^{n_b} ({_{\mathcal{A}}J_{S_i}^T} \cdot m \cdot {_{\mathcal{A}}J_{S_i}} + {_{\mathcal{B}}J_{R_i}^T} \cdot {_{\mathcal{B}}\Theta_{S_i}} \cdot {_{\mathcal{B}}J_{R_i}})$$

$$b = \sum_{i=1}^{n_b} ({_{\mathcal{A}}J_{S_i}^T} m {_{\mathcal{A}}\dot{J}_{S_i}} \dot{q}$$

$$+ {_{\mathcal{B}}J_{R_i}^T}({_{\mathcal{B}}\Theta_{S_i}} \cdot {_{\mathcal{B}}\dot{J}_{R_i}} \cdot \dot{q} + {_{\mathcal{B}}\Omega_{S_i}} \times {_{\mathcal{B}}\Theta_{S_i}} \cdot {_{\mathcal{B}}\Omega_{S_i}}))$$

$$g = \sum_{i=1}^{n_b} (-{_{\mathcal{A}}J_{S_i}^T} \cdot {_{\mathcal{A}}F_{g,i}})$$

### 2.4.1 External Forces & Actuation

For known Forces $F_j$ acting on the system, we can calculate the generalized forces $\tau_{F,ext}$ (due to the external force).

$$\tau_{F,ext} = \sum J_{P,j}^T F_j$$

with the translational (geometric) Jacobian of Point j (i.e. $J_e$ for end effector)
Similar for external Torques $T_j$: $\tau_{T,ext} = \sum J_{R,j}^T T_j$

## 2.5 Joint-Space Dynamic Control

**Joint Impedance Regulation**
In case of torque controlled actuators, we can get a simple PD control law for the desired(*) actuator torque:

$$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$$

This ends in a steady state offset of: $g(q) = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$
**Gravity Compensation:** To compensate for the gravity offset, we simply add an estimaed value $\hat{g}(q)$ to the control law:

$$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q}) + \hat{g}(q)$$

*Note:* $k_d$ and $k_p$ are constant for all configurations ($q$), which reduces the overall performance.
**Inverse Dynamics Control:** A simple way to get dynamic decoupling and motion control is to get estimates $\hat{M}$, $\hat{b}$ and $\hat{g}$ and select the torque with:

$$\tau^* = \hat{M}(q)\ddot{q}^* + \hat{b}(q,\dot{q}) + \hat{g}(q)$$

Then, a common approach selects the desired acceleration according to:

$$\ddot{q}^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$$

which has eigenfrequency $\omega = \sqrt{k_p}$ and Damping $D = \frac{k_d}{2\sqrt{k_p}}$
($D = 1$ critical- , $D < 1$ under-, $D > 1$ over-damped)

### 2.6 Task-Space Dynamic Control

To move to a specific point in Task-Space (Fixed Frame) we need the linear and rotational acceleration of the end-effector:

$$\dot{w}_e = \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} = J_e \ddot{q} + \dot{J}_e \dot{q}$$

#### 2.6.1 Multi-task

Similar to kinematics, we can fulfill multiple tasks:

$$\ddot{q} = \begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^+ \left( \begin{bmatrix} \dot{w}_1 \\ \vdots \\ \dot{w}_{n_t} \end{bmatrix} - \begin{bmatrix} \dot{J}_1 \\ \vdots \\ \dot{J}_{n_t} \end{bmatrix} \dot{q} \right)$$

and the recursive algorithm:

$$\dot{q} = \sum_{i=1}^{n_t} \bar{N}_i \dot{q}_i \text{ with } \ddot{q}_i = (J_i \bar{N}_i)^+ \left( w_i^* - \dot{J}_i \dot{q} - J_i \sum_{k=1}^{i-1} \bar{N}_k \ddot{q}_k \right)$$

#### 2.6.2 End-Effector Dynamics

With $\tau = J_e^T F_e$ we can formulate the end-effector Dynamics:

$$\Lambda_e \dot{w}_e + \mu + p = F_e$$

with $\boxed{\boldsymbol{\Lambda}_\mathbf{e} = (J_e M^{-1} J_e^T)^{-1}, \quad \boldsymbol{\mu} = \Lambda_e J_e M^{-1} b - \Lambda_e \dot{J}_e \dot{q}, \quad \mathbf{p} = \Lambda_e J_e M^{-1} g}$

as the end-effector inertia, centrifugal and gravitational terms in task-space.

**End-Effector Motion Control**
From the above Dynamics, we can get an inversion motion control, like in the joint space:

$$\tau^* = \hat{J}^T (\hat{\Lambda}_e \dot{w}_e^* + \hat{\mu} + \hat{p})$$

together with a control law:

$$\dot{w}_e^* = k_p \begin{pmatrix} r_e^* - r_e \\ \Delta\phi_e \end{pmatrix} + k_d (w_e^* - w_e)$$

For small errors we can approximate:

$$\begin{pmatrix} r_e^* - r_e \\ \Delta\phi_e \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & E_R \end{pmatrix} \begin{pmatrix} r^* - r \\ \chi_R^* - \chi_R \end{pmatrix}$$

#### 2.6.3 Operational Space Control

**Note:** We need to extend the end effector dynamics with a contact Force $F_c$:

$$F_c + \Lambda_e \dot{w}_e + \mu + p = F_e$$

In some situations the robot has to either apply a force or move in a direction. This can be described by two specification matrices for position and orientation:

$$\Sigma_p = \begin{pmatrix} \sigma_{px} & 0 & 0 \\ 0 & \sigma_{py} & 0 \\ 0 & 0 & \sigma_{pz} \end{pmatrix} \quad \Sigma_r = \begin{pmatrix} \sigma_{rx} & 0 & 0 \\ 0 & \sigma_{ry} & 0 \\ 0 & 0 & \sigma_{rz} \end{pmatrix}$$

with $\sigma_i$ either 1(move) or 0(don't).

$$\tau^* = \hat{J}^T (\hat{\Lambda}_e S_M \dot{w}_e^* + S_F F_c + \hat{\mu} + \hat{p})$$

$$S_M = \begin{pmatrix} C^T \Sigma_p C & 0 \\ 0 & C^T \Sigma_r C \end{pmatrix} \quad S_F = \begin{pmatrix} C^T (\mathbb{1} - \Sigma_p) C & 0 \\ 0 & C^T (\mathbb{1} - \Sigma_r) C \end{pmatrix}$$

### 2.7 Least Square Optimisation | Slides 6.5

So far we only considered the optimisation of $min||\ddot{q}||_2$ as result of the Pseudoinverse $J^+$. To optimize another objective, we can formulate the problem in multiple tasks.

$$\left| \begin{matrix} \tau = M\ddot{q} + b + g \\ \dot{w} = J\ddot{q} + \dot{J}\dot{q} \end{matrix} \right| \Rightarrow \left| \begin{matrix} \begin{bmatrix} M & -\mathbb{1} \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} + b + g = 0 \\ \begin{bmatrix} J_e & 0 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} + \dot{J}_e \dot{q} = \dot{w}^* \end{matrix} \right|$$

This has always to be fulfilled, and can be extended by additional objectives.
It can be solves as single (stacked) tasks:

$$\min_{q,\tau} \left\| \begin{bmatrix} M & -\mathbb{1} \\ J_e & 0 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} - \begin{pmatrix} -b - g \\ \dot{w}_e^* - \dot{J}_e \dot{q} \end{pmatrix} \right\|_2$$

Or with different priorities:

$$\min_{q,\tau} \left\| \begin{bmatrix} J_e & 0 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} - (\dot{w}_e^* - \dot{J}_e \dot{q}) \right\|_2$$

$$\text{such that: } \begin{bmatrix} M & -\mathbb{1} \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} - (-b - g) = 0$$

This will exploit the nullspace of the higher priority task to minimize the solution. $\Rightarrow$ Solve with numeric solver

## 3 Floating Base Systems

### 3.1 FB Kinematics

Free floating robots are described by $n_b$ **unactuated** base coordinates $q_b$ and $n_j$ actuated joint coordinates $q_j$.

$$q = \begin{pmatrix} q_b \\ q_j \end{pmatrix} \quad \text{with } q_b = \begin{pmatrix} q_{b_P} \\ q_{b_R} \end{pmatrix} \in \mathbb{R}^3 \times SO(3)$$

The minimal number of generalized coordinates for the base is $n_{b0} = 6$ (3D).

**Generalized Velocity** (often simply written as $\dot{q}$)

$$u = \begin{pmatrix} {}_\mathcal{I}v_B \\ {}_\mathcal{B}\omega_{\mathcal{IB}} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \text{ with mapping } \quad u = E_{fb} \cdot \dot{q}, \quad E_{fb} = \begin{bmatrix} \mathbb{1}_{3\times3} & 0 & 0 \\ 0 & E_{\chi_R} & 0 \\ 0 & 0 & \mathbb{1}_{n_j \times n_j} \end{bmatrix}$$

#### 3.1.1 Forward Kinematics

The position vector of point $Q$ can be expressed via the Base $\mathcal{B}$

$${}_\mathcal{I}r_{IQ}(q) = {}_\mathcal{I}r_{IB}(q) + C_{\mathcal{IB}}(q) \cdot {}_\mathcal{B}r_{BQ}(q)$$

#### 3.1.2 Differential Kinematics

The spacial Jacobian maps $u$ to $v$ and $\omega$:

$$\begin{pmatrix} {}_\mathcal{I}v_Q \\ {}_\mathcal{I}\omega_{\mathcal{IQ}} \end{pmatrix} = {}_\mathcal{I}J_Q(q) \cdot u$$

$${}_\mathcal{I}J_Q(q) = \begin{bmatrix} \mathbb{1}_{3\times3} & -C_{\mathcal{IB}} \cdot [{}_\mathcal{B}r_{BQ}]_\times & C_{\mathcal{IB}} \cdot {}_\mathcal{B}J_{P_{q_j}}(q_j) \\ 0_{3\times3} & C_{\mathcal{IB}} & C_{\mathcal{IB}} \cdot {}_\mathcal{B}J_{R_{q_j}} \end{bmatrix}$$

#### 3.1.3 Contacts & Constraints

Every Point $\mathbf{C_i}$ in contact with the environment imposes **constant** position and **zero** velocity and acceleration. $\rightarrow$ Contact Jac. $J_{C_i}$

$${}_\mathcal{I}J_{C_i} u = 0, \quad {}_\mathcal{I}J_{C_i} \dot{u} + {}_\mathcal{I}\dot{J}_{C_i} u = 0$$

where multiple $J_{C_i}$ can be stacked for multiple contact points.

The $rank(J_c)$ indicates the number of independent contact constraints. The stacked $J_c$ can be split in a body and joint part:

$$J_c = [J_{c,b} \; J_{c,j}]$$

If the rank of $J_{c,b}$ has full rank (=6 in 3D), the joints can move the body in every direction. The difference $rank(J_c) - rank(J_{c,b})$ is the number of **internal kinematic constraints** (i.e. legs can move in respect to each other).

#### 3.1.4 Inverse Kinematics

We apply inverse kinematics, where the ground contact $J_c u = 0$ has the highest priority:

$$J_c u = 0 \quad \Rightarrow \quad u = J_c^+ 0 + \mathcal{N}(J_c) u_0 = N_c u_0$$

Given a demanded motion $w_t$ we can calculate the required velocity: $w_t = J_t u \quad \Rightarrow \quad u = N_c (J_t N_c)^+ w_t$

### 3.2 FB Dynamics

We need to extend the known dynamics with a selection for the torques $\tau$, since the body is unactuated:

$$M(q)\dot{u} + b(q,u) + g(q) = \mathbf{S^T} \tau + J_{ext}(q)^T F_{ext}$$

with $u_j = Su = S \begin{pmatrix} u_b \\ u_j \end{pmatrix} = [0_{n_j \times 6} \quad \mathbb{1}_{n_j \times n_j}] \begin{pmatrix} u_b \\ u_j \end{pmatrix}$

Note: If we have the forces that the robot exerts **on its environment**, we need them to switch the side in the equation:

$$M(q)\dot{u} + b(q,u) + g(q) + \mathbf{J_c(q)^T F_c} = S^T \tau$$

Together with the contact constraints we can calculate the contact Force $F_c$:

$$F_c = (J_c M^{-1} J_C^T)^{-1} (J_c M^{-1} (S^T \tau - b - g) + \dot{J}_c u)$$

#### 3.2.1 Constraint Dynamics

We can define a Null-space matrix for the contact constraints:

$$N_c = \mathbb{1} - M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c$$

This gives the following equations of motion, which are reduced, but **consistent with the constraints** (contact forces):

$$N_c^T (M\dot{u} + b + g) = N_c^T S^T \tau$$

#### 3.2.2 FB Inverse Dynamics

With a desired $\dot{u}^*_{consistent}$ We can invert the equation of motion of above:

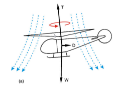
$$\tau^* = (N_c^T S^T)^+ N_c^T (M\dot{u}^* + b + g) + \mathcal{N}(N_c{}^T S^T) \tau_0^*$$

When taking only the first part without a Nullspace ($\tau_0^* = 0$), the solution is the least square minimal torque $\tau^*$ that fulfills the EoM.

## 4 Rotorcrafts

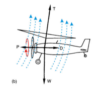
**Overview**

Helicopter



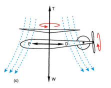
Has a power driven main rotor, which can be tilted

Autogyro



Passive main rotor and a forward facing active propeller. Can't hover.

Gyrodyne

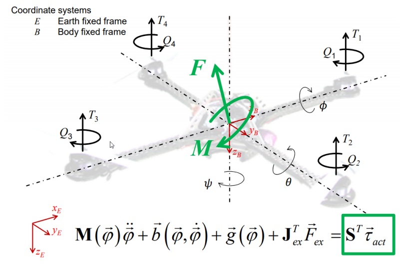


Active main rotor, but can't be tilted. Additional front facing active propeller.

Typical rotorcrafts are: Single Rotors, Multi rotors, Coaxial, Ducted Fan, Omnidirectional Multicopter(movable rotors).

### 4.1 Modelling of Quadrotor

*Modeling and simulations are important, but they must be validated in reality.*




**Structural Properties:**

Arm length $l$, Rotor height $h$, Mass $m$, Inertia $I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

Hub force & rolling moments depend on flight regime and can be neglected in hovering.

**Rotation:**

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} Roll \\ Pitch \\ Yaw \end{pmatrix} \rightarrow \begin{pmatrix} xAxis \\ yAxis \\ zAxis \end{pmatrix} \text{ with the known rot. matrices.}$$

This can be used for a equation for the rotational speed:

$${}_\mathcal{B}\omega = E_r \dot{\chi}_r = E_r \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}, \text{ with } E_r = \begin{bmatrix} 1 & 0 & -sin\theta \\ 0 & cos\phi & sin\phi cos\theta \\ 0 & -sin\phi & cos\phi cos\theta \end{bmatrix}$$

Linearization for small Roll and Pitch ($\phi \approx \theta \approx 0$) results in a **unity matrix** $E_r = \mathbb{1}$

#### 4.1.1 Body Dynamics

$$\begin{bmatrix} m\mathbb{1}_{3\times3} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} {}_\mathcal{B}\dot{v} \\ {}_\mathcal{B}\dot{\omega} \end{bmatrix} + \begin{bmatrix} {}_\mathcal{B}\omega \times m {}_\mathcal{B}v \\ {}_\mathcal{B}\omega \times I {}_\mathcal{B}\omega \end{bmatrix} = \begin{bmatrix} {}_\mathcal{B}F \\ {}_\mathcal{B}M \end{bmatrix}$$

with Forces

$${}_\mathcal{B}F = {}_\mathcal{B}F_G + {}_\mathcal{B}F_{Aero} = C_{EB}^T \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + \sum_{i=1}^{4} {}_\mathcal{B} \begin{pmatrix} 0 \\ 0 \\ -T_i \end{pmatrix}$$

and Hover Moments

$${}_\mathcal{B}M_{\text{Aero}} = {}_\mathcal{B} \begin{pmatrix} l(T_4 - T_2) \\ l(T_1 - T_3) \\ 0 \end{pmatrix} + {}_\mathcal{B} \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} Q_i (-1)^{(i-1)} \end{pmatrix}$$

with Thrust forces $T_i = b_i\omega_{p,i}^2$ and Drag Forces $Q_i = d_i\omega_{p,i}^2$

**Rotational Dynamics** (fist row of Dynamics)

$$I_{xx}\dot\omega_x = \omega_y \cdot \omega_z(I_{yy} - I_{zz}) + l\cdot b(\omega_{p,4}^2 - \omega_{p,2}^2)$$
$$I_{yy}\dot\omega_y = \omega_z \cdot \omega_x(I_{zz} - I_{xx}) + l\cdot b(\omega_{p,1}^2 - \omega_{p,3}^2)$$
$$I_{zz}\dot\omega_z = d(\omega_{p,1}^2 - \omega_{p,2}^2 + \omega_{p,3}^2 - \omega_{p,4}^2)$$

with $\omega_{xyz}$ as entries of the body rotation ${}_B\omega$

⇒ We have full control over all rotational speeds (every equation depends on rotor speeds $w_{p,i}$). Note that $(I_{xx} - I_{yy}) = 0$ (x-y symmetry), which dropped out of the 3rd equation.

**Translational Dynamics** (2nd row of dynamics)

$$m\dot v_x = m(\omega_z \cdot v_y - \omega_y \cdot v_z) - sin\theta mg$$
$$m\dot v_y = m(\omega_x \cdot v_z - \omega_z \cdot v_x) + sin\phi cos\theta mg$$
$$m\dot v_z = m(\omega_y \cdot v_x - \omega_x \cdot v_y) + cos\phi cos\theta mg - b(\omega_{p,1}^2 + \omega_{p,2}^2 + \omega_{p,3}^2 + \omega_{p,4}^2)$$

with gravitational Terms in blue, ${}_BF_G = C_{EB}^T \cdot {}_E\vec n_z mg$

⇒ Only z-Axis can be controlled directly with $\omega_{p,i}$

Note: To be consistent with the lecture notation: $(\omega_x, \omega_y, \omega_z) = (p, q, r)$ and $(v_x, v_y, v_z) = (u, v, w)$

## 4.2 Control of a Quadrotor

The system has 6 DoF, but only 4 Inputs (Motors) → **Under-actuated**!
⇒ Forward Motion requires tipping around Roll and Pitch.

**Define Virtual control inputs:**
By defining a new set of inputs, we can decouple the Dynamic equations

Total Thrust

$$U_1 = b(\omega_{p1}^2 + \omega_{p2}^2 + \omega_{p3}^2 + \omega_{p4}^2)$$

Moments along axis

$$U_2 = l\cdot b(\omega_{p4}^2 - \omega_{p2}^2)$$
$$U_3 = l\cdot b(\omega_{p1}^2 - \omega_{p3}^2)$$
$$U_4 = d(\omega_{p1}^2 - \omega_{p2}^2 + \omega_{p3}^2 - \omega_{p4}^2)$$

which simplify the dynamics from above.

**Linearize Attitude Dynamics**
Linearization around the Equilibrium $(\omega_{x,y,z} = \phi = \theta = U_{2,3,4} = 0; U_1 = mg)$ gives further simplification (Recall: $E_r = \mathbb{1} \Rightarrow \dot\chi_r = [\dot\phi\ \ \dot\theta\ \ \dot\psi]^T$): ⇒ Can use 3 individual controller

$$\boxed{\dot\omega_x = \ddot\phi = \frac{1}{I_{xx}}U_2 \quad \dot\omega_y = \ddot\theta = \frac{1}{I_{yy}}U_3 \quad \dot\omega_z = \ddot\psi = \frac{1}{I_{zz}}U_4}$$

**Altitude Control:**
Dynamics from above, but in *Inertial Frame*:

$$\dot v_z = \ddot z = g - cos\phi cos\theta \frac{1}{m}U_1 =: g - \frac{1}{m}T_z$$

Now we can derive the input $U_1$ for a chosen controller $T_z$ (representing the Thrust in the inertial frame) :

$$T_z = -k_p(z_{des} - z) + k_d\dot z - mg \Rightarrow U_1 = \frac{T_z}{cos\phi cos\theta}$$

**Position Control:**
We can use 3 separate PD Controller to get the trust for x,y and z in the inertial frame.

$$\text{Dynamics} \Rightarrow [\ddot x\ \ \ddot y\ \ \ddot z]^T = \frac{1}{m}[T_x\ \ T_y\ \ T_z]^T + [0\ \ 0\ \ g]^T$$

This must be transformed to get the desired Total thrust as well as roll and pitch angles.

$$T = \sqrt{T_x^2 + T_y^2 + T_z^2} \quad \& \quad \frac{1}{T}C_{E1}^T(z,\psi)\begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix} = \begin{bmatrix}sin\theta cos\phi\\ -sin\phi\\ cos\theta cos\phi\end{bmatrix}$$

**Remark:** $C_{E1}(z,\psi)$ is the rotation matrix around $z$ with angle $\psi$.

**Remark2:** $\begin{bmatrix}sin\theta cos\phi\\ -sin\phi\\ cos\theta cos\phi\end{bmatrix}$ describes the Thrust of the Quadrotor in the Inertial frame (can only apply thrust perpendicular to the rotors). It's calculated with $C_{12}(y,\theta)\cdot C_{2B}(x,\phi)\cdot(0,0,1)^T$

## 4.3 Propeller Aerodynamics

There are 4 main forces generated by the rotor:

**For a Rotor in Hover:**
Thrust Force T
Aerodynamic force perpendicular to rotor plane
$|T| = \frac{\rho}{2}A_P C_T(\omega_P R_P)^2$

Drag Torque Q
Torque around rotor plane
$|Q| = \frac{\rho}{2}A_P C_Q(\omega_P R_P)^2 R_P$

**For a Rotor in forward flight:**
Hub force H
Opposite to horizontal flight direction $V_H$
$|H| = \frac{\rho}{2}A_P C_H(\omega_P R_P)^2$

Rolling Moment R
Around flight direction
$|R| = \frac{\rho}{2}A_P C_R(\omega_P R_P)^2 R_P$

### 4.3.1 Momentum Theory

$\rho$ : fluid density, $\vec V$ : flow speed, $\vec n$ : surface normal,
$dA$ : surface Area patch, $p$ : surface pressure, $E$ : Energy, $P$ : Power

Conservation of fluid mass

$$\iint \rho\vec V \cdot \vec n dA = 0$$

Mass flow inside and outside (closed) control Volume must be equal

Conservation of fluid Momentum:

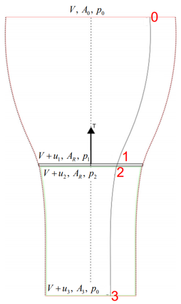
$$\iint p\cdot\vec n dA + \iint(\rho\vec V)\vec V\cdot\vec n dA = \vec F$$

The net Force is the change of momentum of the fluid

Conservation of energy:

$$\iint \frac{1}{2}\rho V^2\vec V\cdot\vec n dA = \frac{dE}{dt} = P$$

Work done on the fluid results in a gain of kinetic energy




**1D Analysis:**

The formulas lead to the following results:
$\rho A_0 V = \rho A_R(V + u_1)$
$= \rho A_R(V + u_2) = \rho A_R(V + u_3)$
$\Rightarrow u_1 = u_2$

$F_{Thrust} = \rho A_R(V + u_1)u_3$
$P_{Thrust} = F_{Thrust}(V + u_1)$
$= \frac{1}{2}\rho A_R(V + u_1)(2V + u_3)u_3$
$\Rightarrow = u_3 = 2u_1$

**In the Hover case** ($V = 0$):
Thrust Force: $F_{Thrust} = 2\rho A_R u_1^2$
Slipstream Tube: $A_0 = \infty \quad A_3 = \frac{A_R}{2}$

Combining $P = F_{Thrust}(\overbrace{V}^{=0} + u_1)$ with $F_{Thrust} = 2\rho A_R u_1^2$ gives the ideal Power to Hover:

$$P = \frac{F_{Thrust}^{3/2}}{\sqrt{2\rho A_R}} = \frac{(mg)^{3/2}}{\sqrt{2\rho A_R}} \text{ with } F_{Thrust} = mg$$

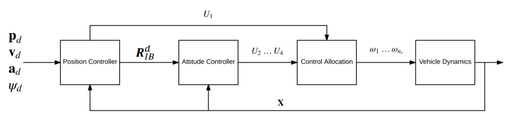
The Power depends on the **Disc Loading** $F_{Thrust}/A_R$

**Defining the rotor efficiency, Figure of Merit**, $FM$

$$FM = \frac{\textit{Ideal power to hover}}{\textit{Actual power to hover}} < 1$$

→ compare different propellers with the same disc loading.

# 5 Case Study: Micro Aerial Vehicles




## 5.1 Control for MAVs

**Virtual Control Input (allocation):**

$$\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix} = A\begin{pmatrix}\omega_1^2\\ \dots\\ \omega_{n_r}^2\end{pmatrix}, \quad A \in \mathbb{R}^{4\times n_r}$$

### 5.1.1 Trajectory Tracking Controller

Error definitions: Position $e_p = p - p_d$, Velocity $e_v = v - v_d$

$$z_B^d = \frac{-K_pe_p - K_ve_v - m(g - a_d)}{||-K_pe_p - K_ve_v - m(g - a_d)||}$$

$${}_Ix_{temp}^d = \begin{pmatrix}cos\psi_d\\sin\psi_d\\0\end{pmatrix} \quad {}_Iy_B^d = \frac{{}_Iz_B^d \times {}_Ix_{temp}^d}{||{}_Iz_B^d \times {}_Ix_{temp}^d||}$$

From this we can construct the Rotation Matrix $\mathbf{R_{IB}^d} = [\mathbf{{}_Iy_B^d} \times \mathbf{{}_Iz_B^d},\ \mathbf{{}_Iy_B^d},\ \mathbf{{}_Iz_B^d}]$

**Note:** ${}_Ix_{temp}^d$ is generally not perpendicular to ${}_Iz_B^d$, so we use some orthogonal projections for the first entry of $\mathbf{R_{IB}^d}$

**Attitude Control**

$$\begin{pmatrix}U_2\\U_3\\U_4\end{pmatrix} = -K_Re_R - K_\omega e_\omega + \omega\times J\omega$$

with errors:

$$\mathbf{e_R} = \frac{1}{2}\left((R_{IB}^d)^T R_{IB} - R_{IB}^T R_{IB}^d\right)^\vee, \mathbf{e}_\omega = \omega - R_{IB}^T R_{IB}^d\omega_d$$

where $(.)^\vee$ maps the cross product matrix to a vector.

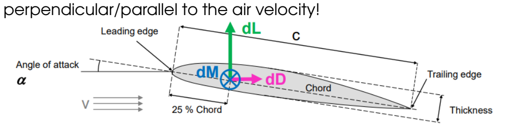
**Trajectory Tracking**
Project Thrust onto Body z-Axis:

$$U_1 = (-K_pe_p - K_ve_v - m(g - a_d))\cdot{}_I\vec n_{z,B}$$

# 6 Fixed Wing UAVs

## 6.1 Basic Principles

For incompressable and non viscous fluids, we can use the Bernoulli Equation: $\frac{v^2}{2} + gh + \frac{p}{\rho} = const$. This results in a Lift and a Drag Force with an additional Moment on the Airfoil. Lift/Drag is always perpendicular/parallel to the air velocity!




**Stall Point:** The angle of attack at which the maximum lift occurs. (Flow will separate for higher $\alpha$)

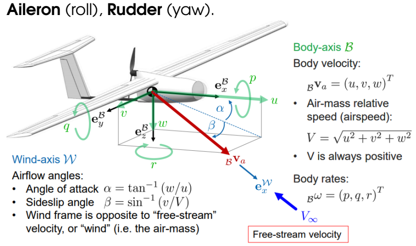
## 6.2 Fixed Wing Kinematics

**Assumptions:** Aerodynamics where we don't enter Stall, neglect side-forces and interference effects

**Inertial frame:** X-Axis: North; Y-Axis: East; Z-Axis: Downwards

**Body-fixed frame:** X-Axis: out of the nose; Y-Axis: out of the **right** wing; Z-Axis: Downwards; The Orientation of the Body Frame is describes as usual in Euler Angles with roll, pitch and yaw → $\phi, \theta, \psi$.

**Control Surfaces:** The standard Control surfaces are **Elevator** (pitch), **Aileron** (roll), **Rudder** (yaw).




**Flight Path Angle:** $\gamma$, defined from horizon to ${}_Iv_a$
**Heading Angle:** $\xi$, defined from North to ${}_Iv_a$
**Course Angle:** $\chi$, defined from North to ${}_Iv = {}_Iv_a + {}_Iv_{wind}$

## 6.3 Fixed Wing Dynamics

### 6.3.1 Forces

**Non Aerodynamics:** Weight, Propeller Thrust.
**Aerodynamic:**

$$\text{Lift: } L = \frac{1}{2}\rho V^2 Sc_L, \quad \text{Drag: } D = \frac{1}{2}\rho V^2 Sc_D$$

with surface $S$.
Note: $c_L$ and $c_D$ are dependent on $\alpha$, Side-Forces: assumed zero

### 6.3.2 Equations of Motion, ($I_{xz}$ is typically small)

$${}_B\dot v_a = \frac{1}{m}\sum{}_BF - {}_B\omega\times{}_Bv_a, \quad {}_I\dot r = C_{IB}\,{}_Bv_a + {}_Iv_w$$

$${}_B\dot\omega = {}_BI^{-1}\left(\sum{}_BM - {}_B\omega\times({}_BI\,{}_B\omega)\right), \quad {}_BI = \begin{bmatrix}I_{xx}&0&I_{xz}\\0&I_{yy}&0\\I_{xz}&0&I_{zz}\end{bmatrix}$$

## 6.4 Cascaded Control Loops

### 6.4.1 Steady Level Turning

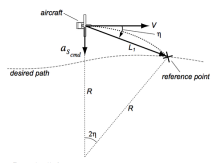
We need ${}_B\dot v_a = {}_B\dot\omega = 0; \theta = \alpha \to \gamma = 0$ and $\phi = const \neq 0$.

- The Lift $L$ increases with $\frac{1}{cos\phi}$ $\leftarrow (L = \frac{mg}{cos\phi})$
- The air speed $V_{min}$ increases with $\sqrt{1/cos\phi}$ $\leftarrow (\frac{1}{2}\rho V^2 Sc_L = L)$

The Heading rate $\xi$ can be found with a force balance with the centripetal force (and $\dot\psi \approx \dot\xi$). Note: $\dot\xi = V/R \Leftrightarrow v = r\cdot\omega$

$$F_{cent} = m\frac{V_G^2}{R} = L\cdot sin\phi \ \&\ L\cdot cos\phi = mg \Rightarrow \dot\psi = \dot\xi = \frac{g}{V}tan\phi$$

**Lateral-directional path**




$sin\eta = \frac{L_1}{2R} \Rightarrow R = \frac{L_1}{2sin\eta}$

$a_{s_{cmd}} = \frac{V^2}{R} = 2\frac{V^2 sin\eta}{L_1} \Rightarrow \dot\xi_d = \frac{a_{s_{cmd}}}{V}$

with the above:
$\phi = tan^{-1}\left(\frac{a_{s_{cmd}}}{g}\right)$

So we can get the needed roll angle for a desired turn with velocity $V$.

### 6.4.2 TECS - Total Energy Control System

$$\dot E_{spec} = \frac{\dot V}{g} + sin\gamma \approx \frac{\dot V}{g} + \gamma, \quad \dot E_{dist} = \gamma - \frac{\dot V}{g}$$

*'Potential energy rate minus kinetic energy rate'*