In the name of God.

Comparison of the deflection of a beam with fixed boundary condition under static loading using analytical and Finite Element Method (FEM) Using ABAQUS(CAE) software

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ABSTRACT:

This study presents a comparison of the deflection behavior of a beam with a fixed boundary condition under static loading using both analytical methods and the Finite Element Method (FEM). The beam was analyzed using classical beam theory and verified through FEM simulations. The results indicate a close agreement between the two approaches, with slight variations due to numerical approximations. The findings underscore the reliability of FEM in solving complex structural problems.

INTRODUCTION & PURPOSE:

Structural analysis is fundamental in engineering, with beams being critical components in many structures. Predicting beam deflection under various loads is essential for ensuring structural safety. Beams with fixed boundary conditions, commonly found in buildings and bridges, present a specific challenge for accurate deflection analysis.

Traditionally, beam deflection has been calculated using analytical methods like the Euler-Bernoulli and Timoshenko beam theories. While these methods offer precise solutions under certain conditions, they may fall short in complex scenarios involving thick beams or irregular loading. The Finite Element Method (FEM), a numerical approach, addresses these limitations by allowing for detailed modeling of complex structures and load conditions.

This study compares the deflection of a beam with a fixed boundary condition under static loading using both analytical methods and FEM. By evaluating the accuracy and reliability of these approaches, the study aims to provide insights into their applicability in engineering practice.

LITERATURE REVIEW:

The deflection of beams is a well-established area of study in structural engineering, with a rich history of analytical and numerical approaches developed to predict and analyze beam behavior under various loading conditions. This literature review explores the foundational theories, the evolution of analytical methods, and the rise of the Finite Element Method (FEM) as a critical tool in modern structural analysis. Additionally, it highlights existing research that compares analytical solutions with FEM, providing context for the present study.

1. Analytical Methods for Beam Deflection

The analytical study of beam deflection dates back to the development of classical beam theories, which have served as the cornerstone for understanding the behavior of beams under load. The Euler-Bernoulli beam theory, proposed by Leonhard Euler and Daniel Bernoulli in the 18th century, is one of the most widely used models for predicting beam deflection. This theory assumes that plane sections of the beam remain plane and perpendicular to the neutral axis during bending, and it neglects the effects of shear deformation. While this theory provides accurate results for slender beams, its limitations become apparent when dealing with thick beams or beams subjected to significant shear forces.

To address these limitations, the Timoshenko beam theory was developed in the early 20th century. This theory, named after the Ukrainian engineer Stephen Timoshenko, accounts for shear deformation and rotational inertia effects, making it more suitable for short or thick beams where the assumptions of the Euler-Bernoulli theory may not hold. The Timoshenko beam theory has been extensively validated through experiments and is considered a more general approach, though it also introduces additional complexity into the calculations.

Both of these theories have been instrumental in providing closed-form solutions for beam deflection, which are invaluable for preliminary design and analysis. These solutions are typically derived from differential equations governing the behavior of beams under various boundary conditions and loadings. For instance, the deflection of a simply supported beam under uniform load is a classic problem solved using these theories, providing engineers with a straightforward means to estimate deflection and ensure structural safety.

However, the limitations of these analytical methods become evident in more complex scenarios. Real-world structures often involve non-uniform loads, varying cross-sections, and non-linear material properties, all of which complicate the use of closed-form solutions. Additionally, the assumptions of small deformations and linear elasticity may not be valid in all cases, leading to discrepancies between analytical predictions and actual behavior. This has prompted the need for more versatile and accurate methods, such as FEM, which can address these complexities.

2. The Finite Element Method (FEM)

The Finite Element Method (FEM) was developed in the mid-20th century as a numerical technique for solving complex structural problems that were previously intractable using analytical methods. The origins of FEM can be traced back to the work of engineers like Richard Courant, who used piecewise polynomial approximations to solve torsion problems, and later researchers like Ray W. Clough, who coined the term "finite element" and applied it to structural analysis.

FEM divides a structure into a finite number of smaller, simpler elements, typically triangles or quadrilaterals in 2D analysis, or tetrahedra and hexahedra in 3D analysis. The behavior of each element is described by a set of equations, which are then assembled into a global system that represents the entire structure. This approach allows FEM to handle complex geometries, material properties, and boundary conditions that are difficult to manage with traditional analytical methods.

One of the major advantages of FEM is its ability to model irregular shapes and complex loading conditions with high accuracy. This is particularly important in modern engineering, where structures often feature intricate designs and are subjected to variable forces. FEM also excels in handling non-linearities, such as large deformations, plasticity, and contact problems, making it a versatile tool for both linear and non-linear analysis.

Numerous studies have validated FEM against experimental results, confirming its accuracy and reliability in predicting structural behavior. For example, FEM has been used to model the deflection of beams with various boundary conditions, including fixed, simply supported, and cantilevered beams. These studies have demonstrated that FEM can closely match experimental data, even in cases where analytical solutions are difficult or impossible to obtain.

Despite its advantages, FEM is not without its challenges. The accuracy of FEM results depends heavily on the quality of the mesh, the choice of element type, and the proper application of boundary conditions and loads. Poor meshing can lead to inaccurate results, particularly in areas with high stress gradients or complex geometries. Additionally, FEM is computationally intensive, requiring significant processing power and time for large or highly detailed models.

3. Comparative Studies of Analytical Methods and FEM

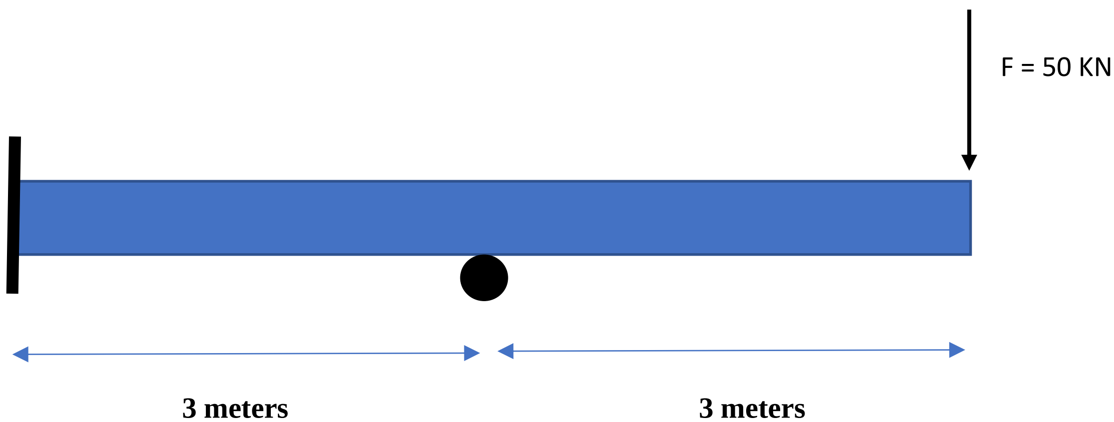
Comparative studies of analytical methods and FEM have been conducted to assess the strengths and limitations of each approach. These studies typically involve solving a benchmark problem using both methods and comparing the results to determine their accuracy and applicability. For example, research by Reddy (2004) compared the deflection of simply supported beams under uniform load using both the Euler-Bernoulli beam theory and FEM. The study found that FEM provided results that were in close agreement with the analytical solution, with differences attributed to the discretization process in FEM.

Another study by Zienkiewicz and Taylor (2000) examined the deflection of beams with various boundary conditions, including fixed-end conditions, using both the Timoshenko beam theory and FEM. The study highlighted the importance of shear deformation in thick beams, showing that the Timoshenko theory and FEM produced similar results, while the Euler-Bernoulli theory underestimated deflection due to its neglect of shear effects.

These studies underscore the importance of choosing the appropriate method for a given problem. While analytical methods are valuable for their simplicity and ease of use, FEM offers greater flexibility and accuracy for complex problems. The choice between these methods depends on the specific requirements of the analysis, including the geometry of the structure, the type of loading, and the desired level of accuracy.

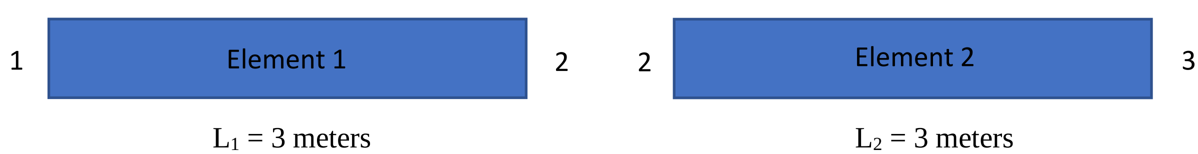
METHODOLOGY & RESULTS

ANALYTICAL METHOD



E = 210 GPa, ν = 0.3, I = 0.0002 m²

I considered L = 3 meters in the relations below.



Considerations: $L_1 = L_2 = L = 3$ meters

$$K_1 = \frac{EI}{L_1^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12L & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \& K_2 = \frac{EI}{L_2^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12L & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$K_{SYS} = \begin{bmatrix} 12 & 6l & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12L & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 2L^2 & -6L & 2L^2 \\ 0 & 0 & -12L & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 6l & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12L & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 2L^2 & -6L & 2L^2 \\ 0 & 0 & -12L & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \times \begin{bmatrix} V_1 \\ \Phi_1 \\ V_2 \\ \Phi_2 \\ V_3 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ 0 \\ -50 \\ 0 \end{bmatrix} \xrightarrow{V_1=\Phi_1=V_2=0} \begin{Bmatrix} 0 \\ -50000 \\ 0 \end{Bmatrix} = \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12L & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \times \begin{Bmatrix} \Phi_2 \\ V_3 \\ \Phi_3 \end{Bmatrix}$$

$$\frac{EI}{L^3}\left(8L^2\Phi_2 - 6LV_3 + 2L^2\Phi_3\right) = 0 \rightarrow 4L\Phi_2 - 3V_3 + L\Phi_3 = 0$$
$$\frac{EI}{L^3}\left(-6L\Phi_2 + 12V_3 + 6L\Phi_3\right) = -50000$$
$$\frac{EI}{L^3}\left(2L^2\Phi_2 - 6LV_3 + 4L^2\Phi_3\right) = 0 \rightarrow L\Phi_2 - 3V_3 + 2L\Phi_3 = 0$$

After solving equations above we have:

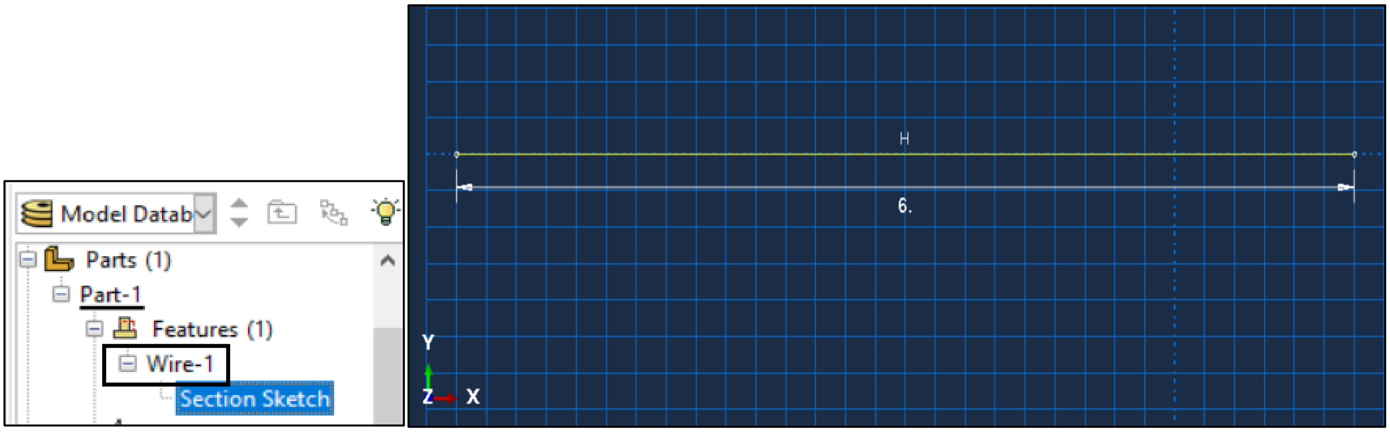
$$\Phi_2 = -0.002679[rad] = -2.679\times10^{-3}[rad]$$
$$V_3 = -0.01875[m] = -1.875\times10^{-2}[m]$$
$$\Phi_3 = -0.008036[rad] = -8.036\times10^{-3}[rad]$$

$$F_1 = 6L(\Phi_2)\times\frac{EI}{L^3} = \frac{6EI}{L^2}(\Phi_2) = \frac{6\times210\times10^9\times2\times10^{-4}}{3^2}(-0.002679) = -75012[N]$$
$$F_2 = \frac{EI}{L^3}(-12V_3 + 6L\Phi_3) = \frac{210\times10^9\times2\times10^{-4}}{3^2}(-12\times(-0.01875)+6\times(-0.008036)) = 124992[N]$$
$$M_1 = \frac{EI}{L^3}(2L^2(\Phi_2)) = \frac{2EI}{L}(\Phi_2) = \frac{2\times210\times10^9\times2\times10^{-4}}{3}\times(-0.002679) = -75012[N.m]$$

These are the exact values extracted with analytical method.

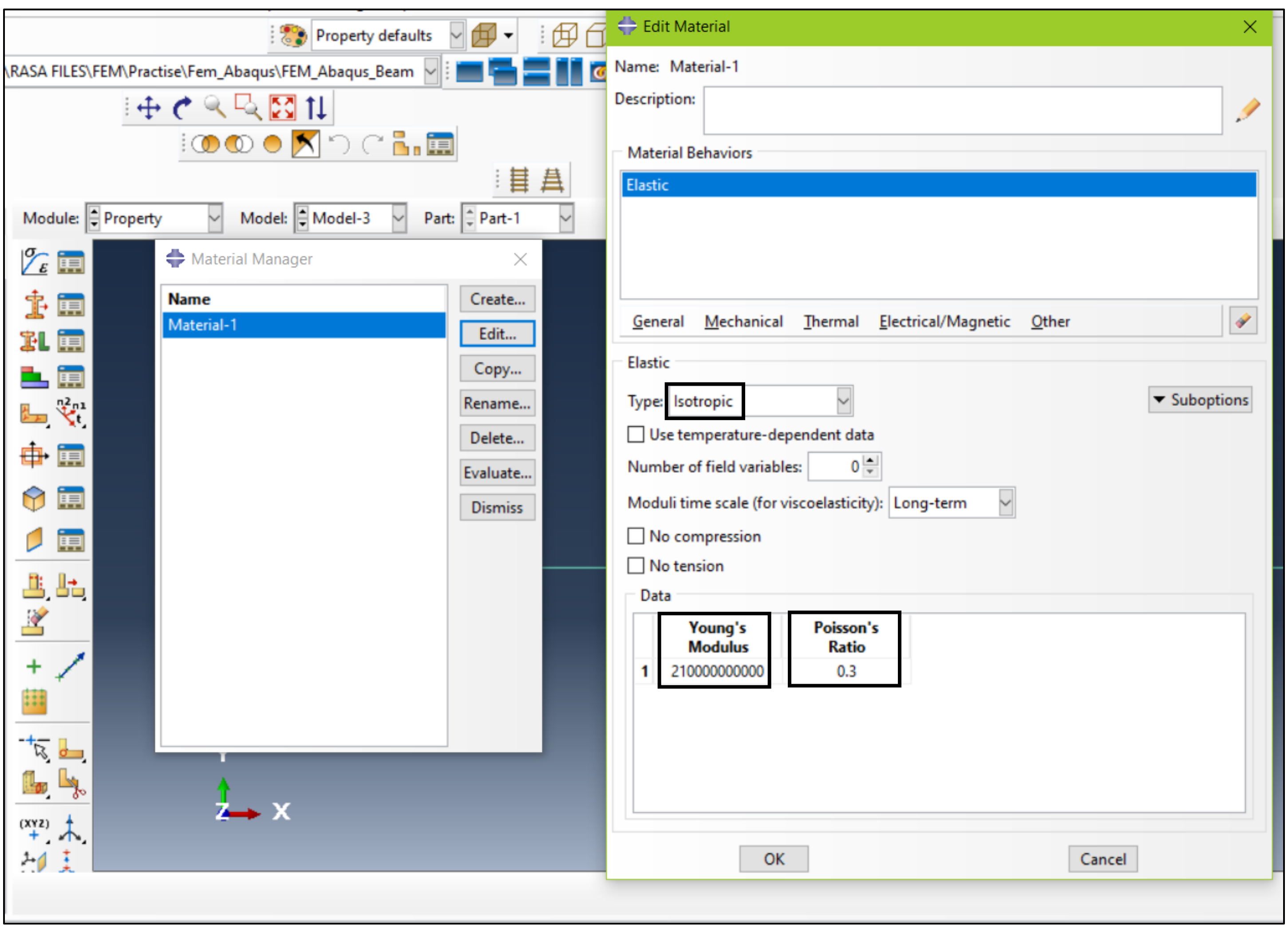
Finite Element Method using Abaqus software:

In order to model this case firstly, I chose 2 dimensional wire feature (2D-wire) with specified dimensions in meters.

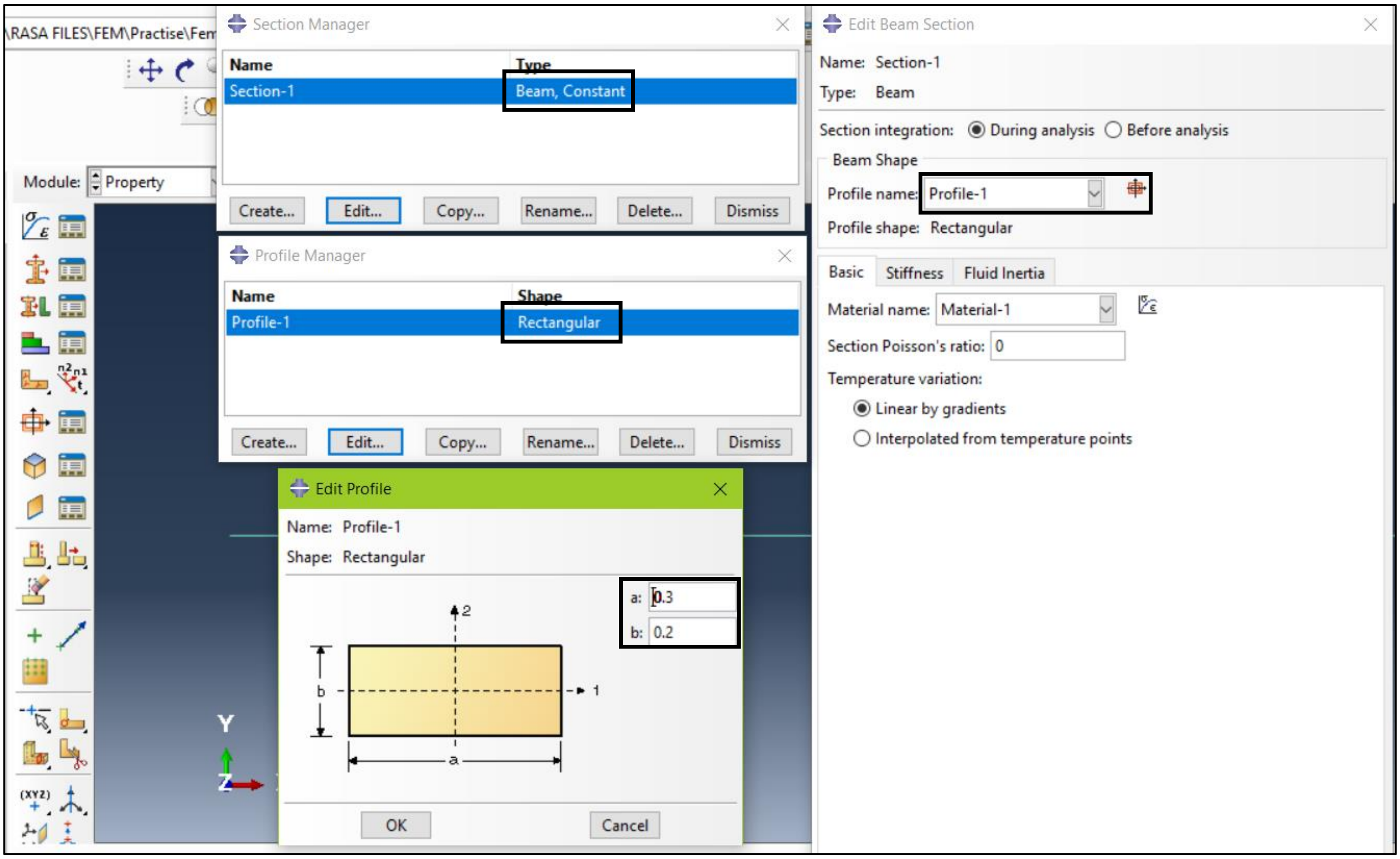


After creating the part of our model we have to assign property to the part that we have created**.**

We have to choose Elasticity/Elastic in the property module, after that we have to select isotropic type and enter Young's Modulus and Poissin's ratio to the material



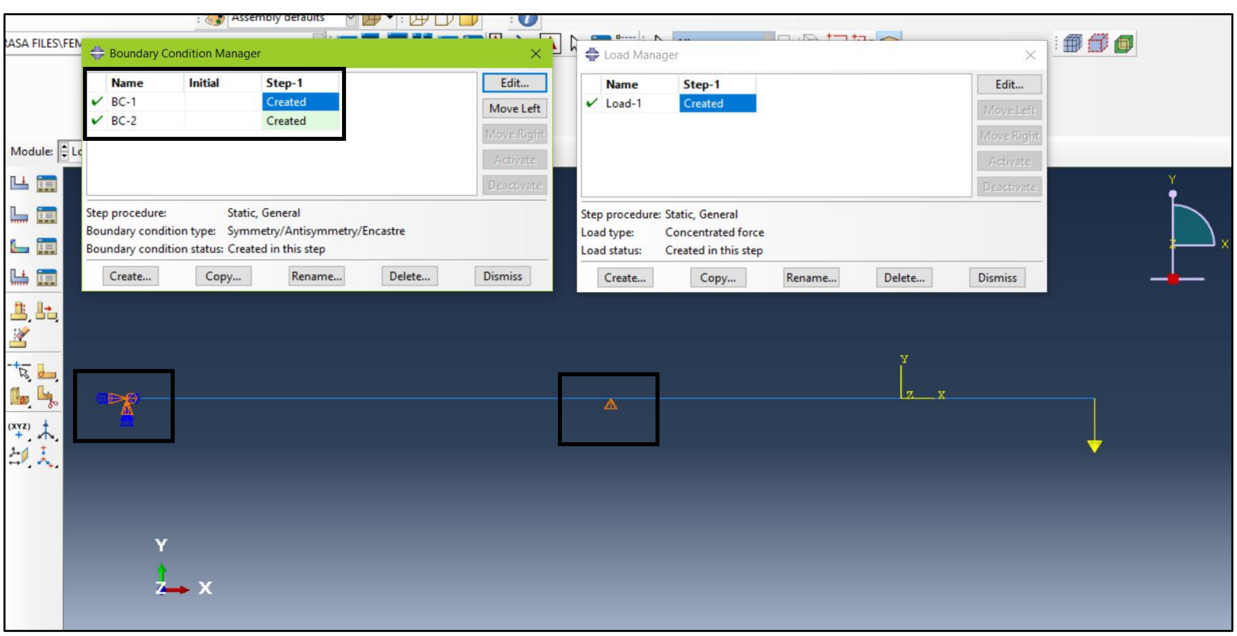
After defining the material we considered, we have to define a section type too, according to the problem we have, type of the section should be 2-D/Beam with specified rectangular profile dimensions. The key-point in specifying profile dimensions is that the second moment of inertia should be equals to specified value of **I = 0.0002 m**^{2}



After doing these, we are going to Assemble our Part, Because we have one part, this Module doesn't need any explanations as well as the last parts.

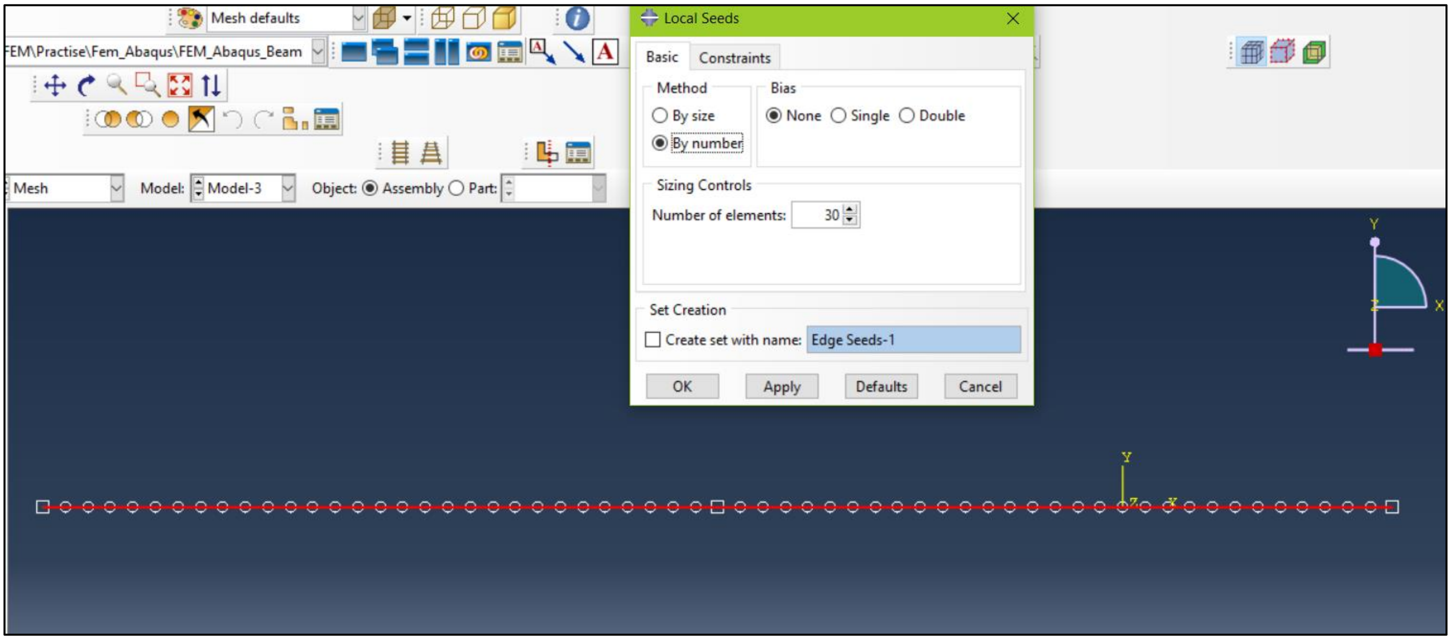
After the Assembly part we should go for the Step Module, according to the situation that problem has, we ave static loading so, we should select **Static/General** option.

After getting done with Step Module, we have to define boundary conditions and loads for the beam with this order:

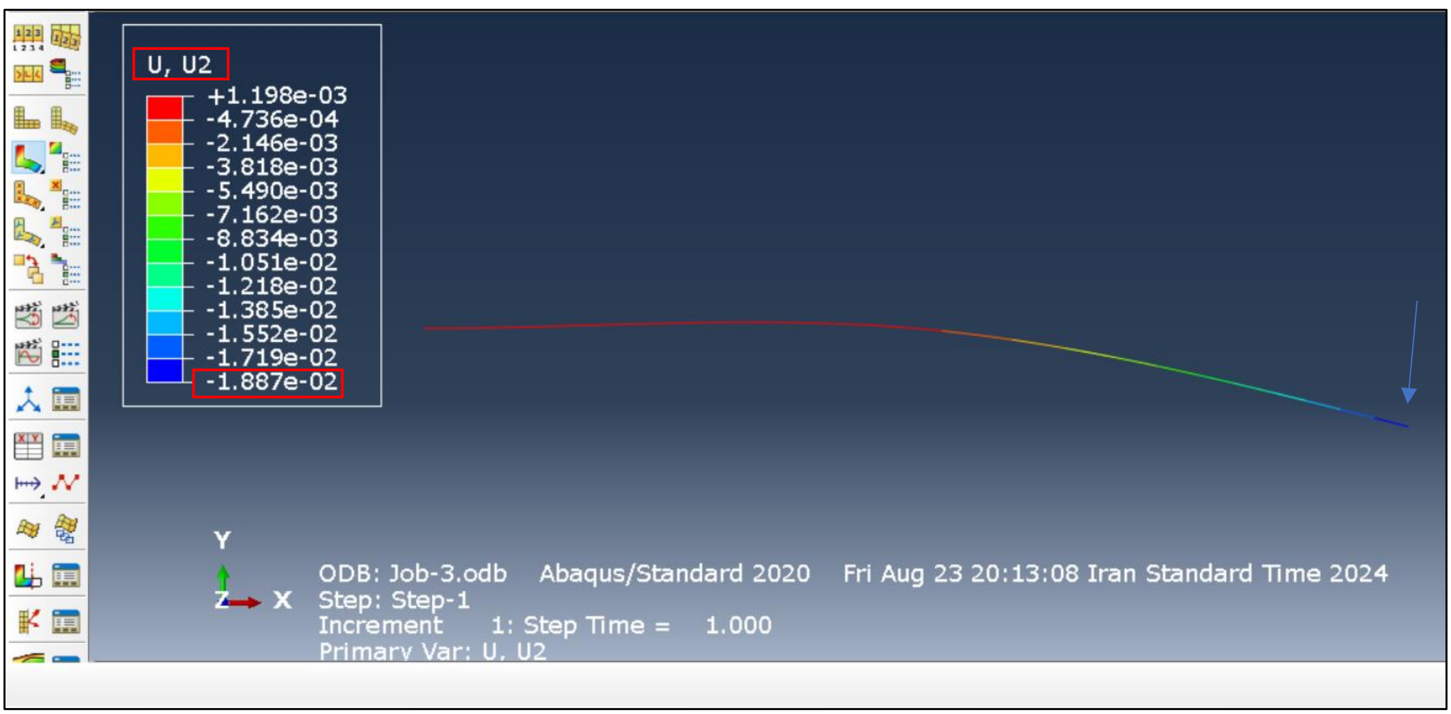


After defining Boundary conditions to our problem we have to mesh our feature.

I choose 30 elements to be created in the whole beam

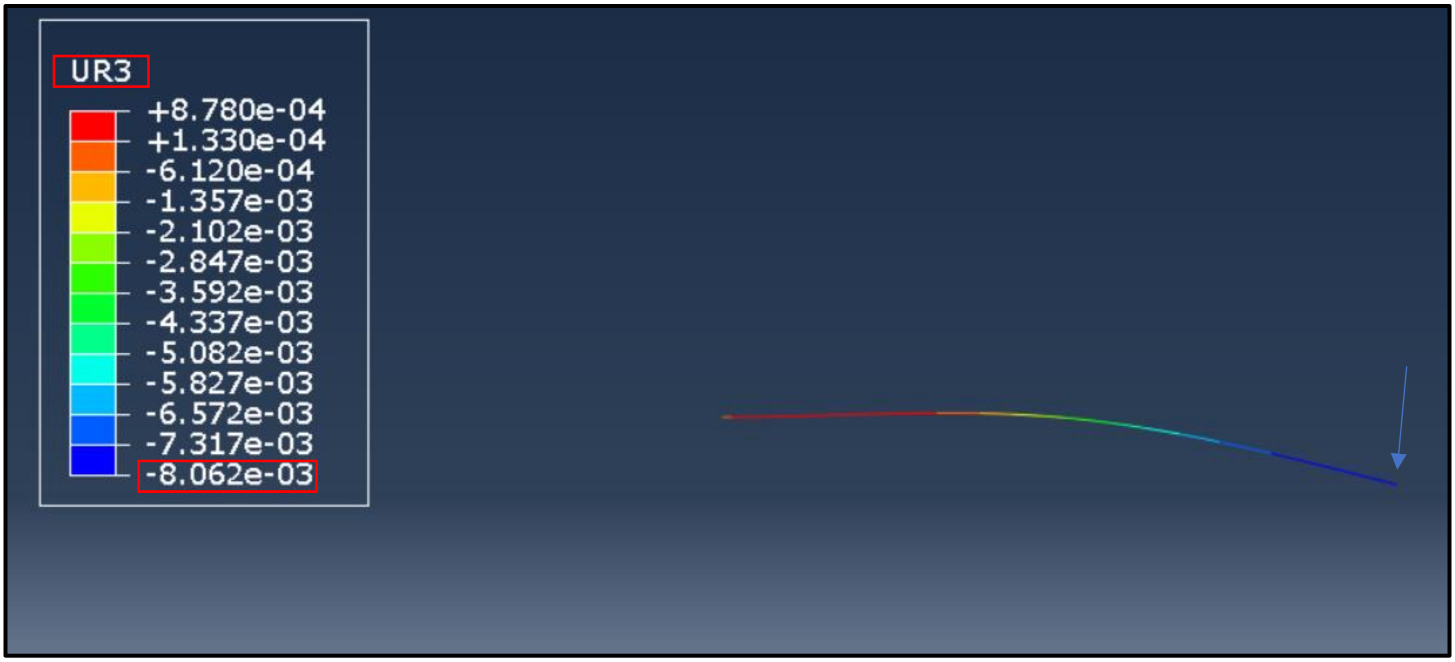


After defining mesh, we have to create Job and then, see the results:



As you see, the blue region is exactly indicating **V_3** which we extracted the value of it equals to $1.875e^{-3}$ [m]. from **Analytical Method** at the last section

The value calculated by Abaqus is $1.887e^{-2}$ [m]. As you can see there is a little difference between the values calculated with two methods and this indicates the power of the Abaqus software



As you see, the blue region is exactly indicating $\boldsymbol{\varphi}_3$ which we extracted the value of it equals to $-8.036e^{-3}$[rad] from **Analytical Method** at the last section.

The value calculated by Abaqus is $-8.062e^{-3}$ [rad]. As you can see there is a little difference between the values calculated with two methods and this indicates the power of the Abaqus software.

CONCLUSION

This study confirms the consistency between analytical methods and FEM in predicting beam deflection under fixed boundary conditions. The slight differences observed are attributed to the inherent assumptions in analytical methods and the numerical nature of FEM. These findings suggest that FEM can be reliably used in practical applications where analytical solutions are not feasible. Future research could explore the impact of different boundary conditions and loading scenarios.