

The software challenge in the Rasch model

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The background of the slide is an aerial photograph of a European city, likely Copenhagen, featuring a dense grid of buildings, a winding river, and a bridge. A large, solid blue circle covers the upper half of the slide, centered over the text.

THE RASCH MODEL

- $\boldsymbol{X}_v = (X_{1v}, \dots, X_{kv}) \in \{0, 1\}^k$ item responses
 - $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k)$ item locations
 - $\theta_v \in \mathbb{R}$ latent variable, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$
- $\theta \mapsto P(X_{iv} = 1 | \theta_v = \theta)$ functional relationship
- Test assumptions - get 'measurement' of θ

Rasch model part of general class of models¹

$$P(\mathbf{X}_v = \mathbf{x} | \theta_v = \theta) = \frac{\exp(\mathbf{x}^T(\mathbf{B}\theta + \mathbf{A}\eta))}{\sum_z \exp(\mathbf{z}^T(\mathbf{B}\theta + \mathbf{A}\eta))}$$

choosing $\mathbf{A} = I$ and $\mathbf{B} = [1 \cdots 1]^T$ yields

$$R := \mathbf{x}^T \mathbf{B} = \sum_{i=1}^k x_i$$

seen to be sufficient for θ

¹Adams, Wu (2007)

https://doi.org/10.1007/978-0-387-49839-3_4

Rasch model part of general class of models

$$P(\mathbf{X}_v = \mathbf{x} | \theta_v = \theta) = \frac{\exp(\mathbf{x}^T(\mathbf{B}\theta + \mathbf{A}\eta))}{\sum_z \exp(\mathbf{z}^T(\mathbf{B}\theta + \mathbf{A}\eta))}$$

choosing $\mathbf{A} = I$ and $\mathbf{B} = [1 \cdots 1]^T$ further yields

$$P(\mathbf{X} = \mathbf{x} | \theta) = \prod_i P(X_i = x_i | \theta)$$

local independence²

²Lazarsfeld, Henry (1968). Latent structure analysis. Boston, MA:
Houghton Mifflin

Conditional probability independent of θ

$$P(\mathbf{X}_v = \mathbf{x} | R_v, \theta_v = \theta) = \frac{\exp(\mathbf{x}^T \mathbf{A}\boldsymbol{\eta})}{\sum_{z: \mathbf{z}^T \mathbf{B} = R} \exp(\mathbf{z}^T \mathbf{A}\boldsymbol{\eta})}$$

Conditional inference yields unbiased estimates $\hat{\boldsymbol{\eta}}$ ³
Historically a computational challenge. Not so anymore^{4,5}

³ Andersen (1970). JRSS B, 32, 283-301.

<http://www.jstor.org/stable/2984535>

⁴ Mair, Hatzinger (2007). Journal of Statistical Software, 20(9), 9.

<https://doi.org/10.18637/jss.v020.i09>

⁵ Christensen (2013). ISRN Computational Mathematics,

<http://doi.org/10.1155/2013/617475>

An aerial photograph of a European city, likely Copenhagen, featuring a dense grid of buildings, a winding river, and a bridge. A large, solid teal circle is overlaid on the center of the image, obscuring the middle portion. Inside this circle, the letters "AMTS" are written in a white, sans-serif font.

AMTS

ABBREVIATED MENTAL TEST SCORE

Age // Time (to nearest hour) // Address

Name of Hospital (or area of town) // Current Year

Date of birth of patient // Month // Date of first world war

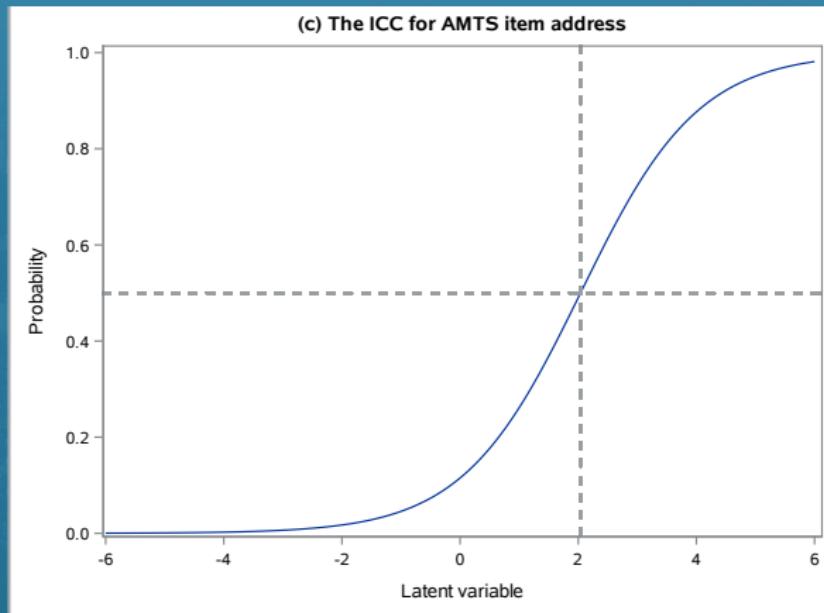
Name of Monarch // Count backwards

6/10 or less abnormal in elderly.⁶

⁶ Hodgkinson HM (1972). Age Ageing. 1(4):233-8

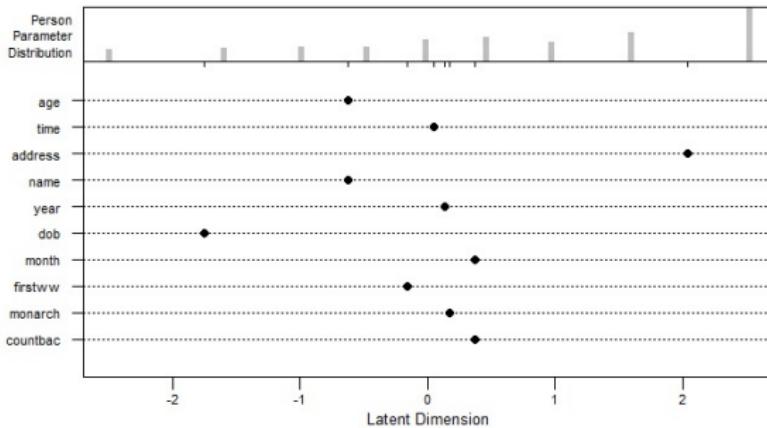
<https://doi.org/10.1093/ageing/afs148>

SAS



R

Person-Item Map



An aerial photograph of a European city, likely Copenhagen, featuring a prominent river (the Amager River) flowing through the urban landscape. The city is densely built with a mix of modern and traditional architecture. A large, solid blue circle is overlaid on the upper half of the image, covering the central area where the text is placed.

RASCH SOFTWARE

- RUMM⁷ (pairwise) conditional estimation of $\hat{\eta}$, ML estimation of $\hat{\theta}$
- WINSTEPS⁸ joint likelihood estimation of $(\hat{\eta}, \hat{\theta})$

Model fit evaluated by plugging in $\hat{\theta}_v$

⁷ Andrich, Sheridan, Luo (2010). RUMM2030 [Computer software and manual]. Perth, Australia.

⁸ Linacre (2017). Winsteps® Rasch measurement computer program. Beaverton, Oregon: Winsteps.com

An aerial photograph of a European city, likely Copenhagen, featuring a dense grid of buildings, a winding river, and a bridge. A large, solid blue circle is positioned in the upper half of the frame, obscuring the top portion of the cityscape.

INDIVIDUAL ITEM FIT

Standard evaluation of individual item fit using

$$Z_{iv} = \frac{X_{iv} - E(X_{iv} | \theta_v = \hat{\theta}_v)}{\sqrt{V(X_{iv} | \theta_v = \hat{\theta}_v)}}$$

that rely on (biased) estimates $\hat{\theta}_v$.⁹ Fit statistics like

$$OUTFIT_i = \frac{1}{N} \sum_{v=1}^N Z_{iv}^2$$

with no established null distribution (early Rasch literature claims of χ^2 distribution.).

⁹ Warm (1989). Psychometrika, 54, 427–450.

It has been suggested that the Wilson-Hilferty cube-root transformation

$$t_i = (OUTFIT_i^{1/3} - 1) \frac{3}{V(OUTFIT_i)} + \frac{V(OUTFIT_i)}{3}$$

has an approximate t distribution and that

$$FitResid_i = \frac{f(\log(N \cdot OUTFIT_i) - \log(f))}{\sqrt{V(N \cdot OUTFIT_i)}}$$

with $f = (Nk - N - k + 1)/k$, has a symmetrical distribution with mean zero and variance one.

Item χ^2 fit statistic in RUMM groups respondents in G 'class intervals' based on the estimated person locations $\hat{\theta}_v$.

Computed as

$$\chi^2(X_i) = \sum_g \frac{\left(\sum_{v \in V_g} X_{vi} - \sum_{v \in V_g} E(X_{vi}) \right)^2}{\sum_{v \in V_g} V(X_{vi})}$$

where V_g denotes the set of respondents in class interval g

still relies on $\hat{\theta}_1, \hat{\theta}_2, \dots$

In SAS:

- Re-write $\sum_{v=1}^N Z_{iv}^2 = \sum_{r=0}^k \sum_{v:R_v=r} Z_{iv}^2$
- replace $X_{iv}|(\theta_v = \hat{\theta}_v)$ by $X_{iv}|(R_v = r)$:

$$\sum_{v=1}^N Z_{iv}^2 = \sum_{r=0}^k \underbrace{\sum_{v:R_v=r} Z_{iv}^2}_{iid}$$

has known null distribution¹⁰ (abandoned¹¹)

¹⁰ Christensen, Kreiner (2013). Rasch Models in Health (chap. 5, pp. 83–104), Wiley <https://doi.org/10.1002/9781118574454.ch5>

¹¹ Wright, Panchapakesan (1969). Educational and Psychological Measurement, 29, 23-48.

	RUMM/WINSTEPS	R/SAS
Overall model fit	total χ^2	CLR ¹²
Individual item fit	OUTFIT item χ^2 :	cond. OUTFIT cond. item χ^2 :

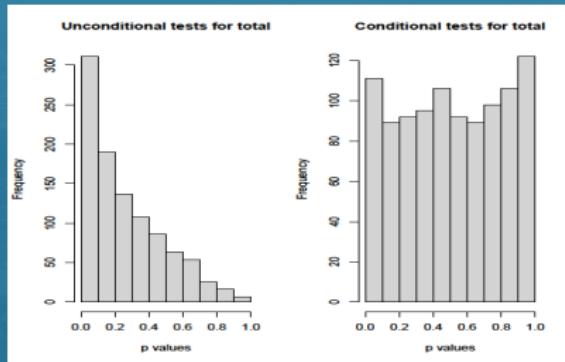
¹²Andersen (1973). *Psychometrika*, 38, 123–140.

<http://doi.org/10.1007/BF02291180>

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OVER-ALL MODEL FIT

Over-all fit¹³



¹³ Müller, Kreiner (2015). Res. Report 6, Department of Biostatistics, University of Copenhagen. https://ifsv.sund.ku.dk/biostat/annualreport/images/2/2f/Research_Report_15-06.pdf

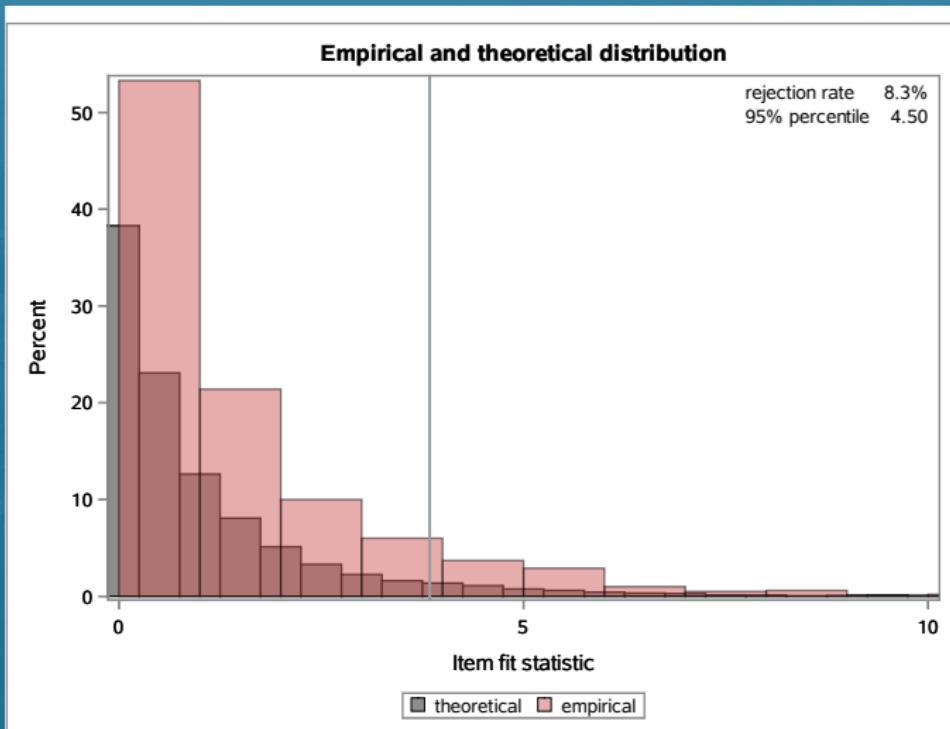
An aerial photograph of a European city, likely Copenhagen, featuring a prominent blue circular graphic in the center. The city's architecture, including numerous buildings with red roofs, is visible in the background. In the foreground, a river or canal flows through the city, lined with trees and some modern infrastructure.

AMTS INDIVIDUAL ITEM FIT

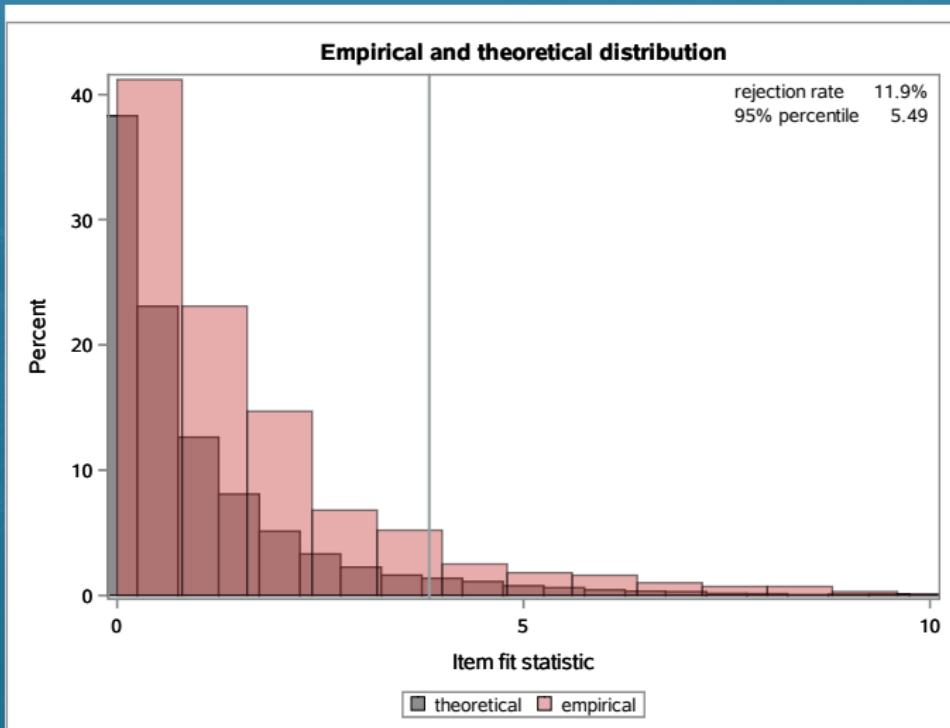
item	chisq	df	p
ADDRESS	1.73	1	0.1889
AGE	2.74	1	0.0976
COUNTBAC	0.98	1	0.3226
DOB	0.60	1	0.4382
FIRSTWW	2.25	1	0.1337
MONARCH	3.39	1	0.0656
MONTH	12.99	1	0.0003
NAME	2.74	1	0.0976
TIME	2.21	1	0.1369
YEAR	6.25	1	0.0124

- Look at the item 'MONTH'
- Simulate 1000 data sets based on $\hat{\eta}$ and $\hat{\theta}$
- Look at empirical distribution of $\chi^2(X_{MONTH})$
 - in data set of original size
 - in data set with twice the number of respondents

item 'MONTH' (sample size N)



item 'MONTH' (sample size $2 \cdot N$)



- Large observed χ^2 values indicate misfit
- Simulations: p -values cannot be trusted
- Makes it difficult to address multiple testing
- Simulations provide tables of critical values
- Fit statistics with known asymptotic distributions are known - are being implemented in SAS/R.

An aerial photograph of a European city, likely Copenhagen, Denmark. The city is built along a large, winding river that cuts through the urban landscape. In the foreground, there's a dense cluster of buildings, possibly residential apartments. A prominent feature is a large, solid blue circle that covers the upper two-thirds of the slide. Inside this circle, the word "SUMMARY" is written in a bold, white, sans-serif font.

SUMMARY

RUMM/WINSTEPS	R/SAS
Rasch model only	Rasch part of toolbox
stand-alone software (sell-by-date)	implementations not perfect
faulty stat. methods	valid methods not readily applicable
	time to meet challenge of teaching Rasch

An aerial photograph of a European city, likely Copenhagen, featuring a prominent river winding through the urban landscape. The city is densely built with a mix of historic and modern architecture. A large, solid blue circle is overlaid on the center of the image, partially obscuring the cityscape. Inside this blue circle, the word "PERSPECTIVES" is written in a bold, white, sans-serif font.

PERSPECTIVES

Perspectives in using simulation

- Appropriate critical value for $OUTFIT_i$?
- Appropriate critical value for $\max_i OUTFIT_i$?
 - is $FitResid_i$ better than χ^2 ?
- Tabulate critical values
- Implement parametric bootstrap

An aerial photograph of a European city, likely Copenhagen, featuring a dense grid of buildings, a winding river, and a bridge. A large, solid blue circle is overlaid on the center of the image, containing the text "THANK YOU".

THANK YOU