Calculate and Update Weights and Bias in a Neural Network

1. Initialization of Weights and Bias

Weights and bias are initialized randomly or using specific strategies (e.g., Xavier initialization). For this example, let's start with the following initialized weights and bias:

$$w_1 = 0.5, \quad w_2 = 0.4, \quad w_3 = 0.3$$

Inputs:

$$x_1 = 0.6, \quad x_2 = 0.2, \quad x_3 = 0.1$$

Bias:

$$b = 0.2$$

2. Calculate the Weighted Sum (Linear Combination)

The weighted sum z is calculated using:

$$z = w_1 \cdot x_1 + w_2 \cdot x_2 + w_3 \cdot x_3 + b$$

Substituting the values:

$$z = 0.5 \cdot 0.6 + 0.4 \cdot 0.2 + 0.3 \cdot 0.1 + 0.2 = 0.61$$

3. Activation Function

Using the **Sigmoid function** for non-linearity:

$$f(z) = \frac{1}{1 + e^{-z}}$$

Substituting $z = 0.61$:

$$f(0.61) \approx 0.648$$

The output of the neuron y is:

$$y = 0.648$$

4. Calculate the Loss

Assuming the **target output** is $y_{\text{true}} = 1$, using the **Mean Squared Error (MSE)** loss function:

$$L = \frac{1}{2}(y - y_{\text{true}})^2$$

Substituting the values:

$$L = \frac{1}{2}(0.648 - 1)^2 \approx 0.062$$

5. Backpropagation (Calculate the Gradient)

To update weights and bias, we need the **gradient of the loss function with respect to each weight and the bias**.

Calculate the Gradient for w_1:

The gradient of the loss L with respect to weight w_1 is:

$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial w_1}$$

Calculate the Gradient for the Bias b:

The gradient of the loss L with respect to the bias b is:

$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial b}$$

- **Calculate $\frac{\partial L}{\partial y}$:**

$$\frac{\partial L}{\partial y} = y - y_{\text{true}} = 0.648 - 1 = -0.352$$

- **Calculate $\frac{\partial y}{\partial z}$ (Sigmoid derivative):**

> $$\frac{\partial y}{\partial z} = f(z) \cdot (1 - f(z)) = 0.648 \cdot (1 - 0.648) \approx 0.228$$

- **Calculate $\frac{\partial z}{\partial b}$:**

> $$\frac{\partial z}{\partial b} = 1$$

6. Update the Weights and Bias (Using Gradient Descent)

Using the gradient descent rule for the weights and bias:

> $$w_i^{\text{new}} = w_i - \eta \cdot \frac{\partial L}{\partial w_i}, \quad b^{\text{new}} = b - \eta \cdot \frac{\partial L}{\partial b}$$

Assume a learning rate $\eta = 0.1$. For the bias:

> $$b^{\text{new}} = 0.2 - 0.1 \cdot (-0.080) = 0.208$$

7. Final Updated Weights and Bias

After one iteration, the updated weights and bias are:

> $$w_1^{\text{new}} = 0.5048, \quad w_2^{\text{new}} = 0.4016, \quad w_3^{\text{new}} = 0.3008, \quad b^{\text{new}} = 0.208$$