# Convergence of Alternating Gradient Descent for Matrix Factorization

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# Abstract

We consider alternating gradient descent (AGD) with fixed step size  $\eta > 0$ , applied to the asymmetric matrix factorization objective. We show that, for a rank-r matrix

$$\mathbf{A} \in \mathbb{R}^{m \times n}, T = \left(\left(\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}\right)^2 \log(1/\epsilon)\right)$$
 iterations of alternating gradient descent

suffice to reach an  $\epsilon$ -optimal factorization  $\|\mathbf{A} - \mathbf{X}_T \mathbf{Y}_T^\intercal\|_F^2 \leq \epsilon \|\mathbf{A}\|_F^2$  with high probability starting from an atypical random initialization. The factors have rank d > r so that  $\mathbf{X}_T \in \mathbb{R}^{m \times d}$  and  $\mathbf{Y}_T \in \mathbb{R}^{m \times d}$ . Experiments suggest that our proposed initialization is not merely of theoretical benefit, but rather significantly improves convergence of gradient descent in practice. Our proof is conceptually simple: a uniform PL-inequality and uniform Lipschitz smoothness constant are guaranteed for a sufficient number of iterations, starting from our random initialization. Our proof method should be useful for extending and simplifying convergence analyses for a broader class of nonconvex low-rank factorization problems.

## 1 Introduction

This paper focuses on the convergence behavior of alternating gradient descent (AGD) on the low-rank matrix factorization objective

$$\min f(\mathbf{X}, \mathbf{Y}) \equiv \frac{1}{2} \|\mathbf{X} \mathbf{Y}^{\mathsf{T}} - \mathbf{A}\|_{\mathrm{F}}^{2} \quad \text{subject to} \quad \mathbf{X} \in \mathbb{R}^{m \times d}, \mathbf{Y} \in \mathbb{R}^{n \times d}. \tag{1}$$

Here, we assume  $m, n \gg d > r = {\rm rank}({\bf A})$ . While there are a multitude of more efficient algorithms for low-rank matrix approximation, this serves as a simple prototype and special case of more complicated nonlinear optimization problems where gradient descent (or stochastic gradient descent) is the method of choice but not well-understood theoretically. Such problems include low-rank tensor factorization using the GCP algorithm descent [HKD20], a stochastic gradient variant of the GCP algorithm [KH20], as well as deep learning optimization.

Surprisingly, the convergence behavior of gradient descent for low-rank matrix factorization is still not optimally understood, in the sense that there is a large gap between theoretical guarantees and empirical performance. We take a step in closing this gap, providing a sharp linear convergence rate from a simple asymmetric random initialization. Precisely, we show that if  $\mathbf{A}$  is rank-r, then a number of iterations  $T = C\left(\frac{d}{d-r}\left(\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}\right)^2\log(1/\epsilon)\right)$  suffices to obtain an  $\epsilon$ -optimal factorization with high probability. Here  $\sigma_k(\mathbf{A})$  denotes the kth singular value of  $\mathbf{A}$ . To the authors' knowledge, this improves on the state-of-art convergence result in the literature [JCD22], which provides an

iteration complexity  $T = C\left(\left(\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}\right)^3 \log(1/\epsilon)\right)$  for gradient descent to reach an  $\epsilon$ -approximate rank-r approximation 1.

Our improved convergence analysis is likely due to our choice of initialization of  $\mathbf{X}_0$ ,  $\mathbf{Y}_0$ , which appears to be new in the literature and is distinct from the standard Gaussian initialization. Specifically, for  $\mathbf{\Phi}_1$  and  $\mathbf{\Phi}_2$  independent Gaussian matrices, we consider an "unbalanced" random initialization of the form  $\mathbf{X}_0 \sim \frac{1}{\sqrt{\eta}} \mathbf{A} \mathbf{\Phi}_1$  and  $\mathbf{Y}_0 \sim \sqrt{\eta} \mathbf{\Phi}_2$ . A crucial feature of this initialization is that the columns of  $\mathbf{X}_0$  are in the column span of  $\mathbf{A}$ , and thus by invariance of the alternating gradient update steps, the columns of  $\mathbf{X}_t$  remain in the column span of  $\mathbf{A}$  throughout the optimization. Because of this, a positive rth singular value of  $\mathbf{X}_t$  serves as a PL-inequality for the optimization process.

The second crucial feature of this initialization is that by Gaussian concentration, the pseudo-condition numbers  $\frac{\sigma_1(\mathbf{X}_0)}{\sigma_r(\mathbf{X}_0)} \sim \frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}$  are comparable with high probability<sup>2</sup>; for a range of step-size  $\eta$ , we show that  $\frac{\sigma_1(\mathbf{X}_t)}{\sigma_r(\mathbf{X}_t)}$  is guaranteed to remain comparable to  $\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}$  for a sufficiently large number of iterations t that we are guaranteed a linear rate of convergence with rate  $\left(\frac{\sigma_r(\mathbf{X}_0)}{\sigma_1(\mathbf{X}_0)}\right)^2$ . We expect that this approach should be of use in the analysis of a broader class of nonconvex optimization problems.

Our analysis crucially depends on an unbalanced initialization of  $\mathbf{X}_0$  and  $\mathbf{Y}_0$ , which is different from balanced random initializations standard in neural network training, which is the standard motivation for studying matrix factorization with gradient descent. The unbalanced initialization is not in contradiction to the known implicit bias of gradient descent towards balanced factorizations [CKL+21, WCZT21, ABC+22, CB22], which is of interest due to the potential link between balanced factorizations and better generalization. Indeed, our experiments strongly suggest that as gradient descent progresses, the unbalanced initial factor matrices become more balanced in that the factors increasingly share the condition number of  $\mathbf{A}$  between them, experiments indicate the rate of linear convergence improves, potentially to proportional to  $(\frac{\sigma_r(\mathbf{X}_0)}{\sigma_1(\mathbf{X}_0)})$  rather than  $(\frac{\sigma_r(\mathbf{X}_0)}{\sigma_1(\mathbf{X}_0)})^2$ ; see Fig. 2.

#### 2 Preliminaries

Consider the square loss applied to the matrix factorization problem (1). The gradients are

$$\nabla_{\mathbf{x}} f(\mathbf{X}, \mathbf{Y}) = (\mathbf{X} \mathbf{Y}^{\mathsf{T}} - \mathbf{A}) \mathbf{Y}, \tag{2a}$$

$$\nabla_{\mathbf{x}} f(\mathbf{X}, \mathbf{Y}) = (\mathbf{X} \mathbf{Y}^{\mathsf{T}} - \mathbf{A})^{\mathsf{T}} \mathbf{X}. \tag{2b}$$

The details are in Appendix B.

We will analyze alternating gradient descent, defined as follows.

Assumption 1 (Alternating Gradient Descent). For fixed stepsize  $\eta > 0$  and initial condition  $(\mathbf{X}_0, \mathbf{Y}_0)$ , the update is

$$\mathbf{X}_{t+1} = \mathbf{X}_t - \eta \nabla_{\mathbf{x}} f(\mathbf{X}_t, \mathbf{Y}_t), \tag{A1a}$$

$$\mathbf{Y}_{t+1} = \mathbf{Y}_t - \eta \nabla_{\mathbf{Y}} f(\mathbf{X}_{t+1}, \mathbf{Y}_t). \tag{A1b}$$

We assume that the iterations are initialized in an asymmetric way, which depends on the step size  $\eta$  and assumes a known upper bound on the spectral norm of **A**. The matrix factorization is of rank d > r, and we also make assumptions about the relationship of d, r, and quantities s,  $\beta$ , and  $\delta$  that will impact the bounds on the probability of finding and  $\epsilon$ -optimal factorization.

<sup>&</sup>lt;sup>1</sup>We note that our results are not precisely directly comparable as our analysis is for alternating gradient descent whereas existing results hold for gradient descent. However, empirically, alternating and non-alternating gradient descent exhibit similar behavior across many experiments

gradient descent exhibit similar behavior across many experiments  ${}^2$ The pseudo-condition number,  $\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}$ , is equivalent to and sometimes discussed as the product of the spectral norms of the matrix and its pseudoinverse, i.e.,  $\|\mathbf{A}\| \|\mathbf{A}^{\dagger}\|$ .

Assumption 2 (Initialization and key quantities). Draw random matrices  $\Phi_1, \Phi_2 \in \mathbb{R}^{n \times d}$  with i.i.d.  $\mathcal{N}(0, 1/d)$  and  $\mathcal{N}(0, 1/n)$  entries, respectively. Fix  $C \geq 1$ ,  $\nu < 1$ , and  $D \leq \frac{C}{9}\nu$ , and let

$$\mathbf{X}_0 = \frac{1}{\eta^{1/2} C \,\sigma_1(\mathbf{A})} \,\mathbf{A} \Phi_1, \quad \text{and} \quad \mathbf{Y}_0 = \eta^{1/2} \,D \,\sigma_1(\mathbf{A}) \,\Phi_2. \tag{A2a}$$

The factor matrices each have d columns where d>r satisfies the following conditions. There exists s>0 such that

$$\sqrt{r} + \sqrt{s} < \sqrt{d}$$
. (A2b)

Define the quantity

$$\rho = 1 - \frac{\sqrt{r} + \sqrt{s}}{\sqrt{d}} \in (0, 1).$$
(A2c)

The number of iterations for convergence to  $\epsilon$ -optimal factorization will ultimately be shown to depend on

$$\beta = \frac{\rho^2 \sigma_r^2(\mathbf{A})}{C^2 \sigma_1^2(\mathbf{A})}.$$
 (A2d)

The quantity s will impact the probability of finding this  $\epsilon$ -optimal factorization. For this purpose, we define

$$\delta = e^{-s/2} + e^{-r/2} + e^{-d/2},\tag{A2e}$$

to be employed later.

Observe that the initialization of  $X_0$  ensures its columns are in the column span of A.

It is useful to define residuals which yield the following relations:

$$\begin{aligned} \mathbf{R}_t &\equiv \mathbf{X}_t \mathbf{Y}_t^\intercal - \mathbf{A}, & f(\mathbf{X}_t, \mathbf{Y}_t) &= \|\mathbf{R}_t\|_{\mathrm{F}}^2, & \nabla_{\!\mathbf{x}} f(\mathbf{X}_t, \mathbf{Y}_t) &= \mathbf{R}_t \mathbf{Y}_t, \\ \bar{\mathbf{R}}_t &= \mathbf{Y}_t \mathbf{X}_{t+1}^\intercal - \mathbf{A}^\intercal & f(\mathbf{X}_{t+1}, \mathbf{Y}_t) &= \|\bar{\mathbf{R}}_t\|_{\mathrm{F}}^2, & \nabla_{\!\mathbf{x}} f(\mathbf{X}_{t+1}, \mathbf{Y}_t) &= \bar{\mathbf{R}}_t \mathbf{X}_{t+1}. \end{aligned}$$

We have conveniently transposed the second residual.

Remark 2.1. Observe that  $\mathbf{R}_0$  and hence  $f(\mathbf{X}_0, \mathbf{Y}_0)$  does not depend on the step size  $\eta$  of  $\sigma_1(\mathbf{A})$  in Assumption 2 since

$$\mathbf{R}_0 = \mathbf{X}_0 \mathbf{Y}_0^\intercal - \mathbf{A} = \mathbf{A} \left( \frac{D}{C} \mathbf{\Phi}_1 \mathbf{\Phi}_2^\intercal - \mathbf{I} \right).$$

Remark 2.2. In Assumption 2, we may assume that  $s \ll 1$ , so for all intents and purposes

$$\rho = 1 - \frac{\sqrt{r}}{\sqrt{d}}$$
 and  $\frac{1}{\rho} = \frac{\sqrt{d}}{\sqrt{d} - \sqrt{r}}$ .

Further, if d=r+1, then  $\frac{1}{\rho^2}\approx 4r^2$ . If  $d=(1+\alpha)r$ , then  $\frac{1}{\rho^2}\approx \left(\frac{\sqrt{1+\alpha}}{\sqrt{1+\alpha}-1}\right)^2$ , and there is no dependency on r.

# 3 Main results

Our first main result gives a sharp guarantee on the number of iterations necessary for alternating gradient descent to be guaranteed to product an  $\epsilon$ -optimal factorization.

**Theorem 3.1** (Main result, informal). For a rank-r matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , set d > r and consider  $\mathbf{X}_0, \mathbf{Y}_0$  randomly initialized as in Assumption 2. For any  $\epsilon > 0$ , there is a step-size  $\eta = \eta(\epsilon) > 0$  for alternating gradient descent as in Assumption 1 such that

$$\|\mathbf{A} - \mathbf{X}_T \mathbf{Y}_T^{\mathsf{T}}\|_{\mathrm{F}}^2 \le \epsilon \quad \text{for all} \quad T \ge C \left( \frac{\sigma_1^2(\mathbf{A})}{\sigma_r^2(\mathbf{A})} \frac{1}{\rho^2} \log \frac{\|\mathbf{A}\|_{\mathrm{F}}^2}{\epsilon} \right)$$

with probability  $1 - \delta$  with respect to the draw of  $\mathbf{X}_0$  and  $\mathbf{Y}_0$  where  $\delta$  is defined in (A2e). Here, C > 0 is an explicit numerical constant.

For more complete theorem statements, see Corollary 5.2 and Corollary 5.3 below. We highlight a few points below.

1. The iteration complexity in Theorem 3.1 is independent of the ambient dimensions n,m. In the edge case d=r+1, it holds that  $\frac{1}{\rho^2} \leq 4r^2$ , and so the iteration complexity depends linearly on r (see Remark 2.2). With mild multiplicative overparameterization  $d=(1+\alpha)r$ , the dependence on r is erased:  $\frac{1}{\rho^2} = \frac{(1+\alpha)}{(\sqrt{1+\alpha}-1)^2}$  Here, the improved convergence guarantee from mild overparameterization is directly traced to the improvement in the condition number of a  $d \times r$  Gaussian random matrix as d grows from r to  $(1+\alpha)r$ . We note that by appealing to high-probability bounds on the smallest and largest singular values of a square Gaussian matrix d=r, we could also derive a result for the case d=r; see e.g. [DS01, Theorem II.13] and [AZL17, Lemma i.A.3].

- 2. Experiments illustrate that initializing  $\mathbf{X}_0$  in the column space of  $\mathbf{A}$ , as well as rescaling  $\mathbf{X}_0$  and  $\mathbf{Y}_0$  asymmetrically, lead to significant practical convergence improvements, and are not just an artifact of our analysis; see Fig. 1.
- 3. The iteration complexity in Theorem 3.1 is conservative. In experiments, the convergence tends to switch from a linear rate proportional to  $\frac{\sigma_r^2(\mathbf{A})}{\sigma_1^2(\mathbf{A})}$  to a faster linear rate proportional to  $\frac{\sigma_r(\mathbf{A})}{\sigma_1(\mathbf{A})}$ .

#### 3.1 Our contribution and prior work

The seminal work of Burer and Monteiro [BM03, BM05] advocated for the general approach of using simple algorithms such as gradient descent directly applied to low-rank factor matrices for solving non-convex optimization problems with low-rank matrix solutions. Initial theoretical work on gradient descent for low-rank factorization problems such as [ZWL15], [TBS+16], [ZL16], [SWW17], [BKS16] did not prove global convergence of gradient descent, but rather local convergence of gradient descent starting from a spectral initialization (that is, an initialization involving SVD computations). In almost all cases, the spectral initialization is the dominant computation, and thus a more global convergence analysis for gradient descent is desirable.

Global convergence for gradient descent for matrix factorization problems without additional explicit regularization was first derived in the symmetric setting, where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is positive semi-definite, and  $f(\mathbf{X}) = \|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathsf{T}}\|_{\mathrm{F}}^2$ , see for example [GHJY15, JJKN17, CCFM19].

For overparametrized symmetric matrix factorization, the convergence behavior and implicit bias towards particular solutions for gradient descent with small step-size and from small initialization was analyzed in the work [GWB<sup>+</sup>17, LMZ18, ACHL19, CGMR20].

The paper [YD21] initiated a study of gradient descent with fixed step-size in the more challenging setting of asymmetric matrix factorization, where  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is rank-r and the objective is  $\|\mathbf{A} - \mathbf{X}\mathbf{Y}^{\mathsf{T}}\|_{\mathrm{F}}^2$ . This work improved on previous work in the setting of gradient flow and gradient descent with decreasing step-size [DHL18]. The paper [YD21] proved an iteration complexity of  $T = \mathcal{O}\left(nd\left(\frac{\sigma_1(\mathbf{A})}{\sigma_1(\mathbf{A})}\right)^4\log(1/\epsilon)\right)$  for reaching an  $\epsilon$ -approximate matrix factorization, starting from small i.i.d. Gaussian initialization for the factors  $\mathbf{X}_0, \mathbf{Y}_0$ . More recently, [JCD22] studied gradient descent for asymmetric matrix factorization, and proved an iteration complexity  $T = \mathcal{O}\left(C_d\left(\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}\right)^3\log(1/\epsilon)\right)$  to reach an  $\epsilon$ -optimal factorization, starting from small i.i.d. Gaussian initialization.

We improve on previous analysis of gradient descent applied to objectives of the form (1), providing an improved iteration complexity  $T = \mathcal{O}\left(\left(\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}\right)^2\log(1/\epsilon)\right)$  to reach an  $\epsilon$ -approximate factorization. There is no dependence on the matrix dimensions in our bound, and the dependence on the rank r disappears if the optimization is mildly over-parameterized, i.e.,  $d = (1+\alpha)r$ . We do note that our results are not directly comparable to previous work as we analyze alternating gradient descent rather than full gradient descent. Our method of proof is conceptually simpler than previous works; in particular, because our initialization  $\mathbf{X}_0$  is in the column span of  $\mathbf{A}$ , we do not require a two-stage analysis and instead can prove a fast linear convergence from the initial iteration.

## 4 Preliminary lemmas

**Lemma 4.1** (Bounding sum of norms of gradients). Consider alternating gradient decent as in Assumption 1. If  $\|\mathbf{Y}_t\|^2 \leq \frac{1}{n}$ , then

$$\|\nabla_{\mathbf{x}} f(\mathbf{X}_t, \mathbf{Y}_t)\|_{\mathbf{F}}^2 \le \frac{2}{n} \left( f(\mathbf{X}_t, \mathbf{Y}_t) - f(\mathbf{X}_{t+1}, \mathbf{Y}_t) \right). \tag{3}$$

If moreover  $\|\mathbf{X}_t\|^2 \leq \frac{2}{\eta}$ , then  $f(\mathbf{X}_t, \mathbf{Y}_t) \leq f(\mathbf{X}_t, \mathbf{Y}_{t-1})$ . Consequently, if  $\|\mathbf{Y}_t\|^2 \leq \frac{1}{\eta}$  for all  $t = 0, \dots, T$ , and  $\|\mathbf{X}_t\|^2 \leq \frac{2}{\eta}$  for all  $t = 0, \dots, T$ , then

$$\sum_{t=0}^{T} \|\nabla_{\mathbf{x}} f(\mathbf{X}_t, \mathbf{Y}_t)\|_{\mathbf{F}}^2 \le \frac{2}{\eta} f(\mathbf{X}_0, \mathbf{Y}_0)$$

$$\tag{4}$$

Likewise, if  $\|\mathbf{X}_{t+1}\|^2 \leq \frac{1}{n}$ , then

$$\|\nabla_Y f(\mathbf{X}_{t+1}, \mathbf{Y}_t)\|_{\mathrm{F}}^2 \le \frac{2}{\eta} (f(\mathbf{X}_{t+1}, \mathbf{Y}_t) - f(\mathbf{X}_{t+1}, \mathbf{Y}_{t+1})).$$
 (5)

If moreover  $\|\mathbf{Y}_t\|^2 \leq \frac{2}{\eta}$ , then  $f(\mathbf{X}_{t+1}, \mathbf{Y}_t) \leq f(\mathbf{X}_t, \mathbf{Y}_t)$ , and so if  $\|\mathbf{X}_{t+1}\|^2 \leq \frac{1}{\eta}$  for all  $t = 0, \dots, T$ , and  $\|\mathbf{Y}_t\|^2 \leq \frac{2}{\eta}$  for all  $t = 0, \dots, T$ , then

$$\sum_{t=0}^{T} \|\nabla_{\mathbf{x}} f(\mathbf{X}_{t+1}, \mathbf{Y}_t)\|_{F}^{2} \le \frac{2}{\eta} f(\mathbf{X}_0, \mathbf{Y}_0)$$
 (6)

*Proof.* Using alternating gradient descent per Assumption 1, we have

$$\bar{\mathbf{R}}_t = \mathbf{R}_t (\mathbf{I} - \eta \mathbf{Y}_t \mathbf{Y}_t^\intercal), \quad \text{and} \quad \mathbf{R}_{t+1} = (\mathbf{I} - \eta \mathbf{X}_{t+1} \mathbf{X}_{t+1}^\intercal) \bar{\mathbf{R}}_t.$$

If  $\|\mathbf{Y}_t\| \leq \frac{1}{\sqrt{\eta}}$ , we have

$$\begin{split} \|\bar{\mathbf{R}}_{t}\|_{\mathrm{F}}^{2} &= \|\mathbf{R}_{t}\|_{\mathrm{F}}^{2} - 2\eta \|\mathbf{R}_{t}\mathbf{Y}_{t}\|_{\mathrm{F}}^{2} + \eta^{2} \|\mathbf{R}_{t}\mathbf{Y}_{t}\mathbf{Y}_{t}^{\mathsf{T}}\|_{\mathrm{F}}^{2} \\ &\leq \|\mathbf{R}_{t}\|_{\mathrm{F}}^{2} - 2\eta \|\mathbf{R}_{t}\mathbf{Y}_{t}\|_{\mathrm{F}}^{2} + \eta^{2} \|\mathbf{Y}_{t}\|^{2} \|\mathbf{R}_{t}\mathbf{Y}_{t}\|_{\mathrm{F}}^{2} \\ &\leq \|\mathbf{R}_{t}\|_{\mathrm{F}}^{2} - \eta \|\mathbf{R}_{t}\mathbf{Y}_{t}\|_{\mathrm{F}}^{2}. \end{split}$$

The first line uses the cyclic invariance of trace, i.e.,  $\operatorname{trace}(\mathbf{Y}_t\mathbf{Y}_t^\intercal\mathbf{R}_t^\intercal\mathbf{R}_t) = \operatorname{trace}(\mathbf{R}_t^\intercal\mathbf{R}_t\mathbf{Y}_t\mathbf{Y}_t^\intercal) = \operatorname{trace}(\mathbf{R}_t\mathbf{Y}_t\mathbf{Y}_t^\intercal\mathbf{R}_t^\intercal)$ . The second lines uses the inequality  $\|\mathbf{A}\mathbf{B}\|_{\mathrm{F}} \leq \|\mathbf{A}\| \|\mathbf{B}\|_{\mathrm{F}}$  (and also the cyclic invariance of trace). The last line uses the assumption on  $\|\mathbf{Y}_t\|$ . Thus, rearranging terms yields

$$\|\mathbf{R}_t \mathbf{Y}_t\|_{\mathrm{F}}^2 \le \frac{1}{n} \Big( \|\mathbf{R}_t\|_{\mathrm{F}}^2 - \|\bar{\mathbf{R}}_t\|_{\mathrm{F}}^2 \Big).$$
 (7)

Now, observe that if  $\|\mathbf{X}_t\|^2 \leq \frac{2}{n}$ , then

$$\|\mathbf{R}_t\|_{\mathrm{F}}^2 \le \|\bar{\mathbf{R}}_{t-1}\|_{\mathrm{F}}^2.$$
 (8)

Using the assumptions on the norms of the factors, we then have

$$\sum_{t=0}^{T} \|\nabla_{\mathbf{x}} f(\mathbf{X}_{t}, \mathbf{Y}_{t})\|_{F}^{2} = \sum_{t=0}^{T} \|\mathbf{R}_{t} \mathbf{Y}_{t}\|_{F}^{2} \leq \frac{1}{\eta} \sum_{t=0}^{T} \left( \|\bar{\mathbf{R}}_{t-1}\|_{F}^{2} - \|\bar{\mathbf{R}}_{t}\|_{F}^{2} \right)$$
$$\leq \frac{1}{\eta} \|\bar{\mathbf{R}}_{0}\|_{F}^{2}.$$

The first inequality comes from (7). The last summation is a telescoping sum.

The second inequality is proven analogously.

**Proposition 4.2** (Bounding singular values of iterates). Consider alternating gradient decent as in Assumption 1. Set  $f_0 := f(\mathbf{X}_0, \mathbf{Y}_0)$ . Set  $T_* = \left\lfloor \frac{1}{32\eta^2 f_0} \right\rfloor$ . Suppose  $\sigma_1^2(\mathbf{X}_0) \le \frac{9}{16\eta}$ ,  $\sigma_1^2(\mathbf{Y}_0) \le \frac{9}{16\eta}$ . Then for all  $0 \le T \le T_*$ ,

1. 
$$\|\mathbf{X}_T\| \leq \frac{1}{\sqrt{\eta}}$$
 and  $\|\mathbf{Y}_T\| \leq \frac{1}{\sqrt{\eta}}$ ,

2. 
$$\sigma_r(\mathbf{X}_0) - \sqrt{2T\eta f_0} \le \sigma_r(\mathbf{X}_T) \le \sigma_1(\mathbf{X}_T) \le \sigma_1(\mathbf{X}_0) + \sqrt{2T\eta f_0}$$
,

3. 
$$\sigma_r(\mathbf{Y}_0) - \sqrt{2T\eta f_0} \le \sigma_r(\mathbf{Y}_T) \le \sigma_1(\mathbf{Y}_T) \le \sigma_1(\mathbf{Y}_0) + \sqrt{2T\eta f_0}$$
.

*Proof.* First, observe that by assumption,  $\|\mathbf{X}_0\|^2$ ,  $\|\mathbf{Y}_0\|^2 \leq \frac{9}{16\eta} \leq \frac{1}{\eta}$ . Now, suppose that that  $\|\mathbf{X}_0\|^2$ ,  $\|\mathbf{Y}_0\|^2 \leq \frac{9}{16\eta}$  and  $\|\mathbf{X}_t\|^2$ ,  $\|\mathbf{Y}_t\|^2 \leq \frac{1}{\eta}$  for  $t = 0, \dots T - 1$ , and  $1 \leq T \leq \lfloor \frac{1}{32\eta^2 f_0} \rfloor$ . Then by Lemma 4.1,

$$\sum_{t=0}^{T-1} \left\| \nabla_{\mathbf{x}} f(\mathbf{X}_t, \mathbf{Y}_t) \right\|_{\mathbf{F}}^2 \le \frac{2}{\eta} f_0. \tag{9}$$

Hence.

$$\|\mathbf{X}_{T} - \mathbf{X}_{0}\| \leq \eta \left\| \sum_{t=0}^{T-1} \nabla_{\mathbf{x}} f(\mathbf{X}_{t}, \mathbf{Y}_{t}) \right\|$$

$$\leq \eta \left\| \sum_{t=0}^{T-1} \nabla_{\mathbf{x}} f(\mathbf{X}_{t}, \mathbf{Y}_{t}) \right\|_{F}$$

$$\leq \eta \sqrt{T \sum_{t=0}^{T-1} \|\nabla_{\mathbf{x}} f(\mathbf{X}_{t}, \mathbf{Y}_{t})\|_{F}^{2}}$$

$$\leq \eta \sqrt{\frac{2T}{\eta}} f_{0}.$$
(10)

Then, for  $T \leq T_*$ ,

$$\|\mathbf{X}_{T}\| \leq \|\mathbf{X}_{0}\| + \|\mathbf{X}_{T} - \mathbf{X}_{0}\|$$

$$\leq \frac{3}{4\sqrt{\eta}} + \sqrt{2T\eta f_{0}}$$

$$\leq \frac{1}{\sqrt{\eta}}.$$
(11)

and so it follows that  $\|\mathbf{X}_t\|^2 \leq \frac{1}{\eta}$  for  $t = 0, \dots T$ . Using Lemma 4.1 again, and repeating the same argument,

$$\|\mathbf{Y}_t\| \le \frac{1}{\sqrt{\eta}}, \quad t = 0, \dots, T.$$

Thus, we can iterate the induction until  $T=T_*=\lfloor\frac{1}{32\eta^2f_0}\rfloor$ , to obtain  $\|\mathbf{X}_t\|^2, \|\mathbf{Y}_t\|^2\leq \frac{1}{\eta}$  for  $t=1,\ldots,T_*$ .

Because  $\|\mathbf{X}_T - \mathbf{X}_0\| \le \sqrt{2\eta T f_0}$  for  $T \le T_* = \lfloor \frac{1}{32\eta^2 f_0} \rfloor$ ,

$$\sigma_r(\mathbf{X}_T) \ge \sigma_r(\mathbf{X}_0) - \|\mathbf{X}_T - \mathbf{X}_0\|; \qquad \sigma_1(\mathbf{X}_T) \le \sigma_1(\mathbf{X}_0) + \|\mathbf{X}_T - \mathbf{X}_0\|.$$

A similar argument applies to achieve the stated bounds for  $\sigma_r(\mathbf{Y}_T)$  and  $\sigma_1(\mathbf{Y}_T)$ .

**Proposition 4.3** (Initialization). Assume  $X_0$  and  $Y_0$  are initialized as in Assumption 2, which fixes  $C \ge 1$ ,  $\nu < 1$ , and  $D \le \frac{C}{9}\nu$ , and consider alternating gradient decent as in Assumption 1. Then with probability at least  $1 - \delta$ , with respect to the random initialization and  $\delta$  defined in (A2e), the following hold:

1. 
$$\frac{1}{\sqrt{\eta}} \frac{\rho}{C} \frac{\sigma_r(\mathbf{A})}{\sigma_1(\mathbf{A})} \le \sigma_r(\mathbf{X}_0),$$

$$2. \ \sigma_1(\mathbf{X}_0) \le \frac{3}{C\sqrt{\eta}},$$

3. 
$$\sigma_1(\mathbf{Y}_0) \leq \frac{\sqrt{\eta} C \nu \sigma_1(\mathbf{A})}{3}$$

4. 
$$f(\mathbf{X}_0, \mathbf{Y}_0) \le \frac{1}{2} (1 + \nu)^2 ||\mathbf{A}||_{\mathrm{F}}^2$$
.

*Proof.* Write the SVD  $\mathbf{A} = \mathbf{U}_{m \times r} \mathbf{\Sigma}_{r \times r} \mathbf{V}_{r \times n}^{\mathsf{T}}$  so that  $\mathbf{A} \mathbf{\Phi}_1 = \mathbf{U}_{m \times r} \mathbf{\Sigma}_{r \times r} (\mathbf{V}^{\mathsf{T}} \mathbf{\Phi}_1)$ . Note that  $\mathbf{V}^{\mathsf{T}} \mathbf{\Phi}_1 \in \mathbb{R}^{r \times d}$  has i.i.d. Gaussian entries  $\mathcal{N}(0, \frac{1}{d})$ . By concentration of measure Proposition A.1, with probability at least  $1 - e^{-s/2}$ ,

$$\left(1 - \frac{\sqrt{r} + \sqrt{s}}{\sqrt{d}}\right) \le \sigma_r(\mathbf{V}^{\intercal} \mathbf{\Phi}_1).$$

Applying concentration of measure Proposition A.1 again, with probability at least  $1 - e^{-r/2} - e^{-d/2}$ ,

$$\sigma_1(\mathbf{\Phi}_1) \le \left(1 + \frac{2\sqrt{r}}{\sqrt{d}}\right) \le 3, \quad \text{and}$$

$$\sigma_1(\mathbf{\Phi}_2) \le \left(1 + \frac{2\sqrt{d}}{\sqrt{m}}\right) \le 3.$$

If these events hold,

$$\sigma_1(\mathbf{V}^{\mathsf{T}}\mathbf{\Phi}_1) \leq \sigma_1(\mathbf{V})\sigma_1(\mathbf{\Phi}_1) \leq 3,$$

and,

$$\begin{split} \frac{\rho}{\sqrt{\eta}C\sigma_1(\mathbf{A})}\sigma_r(\mathbf{A}) &\leq \sigma_r(\mathbf{X}_0) \leq \sigma_1(\mathbf{X}_0) \leq \frac{3}{\sqrt{\eta}C}, \\ \sigma_1(\mathbf{Y}_0) &\leq 3\sqrt{\eta}D\sigma_1(\mathbf{A}) \leq \frac{\sqrt{\eta}C\nu\sigma_1(\mathbf{A})}{3}. \end{split}$$

where the last inequality uses  $D \leq \frac{C\nu}{9}$ . Consequently,

$$1 - \nu \le 1 - \frac{D}{C}\sigma_1(\mathbf{\Phi}_1)\sigma_1(\mathbf{\Phi}_2) \le \left\|\mathbf{I} - \frac{D}{C}\mathbf{\Phi}_1\mathbf{\Phi}_2^{\mathsf{T}}\right\| \le 1 + \frac{D}{C}\sigma_1(\mathbf{\Phi}_1)\sigma_1(\mathbf{\Phi}_2) \le 1 + \nu.$$

Hence,

$$2f(\mathbf{X}_0, \mathbf{Y}_0) = \left\| \mathbf{A} (\mathbf{I} - \frac{D}{C} \mathbf{\Phi}_1 \mathbf{\Phi}_2^{\mathsf{T}}) \right\|_{\mathrm{F}}^2 \le (1 + \nu)^2 \|\mathbf{A}\|_{\mathrm{F}}^2.$$

Combining the previous two propositions gives the following corollary.

**Corollary 4.4.** Assume  $X_0$  and  $Y_0$  are initialized as in Assumption 2, with the stronger assumption that  $C \geq 4$ . Consider alternating gradient decent as in Assumption 1 with

$$\eta \le \frac{9}{4C\nu\sigma_1(\mathbf{A})}.$$

With  $\beta$  as in (A2d) and  $f_0 = f(\mathbf{X}_0, \mathbf{Y}_0)$ , set

$$T = \left| \frac{\beta}{8\eta^2 f_0} \right|.$$

Then with probability at least  $1 - \delta$ , with respect to the random initialization and  $\delta$  defined in (A2e), the following hold for all t = 1, ..., T:

1. 
$$\sigma_r(\mathbf{X}_t) \geq \frac{1}{2} \sqrt{\frac{\beta}{\eta}}$$

2. 
$$\sigma_1(\mathbf{X}_t), \sigma_1(\mathbf{Y}_t) \leq \frac{3}{C\sqrt{\eta}} + \frac{1}{2}\sqrt{\frac{\beta}{\eta}}$$

*Proof.* By Proposition 4.3, we have the following event occurring with the stated probability:

$$\frac{\rho^2 \sigma_r^2(\mathbf{A})}{C^2 \sigma_1^2(\mathbf{A}) \eta} \le \sigma_r^2(\mathbf{X}_0) \le \sigma_1^2(\mathbf{X}_0) \le \frac{9}{16\eta}$$

where the upper bound uses that  $C \geq 4$ . Moreover, using that  $\eta \leq \frac{9}{4C\nu\sigma_1(\mathbf{A})}$ ,

$$\sigma_1^2(\mathbf{Y}_0) \le 9\eta C^2 \sigma_1(\mathbf{A})^2 \le \frac{9}{16\eta}.$$

For  $\beta$  as in (A2d), note that

$$T = \left| \frac{\beta}{8\eta^2 f_0} \right| \le \left| \frac{1}{32\eta^2 f_0} \right|,$$

which means that we can apply Proposition 4.2 up to iteration T, resulting in the bound

$$\sigma_r(\mathbf{X}_t) \ge \sigma_r(\mathbf{X}_0) - \sqrt{2T\eta f_0} \ge \frac{1}{2}\sqrt{\frac{\beta}{\eta}},$$

and, similarly,  $\sigma_1(\mathbf{X}_t), \sigma_1(\mathbf{Y}_t) \leq \frac{3}{C\sqrt{\eta}} + \frac{1}{2}\sqrt{\frac{\beta}{\eta}}$ .

Finally, we use a couple crucial lemmas which apply to our initialization of  $X_0$  and  $Y_0$ .

**Lemma 4.5.** Consider alternating gradient decent as in Assumption 1. If  $ColSpan(\mathbf{X}_0) \subset ColSpan(\mathbf{A})$ , then  $ColSpan(\mathbf{X}_t) \subset ColSpan(\mathbf{A})$  for all t.

*Proof.* Suppose  $ColSpan(\mathbf{X}_t) \subset ColSpan(\mathbf{A})$  and  $ColSpan(\mathbf{Y}_t) \subset RowSpan(\mathbf{A})$ . Then  $ColSpan(\mathbf{X}_t\mathbf{Y}_t^\intercal\mathbf{Y}_t) \subset ColSpan(\mathbf{A})$  and by the update of Assumption 1,

$$\begin{aligned} \operatorname{ColSpan}(\mathbf{X}_{t+1}) &= \operatorname{ColSpan}(\mathbf{X}_t + \eta \mathbf{A} \mathbf{Y}_t - \eta \mathbf{X}_t \mathbf{Y}_t^\intercal \mathbf{Y}_t) \\ &\subset \operatorname{ColSpan}(\mathbf{X}_t) \cup \operatorname{ColSpan}(\mathbf{A} \mathbf{Y}_t) \cup \operatorname{ColSpan}(\mathbf{X}_t \mathbf{Y}_t^\intercal \mathbf{Y}_t) \\ &\subset \operatorname{ColSpan}(\mathbf{A}). \end{aligned} \quad \Box$$

**Lemma 4.6.** If **A** is rank r, and if  $ColSpan(\mathbf{X}_t) \subset ColSpan(\mathbf{A})$  and  $\sigma_r(\mathbf{X}_t) > 0$  then

$$\|\nabla_{\mathbf{Y}} f(\mathbf{X}_t, \mathbf{Y}_{t-1})\|_{\mathbf{F}}^2 \ge 2\sigma_r^2(\mathbf{X}_t) f(\mathbf{X}_t, \mathbf{Y}_{t-1}). \tag{12}$$

*Proof.* If **A** is rank r, if  $ColSpan(\mathbf{X}_t) \subset ColSpan(\mathbf{A})$ , and if  $\sigma_r(\mathbf{X}_t) > 0$ , then  $\mathbf{X}_t$  is rank-r and  $ColSpan(\mathbf{X}_t) = ColSpan(\mathbf{A})$ . In this case, each column of  $(\mathbf{X}_t\mathbf{Y}_t^\intercal - \mathbf{A})$  is in the row span of  $\mathbf{X}_t^\intercal$ , and so

$$\begin{split} \|\nabla_{\mathbf{y}} f(\mathbf{X}_{t}, \mathbf{Y}_{t-1})\|_{\mathbf{F}}^{2} &= \|(\mathbf{X}_{t} \mathbf{Y}_{t-1}^{\mathsf{T}} - \mathbf{A})^{\mathsf{T}} \mathbf{X}_{t}\|_{\mathbf{F}}^{2} \\ &= \|\mathbf{X}_{t}^{\mathsf{T}} (\mathbf{X}_{t} \mathbf{Y}_{t-1}^{\mathsf{T}} - \mathbf{A})\|_{\mathbf{F}}^{2} \\ &\geq \sigma_{r}^{2} (\mathbf{X}_{t}) \|\mathbf{X}_{t} \mathbf{Y}_{t-1}^{\mathsf{T}} - \mathbf{A}\|_{\mathbf{F}}^{2}. \end{split}$$

## 5 Main results

We are now ready to prove the main results.

**Theorem 5.1.** Assume  $X_0$  and  $Y_0$  are initialized as in Assumption 2, with the stronger assumption that  $C \ge 4$ . Consider alternating gradient decent as in Assumption 1 with

$$\eta \le \frac{9}{4C\nu\sigma_1(\mathbf{A})}.$$

With  $\beta$  as in (A2d) and  $f_0 = f(\mathbf{X}_0, \mathbf{Y}_0)$ , set

$$T = \left| \frac{\beta}{8\eta^2 f_0} \right|.$$

Then with probability at least  $1 - \delta$ , with respect to the random initialization and  $\delta$  defined in (A2e), the following hold for all t = 1, ..., T:

$$\|\mathbf{A} - \mathbf{X}_{t} \mathbf{Y}_{t}^{\mathsf{T}}\|_{\mathrm{F}}^{2} \leq 2 \exp(-\beta t/4) f_{0}$$

$$\leq \exp(-\beta t/4) (1 + \nu)^{2} \|\mathbf{A}\|_{\mathrm{F}}^{2}. \tag{13}$$

*Proof.* Corollary 4.4 implies that  $\sigma_r(\mathbf{X}_t)^2 \geq \frac{\beta}{4\eta}$  for  $t=1,\ldots,T$ . Lemmas 4.5 and 4.6 imply since  $\mathbf{X}_0$  is initialized in the column space of  $\mathbf{A}$ ,  $\mathbf{X}_t$  remains in the column space of  $\mathbf{A}$  for all t, and

$$\|\nabla_{\mathbf{Y}} f(\mathbf{X}_{t+1}, \mathbf{Y}_{t})\|_{\mathbf{F}}^{2} = \|(\mathbf{A}^{\mathsf{T}} - \mathbf{Y}_{t} \mathbf{X}_{t+1}^{\mathsf{T}}) \mathbf{X}_{t+1}\|_{\mathbf{F}}^{2}$$

$$\geq \sigma_{r} (\mathbf{X}_{t+1})^{2} \|(\mathbf{A}^{\mathsf{T}} - \mathbf{Y}_{t} \mathbf{X}_{t+1}^{\mathsf{T}})\|_{\mathbf{F}}^{2}$$

$$\geq \frac{\beta}{4\eta} \|\mathbf{A} - \mathbf{X}_{t+1} \mathbf{Y}_{t}^{\mathsf{T}}\|_{\mathbf{F}}^{2}$$

$$= \frac{\beta}{2\eta} f(\mathbf{X}_{t+1}, \mathbf{Y}_{t}).$$
(14)

That is, a lower bound on  $\sigma_r(\mathbf{X}_t)^2$  implies that the gradient step with respect to  $\mathbf{Y}$  satisfies the Polyak-Lojasiewicz (PL)-equality<sup>3</sup>.

We can combine this PL inequality with the Lipschitz bound from Lemma 4.1 to derive the linear convergence rate. Indeed, by (3),

$$f(\mathbf{X}_{t+1}, \mathbf{Y}_{t+1}) - f(\mathbf{X}_{t+1}, \mathbf{Y}_t) \le -\frac{\eta}{2} \|\nabla_{\mathbf{Y}} f(\mathbf{X}_{t+1}, \mathbf{Y}_t)\|_{\mathbf{F}}^2$$
$$\le -\frac{\beta}{4} f(\mathbf{X}_{t+1}, \mathbf{Y}_t).$$

where the final inequality is (14). Consequently,

$$f(\mathbf{X}_T, \mathbf{Y}_T) \le (1 - \frac{\beta}{4}) f(\mathbf{X}_{T-1}, \mathbf{Y}_{T-1})$$
$$\le (1 - \beta/4)^T f(\mathbf{X}_0, \mathbf{Y}_0)$$
$$\le \exp(-\beta T/4) f(\mathbf{X}_0, \mathbf{Y}_0).$$

The final inequality uses Proposition 4.3.

<sup>&</sup>lt;sup>3</sup>A function f satisfies the PL-equality if for all  $\mathbf{x} \in \mathbb{R}^m$ ,  $f(\mathbf{x}) - f(\mathbf{x}^*) \le \frac{1}{2m} \|\nabla f(\mathbf{x})\|^2$ , where  $f(\mathbf{x}^*) = \min_{\mathbf{x}} f(\mathbf{x})$ 

**Corollary 5.2.** Assume  $X_0$  and  $Y_0$  are initialized as in Assumption 2, with the stronger assumptions that  $C \ge 4$  and  $\nu \le \frac{1}{2}$ . Consider alternating gradient decent as in Assumption 1 with

$$\eta \le \frac{\beta}{\sqrt{32f_0 \log(2f_0/\epsilon)}},\tag{15}$$

where  $\beta$  is defined in (A2d) and  $f_0 = f(\mathbf{X}_0, \mathbf{Y}_0)$ . Then with probability at least  $1 - \delta$ , with respect to the random initialization and  $\delta$  defined in (A2e), it holds

$$\|\mathbf{A} - \mathbf{X}_T \mathbf{Y}_T^\intercal\|_{\mathrm{F}}^2 \leq \epsilon \quad \text{at interation} \quad T = \left\lfloor \frac{\beta}{8\eta^2 f_0} \right\rfloor.$$

Here  $\rho$  is defined in (A2c). Using the upper bound for  $\eta$  in (15), the iteration complexity to reach an  $\epsilon$ -optimal loss value is

$$T = \mathcal{O}\left(\left(\frac{\sigma_1(\mathbf{A})}{\sigma_r(\mathbf{A})}\right)^2 \frac{1}{\rho^2} \log\left(\frac{\|\mathbf{A}\|_{\mathrm{F}}^2}{\epsilon}\right)\right).$$

This corollary follows from Theorem 5.1 by solving for  $\eta$  so that the RHS of (13) is at most  $\epsilon$ , and then noting that  $\eta \leq \frac{\beta}{\sqrt{32f_0\log(2f_0/\epsilon)}}$  implies that  $\eta \leq \frac{9}{4C\nu\sigma_1(\mathbf{A})}$  when  $\nu \leq \frac{1}{2}$ , using the lower bound on  $f_0$  from Proposition 4.3.

Using this corollary recursively, we can prove that the loss value remains small for  $T' \ge \lfloor \frac{\beta}{8\eta^2 f_0} \rfloor$ , provided we increase the lower bound on C by a factor of 2.

**Corollary 5.3.** Assume  $X_0$  and  $Y_0$  are initialized as in Assumption 2, with the stronger assumptions that  $C \geq 8$  and  $\nu \leq \frac{1}{2}$ . Let  $\epsilon < 1/16$ , and consider alternating gradient decent as in Assumption 1 with

$$\eta \le \frac{\beta}{\sqrt{32f_0 \log(1/\epsilon)}},\tag{16}$$

where  $\beta$  is defined in (A2d) and  $f_0 = f(\mathbf{X}_0, \mathbf{Y}_0)$ . Then with probability at least  $1 - \delta$ , with respect to the random initialization and  $\delta$  defined in (A2e), it holds that

$$\|\mathbf{A} - \mathbf{X}_{T} \mathbf{Y}_{T}^{\mathsf{T}}\|_{\mathrm{F}}^{2} \leq \epsilon \|\mathbf{A} - \mathbf{X}_{0} \mathbf{Y}_{0}^{\mathsf{T}}\|_{\mathrm{F}}^{2} \qquad for \quad T \geq \left\lfloor \frac{\beta}{8\eta^{2} f_{0}} \right\rfloor;$$

$$\|\mathbf{A} - \mathbf{X}_{T} \mathbf{Y}_{T}^{\mathsf{T}}\|_{\mathrm{F}}^{2} \leq \epsilon^{2} \|\mathbf{A} - \mathbf{X}_{0} \mathbf{Y}_{0}^{\prime}\|_{\mathrm{F}}^{2} \qquad for \quad T \geq \left\lfloor \frac{\beta}{8\eta^{2} f_{0}} \right\rfloor + \left\lfloor \frac{1}{4\epsilon} \frac{\beta}{8\eta^{2} f_{0}} \right\rfloor;$$

$$\vdots$$

$$\|\mathbf{A} - \mathbf{X}_T \mathbf{Y}_T^{\mathsf{T}}\|_{\mathrm{F}}^2 \le \epsilon^k \|\mathbf{A} - \mathbf{X}_0 \mathbf{Y}_0'\|_{\mathrm{F}}^2 \qquad for \quad T \ge \sum_{\ell=0}^{k-1} \left[ \left( \frac{1}{4\epsilon} \right)^{\ell} \frac{\beta}{8\eta^2 f_0} \right].$$

*Proof.* Set  $\beta_1 = \beta$  as in (A2d). Set  $f_{0(1)} = f_0$ .

By Corollary 5.2, iterating (1) for  $T_1 = \lfloor \frac{\beta_1}{8\eta^2 f_{0(1)}} \rfloor$  iterations with step-size

$$\eta \le \frac{\beta_1}{\sqrt{32f_{0(1)}\log(1/\epsilon)}}$$

guarantees that

$$\frac{1}{2} \|\mathbf{A} - \mathbf{X}_{T_1} \mathbf{Y}'_{T_1} \|_{\mathrm{F}}^2 \le f_{0(2)} := \epsilon f_{0(1)};$$
$$\|\sigma_r(\mathbf{X}_{T_1})\|^2 \ge \frac{1}{4} \frac{\beta_1}{\eta}.$$

This means that at time  $T_1$ , we can restart the analysis, and appeal again to Proposition 4.2 with modified parameters

- $f(\mathbf{X}_{T_1}, \mathbf{Y}_{T_1}) \le f_{0_2} := \epsilon f_{0_1}$ ,
- $\beta_2 := \frac{\beta_1}{4}$ .

Corollary 5.2 again guarantees that provided

$$\eta \le \frac{\beta_2}{\sqrt{32f_{0(2)}\log(1/\epsilon)}} = \frac{1}{4\sqrt{\epsilon}} \frac{\beta_1}{\sqrt{32f_{0(1)}\log(1/\epsilon)}}$$
(17)

then  $f(\mathbf{X}_{T_1+T_2}, \mathbf{Y}_{T_1+T_2}) \le \epsilon f(\mathbf{X}_{T_1}, \mathbf{Y}_{T_1}) \le \epsilon^2 f(\mathbf{X}_0, \mathbf{Y}_0)$  where

$$T_2 = \frac{T_1}{4\epsilon}. (18)$$

We have that (17) is satisfied by assumption as we assume  $\epsilon \leq \frac{1}{16}$ . Repeating this inductively, we find that after  $T = T_1 + \dots + T_k = T_1 \sum_{\ell=0}^{k-1} (\frac{1}{4\epsilon})^\ell \leq T_1(\frac{1}{4\epsilon})^k$  iterations, we are guaranteed that  $f(X_T,Y_T) \leq \epsilon^k f(X_0,Y_0)$ . This is valid for any  $k \in \mathbb{N}$  because we may always apply Proposition 4.2 in light of summability and  $C \geq 8$ : for any t,

$$\sigma_{1}(\mathbf{X}_{t}) \leq \sigma_{1}(\mathbf{X}_{0}) + \sqrt{\eta} \sum_{j=1}^{k} \sqrt{2T_{k} f_{0(k)}}$$

$$\leq \frac{3}{8\sqrt{\eta}} + \frac{1}{\sqrt{\eta}} \sum_{j=1}^{k} \sqrt{2(1/(4\epsilon))^{j} \frac{\beta_{1}}{8f_{01}} \epsilon^{j} f_{0(1)}}$$

$$\leq \frac{3}{8\sqrt{\eta}} + \frac{\sqrt{\beta}}{2\sqrt{\eta}} \sum_{j=1}^{k} (1/2)^{j}$$

$$\leq \frac{3 + 4\sqrt{\beta}}{8\sqrt{\eta}} \leq \frac{1}{2\sqrt{\eta}}.$$

## 6 Numerical experiments

We perform an illustrative numerical experiment to demonstrate the effect of different choice of initialization. In Fig. 1, we factorize a rank-5 matrix of size  $100 \times 100$ . The matrix is constructed as  $\mathbf{A} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^{\mathsf{T}}$  with  $\mathbf{U}$  and  $\mathbf{V}$  random  $100 \times 5$  orthonormal matrices and  $\boldsymbol{\Sigma} = \mathrm{diag}(1.000, 0.875, 0.750, 0.625, 0.500)$ . The same matrix is used for all experiments. We consider three initializations:

- 1. Orange. The orange initialization is from Assumption 2. Here,  $X_0$  is in the column space of A. Further, the scale is asymmetric, with  $Y_0$  multiplied by  $\sqrt{\eta}\sigma_1(A)$  and  $X_0$  by the inverse. In this case, the constants are C=4,  $\nu=1\text{e-9}$ , and  $D=C\nu/9$ . This initialization provides a significant "head start" in convergence, moving quickly to fast convergence.
- 2. **Blue.** The blue initialization keeps  $X_0$  in the column space of A but scales both matrices similarly. Here we see that the multiplication by A is helpful but not the sole source of improvement.
- 3. **Green.** The green initialization is the typical initialization: choose  $\mathbf{X}_0$  and  $\mathbf{Y}_0$  as suitably normalized random Gaussians, similarly scaled. From this initialization, the alternating gradient descent has a slow start.

The factorization uses rank d=6=r+1. The step length for all runs is  $\eta=0.0683$ . The rate of convergence is much improved for the initialization from Assumption 2. We explore this further in the next experiment.

The rate of convergence is tied to the ratio of the largest and smallest singular values, which we illustrate in Figs. 2 and 3. In these experiments, we factorize three different rank-5 matrices of size

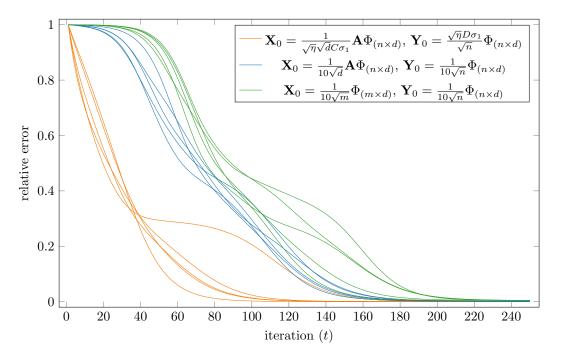


Figure 1: **Effect of initialization.** Five runs each of alternating gradient descent for matrix factorization with matrix of size  $100 \times 100$  (m=n=100) and rank 5 (r=5). All runs use the same matrix, which is constructed such that  $\sigma_1=1$ ,  $\sigma_r=0.5$ , and the remaining singular values are evenly spaced between. For all runs, we have  $\eta=0.0683$  (which is 1e6 times the theoretical value), d=6, C=4,  $\nu=1$ e-10,  $D=C\nu/9$ , and each  $\Phi$  is an indepedent Gaussian random matrix.

 $100 \times 100$  with different singular values ratios. The matrices are each constructed as  $\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^\mathsf{T}$  with  $\mathbf{U}$  and  $\mathbf{V}$  random  $100 \times 5$  orthonormal matrices. The difference is that each has a different set of singular values. All three have  $\sigma_1 = 1$ , and then we have  $\sigma_r = 0.1$  in orange,  $\sigma_r = 0.5$  in blue, and  $\sigma_r = 0.9$  in green. The intermediate singular values are evenly spaced between the specified largest and smallest singular values. The factors are rank d = r + 1 = 6. The step length varies as we use 10000 times the bound (15) with C = 4. The initialization is proposed in Assumption 2 with  $\nu = 1\text{e-}10$ . We run each method five times. The number of iterations until convergence depends on  $\sigma_r^2(\mathbf{A})/\sigma_1^2(\mathbf{A})$ , so we plot reference curves for comparison, as dashed lines. Smaller values of  $\sigma_r(\mathbf{A})/\sigma_1(\mathbf{A})$  indicate worse conditioning of the problem and result in slower convergence. We zoom in on the first few iterations in Fig. 3 to show that this bound is fairly tight initially.

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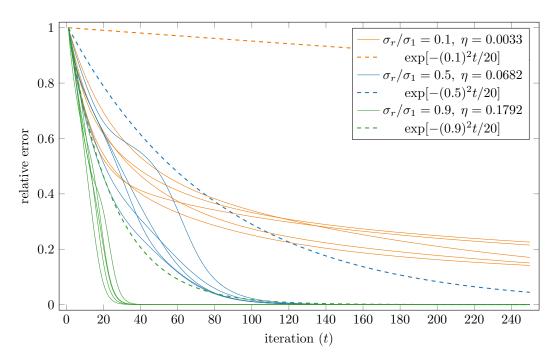


Figure 2: Five runs each of alternating gradient descent for matrix factorization with matrix of size  $100 \times 100$  and rank 5. Each matrix is constructed such that  $\sigma_1 = 1$ ,  $\sigma_r \in \{0.1, 0.5, 0.9\}$ , and the remaining singular values are evenly spaced between. The  $\eta$  is 1e4 times the theoretical value using d=6, C=4, and  $\nu=1$ e-10.

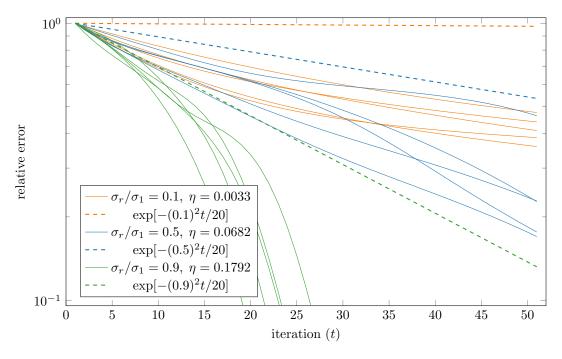


Figure 3: Zoom into the first few iterations of Fig. 2, with the y-axis in log scale.

# A Non-asympototic singular value bounds

**Proposition A.1** ([Ver10]). Let A be an  $d \times r$  matrix whose entries are independently drawn from  $\mathcal{N}(0,1)$ . Then for every  $t \geq 0$ , with probability at least  $1 - \exp(-t^2/2)$ , we have

$$\mathbb{E}\sigma_r(\mathbf{A}) \geq \sqrt{d} - \sqrt{r} - t$$

and for every  $t \ge 0$ , with probability at least  $1 - \exp(-t^2/2)$ , we have

$$\mathbb{E}\sigma_1(\mathbf{A}) \le \sqrt{d} + \sqrt{r} + t$$

If the variance if v, then everything in the above gets divided by  $\sqrt{v}$ .

## **B** Differentiation

Consider the function

$$f(\mathbf{X}, \mathbf{Y}) \equiv \frac{1}{2} \left\| \mathbf{X} \mathbf{Y}^{\mathsf{T}} - \mathbf{A} \right\|_{\mathrm{F}}^{2}.$$

with  $\mathbf{X} \in \mathbb{R}^{m \times d}$  and  $\mathbf{Y} \in \mathbb{R}^{n \times d}$ . Using the Kronecker relation  $\text{vec}(\mathbf{ACB}^{\mathsf{T}}) = (\mathbf{B} \otimes \mathbf{A}) \text{vec}(\mathbf{C})$ , we can express f as a function of  $\text{vec}(\mathbf{X})$ :

$$f(\mathbf{X}, \mathbf{Y}) = \frac{1}{2} \| (\mathbf{Y} \otimes \mathbf{I}_m) \operatorname{vec}(\mathbf{X}) - \operatorname{vec}(\mathbf{A}) \|_2^2.$$

Now we can apply vector calculus to observe

$$\nabla_{\mathbf{x}} f(\mathbf{X}, \mathbf{Y}) = (\mathbf{Y} \otimes \mathbf{I}_m)^{\mathsf{T}} ((\mathbf{Y} \otimes \mathbf{I}_m) \operatorname{vec}(\mathbf{X}) - \operatorname{vec}(\mathbf{A}))$$

$$= (\mathbf{Y}^{\mathsf{T}} \mathbf{Y} \otimes \mathbf{I}_m) \operatorname{vec}(\mathbf{X}) - (\mathbf{Y} \otimes \mathbf{I}_m)^{\mathsf{T}} \operatorname{vec}(\mathbf{A}) \in \mathbb{R}^{md}$$

$$= \mathbf{X} (\mathbf{Y}^{\mathsf{T}} \mathbf{Y}) - \mathbf{A} \mathbf{Y}$$

$$= (\mathbf{X} \mathbf{Y}^{\mathsf{T}} - \mathbf{A}) \mathbf{Y} \in \mathbb{R}^{m \times d}.$$

Applying the same logic to the equivalent  $f(\mathbf{X}, \mathbf{Y}) = \frac{1}{2} \|\mathbf{Y}\mathbf{X}^{\mathsf{T}} - \mathbf{A}^{\mathsf{T}}\|_{\mathrm{F}}^2$  yields

$$\nabla_{\!\scriptscriptstyle{\mathbf{Y}}} f(\mathbf{X}, \mathbf{Y}) = (\mathbf{X} \mathbf{Y}^\intercal - \mathbf{A})^\intercal \mathbf{X} \in \mathbb{R}^{n \times d}.$$

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