Solutions to Part B of Problem Sheet 6

Solution (6.4)

(a) We first write down the matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 & 1.2 & 1.4 & 1.6 & 1.8 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$:

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} 1 & 1.01 & 1.04 & 1.09 & 1.16 & 1.25 & 1.36 & 1.49 & 1.64 & 1.81 & 2 \end{pmatrix}^\top$$

The primal version of this problem is

$$\begin{aligned} \text{minimize} \quad & x_1 + 1.01x_2 + 1.04x_3 + 1.09x_4 + 1.16x_5 + 1.25x_6 + 1.36x_7 \\ & + 1.49x_8 + 1.64x_9 + 1.81x_{10} + 2x_{11} \\ \text{subject to} \quad & 0.2x_2 + 0.4x_3 + 0.6x_4 + 0.8x_5 + x_6 + 1.2x_7 + 1.4x_8 + 1.6x_9 \\ & + 1.8x_{10} + 2x_{11} = 1 \\ & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} = 1 \\ & x_i \geq 0. \end{aligned}$$

(b) The problem has $m = 2$ dual variables y_1 and y_2, so the projection of the trajectory on the $\boldsymbol{y}$ plane can be easily visualized. A naive implementation is shown below. The trajectories are shown in the figures.

In [1]:

```
import numpy as np
import numpy.linalg as la

v = np.linspace(0,1,11)
n = len(v)
A = np.concatenate((2*v.reshape((1,n)), np.ones((1,n))), axis=0)
c = 1+v**2
b = np.array([1,1])
```

Define function F and Jacobian matrix M.

In [2]:

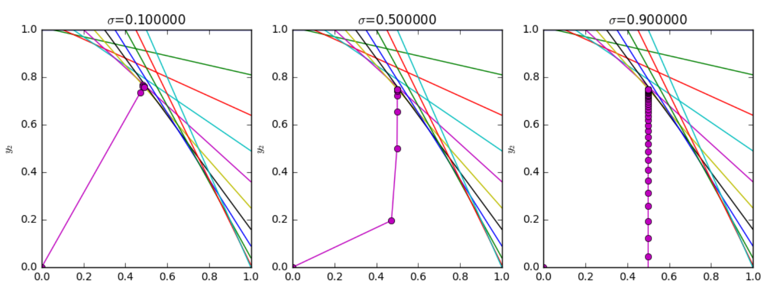
```
def F(x, y, s):
    C1 = np.dot(A.T,y)+s-c
    C2 = np.dot(A,x)-b
    C3 = x*s
    return np.concatenate((C1, C2, C3))

def M(x, y, s):
    return np.asarray(np.bmat([[np.zeros((n,n)), A.T, np.eye(n)],
                        [A, np.zeros((2,2)), np.zeros((2,n))],
                        [np.diag(s), np.zeros((n,2)), np.diag(x)]]))
```

In [3]:

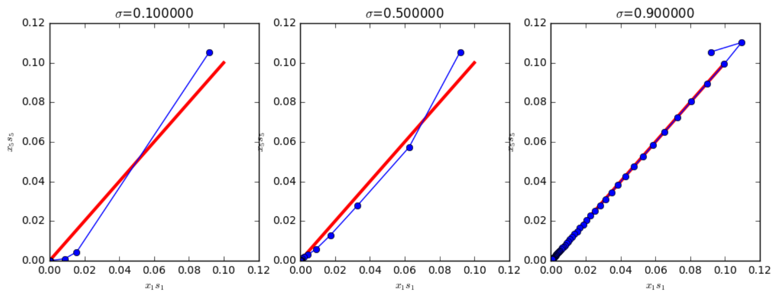
```
x = np.ones(n)/11.
y = np.array([0,0])
s = c-np.dot(A.T, y)
```

In [4]:
```
def longstep(x, y, s, sigma, gamma=1e-3, tol=1e-4):
    mu = 1
    i = 1
    yy = np.zeros((2,50))
    while mu>tol and i<50:
        a = 1
        mu = np.dot(x,s)/11.
        rhs = F(x,y,s)-np.concatenate((np.zeros(n+2), sigma*mu*np.ones(11)))
        delta = -la.solve(M(x,y,s), rhs)
        xs = np.concatenate((x,s))
        deltaxs = np.concatenate((delta[:11], delta[13:]))

        I = np.argmin(xs+deltaxs)
        m = xs[I]+deltaxs[I]
        if m<gamma*mu:
            a = np.amin(-xs[I]/deltaxs[I])

        x = x+a*delta[:11]
        y = y+a*delta[11:13]
        s = s+a*delta[13:]

        yy[:,i] = y
        i+=1
    return yy[:,:i]
```

In [5]:
```
import matplotlib.pyplot as plt
% matplotlib inline
fig, ax = plt.subplots(1,3, figsize=(12, 4))
xx = np.linspace(0,1,100)
sigmas = [0.1, 0.5, 0.9]

for k in range(3):
    yy = longstep(x,y,s,sigmas[k])
    ax[k].set_ylim([0,1])
    for j in range(n):
        ax[k].plot(xx,c[j]-np.dot(A[0,j],xx))
    ax[k].plot(yy[0,:], yy[1,:], '-o')
    ax[k].set_title("$\sigma$={:f}".format(sigmas[k]))
    ax[k].set_xlabel('$y_1$')
    ax[k].set_ylabel('$y_2$')
plt.show()
```



(c) In the figure, the central path is shown as the vertical line in the $\boldsymbol{y}$ plane. The code is exactly the same as above, but instead of recording the trajectory in the y variables, we use the $x_1 s_1$ against the $x_5 s_5$ axis.



Solution (6.5) This problem is a special case of Problem 6.3. We nevertheless solve it from scratch. First compute the gradient of all terms,

$$\nabla(\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}) = 2\boldsymbol{\Sigma}\boldsymbol{x}, \quad \nabla(\boldsymbol{x}^\top \boldsymbol{e}) = \boldsymbol{e}, \quad \nabla(\boldsymbol{x}^\top \boldsymbol{r}) = \boldsymbol{r}.$$

Then the Lagrange equation is given by (we absorbed the factor 2):

$$\boldsymbol{\Sigma}\boldsymbol{x} - \lambda\boldsymbol{e} - \eta\boldsymbol{r} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = \lambda\boldsymbol{\Sigma}^{-1}\boldsymbol{e} + \eta\boldsymbol{\Sigma}^{-1}\boldsymbol{r}. \tag{1}$$

Multiplying with $\boldsymbol{e}^\top$ and $\boldsymbol{r}^\top$ we get λ and η as the solution of a system of equations

$$\begin{aligned} 1 &= \boldsymbol{e}^\top\boldsymbol{x} = \lambda\boldsymbol{e}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{e} + \eta\boldsymbol{e}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{r} \\ \mu &= \boldsymbol{r}^\top\boldsymbol{x} = \lambda\boldsymbol{r}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{e} + \eta\boldsymbol{r}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{r}. \end{aligned}$$

Setting $a = \boldsymbol{e}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{e}$, $b = \boldsymbol{e}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{r}$ and $c = \boldsymbol{r}^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{r}$, this corresponds to the system of equations

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

Using Cramer's rule for the solution of a 2×2 system of equations (or solving this by Gaussian elimination directly), we get

$$\begin{pmatrix} \lambda \\ \eta \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} a\mu - b \\ c - b\mu \end{pmatrix}.$$

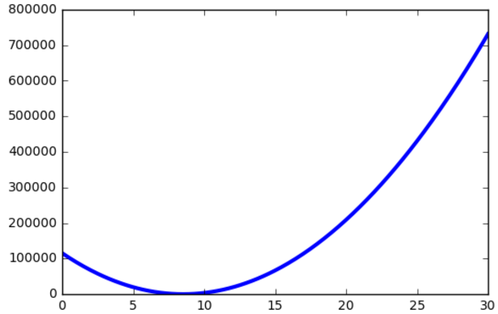
Plugging these into Equation (1), the closed-form solution is

$$\boldsymbol{x} = \frac{1}{ac - b^2} \left(c\boldsymbol{\Sigma}^{-1}\boldsymbol{r} - b\boldsymbol{\Sigma}^{-1}\boldsymbol{e} \right) + \mu \cdot \left(a\boldsymbol{\Sigma}^{-1}\boldsymbol{e} - b\boldsymbol{\Sigma}^{-1}\boldsymbol{r} \right).$$

Note that this is an affine function in μ, the target return. The variance itself is then

$$\boldsymbol{x}^\top\boldsymbol{\Sigma}\boldsymbol{x}.$$

Plotting the variance against the target return gives the following graph As we see, the smallest risk occurs when targeting around 6-8% return.



In [6]:

```
S = np.array([[185, 86.5, 80, 20],
    [86.5, 196, 76, 13.5],
    [80, 76, 411, -19],
    [20, 13.5, -19, 25]])
r = np.array([14,12,15,7])
e = np.ones(4)
```

In [7]:

```
Se = la.solve(S, e)
Sr = la.solve(S, r)
a = np.dot(e, Se)
b = np.dot(e, Sr)
c = np.dot(r, Sr)
d = (c*Sr-b*Se)/(a*c-b**2)
s = (a*Se-b*Sr)/(a*c-b**2)
```

In [8]:

```
mu = np.linspace(0,30,300)
xx = np.outer(d.reshape((len(d),1)),np.ones((len(mu),1)))+np.outer(s.reshape((len(s),1)),mu.reshape((le
temp = np.dot(S,xx)
risk = np.zeros(xx.shape[1])
for i in range(xx.shape[1]):
    risk[i] = np.dot(xx[:,i],temp[:,i])
plt.plot(mu, risk, linewidth=3)
plt.show()
```