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# Lecture 8

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Linear programming is about problems of the form

$$\begin{aligned} & \text{maximize} && \langle \mathbf{c}, \mathbf{x} \rangle \\ & \text{subject to} && \mathbf{A}\mathbf{x} \leq \mathbf{b}, \end{aligned}$$

where  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{c} \in \mathbb{R}^n$ , and  $\mathbf{b} \in \mathbb{R}^m$ , and the inequality sign means inequality in each row. The *feasible set* is the set of all possible candidates,

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

This set can be empty (example:  $x \leq 1$ ,  $-x \leq -2$ ), unbounded (example:  $x \leq 1$ ) or bounded (example:  $x \leq 1$ ,  $-x \leq 0$ ). In any case, it is a convex set. To understand linear programming it is of paramount importance to understand the geometry of the feasible sets of linear programming, also called *polyhedra*.

## 8.1 Linear Programming Duality: a first glance

Suppose we are faced with a linear programming problem and would like to know if the feasible set  $\mathcal{F}$  is empty or not, i.e., if  $\mathbf{A}\mathbf{x} \leq \mathbf{b}$  has a solution. If it is not empty, we can certify that by producing a vector from  $\mathcal{F}$ . To verify that the feasible set is empty is more tricky: we are asked to show that *no* vector lives in  $\mathcal{F}$ . What we can try to do, however, is to show that the assumption of a solution to  $\mathbf{A}\mathbf{x} \leq \mathbf{b}$  would lead to a contradiction. Denote by  $\mathbf{a}_i^\top$  the rows of  $\mathbf{A}$ . Assuming  $\mathbf{x} \in \mathcal{F}$ , then given a vector  $\mathbf{0} \neq \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)^\top$  with  $\lambda_i \geq 0$ , the linear combination satisfies

$$\sum_{i=1}^m \lambda_i \mathbf{a}_i^\top \mathbf{x} \leq \sum_{i=1}^m \lambda_i b_i = \langle \boldsymbol{\lambda}, \mathbf{b} \rangle. \quad (8.1)$$

If we can find parameters  $\boldsymbol{\lambda}$  such that the left-hand side of (8.1) is identically 0 and the right-hand side is strictly negative, then we have found a contradiction and can conclude that no  $\mathbf{x}$  satisfies  $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ . A condition that ensures this is

$$\sum_{i=1}^m \lambda_i \mathbf{a}_i = \mathbf{0}, \quad \langle \boldsymbol{\lambda}, \mathbf{b} \rangle < 0. \quad (8.2)$$

In matrix form,

$$\exists \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{A}^\top \boldsymbol{\lambda} = \mathbf{0}, \langle \boldsymbol{\lambda}, \mathbf{b} \rangle < 0.$$

This condition will still be satisfied if we normalise the vector  $\boldsymbol{\lambda}$  such that  $\sum_{i=1}^m \lambda_i = 1$ , so the statement says that  $\mathbf{0}$  is a convex combination of the vectors defining the equations.

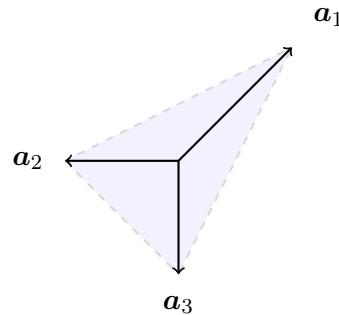
**Example 8.1.** Consider the system

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ -x_1 &\leq -1 \\ -x_2 &\leq -1.5. \end{aligned} \tag{8.3}$$

The transpose matrix  $\mathbf{A}^\top$  and the vector  $\mathbf{b}$  are

$$\mathbf{A}^\top = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -1.5 \end{pmatrix}$$

Drawing the columns of  $\mathbf{A}^\top$  we get We can get the origin as a convex combination



of the vectors  $\mathbf{a}_i$  (drawing a rope around them encloses the origin), and such a combination is given by  $\boldsymbol{\lambda} = (1/3, 1/3, 1/3)$ . Taking the scalar product with the vector  $\mathbf{b}$  we get

$$\langle \boldsymbol{\lambda}, \mathbf{b} \rangle = \frac{1}{3}(2 - 1 - 1.5) = -\frac{1}{6} < 0.$$

This shows that the system (8.3) does not have a solution (a fact that in this simple example can also be seen by drawing a picture).

It turns out that Condition (8.2) is not only sufficient but also necessary, and the separating hyperplane theorem is an essential part of this. We first make a detour in order to better understand the feasible sets, the polyhedra.

## 8.2 Polyhedra

**Definition 8.2.** A polyhedron (plural: polyhedra) is a set defined as the solution of linear equalities and inequalities,

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}, \quad (8.1)$$

where  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ .

More classically, we can write out the equations.

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_d &\leq b_1, \\ &\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_d &\leq b_m. \end{aligned} \quad (8.2)$$

We now introduce some useful terminology and concepts associated to polyhedra, and illustrate them with a few examples. A supporting hyperplane  $H$  of a polyhedron  $P$  is a hyperplane such that  $P \subseteq H_-$ , where  $H_-$  is a halfspace associated to  $H$ . If  $H$  is a supporting hyperplane, then a set of the form  $F = H \cap P$  is called a *face* of  $P$ . In particular, the polyhedron  $P$  is a face. Each of the inequalities  $\langle \mathbf{a}_i, \mathbf{x} \rangle \leq b_i$  in (8.1) defines a supporting hyperplane, and therefore a face. The *dimension* of a face  $F$ ,  $\dim F$ , is the smallest dimension of an affine space containing  $F$ . Faces of dimension  $\dim F = \dim P - 1$  are called *facets*, faces of dimension 0 are vertices, and of dimension 1 edges. A vertex can equivalently be characterised as a point  $\mathbf{x} \in P$  that can not be written as a convex combination of two other points in  $P$ .

**Example 8.3.** Polyhedra in one dimension are the sets  $[a, b]$ ,  $[a, \infty)$ ,  $(-\infty, b]$ ,  $\mathbb{R}$  or  $\emptyset$ , where  $a \leq b$ . Each of them is clearly convex.

**Example 8.4.** The set

$$P = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}.$$

is the polyhedron shown in Figure 8.1. We can write the defining inequalities in standard form  $\mathbf{A}\mathbf{x} \leq \mathbf{b}$  by setting

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This polyhedron has one face of dimension 2 (itself), three facets of dimension 1 (the sides, corresponding to the three equations), and three vertices of dimension 0 (the corners, corresponding any two of the defining equations).

**Example 8.5.** The polyhedron in Albrecht Dürer's famous "Melencolia I", aka the "truncated triangular trapezohedron", can be described using eight inequalities, see Figure 8.2.

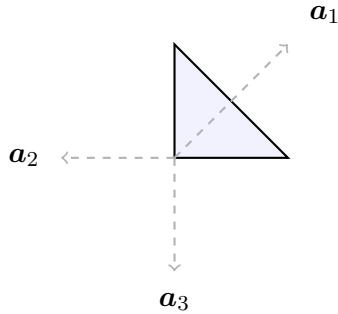


Figure 8.1: A two-dimensional polyhedron and defining equations.



$$\begin{aligned}
 0.7071x_1 - 0.4082x_2 + 0.3773x_3 &\leq 1 \\
 -0.7071x_1 + 0.4082x_2 - 0.3773x_3 &\leq 1 \\
 0.7071x_1 + 0.4082x_2 - 0.3773x_3 &\leq 1 \\
 -0.7071x_1 - 0.4082x_2 + 0.3773x_3 &\leq 1 \\
 0.8165x_2 + 0.3773x_3 &\leq 1 \\
 -0.8165x_2 - 0.3773x_3 &\leq 1 \\
 0.6313x_3 &\leq 1 \\
 -0.6313x_3 &\leq 1
 \end{aligned}$$

Figure 8.2: Dürer's Melencolia and equations defining the polyhedron

We now move to a different characterization of bounded polyhedra. The main result of this lecture is that bounded polytopes can be described completely from knowing their vertices. A polyhedron  $P$  is called bounded, if there exists a ball  $B(\mathbf{0}, r)$  with  $r > 0$  such that  $P \subset B(\mathbf{0}, r)$ . For example, halfspaces are not bounded, but the polytope from Examples 8.4 and 8.5 are.

We first observe the nontrivial fact that a polyhedron has only fin-

**Definition 8.6.** A *polytope* is the convex hull of finitely many points,

$$P = \text{conv}(\{x_1, \dots, x_k\}) = \left\{ \sum_{i=1}^k \lambda_i x_i, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}.$$

**Theorem 8.7.** A bounded polyhedron  $P$  is the convex hull of its vertices.

**Example 8.8.** The triangle in Example 8.4 is the convex hull of the points  $(0, 0)^\top$ ,  $(0, 1)^\top$ , and  $(1, 0)^\top$ .

**Example 8.9.** The Dürer polytope is the convex hull of the following 12 vertices:

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} -1.4142 \\ -0.8165 \\ -0.8835 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1.4142 \\ 0.8165 \\ 0.8835 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -0.8536 \\ -0.4928 \\ -1.5840 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -0.8536 \\ 0.4928 \\ 1.5840 \end{pmatrix}, \\ \mathbf{v}_5 &= \begin{pmatrix} -0.0000 \\ -1.6330 \\ 0.8835 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0.0000 \\ -0.9856 \\ 1.5840 \end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix} -0.0000 \\ 0.9856 \\ -1.5840 \end{pmatrix}, \mathbf{v}_8 = \begin{pmatrix} 0.0000 \\ 1.6330 \\ -0.8835 \end{pmatrix}, \\ \mathbf{v}_9 &= \begin{pmatrix} 0.8536 \\ -0.4928 \\ -1.5840 \end{pmatrix}, \mathbf{v}_{10} = \begin{pmatrix} 0.8536 \\ 0.4928 \\ 1.5840 \end{pmatrix}, \mathbf{v}_{11} = \begin{pmatrix} 1.4142 \\ -0.8165 \\ -0.8835 \end{pmatrix}, \mathbf{v}_{12} = \begin{pmatrix} 1.4142 \\ 0.8165 \\ 0.8835 \end{pmatrix}. \end{aligned}$$

The converse of Theorem 8.7 is also true.

**Theorem 8.10.** *A polytope is a bounded polyhedron.*

The equivalence between polytopes and bounded polyhedra gives a first glimpse into linear programming duality theory, a topic of central importance in both modeling and algorithm design.