Midterm test

Closed book. Attempt all questions. Calculators permitted. 13:00-13:50
Please write your name and student identity number on the front page.

(1) Determine the order of convergence of each of the following sequences (if they converge at all). You may assume $k \geq 1$.

$$\text{(a)}\ \ x_k = \frac{1}{\sqrt{k}}, \quad \text{(b)}\ \ x_k = 1 + (0.2)^{3^k}, \quad \text{(c)}\ \ x_k = k^{-k}, \quad \text{(d)}\ \ x_k = 1$$

[4 marks]

(2) Consider the function on $\mathbb{R}^2$, $f(\boldsymbol{x}) = (2x_1 + x_2^2)^2$. Show that $\boldsymbol{p} = (-1, 0)^\intercal$ is a descent direction at $\boldsymbol{x}_0 = (0, 1)^\intercal$, and find a step length α that minimizes $f(\boldsymbol{x}_0 + \alpha p)$.

[4 marks]

(3) Determine, with justification, which of the following functions is convex ($\ln(x)$ refers to the natural logarithm).

(a) $f(x) = \ln(x)$ for $x > 0$;

(b) $f(x) = \dfrac{1}{x}$ for $x > 0$;

(c) $f(x, y, z) = z^2 - x^2 - y^2$ for $x \in \mathbb{R}$;

(d) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.

You may used criteria for convexity seen in the lecture and problem sessions. [4 marks]

(4) Consider the following linear programming problem

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 \\ \text{subject to} \quad & x_1 \leq 1 \\ & x_2 \leq 2 \\ & 2x_1 + x_2 \geq 2 \end{aligned}$$

(a) Determine the vertices of the polyhedron of feasible points;

(b) Find an optimizer and the optimal value;

(c) Write down the dual to this problem.

[4 marks]

(5) Consider the function

$$f(x, y) = \sqrt{1 + x^2 + y^2}$$

By computing the gradient and Hessian, show that this function is convex and determine the unique minimum. Write down the form of one iteration of Newton's method for this function. [4 marks]