Problem Sheet 3

Problems in Part A will be discussed in class. Problems in Part B come with solutions and should be tried at home.

Part A

(3.1) Recall that a norm was a function $\|\cdot\|$ on $\mathbb{R}^n$ such that

- $\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = \mathbf{0}$;
- $\|\lambda \boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$ for $\lambda \in \mathbb{R}$;
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

The most prominent examples are

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}, \quad \|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|, \quad \|\boldsymbol{x}\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

(a) Show that for any norm, the *norm balls*

$$B(\boldsymbol{p}, r) := \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{p}\| \leq r\}$$

are convex sets.

(b) To a convex set C with $\mathbf{0} \in \mathrm{int} C$ we can associate the *polar* set

$$C^* := \{\boldsymbol{y} \in \mathbb{R}^n : \forall \boldsymbol{x} \in C, \langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq 1\}.$$

Show that the polar of a convex set is again convex.

(c) Describe the polar sets of the norm balls $B(\mathbf{0}, 1)$ for the 1, 2, and ∞ norms.

(3.2) Let $C, D \subseteq \mathbb{R}^n$ be disjoint, non-empty, closed, bounded convex sets. Show that there exists an affine hyperplane H strictly separating C and D (i.e., $C \subset \mathrm{int} H_-$ and $D \subset \mathrm{int} H_+$). Hint: consider the set $C - D = \{\boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x} \in C, \boldsymbol{y} \in D\}$ and use the separation theorem for a convex set and a point from the lecture. Give an example of closed convex sets C and D that cannot be strictly separated by a hyperplane.

(3.3) (Facility location.) Facility location problems concern the location of points (facilities or devices on a circuit) in a region, some of which are required to be connected by links (streets or wires). The objective is to locate some of the points that are not fixed in a way that minimizes the transport cost along the links. The distance can be measured with respect to the usual Euclidean norm or the 1-norm.

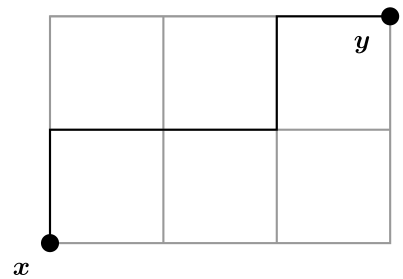


Figure 1: Manhattan distance

(a) Given a rectangular grid, show that the length of a path from coordiante $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by

$$d(\boldsymbol{x},\boldsymbol{y}) := \|\boldsymbol{x}-\boldsymbol{y}\|_1.$$

This is an example of the so-called *Manhattan distance*.

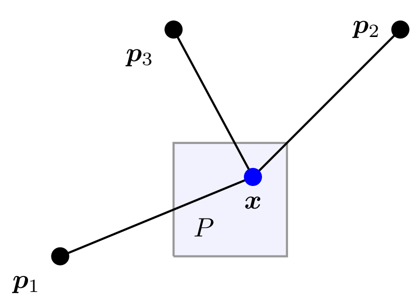
(b) Suppose now that we are given N fixed points $\boldsymbol{p}_1,\dots,\boldsymbol{p}_N$ in on a fine grid in $\mathbb{R}^2$ and would like to find a point $\boldsymbol{x}$ that minimizes the total 1-norm distance

$$\text{minimize} \quad \sum_{i=1}^{N} \|\boldsymbol{p}_i-\boldsymbol{x}\|_1 = \sum_{i=1}^{N} |p_{i,1}-x_1| + |p_{i,2}-x_2|.$$

Assume in addition that the point $\boldsymbol{x}$ is constrained to be in a polyhedral region

$$P = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}\}.$$

Formulate this optimal placement problem as a linear programming problem.

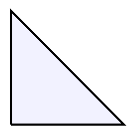


Part B

(3.4) A *polyhedron* P is a convex set defined by a system of linear inequalities

$$P = \{\boldsymbol{x}\in\mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}\},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{b} \in \mathbb{R}^m$. A vertex $\boldsymbol{z}$ of P is a point of P that can not be expressed as a convex combination of distinct points in P. For $\boldsymbol{z} \in P$, let $\boldsymbol{A}_{\boldsymbol{z}}$ denote the matrix consisting of those rows $\boldsymbol{a}_i^\top$ of $\boldsymbol{A}$ where $\boldsymbol{a}_i^\top \boldsymbol{z} = b_i$. For example, in Figure 2 the matrix corresponding to the vertex $\boldsymbol{z} = (1,0)^\top$ is given by



$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{A}_{\boldsymbol{z}} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Figure 2: A triangle

Show that the vertices of P are precisely the points $\boldsymbol{z}$ for which $\boldsymbol{A}_{\boldsymbol{z}}$ has rank n. In particular, every polyhedron has only finitely many vertices.

(3.5) (Text classification) Consider the vocabulary consisting of the words