Solutions to Part B of Problem Sheet 3

Solution (3.4) Let $\boldsymbol{z} \in P$ be a vertex. We prove by contradition that $\boldsymbol{A}_{\boldsymbol{z}}$ has to have rank n. Assume to the contrary that $\mathrm{rank}\boldsymbol{A}_{\boldsymbol{z}} < n$. Then there exists $\boldsymbol{c} \in \mathbb{R}^n$ such that $\boldsymbol{A}_{\boldsymbol{z}}\boldsymbol{c} = \boldsymbol{0}$. Since for rows $\boldsymbol{a}_i^\top$ of $\boldsymbol{A}$ that are not in $\boldsymbol{A}_{\boldsymbol{z}}$ we have (by definition) $\boldsymbol{a}_i^\top \boldsymbol{z} < b_i$, there exists a small $\delta > 0$ such that $\boldsymbol{a}_i^\top(\boldsymbol{z} + \delta\boldsymbol{c}) < b_i$ and $\boldsymbol{a}_i^\top(\boldsymbol{z} - \delta\boldsymbol{c}) < b_i$. Moreover, $\boldsymbol{A}_{\boldsymbol{z}}(\boldsymbol{z} \pm \delta\boldsymbol{c}) = \boldsymbol{A}_{\boldsymbol{z}}\boldsymbol{z}$, since $\boldsymbol{c} \in \ker \boldsymbol{A}_{\boldsymbol{z}}$. It follows that $\boldsymbol{x} = \boldsymbol{z} - \delta\boldsymbol{c}$ adn $\boldsymbol{y} = \boldsymbol{z} + \delta\boldsymbol{c}$ are in P, and since

$$\boldsymbol{z} = \frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}$$

is a convex combination of two points in P, this contradicts the assumption that $\boldsymbol{z}$ is a vertex. We conclude that the rank of $\boldsymbol{A}_{\boldsymbol{z}}$ is n.

Now assume that the rank of $\boldsymbol{A}_{\boldsymbol{z}}$ is n. Again, to get a contradiction, assume that $\boldsymbol{z}$ is not a vertex. Then there exist points $\boldsymbol{z} \neq \boldsymbol{x}, \boldsymbol{y} \in P$ such that $\boldsymbol{z} = (\boldsymbol{x} + \boldsymbol{y})/2$. For every row $\boldsymbol{a}_i^\top$ of $\boldsymbol{A}_{\boldsymbol{z}}$ we therefore have

$$\boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i = \boldsymbol{a}_i^\top \boldsymbol{z} \Rightarrow \boldsymbol{a}_i^\top(\boldsymbol{z} - \boldsymbol{x}) \leq 0, \tag{1}$$

and

$$\boldsymbol{a}_i^\top \boldsymbol{y} \leq b_i = \boldsymbol{a}_i^\top \boldsymbol{z} \Rightarrow \boldsymbol{a}_i^\top(\boldsymbol{z} - \boldsymbol{y}) \leq 0.$$

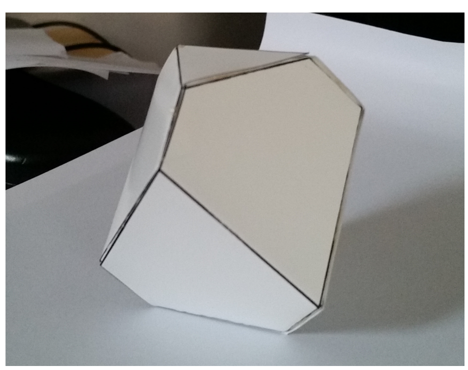
Since $\boldsymbol{z} - \boldsymbol{y} = -(\boldsymbol{z} - \boldsymbol{x})$, the reverse inequality to (1) also holds, and we conclude that

$$\boldsymbol{A}_{\boldsymbol{z}}(\boldsymbol{z} - \boldsymbol{x}) = \boldsymbol{0}.$$

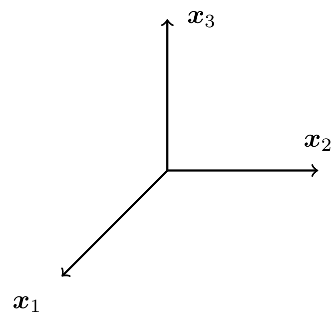
Since $\mathrm{rank}\boldsymbol{A}_{\boldsymbol{z}} = n$, this means that $\boldsymbol{z} = \boldsymbol{x}$, in contradiction to the assumption. This shows that $\boldsymbol{z}$ is in fact a vertex.

Solution (3.5)

(a) This is the solution:



(b) Notice that the equations come in positive/negative pairs: each such pair corresponds to opposite faces. The equations (7) and (8) correspond to the triangles, as



they give conditions only on the x_3 axis. Let's fix a coordinate system oriented as follows (we only care about the orientation, not the absolute positioning!) Then Equation (7) corresponds to the top triangle, (8) on the bottom triangle, Equations (1), (4), (5) describe the upward-facing superman shapes, and (2), (3) and (6) the downward facing ones (because their x_3-coordinate is negative). Taking one of the upward facing shapes as being parallel to the x_1 axis, this corresponds to Equation (5), because of the x_1 term missing there. The upward facing shape to the left of it is then (1) (because of the positive x_1 coordinate), and the other one (4). In summary, the following labels are consistent.

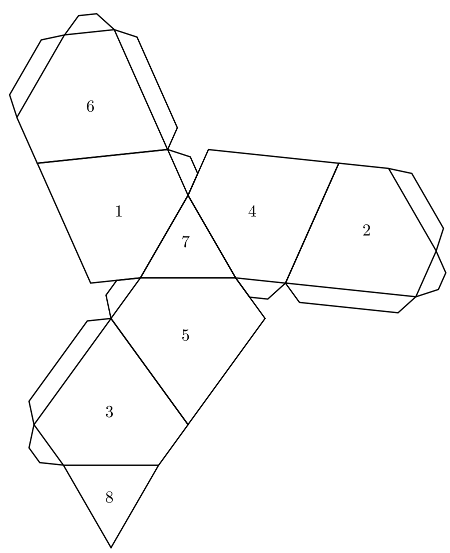


Figure 1: Foldout with supporting hyperplanes

(c) From the above diagram (or better, the assembled polyhedron) we see precisely which hyperplanes intersect to create the vertices:

$$\begin{aligned}&(157),(457),(147),(146),(245),(135),\\&(136),(246),(235),(248),(238),(368)\end{aligned} \tag{2}$$

From this list we get a recipe for determining the vertices from the system of inequalities: for the triple (157) select the equations 1, 5 and 7, and solve the corresponding system of three equations in three unknowns. The solution will be the vertex. Implementing this in Python, we get the vertices

$$\boldsymbol{v}_1=\begin{pmatrix}-1.4142\\-0.8165\\-0.8835\end{pmatrix},\boldsymbol{v}_2=\begin{pmatrix}-1.4142\\0.8165\\0.8835\end{pmatrix},\boldsymbol{v}_3=\begin{pmatrix}-0.8536\\-0.4928\\-1.5840\end{pmatrix},\boldsymbol{v}_4=\begin{pmatrix}-0.8536\\0.4928\\1.5840\end{pmatrix},$$
$$\boldsymbol{v}_5=\begin{pmatrix}-0.0000\\-1.6330\\0.8835\end{pmatrix},\boldsymbol{v}_6=\begin{pmatrix}0.0000\\-0.9856\\1.5840\end{pmatrix},\boldsymbol{v}_7=\begin{pmatrix}-0.0000\\0.9856\\-1.5840\end{pmatrix},\boldsymbol{v}_8=\begin{pmatrix}0.0000\\1.6330\\-0.8835\end{pmatrix},$$
$$\boldsymbol{v}_9=\begin{pmatrix}0.8536\\-0.4928\\-1.5840\end{pmatrix},\boldsymbol{v}_{10}=\begin{pmatrix}0.8536\\0.4928\\1.5840\end{pmatrix},\boldsymbol{v}_{11}=\begin{pmatrix}1.4142\\-0.8165\\-0.8835\end{pmatrix},\boldsymbol{v}_{12}=\begin{pmatrix}1.4142\\0.8165\\0.8835\end{pmatrix}.$$