Solutions to Part A of Problem Sheet 5

Solution (5.1) From the first block of rows we have

$$\boldsymbol{A}^\top \Delta \boldsymbol{y} + \Delta \boldsymbol{s} = \boldsymbol{0} \iff \Delta \boldsymbol{s} = -\boldsymbol{A}^\top \Delta \boldsymbol{y}.$$

From the second block of rows we have

$$\boldsymbol{A}\Delta \boldsymbol{x} = \boldsymbol{0}.$$

Putting these two together, we get

$$\langle \Delta \boldsymbol{x}, \Delta \boldsymbol{s}\rangle = \langle \Delta \boldsymbol{x}, -\boldsymbol{A}^\top \Delta \boldsymbol{y}\rangle = -\langle \boldsymbol{A}\Delta \boldsymbol{x}, \Delta \boldsymbol{y}\rangle = 0,$$

which shows the claim.

Solution (5.2)

(a) The matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

which leads to the function F and the Jacobian DF being

$$F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) = \begin{pmatrix} x_1 + x_2 - 1 \\ y + s_1 - 1 \\ y + s_2 \\ x_1 s_1 \\ x_2 s_2 \end{pmatrix}, \ \boldsymbol{J}F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ s_1 & 0 & 0 & x_1 & 0 \\ 0 & s_2 & 0 & 0 & x_2 \end{pmatrix}$$

The algorithm starts with $(x_1^{(0)}, x_2^{(0)}, y^{(0)}, s_1^{(0)}, s_2^{(0)})$ and the average

$$\mu_0 = \frac{1}{2}(x_1^{(0)} s_1^{(0)} + x_2^{(0)} s_2^{(0)}).$$

Then solve the system of equations

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ s_1^{(k)} & 0 & 0 & x_1^{(k)} & 0 \\ 0 & s_2^{(k)} & 0 & 0 & x_2^{(k)} \end{pmatrix} \begin{pmatrix} \Delta x_1^{(k)} \\ \Delta x_2^{(k)} \\ \Delta y \\ \Delta s_1^{(k)} \\ \Delta s_2^{(k)} \end{pmatrix} = - \begin{pmatrix} x_1^{(k)} + x_2^{(k)} - 1 \\ y^{(k)} + s_1^{(k)} - 1 \\ y^{(k)} + sv_2 \\ x_1^{(k)} s_1^{(k)} - \sigma_k \mu_k \\ x_2^{(k)} s_2^{(k)} - \sigma_k \mu_k \end{pmatrix}$$

and update

$$(x_1^{(k+1)}, x_2^{(k+1)}, y^{(k+1)}, s_1^{(k+1)}, s_2^{(k+1)}) = (x_1^{(k)}, x_2^{(k)}, y^{(k)}, s_1^{(k)}, s_2^{(k)}) + \alpha_k(\Delta x_1, \Delta x_2, \Delta y, \Delta s_1, \Delta s_2).$$

(b) The solution here is obvious: the feasible set is the line segment connecting $(0,1)^\top$ and $(1,0)^\top$, and the optimal value is $\boldsymbol{x}^* = (0,1)^\top$.

(c) Solving

$$F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) = \begin{pmatrix} x_1 + x_2 - 1 \\ y + s_1 - 1 \\ y + s_2 \\ x_1 s_1 \\ x_2 s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

can give the solution (there is more than one)

$$\boldsymbol{x}^* = (1,0)^\top, \boldsymbol{s} = (0,-1)^\top, \boldsymbol{y} = 1.$$

This solution is clearly not a minimizer of the optimization problem.

Solution (5.3) If $\boldsymbol{B}$ is the matrix with the $-y_i \boldsymbol{x}_i$ as rows, and $\boldsymbol{y}$ denotes the vector with the y_i as entries, define $\boldsymbol{A} = [\boldsymbol{B}, -\boldsymbol{y}]$. Let $\boldsymbol{z} = (\boldsymbol{w}, b) \in \mathbb{R}^{n+1}$. Then any solution of the feasibility problem

$$\boldsymbol{A}\boldsymbol{z} < 0$$

has the property that $\langle \boldsymbol{w}, \boldsymbol{x}_i \rangle + b < 0$ if $y_i = -1$ and $\langle \boldsymbol{w}, \boldsymbol{x}_j \rangle + b > 0$ if $y_i = 1$, so we get the separating hyperplane from $\boldsymbol{w}, b$. If the convex hulls of the $\boldsymbol{x}_i$ with $y_i = 1$ and of the $\boldsymbol{x}_j$ with $y_j = -1$ are disjoint, we have two disjoint, bounded, closed convex sets, and these can be separated by a hyperplane. The existence of a separating hyperplane is equivalent to the linear programming feasibility problem having a solution.