Queueing Theory
(Part 5)

Jackson Queueing Networks

Network of M/M/s Queues

- The output of an M/M/s queue with $s\mu>\lambda$ in steady-state is a ***Poisson process*** at rate λ
  - This result may seem surprising at first glance
  - It is due to the properties of exponential distributions
  - This is called the “equivalence property”

Equivalence Property

- Assume that a service facility with s servers and an infinite queue has a Poisson input with parameter λ and the same exponential service-time distribution with parameter μ for each server (the *M/M/s* model), where $s\mu>\lambda$. Then the steady-state <u>*output*</u> of this service facility is <u>also a Poisson process with parameter λ.</u>

Infinite Queues in Series

- Suppose that customers must all receive service at a series of *m* service facilities in a fixed sequence. Assume that each facility has an infinite queue, so that the series of facilities form a system of *infinite queues in series*.
- The *joint probability* of n_1 customers at facility 1, n_2 customers at facility 2, … , then, is the *product* of the individual probabilities obtained in this simple way.

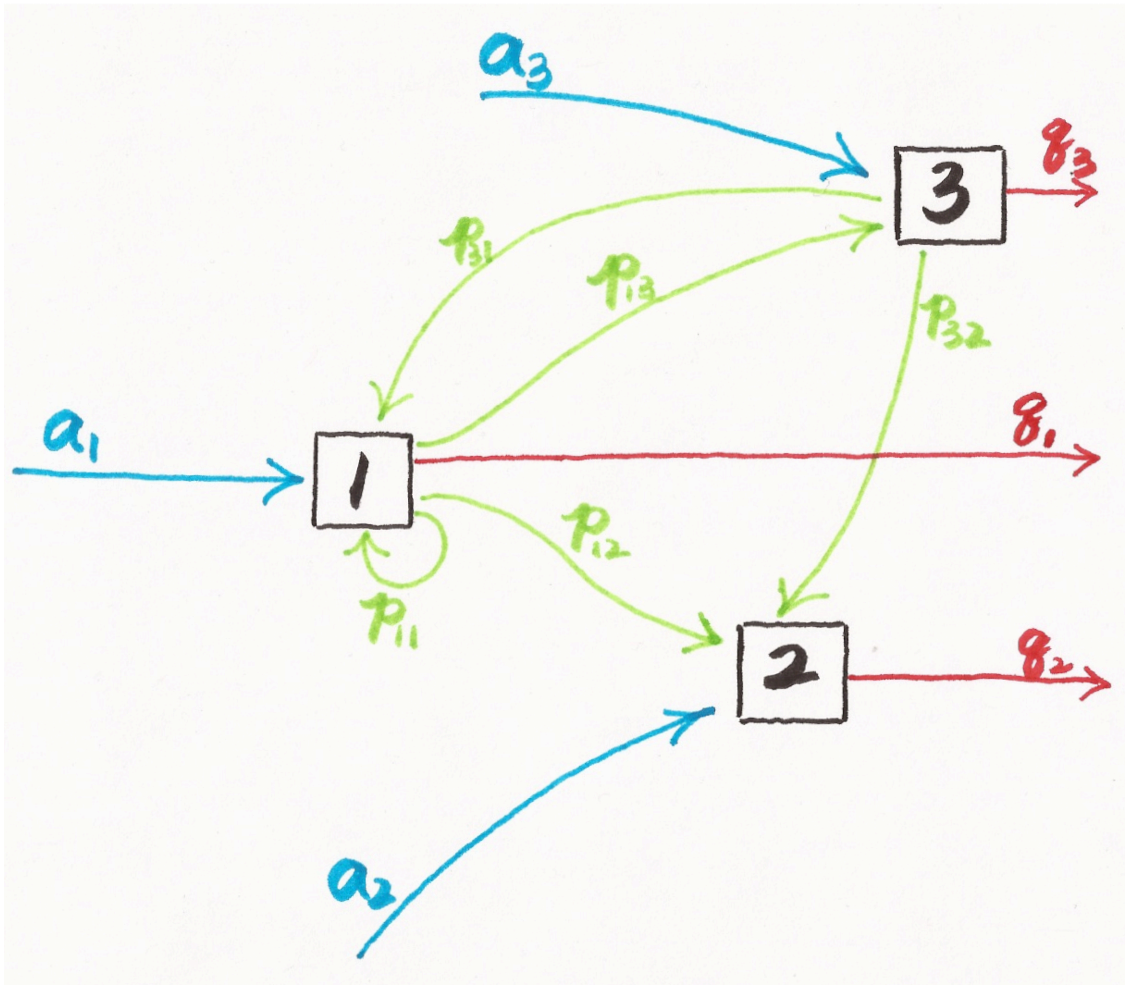
$$P\left\{(N_1, N_2, \ldots, N_m) = (n_1, n_2, \ldots, n_m)\right\} = P_{n_1} P_{n_2} \cdots P_{n_m}$$

What is a Jackson Network ?

- A **Jackson network** is a system of *m* service facilities where facility *i* (*i* = 1, 2, … , *m*) has
  1. An infinite queue
  2. Customers arriving from outside the system according to a Poisson input process with parameter a_i
  3. s_i servers with an exponential service-time distribution with parameter μ_i .
- A customer leaving facility *i* is routed next to facility *j* (*j* = 1, 2, … , *m*) with probability p_{ij} or departs the system with probability q_i where

$$q_i = 1 - \sum_{j=1}^{m} p_{ij}$$

Jackson Network Diagram



Key Property of a Jackson Network

- Any such network has the following key property
  - Under steady-state conditions, each facility j (j = 1, 2, … , m) in a Jackson network behaves as if it were an independent $M/M/s_j$ queueing system with arrival rate

$$\lambda_j = a_j + \sum_{i=1}^{m} \lambda_i p_{ij},$$

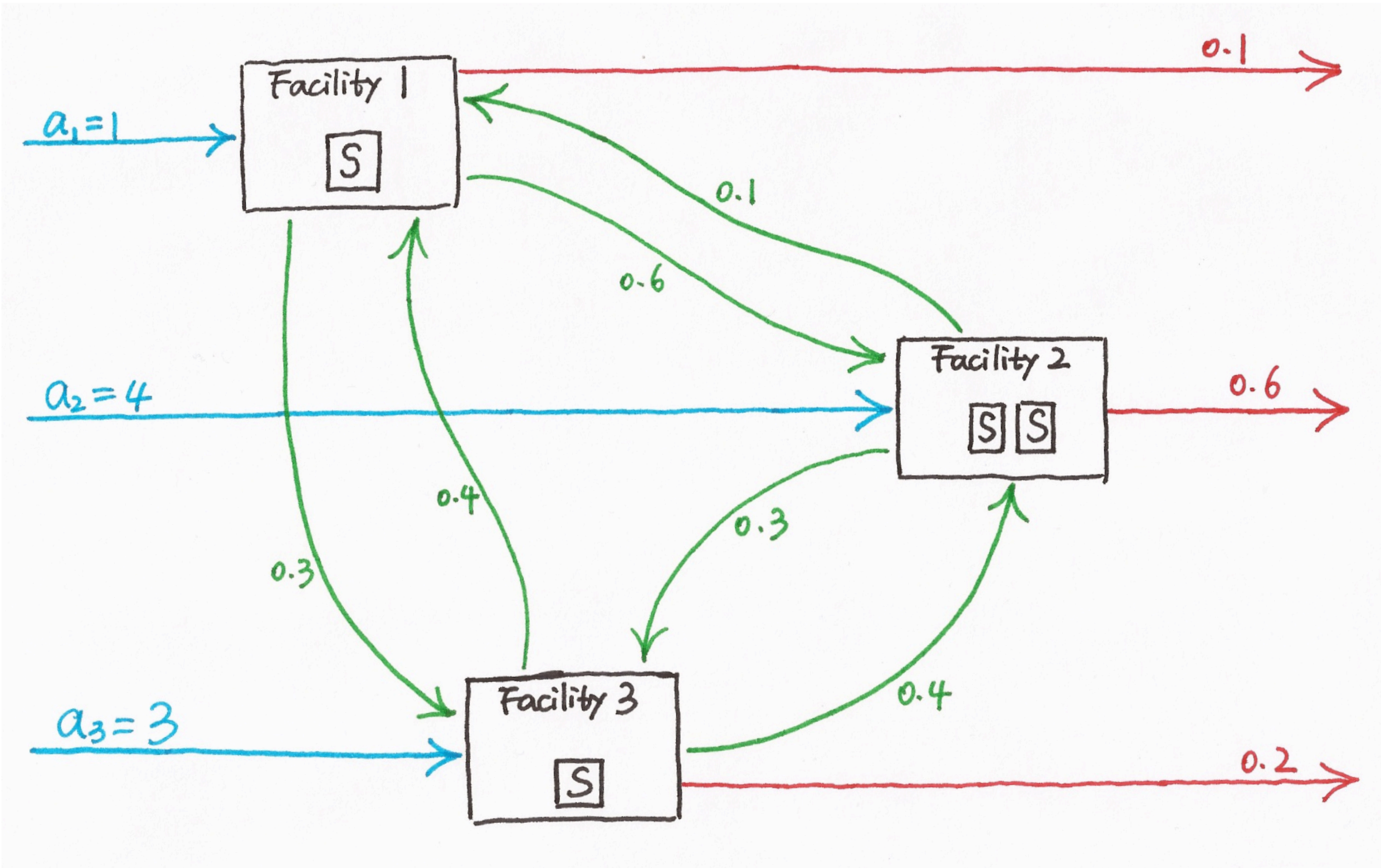
where $s_j \mu_j > \lambda_j$.

Jackson Network Example

- To illustrate these calculations, consider a Jackson network with three service facilities that have the parameters shown in the table below.

Facility j	s_j	μ_j	a_j	p_{ij}		
				$i = 1$	$i = 2$	$i = 3$
$j = 1$	1	10	1	0	0.1	0.4
$j = 2$	2	10	4	0.6	0	0.4
$j = 3$	1	10	3	0.3	0.3	0

Jackson Network Example Diagram



Jackson Network Example (cont'd)

- Plugging into the formula for λ_j for j = 1, 2, 3, we obtain

$$
\begin{aligned}
\lambda_1 &= 1 &&+ 0.1\lambda_2 + 0.4\lambda_3 \\
\lambda_2 &= 4 + 0.6\lambda_1 &&+ 0.4\lambda_3 \\
\lambda_3 &= 3 + 0.3\lambda_1 + 0.3\lambda_2
\end{aligned}
$$

- The simultaneous solution for this system is

$$\lambda_1 = 5, \quad \lambda_2 = 10, \quad \lambda_3 = 7\frac{1}{2}$$

Jackson Network Excel Template

	A	B	C	D	E	F	G	H
1	Template for a Jackson Network							
2								
3		Number of facilities, m =	3					
4								
5					p_{ij}			
6			Facility j	a_j	$i = 1$	$i = 2$	$i = 3$	
7			$j = 1$	1.0000	0.0000	0.1000	0.4000	
8			$j = 2$	4.0000	0.6000	0.0000	0.4000	
9			$j = 3$	3.0000	0.3000	0.3000	0.0000	
10								
11				λ_i	5.0000	10.0000	7.5000	
12								

Jackson Network Example (cont'd)

- Given this simultaneous solution for λ_j, to obtain the distribution of the number of customers $N_i = n_i$ at facility i, note that

$$\rho_i = \frac{\lambda_i}{s_i \mu_i} = \begin{cases} \frac{1}{2} & \text{for } i = 1 \\ \frac{1}{2} & \text{for } i = 2 \\ \frac{3}{4} & \text{for } i = 3 \end{cases}$$

$$P_{n_1} = \frac{1}{2}\left(\frac{1}{2}\right)^{n_1} \quad \text{for facility 1,}$$

$$P_{n_2} = \begin{cases} \frac{1}{3} & \text{for } n_2 = 0 \\ \frac{1}{3} & \text{for } n_2 = 1 \\ \frac{1}{3}\left(\frac{1}{2}\right)^{n_2 - 1} & \text{for } n_2 \geq 2 \end{cases} \quad \text{for facility 2,}$$

$$P_{n_3} = \frac{1}{4}\left(\frac{3}{4}\right)^{n_3} \quad \text{for facility 3.}$$

Jackson Network Example (cont'd)

- The *joint probability* of (n_1, n_2, n_3) then is given simply by the product form solution

$$P\left\{(N_1, N_2, N_3) = (n_1, n_2, n_3)\right\} = P_{n_1} P_{n_2} P_{n_3}$$

- The expected number of customers L_i at facility i

$$L_1 = 1, \quad L_2 = \frac{4}{3}, \quad L_3 = 3$$

- The expected *total* number of customers in the system

$$L = L_1 + L_2 + L_3 = 5\frac{1}{3}$$

Jackson Network Example (cont'd)

- To obtain *W*, the expected *total* waiting time in the system (including service times) for a customer, you cannot simply add the expected waiting times at the respective facilities, because a customer does not necessarily visit each facility exactly once.
- However, Little's formula can still be used for the entire network,

$$W = \frac{L}{a_1 + a_2 + a_3} = \frac{16/3}{1+4+3} = \frac{2}{3}$$