SIMG-716
Linear Imaging Mathematics I, Handout 05

1 1-D STOCHASTIC FUNCTIONS – NOISE

- "noise" = function whose amplitude is is derived from a random or a *stochastic* process (i.e., not deterministic)
- Deterministic: f at x specified completely by some parameters (width, amplitude, etc.)
- Stochastic function n at x selected from distribution that describes the probability of occurrence
  - Only statistical averages of signal amplitude are specified
- Example of "discrete" process: number of individual photons measured by detector in time interval Δt
  - must be integer
  - number counted during disjoint intervals of equal length exhibit a degree of "scatter" about some mean value.
- Example of "continuous" stochastic process
  - spatial distribution of photon arrivals from an optical system; the location $[x_n, y_n]$
- Notation: $n[x]$ instead of $f[x]$; "n" is used to indicate its "noisy" character.
  - potential confusion because n also specifies an integer coordinate, *e.g.*, $f[n]$
- Complex-valued noise constructed by adding two random variables at each coordinate after weighting one by i:

$$n[x] = n_1[x] + i\ n_2[x]$$

1.1 PROBABILITY DISTRIBUTION

- P_n = probability of the integer value n for a discrete distribution
- $p[n]$ = probability density of a real-valued amplitude n derived from a continuous distribution
  - probability that continuous variable n lies within a specific interval is the integral of $p[n]$ over that interval, e.g.

$$P(a \leq n \leq b) = \int_a^b p[n]\ dn$$

1.2 THE MOMENTS of a PROBABILITY DISTRIBUTION

- "distribution" of probability about origin of coordinates determined by moments of $p[n]$
- k^{th} moment is area of product of $p[n]$ and a weight factor n^k:

$$m_k\{p[n]\} \equiv \int_{-\infty}^{+\infty} n^k\ p[n]\ dn$$

- "Zeroth moment" evaluated by setting $k = 0$; is projection of unit constant onto $p[n]$, which must be the unit area of $p[n]$:

$$m_0\{p[n]\} = \int_{-\infty}^{+\infty} n^0 \; p[n] \; dn = \int_{-\infty}^{+\infty} p[n] \; dn = \left\langle n^0 \right\rangle = 1$$

- First moment is projection of n^1 onto $p[n]$
  - "amplifies" contributions from large values of n.
  - Mean value of amplitude n selected from probability distribution $p[n]$
  - Notations: $\langle n \rangle$, $\hat{n}$, and μ

$$m_1\{p[n]\} = \int_{-\infty}^{+\infty} n^1 \; p[n] \; dn = \left\langle n^1 \right\rangle = \langle n \rangle$$

  - $\langle n \rangle$ is value of n that divides $p[n]$ into two equal parts

1.3 Central Moments

- expected values of $(n - \langle n \rangle)^k$
- measure weighted variation of $p[n]$ about mean:

$$\mu_k\{p[n]\} \equiv \int_{-\infty}^{+\infty} (n - \langle n \rangle)^k \; p[n] \; dn$$

- first central moment $\mu_1 = 0$.
- second moment is projection of n^2 onto $p[n]$:

$$m_2\{p[n]\} = \int_{-\infty}^{+\infty} n^2 \; p[n] \; dn \equiv \left\langle n^2 \right\rangle$$

- second central moment of $p[n]$ is variance σ^2
  - measures "spread" of noise probability about mean value:

$$\mu_2\{p[n]\} \equiv \int_{-\infty}^{+\infty} (n - \langle n \rangle)^2 \; p[n] \; dn \equiv \sigma^2$$

- Expand $(n - \langle n \rangle)^2$ and evaluating three resulting integrals to obtain important relationship among the mean, variance, and $\left\langle n^2 \right\rangle$

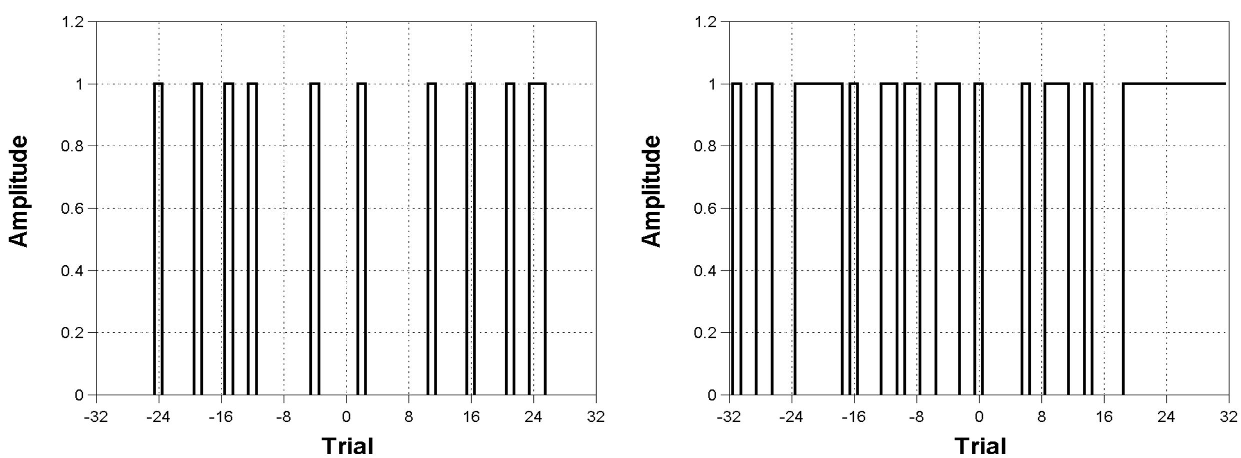
$$\sigma^2 = \left\langle n^2 \right\rangle - \langle n \rangle^2$$

1.4 DISCRETE PROBABILITY LAWS

- model processes that have discrete (and often binary) outcomes
- particularly useful when constructing models of such imaging processes as photon absorption by sensor.
- Simplest type of discrete probability law applies to events that have only two possible outcomes
  - success or failure
  - true or false
  - on or off
  - head or tail.
- Individual implementation of binary event is "Bernoulli trial"
- Collective realizations of many such events described by binomial and Poisson distributions.

1.4.1 BERNOULLI TRIALS

- Consider flux of photons onto "imperfect" absorbing material
  - Individual photons may be absorbed or not
  - Testing of absorption of a photon is a Bernoulli trial
  - "successful" absorption indicated by 1, and "failure" by 0
  - Statistics of string of Bernoulli trials specified by probability of "success" (outcome "1", denoted by "p")
    - $0 \leq p \leq 1$
  - Probability of "failure" (outcome "0") is $1 - p$, often denoted by "q".
- Relative probabilities for particular absorber determined from physical model of interaction or from observed results of large number of Bernoulli trials
- "Images" of independent Bernoulli trials for different values of p:



Examples of 64 Bernoulli trials: (a) $p = 0.2$, (b) $p = 0.5$.

1.4.2 MULTIPLE BERNOULLI TRIALS – THE BINOMIAL PROBABILITY LAW

- N Bernoulli trials where probability of outcome "1" is p and the probability of outcome "0" is $1-p$
- Number of possible distinct outcomes is 2^N
- Relative likelihood of outcomes is determined by p.
  - Probability of specific sequence $101010 \cdots 10$ (alternating "1" and "0", assuming N is even)

$$p\,(10101010\cdots101010) = p\cdot q\cdot p\cdot q\cdot\ \cdots\ \cdot p\cdot q = p^{\frac{N}{2}}\cdot q^{\frac{N}{2}}$$

- Equal numbers of "successes" and "failures".
- Probability of a different sequence where the first $\frac{N}{2}$ outcomes are "1" and the remaining $\frac{N}{2}$ outcomes are "0" is same:

$$p\,(1111111\cdots0000000) = p\cdot p\cdot p\cdot p\cdot\ \cdots\cdot q\cdot q\cdot q\cdot q\cdot q = p^{\frac{N}{2}}\cdot q^{\frac{N}{2}}$$

- Two distinct outcomes have same number of "successes" and "failures", and therefore have identical histograms.
- Outcome of N trials with exactly n "successes". The probability of a specific such outcome is:

$$p_1\cdot p_2\cdot p_3\cdot\cdots\cdot p_n\cdot q_1\cdot q_2\cdot q_3\cdot\cdots\cdot q_{N-n} = p^n\cdot q^{N-n}$$

- In many applications, order of arrangement is not significant; only total number of successes n matters.
- Compute number of possible combinations of n successes in N trials.
- It is straightforward to show that this number is the "binomial coefficient":

$$\frac{N!}{(N-n)!\;n!} \equiv \begin{pmatrix} N \\ n \end{pmatrix}$$

- probability of n "successes" in N trials is:

$$P_n = \frac{N!}{n!\;(N-n)!}\;p^n\;\;(1-p)^{N-n} = \begin{pmatrix} N \\ n \end{pmatrix}\;p^n\;\;[1-p]\;\;[N-n]$$

- Consider a coin flip with two outcomes, H and T.
  - $N=4 \Longrightarrow$number combinations with $n=2$ heads is $\frac{4!}{2!2!}=6$
  - *HHTT*, *HTHT*, *THHT*, *THTH*, *HTTH*, and *TTHH*
- If H and T equally likely $(p=q=0.5)$ $\Longrightarrow$probability of two heads in four flips is $P_2 = 6\,(0.5)^2\,(0.5)^2 = 0.375$.
- Number of realizations of no heads in four flips is $\frac{4!}{0!4!}=1$, Probability of no heads in four flips is $P_0 = 1\,(0.5)^0\,(0.5)^4 = 0.0625$.
- Binomial law also applies to cases where $p\neq q$
  - Flipping "unfair" coin
  - $p=0.75 \Longrightarrow$probability that four flips would produce two heads is $P_2 = 6\,(0.75)^2\,(0.25)^2 \simeq 0.211$

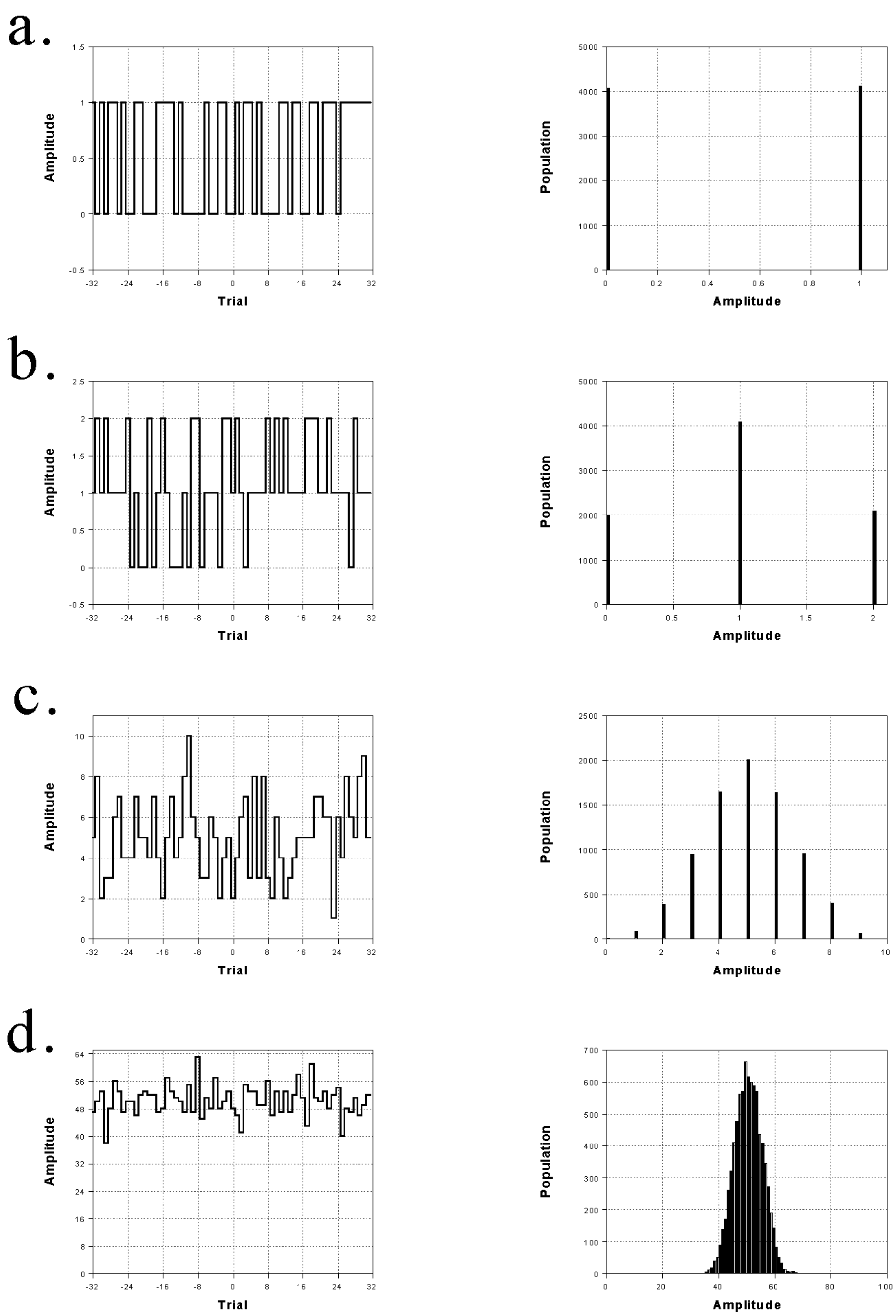
  - less than probability of two heads in four flips of fair coin.

- Mean and variance of number of outcomes with individual probability p in experiments obtained by substitution

$$\langle n \rangle = Np$$

$$\sigma^2 = N\ p\ [1 - p]$$

- "Shape" of histogram is approximately Gaussian.
  - Gaussian distribution is continuous
  - Binomial distribution is discrete
  - Observation suggests that samples obtained from large number of independent Bernoulli trials with probability p may be approximately generated by thresholding values generated by a Gaussian distribution.



Realizations of N Bernoulli trials at 8192 samples with $p = 0.5$ and the resulting histograms: (a) $N = 1$ trial per sample, two (nearly) equally likely outcomes; (b) $N = 2$; (c) $N = 10$; and (d) $N = 100$. The histogram approaches a Gaussian function for large N.

1.4.3 POISSON PROBABILITY LAW

- Approximation to binomial law for large numbers of rarely occurring events, i.e., $N >> 1$ and $p \to 0$
- Mean number of events is $\langle n \rangle = Np$, denoted by λ
- Form of Poisson law obtained by substituting into the binomial law in limit $N \to \infty$:

$$p_n = \lim_{N\to\infty} \left\{ \left(\begin{array}{c} N \\ n \end{array} \right) \left[\frac{\lambda}{N} \right]^n \left[1 - \frac{\lambda}{N} \right] [N-n] \right\}$$

- Take the natural logarithm of both sides to obtain:

$$\begin{aligned} \log_e [p_n] &= \lim_{N\to\infty} \left\{ \log_e \left[\left(\begin{array}{c} N \\ n \end{array} \right) \left[\frac{\lambda}{N} \right]^n \left[1 - \frac{\lambda}{N} \right] [N-n] \right] \right\} \\ &= \lim_{N\to\infty} \left\{ \log_e \left[\left(\begin{array}{c} N \\ n \end{array} \right) \left[\frac{\lambda}{N} \right]^n \right] \right\} + \lim_{N\ \to\infty} \left\{ [N-n]\ \ \log_e \left[1 - \frac{\lambda}{N} \right] \right\} \\ &= \lim_{N\to\infty} \left\{ \log_e \left[\left(\frac{N!}{[N-n]!\ n!} \right) \left[\frac{\lambda}{N} \right]^n \right] \right\} + \lim_{N\ \to\infty} \left\{ \frac{\log_e \left[1 - \frac{\lambda}{N} \right]}{[N-n]^{-1}} \right\} \end{aligned}$$

  - Use fact that n is small to evaluate first additive term:

$$\lim_{N\to\infty} \left\{ \log_e \left[\frac{N\ (N-1)\ \ (N-2)\cdots(N-n+1)}{n!} \left(\frac{\lambda}{N} \right)^n \right] \right\} \simeq \log_e \left\{ \frac{N^n}{n!} \left(\frac{\lambda}{N} \right)^n \right\}$$

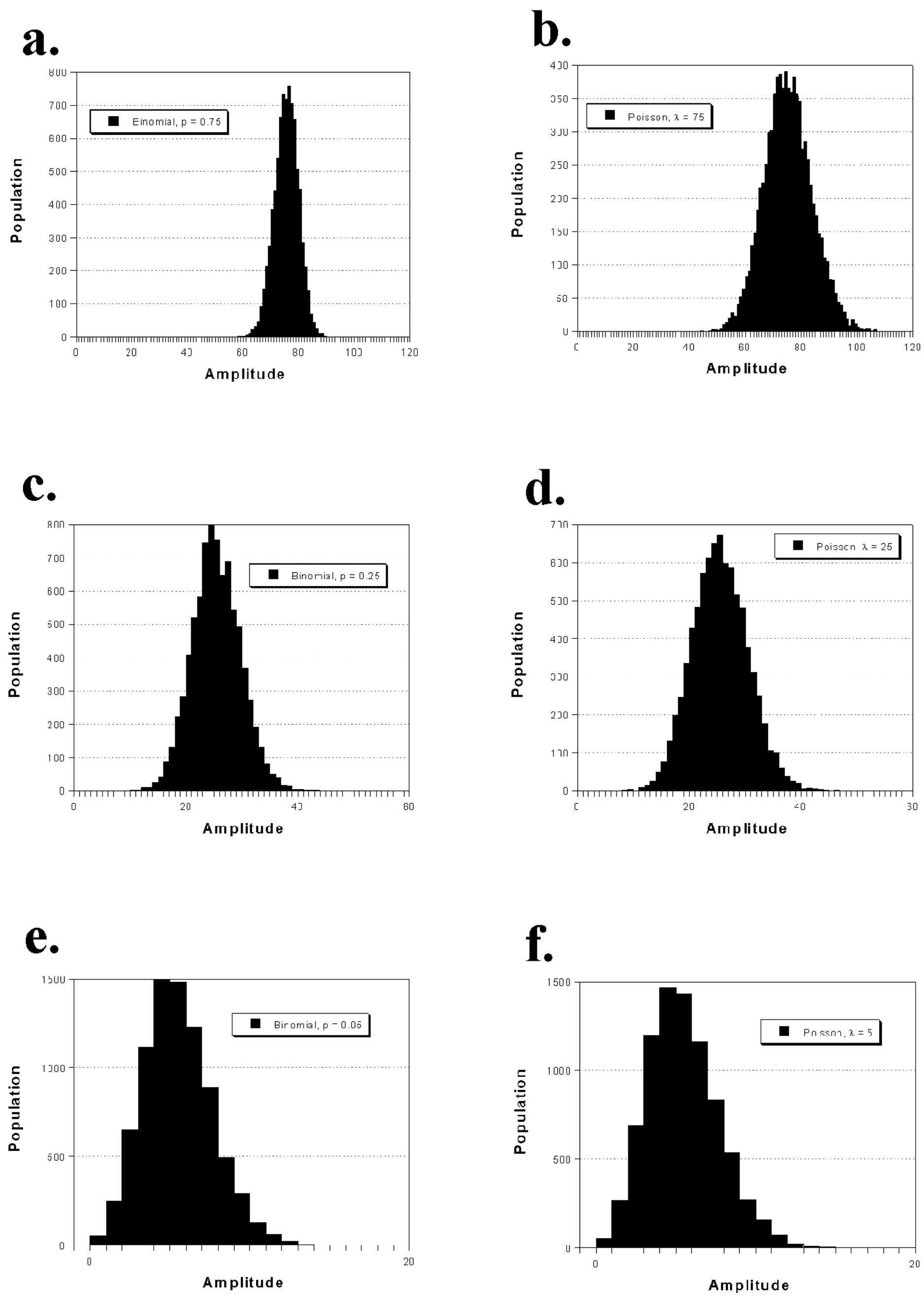
  - Second term evaluated by recognizing as ratio of two terms that both approach zero in the limit
  - Apply l'Hôspital's rule:

$$\begin{aligned} \lim_{N\to\infty} \left\{ \frac{\log_e \left(1 - \frac{\lambda}{N}\right)}{(N-n)^{-1}} \right\} &= \lim_{N\to\infty} \left\{ \frac{\frac{d}{dN}\left(\log_e \left(1 - \frac{\lambda}{N}\right)\right)}{\frac{d}{dN}(N-n)^{-1}} \right\} \\ &= \lim_{N\to\infty} \left\{ \frac{\left(1 - \frac{\lambda}{N}\right)^{-1} \left(\frac{\lambda}{N^2}\right)}{-(N-n)^{-2}} \right\} \\ &= \lim_{N\to\infty} \left\{ -\lambda \left(\frac{N-n}{N} \right)^2 \left(1 - \frac{\lambda}{N} \right)^{-1} \right\} \end{aligned}$$

  - Collect terms:

$$\log_e [p_n] = \log_e \left[\frac{\lambda^n}{n!} \right] - \lambda \Longrightarrow p_n = \left[\frac{\lambda^n}{n!} \right]\ e^{-\lambda}$$

  - Poisson distribution is particular limiting case of binomial distribution
- Mean, variance, and third central moment of Poisson distribution are identically λ



Comparison of binomial and Poisson random variables, $N = 100$: (a) binomial, $p = 0.75$, $\langle n \rangle = 75.05$, $\sigma^2 = 18.68$; (b) Poisson, $\lambda = 75$, $\langle n \rangle = 74.86$, $\sigma^2 = 74.05$; (c) binomial, $p = 0.25$, $\langle n \rangle = 24.93$, $\sigma^2 = 18.77$; (d) Poisson, $\lambda = 25$, $\langle n \rangle = 25.01$, $\sigma^2 = 24.85$; (e) binomial, $p = 0.05$, $\langle n \rangle = 5.00$, $\sigma^2 = 4.71$; (f) Poisson, $\lambda = 5$, $\langle n \rangle = 4.97$, $\sigma^2 = 4.97$.

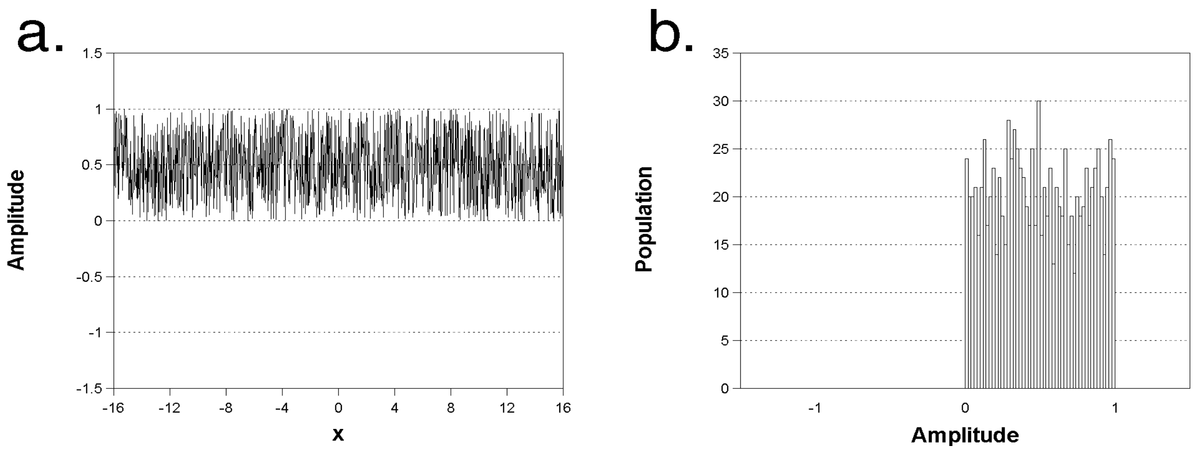
1.5 CONTINUOUS PROBABILITY DISTRIBUTIONS

1.5.1 UNIFORM DISTRIBUTION

- generates most intuitive type of noise
  - amplitude n equally likely to occur within any finite interval of equal size.

$$p_{Uniform}[n] = \frac{1}{|b|} RECT\left[\frac{n - \langle n \rangle}{b}\right]$$

  - b is width of allowed values of n
  - $\langle n \rangle$ is mean value
  - Multiplicative scale factor b^{-1} ensures unit area
  - Variance of uniform distribution is $\sigma^2 = \frac{b^2}{12}$.



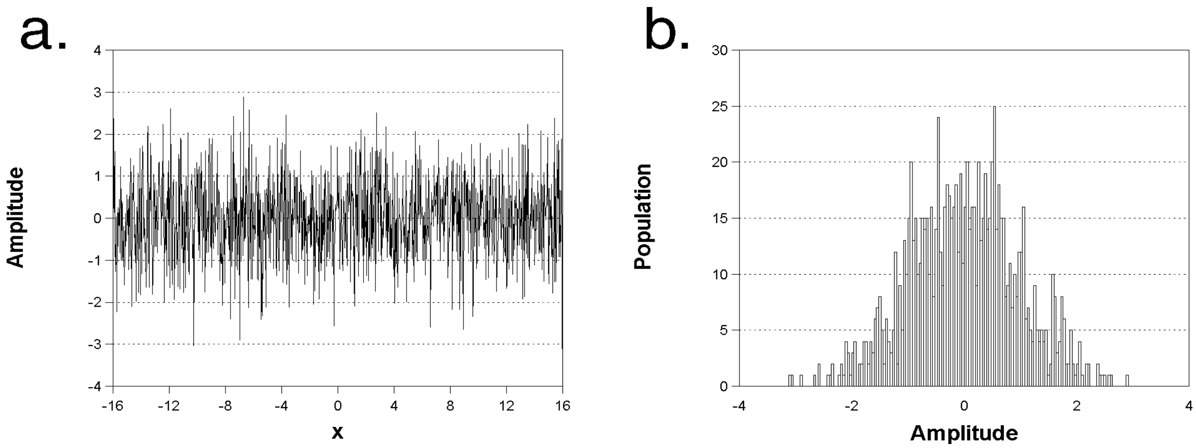
Uniformly distributed random variable on interval $[0, 1)$ with $\mu = 0.5$ and $\sigma^2 = \frac{1}{12}$: (a) sample, (b) histogram.

1.5.2 NORMAL DISTRIBUTION

- Familiar symmetric "bell curve" of probability
- Most applicable of all probability laws
- $\langle n \rangle$ = most likely amplitude (peak of the probability density)
- Probability that amplitude will differ from mean progressively decreases as the value moves away from $\langle n \rangle$
- Probability density function is Gaussian function with width parameter b proportional to standard deviation σ of probability distribution

$$p_{Normal}[n] = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(n-\langle n \rangle)^2}{2\sigma^2}}$$

- Leading factor $\left(\sqrt{2\pi}\right)^{-1}$ ensures that area of the probability density function is unity



Samples of a random variable generated by a normal distribution with $\langle n \rangle = 0, \sigma^2 = 1$: (a) samples, (b) histogram.

Central-Limit Theorem:

- Cascade of stochastic processes derived from (nearly) arbitrary set of probability density functions generates a normal distribution
- Central-limit theorem ensures that probability law of outputs is generally Gaussian, to good approximation.

1.5.3 RAYLEIGH DISTRIBUTION

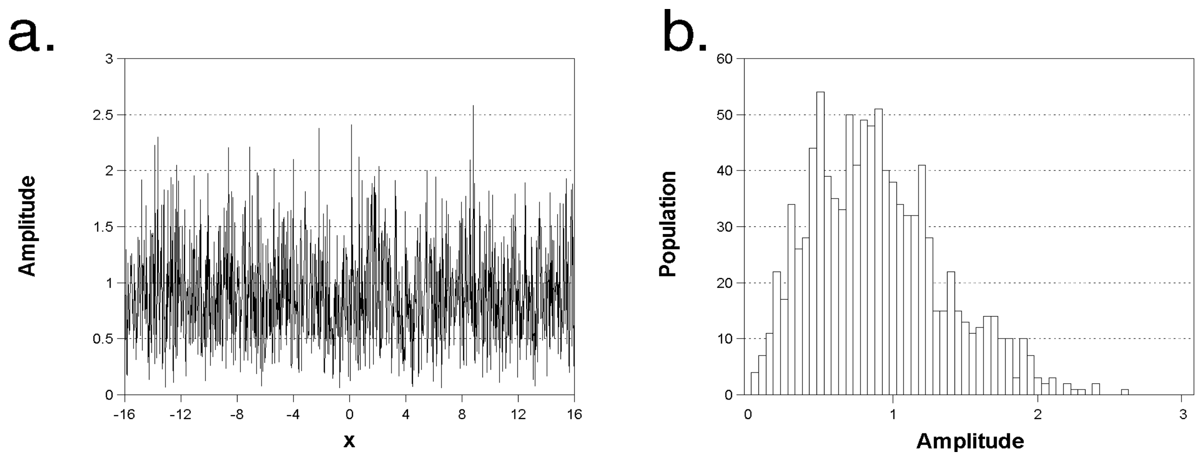
- Imaging applications that involve Fourier transforms of distributions of complex-valued random variables
  - description of Fraunhofer diffraction from a random scatterer
  - computer-generated holography.
- Distribution of magnitude where where real and imaginary parts are random variables selected from same Gaussian distribution.
- probability density function characterized by single parameter a:

$$p_{Rayleigh}[n] = \frac{n}{a^2} e^{-\left(\frac{n^2}{2a^2}\right)} STEP[n]$$

- $STEP$ function ensures that allowed amplitudes n must be nonnegative
- Mean $\langle n \rangle$ and variance σ^2 of the Rayleigh distribution must be functions of the parameter a:

$$\langle n \rangle = \sqrt{\frac{\pi}{2}} a \simeq 1.25a$$

$$\sigma^2 = \left(2 - \frac{\pi}{2}\right) a^2 \simeq 0.429a^2, \ \sigma \simeq 0.655a$$



Rayleigh-distributed random variable generated from Gaussian-distributed random variables in quadrature, each with $\langle n \rangle = 0, \sigma^2 = 1$: (a) sample, (b) histogram. Resulting mean and variance are $\langle n \rangle = 1.270, \sigma^2 = 0.424$.