Past Paper: Partial Differential Equations

Midterm Exam, 2024

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Question 1.

Classify the following equation and reduce it to the canonical form

$$u_{xy} + yu_{yy} + \sin(x+y) = 0.$$

Solution. We start by swapping the variables $x \leftrightarrow y$ to get the transformed PDE

$$xu_{xx} + u_{xy} + \sin(x+y) = 0. \tag{$*$}$$

Now, $A = x$, $B = 1$, $C = D = E = 0$. Since, $B^2 - 4AC = 1 > 0$, this is a hyperbolic PDE. So, the corresponding characteristic equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{1 \pm 1}{2x}$$

gives

$$\frac{dy}{dx} = \frac{1}{x} \implies k + y = \ln x \implies xe^{-y} = c_1, \quad \frac{dy}{dx} = 0 \implies y = c_2.$$

Here, k, c_1 and c_2 are constants. These lead to $\xi = xe^{-y} = c_1$ and $\eta = y = c_2$. As a result

$$\begin{aligned} B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = e^{-y}, \\ D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = -e^{-y}, \\ E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0. \end{aligned}$$

(Also, $A^* = C^* = 0$ because the PDE is hyperbolic.)

As a result, equation ($*$) implies

$$e^{-y}u_{\xi\eta} - e^{-y}u_\xi = \sin(x+y) \implies u_{\xi\eta} - u_\xi = e^\eta \sin(\eta + \xi e^\eta).$$

Here, we used $y = \eta$ and $x = \xi e^y = \xi e^\eta$.

Overall, the canonical form of the given PDE is

$$u_{\xi\eta} = u_\xi + e^\eta \sin(\eta + \xi e^\eta)$$

with $\xi = ye^{-x}$ and $\eta = x$ (in terms of the original variables).

Question 2.

Apply a linear transformation $\xi = x + by$ and $\eta = x + dy$ to transform the Euler equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

into canonical form, where b, d A, B and C are constants.

Solution. Solution

Question 3.

Determine the solution of the equation

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$$

with the data $x+y=0$ and $u=1$.

Solution. Solution

Question 4.

Show that the general solution of a first-order quasilinear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

is $f(\varphi,\psi)=0$, where f is an arbitrary function of $\varphi(x,y,u)$ and $\psi(x,y,u)$, and $\varphi(x,y,u)=c_1$ and $\psi(x,y,u)=c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

Solution. Solution.

Question 5.

Verify that the function

$$u = \varphi(xy) + x\psi\left(\frac{y}{x}\right)$$

is the general solution of the equation

$$x^2u_{xx} - y^2u_{yy} = 0.$$

Solution. We have

$$\begin{aligned}
u_x &= y\varphi'(xy) + \psi\left(\frac{y}{x}\right) - x\left(\frac{y}{x^2}\right)\psi'\left(\frac{y}{x}\right) = y\varphi'(xy) + \psi\left(\frac{y}{x}\right) - \frac{y}{x}\psi'\left(\frac{y}{x}\right)\\
u_y &= x\varphi'(xy) + x\left(\frac{1}{x}\right)\psi'\left(\frac{y}{x}\right) = x\varphi'(xy) + \psi'\left(\frac{y}{x}\right)\\
u_{xx} &= y^2\varphi''(xy) - \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) + \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) + \frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right) = y^2\varphi''(xy) + \frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right)\\
u_{yy} &= x^2\varphi''(xy) + \frac{1}{x}\psi''\left(\frac{y}{x}\right)
\end{aligned}$$

Therefore,

$$x^2u_{xx} - y^2u_{yy} = x^2y^2\varphi''(xy) + \frac{y^2}{x}\psi''\left(\frac{y}{x}\right) - x^2y^2\varphi''(xy) + \frac{y^2}{x}\psi''\left(\frac{y}{x}\right) = 0$$

Question 6.

Determine the solution of the initial boundary value problem

$$\begin{aligned} u_{tt} &= 4u_{xx}, && 0 < x < 1, \quad t > 0, \\ u(x,0) &= 0, && 0 \le x \le 1, \\ u_t(x,0) &= x(1-x), && 0 \le x \le 1, \\ u(0,t) = 0,\ u(1,t) &= 0, && t \ge 0. \end{aligned}$$

Solution. This is the wave equation ($c = 2$) on a bounded spatial interval $[0,1]$. The general solution is $u(x,t) = \varphi(x+2t) + \psi(x-2t)$ with

$$\varphi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^\eta g(\tau)\,d\tau + \frac{K}{2}, \quad \psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^\eta g(\tau)\,d\tau - \frac{K}{2}.$$

for $0 \le \eta \le 1$. Using $f(\eta) = 0$ and $g(\eta) = \eta(1-\eta)$ gives,

$$\varphi(\eta) = \frac{1}{4}\left(\frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right) + \frac{K}{2}, \quad 0 \le \eta \le 1, \tag{1}$$

$$\psi(\eta) = -\frac{1}{4}\left(\frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right) - \frac{K}{2}, \quad 0 \le \eta \le 1. \tag{2}$$

The boundary conditions give

$$u(0,t) = 0 \implies \psi(-2t) = -\varphi(2t) \implies \psi(\eta) = -\varphi(-\eta), \tag{$*$}$$

$$u(1,t) = 0 \implies \phi(1+2t) = -\varphi(1-2t) \implies \psi(\eta) = -\varphi(2-\eta). \tag{$**$}$$

Now, using ($*$) in (1) gives,

$$\psi(\eta) = -\varphi(-\eta) = -\frac{1}{4}\left(\frac{1}{2}\eta^2 + \frac{1}{3}\eta^3\right) - \frac{K}{2}. \tag{3}$$

This is valid for $0 \le -\eta \le 1 \implies -1 \le \eta \le 0$. So, the domain of ψ has been extended.

Similarly, using ($**$) in (2) gives,

$$\varphi(\eta) = -\psi(2-\eta) = \frac{1}{4}\left(\frac{1}{2}(2-\eta)^2 - \frac{1}{3}(2-\eta)^3\right) + \frac{K}{2}. \tag{4}$$

This is valid for $0 \le 2-\eta \le 1 \implies 1 \le \eta \le 2$. So, the domain of φ has been extended.

We repeat this process to extend the domain further: using ($*$) in (3) and ($**$) in (4).