Past Paper: Partial Differential Equations

Midterm Exam, 2024

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Question 1.

Classify the following equation and reduce it to the canonical form

$$u_{xy} + yu_{yy} + \sin(x+y) = 0.$$

Solution. We start by making a change of variables $x \leftrightarrow y$ to get the transformed PDE

$$xu_{xx} + u_{xy} + \sin(x+y) = 0. \tag{$*$}$$

Now, $A = x$, $B = 1$, $C = D = E = 0$. Since, $B^2 - 4AC = 1 > 0$, this is a hyperbolic PDE. So, the corresponding characteristic equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{1 \pm 1}{2x}$$

gives

$$\frac{dy}{dx} = \frac{1}{x} \implies k + y = \ln x \implies xe^{-y} = c_1, \quad \frac{dy}{dx} = 0 \implies y = c_2.$$

These lead to $\xi = xe^{-y} = c_1$ and $\eta = y = c_2$. As a result

$$\begin{aligned} B^* &= EQQ = EQQ \\ D^* &= EQQ = EQQ \\ E^* &= EQQ = EQQ. \end{aligned}$$

(The coefficients $A^* = C^* = 0$ because the PDE is hyperbolic.)

As a result, equation $(*)$ implies

$$e^y u_{\xi\eta} - e^y u_{\xi\xi} = \sin(x+y) \implies u_{\xi\eta} - u_{\xi\xi} = e^{-\eta}\sin(\eta + \xi e^\eta).$$

Here, we used $y = \eta$ and $x = \xi e^y = \xi e^\eta$.

Overall, the canonical form of the given PDE is

$$u_{\xi\eta} = u_{\xi\xi} + e^{-\eta}\sin(\eta + \xi e^\eta)$$

with $\xi = ye^{-x}$ and $\eta = x$ (in the original variables).

Question 2.

Apply a linear transformation $\xi = x + by$ and $\eta = x + dy$ to transform the Euler equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

into canonical form, where b, d A, B and C are constants.

Solution. Solution

Question 3.

Determine the solution of the equation

$$x(y^2+u)u_x - y(x^2+u)u_y = (x^2-y^2)u$$

with the data $x+y=0$ and $u=1$.

Solution. Solution

Question 4.

Show that the general solution of a first-order quasilinear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

is $f(\varphi,\psi)=0$, where f is an arbitrary function of $\varphi(x,y,u)$ and $\psi(x,y,u)$, and $\varphi(x,y,u)=c_1$ and $\psi(x,y,u)=c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a}=\frac{dy}{b}=\frac{du}{c}.$$

Solution. Solution.

Question 5.

Verify that the function

$$u=\varphi(xy)+x\psi\left(\frac{y}{x}\right)$$

is the general solution of the equation

$$x^2u_{xx}-y^2u_{yy}=0.$$

Solution. We have

$$\begin{aligned}
u_x &= y\varphi'(xy)+\psi\left(\frac{y}{x}\right)-x\left(\frac{y}{x^2}\right)\psi'\left(\frac{y}{x}\right)=y\varphi'(xy)+\psi\left(\frac{y}{x}\right)-\frac{y}{x}\psi'\left(\frac{y}{x}\right)\\
u_y &= x\varphi'(xy)+x\left(\frac{1}{x}\right)\psi'\left(\frac{y}{x}\right)=x\varphi'(xy)+\psi'\left(\frac{y}{x}\right)\\
u_{xx} &= y^2\varphi''(xy)-\frac{y}{x^2}\psi'\left(\frac{y}{x}\right)+\frac{y}{x^2}\psi'\left(\frac{y}{x}\right)+\frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right)=y^2\varphi''(xy)+\frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right)\\
u_{yy} &= x^2\varphi''(xy)+\frac{1}{x}\psi''\left(\frac{y}{x}\right)
\end{aligned}$$

Therefore,

$$x^2u_{xx}-y^2u_{yy}=x^2y^2\varphi''(xy)+\frac{y^2}{x}\psi''\left(\frac{y}{x}\right)-x^2y^2\varphi''(xy)+\frac{y^2}{x}\psi''\left(\frac{y}{x}\right)=0$$

Question 6.

Determine the solution of the initial boundary value problem

$$\begin{gathered}
u_{tt}=4u_{xx},\quad 0<x<1,\quad t>0,\\
u(x,0)=0,\quad 0\le x\le 1,\\
u_t(x,0)=x(1-x),\quad 0\le x\le 1,\\
u(0,t)=0,\quad u(1,t)=0,\quad t\ge 0.
\end{gathered}$$

Solution. Solution