PHY 443: Quantum Field Theory

Brief lecture notes

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Textbook: Modern Particle Physics, Mark Thomson

1 The Klein-Gordon Equation

2 The Dirac Equation

DERIVATION

CONVERSION TO THE COVARIANT FORM

The Dirac equation is covariant form is

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \tag{1}$$

THE VERSION WITH $\hbar$ and c.

2.1 Adjoint Dirac Equation

In order to construct the adjoint Dirac equation we take the Hermitian adjoint of the full equation, carefully reserving the order of the matrix multiplications

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0 \implies -i\partial_{\mu}\psi^{\dagger}(\gamma^{\mu})^{\dagger} - m\psi^{\dagger} = 0. \tag{2}$$

Note that $(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0$. Multiply this equation by γ^0 from the right to get

$$-i\partial_{\mu}\psi^{\dagger}\gamma^0\gamma^{\mu}\gamma^0\gamma^0 - m\psi^{\dagger}\gamma^0 = 0. \tag{3}$$

Now, by using $\left(\gamma^0\right)^2 = I$ and defining $\overline{\psi} = \psi^{\dagger}\gamma^0$ we obtain

$$-i\partial_{\mu}\overline{\psi}\gamma^{\mu} - m\overline{\psi} = 0. \tag{4}$$

This is the adjoint Dirac equation, and the new object $\overline{\psi}$ is called the adjoint Dirac spinor.

2.2 Dirac Current

Following the standard procedure we take the difference of the Dirac equation pre-multiplied by $\overline{\psi}$ and the adjoint Dirac equation post-multiplied by ψ,

$$\overline{\psi}(i\gamma^{\mu}\partial_{\mu}\psi - m\psi) - \left(-i\partial_{\mu}\overline{\psi}\gamma^{\mu} - m\overline{\psi}\right)\psi = 0 \tag{5}$$

Expanding this while maintaining the order in which each factor appears, we get

$$i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi + i\partial_{\mu}\overline{\psi}\gamma^{\mu}\psi - m\overline{\psi}\psi = 0 \implies i\left(\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi + \partial_{\mu}\overline{\psi}\gamma^{\mu}\psi\right) = 0 \tag{6}$$

Now, $\partial_\mu\Big(\overline{\psi}\gamma^\mu\psi\Big) = \overline{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\overline{\psi}\gamma^\mu\psi$ because γ^μ are constant matrices. Therefore, we obtain

$$i\partial_\mu\Big(\overline{\psi}\gamma^\mu\psi\Big) = 0. \tag{7}$$

From this we identify a conserved current

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \overline{\psi}\gamma^\mu\psi \tag{8}$$

This is known as the Dirac (vector) 4-current. It is the conserved current associated with the U(1) symmetry of the Dirac equation. We can separate it into a probability density and 3-current by comparing with $j^\mu = (c\rho, \mathbf{J})$. In particular,

$$\rho = \frac{1}{c}\overline{\psi}\gamma^0\psi = \frac{1}{c}\psi^\dagger\psi \quad \text{and} \quad \mathbf{J} = \overline{\psi}\boldsymbol{\gamma}\psi = \psi^\dagger\boldsymbol{\alpha}\psi \tag{9}$$

This expression for ρ shows that the probability density associated with the Dirac equation is non-negative because

$$\rho = \frac{1}{c}\psi^\dagger\psi = \frac{1}{c}\Big(|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2\Big) \geq 0. \tag{10}$$

3 Gamma Matrices

4 Spin Angular Momentum

In the Heisenberg picture of quantum mechanics, the operators have time dependence and their expectation value evolves according to the Heisenberg equation

$$\frac{d\langle\hat{O}\rangle}{dt} = i\,\langle\psi|\,[\hat{H},\hat{O}]\,|\psi\rangle\,. \tag{11}$$

As a result, the expectation value $\langle\hat{O}\rangle$ of an operator $\hat{O}$ is conserved when it commutes with the associated Hamiltonian

$$\frac{d\langle\hat{O}\rangle}{dt} = 0 \iff [H,O] = 0 \tag{12}$$

and not conserved otherwise.

We shall later discuss that fermionic particles satisfy the Dirac equation. At this stage, we can ask whether certain physical quantities are conserved for such particles. Of interest is the (orbital) angular momentum $\mathbf{L}$. We can construct its quantum operator through the correspondence principle

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \to \hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}. \tag{13}$$

In particular $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$.

To check if the expectation value of $\hat{\mathbf{L}}$ is conserved for fermions we must check if it commutes with the Dirac Hamiltonian $\hat{H}_D = c\boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta$.

We find that $[\hat{H}_D, \hat{\mathbf{L}}] = -i\hbar c(\boldsymbol{\alpha}\times\mathbf{p}) \neq 0$. Therefore, $\langle\hat{\mathbf{L}}\rangle$ is not conserved. (Full calculations in the appendix.)

One explanation for this could be that we failed to account for the *total* angular momentum. This leads to the postulate of a 'spin' angular momentum, $\mathbf{S}$, so that the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is conserved. That is

$$[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}] = 0. \tag{14}$$

The operator corresponding to the spin angular momentum has the form

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\boldsymbol{\Sigma} \tag{15}$$

where the components of $\boldsymbol{\Sigma}$ are 4×4 matrices (in block form)

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \quad \text{for } i = x, y, z. \tag{16}$$

Here σ_i are the 2×2 Pauli matrices satisfying $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$.

We find that $[\hat{H}_D, \hat{\mathbf{S}}] = i\hbar c(\boldsymbol{\alpha} \times \mathbf{p})$. (Full calculations in the appendix.)

Consequently,

$$[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}] = [\hat{H}_D, \hat{\mathbf{L}}] + [\hat{H}_D, \hat{\mathbf{S}}] - i\hbar c(\boldsymbol{\alpha} \times \mathbf{p}) + i\hbar c(\boldsymbol{\alpha} \times \mathbf{p}) = 0, \tag{17}$$

preserving the conservation of the total angular momentum $\mathbf{J}$ for particles satisfying the Dirac equation.

5 Solutions of the Klein-Gordon Equation

6 Solutions of the Dirac Equation

Using the plane wave ansatz $\psi(\mathbf{x}, t) = u(E, \mathbf{p}) \exp(-ip_\mu x^\mu)$ we can reduce the Dirac equation into an equation for $u(E, \mathbf{p})$,

$$\begin{aligned}
&i\gamma^\mu \partial_\mu \psi - m\psi = 0 && (18)\\
&\quad\implies i\gamma^\mu \partial_\mu u(E, \mathbf{p}) \exp(-ip_\mu x^\mu) - m u(E, \mathbf{p}) \exp(-ip_\mu x^\mu) = 0 && (19)\\
&\quad\implies i\gamma^\mu u(E, \mathbf{p})(-ip_\mu) \exp(-ip_\mu x^\mu) - m u(E, \mathbf{p}) \exp(-ip_\mu x^\mu) = 0 && (20)\\
&\quad\implies \gamma^\mu p_\mu u(E, \mathbf{p}) \exp(-ip_\mu x^\mu) - m u(E, \mathbf{p}) \exp(-ip_\mu x^\mu) = 0 && (21)\\
&\quad\implies (\gamma^\mu p_\mu - m) u(E, \mathbf{p}) \exp(-ip_\mu x^\mu) = 0 && (22)\\
&\quad\implies (\gamma^\mu p_\mu - m) u(E, \mathbf{p}) = 0. && (23)
\end{aligned}$$

6.1 Free Particle Solutions with $\mathbf{p} = 0$

We start by considering the case where $\mathbf{p} = 0$. Then, we have

$$\Big(\gamma^0 E - m\Big) u(E, 0) = 0, \tag{24}$$

or equivalently,

$$\gamma^0 u(E, 0) = \frac{m}{E} u(E, 0), \tag{25}$$

which is an eigenvalue equation; m/E correspond to the eigenvalues of γ^0 and $u(E, 0)$ are the associated eigenvectors. Since, $\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$ in the Pauli-Dirac representation, we immediately note that the possible eigenvalues are ± 1 (repeated twice). In other words, there are two solutions with $E = +m$ and two solutions with $E = -m$.

For the eigenvalue $+1$, that is for $E = +m$, the corresponding eigenvectors are

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{26}$$

Likewise, for $E = -m$, we obtain the eigenvectors

$$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{27}$$

Consequently, the full plane wave solutions for a stationary massive Dirac particle are

$$\psi_1 = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \qquad \psi_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \tag{28}$$

$$\psi_3 = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}, \qquad \psi_4 = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}, \tag{29}$$

with N as a normalisation factor.

6.2 General Free Particle Solutions

7 Normalisation

8 Antiparticles

How to interpret the negative energy solutions?

8.1 Dirac Hole Theory

Dirac proposed the dirac hole concept. The idea being that the negative energy states are physical but always filled, so physical particles such as electrons cannot acquire negative energies due to the Pauli-exclusionn principle. In a particles from this sea of filled states is excited, it can move out of the negative states, leaving behind a hole. The hole can be interpreted as a particle in its own right, with the same mass but the sign of all the quantum numbers and charges are swapped. Such a hole is then called an antiparticle.

There were several issues with this idea. Firstly, the Pauli exclusion principle only applied to fermionic particles but bosonic particles also have negative energy solution (antiparticles) whose existence is not explained by this construction. Moreover, even the positive energy bosonic states suffer from instability (they can indefinitely lose energy and fall to lower energy levels).

8.2 Feynman-Stückelberg Interpretation

In the previous section we found four solutions plane wave solution of the Dirac equation that were moving forward in time. Two of these contained negative energies. One method for converting them into positive energy solutions is by interpreting them as positive energy solutions moving backwards in time

$$e^{-i(Et)} = e^{-i((-E)(-t))} \qquad E < 0. \tag{30}$$

This proposal is known as the Feynman-Stückelberg interpretation.

Formally, we consider the solutions to the adjoint Dirac equation

$$-i\partial_\mu \overline{\psi}\gamma^\mu - m\overline{\psi} = 0 \tag{31}$$

using the same plane wave ansatz

$$\overline{\psi} = \overline{u}(E, \mathbf{P})e^{-ip_\mu x^\mu}. \tag{32}$$

(This is equivalent to seeking time-reversed solution $v(E, \mathbf{p}) \exp(+ip_\mu x^\mu)$ of the original Dirac equation.)

Following the same procedure as described in the previous section, we obtain

$$SOLUTIONSFOR\overline{\psi}, \tag{33}$$

where again $\overline{\psi}_1$ and $\overline{\psi}_2$ have $E > 0$, while $\overline{\psi}_3$ and $\overline{\psi}_4$ have $E < 0$.

A Commutation Relations of the Dirac Hamiltonian and Angular Momentum Operators