Past Paper: Calculus of Variations

Midterm Exam, 2024

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Question 1 (a).

If $\alpha(x)$ is continuous in $[a, b]$ and if $\int_a^b \alpha(x)h(x)\,dx = 0$ for every $h \in D_1(a, b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$, for all $x \in [a, b]$.

(Note: $D_1(a, b)$ consists of all the continuous functions defined on the interval $[a, b]$ whose first order derivative is also continuous.)

Solution. For a contradiction, suppose $\alpha(x_0) \neq 0$ for some $x_0 \in (a, b)$.

Without loss of generality, assume that $\alpha(x_0) > 0$. By the continuity of α, there exists an interval $I = (x_a, x_b) \subseteq (a, b)$ with $x_0 \in I$ such that $\alpha(x) > 0$ for all $x \in I$.

Construct the function $h : [a, b] \to \mathbb{R}$ with

$$h(x) = \begin{cases} (x - x_a)^2(x - x_b)^2, & x \in (x_a, x_b) \\ 0, & \text{otherwise} \end{cases}$$

This $h(x)$ is continuous, and its first derivative is also continuous everywhere including $x = x_a$ and $x = x_b$. Moreover, $h(a) = h(b) = 0$. Now,

$$\begin{aligned} \int_a^b \alpha(x)h(x)\,dx &= \int_a^{x_a} \alpha(x)h(x)\,dx + \int_{x_a}^{x_b} \alpha(x)h(x)\,dx + \int_{x_b}^{b} \alpha(x)h(x)\,dx \\ &= \int_a^{x_a} \alpha(x)0\,dx + \int_{x_a}^{x_b} \alpha(x)(x - x_a)^2(x - x_b)^2\,dx + \int_{x_b}^{b} \alpha(x)0\,dx \\ &= \int_{x_a}^{x_b} \alpha(x)(x - x_a)^2(x - x_b)^2\,dx > 0 \end{aligned}$$

because each term in the integrand is strictly positive. This contradicts the hypothesis that $\int_a^b \alpha(x)h(x)\,dx = 0$. Therefore, our assumption $\alpha(x_0) \neq 0$ is wrong, and indeed $\alpha(x) = 0$ for all $x \in [a, b]$.

(Note: Taking $\alpha(x_0) < 0$ leads to the same conclusion due to $\int_a^b \alpha(x)h(x)\,dx < 0$.)

Question 1 (b).

Find the extremal of the functional $J[y(x)] = \int_0^1 (xy' + y'^2)\,dx$, $y(0) = 0$, $y(1) = 1$.

Solution. The Euler-Lagrange equation for this functional is

$$F_y - \frac{d}{dx}F_{y'} = 0$$

with $F = xy' + y'^2$. As $F_y = 0$ and $F_{y'} = x + 2y'$, the Euler-Lagrange reduces to

$$0 - \frac{d}{dx}(x + 2y') = 0 \implies \frac{d}{dx}(x + 2y') = 0 \implies x + 2y' = c$$

where c is a constant. Re-arranging and integrating this equation gives

$$x + 2y' = c \implies y' = \frac{c}{2} - x \implies y = k + \frac{c}{2}x - \frac{1}{4}x^2$$

where k is another constant. Now, $y(0) = 0$ implies $k = 0$ and $y(1) = 1$ gives $c = 5/2$. As a result, the required extremal is

$$y = \frac{5}{4}x - \frac{1}{4}x^2$$

Question 2 (a).

Show that the Euler equation of the functional $J[y(x)] = \int_a^b F(x, y, y')\,dx$ has the first integral/conserved quantity $F - y'F_{y'} = \text{const.}$, if the integrand does not depend on x.

Solution. Firstly note that the Euler-Lagrange equation for this functional is

$$F_y - \frac{d}{dx}F_{y'} = 0.$$

If F is independent of x then $F_x = 0$. Therefore,

$$\begin{aligned}
\frac{d}{dx}(F - y'F_{y'}) &= \frac{dF}{dx} - y''F_{y'} - y'\frac{d}{dx}F_{y'} \\
&= F_x + y'F_y + y''F_{y'} - y''F_{y'} - y'\frac{d}{dx}F_{y'} \\
&= y'F_y - y'\frac{d}{dx}F_{y'} \\
&= y'\left(F_y - \frac{d}{dx}F_{y'}\right) \\
&= 0
\end{aligned}$$

In the last step we used the Euler equation. Consequently, $F - y'F_{y'} = \text{constant}$.

Question 2 (b).

Find the extremal of the functional $J[y(x)] = \int_0^{\pi/2} (y'' - y'^2 + x^2)\,dx$, subject to $y(0) = 1$, $y'(0) = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = -1$.

Solution. The Euler-Lagrange equation for this functional is

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$$

with $F = y'' - y'^2 + x^2$. Since, $F_y = 0$, $F_{y'} = -2y'$ and $F_{y''} = 1$, the Euler-Lagrange reduces to

$$0 + 2\frac{d}{dx}(y') + \frac{d^2}{dx^2}(1) = 0 \implies \frac{d}{dx}(y') = 0 \implies y' = c_1, \implies y = c_0 + c_1 x,$$

where c_0, c_1 are constants.

This is not compatible with the boundary conditions, because $y'(0) = 0$ implies $c_1 = 0$, while $y'(\pi/2) = -1$ requires $c_1 = -1$; we cannot have $c_1 = 0$ and $c_1 = -1$ simultaneously.

Therefore, this variational problem has no solution.

Question 3.

Find the geodesics on the plane $r(x, y) = (x, y, 0)$.

Solution. A geodesic between two points (a, A) and (b, B) is a curve of shortest length passing through the given two points. The plane $r(x, y) = (x, y, 0)$ corresponds to the xy-plane. Therefore, the complete variation problem is to find the curve of shortest length $y = y(x)$ passing through (a, A) and (b, B). The length of such a curve is

$$L[y] = \int_a^b ds = \int_a^b \sqrt{1 + y'^2}\, dx.$$

The Euler-Lagrange equation for this functional is

$$F_y - \frac{d}{dx} F_{y'} = 0$$

with $F = \sqrt{1 + y'^2}$. Using $F_y = 0$ and $F_{y'} = y'(1 + y'^2)^{-1/2}$, the Euler-Lagrange reduces to

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0 \implies \frac{y'}{\sqrt{1 + y'^2}} = k \implies y' = \frac{k}{\sqrt{1 - k^2}} = m, \implies y = mx + c,$$

where m, c, k are constants. Thus, the required extremals are straight line segments.

Imposing the condition $y(a) = A$ and $y(b) = B$ leads to

$$m = \frac{B - A}{b - a} \quad \text{and} \quad c = A - \left(\frac{B - A}{b - a}\right) a.$$

If $a = b$, then we get the straight line segment $x = a$, $A \leq y \leq B$.