

# Past Paper: Differential Geometry

Midterm Exam, 2024

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## Question 1.

Let  $\alpha(t) = (t, t^2, t^3)$  be a curve in the Euclidean space. Calculate the following parameters for the given curve: unit tangent vector, unit normal vector, unit binormal vector, curvature, and torsion.

*Solution.* Firstly, note that

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}, \quad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|^2}, \quad \mathbf{T} = \frac{\alpha'}{|\alpha'|}, \quad \mathbf{B} = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}, \quad \mathbf{N} = \mathbf{B} \times \mathbf{T}.$$

Now,  $\alpha'(t) = (1, 2t, 3t^2)$ ,  $\alpha''(t) = (0, 2, 6t)$  and  $\alpha'''(t) = (0, 0, 6)$ . Therefore,

$$\alpha' \times \alpha'' = (6t^2, -6t, 2), \quad |\alpha' \times \alpha''| = \sqrt{4 + 36t^2 + 36t^4} = 2\sqrt{1 + 9t^2 + 9t^4},$$

and  $|\alpha'| = \sqrt{1 + 4t^2 + 9t^4}$ . As a result,

$$\begin{aligned} \kappa &= \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(\sqrt{1 + 4t^2 + 9t^4})^3} = 2 \left( \frac{1 + 9t^2 + 9t^4}{(1 + 4t^2 + 9t^4)^3} \right)^{1/2} \\ \tau &= \frac{(6t^2, -6t, 2) \cdot (0, 0, 6)}{(\sqrt{4 + 36t^2 + 36t^4})^2} = \frac{12}{4 + 36t^2 + 36t^4} = \frac{3}{1 + 9t^2 + 9t^4} \\ \mathbf{T} &= \frac{(1, 2t, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}} \\ \mathbf{B} &= \frac{(6t^2, -6t, 2)}{2\sqrt{1 + 9t^2 + 9t^4}} = \frac{(3t^2, -3t, 1)}{\sqrt{1 + 9t^2 + 9t^4}}. \end{aligned}$$

Lastly,  $(3t^2, -3t, 1) \times (1, 2t, 3t^2) = ()$  gives

$$\mathbf{N} = \frac{()} {\sqrt{(1 + 9t^2 + 9t^4)(1 + 4t^2 + 9t^4)}}.$$

## Question 2 (i).

State and prove the Wirtinger's inequality.

*Solution.*

**Statement:** Let  $F : [0, \pi] \rightarrow \mathbb{R}$  be a smooth function with  $F(0) = F(\pi) = 0$ . Then,

$$\int_0^\pi \left( \frac{dF}{dt} \right)^2 dt \geq \int_0^\pi F^2 dt$$

and the equality holds if and only if  $F(t) = K \sin t$  for all  $t \in [0, \pi]$  and  $K$  is a constant.

**Proof:** Let  $G(t) = F(t)/\sin(t)$ . This is well-defined because  $\sin(t)$  is non-zero for  $t \in (0, \pi)$ , and smooth because it is a quotient of two smooth functions. Writing  $F(t) = G(t)\sin(t)$  we get,

$$\begin{aligned} \int_0^\pi \left( \frac{dF}{dt} \right)^2 dt &= \int_0^\pi (G' \sin t + G \cos t)^2 dt \\ &= \int_0^\pi \{ G'^2 \sin^2 t + G^2 \cos^2 t + 2GG' \sin t \cos t \} dt \\ &= \int_0^\pi \{ G'^2 \sin^2 t + G^2 \cos^2 t \} dt + \int_0^\pi \{ 2GG' \sin t \cos t \} dt. \end{aligned}$$

Writing  $2GG' = (G^2)'$  and  $\sin t \cos t = \frac{1}{2} \sin 2t$  gives

$$\int_0^\pi \{ 2GG' \sin t \cos t \} dt = \int_0^\pi \left\{ (G^2)' \times \frac{1}{2} \sin 2t \right\} dt = \frac{1}{2} \int_0^\pi \{ (G^2)' \sin 2t \} dt.$$

Integrating this by parts, leads to

$$\frac{1}{2} \int_0^\pi \{ (G^2)' \sin 2t \} dt = \frac{1}{2} G(t)^2 \sin(2t) \Big|_0^\pi - \int_0^\pi G^2(t) \cos 2t dt = - \int_0^\pi G^2(t) \cos 2t dt.$$

By using  $\cos(2t) = \cos^2 t - \sin^2 t$ , we obtain

$$\frac{1}{2} \int_0^\pi \{ (G^2)' \sin 2t \} dt = - \int_0^\pi G^2(t) (\cos^2 t - \sin^2 t) dt.$$

Consequently,

$$\begin{aligned} \int_0^\pi \left( \frac{dF}{dt} \right)^2 dt &= \int_0^\pi \{ G'^2 \sin^2 t + G^2 \cos^2 t \} dt + \int_0^\pi \{ 2GG' \sin t \cos t \} dt \\ &= \int_0^\pi \{ G'^2 \sin^2 t + G^2 \cos^2 t \} dt - \int_0^\pi G^2(t) (\cos^2 t - \sin^2 t) dt \\ &= \int_0^\pi \{ G'^2 \sin^2 t + G^2 \sin^2 t \} dt \\ &= \int_0^\pi (G' \sin t)^2 dt + \int_0^\pi F^2 dt. \end{aligned}$$

Therefore,

$$\int_0^\pi \left( \frac{dF}{dt} \right)^2 dt - \int_0^\pi F^2 dt = \int_0^\pi (G' \sin t)^2 dt \geq 0$$

since  $(G' \sin t)^2 \geq 0$ . We find that the equality holds if and only if  $G' \equiv 0$  (because  $\sin^2 t > 0$  for  $t \in (0, \pi)$ ). That is, the equality holds if and only if  $G(t) = K$  for some constant  $K$  and  $F(t) = K \sin t$ .

## Question 2 (ii).

Using the Wirtinger's inequality, prove the isoperimetric inequality.

*Solution.*

**Statement:** Let  $\gamma$  be a simple closed curve in the plane with length  $\ell(\gamma)$ , and enclosing an area  $A(\gamma)$ . Then,

$$A(\gamma) \leq \frac{1}{4\pi} \ell(\gamma)^2$$

and the equality holds if and only if  $\gamma$  is a circle.

**Proof:** Without loss of generality, we assume that the curve is parametrised by arc-length, so  $\gamma : [0, \ell] \rightarrow \mathbb{R}^2$  with  $\gamma = \gamma(s)$ . Also, we translate the curve so that  $\gamma(0) = 0$ ; this doesn't effect the value of  $\ell(\gamma)$  and  $A(\gamma)$ . Also note that  $\gamma(\ell) = \gamma(0)$  because it is a closed curve.

We reparametrize the curve using the parameter  $t = \pi s/\ell$ . Then,  $t \in [0, \pi]$  and  $ds/dt = \ell/\pi$ . We can then write the curve in coordinates as  $\gamma(t) = (x(t), y(t))$ . Switching to polar coordinates  $x = r \cos \theta$  and  $y = r \sin \theta$  we find  $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2$  and  $x\dot{y} - y\dot{x} = r^2 \dot{\theta}$  because  $\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta$  and  $\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta$ .

Therefore,

$$\dot{r}^2 + r^2 \dot{\theta}^2 = \left( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right) = \left( \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 \right) \left( \frac{ds}{dt} \right)^2 = \frac{\ell^2}{\pi^2}$$

as  $(dx/ds)^2 + (dy/ds)^2 = 1$ . This allows us to write

$$\frac{\ell^2}{4\pi^2} = \frac{1}{4} \int_0^\pi (\dot{r}^2 + r^2 \dot{\theta}^2) dt.$$

Also,  $A = \frac{1}{2} \int_0^\pi (x\dot{y} - y\dot{x}) dt = \frac{1}{2} \int_0^\pi r^2 \dot{\theta} dt$ . Consequently,

$$\begin{aligned} \frac{\ell^2}{4\pi} - A &= \frac{1}{4} \int_0^\pi (\dot{r}^2 + r^2 \dot{\theta}^2 - 2r^2 \dot{\theta}) dt \\ &= \frac{1}{4} \int_0^\pi (\dot{r}^2 + r^2 \dot{\theta}^2 - 2r^2 \dot{\theta} + r^2 - r^2) dt \\ &= \frac{1}{4} \int_0^\pi (r^2 (\dot{\theta}^2 - 2\dot{\theta} + 1) + \dot{r}^2 - r^2) dt \\ &= \frac{1}{4} \int_0^\pi r^2 (\dot{\theta} - 1)^2 dt + \frac{1}{4} \int_0^\pi (\dot{r}^2 - r^2) dt. \end{aligned} \quad (*)$$

In equation (\*), the first term is non-negative because the integrand is non-negative, and the second is non-negative by Wirtinger's inequality. Consequently,  $\frac{\ell^2}{4\pi} - A \geq 0$ . Or equivalently,

$$A \leq \frac{1}{4\pi} \ell^2.$$

Also, the first term in equation (\*) is zero precisely when  $\dot{\theta} = 1$  (i.e.  $\theta = t + \theta_0$  with  $\theta_0$  constant), while the second term becomes zero when  $r = K \sin \theta$  for some constant  $K$ . Collectively, both of these terms vanish when  $r = K \sin(t + \theta_0)$ . This is the equation of a circle with diameter  $K$  and passing through the origin. Therefore,  $A = \ell^2/4\pi$  if and only if the curve is a circle.

### Question 3 (i).

For a unit speed curve  $\beta(s)$ , show that  $\beta'' \cdot \beta''' \times \beta'''' = \kappa^5 \frac{d}{ds} \left( \frac{\tau}{\kappa} \right)$ , where  $\kappa$  and  $\tau$  are curvature and torsion of the curve  $\beta(s)$ .

*Solution.* Solution

### Question 3 (ii).

Reparametrize the curve  $\alpha(t) = e^t(\cos t, \sin t, 1)$  by its arc-length.

*Solution.* Solution

**Question 4 (i).**

Show that the curve  $\beta(s) = \frac{1}{2} \left( s + \sqrt{s^2 + 1}, \left( s + \sqrt{s^2 + 1} \right)^{-1}, \sqrt{2} \ln \left( s + \sqrt{s^2 + 1} \right) \right)$  has unit speed.

*Solution.* Solution

**Question 4 (ii).**

Calculate the curvature of the curve  $\beta(s)$  given in Q4(i).

*Solution.* Solution