Past Paper: Numerical Analysis-I

Midterm Exam, 2024

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Question 1.

Convert the following numbers to base 10.

(a) $x = (10010110)_2$

(b) $x = (1011)_2$

(c) $x = (777)_8$

Solution.

(a) $x = (10010110)_2 = 0(1)+1(2)+1(4)+0(8)+1(16)+0(32)+0(64)+1(128) = 150.$

(b) $x = (1011)_2 = 1(1) + 1(2) + 0(4) + 1(8) = 11.$

(c) $x = (777)_8 = 7(1) + 7(8) + 7(64) = 511.$

Question 2.

Determine the LU factorization of the given matrix using Crout's method

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}$$

Solution. We take

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Then, $A = LU$ gives

$$\begin{gathered} \ell_{11} = 2, \quad \ell_{21} = 8, \quad \ell_{31} = 4, \\ \ell_{11}u_{12} = 1 \implies u_{12} = 0.5, \quad \ell_{11}u_{13} = 4 \implies u_{13} = 2 \\ \ell_{21}u_{12} + \ell_{22} = -3 \implies \ell_{22} = -7, \quad \ell_{31}u_{12} + \ell_{32} = 11 \implies \ell_{32} = 9 \\ \ell_{21}u_{13} + \ell_{22}u_{23} = 2 \implies u_{23} = 2, \quad \ell_{31}u_{13} + \ell_{32}u_{23} + \ell_{33} = -1 \implies \ell_{33} = -27. \end{gathered}$$

Overall,

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -7 & 0 \\ 4 & 9 & -27 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 3.

Apply the Gauss-Seidel method to solve the following system of equations.

$$4x - y + z = 12, \quad x - 2y + 4z = 5, \quad -x + 4y - 2z = -1.$$

Perform two iterations with 5 decimal place digit calculations.

Solution. We arrange this system as

$$\begin{aligned} 4x - y + z &= 12 \\ -x + 4y - 2z &= -1 \\ x - 2y + 4z &= 5 \end{aligned} \quad \rightarrow \quad \underbrace{\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 12 \\ -1 \\ 5 \end{bmatrix}}_{B}$$

to ensure $|a_{11}| > |a_{12}| + |a_{13}|$, $|a_{22}| > |a_{21}| + |a_{23}|$ and $|a_{33}| > |a_{31}| + |a_{32}|$.

The Gauss-Seidel method for this system gives

$$\begin{aligned} x_{n+1} &= \frac{1}{a_{11}}(b_1 - a_{12}y_n - a_{13}z_n) = \frac{1}{4}(12 + y_n - z_n), \\ y_{n+1} &= \frac{1}{a_{22}}(b_2 - a_{21}x_{n+1} - a_{23}z_n) = \frac{1}{4}(-1 + x_{n+1} + 2z_n), \\ z_{n+1} &= \frac{1}{a_{33}}(b_3 - a_{31}x_{n+1} - a_{32}y_{n+1}) = \frac{1}{4}(5 - x_{n+1} + 2y_{n+1}). \end{aligned}$$

So, starting with $x_0 = y_0 = z_0 = 0$, we have:

<u>1st Iteration:</u>

$$\begin{aligned} x_1 &= \frac{1}{4}(12 + 0 - 0) = 3.00000, \\ y_1 &= \frac{1}{4}(-1 + 3 + 0) = 0.50000, \\ z_1 &= \frac{1}{4}(5 - 3 + 2(0.5)) = 0.75000. \end{aligned}$$

<u>2nd Iteration:</u>

$$\begin{aligned} x_2 &= \frac{1}{4}(12 + 0.5 - 0.75) = 2.93750, \\ y_2 &= \frac{1}{4}(-1 + 2.9375 + 2(0.75)) = 0.85938, \\ z_2 &= \frac{1}{4}(5 - 2.9375 + 2(0.85938)) = 0.94532. \end{aligned}$$

Question 4 (a).

Derive the formula for the false-position method.

Solution. Suppose the equation $f(x) = 0$ has a root x_r in the interval $[x_\ell, x_u]$. That means $f(x_r) = 0$. In order for the root to lie in the interval (x_ℓ, x_u), we must have $f(x_\ell)f(x_u) < 0$. So, without loss of generality, we take $f(x_\ell) < 0$ and $f(x_u) > 0$.

By the method of similar triangles, we obtain

$$\frac{f(x_u)}{x_u - x_r} = \frac{-f(x_\ell)}{x_r - x_\ell}$$

which can be re-expressed as

$$(x_r - x_\ell)f(x_u) = -(x_u - x_r)f(x_\ell) \implies x_r(f(x_u) - f(x_\ell)) = x_\ell f(x_u) - x_u f(x_\ell).$$

Therefore, if the root x_r lies within the interval $[x_\ell, x_u]$, then it can be iteratively estimated as

$$x_r = \frac{x_\ell f(x_u) - x_u f(x_\ell)}{f(x_u) - f(x_\ell)}.$$

Question 4 (b).

Apply the method of false position to find a real root of the equation $x^4 - 11x + 8 = 0$ within the interval $[1, 2]$. Perform five decimal place digit calculations. Apply two iterations.

Solution. Let $f(x) = x^4 - 11x + 8$.

1st Iteration: We set $x_\ell = 1$ and $x_u = 2$. This gives $f(x_\ell) = -2$ and $f(x_u) = 2$. So, $f(x_\ell)f(x_u) < 0$ and a root lies in the interval $[1, 2]$. Then,

$$x_r = \frac{x_\ell f(x_u) - x_u f(x_\ell)}{f(x_u) - f(x_\ell)} = \frac{(1)(2) - (2)(-2)}{2 + 2} = 1.50000.$$

2nd Iteration: Next, $f(x_r) = -3.4375 < 0$. Therefore, $f(x_r)f(x_u) < 0$ and the root lies in the interval $[1.5, 2]$. So, we set $x_\ell = 1.5$ and $x_u = 2$. Then,

$$x_r = \frac{x_\ell f(x_u) - x_u f(x_\ell)}{f(x_u) - f(x_\ell)} = \frac{(1.5)(2) - (2)(-3.4375)}{2 + 3.4375} = 1.81609.$$

Question 5 (a).

Derive Newton-Raphson method.

Solution. Suppose the equation $f(x) = 0$ has a root at $x = x_r$. That means $f(x_r) = 0$. We can Taylor expand $f(x)$ near x_r to get

$$f(x_r) = f(x) + (x_r - x)f'(x) + \mathcal{O}(2).$$

Now, working up to linear order and using $f(x_r) = 0$ gives $f(x) + (x_r - x)f'(x) = 0$, which can be re-arrange as

$$x_r = x - \frac{f(x)}{f'(x)}.$$

This gives an iterative formula for root estimation, with an initial guess $x = x_0$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Question 5 (b).

Evaluate $\sqrt{29}$ by Newton-Raphson method. Carry out five decimal place digit calculations. Stop the calculations when five decimal digits match with each other. Take an initial guess as $x_0 = 3.3$.

Solution. Take $f(x) = x^2 - 29$. Then, $f(x) = 0 \implies x = \pm\sqrt{29}$. So, we apply the Newton-Raphson method to estimate $f(x) = 0$. The corresponding formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

with $f'(x) = 2x$. As a result, we have

$$\begin{aligned} x_0 &= 3.3 \\ x_1 &= 6.0439394\ldots \\ x_2 &= 5.4210672\ldots \\ x_3 &= 5.3852837\ldots \\ x_4 &= 5.3851648\ldots \\ x_5 &= 5.3851648\ldots \end{aligned}$$

Therefore, $\sqrt{29} = 5.38516$ to 5 decimal digits.

Question 6.

Convert the system into an iterative form suitable for Newton-Raphson method then solve the following

$$x - y^3 - 1 = 0, \quad \sin(x) + y - 1 = 0.$$

Use the initial guess as $(x_0, y_0) = (0, 0)$. Apply one iteration only.

Solution. We have $f_1(x, y) = x - y^3 - 1$ and $f_2(x, y) = \sin(x) + y - 1$. The iterative formula for the Newton-Raphson method is

$$x_{n+1} = x_n + \left. \frac{-f_1 \dfrac{\partial f_2}{\partial y} + f_2 \dfrac{\partial f_1}{\partial y}}{J(f_1, f_2)} \right|_{(x_n, y_n)}$$

$$y_{n+1} = y_n + \left. \frac{-f_2 \dfrac{\partial f_1}{\partial x} + f_1 \dfrac{\partial f_2}{\partial x}}{J(f_1, f_2)} \right|_{(x_n, y_n)}$$

where

$$J(f_1, f_2) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial x}. \tag{1}$$

For the given f_1 and f_2 we have

$$\frac{\partial f_1}{\partial x} = 1, \quad \frac{\partial f_1}{\partial y} = -3y^2, \quad \frac{\partial f_2}{\partial x} = \cos x, \quad \frac{\partial f_2}{\partial y} = 1, \tag{2}$$

and $J(f_1, f_2) = 1 + 3y^2 \cos x$. Starting with $(x_0, y_0) = (0, 0)$ we get

$$x_1 = \left. \frac{-(x - y^3 - 1)(1) + (\sin(x) + y - 1)(-3y^2)}{1 + 3y^2 \cos x} \right|_{(0,0)} = 1$$

$$y_1 = \left. \frac{-(\sin(x) + y - 1)(1) + (x - y^3 - 1)(\cos x)}{1 + 3y^2 \cos x} \right|_{(0,0)} = 0.$$