Past Paper: Differential Geometry

Midterm Exam, 2024

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Question 1.

Let $\alpha(t) = (t, t^2, t^3)$ be a curve in the Euclidean space. Calculate the following parameters for the given curve: unit tangent vector, unit normal vector, unit binormal vector, curvature, and torsion.

Solution. Firstly, note that

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}, \quad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|^2}, \quad \mathbf{T} = \frac{\alpha'}{|\alpha'|}, \quad \mathbf{B} = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}, \quad \mathbf{N} = \mathbf{B} \times \mathbf{T}.$$

Now, $\alpha'(t) = (1, 2t, 3t^2)$, $\alpha''(t) = (0, 2, 6t)$ and $\alpha'''(t) = (0, 0, 6)$. Therefore,

$$\alpha' \times \alpha'' = (6t^2, -6t, 2), \quad |\alpha' \times \alpha''| = \sqrt{4 + 36t^2 + 36t^4} = 2\sqrt{1 + 9t^2 + 9t^4},$$

and $|\alpha'| = \sqrt{1 + 4t^2 + 9t^4}$. As a result,

$$\begin{aligned}
\kappa &= \frac{2\sqrt{1 + 9t^2 + 9t^4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3} = 2\left(\frac{1 + 9t^2 + 9t^4}{(1 + 4t^2 + 9t^4)^3}\right)^{1/2} \\
\tau &= \frac{(6t^2, -6t, 2) \cdot (0, 0, 6)}{\left(\sqrt{4 + 36t^2 + 36t^4}\right)^2} = \frac{12}{4 + 36t^2 + 36t^4} = \frac{3}{1 + 9t^2 + 9t^4} \\
\mathbf{T} &= \frac{(1, 2t, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}} \\
\mathbf{B} &= \frac{(6t^2, -6t, 2)}{2\sqrt{1 + 9t^2 + 9t^4}} = \frac{(3t^2, -3t, 1)}{\sqrt{1 + 9t^2 + 9t^4}}.
\end{aligned}$$

Lastly, $(3t^2, -3t, 1) \times (1, 2t, 3t^2) = (-2t - 9t^3, 1 - 9t^4, 3t + 6t^3)$ gives

$$\mathbf{N} = \frac{(-2t - 9t^3, 1 - 9t^4, 3t + 6t^3)}{\sqrt{(1 + 9t^2 + 9t^4)(1 + 4t^2 + 9t^4)}}.$$

Question 2 (i).

State and prove the Wirtinger's inequality.

Solution.
Statement: Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function with $F(0) = F(\pi) = 0$. Then,

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi F^2 \, dt$$

and the equality holds if and only if $F(t) = K \sin t$ for all $t \in [0, \pi]$ and K is a constant.

Proof: Let $G(t) = F(t)/\sin(t)$. This is well-defined because $\sin(t)$ is non-zero for $t \in (0, \pi)$, and smooth because it is a quotient of two smooth functions. Writing $F(t) = G(t)\sin(t)$ we get,

$$\begin{aligned}
\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt &= \int_0^\pi (G' \sin t + G \cos t)^2 \, dt \\
&= \int_0^\pi \Big\{ G'^2 \sin^2 t + G^2 \cos^2 t + 2GG' \sin t \cos t \Big\} \, dt \\
&= \int_0^\pi \Big\{ G'^2 \sin^2 t + G^2 \cos^2 t \Big\} + \int_0^\pi \{ 2GG' \sin t \cos t \} \, dt.
\end{aligned}$$

Writing $2GG' = (G^2)'$ and $\sin t \cos t = \frac{1}{2} \sin 2t$ gives

$$\int_0^\pi \{2GG' \sin t \cos t\} \, dt = \int_0^\pi \Big\{ (G^2)' \times \frac{1}{2} \sin 2t \Big\} \, dt. = \frac{1}{2} \int_0^\pi \Big\{ (G^2)' \sin 2t \Big\} \, dt.$$

Integrating this by parts, leads to

$$\frac{1}{2} \int_0^\pi \Big\{ (G^2)' \sin 2t \Big\} \, dt = \frac{1}{2} G(t)^2 \sin(2t) \Big|_0^\pi - \int_0^\pi G^2(t) \cos 2t \, dt = - \int_0^\pi G^2(t) \cos 2t \, dt.$$

By using $\cos(2t) = \cos^2 t - \sin^2 t$, we obtain

$$\frac{1}{2} \int_0^\pi \Big\{ (G^2)' \sin 2t \Big\} \, dt = - \int_0^\pi G^2(t) \Big(\cos^2 t - \sin^2 t \Big) \, dt.$$

Consequently,

$$\begin{aligned}
\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt &= \int_0^\pi \Big\{ G'^2 \sin^2 t + G^2 \cos^2 t \Big\} + \int_0^\pi \{ 2GG' \sin t \cos t \} \, dt \\
&= \int_0^\pi \Big\{ G'^2 \sin^2 t + G^2 \cos^2 t \Big\} - \int_0^\pi G^2(t) \Big(\cos^2 t - \sin^2 t \Big) \, dt \\
&= \int_0^\pi \Big\{ G'^2 \sin^2 t + G^2 \sin^2 t \Big\} \, dt \\
&= \int_0^\pi (G' \sin t)^2 \, dt + \int_0^\pi F^2 \, dt.
\end{aligned}$$

Therefore,

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt - \int_0^\pi F^2 \, dt = \int_0^\pi (G' \sin t)^2 \, dt \geq 0$$

since $(G' \sin t)^2 \geq 0$. We find that the equality holds if and only if $G' \equiv 0$ (because $\sin^2 t > 0$ for $t \in (0, \pi)$). That is, the equality holds if and only if $G(t) = K$ for some constant K and $F(t) = K \sin t$.

Question 2 (ii).

Using the Wirtinger's inequality, prove the isoperimetric inequality.

Solution.

Statement: Let γ be a simple closed curve in the plane with length $\ell(\gamma)$, and enclosing an area $A(\gamma)$. Then,

$$A(\gamma) \leq \frac{1}{4\pi} \ell(\gamma)^2$$

and the equality holds if and only if γ is a circle.

Proof: Without loss of generality, we assume that the curve is parametrised by arc-length, so $\gamma : [0, \ell] \to \mathbb{R}^2$ with $\gamma = \gamma(s)$. Also, we translate the curve so that $\gamma(0) = 0$; this doesn't effect the value of $\ell(\gamma)$ and $A(\gamma)$. Also note that $\gamma(\ell) = \gamma(0)$ because it is a closed curve.

We reparametrize the curve using the parameter $t = \pi s/\ell$. Then, $t \in [0, \pi]$ and $ds/dt = \ell/\pi$. We can then write the curve in coordinates as $\gamma(t) = (x(t), y(t))$. Switching to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ we find $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$ and $x\dot{y} - y\dot{x} = r^2\dot{\theta}$ because $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$.

Therefore,

$$\dot{r}^2 + r^2\dot{\theta}^2 = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) = \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)\left(\frac{ds}{dt}\right)^2 = \frac{\ell^2}{\pi^2}$$

as $(dx/ds)^2 + (dy/ds)^2 = 1$. This allows us to write

$$\frac{\ell^2}{4\pi^2} = \frac{1}{4}\int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2\right) dt.$$

Also, $A = \frac{1}{2}\int_0^\pi (x\dot{y} - y\dot{x})\, dt = \frac{1}{2}\int_0^\pi r^2\dot{\theta}\, dt$. Consequently,

$$\begin{aligned}
\frac{\ell^2}{4\pi} - A &= \frac{1}{4}\int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta}\right) dt \\
&= \frac{1}{4}\int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta} + r^2 - r^2\right) dt \\
&= \frac{1}{4}\int_0^\pi \left(r^2(\dot{\theta}^2 - 2\dot{\theta} + 1) + \dot{r}^2 - r^2\right) dt \\
&= \frac{1}{4}\int_0^\pi r^2(\dot{\theta} - 1)^2\, dt + \frac{1}{4}\int_0^\pi \left(\dot{r}^2 - r^2\right) dt. \qquad (*)
\end{aligned}$$

In equation $(*)$, the first term is non-negative because the integrand is non-negative, and the second is non-negative by Wirtinger's inequality. Consequently, $\frac{\ell^2}{4\pi} - A \geq 0$. Or equivalently,

$$A \leq \frac{1}{4\pi}\ell^2.$$

Also, the first term in equation $(*)$ is zero precisely when $\dot{\theta} = 1$ (i.e. $\theta = t + \theta_0$ with θ_0 constant), while the second term becomes zero when $r = K\sin\theta$ for some constant K. Collectively, both of these terms vanish when $r = K\sin(t + \theta_0)$. This is the equation of a circle with diameter K and passing through the origin. Therefore, $A = \ell^2/4\pi$ if and only if the curve is a circle.

Question 3 (i).

For a unit speed curve $\beta(s)$, show that $\beta'' \cdot \beta''' \times \beta'''' = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right)$, where κ and τ are curvature and torsion of the curve $\beta(s)$.

Solution. For a unit speed curve $\beta'' = \kappa\mathbf{N}$ and $\beta''' = -\kappa^2\mathbf{T} + \kappa'\mathbf{N} + \tau\kappa\mathbf{B}$. So,

$$\beta^{(4)} = -3\kappa\dot{\kappa}\mathbf{T} + (\dot{\tau}\kappa + 2\tau\dot{\kappa})\mathbf{B} + \left(\ddot{\kappa} - \kappa^3 - \tau^2\kappa\right)\mathbf{N}.$$

Frenet-Serret equations were used to simplify the previous expression. Now, by the cyclic property of the triple scalar product, $\beta'' \cdot \beta''' \times \beta^{(4)} = \beta^{(4)} \cdot \beta'' \times \beta'''$. And

$$\beta'' \times \beta''' = \kappa \mathbf{N} \times \Big(-\kappa^2 \mathbf{T} + \kappa' \mathbf{N} + \tau\kappa \mathbf{B}\Big) = \tau\kappa^2 \mathbf{T} + \kappa^3 \mathbf{B}$$

where we used $\mathbf{N} \times \mathbf{T} = -\mathbf{B}$ and $\mathbf{N} \times \mathbf{B} = \mathbf{T}$. Consequently,

$$\beta^{(4)} \cdot \beta'' \times \beta''' = \Big(-3\kappa^3\tau\dot{\kappa}\Big) + \Big(\kappa^4\dot{\tau} + 2\kappa^3\tau\dot{\kappa}\Big) = \kappa^4\dot{\tau} - \kappa^3\tau\dot{\kappa}.$$

We can re-write this as

$$\beta'' \cdot \beta''' \times \beta^{(4)} = \beta^{(4)} \cdot \beta'' \times \beta''' = \kappa^4\dot{\tau} - \kappa^3\tau\dot{\kappa} = \kappa^5\left(\frac{\dot{\tau}\kappa}{\kappa^2} - \frac{\tau\dot{\kappa}}{\kappa^2}\right) = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

Question 3 (ii).

Reparametrize the curve $\alpha(t) = e^t(\cos t, \sin t, 1)$ by its arc-length.

Solution. Differentiating $\alpha(t) = \big(e^t \cos t, e^t \sin t, e^t\big)$ gives,

$$\alpha'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t) = e^t(\cos t - \sin t, \cos t + \sin t, 1).$$

So, $|\alpha'|^2 = e^{2t}\big(\cos^2 t + \sin^2 t - 2\cos t \sin t + \cos^2 t + \sin^2 t + 2\cos t \sin t + 1\big) = 3e^{2t}$. Therefore, $|\alpha'(t)| = \sqrt{3}e^t$ and,

$$s(t) = \int_0^t |\alpha'(u)|\, du = \sqrt{3}\int_0^t e^u\, du = \sqrt{3}e^u\Big|_0^t = \sqrt{3}(e^t - 1).$$

Inverting this to make t the subject,

$$s(t) = \sqrt{3}(e^t - 1) \implies e^t = 1 + \frac{s}{\sqrt{3}} \implies t = \ln\left(1 + \frac{s}{\sqrt{3}}\right)$$

So, the arc-length paremetrisation of α is

$$\alpha(s) = \left(1 + \frac{s}{\sqrt{3}}\right)\cos\left(\ln\left(1 + \frac{s}{\sqrt{3}}\right)\right)\hat{\mathbf{i}} + \left(1 + \frac{s}{\sqrt{3}}\right)\sin\left(\ln\left(1 + \frac{s}{\sqrt{3}}\right)\right)\hat{\mathbf{j}} + \left(1 + \frac{s}{\sqrt{3}}\right)\hat{\mathbf{k}}.$$

Question 4 (i).

Show that the curve $\beta(s) = \frac{1}{2}\left(s + \sqrt{s^2+1}, \left(s + \sqrt{s^2+1}\right)^{-1}, \sqrt{2}\ln\left(s + \sqrt{s^2+1}\right)\right)$ has unit speed.

Solution. Differentiating $\beta(s)$ gives,

$$\begin{aligned}
\beta'(s) &= \frac{1}{2}\left(1 + \frac{s}{\sqrt{s^2+1}}, -\frac{1 + \frac{s}{\sqrt{s^2+1}}}{\left(s + \sqrt{s^2+1}\right)^2}, \frac{\sqrt{2}\left(1 + \frac{s}{\sqrt{s^2+1}}\right)}{s + \sqrt{s^2+1}}\right) \\
&= \frac{1}{2}\left(\frac{s + \sqrt{s^2+1}}{\sqrt{s^2+1}}, -\frac{1}{\sqrt{s^2+1}\left(s + \sqrt{s^2+1}\right)}, \frac{\sqrt{2}}{\sqrt{s^2+1}}\right) \\
&= \frac{1}{2}\frac{1}{\sqrt{s^2+1}}\left(s + \sqrt{s^2+1}, -\frac{1}{\left(s + \sqrt{s^2+1}\right)}, \sqrt{2}\right) \\
&= \frac{1}{2\sqrt{s^2+1}}\left(s + \sqrt{s^2+1}, s - \sqrt{s^2+1}, \sqrt{2}\right).
\end{aligned}$$

Here we used

$$1 + \frac{s}{\sqrt{s^2+1}} = \frac{s+\sqrt{s^2+1}}{\sqrt{s^2+1}}.$$

in the second line, and $-\left(s+\sqrt{s^2+1}\right)^{-1} = s - \sqrt{s^2+1}$ in the last line.

Now, since $\left(s+\sqrt{s^2+1}\right)^2 + \left(s-\sqrt{s^2+1}\right)^2 = 4s^2+2$, we obtain

$$|\beta'(s)|^2 = \frac{1}{4(s^2+1)}\left(4s^2+2+2\right) = \frac{4(s^2+1)}{4(s^2+1)} = 1.$$

Thus, $|\beta'(s)| = 1$. In other words, $\beta(s)$ is a unit-speed parametrised curve.

Question 4 (ii).

Calculate the curvature of the curve $\beta(s)$ given in Q4(i).

Solution. Since $\beta = \beta(s)$ is unit-speed parameterised, its curvature is $\kappa = |\beta''(s)|$. Differentiating

$$\beta'(s) = \frac{1}{2\sqrt{s^2+1}}\left(s+\sqrt{s^2+1}, s-\sqrt{s^2+1}, \sqrt{2}\right) = \frac{1}{2}\left(\frac{s}{\sqrt{s^2+1}}+1, \frac{s}{\sqrt{s^2+1}}-1, \frac{\sqrt{2}}{\sqrt{s^2+1}}\right).$$

with respect to s gives, (carefully using the product rule)

$$\begin{aligned}
\beta''(s) &= \frac{1}{2}\left(\frac{\sqrt{s^2+1}-s^2(s^2+1)^{-1/2}}{s^2+1}, \frac{\sqrt{s^2+1}-s^2(s^2+1)^{-1/2}}{s^2+1}, \frac{-\sqrt{2}s(s^2+1)^{-1/2}}{s^2+1}\right) \\
&= \frac{1}{2}\left(\frac{1}{(s^2+1)^{3/2}}, \frac{1}{(s^2+1)^{3/2}}, -\frac{s\sqrt{2}}{(s^2+1)^{3/2}}\right) \\
&= \frac{1}{2(s^2+1)^{3/2}}\left(1,1,-s\sqrt{2}\right)
\end{aligned}$$

Now, $\left|\left(1,1,-s\sqrt{2}\right)\right|^2 = 1+1+2s^2 = 2(1+s^2)$, therefore,

$$\kappa = |\beta''| = \frac{\sqrt{2}(1+s^2)^{1/2}}{2(s^2+1)^{3/2}} = \frac{1}{\sqrt{2}(s^2+1)}$$