Past Paper: Partial Differential Equations

Midterm Exam, 2024

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Question 1.

Classify the following equation and reduce it to the canonical form

$$u_{xy} + yu_{yy} + \sin(x+y) = 0.$$

Solution. We start by making a change of variables $x \leftrightarrow y$ to get the transformed PDE

$$xu_{xx} + u_{xy} + \sin(x+y) = 0. (*)$$

Now, A = x, B = 1, C = D = E = 0. Since, $B^2 - 4AC = 1 > 0$, this is a hyperbolic PDE. So, the corresponding characteristic equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{1 \pm 1}{2x}$$

gives

$$\frac{dy}{dx} = \frac{1}{x} \implies k + y = \ln x \implies xe^{-y} = c_1, \quad \frac{dy}{dx} = 0 \implies y = c_2.$$

These lead to $\xi = xe^{-y} = c_1$ and $\eta = y = c_2$. As a result

$$B^* = EQQ = EQQ$$

$$D^* = EQQ = EQQ$$

$$E^* = EQQ = EQQ.$$

(The coefficients $A^* = C^* = 0$ because the PDE is hyperbolic.)

As a result, equation (*) implies

$$e^y u_{\xi\eta} - e^y u_{\xi\xi} = \sin(x+y) \implies u_{\xi\eta} - u_{\xi\xi} = e^{-\eta} \sin(\eta + \xi e^{\eta}).$$

Here, we used $y = \eta$ and $x = \xi e^y = \xi e^{\eta}$.

Overall, the canonical form of the given PDE is

$$u_{\xi\eta} = u_{\xi\xi} + e^{-\eta}\sin(\eta + \xi e^{\eta})$$

with $\xi = ye^{-x}$ and $\eta = x$ (in the original variables).

Question 2.

Apply a linear transformation $\xi = x + by$ and $\eta = x + dy$ to transform the Euler equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

into canonical form, where b, d A, B and C are constants.

Solution. Solution

Question 3.

Determine the solution of the equation

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the data x + y = 0 and u = 1.

Solution. Solution

Question 4.

Show that the general solution of a first-order quasilinear partial differential equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

is $f(\varphi, \psi) = 0$, where f is an arbitrary function of $\varphi(x, y, u)$ and $\psi(x, y, u)$, and $\varphi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$
.

Solution. Solution.

Question 5.

Verify that the function

$$u = \varphi(xy) + x\psi\left(\frac{y}{x}\right)$$

is the general solution of the equation

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

Solution. We have

$$u_x = y\varphi'(xy) + \psi\left(\frac{y}{x}\right) - x\left(\frac{y}{x^2}\right)\psi'\left(\frac{y}{x}\right) = y\varphi'(xy) + \psi\left(\frac{y}{x}\right) - \frac{y}{x}\psi'\left(\frac{y}{x}\right)$$

$$u_y = x\varphi'(xy) + x\left(\frac{1}{x}\right)\psi'\left(\frac{y}{x}\right) = x\varphi'(xy) + \psi'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\varphi''(xy) - \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) + \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) + \frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right) = y^2\varphi''(xy) + \frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right)$$

$$u_{yy} = x^2\varphi''(xy) + \frac{1}{x}\psi''\left(\frac{y}{x}\right)$$

Therefore,

$$x^{2}u_{xx} - y^{2}u_{yy} = x^{2}y^{2}\varphi''(xy) + \frac{y^{2}}{x}\psi''\left(\frac{y}{x}\right) - x^{2}y^{2}\varphi''(xy) + \frac{y^{2}}{x}\psi''\left(\frac{y}{x}\right) = 0$$

Question 6.

Determine the solution of the initial boundary value problem

$$u_{tt} = 4u_{xx},$$
 $0 < x < 1, t > 0,$
 $u(x,0) = 0,$ $0 \le x \le 1,$
 $u_t(x,0) = x(1-x),$ $0 \le x \le 1,$
 $u(0,t) = 0, u(1,t) = 0,$ $t \ge 0.$

Solution. This is the wave equation (c=2) on a bounded spatial interval [0,1]. The general solution is $u(x,t) = \varphi(x+2t) + \psi(x-2t)$ with

$$\varphi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c} \int_0^{\eta} g(\tau) d\tau + \frac{K}{2}, \quad \psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c} \int_0^{\eta} g(\tau) d\tau - \frac{K}{2}.$$

for $0 \le \eta \le 1$. Using $f(\eta) = 0$ and $g(\eta) = \eta(1 - \eta)$ gives,

$$\varphi(\eta) = \frac{1}{4} \left(\frac{1}{2} \eta^2 - \frac{1}{3} \eta^3 \right) + \frac{K}{2}, \quad 0 \le \eta \le 1, \tag{1}$$

$$\psi(\eta) = -\frac{1}{4} \left(\frac{1}{2} \eta^2 - \frac{1}{3} \eta^3 \right) - \frac{K}{2}, \quad 0 \le \eta \le 1.$$
 (2)

The boundary conditions give

$$u(0,t) = 0 \implies \psi(-2t) = -\varphi(2t) \implies \psi(\eta) = -\varphi(-\eta),$$
 (*)

$$u(1,t) = 0 \implies \phi(1+2t) = -\varphi(1-2t) \implies \psi(\eta) = -\varphi(2-\eta). \tag{**}$$

Now, using (*) in (1) gives,

$$\psi(\eta) = -\varphi(-\eta) = -\frac{1}{4} \left(\frac{1}{2} \eta^2 + \frac{1}{3} \eta^3 \right) - \frac{K}{2}.$$
 (3)

This is valid for $0 \le -\eta \le 1 \implies -1 \le \eta \le 0$. So, the domain of ψ has been extended. Similarly, using (**) in (2) gives,

$$\varphi(\eta) = -\psi(2 - \eta) = \frac{1}{4} \left(\frac{1}{2} (2 - \eta)^2 - \frac{1}{3} (2 - \eta)^3 \right) + \frac{K}{2}.$$
 (4)

This is valid for $0 \le 2 - \eta \le 1 \implies 1 \le \eta \le 2$. So, the domain of φ has been extended. We repeat this process to extend the domain further: using (*) in (3) and (**) in (4).