Past Paper: Partial Differential Equations

Midterm Exam, 2024

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Question 1.

Classify the following equation and reduce it to the canonical form

$$u_{xy} + yu_{yy} + \sin(x+y) = 0.$$

Solution. We start by swapping the variables $x \leftrightarrow y$ to get the transformed PDE

$$xu_{xx} + u_{xy} + \sin(x+y) = 0. \tag{$*$}$$

Now, $A = x$, $B = 1$, $C = D = E = 0$. Since, $B^2 - 4AC = 1 > 0$, this is a hyperbolic PDE. So, the corresponding characteristic equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{1 \pm 1}{2x}$$

gives

$$\frac{dy}{dx} = \frac{1}{x} \implies k + y = \ln x \implies xe^{-y} = c_1, \quad \frac{dy}{dx} = 0 \implies y = c_2.$$

Here, k, c_1 and c_2 are constants. These lead to $\xi = xe^{-y} = c_1$ and $\eta = y = c_2$. As a result

$$\begin{aligned}
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = e^{-y}, \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = -e^{-y}, \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0.
\end{aligned}$$

(Also, $A^* = C^* = 0$ because the PDE is hyperbolic.)

As a result, equation ($*$) implies

$$e^{-y}u_{\xi\eta} - e^{-y}u_\xi = \sin(x+y) \implies u_{\xi\eta} - u_\xi = e^\eta \sin(\eta + \xi e^\eta).$$

Here, we used $y = \eta$ and $x = \xi e^y = \xi e^\eta$.

Overall, the canonical form of the given PDE is

$$u_{\xi\eta} = u_\xi + e^\eta \sin(\eta + \xi e^\eta)$$

with $\xi = ye^{-x}$ and $\eta = x$ (in terms of the original variables).

Question 2.

Apply a linear transformation $\xi = x + by$ and $\eta = x + dy$ to transform the Euler equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

into canonical form, where b, d A, B and C are constants.

Solution. We note that this is a second order linear PDE with constant coefficients such that $D = E = 0$. Under the change of variables $\xi = x + by$ and $\eta = x + dy$, these coefficients transform to

$$\begin{aligned}
A^* &= A\xi_x^2 + (2B)\xi_x\xi_y + C\xi_y^2 = A + 2Bb + Cb^2, \\
B^* &= 2A\xi_x\eta_x + (2B)(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 2A + 2B(b+d) + 2Cbd, \\
C^* &= A\eta_x^2 + (2B)\eta_x\eta_y + C\eta_y^2 = A + 2Bd + Cd^2, \\
D^* &= A\xi_{xx} + (2B)\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 0, \\
E^* &= A\eta_{xx} + (2B)\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0.
\end{aligned}$$

The resulting transformed PDE is

$$\left(A + 2Bb + Cb^2\right)u_{\xi\xi} + 2(A + B(b+d) + Cbd)u_{\xi\eta} + \left(A + 2Bd + Cd^2\right)u_{\eta\eta} = 0.$$

The canonical form depends on the solutions of $A^* = 0$; i.e. $A + 2Bb + Cb^2 = 0$. This gives

$$A + 2Bb + Cb^2 \implies b = \frac{-B \pm \sqrt{B^2 - AC}}{C}.$$

If $B^2 > AC$, then let

$$b = \frac{-B + \sqrt{B^2 - AC}}{C} \quad \text{and} \quad d = \frac{-B - \sqrt{B^2 - AC}}{C}.$$

This eliminates A^* and C^* and reduces the transformed PDE to the hyperbolic canonical form $2(A + B(b+d) + Cbd)u_{\xi\eta} = 0 \implies u_{\xi\eta} = 0$.

If $B^2 = AC$, then let $b = -B/C$. We note that

$$A + B(b+d) + Cbd = A - \frac{B^2}{C} + Bd - Bd = A - A = 0.$$

This eliminates A^* and B^* and reduces the transformed PDE to the parabolic canonical form $(A + 2Bd + Cd^2)u_{\eta\eta} = 0 \implies u_{\eta\eta} = 0$.

If $B^2 < AC$, then let

$$b = \frac{-B + i\sqrt{AC - B^2}}{C}, \quad \alpha = \operatorname{Re}\xi = x - \frac{B}{C}, \quad \beta = \operatorname{Im}\xi = \frac{\sqrt{AC - B^2}}{C}.$$

We find that this leads to $u_{\alpha\alpha} + u_{\beta\beta} = 0$, the elliptic canonical form.

Question 3.

Determine the solution of the equation

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the data $x + y = 0$ and $u = 1$.

Solution. The characteristic equations are

$$\frac{dx}{xy^2 + xu} = \frac{dy}{-yx^2 - yu} = \frac{du}{(x^2 - y^2)u} = \frac{xdx + ydy}{(x^2 - y^2)u} = \frac{\frac{dx}{x} + \frac{dy}{y}}{y^2 - x^2}$$

Therefore,

$$\begin{aligned}
\frac{du}{(x^2-y^2)u} = \frac{xdx+ydy}{(x^2-y^2)u} &\implies du = xdx+ydy \\
&\implies u = \frac{1}{2}x^2+\frac{1}{2}y^2+k \\
&\implies 2u-x^2-y^2 = c_1.
\end{aligned}$$

And,

$$\begin{aligned}
\frac{du}{(x^2-y^2)u} = \frac{\frac{dx}{x}+\frac{dy}{y}}{y^2-x^2} &\implies -\frac{du}{u} = \frac{dx}{x}+\frac{dy}{y} \\
&\implies \log|u|+\log|x|+\log|y| = k \\
&\implies uxy = c_2.
\end{aligned}$$

Here, c_1, c_2 and k are constants. The general solution is $f(2u-x^2-y^2, uxy) = 0$ where f is an arbitrary function.

We are given that $x+y=0$ when $u=1$. This means, $y=-x$ when $u=1$. So,

$$\begin{gathered}
c_2 = uxy = x(-x) = -x^2, \\
c_1 = 2u-x^2-y^2 = 2-x^2-x^2 = 2-2x^2 = 2+2c_2.
\end{gathered}$$

Therefore, $c_1 = 2+2c_2 \implies 2u-x^2-y^2 = 2(1+uxy)$, which can be expressed as

$$u = \frac{2+x^2+y^2}{2-2xy}.$$

Question 4.

Show that the general solution of a first-order quasilinear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

is $f(\varphi,\psi) = 0$, where f is an arbitrary function of $\varphi(x,y,u)$ and $\psi(x,y,u)$, and $\varphi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

Solution. By equating the ratios in the characteristic equations to a common parameter

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} = dt$$

we can write $dx = adt$, $dy = bdt$ and $du = cdt$. Now,

$$\begin{aligned}
\phi(x,y,u) = c_1 &\implies d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0 \implies a\phi_x + b\phi_y + c\phi_u = 0, \\
\psi(x,y,u) = c_2 &\implies d\psi = \psi_x dx + \psi_y dy + \psi_u du = 0 \implies a\psi_x + b\psi_y + c\psi_u = 0.
\end{aligned}$$

Simultaneously solving these two equations (for a, b, c) gives

$$\frac{a}{\begin{vmatrix} \phi_y & \phi_u \\ \psi_y & \psi_u \end{vmatrix}} = \frac{-b}{\begin{vmatrix} \phi_x & \phi_u \\ \psi_x & \psi_u \end{vmatrix}} = \frac{c}{\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}} = \lambda.$$

By defining the notation $\frac{\partial(\phi,\psi)}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}$ these ratios can be rewritten as

$$\frac{a}{\dfrac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\dfrac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\dfrac{\partial(\phi,\psi)}{\partial(x,y)}} = \lambda.$$

Here we used $-\begin{vmatrix} \phi_u & \phi_x \\ \psi_u & \psi_x \end{vmatrix} = \begin{vmatrix} \phi_x & \phi_u \\ \psi_x & \psi_u \end{vmatrix}$. Consequently,

$$\frac{\partial(\phi,\psi)}{\partial(y,u)} = \frac{a}{\lambda}, \quad \frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{b}{\lambda}, \quad \frac{\partial(\phi,\psi)}{\partial(x,y)} = \frac{c}{\lambda}.$$

Using the expressions found above, we get

$$au_x + bu_y = c \iff u_x \frac{\partial(\phi,\psi)}{\partial(y,u)} + u_y \frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}.$$

Now, since $f(\phi,\psi) = 0$ satisfies the PDE

$$u_x \frac{\partial(\phi,\psi)}{\partial(y,u)} + u_y \frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)},$$

we conclude that $f(\phi,\psi) = 0$ satisfies the PDE $au_x + bu_y = c$.

Question 5.

Verify that the function

$$u = \varphi(xy) + x\psi\left(\frac{y}{x}\right)$$

is the general solution of the equation

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

Solution. We have

$$\begin{aligned}
u_x &= y\varphi'(xy) + \psi\left(\frac{y}{x}\right) - x\left(\frac{y}{x^2}\right)\psi'\left(\frac{y}{x}\right) = y\varphi'(xy) + \psi\left(\frac{y}{x}\right) - \frac{y}{x}\psi'\left(\frac{y}{x}\right) \\
u_y &= x\varphi'(xy) + x\left(\frac{1}{x}\right)\psi'\left(\frac{y}{x}\right) = x\varphi'(xy) + \psi'\left(\frac{y}{x}\right) \\
u_{xx} &= y^2\varphi''(xy) - \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) + \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) + \frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right) = y^2\varphi''(xy) + \frac{y^2}{x^3}\psi''\left(\frac{y}{x}\right) \\
u_{yy} &= x^2\varphi''(xy) + \frac{1}{x}\psi''\left(\frac{y}{x}\right)
\end{aligned}$$

Therefore,

$$x^2 u_{xx} - y^2 u_{yy} = x^2y^2\varphi''(xy) + \frac{y^2}{x}\psi''\left(\frac{y}{x}\right) - x^2y^2\varphi''(xy) + \frac{y^2}{x}\psi''\left(\frac{y}{x}\right) = 0$$

Question 6.

Determine the solution of the initial boundary value problem

$$\begin{aligned} u_{tt} &= 4u_{xx}, && 0 < x < 1, \quad t > 0, \\ u(x,0) &= 0, && 0 \le x \le 1, \\ u_t(x,0) &= x(1-x), && 0 \le x \le 1, \\ u(0,t) &= 0,\ u(1,t) = 0, && t \ge 0. \end{aligned}$$

Solution. This is the wave equation ($c = 2$) on a bounded spatial interval $[0,1]$. The general solution is $u(x,t) = \varphi(x+2t) + \psi(x-2t)$ with

$$\varphi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^\eta g(\tau)\,d\tau + \frac{K}{2}, \quad \psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^\eta g(\tau)\,d\tau - \frac{K}{2}.$$

for $0 \le \eta \le 1$. Using $f(\eta) = 0$ and $g(\eta) = \eta(1-\eta)$ gives,

$$\varphi(\eta) = \frac{1}{4}\left(\frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right) + \frac{K}{2}, \quad 0 \le \eta \le 1, \tag{1}$$

$$\psi(\eta) = -\frac{1}{4}\left(\frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right) - \frac{K}{2}, \quad 0 \le \eta \le 1. \tag{2}$$

The boundary conditions give

$$u(0,t) = 0 \implies \psi(-2t) = -\varphi(2t) \implies \psi(\eta) = -\varphi(-\eta), \tag{$*$}$$

$$u(1,t) = 0 \implies \varphi(1+2t) = -\psi(1-2t) \implies \varphi(\eta) = -\psi(2-\eta). \tag{$**$}$$

Now, using ($*$) in (1) gives,

$$\psi(\eta) = -\varphi(-\eta) = -\frac{1}{4}\left(\frac{1}{2}\eta^2 + \frac{1}{3}\eta^3\right) - \frac{K}{2}. \tag{3}$$

This is valid for $0 \le -\eta \le 1 \implies -1 \le \eta \le 0$. So, the domain of ψ has been extended.

Similarly, using ($**$) in (2) gives,

$$\varphi(\eta) = -\psi(2-\eta) = \frac{1}{4}\left(\frac{1}{2}(2-\eta)^2 - \frac{1}{3}(2-\eta)^3\right) + \frac{K}{2}. \tag{4}$$

This is valid for $0 \le 2-\eta \le 1 \implies 1 \le \eta \le 2$. So, the domain of φ has been extended.

We repeat this process to extend the domain further: using ($*$) in (3) and ($**$) in (4).