Past Paper: Group Theory I

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Question 1 (i).

Prove that the order of a cyclic group is equal to the order of its generator.

Solution. Let G be a cyclic group generated by a; $G = \langle a \rangle$.

If $O(a)$ is infinite then a^n and a^m are distinct for all $n \neq m$, because otherwise,

$$a^n = a^m \implies a^{n-m} = e \implies O(a) \leq |n - m|$$

which is a contradiction because $O(a) = +\infty$. Now, since each a^n is different from a^{n+1}, we have infinitely many elements in the set $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$. Consequently, $G = \langle a \rangle$ has infinitely many elements; i.e. $|G|$ is also infinite.

Next, consider the case when $O(a) = n$ is finite. By definition $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$. We can write $k = qn + r$ for $q, r \in \mathbb{Z}$ and $0 \leq r < n$. So, for all $k \in \mathbb{Z}$,

$$a^k = a^{qn+r} = a^{qn}a^r = (a^n)^q = ea^r = a^r.$$

Therefore, $\langle a \rangle = \{a^r \mid 0 \leq r < n\} = \{e, a, a^2, \ldots, a^{n-1}\}$. Again, take any $a^p, a^q \in \langle a \rangle$ with $p \neq q$, then

$$a^p = a^q \implies a^{p-q} = e \implies O(a) \leq |p - q| \leq n - 1$$

which is not possible because $O(a) = n$. Therefore, each element in this set is distinct, and we get $|G| = |\langle a \rangle| = n$.

Question 1 (ii).

Let $G = \langle a \rangle$ be a finite cyclic group of order n. Then prove that an element a^k is a generator if and only if $\gcd(n, k) = 1$.

Solution. Suppose a^k is a generator of G. Then, we can write a as a power of a^k, say, $a = (a^k)^m = a^{km}$ for some $m \in \mathbb{Z}$.

Then, $a = a^{km} \implies aa^{-km} = e \implies a^{1-km} = e$. So, $n \mid 1 - km$. That is $\exists q \in \mathbb{Z}$ such that $1 - km = qn$. We can re-arrange this to get $qn + km = 1$. From number theory, this implies $\gcd(n, k) = 1$ (Bezout's lemma).

Conversely, suppose $\gcd(n, k) = 1$. Then, there exist integers x, y such that $xk + yn = 1$. So, $a = a^1 = a^{xk+yn} = a^{xk}a^{yn} = (a^k)^x(a^n)^y = (a^k)^x e^y = (a^k)^x$.

Now, for all $b \in G = \langle a \rangle$, we have $b = a^r$ for some $r \in Z$. Therefore, we can write it as a power of a^k as $b = a^r = ((a^k)^x)^r = (a^k)^{xr}$. So, a^k also generates G.

Question 2 (i).

Let H be a subgroup of G. Let $\sim$ be a relation on G defined by $a \sim b$ if and only if $a^{-1}b \in H$. Then show that $\sim$ is an equivalence relation on G. Also prove $[a] = aH$.

Solution. We check the three conditions for being an equivalence relation.

(Reflexive.) Take $a \in G$. Then, $a^{-1}a = e$ and $e \in H$ because $H \leq G$. So, $a \sim a$.

(Symmetric.) If $a \sim b$, then $a^{-1}b \in H$. As $H \leq G$, the element $\left(a^{-1}b\right)^{-1} \in H$. And $\left(a^{-1}b\right)^{-1} = b^{-1}(a^{-1})^{-1} = b^{-1}a$. That is, $b^{-1}a \in H$. So, $b \sim a$.

(Transitive.) If $a \sim b$ and $b \sim c$, then $a^{-1}b \in H$ and $b^{-1}c \in H$. As $H \leq G$, closure and associativity gives $\left(a^{-1}b\right)\left(b^{-1}c\right) = a^{-1}(bb^{-1})c = a^{-1}c \in H$. So, $a \sim c$.

Next, $[a] = \{b \in G \,|\, b \sim a\} = \{b \in G \,|\, a \sim b\} = \{b \in G \,|\, a^{-1}b \in H\}$. And, $a^{-1}b \in H$ means $a^{-1}b = h$ for some $h \in H$. As a result, $b = ah$ and we can write

$$[a] = \{b \in G \,|\, a^{-1}b \in H\} = \{ah \in G \,|\, h \in H\} = aH.$$

Question 2 (ii).

Let G be a group and H be a subgroup of G. If $a, b \in G$, then prove that $aH = bH$ if and only if $a^{-1}b \in H$.

Solution. Suppose $aH = bH$. As $H \leq G$, we have $e \in H$. So, $b = be \in bH = aH$. Thus, we can write $b = ah$ for some $h \in H$. Consequently, $a^{-1}b = h \in H$.

Conversely, let $a^{-1}b \in H$. This means $a^{-1}b = h$ for some $h \in H$. As a result, $b = ah$. Therefore,

$$bH = \{bk \,|\, k \in H\} = \{(ah)k \,|\, k \in H\} = \{ah' \,|\, h' \in H\} = aH.$$

Here we used the associativity and closure property because $H \leq G$.

Question 2 (iii).

Let G be a group and $a \in G$ such that $O(a) = n$. If m is an integer such that $a^m = e$, then prove that n divides m.

Solution. By the division algorithm, $m = qn + r$ for $q, r \in \mathbb{Z}$ and $0 \leq r < n$. So,

$$a^m = a^{nq+r} = (a^{nq})(a^r) = ((a^n)^q)(a^r) = (e^q)(a^r) = e(a^r) = a^r.$$

Now, $a^m = e \implies a^r = e$. However, $r < n$ and n is the least positive integer for which $a^n = e$. Therefore, r must be zero (if it was positive then it would contradict the minimality of n). Therefore, $m = qn$. That is $n \mid m$.

Question 3 (i).

Let $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$ be the group of integers modulo 12. Then find the order of each element of $\mathbb{Z}_{12}$.

Solution. The order of each element in $\mathbb{Z}_{12}$ is

$O(0) = 1$	$O(1) = 12$	$O(2) = 6$	$O(3) = 4$
$O(4) = 3$	$O(5) = 12$	$O(6) = 2$	$O(7) = 12$
$O(8) = 3$	$O(9) = 4$	$O(10) = 6$	$O(11) = 12$

Question 3 (ii).

Find all subgroups of the cyclic group $C_{12} = \{1, a, a^2, \ldots, a^{11} \,|\, a^{12} = 1\}$.

Solution. We arrange these by the GCD of $n = 12$ and the order of the generator.

$$\begin{aligned}
&\gcd = 1: && \langle 1 \rangle = \{1\} \\
&\gcd = 2: && \langle a^6 \rangle = \{1, a^6\} \\
&\gcd = 3: && \langle a^4 \rangle = \langle a^8 \rangle = \{1, a^4, a^8\} \\
&\gcd = 4: && \langle a^3 \rangle = \langle a^9 \rangle = \{1, a^3, a^6, a^9\} \\
&\gcd = 6: && \langle a^2 \rangle = \langle a^{10} \rangle = \{1, a^2, a^4, a^6, a^8, a^{10}\} \\
&\gcd = 12: && \langle a \rangle = \langle a^{11} \rangle = \langle a^5 \rangle = \langle a^7 \rangle = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}\}
\end{aligned}$$

Question 3 (iii).

Let $G = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{Z}\}$ be a group under addition of matrices, and take $H = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a + b + c + d = 1 \in \mathbb{Z}\}$. Prove or disprove that H is a subgroup of G.

Solution. H is not a subgroup of G because it doesn't contain the identity element. That is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \notin H$ since $0 + 0 + 0 + 0 \neq 1$.

Question 4 (i).

Let $*$ be a binary operation on $\mathbb{Q}^+$ defined by $a * b = \frac{ab}{2}$. Then show that $\mathbb{Q}^+$ is an Abelian group with respect to $*$. ($\mathbb{Q}^+$ denotes the set of positive rational numbers.)

Solution. We check the group axioms.

(Closure.) Take $a, b \in \mathbb{Q}^+$. In particular, $a, b \in \mathbb{Q}$ and $a, b > 0$. So, $ab \in \mathbb{Q}$ and $\frac{ab}{2} \in \mathbb{Q}$. Moreover, $ab > 0$ and $\frac{ab}{2} > 0$. That means $\frac{ab}{2} \in \mathbb{Q}^+$. Therefore, $a * b = \frac{ab}{2} \in \mathbb{Q}^+$.

(Associativity.) Take $a, b, c \in Q^+$. Then,

$$\begin{aligned}
a * (b * c) &= a * \left(\frac{bc}{2}\right) = \frac{a(\frac{bc}{2})}{2} = \frac{abc}{4}, \\
(a * b) * c &= \left(\frac{ab}{2}\right) * c = \frac{(\frac{ab}{2})c}{2} = \frac{abc}{4}.
\end{aligned}$$

As a result, $a * (b * c) = (a * b) * c$.

(Identity.) The element $e = 2 \in \mathbb{Q}^+$ serves as the identity element because

$$a * 2 = \frac{(a)(2)}{2} = a \quad \text{and} \quad 2 * a = \frac{(2)(a)}{2} = a.$$

(Inverse.) For any $a \in \mathbb{Q}^+$ take $a^{-1} = 4/a$. Then, $a^{-1} \in \mathbb{Q}^+$ because $a > 0$, and

$$a * \frac{4}{a} = \frac{(a)(\frac{4}{a})}{2} = 2 = e \quad \text{and} \quad \frac{4}{a} * a = \frac{(\frac{4}{a})(a)}{2} = 2 = e.$$

(Commutativity.) For any $a, b \in \mathbb{Q}^+$, $a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$ because the usual multiplication of rational numbers is commutative, $ab = ba$.

Question 4 (ii).

Find all cyclic subgroups of $D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$, $a^4 = b^2 = 1$, $ab = ba^3$.

Solution. The cyclic subgroups of D_4 are

$$\langle 1 \rangle = \{1\}, \qquad \langle a \rangle = \langle a^3 \rangle = \{1, a, a^2, a^3\}, \qquad \langle a^2 \rangle = \{1, a^2\},$$
$$\langle b \rangle = \{1, b\}, \qquad \langle ba \rangle = \{1, ba\}, \qquad \langle ba^2 \rangle = \{1, ba^2\}, \qquad \langle ba^3 \rangle = \{1, ba^3\}.$$